

# UNILATERAL GRADIENT FLOW OF THE AMBROSIO-TORTORELLI FUNCTIONAL BY MINIMIZING MOVEMENTS

JEAN-FRANÇOIS BABADJIAN

*Université Pierre et Marie Curie - Paris 6  
CNRS, UMR 7598 Laboratoire J.-L. Lions  
Paris, France*

`babadjian@ann.jussieu.fr`

VINCENT MILLOT

*Université Paris Diderot - Paris 7  
CNRS, UMR 7598 Laboratoire J.-L. Lions  
Paris, France*

`millot@ljl1.univ-paris-diderot.fr`

**Abstract.** Motivated by models of fracture mechanics, this paper is devoted to the analysis of unilateral gradient flows of the Ambrosio-Tortorelli functional, where unilaterality comes from an irreversibility constraint on the fracture density. In the spirit of gradient flows in metric spaces, such evolutions are defined in terms of curves of maximal unilateral slope, and are constructed by means of implicit Euler schemes. An asymptotic analysis in the Mumford-Shah regime is also carried out. It shows the convergence towards a generalized heat equation outside a time increasing crack set.

*Keywords:* Gradient flow,  $\Gamma$ -convergence, free discontinuity problems, functions of bounded variation, Mumford-Shah

## 1. Introduction

Many free discontinuity problems are variational in nature and involve two unknowns, a function  $u$  and a discontinuity set  $\Gamma$  across which  $u$  may jump. The most famous example is certainly the minimization of the MUMFORD-SHAH (MS) functional introduced in [36] to approach image segmentation. It is defined by

$$\frac{1}{2} \int_{\Omega \setminus \Gamma} |\nabla u|^2 dx + \mathcal{H}^{N-1}(\Gamma) + \frac{\beta}{2} \int_{\Omega} (u - g)^2 dx,$$

where  $\Omega \subset \mathbb{R}^N$  is a bounded open set,  $\mathcal{H}^{N-1}$  is the  $(N - 1)$ -dimensional Hausdorff measure,  $\beta > 0$  is a fidelity (constant) factor, and  $g \in L^\infty(\Omega)$  stands for the grey level of the original image. In the resulting minimization process, we end up with a segmented image  $u : \Omega \setminus \Gamma \rightarrow \mathbb{R}$  and a set of contours  $\Gamma \subset \Omega$ . To efficiently tackle this problem, a weak formulation in the space of *Special functions of Bounded Variation* has been suggested and solved in [21], where the set  $\Gamma$  is replaced by the jump set  $J_u$  of  $u$ . The new energy is defined for  $u \in SBV^2(\Omega)$  by

$$\frac{1}{2} \int_{\Omega} |\nabla u|^2 dx + \mathcal{H}^{N-1}(J_u) + \frac{\beta}{2} \int_{\Omega} (u - g)^2 dx, \quad (1.1)$$

where  $\nabla u$  is now intended to be the measure theoretic gradient of  $u$ .

A related model based on Mumford-Shah type functionals has been introduced by FRANCFORT & MARIGO in [26] (see also [8]) to describe quasi-static crack propagation inside elastic bodies. It is a variational model relying on three fundamental principles:

- the fractured body must stay in elastic equilibrium at each time (*quasi-static hypothesis*);
- the crack can only grow (*irreversibility constraint*);
- an energy balance holds.

In the anti-plane setting, the equilibrium and irreversibility principles lead us to look for constrained critical points (or local minimizers) at each time of the Mumford-Shah functional, where  $u$  stands now for the scalar displacement while  $\Gamma$  is the crack. Unfortunately, there is no canonical notion of local minimality since the family of all admissible cracks is not endowed with a natural topology. The research of local minimizers of such energies has consequently become a great challenge, and a lot of works in that direction have considered

global minimizers instead, see [17,14,24]. In the discrete setting, one looks at each time step for a pair  $(u_i, \Gamma_i)$  minimizing

$$(u, \Gamma) \mapsto \frac{1}{2} \int_{\Omega \setminus \Gamma} |\nabla u|^2 dx + \mathcal{H}^{N-1}(\Gamma),$$

among all cracks  $\Gamma \supset \Gamma_{i-1}$  and all displacements  $u : \Omega \setminus \Gamma \rightarrow \mathbb{R}$  satisfying an updated boundary condition, where  $\Gamma_{i-1}$  is the crack found at the previous time step. A first attempt to local minimization has been carried out in [18] where a variant of this model is considered. At each time step the  $L^2(\Omega)$ -distance to the previous displacement is penalized. More precisely, denoting by  $u_{i-1}$  the displacement at the previous time step, one looks for minimizers of

$$(u, \Gamma) \mapsto \frac{1}{2} \int_{\Omega \setminus \Gamma} |\nabla u|^2 dx + \mathcal{H}^{N-1}(\Gamma) + \lambda \|u - u_{i-1}\|_{L^2(\Omega)}^2, \quad (1.2)$$

on the same class of competitors than before, where  $\lambda > 0$  is a fixed parameter. We emphasize that this formulation only involves some kind of local minimality with respect to the displacement. A notion of stability which implies another local minimality criterion has been introduced in [32]. It focuses on what the author calls “accessibility between two states”. In the case of global minimization, when passing from one discrete time to the next, all states are accessible. From the point of view of [18], a state  $u$  is accessible from  $u_{i-1}$  if and only if there is a certain gradient flow beginning at  $u_{i-1}$  which approaches  $u$  in the long-time limit. The main idea in [32] is that a state  $u$  is accessible from  $u_{i-1}$  if and only if both states can be connected through a continuous path for which the total energy is never increased more than a fixed amount.

While static free discontinuity problems start to be well understood, many questions remain open concerning their evolutionary version. Apart from the quasi-static case, the closest evolution problem to statics consists in finding a steepest gradient descent of the energy, and thus in solving a gradient flow type equation. A major difficulty in this setting is to define a suitable notion of gradient since the functional is neither regular nor convex, and standard theories such as maximal monotone operators [11] do not apply. To overcome the use of differential or subdifferential, a general theory of gradient flows in metric spaces has been introduced in [22] where the notion of gradient is replaced by the notion of slope, and the standard gradient flow equation is recast in terms of curves of maximal slope (see [4] for a detailed description of this subject). The construction of such generalized evolution is usually performed by means of an implicit Euler scheme (corresponding to a discrete-in-time steepest descent) whose limits are referred to as DE GIORGI’s *minimizing movements*, see [1,20]. The minimizing movements of the Mumford-Shah functional have been first considered in [2], and further developed in [12]. Motivated by the crack growth model, the authors actually apply the iterative scheme with respect to the variable  $u$  while minimizing the energy with respect to  $\Gamma$  under the constraint of irreversibility. More precisely, they minimize energy (1.2) exactly as in [18] with  $\lambda$  replaced by  $(2\delta)^{-1}$ , where  $\delta > 0$  is the time step. Showing compactness of the resulting discrete evolution as  $\delta \rightarrow 0$ , they obtain existence of “unilateral” minimizing movements of the Mumford-Shah energy (we add here the adjective unilateral to underline the irreversibility constraint on the evolution). In any space dimension, the limiting displacement  $u(t)$  satisfies some kind of heat equation (in a very weak sense), and an energy inequality with respect to the initial time holds. Assuming that admissible cracks are compact and connected, they improve the result in two dimensions showing that  $u(t)$  solves a true heat equation in a fractured space-time domain, and that the energy inequality holds between arbitrary times. With the (probable) aim of relating the unilateral minimizing movements of [2,12] to some generalized gradient flow, a notion of unilateral slope for the Mumford-Shah functional has been introduced in [19]. Up to our knowledge, no precise relation between these objects has been found yet.

The Mumford-Shah functional enjoys good variational approximation properties by means of regular energies. Constructing gradient flows for these regularized energies and taking the limit in the approximation parameter could be a way to derive a generalized gradient flow for MS. It was actually the path followed in [30] where a gradient flow equation for the one-dimensional Mumford-Shah functional is obtained as a limit of ordinary differential equations derived from a non-local approximation of MS. Many other approximations are available, and the most famous one is certainly the AMBROSIO-TORTORELLI functional defined for  $(u, \rho) \in [H^1(\Omega)]^2$  by

$$AT_\varepsilon(u, \rho) := \frac{1}{2} \int_{\Omega} (\eta_\varepsilon + \rho^2) |\nabla u|^2 dx + \frac{1}{2} \int_{\Omega} \left( \varepsilon |\nabla \rho|^2 + \frac{1}{\varepsilon} (1 - \rho)^2 \right) dx + \frac{\beta}{2} \int_{\Omega} (u - g)^2 dx.$$

The idea is to replace the discontinuity set  $\Gamma$  by a (diffuse) phase field variable, denoted by  $\rho : \Omega \rightarrow [0, 1]$ , which is “smooth” and identically 0 in a  $\varepsilon$ -neighborhood of  $\Gamma$ . Such energies are of great importance for numerical

simulations in imaging or brittle fracture, see [7,8]. From the mechanical point of view, it is interpreted as a non-local damage approximation of fracture models, where  $\rho$  represents a fracture density. The approximation result of [5,6] (see also [28]) states that  $AT_\varepsilon$   $\Gamma$ -converges as  $\varepsilon \rightarrow 0$  to MS (in the form (1.1)) with respect to a suitable topology. For the static problem, it implies the convergence of  $AT_\varepsilon$ -minimizers towards MS-minimizers by standard results from  $\Gamma$ -convergence theory. However, the convergence of general critical points is a priori not guaranteed. Positive results in this direction have been obtained in [25,34] for the one-dimensional case. Note that it is also of numerical interest to investigate the behavior of general critical points of  $AT_\varepsilon$ . Indeed, while existence of minimizers is easy to prove, it is not straightforward to compute them numerically since the functional  $AT_\varepsilon$  is not globally convex (but only separately convex). The algorithm used in [8] consists in performing alternate minimizations in each variable, and it only provides critical points of  $AT_\varepsilon$ .

The Ambrosio-Tortorelli approximation of quasi-static crack evolution is considered in [29], where the irreversibility constraint translates into the decrease monotonicity of the phase field  $t \mapsto \rho(t)$ . The main result of [29] concerns the convergence of this regularized model towards the original one in [24]. Motivated by the formulation of a model of fracture dynamics, a hyperbolic evolution related to the Ambrosio-Tortorelli functional is also studied in [33], but the asymptotic behavior of solutions as  $\varepsilon \rightarrow 0$  is left open. A first step in that direction is made in [15] where the analysis of a wave equation on a domain with growing cracks is performed. Concerning parabolic type evolutions, a full gradient flow of the Ambrosio-Tortorelli functional is numerically investigated in [27] for the purposes of image segmentation and inpainting.

The object of the present article is to present a new notion of *unilateral gradient flow* for the Ambrosio-Tortorelli functional taking into account the irreversibility constraint on the phase field variable. Motivated by [19], we adopt the general framework of gradient flows in metric spaces [4], defined here as *curves of maximal unilateral slope* (see Definition 3.7). Existence is obtained through the minimizing movements method, where the discrete Euler scheme is precisely the Ambrosio-Tortorelli regularization of the one studied in [2,12]. As in [29], the irreversibility of the process is encoded into the decrease monotonicity of the phase field variable, and leads to constrained minimization problems. At the discrete time level, given an initial data  $(u_0, \rho_0)$ , one recursively defines pairs  $(u_i, \rho_i)$  by minimizing at each time  $t_i \sim i\delta$ ,

$$(u, \rho) \mapsto AT_\varepsilon(u, \rho) + \frac{1}{2\delta} \|u - u_{i-1}\|_{L^2(\Omega)}^2, \quad (1.3)$$

among all  $u$  and  $\rho \leq \rho_{i-1}$ , where  $(u_{i-1}, \rho_{i-1})$  is a pair found at the previous time step. The objective is then to pass to the limit as the time step  $\delta$  tends to 0. A main difficulty is to deal with the asymptotics of the obstacle problems in the  $\rho$  variable. It is known that such problems are not stable with respect to weak  $H^1(\Omega)$ -convergence, and that “strange terms” of capacitary type may appear [13,16]. However, having uniform convergence of obstacles would be enough to rule out this situation. For that reason, instead of  $AT_\varepsilon$ , we consider a modified Ambrosio-Tortorelli functional with  $p$ -growth in  $\nabla\rho$  with  $p > N$ . By the Sobolev Imbedding Theorem, with such a functional in hand, uniform convergence on the  $\rho$  variable is now ensured. We define for every  $(u, \rho) \in H^1(\Omega) \times W^{1,p}(\Omega)$ ,

$$\mathcal{E}_\varepsilon(u, \rho) := \frac{1}{2} \int_\Omega (\eta_\varepsilon + \rho^2) |\nabla u|^2 dx + \int_\Omega \left( \frac{\varepsilon^{p-1}}{p} |\nabla \rho|^p + \frac{\alpha}{p'\varepsilon} |1 - \rho|^p \right) dx + \frac{\beta}{2} \int_\Omega (u - g)^2 dx,$$

where  $\alpha > 0$  is a suitable normalizing factor defined in (2.2). Note that an immediate adaptation of [28] shows that  $\mathcal{E}_\varepsilon$  is still an approximation of MS in the sense of  $\Gamma$ -convergence.

Considering the incremental scheme (1.3) with  $\mathcal{E}_\varepsilon$  instead of  $AT_\varepsilon$ , we prove that the discrete evolutions strongly converge as  $\delta \rightarrow 0$  to continuous evolutions  $t \mapsto (u_\varepsilon(t), \rho_\varepsilon(t))$  that we call *unilateral minimizing movements* (see Definition 3.4). These limiting evolutions turn out to be  $L^2(\Omega)$ -steepest descents of  $\mathcal{E}_\varepsilon$  with respect to  $u$  in the direction of non-increasing  $\rho$ 's, i.e., curves of maximal unilateral slope in the spirit of [4,19] (see Theorems 5.1 & 5.2). The Euler-Lagrange equation for  $u_\varepsilon$  is given by

$$\begin{cases} \partial_t u_\varepsilon - \operatorname{div}((\eta_\varepsilon + \rho_\varepsilon^2) \nabla u_\varepsilon) + \beta(u_\varepsilon - g) = 0 & \text{in } \Omega \times (0, +\infty), \\ \frac{\partial u_\varepsilon}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, +\infty), \end{cases} \quad (1.4)$$

while the irreversibility and minimality conditions for  $\rho_\varepsilon$  are

$$\begin{cases} t \mapsto \rho_\varepsilon(t) \text{ is non-increasing,} \\ \mathcal{E}_\varepsilon(u_\varepsilon(t), \rho_\varepsilon(t)) \leq \mathcal{E}_\varepsilon(u_\varepsilon(t), \rho) \text{ for every } t \geq 0 \text{ and } \rho \in W^{1,p}(\Omega) \text{ such that } \rho \leq \rho_\varepsilon(t) \text{ in } \Omega. \end{cases} \quad (1.5)$$

The system (1.4)-(1.5) is supplemented with the initial condition

$$(u_\varepsilon(0), \rho_\varepsilon(0)) = (u_0, \rho_0) \quad \text{in } \Omega.$$

In addition, we prove that the bulk and diffuse surface energies, defined by

$$t \mapsto \frac{1}{2} \int_{\Omega} (\eta_\varepsilon + \rho_\varepsilon(t)^2) |\nabla u_\varepsilon(t)|^2 dx$$

and

$$t \mapsto \int_{\Omega} \left( \frac{\varepsilon^{p-1}}{p} |\nabla \rho_\varepsilon(t)|^p + \frac{\alpha}{p' \varepsilon} |1 - \rho_\varepsilon(t)|^p \right) dx$$

are respectively non-increasing and non-decreasing, a fact which is meaningful from the mechanical point of view. Moreover, the total energy is non-increasing, and it satisfies the following inequality: for a.e.  $s \in [0, +\infty)$  and every  $t \geq s$ ,

$$\mathcal{E}_\varepsilon(u_\varepsilon(t), \rho_\varepsilon(t)) + \int_s^t \|\partial_t u_\varepsilon(r)\|_{L^2(\Omega)}^2 dr \leq \mathcal{E}_\varepsilon(u_\varepsilon(s), \rho_\varepsilon(s)).$$

Note that the inequality above is reminiscent of gradient flow type equations, and that it usually reduces to equality whenever the flow is regular enough. In any case, an energy equality would be equivalent to the absolute continuity in time of the total energy. The reverse inequality might be obtained through an abstract infinite-dimensional chain-rule formula in the spirit of [37]. In our case, if we formally differentiate in time the total energy, we obtain

$$\frac{d}{dt} \mathcal{E}_\varepsilon(u_\varepsilon(t), \rho_\varepsilon(t)) = \langle \partial_u \mathcal{E}_\varepsilon(u_\varepsilon(t), \rho_\varepsilon(t)), \partial_t u_\varepsilon(t) \rangle + \langle \partial_\rho \mathcal{E}_\varepsilon(u_\varepsilon(t), \rho_\varepsilon(t)), \partial_t \rho_\varepsilon(t) \rangle.$$

From (1.5) we could expect that

$$\langle \partial_\rho \mathcal{E}_\varepsilon(u_\varepsilon(t), \rho_\varepsilon(t)), \partial_t \rho_\varepsilon(t) \rangle = 0, \quad (1.6)$$

which would lead, together with (1.4), to the energy equality. Now observe that (1.6) is precisely the regularized version of Griffith's criterion stating that a crack evolves if and only if the release of bulk energy is compensated by the increase of surface energy (see e.g. [8, Section 2.1]). Unfortunately, the chain-rule above is not available since we do not have enough control on the time regularity of  $\rho_\varepsilon$ . In the quasi-static case, one observes discontinuous time evolutions for the surface energy. Since the evolution law for  $\rho_\varepsilon$  is quite similar to the quasi-static case (see [29]), we also expect here time discontinuities for the diffuse surface energy. Adding a parabolic regularization in  $\rho$  could be a way to improve the time regularity, but unfortunately, it would also break the increase monotonicity of the surface energy.

A natural continuation to the qualitative analysis of Ambrosio-Tortorelli minimizing movements is now to understand their limiting behavior as  $\varepsilon \rightarrow 0$ . We stress that the general theory on  $\Gamma$ -convergence of gradient flows as presented in [38,39] does not apply here since it requires a well defined gradient structure for the  $\Gamma$ -limit. A specific analysis thus seems to be necessary. In doing so, we show that  $(u_\varepsilon, \rho_\varepsilon)$  tends to  $(u_*, 1)$  for some mapping  $t \mapsto u_*(t)$  taking values in  $SBV^2(\Omega)$ , and solving (in a weak sense) the equation

$$\begin{cases} \partial_t u_* - \operatorname{div}(\nabla u_*) + \beta(u_* - g) = 0 & \text{in } \Omega \times (0, +\infty), \\ \nabla u_* \cdot \nu = 0 & \text{on } \partial\Omega \times (0, +\infty), \\ u_*(0) = u_0. \end{cases} \quad (1.7)$$

In addition, using the monotonicity of the diffuse surface energy, we are able to pass to the limit in (1.5). It yields the existence of a non-decreasing family of rectifiable subsets  $\{\Gamma(t)\}_{t \geq 0}$  of  $\bar{\Omega}$  such that  $J_{u_*(t)} \tilde{\subset} \Gamma(t)$  for every  $t \geq 0$ , and for which the following energy inequality holds at any time:

$$\begin{aligned} \frac{1}{2} \int_{\Omega} |\nabla u_*(t)|^2 dx + \mathcal{H}^{N-1}(\Gamma(t)) + \frac{\beta}{2} \int_{\Omega} (u_*(t) - g)^2 dx + \int_0^t \|u'_*(s)\|_{L^2(\Omega)}^2 ds \\ \leq \frac{1}{2} \int_{\Omega} |\nabla u_0|^2 dx + \frac{\beta}{2} \int_{\Omega} (u_0 - g)^2 dx. \end{aligned}$$

Comparing our result with [12], we find that  $u_*$  solves the same generalized heat equation with an improvement in the energy inequality where an increasing family of cracks appears. The optimality of this inequality and the

convergence of energies remain open problems. Note that the convergence of the bulk energy usually follows by taking the solution as test function in the equation. In our case it asks the question whether  $SBV^2(\Omega)$  functions whose jump set is contained in  $J_{u_*(t)}$  can be used in the variational formulation of (1.7). It would yield a weak form of the relation

$$((u_*^+(t) - u_*^-(t)) \frac{\partial u_*(t)}{\partial \nu} = 0 \quad \text{on } \Gamma(t),$$

where  $u_*^\pm(t)$  denote the one-sided traces of  $u_*(t)$  on  $\Gamma(t)$ . This is indeed the missing equation to complement (1.7), and it is intimately related to the finiteness of the unilateral slope of the Mumford-Shah functional evaluated at  $(u_*(t), \Gamma(t))$  (see [19, Proposition 1.1], and Section 6.3).

The paper is organized as follows. In Section 2, we define in details the Ambrosio-Tortorelli and Mumford-Shah functionals as well as the functional setting of the problem. In the first part of Section 3, we introduce two implicit Euler schemes generating two different types of unilateral minimizing movements. The first one is based on global minimization, while the second one relies on alternate minimization as in [7,8]. In the second part of Section 3, we define and analyse the notions of unilateral slope and curve of maximal unilateral slope for the Ambrosio-Tortorelli functional. In Section 4, we establish compactness of discrete evolutions when the time step tends to zero. In order to get sharp energy inequalities, we are led to extend the concept of De Giorgi's interpolation to our unilateral setting. Then we study in Section 5 some qualitative properties of unilateral minimizing movements, showing in particular that they are curves of maximal unilateral slope. Finally, Section 6 is devoted to the asymptotic analysis as  $\varepsilon \rightarrow 0$ , where we show that unilateral minimizing movements of the Ambrosio-Tortorelli functional converge to solutions of the generalized heat equation (1.7) outside time-increasing cracks.

**Notations.** For an open set  $U \subset \mathbb{R}^N$ , we denote by  $\mathcal{M}(U; \mathbb{R}^m)$  the space of all finite  $\mathbb{R}^m$ -valued Radon measures on  $U$ , i.e., the topological dual of the space  $\mathcal{C}_0(U; \mathbb{R}^m)$  of all  $\mathbb{R}^m$ -valued continuous functions vanishing on  $\partial U$ . For  $m = 1$  we simply write  $\mathcal{M}(U)$ . The Lebesgue measure in  $\mathbb{R}^N$  is denoted by  $\mathcal{L}^N$ , while  $\mathcal{H}^{N-1}$  stands for the  $(N-1)$ -dimensional Hausdorff measure. If  $B_1$  is the open unit ball in  $\mathbb{R}^N$ , we write  $\omega_N := \mathcal{L}^N(B_1)$ . We use the notations  $\tilde{\subset}$  and  $\tilde{=}$  for inclusions or equalities between sets up to  $\mathcal{H}^{N-1}$ -negligible sets. For two real numbers  $a$  and  $b$ , we denote by  $a \wedge b$  and  $a \vee b$  the minimum and maximum value between  $a$  and  $b$ , respectively, and  $a^+ := a \vee 0$ .

## 2. Preliminaries

**Absolutely continuous functions.** Throughout the paper, we consider the integration theory for Banach space valued functions in the sense of Bochner. All standard definitions and results we shall use can be found in [11, Appendix] (see also [23]). We just recall here some basic facts. If  $X$  denotes a Banach space, we say that a mapping  $u : [0, +\infty) \rightarrow X$  is absolutely continuous, and we write  $u \in AC([0, +\infty); X)$ , if there exists  $m \in L^1(0, +\infty)$  such that

$$\|u(s) - u(t)\|_X \leq \int_s^t m(r) dr \quad \text{for every } t \geq s \geq 0. \quad (2.1)$$

If the space  $X$  turns out to be reflexive, then any map  $u \in AC([0, +\infty); X)$  is (strongly) derivable almost everywhere. More precisely, for a.e.  $t \in (0, +\infty)$ , there exists  $u'(t) \in X$  such that

$$\frac{u(t) - u(s)}{t - s} \rightarrow u'(t) \quad \text{strongly in } X \text{ as } s \rightarrow t.$$

Moreover  $u' \in L^1(0, +\infty; X)$ ,  $u'$  coincides with distributional derivative of  $u$ , and the Fundamental Theorem of Calculus holds, i.e.,

$$u(t) - u(s) = \int_s^t u'(r) dr \quad \text{for every } t \geq s \geq 0.$$

If further the function  $m$  in (2.1) belongs to  $L^2(0, +\infty)$ , then we write  $u \in AC^2([0, +\infty); X)$ , and in that case we have  $u' \in L^2(0, +\infty; X)$ .

**Special functions of bounded variation.** For an open set  $U \subset \mathbb{R}^N$ , we denote by  $BV(U)$  the space of functions of bounded variation, i.e., the space of all functions  $u \in L^1(U)$  whose distributional gradient  $Du$

belongs to  $\mathcal{M}(U; \mathbb{R}^N)$ . We shall also consider the subspace  $SBV(U)$  of special functions of bounded variation made of functions  $u \in BV(U)$  whose derivative  $Du$  can be decomposed as

$$Du = \nabla u \mathcal{L}^N + (u^+ - u^-) \nu_u \mathcal{H}^{N-1} \llcorner J_u.$$

In the previous expression,  $\nabla u$  is the Radon-Nikodým derivative of  $Du$  with respect to  $\mathcal{L}^N$ , and it is called approximate gradient of  $u$ . The Borel set  $J_u$  is the (approximate) jump set of  $u$ . It is a countably  $\mathcal{H}^{N-1}$ -rectifiable subset of  $U$  oriented by the (normal) direction of jump  $\nu_u : J_u \rightarrow \mathbb{S}^{N-1}$ , and  $u^\pm$  are the one-sided approximate limits of  $u$  on  $J_u$  according to  $\nu_u$ , see [3]. We say that a measurable set  $E$  has finite perimeter in  $U$  if  $\chi_E \in BV(U)$ , and we denote by  $\partial^* E$  its reduced boundary. We also denote by  $GSBV(U)$  the space of all measurable functions  $u : U \rightarrow \mathbb{R}$  such that  $(-M \vee u) \wedge M \in SBV(U)$  for all  $M > 0$ . Again, we refer to [3] for an exhaustive treatment on the subject. Finally we define the spaces

$$SBV^2(U) := \{u \in SBV(U) \cap L^2(U) : \nabla u \in L^2(U; \mathbb{R}^N) \text{ and } \mathcal{H}^{N-1}(J_u) < \infty\},$$

and

$$GSBV^2(U) := \{u \in GSBV(U) \cap L^2(U) : \nabla u \in L^2(U; \mathbb{R}^N) \text{ and } \mathcal{H}^{N-1}(J_u) < \infty\}.$$

Note that, according to the chain rule formula for real valued  $BV$ -functions, we have the inclusion  $SBV^2(U) \cap L^\infty(U) \subset GSBV^2(U)$  (see e.g. [3, Theorem 3.99]).

The following proposition will be very useful to derive a lower estimate for the Ambrosio-Tortorelli functional. It is a direct consequence of the proof of [9, Theorem 10.6] (see [10, Theorem 16] for the original proof).

**Proposition 2.1.** *Let  $\Omega \subset \mathbb{R}^N$  be a bounded open set, let  $\{u_n\}_{n \in \mathbb{N}} \subset H^1(\Omega) \cap L^\infty(\Omega)$  be such that  $\sup_{n \in \mathbb{N}} \|u_n\|_{L^\infty(\Omega)} < \infty$ , and let  $\{E_n\}_{n \in \mathbb{N}}$  be a sequence of subsets of  $\Omega$  of finite perimeter in  $\Omega$  such that  $\sup_{n \in \mathbb{N}} \mathcal{H}^{N-1}(\partial^* E_n \cap \Omega) < \infty$ . Assume that  $u_n \rightarrow u$  strongly in  $L^2(\Omega)$ , and that  $\mathcal{L}^N(E_n) \rightarrow 0$ . Setting  $\tilde{u}_n := (1 - \chi_{E_n})u_n \in SBV^2(\Omega) \cap L^\infty(\Omega)$ , assume in addition that  $\sup_{n \in \mathbb{N}} \|\nabla \tilde{u}_n\|_{L^2(\Omega; \mathbb{R}^N)} < \infty$ . Then  $u \in SBV^2(\Omega) \cap L^\infty(\Omega)$ , and*

$$\begin{cases} \tilde{u}_n \rightarrow u \text{ strongly in } L^2(\Omega), \\ \tilde{u}_n \rightharpoonup u \text{ weakly}^* \text{ in } L^\infty(\Omega), \\ \nabla \tilde{u}_n \rightharpoonup \nabla u \text{ weakly in } L^2(\Omega; \mathbb{R}^N), \\ 2 \mathcal{H}^{N-1}(J_u) \leq \liminf_{n \rightarrow \infty} \mathcal{H}^{N-1}(\partial^* E_n \cap \Omega). \end{cases}$$

**The Ambrosio-Tortorelli & Mumford-Shah functionals.** Let us now consider a bounded open subset  $\Omega$  of  $\mathbb{R}^N$ ,  $p > N$ ,  $\beta > 0$ , and  $g \in L^\infty(\Omega)$ . For  $\varepsilon > 0$  and  $\eta_\varepsilon \in (0, 1)$ , we define the Ambrosio-Tortorelli functional  $\mathcal{E}_\varepsilon : L^2(\Omega) \times L^p(\Omega) \rightarrow [0, +\infty]$  by

$$\mathcal{E}_\varepsilon(u, \rho) := \begin{cases} \frac{1}{2} \int_\Omega (\eta_\varepsilon + \rho^2) |\nabla u|^2 dx + \int_\Omega \left( \frac{\varepsilon^{p-1}}{p} |\nabla \rho|^p + \frac{\alpha}{p' \varepsilon} |1 - \rho|^p \right) dx & \text{if } (u, \rho) \in H^1(\Omega) \times W^{1,p}(\Omega), \\ + \frac{\beta}{2} \int_\Omega (u - g)^2 dx & \\ +\infty & \text{otherwise,} \end{cases}$$

where  $p' := p/(p-1)$  and  $\alpha$  is the normalizing factor given by

$$\alpha := \left( 2 \int_0^1 (1-s)^{p/p'} ds \right)^{-p'} = \left( \frac{p}{2} \right)^{p'}. \quad (2.2)$$

The Mumford-Shah functional  $\mathcal{E} : L^2(\Omega) \rightarrow [0, +\infty]$  is in turn defined by

$$\mathcal{E}(u) := \begin{cases} \frac{1}{2} \int_\Omega |\nabla u|^2 dx + \mathcal{H}^{N-1}(J_u) + \frac{\beta}{2} \int_\Omega (u - g)^2 dx & \text{if } u \in GSBV^2(\Omega), \\ +\infty & \text{otherwise.} \end{cases} \quad (2.3)$$

It is well known by now that the Ambrosio-Tortorelli functional approximates as  $\varepsilon \rightarrow 0$  the Mumford-Shah functional in the sense of  $\Gamma$ -convergence, as stated in the following result, see [5,28]. Let us mention that

Theorem 2.2 is not precisely a direct consequence of [5,28]. In [5], the case  $p = 2$  is addressed, while [28] deals with energies having the same  $p$ -growth in  $\nabla u$  and  $\nabla \rho$  (recall that  $p > N \geq 2$ ). However, a careful inspection of the proof of [28, Theorem 3.1] shows that the  $\Gamma$ -convergence result still holds for  $\mathcal{E}_\varepsilon$ .

**Theorem 2.2.** *Assume that  $\eta_\varepsilon = o(\varepsilon)$ . Then  $\mathcal{E}_\varepsilon$   $\Gamma$ -converges as  $\varepsilon \rightarrow 0$  (with respect to the strong  $L^2(\Omega) \times L^p(\Omega)$ -topology) to the functional  $\mathcal{E}_0$  defined by*

$$\mathcal{E}_0(u, \rho) := \begin{cases} \mathcal{E}(u) & \text{if } u \in GSBV^2(\Omega) \text{ and } \rho = 1 \text{ in } \Omega, \\ +\infty & \text{otherwise.} \end{cases}$$

### 3. Unilateral minimizing movements & curves of maximal unilateral slope

#### 3.1. The discrete evolution schemes

Throughout the paper, we shall say that a sequence of time steps  $\boldsymbol{\delta} := \{\delta^i\}_{i \in \mathbb{N}^*}$  is a partition of  $[0, +\infty)$  if

$$\delta^i > 0, \quad \sup_{i \geq 1} \delta^i < +\infty, \quad \text{and} \quad \sum_{i \geq 1} \delta^i = +\infty.$$

To a partition  $\boldsymbol{\delta}$  we associate the sequence of discrete times  $\{t^i\}_{i \in \mathbb{N}}$  given by  $t^0 := 0$ ,  $t^i := \sum_{j=1}^i \delta^j$  for  $i \geq 1$ , and we define the *time step length* by

$$|\boldsymbol{\delta}| := \sup_{i \geq 1} \delta^i.$$

To an initial datum  $u_0 \in H^1(\Omega) \cap L^\infty(\Omega)$ , we shall *always* associate (for simplicity) the initial state  $\rho_0^\varepsilon$  determined by

$$\rho_0^\varepsilon := \operatorname{argmin}_{\rho \in W^{1,p}(\Omega)} \mathcal{E}_\varepsilon(u_0, \rho). \quad (3.1)$$

It is standard to check that the above minimization problem has a unique solution (by coercivity and strict convexity of the functional  $\mathcal{E}_\varepsilon(u_0, \cdot)$ ), and it follows by minimality that  $0 \leq \rho_0^\varepsilon \leq 1$ . Given a partition  $\boldsymbol{\delta}$  of  $[0, +\infty)$ , we now introduce two discrete evolution schemes starting from  $(u_0, \rho_0^\varepsilon)$ .

**Scheme 1 (global minimization):** Set  $(u^0, \rho^0) := (u_0, \rho_0^\varepsilon)$ , and select recursively for all integer  $i \geq 1$ ,

$$(u^i, \rho^i) \in \operatorname{argmin} \left\{ \mathcal{E}_\varepsilon(u, \rho) + \frac{1}{2\delta^i} \|u - u^{i-1}\|_{L^2(\Omega)}^2 : (u, \rho) \in H^1(\Omega) \times W^{1,p}(\Omega), \rho \leq \rho^{i-1} \text{ in } \Omega \right\}. \quad (3.2)$$

**Scheme 2 (alternate minimization):** Set  $(u^0, \rho^0) := (u_0, \rho_0^\varepsilon)$ , and define recursively for all integer  $i \geq 1$ ,

$$\begin{cases} u^i := \operatorname{argmin} \left\{ \mathcal{E}_\varepsilon(u, \rho^{i-1}) + \frac{1}{2\delta^i} \|u - u^{i-1}\|_{L^2(\Omega)}^2 : u \in H^1(\Omega) \right\}, \\ \rho^i := \operatorname{argmin} \left\{ \mathcal{E}_\varepsilon(u^i, \rho) : \rho \in W^{1,p}(\Omega), \rho \leq \rho^{i-1} \text{ in } \Omega \right\}. \end{cases}$$

While it is straightforward to check that the minimization problems in Scheme 2 admit (unique) solutions, the well-posedness of Scheme 1 requires some little care. Since the sublevel sets of  $\mathcal{E}_\varepsilon$  are clearly relatively compact for the sequential weak  $H^1(\Omega) \times W^{1,p}(\Omega)$ -topology, one may apply the Direct Method of Calculus of Variations to solve (3.2). We only need to show that the constraint in (3.2) is closed, and that  $\mathcal{E}_\varepsilon$  is lower semicontinuous with respect to weak convergence.

**Lemma 3.1.** *Let  $\{(u_n, \rho_n)\}_{n \in \mathbb{N}} \subset H^1(\Omega) \times W^{1,p}(\Omega)$  be such that  $(u_n, \rho_n) \rightharpoonup (u, \rho)$  weakly in  $H^1(\Omega) \times W^{1,p}(\Omega)$ . Then,*

$$\mathcal{E}_\varepsilon(u, \rho) \leq \liminf_{n \rightarrow \infty} \mathcal{E}_\varepsilon(u_n, \rho_n). \quad (3.3)$$

Moreover, if for each  $n \in \mathbb{N}$ ,  $\rho_n \leq \bar{\rho}$  in  $\Omega$  for some  $\bar{\rho} \in W^{1,p}(\Omega)$ , then  $\rho \leq \bar{\rho}$  in  $\Omega$ . Finally, assuming that  $\mathcal{E}_\varepsilon(u_n, \rho_n) \rightarrow \mathcal{E}_\varepsilon(u, \rho)$  as  $n \rightarrow \infty$ , then  $(u_n, \rho_n) \rightarrow (u, \rho)$  strongly in  $H^1(\Omega) \times W^{1,p}(\Omega)$ .

**Proof.** *Step 1.* The sequence  $\{(u_n, \rho_n)\}$  being weakly convergent, it is bounded in  $H^1(\Omega) \times W^{1,p}(\Omega)$ . Therefore  $\rho_n \rightarrow \rho$  in  $\mathcal{C}^0(\bar{\Omega})$  by the Sobolev Imbedding Theorem. Hence  $\rho \leq \bar{\rho}$  in  $\Omega$  whenever  $\rho_n \leq \bar{\rho}$  in  $\Omega$  for every  $n \in \mathbb{N}$ . Then  $\rho_n \nabla u_n \rightharpoonup \rho \nabla u$  weakly in  $L^2(\Omega)$ , and consequently,

$$\int_{\Omega} (\eta_{\varepsilon} + \rho^2) |\nabla u|^2 dx \leq \liminf_{n \rightarrow \infty} \int_{\Omega} (\eta_{\varepsilon} + \rho_n^2) |\nabla u_n|^2 dx.$$

Since all other terms in  $\mathcal{E}_{\varepsilon}$  are clearly lower semicontinuous with respect to the weak convergence in  $H^1(\Omega) \times W^{1,p}(\Omega)$ , we have proved (3.3).

*Step 2.* Let us now assume that  $\mathcal{E}_{\varepsilon}(u_n, \rho_n) \rightarrow \mathcal{E}_{\varepsilon}(u, \rho)$ . We first claim that

$$\int_{\Omega} (\eta_{\varepsilon} + \rho^2) |\nabla u|^2 dx = \lim_{n \rightarrow \infty} \int_{\Omega} (\eta_{\varepsilon} + \rho_n^2) |\nabla u_n|^2 dx. \quad (3.4)$$

Indeed, assume by contradiction that for a subsequence  $\{n_j\}$  we have

$$\int_{\Omega} (\eta_{\varepsilon} + \rho^2) |\nabla u|^2 dx < \liminf_{j \rightarrow \infty} \int_{\Omega} (\eta_{\varepsilon} + \rho_{n_j}^2) |\nabla u_{n_j}|^2 dx.$$

Using the fact that  $u_n \rightarrow u$  strongly in  $L^2(\Omega)$ , we deduce from Step 1 that

$$\begin{aligned} & \lim_{j \rightarrow \infty} \mathcal{E}_{\varepsilon}(u_{n_j}, \rho_{n_j}) \\ & \geq \liminf_{j \rightarrow \infty} \frac{1}{2} \int_{\Omega} (\eta_{\varepsilon} + \rho_{n_j}^2) |\nabla u_{n_j}|^2 dx + \liminf_{j \rightarrow \infty} \int_{\Omega} \left( \frac{\varepsilon^{p-1}}{p} |\nabla \rho_{n_j}|^p + \frac{\alpha}{p'\varepsilon} |1 - \rho_{n_j}|^p \right) dx + \frac{\beta}{2} \int_{\Omega} (u - g)^2 dx \\ & > \mathcal{E}_{\varepsilon}(u, \rho), \end{aligned}$$

which is impossible. Therefore (3.4) holds. Then, combining the convergence of  $\mathcal{E}_{\varepsilon}(u_n, \rho_n)$  with (3.4), we deduce that  $\|\rho_n\|_{W^{1,p}(\Omega)} \rightarrow \|\rho\|_{W^{1,p}(\Omega)}$ , whence the strong  $W^{1,p}(\Omega)$ -convergence of  $\rho_n$ .

It now remains to show that  $u_n \rightarrow u$  strongly in  $H^1(\Omega)$ . Using the uniform convergence of  $\rho_n$  established in Step 1, we first estimate

$$\int_{\Omega} |\rho^2 - \rho_n^2| |\nabla u_n|^2 dx \leq \left( \sup_{k \in \mathbb{N}} \|\nabla u_k\|_{L^2(\Omega; \mathbb{R}^N)} \right) \|\rho^2 - \rho_n^2\|_{L^{\infty}(\Omega)} \xrightarrow{n \rightarrow \infty} 0.$$

Then we infer from (3.4) that

$$\int_{\Omega} (\eta_{\varepsilon} + \rho^2) |\nabla u_n|^2 dx = \int_{\Omega} (\eta_{\varepsilon} + \rho_n^2) |\nabla u_n|^2 dx + \int_{\Omega} (\rho^2 - \rho_n^2) |\nabla u_n|^2 dx \xrightarrow{n \rightarrow \infty} \int_{\Omega} (\eta_{\varepsilon} + \rho^2) |\nabla u|^2 dx.$$

Consequently  $\|u_n\|_{H^1(\Omega)} \rightarrow \|u\|_{H^1(\Omega)}$ , whence the strong  $H^1(\Omega)$ -convergence of  $u_n$ .  $\square$

**Remark 3.2.** Let us now briefly comment both algorithms. The first scheme is very much in the spirit of classical minimizing movements, and it is the analogue of the algorithm introduced in [12] for studying minimizing movements of the Mumford-Shah functional constrained to irreversible crack growth. Scheme 1 can be seen as a regularization of the one in [12] since the Ambrosio-Tortorelli functional  $\Gamma$ -converges as  $\varepsilon \rightarrow 0$  to the Mumford-Shah functional (see [5,28]). However, a practical drawback of Scheme 1 is that the pair  $(u^i, \rho^i)$  obtained at each time step might not be unique since  $\mathcal{E}_{\varepsilon}$  is not strictly convex (although it is separately strictly convex). This lack of uniqueness may generate some troubles from the point of view of numerical approximations. For that reason, it is of importance to consider an alternate scheme which uniquely defines the pair  $(u^i, \rho^i)$ . It turns out that Scheme 2 is actually the algorithm used in numerical experiments for quasi-static evolution in brittle fracture (see [7,8]). We shall prove that both schemes give rise to the same time continuous model, once the time step length  $|\delta|$  is sent to zero.

We state below some important pointwise estimates on the iterates  $\{(u^i, \rho^i)\}_{i \in \mathbb{N}}$  which easily follow from minimality and standard truncation arguments.

**Lemma 3.3.** *Let  $\{(u^i, \rho^i)\}_{i \in \mathbb{N}}$  be a sequence obtained from either Scheme 1 or Scheme 2. Then, for every  $i \in \mathbb{N}$ ,*

$$\|u^i\|_{L^{\infty}(\Omega)} \leq \max\{\|u_0\|_{L^{\infty}(\Omega)}, \|g\|_{L^{\infty}(\Omega)}\} \quad \text{and} \quad 0 \leq \rho^{i+1} \leq \rho^i \leq 1 \quad \text{in } \Omega. \quad (3.5)$$



From a sequence of iterates  $\{(u^i, \rho^i)\}_{i \in \mathbb{N}}$  obtained from either Scheme 1 or Scheme 2 and a partition  $\delta$  of  $[0, +\infty)$ , we associate a **discrete trajectory**  $(u_\delta, \rho_\delta) : [0, +\infty) \rightarrow H^1(\Omega) \times W^{1,p}(\Omega)$  defined as the *left continuous piecewise constant interpolation* of the  $(u^i, \rho^i)$ 's below. More precisely, we set

$$u_\delta(0) = u_0, \quad \rho_\delta(0) = \rho_0^\varepsilon,$$

and for  $t > 0$ ,

$$\begin{cases} u_\delta(t) := u^i & \text{if } t \in (t^{i-1}, t^i]. \end{cases} \quad (3.6)$$

By analogy with the standard notion of minimizing movements, we now introduce the following definition.

**Definition 3.4 (Unilateral - Alternate - Minimizing Movements).** Let  $u_0 \in H^1(\Omega) \cap L^\infty(\Omega)$ . We say that a pair  $(u, \rho) : [0, +\infty) \rightarrow L^2(\Omega) \times L^p(\Omega)$  is a (*generalized*) *unilateral minimizing movement* (resp. a (*generalized*) *unilateral alternate minimizing movement*) for  $\mathcal{E}_\varepsilon$  starting from  $(u_0, \rho_0^\varepsilon)$  if there exist a sequence  $\{\delta_k\}_{k \in \mathbb{N}}$  of partitions of  $[0, +\infty)$  satisfying  $|\delta_k| \rightarrow 0$ , and associated discrete trajectories  $\{(u_{\delta_k}, \rho_{\delta_k})\}_{k \in \mathbb{N}}$  obtained from Scheme 1 (resp. Scheme 2) such that

$$(u_{\delta_k}(t), \rho_{\delta_k}(t)) \xrightarrow[k \rightarrow \infty]{} (u(t), \rho(t)) \quad \text{strongly in } L^2(\Omega) \times L^p(\Omega) \text{ for every } t \geq 0.$$

We denote by  $GUMM(u_0, \rho_0^\varepsilon)$  (resp.  $GUAMM(u_0, \rho_0^\varepsilon)$ ) the collection of all (generalized) unilateral minimizing movements (resp. (generalized) unilateral alternate minimizing movements) for  $\mathcal{E}_\varepsilon$  starting from  $(u_0, \rho_0^\varepsilon)$ .

**Remark 3.5.** At this stage we do not claim that the collections  $GUMM(u_0, \rho_0^\varepsilon)$  and  $GUAMM(u_0, \rho_0^\varepsilon)$  are not empty. This will be proved in the next section through a compactness result on discrete trajectories (see Lemmas 4.9 & 4.11, and Corollary 4.14).

### 3.2. Curves of maximal unilateral slope

In the spirit of [4], we introduce in this section the notion of  $L^2(\Omega)$ -unilateral gradient flow for the Ambrosio-Tortorelli functional in terms of *curves of maximal unilateral slope*, accounting for the quasi-stationarity and the decrease monotonicity constraint on the phase field variable  $\rho$ . To this aim, we first define the unilateral slope of  $\mathcal{E}_\varepsilon$  which is analogous to the one introduced in [19] for the Mumford-Shah functional.

**Definition 3.6.** The *unilateral slope* of  $\mathcal{E}_\varepsilon$  at  $(u, \rho) \in H^1(\Omega) \times W^{1,p}(\Omega)$  is defined by

$$|\partial \mathcal{E}_\varepsilon|(u, \rho) := \limsup_{v \rightarrow u} \sup_{\hat{\rho}} \left\{ \frac{(\mathcal{E}_\varepsilon(u, \rho) - \mathcal{E}_\varepsilon(v, \hat{\rho}))^+}{\|v - u\|_{L^2(\Omega)}} : \hat{\rho} \in W^{1,p}(\Omega), \hat{\rho} \leq \rho \text{ in } \Omega \right\},$$

where the convergence  $v \rightarrow u$  holds in  $L^2(\Omega)$ . The functional  $|\partial \mathcal{E}_\varepsilon|$  is then extended to  $L^2(\Omega) \times L^p(\Omega)$  by setting  $|\partial \mathcal{E}_\varepsilon|(u, \rho) := +\infty$  for  $(u, \rho) \notin H^1(\Omega) \times W^{1,p}(\Omega)$ .

The unilateral slope being defined, we can now define curves of maximal unilateral slope for the Ambrosio-Tortorelli functional.

**Definition 3.7.** We say that a pair  $(u, \rho) : (a, b) \rightarrow L^2(\Omega) \times L^p(\Omega)$  is a *curve of maximal unilateral slope* for  $\mathcal{E}_\varepsilon$  if  $u \in AC^2(a, b; L^2(\Omega))$ ,  $\rho$  is non-increasing, and if there exists a non-increasing function  $\lambda : (a, b) \rightarrow [0, +\infty)$  such that for a.e.  $t \in (a, b)$ ,  $\mathcal{E}_\varepsilon(u(t), \rho(t)) = \lambda(t)$ , and

$$\lambda'(t) \leq -\frac{1}{2} \|u'(t)\|_{L^2(\Omega)}^2 - \frac{1}{2} |\partial \mathcal{E}_\varepsilon|^2(u(t), \rho(t)). \quad (3.7)$$

Our definition of curve of maximal unilateral slope is motivated by the following proposition which parallels [4, Theorem 1.2.5].

**Proposition 3.8.** *If  $(u, \rho) : (a, b) \rightarrow L^2(\Omega) \times L^p(\Omega)$  is a curve of maximal unilateral slope for  $\mathcal{E}_\varepsilon$ , then*

$$\|u'(t)\|_{L^2(\Omega)} = |\partial \mathcal{E}_\varepsilon|(u(t), \rho(t)) \quad \text{for a.e. } t \in (a, b). \quad (3.8)$$

Moreover, if  $t \mapsto \mathcal{E}_\varepsilon(u(t), \rho(t))$  is absolutely continuous on  $(a, b)$ , then

$$\mathcal{E}_\varepsilon(u(t), \rho(t)) + \int_s^t \|u'(r)\|_{L^2(\Omega)}^2 dr = \mathcal{E}_\varepsilon(u(s), \rho(s)) \quad \text{for every } s \text{ and } t \in (a, b) \text{ with } s \leq t.$$

**Proof.** Let  $\lambda$  be as in Definition 3.7. Since  $\lambda$  is non-increasing,  $\lambda$  has finite pointwise variation in  $(a, b)$ . Let us consider the set

$$A := \{t \in (a, b) : \mathcal{E}_\varepsilon(u(t), \rho(t)) = \lambda(t), \lambda \text{ and } u \text{ are derivable at } t\},$$

and observe that  $\mathcal{L}^1((a, b) \setminus A) = 0$ .

Let  $t \in A$ . Since  $\lambda$  is non-increasing, we have  $\lambda'(t) \leq 0$ , and thus

$$|\lambda'(t)| = -\lambda'(t) = \lim_{s \downarrow t, s \in A} \frac{\lambda(t) - \lambda(s)}{s - t} = \lim_{s \downarrow t, s \in A} \frac{\mathcal{E}_\varepsilon(u(t), \rho(t)) - \mathcal{E}_\varepsilon(u(s), \rho(s))}{s - t}.$$

Using the fact that  $\rho(s) \leq \rho(t)$  when  $s > t$  (by the non-increasing property of  $t \mapsto \rho(t)$ ) and the strong  $L^2(\Omega)$ -continuity of  $u$ , we infer that

$$|\lambda'(t)| \leq \limsup_{s \downarrow t} \sup_{s \in A} \sup_{\hat{\rho} \leq \rho(t)} \frac{(\mathcal{E}_\varepsilon(u(t), \rho(t)) - \mathcal{E}_\varepsilon(u(s), \hat{\rho}))^+}{\|u(s) - u(t)\|_{L^2(\Omega)}} \frac{\|u(s) - u(t)\|_{L^2(\Omega)}}{s - t} \leq |\partial \mathcal{E}_\varepsilon|(u(t), \rho(t)) \|u'(t)\|_{L^2(\Omega)}.$$

On the other hand,  $|\lambda'(t)| \geq \frac{1}{2} \|u'(t)\|_{L^2(\Omega)}^2 + \frac{1}{2} |\partial \mathcal{E}_\varepsilon|^2(u(t), \rho(t))$  by (3.7), and (3.8) follows as well as the fact that  $\lambda'(t) = -\|u'(t)\|_{L^2(\Omega)}^2$ .

Finally, if  $t \mapsto \mathcal{E}_\varepsilon(u(t), \rho(t))$  is absolutely continuous on  $(a, b)$ , then for every  $s, t \in (a, b)$  with  $s \leq t$ ,

$$\mathcal{E}_\varepsilon(u(t), \rho(t)) - \mathcal{E}_\varepsilon(u(s), \rho(s)) = \int_s^t \lambda'(r) dr = - \int_s^t \|u'(r)\|_{L^2(\Omega)}^2 dr,$$

which completes the proof of the proposition.  $\square$

We state below necessary and sufficient conditions for the finiteness of the slope, as well as an explicit formula to represent it. This is one of the milestone of our entire analysis.

**Proposition 3.9.** *Assume that  $\Omega$  has a  $\mathcal{C}^{1,1}$ -boundary, and let  $D(|\partial \mathcal{E}_\varepsilon|)$  be the proper domain of  $|\partial \mathcal{E}_\varepsilon|$ . Then,*

$$D(|\partial \mathcal{E}_\varepsilon|) = \left\{ (u, \rho) \in H^2(\Omega) \times W^{1,p}(\Omega) : \frac{\partial u}{\partial \nu} = 0 \text{ in } H^{1/2}(\partial \Omega), \text{ and} \right. \\ \left. \mathcal{E}_\varepsilon(u, \rho) \leq \mathcal{E}_\varepsilon(u, \hat{\rho}) \text{ for all } \hat{\rho} \in W^{1,p}(\Omega) \text{ such that } \hat{\rho} \leq \rho \text{ in } \Omega \right\}. \quad (3.9)$$

In addition, for  $(u, \rho) \in D(|\partial \mathcal{E}_\varepsilon|)$ ,

$$|\partial \mathcal{E}_\varepsilon|(u, \rho) = \left\| \operatorname{div}((\eta_\varepsilon + \rho^2) \nabla u) - \beta(u - g) \right\|_{L^2(\Omega)}, \quad (3.10)$$

and

$$\|u\|_{H^2(\Omega)} \leq C_\varepsilon (1 + \|\nabla \rho\|_{L^p(\Omega; \mathbb{R}^N)})^\alpha (|\partial \mathcal{E}_\varepsilon|(u, \rho) + \beta \|u - g\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)}),$$

where  $\alpha \in \mathbb{N}$  is the smallest integer larger than or equal to  $p/(p - N)$ , and  $C_\varepsilon$  only depends on  $\eta_\varepsilon, p, N$ , and  $\Omega$ .

The proof of Proposition 3.9 is based on the following auxiliary regularity result.

**Lemma 3.10.** *Assume that  $\Omega$  has a  $\mathcal{C}^{1,1}$ -boundary. For  $f \in L^2(\Omega)$  and  $\rho \in W^{1,p}(\Omega)$ , let  $u \in H^1(\Omega)$  be a solution of*

$$\begin{cases} -\operatorname{div}((\eta_\varepsilon + \rho^2) \nabla u) = f & \text{in } H^{-1}(\Omega), \\ (\eta_\varepsilon + \rho^2) \nabla u \cdot \nu = 0 & \text{in } H^{-1/2}(\partial \Omega). \end{cases} \quad (3.11)$$

Then  $u \in H^2(\Omega)$ ,  $\frac{\partial u}{\partial \nu} = 0$  in  $H^{1/2}(\partial \Omega)$ , and

$$\|u\|_{H^2(\Omega)} \leq C_\varepsilon (1 + \|\nabla \rho\|_{L^p(\Omega; \mathbb{R}^N)})^\alpha (\|f\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)}),$$

where  $\alpha \in \mathbb{N}$  is as in Proposition 3.9, and  $C_\varepsilon$  only depends on  $\eta_\varepsilon, p, N$ , and  $\Omega$ .

**Proof.** *Step 1.* We claim that

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{in } H^{-1/2}(\partial \Omega). \quad (3.12)$$

To prove this claim, we first rewrite the equation as

$$-\Delta u = \frac{2\rho}{\eta_\varepsilon + \rho^2} \nabla \rho \cdot \nabla u + \frac{f}{\eta_\varepsilon + \rho^2} \quad \text{in } \mathcal{D}'(\Omega). \quad (3.13)$$

Hence  $\Delta u \in L^q(\Omega)$  with  $q := 2p/(p+2)$  by Hölder's inequality. Then we observe that  $q' := q/(q-1) < 2^*$  since  $p > N$ , so that  $H^1(\Omega) \hookrightarrow L^{q'}(\Omega)$  by the Sobolev Imbedding. Hence the linear mapping

$$\varphi \in H^1(\Omega) \mapsto \int_{\Omega} (\nabla u \cdot \nabla \varphi + (\Delta u)\varphi) dx$$

is well defined and continuous. Consequently,  $u$  admits a (weak) normal derivative  $\frac{\partial u}{\partial \nu}$  on  $\partial\Omega$  which belongs to the dual space  $H^{-1/2}(\partial\Omega)$ , and for any  $\varphi \in H^1(\Omega)$ ,

$$\begin{aligned} \left\langle \frac{\partial u}{\partial \nu}, \varphi|_{\partial\Omega} \right\rangle_{(H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega))} &= \int_{\Omega} (\nabla u \cdot \nabla \varphi + (\Delta u)\varphi) dx \\ &= \int_{\Omega} (\eta_\varepsilon + \rho^2) \nabla u \cdot \nabla \left( \frac{\varphi}{\eta_\varepsilon + \rho^2} \right) dx + \int_{\Omega} \left( \Delta u + \frac{2\rho}{\eta_\varepsilon + \rho^2} \nabla \rho \cdot \nabla u \right) \varphi dx. \end{aligned}$$

We observe that in the second equality above, we have used the fact that  $\frac{\varphi}{\eta_\varepsilon + \rho^2} \in H^1(\Omega)$  whenever  $\varphi \in H^1(\Omega)$ . Indeed,

$$\nabla \left( \frac{\varphi}{\eta_\varepsilon + \rho^2} \right) = \frac{\nabla \varphi}{\eta_\varepsilon + \rho^2} - \frac{2\rho \varphi \nabla \rho}{(\eta_\varepsilon + \rho^2)^2} \in L^2(\Omega),$$

since  $\varphi \in L^{2^*}(\Omega)$ ,  $\rho \in L^\infty(\Omega)$ , and  $\nabla \rho \in L^p(\Omega)$  with  $p > N$ . In view of (3.11) we have

$$\int_{\Omega} (\eta_\varepsilon + \rho^2) \nabla u \cdot \nabla \left( \frac{\varphi}{\eta_\varepsilon + \rho^2} \right) dx = \int_{\Omega} \frac{f\varphi}{\eta_\varepsilon + \rho^2} dx,$$

and by (3.13),

$$\int_{\Omega} \left( \Delta u + \frac{2\rho}{\eta_\varepsilon + \rho^2} \nabla \rho \cdot \nabla u \right) \varphi dx = - \int_{\Omega} \frac{f\varphi}{\eta_\varepsilon + \rho^2} dx,$$

from which (3.12) follows.

*Step 2.* We now prove that  $u \in H^2(\Omega)$ . By the previous step,  $u \in H^1(\Omega)$  satisfies

$$\begin{cases} \Delta u \in L^q(\Omega), \\ \frac{\partial u}{\partial \nu} = 0 \text{ in } H^{-1/2}(\partial\Omega). \end{cases}$$

By elliptic regularity (see e.g. [31, Proposition 2.5.2.3 & Theorem 2.3.3.6]), we deduce that  $u \in W^{2,q_0}(\Omega)$  with  $q_0 := q = \frac{2p}{p+2}$ , and

$$\|u\|_{W^{2,q_0}(\Omega)} \leq C (\|\Delta u\|_{L^{q_0}(\Omega)} + \|u\|_{L^{q_0}(\Omega)}),$$

for some constant  $C > 0$  only depending on  $N$ ,  $p$ , and  $\Omega$ . Observing that the function  $t \mapsto t/(\eta_\varepsilon + t^2)$  is bounded, we derive from (3.13) and Hölder's inequality that

$$\begin{aligned} \|u\|_{W^{2,q_0}(\Omega)} &\leq C_\varepsilon (\|f\|_{L^2(\Omega)} + \|\nabla \rho\|_{L^p(\Omega; \mathbb{R}^N)} \|\nabla u\|_{L^2(\Omega; \mathbb{R}^N)} + \|u\|_{L^2(\Omega)}) \\ &\leq C_\varepsilon (1 + \|\nabla \rho\|_{L^p(\Omega; \mathbb{R}^N)}) (\|f\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)}), \end{aligned}$$

where we used the fact that  $q_0 < 2$ . By the Sobolev Imbedding, we have  $u \in W^{1,q_0^*}(\Omega)$ , and thus  $\nabla u \cdot \nabla \rho \in L^{q_1}(\Omega)$  with

$$\frac{1}{q_1} = \frac{1}{p} + \frac{1}{q_0^*}, \quad \text{i.e.,} \quad q_1 := \frac{2Np}{(N-2)p + 4N}.$$

Note that  $q_1 \geq 2$  if and only if  $p \geq 2N$ , so we have to distinguish the case  $p \geq 2N$  from the case  $p < 2N$ .

*Case 1).* Let us first assume that  $p \geq 2N$ . Then  $\nabla u \cdot \nabla \rho \in L^2(\Omega)$  with

$$\|\nabla u \cdot \nabla \rho\|_{L^2(\Omega)} \leq C \|\nabla u \cdot \nabla \rho\|_{L^{q_1}(\Omega)} \leq C \|\nabla \rho\|_{L^p(\Omega; \mathbb{R}^N)} \|\nabla u\|_{L^{q_0^*}(\Omega; \mathbb{R}^N)} \leq C \|\nabla \rho\|_{L^p(\Omega; \mathbb{R}^N)} \|u\|_{W^{2,q_0}(\Omega)}.$$

Using again (3.12)-(3.13) and the elliptic regularity, we infer that  $u \in H^2(\Omega)$  with the estimate

$$\begin{aligned} \|u\|_{H^2(\Omega)} &\leq C(\|\Delta u\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}) \\ &\leq C_\varepsilon(\|f\|_{L^2(\Omega)} + \|\nabla\rho\|_{L^p(\Omega;\mathbb{R}^N)}\|u\|_{W^{2,q_0}(\Omega)} + \|u\|_{L^2(\Omega)}) \\ &\leq C_\varepsilon(1 + \|\nabla\rho\|_{L^p(\Omega;\mathbb{R}^N)})^2(\|f\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)}). \end{aligned}$$

*Case 2).* If  $p < 2N$  then  $q_1 < 2$ , and we have  $u \in W^{2,q_1}(\Omega)$  by (3.12)-(3.13) and elliptic regularity, with the estimate

$$\begin{aligned} \|u\|_{W^{2,q_1}(\Omega)} &\leq C_\varepsilon(\|f\|_{L^2(\Omega)} + \|\nabla\rho\|_{L^p(\Omega)}\|u\|_{W^{2,q_0}(\Omega)} + \|u\|_{L^2(\Omega)}) \\ &\leq C_\varepsilon(1 + \|\nabla\rho\|_{L^p(\Omega)})^2(\|f\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)}). \end{aligned} \quad (3.14)$$

In particular,  $\nabla u \in L^{q_1^*}(\Omega)$  by the Sobolev Imbedding since  $q_1 < 2 \leq N$ . We then continue the process by setting

$$\frac{1}{q_i} := \frac{1}{p} + \frac{1}{q_{i-1}^*}, \quad \text{i.e.,} \quad q_i := \frac{2Np}{(N-2i)p + 2(i+1)N}$$

as long as  $q_{i-1} < 2$ , that is  $i < \alpha$ . Since  $q_{\alpha-1} \geq 2$ , iterating estimates of the form (3.14) we obtain

$$\begin{aligned} \|u\|_{H^2(\Omega)} &\leq C_\varepsilon(\|f\|_{L^2(\Omega)} + \|\nabla\rho\|_{L^p(\Omega;\mathbb{R}^N)}\|u\|_{W^{2,q_{\alpha-2}}(\Omega)} + \|u\|_{L^2(\Omega)}) \\ &\leq C_\varepsilon(1 + \|\nabla\rho\|_{L^p(\Omega;\mathbb{R}^N)})^\alpha(\|f\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)}), \end{aligned}$$

and the proof is complete.  $\square$

**Proof of Proposition 3.9.** *Step 1.* Let us consider a pair  $(u, \rho)$  such that  $|\partial\mathcal{E}_\varepsilon|(u, \rho) < \infty$ . For  $\varphi \in H^1(\Omega)$  with  $\varphi \neq 0$ , we estimate

$$|\partial\mathcal{E}_\varepsilon|(u, \rho) \geq \limsup_{\delta \downarrow 0} \frac{\mathcal{E}_\varepsilon(u, \rho) - \mathcal{E}_\varepsilon(u - \delta\varphi, \rho)}{\delta\|\varphi\|_{L^2(\Omega)}} \geq \frac{1}{\|\varphi\|_{L^2(\Omega)}} \left( \int_\Omega (\eta_\varepsilon + \rho^2)\nabla u \cdot \nabla\varphi \, dx + \beta \int_\Omega (u - g)\varphi \, dx \right). \quad (3.15)$$

By density of  $H^1(\Omega)$  in  $L^2(\Omega)$  and the Riesz representation Theorem in  $L^2(\Omega)$ , we deduce that there exists  $\tilde{f} \in L^2(\Omega)$  such that

$$\int_\Omega (\eta_\varepsilon + \rho^2)\nabla u \cdot \nabla\varphi \, dx + \beta \int_\Omega (u - g)\varphi \, dx = \int_\Omega \tilde{f}\varphi \, dx$$

for all  $\varphi \in H^1(\Omega)$ . Hence  $u$  solves (3.11) with  $f = \tilde{f} - \beta(u - g)$ . We then infer from Lemma 3.10 that  $u \in H^2(\Omega)$ , and that  $\frac{\partial u}{\partial\nu} = 0$  in  $H^{1/2}(\partial\Omega)$ . Next, taking  $\varphi \in H^1(\Omega)$  such that  $\|\varphi\|_{L^2(\Omega)} = 1$ , integrating by parts in (3.15), and passing to the supremum over all such  $\varphi$ 's yields the lower bound

$$|\partial\mathcal{E}_\varepsilon|(u, \rho) \geq \|\operatorname{div}((\eta_\varepsilon + \rho^2)\nabla u) - \beta(u - g)\|_{L^2(\Omega)}.$$

We now claim that the following minimality property for  $\rho$  holds:

$$\mathcal{E}_\varepsilon(u, \rho) \leq \mathcal{E}_\varepsilon(u, \hat{\rho}) \quad \text{for all } \hat{\rho} \in W^{1,p}(\Omega) \text{ such that } \hat{\rho} \leq \rho \text{ in } \Omega. \quad (3.16)$$

Since  $|\partial\mathcal{E}_\varepsilon|(u, \rho) < +\infty$  we can find sequences  $\{v_n\} \subset H^1(\Omega)$  and  $\{\rho_n\} \subset W^{1,p}(\Omega)$  such that  $v_n \rightarrow u$  strongly in  $L^2(\Omega)$ ,

$$\rho_n = \operatorname{argmin} \{ \mathcal{E}_\varepsilon(v_n, \hat{\rho}) : \hat{\rho} \in W^{1,p}(\Omega), \hat{\rho} \leq \rho \text{ in } \Omega \} \quad \text{for each } n \in \mathbb{N},$$

$$\mathcal{E}_\varepsilon(v_n, \rho_n) \leq \mathcal{E}_\varepsilon(u, \rho), \quad (3.17)$$

and

$$\limsup_{n \rightarrow \infty} \frac{\mathcal{E}_\varepsilon(u, \rho) - \mathcal{E}_\varepsilon(v_n, \rho_n)}{\|v_n - u\|_{L^2(\Omega)}} \leq |\partial\mathcal{E}_\varepsilon|(u, \rho). \quad (3.18)$$

By (3.17) the sequence  $\{\nabla v_n\}$  is uniformly bounded in  $L^2(\Omega; \mathbb{R}^N)$ . Hence, for a suitable subsequence (not relabeled),

$$|\nabla v_n|^2 \mathcal{L}^N \llcorner \Omega \rightharpoonup |\nabla u|^2 \mathcal{L}^N \llcorner \Omega + \mu$$

weakly\* in  $\mathcal{M}(\mathbb{R}^N)$  for some nonnegative Radon measure  $\mu \in \mathcal{M}(\mathbb{R}^N)$  supported in  $\bar{\Omega}$ . Let us now consider the following functionals on  $W^{1,p}(\Omega)$  defined by

$$\mathcal{F}_n(\hat{\rho}) := \begin{cases} \mathcal{E}_\varepsilon(v_n, \hat{\rho}) & \text{if } \hat{\rho} \leq \rho, \\ +\infty & \text{otherwise,} \end{cases} \quad \text{and} \quad \mathcal{F}(\hat{\rho}) := \begin{cases} \mathcal{E}_\varepsilon(u, \hat{\rho}) + \frac{1}{2} \int_{\bar{\Omega}} (\eta_\varepsilon + \hat{\rho}^2) d\mu & \text{if } \hat{\rho} \leq \rho, \\ +\infty & \text{otherwise.} \end{cases}$$

Note that by the Sobolev Imbedding  $W^{1,p}(\Omega) \hookrightarrow \mathcal{C}^0(\bar{\Omega})$ , the functional  $\mathcal{F}$  is well defined on the space  $W^{1,p}(\Omega)$ .

We claim that  $\mathcal{F}_n$   $\Gamma$ -converges to  $\mathcal{F}$  for the sequential weak  $W^{1,p}(\Omega)$ -topology. Indeed, the upper bound is immediate since  $\mathcal{F}_n(\hat{\rho}) \rightarrow \mathcal{F}(\hat{\rho})$  for each  $\hat{\rho} \in W^{1,p}(\Omega)$ . For what concerns the lower bound, if  $\{\hat{\rho}_n\} \subset W^{1,p}(\Omega)$  is such that  $\liminf_n \mathcal{F}_n(\hat{\rho}_n) < \infty$  and  $\hat{\rho}_n \rightharpoonup \hat{\rho}$  weakly in  $W^{1,p}(\Omega)$ , then for a subsequence  $\{n_k\}$  we have  $\lim_k \mathcal{F}_{n_k}(\hat{\rho}_{n_k}) = \liminf_n \mathcal{F}_n(\hat{\rho}_n)$ , and  $\hat{\rho}_{n_k} \rightarrow \hat{\rho}$  in  $\mathcal{C}^0(\bar{\Omega})$  by the compact imbedding  $W^{1,p}(\Omega) \hookrightarrow \mathcal{C}^0(\bar{\Omega})$ . Consequently  $\hat{\rho} \leq \rho$  in  $\Omega$ , and

$$\int_{\Omega} \hat{\rho}_{n_k}^2 |\nabla v_{n_k}|^2 dx \rightarrow \int_{\Omega} \hat{\rho}^2 |\nabla u|^2 dx + \int_{\bar{\Omega}} \hat{\rho}^2 d\mu.$$

Since the remaining terms in the energy  $\mathcal{F}_n$  are independent of  $n$  and lower semicontinuous for the weak  $W^{1,p}(\Omega)$ -convergence, we deduce that

$$\mathcal{F}(\hat{\rho}) \leq \liminf_{n \rightarrow \infty} \mathcal{F}_n(\hat{\rho}_n),$$

and the  $\Gamma$ -convergence is proved.

Since the sublevel sets of  $\mathcal{F}_n$  are relatively compact for the sequential weak  $W^{1,p}(\Omega)$ -topology (uniformly in  $n$ ), we infer from the  $\Gamma$ -convergence of  $\mathcal{F}_n$  towards  $\mathcal{F}$  that

$$\mathcal{E}_\varepsilon(v_n, \rho_n) = \min_{W^{1,p}(\Omega)} \mathcal{F}_n \rightarrow \min_{W^{1,p}(\Omega)} \mathcal{F}.$$

On the other hand, by (3.17) and (3.18) we have  $\mathcal{E}_\varepsilon(v_n, \rho_n) \rightarrow \mathcal{E}_\varepsilon(u, \rho)$  from which we deduce that

$$\mathcal{E}_\varepsilon(u, \rho) = \min_{W^{1,p}(\Omega)} \mathcal{F} = \min \{ \mathcal{F}(\hat{\rho}) : \hat{\rho} \in W^{1,p}(\Omega), \hat{\rho} \leq \rho \text{ in } \Omega \}.$$

We conclude from this last relation that  $\mu = 0$  and that (3.16) holds.

*Step 2.* Conversely, we show that if a pair  $(u, \rho)$  belongs to the set in the right hand side of (3.9), then  $|\partial \mathcal{E}_\varepsilon|(u, \rho) < \infty$  and  $|\partial \mathcal{E}_\varepsilon|(u, \rho) \leq \|\operatorname{div}((\eta_\varepsilon + \rho^2)\nabla u) - \beta(u - g)\|_{L^2(\Omega)}$ .

Consider a pair  $(u, \rho) \in H^2(\Omega) \times W^{1,p}(\Omega)$  satisfying  $\frac{\partial u}{\partial \nu} = 0$  in  $H^{1/2}(\partial\Omega)$  and

$$\mathcal{E}_\varepsilon(u, \rho) \leq \mathcal{E}_\varepsilon(u, \hat{\rho})$$

for all  $\hat{\rho} \in W^{1,p}(\Omega)$  such that  $\hat{\rho} \leq \rho$  in  $\Omega$ . Note that  $u \in W^{1,r}(\Omega)$  for every  $r \leq 2^*$  by the Sobolev Imbedding, and since  $p > N$ , the product  $\nabla u \cdot \nabla \rho$  belongs to  $L^2(\Omega)$  and  $\rho \in L^\infty(\Omega)$ . Hence,

$$\operatorname{div}((\eta_\varepsilon + \rho^2)\nabla u) = (\eta_\varepsilon + \rho^2)\Delta u + 2\rho\nabla\rho \cdot \nabla u \in L^2(\Omega),$$

and consequently, it is enough to check that

$$|\partial \mathcal{E}_\varepsilon|(u, \rho) \leq \|\operatorname{div}((\eta_\varepsilon + \rho^2)\nabla u) - \beta(u - g)\|_{L^2(\Omega)}.$$

Consider a sequence  $\{v_n\} \subset H^1(\Omega)$  converging strongly to  $u$  in  $L^2(\Omega)$  such that

$$|\partial \mathcal{E}_\varepsilon|(u, \rho) = \lim_{n \rightarrow \infty} \sup \left\{ \frac{(\mathcal{E}_\varepsilon(u, \rho) - \mathcal{E}_\varepsilon(v_n, \hat{\rho}))^+}{\|v_n - u\|_{L^2(\Omega)}} : \hat{\rho} \in W^{1,p}(\Omega), \hat{\rho} \leq \rho \text{ in } \Omega \right\},$$

and let

$$\rho_n = \operatorname{argmin} \{ \mathcal{E}_\varepsilon(v_n, \hat{\rho}) : \hat{\rho} \in W^{1,p}(\Omega) \text{ such that } \hat{\rho} \leq \rho \text{ in } \Omega \}.$$

Then

$$\sup_{\hat{\rho} \leq \rho} (\mathcal{E}_\varepsilon(u, \rho) - \mathcal{E}_\varepsilon(v_n, \hat{\rho}))^+ \leq (\mathcal{E}_\varepsilon(u, \rho) - \mathcal{E}_\varepsilon(v_n, \rho_n))^+,$$

so that

$$|\partial \mathcal{E}_\varepsilon|(u, \rho) = \lim_{n \rightarrow \infty} \frac{(\mathcal{E}_\varepsilon(u, \rho) - \mathcal{E}_\varepsilon(v_n, \rho_n))^+}{\|v_n - u\|_{L^2(\Omega)}}. \quad (3.19)$$

If for infinitely many  $n$ 's we have  $\mathcal{E}_\varepsilon(v_n, \rho_n) > \mathcal{E}_\varepsilon(u, \rho)$ , then  $|\partial\mathcal{E}_\varepsilon|(u, \rho) = 0$  and there is nothing to prove. Hence we can assume without loss of generality that  $\mathcal{E}_\varepsilon(v_n, \rho_n) \leq \mathcal{E}_\varepsilon(u, \rho)$ . In particular,  $\{\rho_n\}$  is uniformly bounded in  $W^{1,p}(\Omega)$ , and  $\{v_n\}$  is uniformly bounded in  $H^1(\Omega)$ . As a consequence, for a subsequence  $v_n \rightharpoonup u$  weakly in  $H^1(\Omega)$  and  $\rho_n \rightharpoonup \rho_*$  weakly in  $W^{1,p}(\Omega)$ . From Lemma 3.1 we infer that  $\rho_* \leq \rho$  in  $\Omega$ , and

$$\mathcal{E}_\varepsilon(u, \rho_*) \leq \liminf_{n \rightarrow \infty} \mathcal{E}_\varepsilon(v_n, \rho_n) \leq \limsup_{n \rightarrow \infty} \mathcal{E}_\varepsilon(v_n, \rho_n) \leq \mathcal{E}_\varepsilon(u, \rho). \quad (3.20)$$

By the minimality property of  $\rho$ , we have that  $\mathcal{E}_\varepsilon(u, \rho) \leq \mathcal{E}_\varepsilon(u, \rho_*)$  which leads to  $\mathcal{E}_\varepsilon(u, \rho) = \mathcal{E}_\varepsilon(u, \rho_*)$ . By uniqueness of the minimizer (due to the strict convexity of  $\mathcal{E}_\varepsilon(u, \cdot)$ ), we deduce that  $\rho_* = \rho$ . Then Lemma 3.1 and (3.20) with  $\rho_* = \rho$  shows that  $\rho_n \rightarrow \rho$  strongly in  $W^{1,p}(\Omega)$ .

We now estimate

$$\begin{aligned} \mathcal{E}_\varepsilon(u, \rho) - \mathcal{E}_\varepsilon(v_n, \rho_n) &\leq \mathcal{E}_\varepsilon(u, \rho_n) - \mathcal{E}_\varepsilon(v_n, \rho_n) \\ &\leq \int_{\Omega} (\eta_\varepsilon + \rho_n^2) \nabla u \cdot (\nabla u - \nabla v_n) \, dx + \beta \int_{\Omega} (u - g)(u - v_n) \, dx \\ &= - \int_{\Omega} (u - v_n) (\operatorname{div}((\eta_\varepsilon + \rho_n^2) \nabla u) - \beta(u - g)) \, dx. \end{aligned}$$

Note that in the last equality, there is no boundary term since  $\frac{\partial u}{\partial \nu} = 0$  in  $H^{1/2}(\partial\Omega)$ . Moreover, since  $u \in H^2(\Omega)$  and  $\rho_n \in W^{1,p}(\Omega)$ , we have  $\operatorname{div}((\eta_\varepsilon + \rho_n^2) \nabla u) \in L^2(\Omega)$ . Applying Cauchy-Schwarz Inequality we obtain

$$\frac{\mathcal{E}_\varepsilon(u, \rho) - \mathcal{E}_\varepsilon(v_n, \rho_n)}{\|v_n - u\|_{L^2(\Omega)}} \leq \|\operatorname{div}((\eta_\varepsilon + \rho_n^2) \nabla u) - \beta(u - g)\|_{L^2(\Omega)}.$$

Since  $H^2(\Omega) \hookrightarrow W^{1,r}(\Omega)$  for every  $r \leq 2^*$  and  $\rho_n \rightarrow \rho$  strongly in  $W^{1,p}(\Omega)$ , we get that

$$\operatorname{div}((\eta_\varepsilon + \rho_n^2) \nabla u) = (\eta_\varepsilon + \rho_n^2) \Delta u + 2\rho_n \nabla \rho_n \cdot \nabla u \xrightarrow{n \rightarrow \infty} (\eta_\varepsilon + \rho^2) \Delta u + 2\rho \nabla \rho \cdot \nabla u = \operatorname{div}((\eta_\varepsilon + \rho^2) \nabla u)$$

strongly in  $L^2(\Omega)$ . Hence

$$\lim_{n \rightarrow \infty} \frac{\mathcal{E}_\varepsilon(u, \rho) - \mathcal{E}_\varepsilon(v_n, \rho_n)}{\|v_n - u\|_{L^2(\Omega)}} \leq \|\operatorname{div}((\eta_\varepsilon + \rho^2) \nabla u) - \beta(u - g)\|_{L^2(\Omega)}.$$

Together with (3.19), this last estimate gives the desired upper bound for the slope  $|\partial\mathcal{E}_\varepsilon|(u, \rho)$ .

*Step 3.* Let  $(u, \rho) \in D(|\partial\mathcal{E}_\varepsilon|)$ . By the previous steps,  $\tilde{f} := -\operatorname{div}((\eta_\varepsilon + \rho^2) \nabla u) + \beta(u - g) \in L^2(\Omega)$  and  $u$  solves (3.11) with  $f = \tilde{f} - \beta(u - g)$ . Applying Lemma 3.10 we find that

$$\begin{aligned} \|u\|_{H^2(\Omega)} &\leq C_\varepsilon (1 + \|\nabla \rho\|_{L^p(\Omega; \mathbb{R}^N)})^\alpha (\|f\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)}) \\ &\leq C_\varepsilon (1 + \|\nabla \rho\|_{L^p(\Omega; \mathbb{R}^N)})^\alpha (|\partial\mathcal{E}_\varepsilon|(u, \rho) + \beta\|u - g\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)}), \end{aligned}$$

and the proof is complete.  $\square$

The expression of the slope and the characterization of its domain provided by Proposition 3.9 enables one to show the lower semicontinuity of  $|\partial\mathcal{E}_\varepsilon|$  along sequences with uniformly bounded energy.

**Proposition 3.11.** *Assume that  $\Omega$  has a  $\mathcal{C}^{1,1}$ -boundary. Let  $\{(u_n, \rho_n)\}_{n \in \mathbb{N}} \subset L^2(\Omega) \times L^p(\Omega)$  be such that  $\sup_{n \in \mathbb{N}} \mathcal{E}_\varepsilon(u_n, \rho_n) < \infty$  and  $(u_n, \rho_n) \rightarrow (u, \rho)$  strongly in  $L^2(\Omega) \times L^p(\Omega)$ . Then,*

$$|\partial\mathcal{E}_\varepsilon|(u, \rho) \leq \liminf_{n \rightarrow \infty} |\partial\mathcal{E}_\varepsilon|(u_n, \rho_n).$$

**Proof.** Let us assume without loss of generality that  $\liminf_n |\partial\mathcal{E}_\varepsilon|(u_n, \rho_n) < \infty$ , and extract a subsequence  $\{n_k\}$  such that

$$\liminf_{n \rightarrow \infty} |\partial\mathcal{E}_\varepsilon|(u_n, \rho_n) = \lim_{k \rightarrow \infty} |\partial\mathcal{E}_\varepsilon|(u_{n_k}, \rho_{n_k}).$$

Since  $\mathcal{E}_\varepsilon(u_{n_k}, \rho_{n_k})$  is uniformly bounded with respect to  $k$ , we deduce that the sequence  $\{(u_{n_k}, \rho_{n_k})\}$  is uniformly bounded in  $H^1(\Omega) \times W^{1,p}(\Omega)$ . Moreover  $(u_{n_k}, \rho_{n_k}) \in D(|\partial\mathcal{E}_\varepsilon|)$ , and as a consequence of Proposition 3.9, we deduce that  $\{u_{n_k}\}$  is uniformly bounded in  $H^2(\Omega)$ , and that  $\frac{\partial u_{n_k}}{\partial \nu} = 0$  in  $H^{1/2}(\partial\Omega)$ . Whence  $\rho_{n_k} \rightharpoonup \rho$  weakly in  $W^{1,p}(\Omega)$ ,  $u_{n_k} \rightharpoonup u$  weakly in  $H^2(\Omega)$  for a (not relabeled) subsequence, and  $\frac{\partial u}{\partial \nu} = 0$  in  $H^{1/2}(\partial\Omega)$ . By the Sobolev Imbedding we get that  $\rho_{n_k} \rightarrow \rho$  in  $\mathcal{C}^0(\overline{\Omega})$ , while  $u_{n_k} \rightarrow u$  strongly in  $H^1(\Omega)$ . Thanks to the uniform

convergence of  $\rho_{n_k}$  to  $\rho$ , we may argue as in the proof of Proposition 3.9, Step 1 with minor modifications, to show that the functionals  $\mathcal{F}_k : W^{1,p}(\Omega) \rightarrow [0, +\infty]$  defined by

$$\mathcal{F}_k(\hat{\rho}) := \begin{cases} \mathcal{E}_\varepsilon(u_{n_k}, \hat{\rho}) & \text{if } \hat{\rho} \leq \rho_{n_k}, \\ +\infty & \text{otherwise,} \end{cases} \quad (3.21)$$

$\Gamma$ -converge (with respect to the sequential weak  $W^{1,p}(\Omega)$ -topology) to the functional  $\mathcal{F} : W^{1,p}(\Omega) \rightarrow [0, +\infty]$  given by

$$\mathcal{F}(\hat{\rho}) := \begin{cases} \mathcal{E}_\varepsilon(u, \hat{\rho}) & \text{if } \hat{\rho} \leq \rho, \\ +\infty & \text{otherwise.} \end{cases} \quad (3.22)$$

Indeed, the lower bound inequality can be obtained as in the proof of Proposition 3.9, Step 1, while the upper bound requires the following argument. Given  $\hat{\rho} \in W^{1,p}(\Omega)$  satisfying  $\hat{\rho} \leq \rho$  in  $\Omega$ , we set  $\hat{\rho}_\delta := \hat{\rho} - \delta$  where  $\delta > 0$  is small. Since  $\rho_{n_k} \rightarrow \rho$  uniformly in  $\bar{\Omega}$ , we have  $\hat{\rho}_\delta \leq \rho_{n_k}$  in  $\Omega$  whenever  $k$  large enough (depending only on  $\delta$ ). Hence,

$$\lim_{\delta \downarrow 0} \limsup_{k \rightarrow \infty} \mathcal{F}_k(\hat{\rho}_\delta) \leq \mathcal{F}(\hat{\rho}),$$

and we obtain from  $\{\hat{\rho}_\delta\}_{\delta > 0}$  a suitable recovery sequence for  $\hat{\rho}$  through a diagonalization argument.

Since

$$\rho_{n_k} = \operatorname{argmin}_{\hat{\rho} \in W^{1,p}(\Omega)} \mathcal{F}_k(\hat{\rho}),$$

and  $\rho_{n_k} \rightharpoonup \rho$  weakly in  $W^{1,p}(\Omega)$ , we infer from the  $\Gamma$ -convergence of  $\mathcal{F}_k$  toward  $\mathcal{F}$  that

$$\rho = \operatorname{argmin}_{\hat{\rho} \in W^{1,p}(\Omega)} \mathcal{F}(\hat{\rho}).$$

By the expression of the domain of the slope provided by Proposition 3.9, we infer that  $(u, \rho) \in D(|\partial\mathcal{E}_\varepsilon|)$ . From the established convergences of  $(u_{n_k}, \rho_{n_k})$  we deduce that

$$\begin{aligned} \operatorname{div}((\eta_\varepsilon + \rho_{n_k}^2)\nabla u_{n_k}) &= (\eta_\varepsilon + \rho_{n_k}^2)\Delta u_{n_k} + 2\rho_{n_k}\nabla\rho_{n_k} \cdot \nabla u_{n_k} \\ &\rightarrow (\eta_\varepsilon + \rho^2)\Delta u + 2\rho\nabla\rho \cdot \nabla u = \operatorname{div}((\eta_\varepsilon + \rho^2)\nabla u) \end{aligned}$$

weakly in  $L^2(\Omega)$ . Using now the expression of the slope given by Proposition 3.9, we conclude

$$\begin{aligned} \liminf_{n \rightarrow \infty} |\partial\mathcal{E}_\varepsilon|(u_n, \rho_n) &= \lim_{k \rightarrow \infty} |\partial\mathcal{E}_\varepsilon|(u_{n_k}, \rho_{n_k}) \\ &= \lim_{k \rightarrow \infty} \|\operatorname{div}((\eta_\varepsilon + \rho_{n_k}^2)\nabla u_{n_k}) - \beta(u_{n_k} - g)\|_{L^2(\Omega)} \\ &\geq \|\operatorname{div}((\eta_\varepsilon + \rho^2)\nabla u) - \beta(u - g)\|_{L^2(\Omega)} = |\partial\mathcal{E}_\varepsilon|(u, \rho), \end{aligned}$$

which ends the proof.  $\square$

We next prove that the energy is continuous along convergent sequences with uniformly bounded slope.

**Proposition 3.12.** *Assume that  $\Omega$  has a  $\mathcal{C}^{1,1}$ -boundary. Let  $\{(u_n, \rho_n)\}_{n \in \mathbb{N}} \subset L^2(\Omega) \times L^p(\Omega)$  be such that*

$$\sup_{n \in \mathbb{N}} \{\mathcal{E}_\varepsilon(u_n, \rho_n) + |\partial\mathcal{E}_\varepsilon|(u_n, \rho_n)\} < \infty,$$

*and  $(u_n, \rho_n) \rightarrow (u, \rho)$  strongly in  $L^2(\Omega) \times L^p(\Omega)$ . Then  $\mathcal{E}_\varepsilon(u_n, \rho_n) \rightarrow \mathcal{E}_\varepsilon(u, \rho)$  as  $n \rightarrow \infty$ .*

**Proof.** Arguing as in the proof of Proposition 3.11, we have  $u_n \rightharpoonup u$  weakly in  $H^2(\Omega)$  and  $\rho_n \rightharpoonup \rho$  weakly in  $W^{1,p}(\Omega)$  with  $(u, \rho) \in D(|\partial\mathcal{E}_\varepsilon|)$ . By the Sobolev Imbedding,  $\rho_n \rightarrow \rho$  in  $\mathcal{C}^0(\bar{\Omega})$  and  $u_n \rightarrow u$  strongly in  $H^1(\Omega)$ . Hence the functional  $\mathcal{F}_n : W^{1,p}(\Omega) \rightarrow [0, +\infty]$  defined by (3.21) (with  $n$  in place of  $n_k$ )  $\Gamma$ -converges (with respect to the sequential weak  $W^{1,p}(\Omega)$ -topology) to the functional  $\mathcal{F} : W^{1,p}(\Omega) \rightarrow [0, +\infty]$  given by (3.22). By the convergence of the minimum values, we infer that

$$\mathcal{E}_\varepsilon(u_n, \rho_n) = \min_{W^{1,p}(\Omega)} \mathcal{F}_n \xrightarrow{n \rightarrow \infty} \min_{W^{1,p}(\Omega)} \mathcal{F} = \mathcal{E}_\varepsilon(u, \rho),$$

and the proposition is proved.  $\square$

#### 4. Compactness of discrete trajectories

The purpose of this section is to obtain compactness properties of discrete-in-time evolutions  $\{(u_{\delta_k}, \rho_{\delta_k})\}_{k \in \mathbb{N}}$  associated to a sequence  $\{\delta_k\}_{k \in \mathbb{N}}$  of partitions of  $[0, +\infty)$  as the time step length  $|\delta_k|$  tends to zero. In a first subsection we shall define a suitable notion of De Giorgi interpolants in order to derive from Scheme 1 a precise discrete energy identity (see (4.11)). This interpolation procedure extends the standard De Giorgi interpolation as defined in [4] to our unilateral setting. For Scheme 2, a direct computation shall provide an optimal energy inequality (see (4.12)). From these energy estimates and from the regularity result obtained in Proposition 3.9 (concerning the slope of the Ambrosio-Tortorelli functional), we will derive *a priori* estimates leading to the compactness of  $\{(u_{\delta_k}, \rho_{\delta_k})\}_{k \in \mathbb{N}}$  in various functional spaces.

Throughout this section we fix an arbitrary  $u_0 \in H^1(\Omega) \cap L^\infty(\Omega)$ , and we consider the function  $\rho_0^\varepsilon$  determined by (3.1). A sequence  $\{\delta_k\}_{k \in \mathbb{N}}$  of partitions of  $[0, +\infty)$  satisfying  $|\delta_k| \rightarrow 0$  is also given. We write

$$\delta_k =: \{\delta_k^i\}_{i \in \mathbb{N}^*}, \quad t_k^0 := 0, \quad \text{and} \quad t_k^i := \sum_{j=1}^i \delta_k^j \quad \text{for } i \geq 1.$$

For each  $k \in \mathbb{N}$  we consider a discrete trajectory  $(u_k, \rho_k) : [0, +\infty) \rightarrow H^1(\Omega) \times W^{1,p}(\Omega)$  associated to  $\delta_k$  which is obtained from either Scheme 1 or Scheme 2 (and (3.6)). We also assume that every element of the resulting sequence  $\{(u_k, \rho_k)\}_{k \in \mathbb{N}}$  arises from the same scheme as in Definition 3.4. To simplify the notation, we write

$$(u_k^i, \rho_k^i) := (u_k(t_k^i), \rho_k(t_k^i)).$$

We next define for every  $k \in \mathbb{N}$  a further piecewise constant interpolation  $\rho_k^- : [0, +\infty) \rightarrow W^{1,p}(\Omega)$  from the iterates  $\{\rho_k^i\}_{i \in \mathbb{N}}$  setting  $\rho_k^-(0) = \rho_0^\varepsilon$ , and for  $t > 0$ ,

$$\rho_k^-(t) := \rho_k^{i-1} \quad \text{if } t \in (t_k^{i-1}, t_k^i]. \quad (4.1)$$

We also consider the piecewise affine interpolation  $v_k : [0, +\infty) \rightarrow H^1(\Omega)$  of the  $u_k^i$ 's defined for each  $k \in \mathbb{N}$  by

$$v_k(t) := u_k^{i-1} + \frac{t - t_k^{i-1}}{\delta_k^i} (u_k^i - u_k^{i-1}) \quad \text{if } t \in [t_k^{i-1}, t_k^i]. \quad (4.2)$$

##### 4.1. De Giorgi interpolants for Scheme 1

We present in this subsection a notion of unilateral De Giorgi interpolation very much in the spirit of [4, Chapter 3]. Up to minor modifications due to the unilateral constraint, the proofs of the results below follow closely their analogues in [4, Chapter 3]. For clarity reasons we have decided to present all the details.

Let us fix  $k \in \mathbb{N}$ , and assume that the iterates  $\{(u_k^i, \rho_k^i)\}_{i \in \mathbb{N}}$  are obtained from Scheme 1. For an integer  $i \geq 1$  and  $\tau > 0$ , we define  $\Phi_{i,k}(\tau; \cdot) : H^1(\Omega) \times W^{1,p}(\Omega) \rightarrow [0, +\infty]$  by

$$\Phi_{i,k}(\tau; u, \rho) := \mathcal{E}_\varepsilon(u, \rho) + \frac{1}{2\tau} \|u - u_k^{i-1}\|_{L^2(\Omega)}^2.$$

Then we consider the function  $\phi_{i,k} : (0, +\infty) \rightarrow [0, +\infty)$  given by

$$\phi_{i,k}(\tau) := \inf \left\{ \Phi_{i,k}(\tau; u, \rho) : (u, \rho) \in H^1(\Omega) \times W^{1,p}(\Omega), \rho \leq \rho_k^{i-1} \text{ in } \Omega \right\}.$$

To each  $\phi_{i,k}(\tau)$  we associate the (resolvent) set  $\mathcal{J}_{i,k}(\tau) \subset H^1(\Omega) \times W^{1,p}(\Omega)$  defined as

$$\mathcal{J}_{i,k}(\tau) := \operatorname{argmin} \Phi_{i,k}(\tau; \cdot).$$

We observe that  $\mathcal{J}_{i,k}(\tau) \neq \emptyset$  by the argument used in the proof of Lemma 3.1, and that  $(u_k^i, \rho_k^i) \in \mathcal{J}_{i,k}(\delta_k^i)$ .

**Definition 4.1 (De Giorgi interpolation).** We consider  $(\tilde{u}_k, \tilde{\rho}_k) : [0, +\infty) \rightarrow H^1(\Omega) \times W^{1,p}(\Omega)$  an arbitrary interpolation of the iterates  $\{(u_k^i, \rho_k^i)\}_{i \in \mathbb{N}}$  (obtained from Scheme 1) satisfying

$$\begin{cases} (\tilde{u}_k(t_k^i), \tilde{\rho}_k(t_k^i)) = (u_k^i, \rho_k^i) & \text{for } i \in \mathbb{N}, \\ (\tilde{u}_k(t), \tilde{\rho}_k(t)) \in \mathcal{J}_{i,k}(\tau) & \text{if } t = t_k^{i-1} + \tau \in (t_k^{i-1}, t_k^i). \end{cases}$$

Let us emphasize that, at a formal level, a De Giorgi interpolation corresponds to a unilateral minimal energy path connecting the discrete values  $\{(u_k^i, \rho_k^i)\}_{i \in \mathbb{N}}$ . To investigate the main properties of  $(\tilde{u}_k, \tilde{\rho}_k)$  it therefore of



importance to look for the regularity of the energy along such interpolation. To this purpose it is useful to introduce the functions  $d_{i,k}^\pm : (0, +\infty) \rightarrow [0, +\infty)$  defined by

$$d_{i,k}^+(\tau) := \sup_{(u_\tau, \rho_\tau) \in \mathcal{J}_{i,k}(\tau)} \|u_\tau - u_k^{i-1}\|_{L^2(\Omega)} \quad \text{and} \quad d_{i,k}^-(\tau) := \inf_{(u_\tau, \rho_\tau) \in \mathcal{J}_{i,k}(\tau)} \|u_\tau - u_k^{i-1}\|_{L^2(\Omega)}.$$

First notice that

$$|d_{i,k}^\pm(\tau)| \leq C, \quad (4.3)$$

for some constant  $C > 0$  independent of  $k, i$ , and  $\tau$ . Indeed, we can argue as in Lemma 3.3 to obtain the uniform bound

$$\|u_\tau\|_{L^\infty(\Omega)} \leq \max\{\|u_0\|_{L^\infty(\Omega)}, \|g\|_{L^\infty(\Omega)}\} \quad \text{for every } \tau \geq 0 \text{ and every } (u_\tau, \rho_\tau) \in \mathcal{J}_{i,k}(\tau),$$

from which estimate (4.3) follows.

**Lemma 4.2.** *Let  $i \geq 1$ ,  $0 < \tau_1 < \tau_2$ , and  $(u_{\tau_j}, \rho_{\tau_j}) \in \mathcal{J}_{i,k}(\tau_j)$  for  $j = 1, 2$ . We have*

- (i)  $\phi_{i,k}(\tau_2) \leq \phi_{i,k}(\tau_1) \leq \mathcal{E}_\varepsilon(u_k^{i-1}, \rho_k^{i-1})$ ;
- (ii)  $\|u_{\tau_1} - u_k^{i-1}\|_{L^2(\Omega)} \leq \|u_{\tau_2} - u_k^{i-1}\|_{L^2(\Omega)}$ ;
- (iii)  $d_{i,k}^+(\tau_1) \leq d_{i,k}^-(\tau_2) \leq d_{i,k}^+(\tau_2)$ .

In addition, there exists an (at most) countable set  $\mathcal{N}_{i,k} \subset (0, +\infty)$  such that

$$d_{i,k}^+(\tau) = d_{i,k}^-(\tau) \quad \text{for every } \tau \in (0, +\infty) \setminus \mathcal{N}_{i,k}. \quad (4.4)$$

Finally,

$$\lim_{\tau \downarrow 0} \phi_{i,k}(\tau) = \mathcal{E}_\varepsilon(u_k^{i-1}, \rho_k^{i-1}). \quad (4.5)$$

**Proof.** It is straightforward to check (i), which shows that  $\tau \mapsto \phi_{i,k}(\tau)$  is non-increasing. Then, by minimality of  $(u_{\tau_j}, \rho_{\tau_j})$ , we easily estimate

$$\begin{aligned} \frac{\|u_{\tau_1} - u_k^{i-1}\|_{L^2(\Omega)}^2}{2\tau_1} + \mathcal{E}_\varepsilon(u_{\tau_1}, \rho_{\tau_1}) &\leq \frac{\|u_{\tau_2} - u_k^{i-1}\|_{L^2(\Omega)}^2}{2\tau_1} + \mathcal{E}_\varepsilon(u_{\tau_2}, \rho_{\tau_2}) \\ &= \frac{\tau_2 - \tau_1}{2\tau_1\tau_2} \|u_{\tau_2} - u_k^{i-1}\|_{L^2(\Omega)}^2 + \phi_{i,k}(\tau_2) \leq \frac{\tau_2 - \tau_1}{2\tau_1\tau_2} \|u_{\tau_2} - u_k^{i-1}\|_{L^2(\Omega)}^2 + \frac{\|u_{\tau_1} - u_k^{i-1}\|_{L^2(\Omega)}^2}{2\tau_2} + \mathcal{E}_\varepsilon(u_{\tau_1}, \rho_{\tau_1}), \end{aligned}$$

and then derive

$$\frac{\tau_2 - \tau_1}{2\tau_1\tau_2} \|u_{\tau_1} - u_k^{i-1}\|_{L^2(\Omega)}^2 \leq \frac{\tau_2 - \tau_1}{2\tau_1\tau_2} \|u_{\tau_2} - u_k^{i-1}\|_{L^2(\Omega)}^2,$$

whence (ii) and (iii).

As a consequence of (ii), the functions  $\tau \mapsto d_{i,k}^-(\tau)$  are non-decreasing, and therefore continuous outside an (at most) countable set  $\mathcal{N}_{i,k} \subset (0, +\infty)$ . Then, for  $\tau \in (0, +\infty) \setminus \mathcal{N}_{i,k}$ ,

$$d_{i,k}^-(\tau) \leq d_{i,k}^+(\tau) \leq \lim_{s \downarrow \tau} d_{i,k}^-(s) = d_{i,k}^-(\tau),$$

and (4.4) is proved.

Let us now consider an arbitrary sequence  $\tau_n \downarrow 0$ , and  $(u_{\tau_n}, \rho_{\tau_n}) \in \mathcal{J}_{i,k}(\tau_n)$ . From (i) we first deduce that

$$\mathcal{E}_\varepsilon(u_{\tau_n}, \rho_{\tau_n}) \leq \mathcal{E}_\varepsilon(u_k^{i-1}, \rho_k^{i-1}) \quad \text{and} \quad \|u_{\tau_n} - u_k^{i-1}\|_{L^2(\Omega)}^2 \leq 2\tau_n \mathcal{E}_\varepsilon(u_k^{i-1}, \rho_k^{i-1}) \xrightarrow{n \rightarrow \infty} 0.$$

Hence we can find a subsequence (not relabeled) such that  $(u_{\tau_n}, \rho_{\tau_n}) \rightharpoonup (u_k^{i-1}, \rho_*)$  weakly in  $H^1(\Omega) \times W^{1,p}(\Omega)$  for some  $\rho_* \in W^{1,p}(\Omega)$ . From Lemma 3.1 we derive that  $\rho_* \leq \rho_k^{i-1}$  in  $\Omega$ , and

$$\mathcal{E}_\varepsilon(u_k^{i-1}, \rho_*) \leq \liminf_{n \rightarrow \infty} \mathcal{E}_\varepsilon(u_{\tau_n}, \rho_{\tau_n}) \leq \liminf_{n \rightarrow \infty} \phi_{i,k}(\tau_n) \leq \limsup_{n \rightarrow \infty} \phi_{i,k}(\tau_n) \leq \mathcal{E}_\varepsilon(u_k^{i-1}, \rho_k^{i-1}).$$

On the other hand, we have  $\mathcal{E}_\varepsilon(u_k^{i-1}, \rho_*) \geq \mathcal{E}_\varepsilon(u_k^{i-1}, \rho_k^{i-1})$  by minimality of  $\rho_k^{i-1}$ , and thus  $\phi_{i,k}(\tau_n) \rightarrow \mathcal{E}_\varepsilon(u_k^{i-1}, \rho_k^{i-1})$  as  $n \rightarrow \infty$ . Then (4.5) follows through the standard argument on the uniqueness of the limit.  $\square$

**Remark 4.3.** It follows from the previous lemma that the functions  $\tau \mapsto d_{i,k}^\pm(\tau)$  are Borel measurable.

We next intend to obtain a discrete energy equality involving the term  $\frac{d_{i,k}^+(\tau)}{\tau}$  acting as a “discrete time derivative” of  $u_k$ .

**Proposition 4.4.** *For each integer  $i \geq 1$ , the function  $\tau \mapsto \phi_{i,k}(\tau)$  is locally Lipschitz in  $(0, +\infty)$ , and*

$$\frac{d\phi_{i,k}}{d\tau}(\tau) = -\frac{(d_{i,k}^+(\tau))^2}{2\tau^2} = -\frac{(d_{i,k}^-(\tau))^2}{2\tau^2} \quad \text{for every } \tau \in (0, +\infty) \setminus \mathcal{N}_{i,k}. \quad (4.6)$$

In particular, for every  $(u_\tau, \rho_\tau) \in \mathcal{J}_{i,k}(\tau)$ ,

$$\mathcal{E}_\varepsilon(u_\tau, \rho_\tau) + \frac{1}{2\tau} \|u_\tau - u_k^{i-1}\|_{L^2(\Omega)}^2 + \frac{1}{2} \int_0^\tau \left| \frac{d_{i,k}^+(\sigma)}{\sigma} \right|^2 d\sigma = \mathcal{E}_\varepsilon(u_k^{i-1}, \rho_k^{i-1}), \quad (4.7)$$

and for each  $i \geq 1$ ,

$$\mathcal{E}_\varepsilon(u_k^i, \rho_k^i) + \frac{1}{2} \int_{t_k^{i-1}}^{t_k^i} \|v'_k(s)\|_{L^2(\Omega)}^2 ds + \frac{1}{2} \int_0^{\delta_k^i} \left| \frac{d_{i,k}^+(\sigma)}{\sigma} \right|^2 d\sigma = \mathcal{E}_\varepsilon(u_k^{i-1}, \rho_k^{i-1}). \quad (4.8)$$

**Proof.** Let us fix arbitrary  $0 < \tau_1 < \tau_2$  and  $(u_{\tau_j}, \rho_{\tau_j}) \in \mathcal{J}_{i,k}(\tau_j)$  for  $j = 1, 2$ . We first observe that

$$\phi_{i,k}(\tau_1) - \phi_{i,k}(\tau_2) \leq \Phi_{i,k}(\tau_1; u_{\tau_2}, \rho_{\tau_2}) - \Phi_{i,k}(\tau_2; u_{\tau_2}, \rho_{\tau_2}) = \frac{\tau_2 - \tau_1}{2\tau_1\tau_2} \|u_{\tau_2} - u_k^{i-1}\|_{L^2(\Omega)}^2.$$

Similarly, we obtain that

$$\phi_{i,k}(\tau_1) - \phi_{i,k}(\tau_2) \geq \frac{\tau_2 - \tau_1}{2\tau_1\tau_2} \|u_{\tau_1} - u_k^{i-1}\|_{L^2(\Omega)}^2.$$

From the arbitrariness of  $(u_{\tau_j}, \rho_{\tau_j})$  we infer that

$$\frac{(d_{i,k}^+(\tau_1))^2}{2\tau_1\tau_2} \leq \frac{\phi_{i,k}(\tau_1) - \phi_{i,k}(\tau_2)}{\tau_2 - \tau_1} \leq \frac{(d_{i,k}^-(\tau_2))^2}{2\tau_1\tau_2},$$

which shows, together with (4.3), that  $\tau \mapsto \phi_{i,k}(\tau)$  is locally Lipschitz in  $(0, +\infty)$ . For  $\tau \in (0, +\infty) \setminus \mathcal{N}_{i,k}$ , we can pass to the limit as  $\tau_1 \rightarrow \tau$  and  $\tau_2 \rightarrow \tau$  in the inequality above to derive (4.6).

Then, for  $0 < \tau_0 < \tau$ , integrating (4.6) between  $\tau_0$  and  $\tau$  yields

$$\phi_{i,k}(\tau) + \frac{1}{2} \int_{\tau_0}^\tau \left| \frac{d_{i,k}^+(\sigma)}{\sigma} \right|^2 d\sigma = \phi_{i,k}(\tau_0).$$

In view of (4.5), it now remains to let  $\tau_0 \downarrow 0$  to recover (4.7). Finally for each integer  $i \geq 1$ , we have  $(u_k^i, \rho_k^i) \in \mathcal{J}_{i,k}(\delta_k^i)$ , and so identity (4.7) yields (4.8).  $\square$

We end this subsection by estimating the slope of the Ambrosio-Tortorelli functional at the De Giorgi interpolants in terms of the “discrete time derivative” of  $u_k$ .

**Lemma 4.5.** *For an integer  $i \geq 1$ , and  $\tau > 0$ , let  $(u_\tau, \rho_\tau) \in \mathcal{J}_{i,k}(\tau)$ . Then  $(u_\tau, \rho_\tau) \in D(|\partial\mathcal{E}_\varepsilon|)$ , and*

$$|\partial\mathcal{E}_\varepsilon|(u_\tau, \rho_\tau) \leq \frac{1}{\tau} \|u_\tau - u_k^{i-1}\|_{L^2(\Omega)}.$$

In particular, for every  $t > 0$  we have  $(\tilde{u}_k(t), \tilde{\rho}_k(t)) \in D(|\partial\mathcal{E}_\varepsilon|)$ , and

$$|\partial\mathcal{E}_\varepsilon|(\tilde{u}_k(t), \tilde{\rho}_k(t)) \leq G_k(t), \quad (4.9)$$

where  $G_k : (0, +\infty) \rightarrow [0, +\infty)$  is defined by

$$G_k(t) := \frac{d_{i,k}^+(\tau)}{\tau} \quad \text{if } t = t_k^{i-1} + \tau \in (t_k^{i-1}, t_k^i]. \quad (4.10)$$

**Proof.** Let  $(v, \rho) \in H^1(\Omega) \times W^{1,p}(\Omega)$  be such that  $\rho \leq \rho_k^{i-1}$  in  $\Omega$ . Then, by minimality of  $(u_\tau, \rho_\tau)$ ,

$$\begin{aligned} \mathcal{E}_\varepsilon(u_\tau, \rho_\tau) - \mathcal{E}_\varepsilon(v, \rho) &\leq \frac{1}{2\tau} \|v - u_k^{i-1}\|_{L^2(\Omega)}^2 - \frac{1}{2\tau} \|u_\tau - u_k^{i-1}\|_{L^2(\Omega)}^2 \\ &\leq \frac{1}{2\tau} (\|v - u_k^{i-1}\|_{L^2(\Omega)} + \|u_\tau - u_k^{i-1}\|_{L^2(\Omega)}) \|v - u_\tau\|_{L^2(\Omega)}. \end{aligned}$$

Dividing this inequality by  $\|v - u_\tau\|_{L^2(\Omega)}$  and taking the supremum over  $\rho$ , we derive that

$$\sup_{\rho \leq \rho_\tau} \frac{(\mathcal{E}_\varepsilon(u_\tau, \rho_\tau) - \mathcal{E}_\varepsilon(v, \rho))^+}{\|v - u_\tau\|_{L^2(\Omega)}} \leq \sup_{\rho \leq \rho_k^{i-1}} \frac{(\mathcal{E}_\varepsilon(u_\tau, \rho_\tau) - \mathcal{E}_\varepsilon(v, \rho))^+}{\|v - u_\tau\|_{L^2(\Omega)}} \leq \frac{1}{2\tau} (\|v - u_k^{i-1}\|_{L^2(\Omega)} + \|u_\tau - u_k^{i-1}\|_{L^2(\Omega)}).$$

Taking the lim sup as  $v \rightarrow u_\tau$  in  $L^2(\Omega)$  then yields the desired result.  $\square$

#### 4.2. A priori estimates and energy inequalities

We first state *a priori* estimates based on discrete energy (in)equalities, which will later be essential to get compactness properties of the discrete trajectories.

**Lemma 4.6.** *If the discrete trajectory  $(u_k, \rho_k)$  is obtained from Scheme 1, then for every  $i \in \mathbb{N}$ ,*

$$\mathcal{E}_\varepsilon(u_k^i, \rho_k^i) + \frac{1}{2} \int_0^{t_k^i} \|v'_k(s)\|_{L^2(\Omega)}^2 ds + \frac{1}{2} \int_0^{t_k^i} |G_k(s)|^2 ds = \mathcal{E}_\varepsilon(u_0, \rho_0^\varepsilon). \quad (4.11)$$

**Proof.** In view of the definition (4.10) of  $G_k$ , iterating (4.8) from  $j = 1$  to  $j = i$  yields (4.11).  $\square$

For Scheme 2, we only obtain an (optimal) energy inequality between two arbitrary discrete times.

**Lemma 4.7.** *If the discrete trajectory  $(u_k, \rho_k)$  is obtained from Scheme 2, then for every integers  $j \geq i$ ,*

$$\mathcal{E}_\varepsilon(u_k^j, \rho_k^j) + \int_{t_k^i}^{t_k^j} \|v'_k(s)\|_{L^2(\Omega)}^2 ds \leq \mathcal{E}_\varepsilon(u_k^i, \rho_k^i). \quad (4.12)$$

**Proof.** By minimality, for each  $\ell \in \{i+1, \dots, j\}$ ,  $u_k^\ell$  satisfies

$$\int_\Omega \frac{u_k^\ell - u_k^{\ell-1}}{\delta_k^\ell} \varphi dx + \int_\Omega (\eta_\varepsilon + (\rho_k^{\ell-1})^2) \nabla u_k^\ell \cdot \nabla \varphi dx + \beta \int_\Omega (u_k^\ell - g) \varphi dx = 0$$

for all  $\varphi \in H^1(\Omega)$ . In view of this equation and the minimality of  $\rho_k^\ell$ , we first compute

$$\begin{aligned} \mathcal{E}_\varepsilon(u_k^{\ell-1}, \rho_k^{\ell-1}) &= \mathcal{E}_\varepsilon(u_k^\ell, \rho_k^{\ell-1}) + \int_\Omega (\eta_\varepsilon + (\rho_k^{\ell-1})^2) \nabla u_k^\ell \cdot \nabla (u_k^{\ell-1} - u_k^\ell) dx \\ &\quad + \beta \int_\Omega (u_k^\ell - g)(u_k^{\ell-1} - u_k^\ell) dx + \frac{1}{2} \int_\Omega (\eta_\varepsilon + (\rho_k^{\ell-1})^2) |\nabla (u_k^{\ell-1} - u_k^\ell)|^2 dx + \frac{\beta}{2} \int_\Omega (u_k^{\ell-1} - u_k^\ell)^2 dx, \end{aligned}$$

and then estimate

$$\mathcal{E}_\varepsilon(u_k^{\ell-1}, \rho_k^{\ell-1}) \geq \mathcal{E}_\varepsilon(u_k^\ell, \rho_k^{\ell-1}) + \delta_k^\ell \int_\Omega \left| \frac{u_k^\ell - u_k^{\ell-1}}{\delta_k^\ell} \right|^2 dx \geq \mathcal{E}_\varepsilon(u_k^\ell, \rho_k^\ell) + \int_{t_k^{\ell-1}}^{t_k^\ell} \|v'_k(s)\|_{L^2(\Omega)}^2 ds. \quad (4.13)$$

Iterating this last inequality from  $\ell = i+1$  to  $\ell = j$  yields (4.12).  $\square$

Proposition 3.9 then provides an additional  $H^2(\Omega)$  estimate uniform in  $k$  for both schemes.

**Proposition 4.8.** *Assume that  $\Omega$  has a  $\mathcal{C}^{1,1}$ -boundary. Then  $u_k$  belongs to  $L^2_{\text{loc}}([0, +\infty); H^2(\Omega))$ ,  $\frac{\partial u_k}{\partial \nu} = 0$  in  $L^2(0, +\infty; H^{1/2}(\partial\Omega))$ , and for every  $T > 0$ ,*

$$\int_0^T \|u_k(t)\|_{H^2(\Omega)}^2 dt \leq C_\varepsilon(T+1), \quad (4.14)$$

for some constant  $C_\varepsilon > 0$  independent of  $k$ .

**Proof.** *Step 1.* We first consider the case where  $u_k$  is obtain from Scheme 1. Let  $t > 0$  such that  $t \in (t_k^{i-1}, t_k^i)$  for some integer  $i \geq 1$ . By minimality  $u_k(t)$  solves

$$\begin{cases} -\operatorname{div}((\eta_\varepsilon + \rho_k^2(t)) \nabla u_k(t)) = -v'_k(t) - \beta(u_k(t) - g) & \text{in } H^{-1}(\Omega), \\ (\eta_\varepsilon + \rho_k^2(t)) \nabla u_k(t) \cdot \nu = 0 & \text{in } H^{-1/2}(\partial\Omega). \end{cases} \quad (4.15)$$

From Lemma 3.10 we deduce that  $u_k(t) \in H^2(\Omega)$ ,  $\frac{\partial u_k(t)}{\partial \nu} = 0$  in  $H^{1/2}(\partial\Omega)$ , and the estimate

$$\|u_k(t)\|_{H^2(\Omega)} \leq C_\varepsilon(1 + \|\nabla \rho_k(t)\|_{L^p(\Omega; \mathbb{R}^N)})^\alpha (\|v'_k(t)\|_{L^2(\Omega)} + \beta \|u_k(t) - g\|_{L^2(\Omega)} + \|u_k(t)\|_{H^1(\Omega)}).$$

In view of (4.11) and Lemma 3.3, we deduce that

$$\|u_k(t)\|_{H^2(\Omega)}^2 \leq C_\varepsilon (\|v'_k(t)\|_{L^2(\Omega)}^2 + 1),$$

for some constant  $C_\varepsilon > 0$  independent of  $k$ , and (4.14) follows integrating this last inequality between 0 and  $T$ .

*Step 2.* In the case where  $u_k$  is obtained from Scheme 2, we simply repeat the argument of Step 1 replacing  $\rho_k$  by  $\rho_k^-$ , and using (4.12) instead of (4.11).  $\square$

### 4.3. Compactness

We are now in position to establish some compactness results for the sequences  $\{u_k\}_{k \in \mathbb{N}}$  and  $\{\rho_k\}_{k \in \mathbb{N}}$ . We start with  $\{\rho_k\}_{k \in \mathbb{N}}$  and  $\{\rho_k^-\}_{k \in \mathbb{N}}$ .

**Lemma 4.9.** *There exist a subsequence  $k_n \rightarrow \infty$  and a strongly measurable map  $\rho_\varepsilon : [0, +\infty) \rightarrow W^{1,p}(\Omega)$  such that  $\rho_{k_n}(t) \rightharpoonup \rho_\varepsilon(t)$  weakly in  $W^{1,p}(\Omega)$  for every  $t \geq 0$ . In addition,  $\rho_\varepsilon \in L^\infty(0, +\infty; W^{1,p}(\Omega))$ ,  $\rho_\varepsilon(0) = \rho_0^\varepsilon$ , and  $0 \leq \rho_\varepsilon(t) \leq \rho_\varepsilon(s) \leq 1$  in  $\Omega$  for every  $t \geq s \geq 0$ .*

**Proof.** By Lemma 3.3,  $\rho_k : [0, +\infty) \rightarrow L^1(\Omega)$  is monotone non-increasing, and  $0 \leq \rho_k(t) \leq 1$  in  $\Omega$  for every  $t \geq 0$ . By a generalized version of Helly's selection principle (see [35, Theorem 3.2]), we deduce that there exists a subsequence  $k_n \rightarrow \infty$  and a map  $\rho_\varepsilon : [0, +\infty) \rightarrow L^1(\Omega)$  such that  $\rho_{k_n}(t) \rightharpoonup \rho_\varepsilon(t)$  weakly in  $L^1(\Omega)$  for every  $t \geq 0$ . On the other hand, since

$$\mathcal{E}_\varepsilon(u_0, \rho_0^\varepsilon) \leq \mathcal{E}_\varepsilon(u_0, 1) \leq \|\nabla u_0\|_{L^2(\Omega; \mathbb{R}^N)}^2 + \frac{\beta}{2} \|u_0 - g\|_{L^2(\Omega)}^2,$$

we derive from Lemmas 4.6 & 4.7 that

$$\sup_{t \geq 0} \|\rho_k(t)\|_{W^{1,p}(\Omega)} \leq C_\varepsilon,$$

for some constant  $C_\varepsilon > 0$  independent of  $k$ . Therefore,  $\rho_{k_n}(t) \rightharpoonup \rho_\varepsilon(t)$  weakly in  $W^{1,p}(\Omega)$ , and  $\rho_{k_n}(t) \rightarrow \rho_\varepsilon(t)$  in  $\mathcal{C}^0(\bar{\Omega})$  for every  $t \geq 0$  by the Sobolev Imbedding Theorem. In particular  $\rho_\varepsilon(t) \in W^{1,p}(\Omega)$  for every  $t \geq 0$ , and by lower semicontinuity,

$$\sup_{t \geq 0} \|\rho_\varepsilon(t)\|_{W^{1,p}(\Omega)} \leq C_\varepsilon.$$

Moreover, since  $0 \leq \rho_k(t) \leq \rho_k(s) \leq 1$  in  $\Omega$  whenever  $s \leq t$ , we deduce from the uniform convergence that  $0 \leq \rho_\varepsilon(t) \leq \rho_\varepsilon(s) \leq 1$  in  $\Omega$  for every  $t \geq s \geq 0$ .

Since  $\rho_\varepsilon : [0, +\infty) \rightarrow W^{1,p}(\Omega)$  is a pointwise weak limit of a sequence of measurable (locally) simple functions, we deduce that  $\rho_\varepsilon : [0, +\infty) \rightarrow W^{1,p}(\Omega)$  is weakly measurable, hence strongly measurable thanks to the separability of  $W^{1,p}(\Omega)$  and Pettis Theorem.  $\square$

Through the same argument we obtain the convergence of the sequence  $\{\rho_k^-\}_{k \in \mathbb{N}}$  (defined in (4.1)).

**Lemma 4.10.** *Let  $\{k_n\}_{n \in \mathbb{N}}$  be the subsequence given by Lemma 4.9. There exist a further subsequence (not relabeled) and a strongly measurable map  $\rho_\varepsilon^- : [0, +\infty) \rightarrow W^{1,p}(\Omega)$  such that  $\rho_{k_n}^-(t) \rightharpoonup \rho_\varepsilon^-(t)$  weakly in  $W^{1,p}(\Omega)$  for every  $t \geq 0$ . In addition,  $\rho_\varepsilon^- \in L^\infty(0, +\infty; W^{1,p}(\Omega))$ ,  $\rho_\varepsilon^-(0) = \rho_0^\varepsilon$ , and  $0 \leq \rho_\varepsilon^-(t) \leq \rho_\varepsilon^-(s) \leq 1$  in  $\Omega$  for every  $t \geq s \geq 0$ .*

We next derive some compactness results for the sequences  $\{u_k\}_{k \in \mathbb{N}}$  and  $\{v_k\}_{k \in \mathbb{N}}$  (defined by (4.2)).

**Lemma 4.11.** *Let  $\{k_n\}_{n \in \mathbb{N}}$  be the subsequence given by Lemma 4.10. There exist a further subsequence (not relabeled) and a strongly measurable map  $u_\varepsilon : [0, +\infty) \rightarrow H^1(\Omega)$  such that  $u_{k_n}(t) \rightharpoonup u_\varepsilon(t)$  and  $v_{k_n}(t) \rightarrow u_\varepsilon(t)$  weakly in  $H^1(\Omega)$  for every  $t \geq 0$ . In addition,*

- (i)  $u_\varepsilon(0) = u_0$ ;
- (ii)  $\|u_\varepsilon(t)\|_{L^\infty(\Omega)} \leq \max\{\|u_0\|_{L^\infty(\Omega)}, \|g\|_{L^\infty(\Omega)}\}$  for every  $t \geq 0$ ;
- (iii)  $u_\varepsilon \in L^\infty(0, +\infty; H^1(\Omega)) \cap L^2_{\text{loc}}([0, +\infty); H^2(\Omega))$ ;
- (iv)  $\frac{\partial u_\varepsilon}{\partial \nu} = 0$  in  $L^2(0, +\infty; H^{1/2}(\partial\Omega))$ ;

(v)  $u_\varepsilon \in AC^2([0, +\infty); L^2(\Omega))$  and

$$\int_0^{+\infty} \|u'_\varepsilon(t)\|_{L^2(\Omega)}^2 dt \leq \|\nabla u_0\|_{L^2(\Omega; \mathbb{R}^N)}^2 + \frac{\beta}{2} \|u_0 - g\|_{L^2(\Omega)}^2;$$

(vi)  $v'_{k_n} \rightharpoonup u'_\varepsilon$  weakly in  $L^2(0, +\infty; L^2(\Omega))$ .

**Proof.** We start by establishing the compactness of the sequence  $\{v_k\}$ . First Lemma 3.3 yields for every  $t \geq 0$ ,

$$\|v_{k_n}(t)\|_{L^\infty(\Omega)} = \|u_{k_n}(t)\|_{L^\infty(\Omega)} \leq \max\{\|u_0\|_{L^\infty(\Omega)}, \|g\|_{L^\infty(\Omega)}\}. \quad (4.16)$$

Then, combining the energy (in)equalities (4.11) for Scheme 1 or (4.12) for Scheme 2 together with (4.2), we infer that

$$\sup_{t \geq 0} \|\nabla v_{k_n}(t)\|_{L^2(\Omega; \mathbb{R}^N)} \leq \sup_{t \geq 0} \|\nabla u_{k_n}(t)\|_{L^2(\Omega; \mathbb{R}^N)} \leq C_\varepsilon, \quad (4.17)$$

for some constant  $C_\varepsilon > 0$  independent of  $k_n$ . Consequently, for every  $T > 0$  the set  $\bigcup_n v_{k_n}([0, T])$  is relatively compact in  $L^2(\Omega)$ . On the other hand, (4.11) and (4.12) yield

$$\int_0^{+\infty} \|v'_{k_n}(r)\|_{L^2(\Omega)}^2 dr \leq \mathcal{E}_\varepsilon(u_0, \rho_0^\varepsilon) \leq \mathcal{E}_\varepsilon(u_0, 1) \leq \|\nabla u_0\|_{L^2(\Omega; \mathbb{R}^N)}^2 + \frac{\beta}{2} \|u_0 - g\|_{L^2(\Omega)}^2. \quad (4.18)$$

Since for any  $t \geq s \geq 0$  we have

$$\|v_{k_n}(t) - v_{k_n}(s)\|_{L^2(\Omega)} \leq \int_s^t \|v'_{k_n}(r)\|_{L^2(\Omega)} dr, \quad (4.19)$$

we deduce from (4.18) and Cauchy-Schwarz inequality that

$$\|v_{k_n}(t) - v_{k_n}(s)\|_{L^2(\Omega)}^2 \leq (t-s) \int_s^t \|v'_{k_n}(r)\|_{L^2(\Omega)}^2 dr \leq (t-s) \left( \|\nabla u_0\|_{L^2(\Omega; \mathbb{R}^N)}^2 + \frac{\beta}{2} \|u_0 - g\|_{L^2(\Omega)}^2 \right). \quad (4.20)$$

By the Arzela-Ascoli Theorem we can find a subsequence of  $\{k_n\}$  (not relabeled) such that

$$v_{k_n} \rightarrow u_\varepsilon \quad \text{in } \mathcal{C}^0([0, T]; L^2(\Omega)) \text{ for every } T > 0, \quad (4.21)$$

for some  $u_\varepsilon \in \mathcal{C}^{0,1/2}([0, +\infty); L^2(\Omega))$ . In particular,  $v_{k_n}(t) \rightarrow u_\varepsilon(t)$  strongly in  $L^2(\Omega)$  for every  $t \geq 0$ , which yields (i) since  $v_{k_n}(0) = u_0$ .

On the other hand, in view of estimates (4.16) and (4.17), we obtain (ii) and the fact that  $v_{k_n}(t) \rightharpoonup u_\varepsilon(t)$  weakly in  $H^1(\Omega)$  for every  $t \geq 0$ . By lower semicontinuity we also deduce from (4.17) that

$$\sup_{t \geq 0} \|u_\varepsilon(t)\|_{H^1(\Omega)} \leq C_\varepsilon. \quad (4.22)$$

We next show the compactness of the sequence  $\{u_{k_n}\}$ . Let us now consider an arbitrary  $t > 0$ . For each  $n \in \mathbb{N}$  there is a unique  $i \in \mathbb{N}$  such that  $t \in (t_{k_n}^{i-1}, t_{k_n}^i]$ . We then have  $u_{k_n}(t) = u_{k_n}(t_{k_n}^i) = v_{k_n}(t_{k_n}^i)$ . Consequently, by (4.20),

$$\begin{aligned} \|u_{k_n}(t) - u_\varepsilon(t)\|_{L^2(\Omega)} &= \|v_{k_n}(t_{k_n}^i) - u_\varepsilon(t)\|_{L^2(\Omega)} \\ &\leq \|v_{k_n}(t_{k_n}^i) - v_{k_n}(t)\|_{L^2(\Omega)} + \|v_{k_n}(t) - u_\varepsilon(t)\|_{L^2(\Omega)} \\ &\leq C\sqrt{|\delta_{k_n}^i|} + \|v_{k_n}(t) - u_\varepsilon(t)\|_{L^2(\Omega)} \xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

Hence  $u_{k_n}(t) \rightarrow u_\varepsilon(t)$  strongly in  $L^2(\Omega)$  for every  $t \geq 0$ , and in view of (4.17) we infer that  $u_{k_n}(t) \rightharpoonup u_\varepsilon(t)$  weakly in  $H^1(\Omega)$  for every  $t \geq 0$ . The mappings  $t \mapsto u_{k_n}(t)$  being (locally) simple and measurable, we conclude as in the proof of Lemma 4.9 that  $u_\varepsilon : [0, +\infty) \rightarrow H^1(\Omega)$  is strongly measurable. Moreover  $u_\varepsilon \in L^\infty(0, +\infty; H^1(\Omega))$  by (4.22). By the pointwise strong  $L^2(\Omega)$ -convergence of  $u_{k_n}$  towards  $u_\varepsilon$  and the dominated convergence theorem, we have  $u_{k_n} \rightarrow u_\varepsilon$  strongly in  $L^2_{\text{loc}}(0, +\infty; L^2(\Omega))$ . On the other hand,  $\{u_{k_n}\}$  is bounded in  $L^2(0, T; H^2(\Omega))$  for every  $T > 0$  by (4.14). Hence  $u_{k_n} \rightharpoonup u_\varepsilon$  weakly in  $L^2_{\text{loc}}(0, +\infty; H^2(\Omega))$  which shows in particular that  $u_\varepsilon \in L^2_{\text{loc}}(0, +\infty; H^2(\Omega))$ . Item (iii) is thus proved.

We now prove that  $u_\varepsilon$  satisfies the Neumann boundary condition (iv). To this purpose let us fix  $T > 0$  and an arbitrary  $\varphi \in L^2(0, T; H^1(\Omega))$ . By Proposition 4.8 we have  $\frac{\partial u_{k_n}}{\partial \nu} = 0$  in  $L^2(0, T; H^{1/2}(\partial\Omega))$ , so that

$$\int_0^T \int_\Omega (-\Delta u_\varepsilon) \varphi dx dt = \lim_{n \rightarrow \infty} \int_0^T \int_\Omega (-\Delta u_{k_n}) \varphi dx dt = \lim_{n \rightarrow \infty} \int_0^T \int_\Omega \nabla u_{k_n} \cdot \nabla \varphi dx dt = \int_0^T \int_\Omega \nabla u_\varepsilon \cdot \nabla \varphi dx dt.$$

From the arbitrariness of  $\varphi$  and  $T$ , we conclude that  $\frac{\partial u_\varepsilon}{\partial \nu} = 0$  in  $L^2(0, T; H^{1/2}(\partial\Omega))$  for every  $T > 0$ .

It remains to show the absolute continuity in time of  $u_\varepsilon$ . We note that (4.18) tells us that the functions  $A_{k_n} : t \in (0, +\infty) \mapsto \|v'_{k_n}(t)\|_{L^2(\Omega)}$  are bounded in  $L^2(0, +\infty)$ . Hence we can find a further subsequence (not relabeled) such that  $A_{k_n} \rightharpoonup A$  weakly in  $L^2(0, +\infty)$ , for a non-negative function  $A \in L^2(0, +\infty)$  satisfying

$$\int_0^{+\infty} A^2(t) dt \leq \|\nabla u_0\|_{L^2(\Omega; \mathbb{R}^N)}^2 + \frac{\beta}{2} \|u_0 - g\|_{L^2(\Omega)}^2.$$

Letting  $n \rightarrow \infty$  in (4.19), we conclude that for every  $t \geq s \geq 0$ ,

$$\|u_\varepsilon(t) - u_\varepsilon(s)\|_{L^2(\Omega)} \leq \int_s^t A(r) dr,$$

which shows that  $u_\varepsilon \in AC^2([0, +\infty); L^2(\Omega))$ , whence (v).

Now, since  $\{v'_{k_n}\}$  is bounded in  $L^2(0, +\infty; L^2(\Omega))$ , up to a subsequence,  $\{v'_{k_n}\}$  converges weakly in  $L^2(0, +\infty; L^2(\Omega))$  to some element in  $L^2(0, +\infty; L^2(\Omega))$  which has to agree with  $u'_\varepsilon$  by (4.21). This implies that (vi) holds.  $\square$

**Remark 4.12.** As a consequence of (iii) and (v) in the previous lemma,  $u_\varepsilon : [0, +\infty) \rightarrow H^1(\Omega)$  is weakly continuous.

We finally state a compactness result for the De Giorgi interpolants  $\{\tilde{u}_k\}_{k \in \mathbb{N}}$  (see Definition 4.1) in the case of Scheme 1.

**Lemma 4.13.** *If the discrete trajectories  $\{(u_k, \rho_k)\}_{k \in \mathbb{N}}$  are obtained from Scheme 1, and  $\{k_n\}_{n \in \mathbb{N}}$  is the subsequence obtained in Lemma 4.11, then  $\tilde{u}_{k_n}(t) \rightharpoonup u_\varepsilon(t)$  weakly in  $H^1(\Omega)$  for every  $t \geq 0$ .*

**Proof.** Let  $t > 0$  with  $t \in (t_{k_n}^{i-1}, t_{k_n}^i]$ . In view of (4.7), (4.11), and (4.20), we have

$$\|\tilde{u}_{k_n}(t) - v_{k_n}(t)\|_{L^2(\Omega)} \leq \|\tilde{u}_{k_n}(t) - u_{k_n}^{i-1}\|_{L^2(\Omega)} + \|v_{k_n}(t) - v_{k_n}(t_{k_n}^{i-1})\|_{L^2(\Omega)} \leq C\sqrt{|\delta_{k_n}|},$$

and thus  $\tilde{u}_{k_n}(t) \rightarrow u_\varepsilon(t)$  strongly in  $L^2(\Omega)$  by Lemma 4.11. On the other hand, (4.7) and (4.11) also show that

$$\|\nabla \tilde{u}_{k_n}(t)\|_{L^2(\Omega; \mathbb{R}^N)} \leq C_\varepsilon,$$

and thus  $\tilde{u}_{k_n}(t) \rightharpoonup u_\varepsilon(t)$  weakly in  $H^1(\Omega)$ .  $\square$

As an immediate consequence of Lemmas 4.9 & 4.11, we obtain that unilateral (alternate or not) minimizing movements starting from  $(u_0, \rho_0^\varepsilon)$  do exist.

**Corollary 4.14.** *The collections  $GUMM(u_0, \rho_0^\varepsilon)$  and  $GUAMM(u_0, \rho_0^\varepsilon)$  are not empty.*

## 5. Convergence of discrete trajectories

The object of this section is to provide more accurate information on generalized unilateral (alternate) minimizing movements, and to prove that they are solutions of the unilateral gradient flow of the Ambrosio-Tortorelli functional in sense of Definition 3.7. The main results of this section can be summarized in the following theorems.

**Theorem 5.1.** *Assume that  $\Omega$  has a  $\mathcal{C}^{1,1}$ -boundary. Let  $u_0 \in H^1(\Omega) \cap L^\infty(\Omega)$  and let  $\rho_0^\varepsilon$  be given by (3.1). Any  $(u_\varepsilon, \rho_\varepsilon) \in GUMM(u_0, \rho_0^\varepsilon)$  is a curve of maximal unilateral slope for  $\mathcal{E}_\varepsilon$ . More precisely,*

$$u_\varepsilon \in AC^2([0, +\infty); L^2(\Omega)) \cap L^\infty(0, +\infty; H^1(\Omega)) \cap L^2_{\text{loc}}([0, +\infty); H^2(\Omega)),$$

$$\rho_\varepsilon \in L^\infty(0, +\infty; W^{1,p}(\Omega)), \quad 0 \leq \rho_\varepsilon(t) \leq \rho_\varepsilon(s) \leq 1 \text{ for every } t \geq s \geq 0,$$

and

$$\begin{cases} u'_\varepsilon = \operatorname{div}((\eta_\varepsilon + \rho_\varepsilon^2)\nabla u_\varepsilon) - \beta(u_\varepsilon - g) & \text{in } L^2(0, +\infty; L^2(\Omega)), \\ \frac{\partial u_\varepsilon}{\partial \nu} = 0 & \text{in } L^2(0, +\infty; H^{1/2}(\partial\Omega)), \\ u_\varepsilon(0) = u_0, \end{cases}$$

with

$$\begin{cases} \mathcal{E}_\varepsilon(u_\varepsilon(t), \rho_\varepsilon(t)) \leq \mathcal{E}_\varepsilon(u_\varepsilon(t), \rho) \text{ for every } t \geq 0 \text{ and } \rho \in W^{1,p}(\Omega) \text{ such that } \rho \leq \rho_\varepsilon(t) \text{ in } \Omega, \\ \rho_\varepsilon(0) = \rho_0^\varepsilon. \end{cases}$$

Moreover,  $t \mapsto \mathcal{E}_\varepsilon(u_\varepsilon(t), \rho_\varepsilon(t))$  has finite pointwise variation in  $[0, +\infty)$ , and there exists an (at most) countable set  $\mathcal{N}_\varepsilon \subset (0, +\infty)$  such that

- (i)  $(u_\varepsilon, \rho_\varepsilon) : [0, +\infty) \setminus \mathcal{N}_\varepsilon \rightarrow H^1(\Omega) \times W^{1,p}(\Omega)$  is strongly continuous;
- (ii) for every  $s \in [0, +\infty) \setminus \mathcal{N}_\varepsilon$ , and every  $t \geq s$ ,

$$\mathcal{E}_\varepsilon(u_\varepsilon(t), \rho_\varepsilon(t)) + \int_s^t \|u'_\varepsilon(r)\|_{L^2(\Omega)}^2 dr \leq \mathcal{E}_\varepsilon(u_\varepsilon(s), \rho_\varepsilon(s)).$$

**Theorem 5.2.** Assume that  $\Omega$  has a  $\mathcal{C}^{1,1}$ -boundary. Let  $u_0 \in H^1(\Omega) \cap L^\infty(\Omega)$  and let  $\rho_0^\varepsilon$  be given by (3.1). Let  $(u_\varepsilon, \rho_\varepsilon) \in GUAMM(u_0, \rho_0^\varepsilon)$ , and assume that for each  $t \geq 0$ ,  $(u_\varepsilon(t), \rho_\varepsilon(t))$  is a strong  $L^2(\Omega) \times L^p(\Omega)$ -limit of some discrete trajectories  $\{(u_k(t), \rho_k(t))\}_{k \in \mathbb{N}}$  obtained from Scheme 2, and some sequence of partitions  $\{\delta_k\}_{k \in \mathbb{N}}$  of  $[0, +\infty)$  satisfying  $|\delta_k| \rightarrow 0$  and

$$\sup_{k \in \mathbb{N}} \left( \sup_{i \geq 1} \frac{\delta_k^{i+1}}{\delta_k^i} \right) < \infty. \quad (5.1)$$

Then all the conclusions of Theorem 5.1 hold for  $(u_\varepsilon, \rho_\varepsilon)$ .

The entire section is devoted to the proof of those theorems. To this purpose, we consider for the rest of this section an open set  $\Omega$  with  $\mathcal{C}^{1,1}$ -boundary, an arbitrary element  $(u_\varepsilon, \rho_\varepsilon)$  in  $GUMM(u_0, \rho_0^\varepsilon)$  or  $GUAMM(u_0, \rho_0^\varepsilon)$ , a sequence  $\{\delta_k\}_{k \in \mathbb{N}}$  of partitions of  $[0, +\infty)$  satisfying  $|\delta_k| \rightarrow 0$ , and discrete trajectories  $\{(u_k, \rho_k)\}_{k \in \mathbb{N}}$  associated to  $\{\delta_k\}_{k \in \mathbb{N}}$  such that

$$(u_k(t), \rho_k(t)) \xrightarrow[k \rightarrow \infty]{} (u_\varepsilon(t), \rho_\varepsilon(t)) \quad \text{strongly in } L^2(\Omega) \times L^p(\Omega) \text{ for every } t \geq 0.$$

If  $(u_\varepsilon, \rho_\varepsilon) \in GUMM(u_0, \rho_0^\varepsilon)$  the discrete trajectories  $\{(u_k, \rho_k)\}_{k \in \mathbb{N}}$  are obtained from Scheme 1, while  $\{(u_k, \rho_k)\}_{k \in \mathbb{N}}$  are obtained from Scheme 2 if  $(u_\varepsilon, \rho_\varepsilon) \in GUAMM(u_0, \rho_0^\varepsilon)$ . In addition, we require that (5.1) holds if  $(u_\varepsilon, \rho_\varepsilon) \in GUAMM(u_0, \rho_0^\varepsilon)$ . Finally, extracting a subsequence if necessary, we may assume that all the results of Section 4 hold. For consistency we shall keep the notation of Section 4.

The plan of the proof is as follows. Let us first define the (diffuse) surface energy at a time  $t \geq 0$  by

$$\mathfrak{S}_\varepsilon(t) := \int_\Omega \left( \frac{\varepsilon^{p-1}}{p} |\nabla \rho_\varepsilon(t)|^p + \frac{\alpha}{p' \varepsilon} (1 - \rho_\varepsilon(t))^p \right) dx, \quad (5.2)$$

and the bulk energy

$$\mathfrak{B}_\varepsilon(t) := \frac{1}{2} \int_\Omega (\eta_\varepsilon + \rho_\varepsilon^2(t)) |\nabla u_\varepsilon(t)|^2 dx + \frac{\beta}{2} \int_\Omega (u_\varepsilon(t) - g)^2 dx. \quad (5.3)$$

In subsection 5.1, we establish a first minimality property for the phase field variable  $\rho_\varepsilon$  leading to monotonicity and continuity properties of the surface energy  $\mathfrak{S}_\varepsilon$  (Proposition 5.3), and then to the strong  $W^{1,p}(\Omega)$ -continuity of  $t \mapsto \rho_\varepsilon(t)$  outside a countable subset of  $(0, +\infty)$ . In subsection 5.2 we start by proving that  $u_\varepsilon$  satisfies the inhomogeneous heat equation. Exploiting a semi-group property for this equation, we show that the bulk energy  $\mathfrak{B}_\varepsilon$  has also monotonicity and continuity properties (Proposition 5.8) from which the strong  $H^1(\Omega)$ -continuity of  $t \mapsto u_\varepsilon(t)$  outside a countable subset of  $(0, +\infty)$  follows (Corollary 5.10). Then Subsection 5.3 is devoted to strong convergence results for the sequence  $\{(u_k(t), \rho_k(t))\}_{k \in \mathbb{N}}$  in  $H^1(\Omega) \times W^{1,p}(\Omega)$  (Lemma 5.12 & Proposition 5.13). As a consequence of these strong convergences, a stronger minimality property at every time for  $\rho_\varepsilon$  is obtained. Finally, we show in Subsection 5.4 that  $(u_\varepsilon, \rho_\varepsilon)$  is a curve of maximal unilateral slope (Proposition 5.14) and, as a byproduct, that an energy inequality is valid between (almost every) two arbitrary times (Corollary 5.15).

### 5.1. Time continuity for the phase field variable and the surface energy

We first establish several properties of the limiting phase field  $\rho_\varepsilon$ , starting from a (weak) minimality principle as stated in the following proposition.

**Proposition 5.3.** *For every  $t \geq 0$ ,*

$$\int_{\Omega} \left( \frac{\varepsilon^{p-1}}{p} |\nabla \rho_\varepsilon(t)|^p + \frac{\alpha}{p'\varepsilon} (1 - \rho_\varepsilon(t))^p \right) dx \leq \int_{\Omega} \left( \frac{\varepsilon^{p-1}}{p} |\nabla \rho|^p + \frac{\alpha}{p'\varepsilon} (1 - \rho)^p \right) dx$$

for all  $\rho \in W^{1,p}(\Omega)$  such that  $\rho \leq \rho_\varepsilon(t)$  in  $\Omega$ . In particular, the surface energy  $\mathfrak{S}_\varepsilon$  defined in (5.2) is non-decreasing on  $[0, +\infty)$ , and thus continuous outside an (at most) countable set  $\mathcal{S}_\varepsilon \subset [0, +\infty)$ .

**Proof.** Fix  $t > 0$  and let  $i \in \mathbb{N}$  be such that  $t \in (t_k^{i-1}, t_k^i]$ . Consider a function  $\rho \in W^{1,p}(\Omega)$  such that  $\rho \leq \rho_\varepsilon(t)$  in  $\Omega$ , and define  $\hat{\rho}_k := \rho \wedge \rho_k(t)$ . Then  $\hat{\rho}_k \in W^{1,p}(\Omega)$  and  $\hat{\rho}_k \leq \rho_k(t) \leq \rho_k^{i-1}$ . By the minimality properties of the pair  $(u_k(t), \rho_k(t))$  obtained by either Scheme 1 or 2,

$$\mathcal{E}_\varepsilon(u_k(t), \rho_k(t)) \leq \mathcal{E}_\varepsilon(u_k(t), \hat{\rho}_k),$$

and since  $\hat{\rho}_k \leq \rho_k(t)$ ,

$$\int_{\Omega} \left( \frac{\varepsilon^{p-1}}{p} |\nabla \rho_k(t)|^p + \frac{\alpha}{p'\varepsilon} (1 - \rho_k(t))^p \right) dx \leq \int_{\Omega} \left( \frac{\varepsilon^{p-1}}{p} |\nabla \hat{\rho}_k|^p + \frac{\alpha}{p'\varepsilon} (1 - \hat{\rho}_k)^p \right) dx. \quad (5.4)$$

Let us now define the measurable sets  $A_k := \{\rho \leq \rho_k(t)\}$ . By definition of  $\hat{\rho}_k$ , we have

$$\int_{\Omega} |\nabla \hat{\rho}_k|^p dx = \int_{A_k} |\nabla \rho|^p dx + \int_{\Omega \setminus A_k} |\nabla \rho_k(t)|^p dx,$$

and thanks to (5.4), we infer that

$$\frac{\varepsilon^{p-1}}{p} \int_{A_k} |\nabla \rho_k(t)|^p dx + \frac{\alpha}{p'\varepsilon} \int_{\Omega} (1 - \rho_k(t))^p dx \leq \frac{\varepsilon^{p-1}}{p} \int_{A_k} |\nabla \rho|^p dx + \frac{\alpha}{p'\varepsilon} \int_{\Omega} (1 - \hat{\rho}_k)^p dx. \quad (5.5)$$

Since  $\rho_k(t) \rightarrow \rho_\varepsilon(t)$  strongly in  $L^p(\Omega)$  and  $\rho \leq \rho_\varepsilon(t)$  in  $\Omega$ , we deduce that  $\mathcal{L}^N(\Omega \setminus A_k) \rightarrow 0$ . As a consequence,

$$\int_{A_k} |\nabla \rho|^p dx \rightarrow \int_{\Omega} |\nabla \rho|^p dx,$$

and  $\chi_{A_k} \nabla \rho_k(t) \rightarrow \nabla \rho_\varepsilon(t)$  weakly in  $L^p(\Omega; \mathbb{R}^N)$  which in turn leads to

$$\liminf_{k \rightarrow \infty} \int_{A_k} |\nabla \rho_k(t)|^p dx \geq \int_{\Omega} |\nabla \rho_\varepsilon(t)|^p dx.$$

Passing to the limit in (5.5) as  $k \rightarrow \infty$  yields

$$\int_{\Omega} \left( \frac{\varepsilon^{p-1}}{p} |\nabla \rho_\varepsilon(t)|^p + \frac{\alpha}{p'\varepsilon} (1 - \rho_\varepsilon(t))^p \right) dx \leq \int_{\Omega} \left( \frac{\varepsilon^{p-1}}{p} |\nabla \rho|^p + \frac{\alpha}{p'\varepsilon} (1 - \rho)^p \right) dx.$$

In particular, taking  $\rho = \rho_\varepsilon(s)$  with  $s \geq t$  leads to the desired monotonicity property for the function  $\mathfrak{S}_\varepsilon$ .  $\square$

At this stage we do not have any a priori time-regularity for  $t \mapsto \rho_\varepsilon(t)$  except that it is non-increasing, and thus it has finite pointwise variation (with values in  $L^1(\Omega)$ ). In the following result we show that this mapping is actually strongly continuous in  $W^{1,p}(\Omega)$  outside a countable subset of  $(0, +\infty)$  containing the discontinuity points of the surface energy  $\mathfrak{S}_\varepsilon$ .

**Lemma 5.4.** *There exists an (at most) countable set  $\mathcal{R}_\varepsilon \subset (0, +\infty)$  containing  $\mathcal{S}_\varepsilon$  such that the mapping  $t \mapsto \rho_\varepsilon(t)$  is strongly continuous in  $W^{1,p}(\Omega)$  on  $[0, +\infty) \setminus \mathcal{R}_\varepsilon$ . In particular,  $\rho_\varepsilon$  is strongly continuous at  $t = 0$ .*

**Proof.** Let  $\mathcal{R}_\varepsilon$  be the union of the set  $\mathcal{S}_\varepsilon$  given by Proposition 5.3 and the set of all discontinuity points of

$$t \mapsto \int_{\Omega} \rho_\varepsilon(t) dx. \quad (5.6)$$

Note that  $\mathcal{R}_\varepsilon$  is at most countable by the decreasing property of the latter function. Let  $t \in [0, +\infty) \setminus \mathcal{R}_\varepsilon$ , we claim that  $\rho_\varepsilon$  is strongly continuous in  $W^{1,p}(\Omega)$  at  $t$ . Consider a sequence  $t_n \rightarrow t$  and extract a subsequence  $\{t_{n_j}\} \subset \{t_n\}$  such that  $\rho_\varepsilon(t_{n_j}) \rightarrow \rho_\star$  weakly in  $W^{1,p}(\Omega)$  for some  $\rho_\star \in W^{1,p}(\Omega)$ . Upon extracting a further



subsequence, we may assume without loss of generality that  $t_{n_j} > t$  for each  $j \in \mathbb{N}$  (the other case  $t_{n_j} < t$  can be treated in a similar way). Then  $\rho_\varepsilon(t_{n_j}) \leq \rho_\varepsilon(t)$  in  $\Omega$ , and passing to the limit yields  $\rho_\star \leq \rho_\varepsilon(t)$  in  $\Omega$ . On the other hand, by our choice of  $t$  as a continuity point of the mapping (5.6), we have

$$\int_{\Omega} \rho_\varepsilon(t) dx = \lim_{j \rightarrow \infty} \int_{\Omega} \rho_\varepsilon(t_{n_j}) dx = \int_{\Omega} \rho_\star dx,$$

and thus  $\rho_\star = \rho_\varepsilon(t)$ . As a consequence, the limit is independent of the choice of the subsequence, and the full sequence  $\{\rho_\varepsilon(t_n)\}$  weakly converges to  $\rho_\varepsilon(t)$  in  $W^{1,p}(\Omega)$ . Finally, using the fact that  $t$  is a continuity point of  $\mathfrak{S}_\varepsilon$ , we get that  $\mathfrak{S}_\varepsilon(t_n) \rightarrow \mathfrak{S}_\varepsilon(t)$ , and thus  $\|\rho_\varepsilon(t_n)\|_{W^{1,p}(\Omega)} \rightarrow \|\rho_\varepsilon(t)\|_{W^{1,p}(\Omega)}$ . We then deduce that  $\rho_\varepsilon(t_n) \rightarrow \rho_\varepsilon(t)$  strongly in  $W^{1,p}(\Omega)$ .

It now remains to show that  $\rho_\varepsilon$  is continuous at  $t = 0$ . Let  $t_j \downarrow 0$  be an arbitrary sequence. By Remark 4.12 we have  $u_\varepsilon(t_j) \rightharpoonup u_0$  weakly in  $H^1(\Omega)$ . By Lemma 4.9,  $\rho_\varepsilon \in L^\infty(0, +\infty; W^{1,p}(\Omega))$ , and we can extract a (not relabeled) subsequence such that  $\rho_\varepsilon(t_j) \rightharpoonup \rho_\star$  weakly in  $W^{1,p}(\Omega)$  for some  $\rho_\star \in W^{1,p}(\Omega)$ . According to the energy inequality proved in Lemma 4.6 & 4.7, we have  $\mathcal{E}_\varepsilon(u_k(t_j), \rho_k(t_j)) \leq \mathcal{E}_\varepsilon(u_0, \rho_0^\varepsilon)$  for all  $j \in \mathbb{N}$  and all  $k \in \mathbb{N}$ . Now we apply Lemma 3.1 to pass to the limit first as  $k \rightarrow \infty$  and then as  $j \rightarrow \infty$ , which yields  $\mathcal{E}_\varepsilon(u_0, \rho_\star) \leq \mathcal{E}_\varepsilon(u_0, \rho_0^\varepsilon)$ . From the minimality property (3.1) satisfied by  $\rho_0^\varepsilon$ , we deduce that  $\mathcal{E}_\varepsilon(u_0, \rho_\star) = \mathcal{E}_\varepsilon(u_0, \rho_0^\varepsilon)$ . By uniqueness of the solution of the minimization problem (3.1), we have  $\rho_\star = \rho_0^\varepsilon$ . Moreover, we infer from the discussion above that  $\lim_j \mathcal{E}_\varepsilon(u_\varepsilon(t_j), \rho_\varepsilon(t_j)) = \mathcal{E}_\varepsilon(u_0, \rho_0^\varepsilon)$ , which implies that  $\rho_\varepsilon(t_j) \rightarrow \rho_0^\varepsilon$  strongly in  $W^{1,p}(\Omega)$  by Lemma 3.1. This convergence holds for the full sequence  $\{t_j\}$  by uniqueness of the limit.  $\square$

Thanks to the just established continuity of  $t \mapsto \rho_\varepsilon(t)$ , we deduce that  $\rho_\varepsilon^-$  and  $\rho_\varepsilon$  actually coincide almost everywhere in time whenever (5.1) holds.

**Corollary 5.5.** *Assume that (5.1) holds. Then there exists an  $\mathcal{L}^1$ -negligible set  $\mathcal{M}_\varepsilon \subset [0, +\infty)$  such that  $\rho_\varepsilon^-(t) = \rho_\varepsilon(t)$  for every  $t \in [0, +\infty) \setminus \mathcal{M}_\varepsilon$ .*

**Proof.** *Step 1.* Let us consider the function  $\ell_k : [0, +\infty) \rightarrow [0, +\infty)$  defined by

$$\ell_k(t) := \begin{cases} 0 & \text{if } t \in [0, t_k^1], \\ t_k^{i-1} + \frac{\delta_k^i}{\delta_k^{i+1}}(t - t_k^i) & \text{if } t \in (t_k^i, t_k^{i+1}] \text{ with } i \geq 1. \end{cases}$$

Notice that

$$\sup_{t \geq 0} |\ell_k(t) - t| \leq 3|\delta_k| \xrightarrow[k \rightarrow \infty]{} 0.$$

Setting

$$\rho_\varepsilon^k(t) := \rho_\varepsilon(\ell_k(t)),$$

we infer from Lemma 5.4 that  $\rho_\varepsilon^k(t) \rightarrow \rho_\varepsilon(t)$  strongly in  $L^1(\Omega)$  for every  $t \in [0, +\infty) \setminus \mathcal{R}_\varepsilon$ . Since  $0 \leq \rho_\varepsilon \leq 1$ , by dominated convergence we have  $\rho_\varepsilon^k \rightarrow \rho_\varepsilon$  strongly in  $L^1(0, T; L^1(\Omega))$  for every  $T > 0$ . Similarly, by (3.5) we have that  $\rho_k \rightarrow \rho_\varepsilon$  and  $\rho_k^- \rightarrow \rho_\varepsilon^-$  strongly in  $L^1(0, T; L^1(\Omega))$  for every  $T > 0$ . Given  $T > 0$  arbitrary, we estimate

$$\|\rho_\varepsilon^- - \rho_\varepsilon\|_{L^1(0, T; L^1(\Omega))} \leq \|\rho_\varepsilon^- - \rho_k^-\|_{L^1(0, T; L^1(\Omega))} + \|\rho_k^- - \rho_\varepsilon^k\|_{L^1(0, T; L^1(\Omega))} + \|\rho_\varepsilon^k - \rho_\varepsilon\|_{L^1(0, T; L^1(\Omega))} \xrightarrow[k \rightarrow \infty]{} 0.$$

Indeed, observing that  $\rho_k^-(t) = \rho_k(\ell_k(t))$  and  $\ell_k(t) \leq t$ , we deduce from (5.1) that

$$\begin{aligned} \int_0^T \|\rho_k^-(t) - \rho_\varepsilon^k(t)\|_{L^1(\Omega)} dt &= \int_0^T \|\rho_k(\ell_k(t)) - \rho_\varepsilon(\ell_k(t))\|_{L^1(\Omega)} dt \\ &\leq \left( \sup_{i \geq 1} \frac{\delta_k^{i+1}}{\delta_k^i} \right) \int_0^T \|\rho_k(t) - \rho_\varepsilon(t)\|_{L^1(\Omega)} dt \xrightarrow[k \rightarrow \infty]{} 0. \end{aligned}$$

Hence  $\|\rho_\varepsilon^- - \rho_\varepsilon\|_{L^1(0, T; L^1(\Omega))} = 0$  for every  $T > 0$ , whence  $\rho_\varepsilon^-(t) = \rho_\varepsilon(t)$  for all  $t \in [0, +\infty) \setminus \mathcal{M}_\varepsilon$  for some  $\mathcal{L}^1$ -negligible set  $\mathcal{M}_\varepsilon \subset [0, +\infty)$ .  $\square$

## 5.2. Time continuity for $u_\varepsilon$ and the bulk energy

We start proving the following convergence result which is the main point to show that  $u_\varepsilon$  solves the inhomogeneous heat equation.

**Proposition 5.6.** *If  $(u_\varepsilon, \rho_\varepsilon) \in GU MM(u_0, \rho_0^\varepsilon)$  then*

$$\operatorname{div}((\eta_\varepsilon + \rho_k^2) \nabla u_k) \xrightarrow[k \rightarrow \infty]{} \operatorname{div}((\eta_\varepsilon + \rho_\varepsilon^2) \nabla u_\varepsilon) \quad \text{weakly in } L_{\text{loc}}^2([0, +\infty); L^2(\Omega)),$$

while

$$\operatorname{div}((\eta_\varepsilon + (\rho_k^-)^2) \nabla u_k) \xrightarrow[k \rightarrow \infty]{} \operatorname{div}((\eta_\varepsilon + \rho_\varepsilon^2) \nabla u_\varepsilon) \quad \text{weakly in } L_{\text{loc}}^2([0, +\infty); L^2(\Omega)),$$

if  $(u_\varepsilon, \rho_\varepsilon) \in GU AMM(u_0, \rho_0^\varepsilon)$ .

**Proof.** *Step 1.* We first consider the case where  $(u_\varepsilon, \rho_\varepsilon) \in GU MM(u_0, \rho_0^\varepsilon)$ . For  $t > 0$  such that  $t \in (t_k^{i-1}, t_k^i]$  for some integer  $i \geq 1$ , the function  $u_k(t)$  solves (4.15). Hence  $\operatorname{div}((\eta_\varepsilon + \rho_k^2(t)) \nabla u_k(t)) \in L^2(\Omega)$  with the estimate

$$\|\operatorname{div}((\eta_\varepsilon + \rho_k^2(t)) \nabla u_k(t))\|_{L^2(\Omega)} \leq \|v_k'(t)\|_{L^2(\Omega)} + \beta \|u_k(t) - g\|_{L^2(\Omega)}.$$

In view of (3.5) and (4.18), we deduce that for every  $T > 0$ ,

$$\int_0^T \|\operatorname{div}((\eta_\varepsilon + \rho_k^2(t)) \nabla u_k(t))\|_{L^2(\Omega)}^2 dt \leq C \int_0^T \|v_k'(t)\|_{L^2(\Omega)}^2 dt + CT \leq C(T+1),$$

for some constant  $C > 0$  independent of  $k$  and  $T$ . Hence we can find a subsequence (not relabeled) such that  $\operatorname{div}((\eta_\varepsilon + \rho_k^2) \nabla u_k) \rightharpoonup \Theta$  weakly in  $L_{\text{loc}}^2([0, +\infty); L^2(\Omega))$ . Let  $\varphi \in \mathcal{C}_c^\infty([0, T] \times \Omega)$ . We have

$$\int_0^T \int_\Omega \operatorname{div}((\eta_\varepsilon + \rho_k^2) \nabla u_k) \varphi dx dt = - \int_0^T \int_\Omega (\eta_\varepsilon + \rho_k^2) \nabla u_k \cdot \nabla \varphi dx dt. \quad (5.7)$$

From the convergences established in Lemmas 4.9 and 4.11 we have

$$\int_\Omega (\eta_\varepsilon + \rho_k^2(t)) \nabla u_k(t) \cdot \nabla \varphi(t) dx \xrightarrow[k \rightarrow \infty]{} \int_\Omega (\eta_\varepsilon + \rho_\varepsilon^2(t)) \nabla u_\varepsilon(t) \cdot \nabla \varphi(t) dx \quad \text{for every } t \geq 0.$$

Then, from (3.5) and (4.17) we deduce that

$$\int_0^T \int_\Omega (\eta_\varepsilon + \rho_k^2) \nabla u_k \cdot \nabla \varphi dx dt \xrightarrow[k \rightarrow \infty]{} \int_0^T \int_\Omega (\eta_\varepsilon + \rho_\varepsilon^2) \nabla u_\varepsilon \cdot \nabla \varphi dx dt,$$

by dominated convergence. Therefore, letting  $k \rightarrow \infty$  in (5.7) yields

$$\int_0^T \int_\Omega \Theta \varphi dx dt = - \int_0^T \int_\Omega (\eta_\varepsilon + \rho_\varepsilon^2) \nabla u_\varepsilon \cdot \nabla \varphi dx dt$$

from which we deduce that  $\Theta = \operatorname{div}((\eta_\varepsilon + \rho_\varepsilon^2) \nabla u_\varepsilon)$ , and this first step is complete.

*Step 2.* The case where  $(u_\varepsilon, \rho_\varepsilon) \in GU AMM(u_0, \rho_0^\varepsilon)$  is essentially the same, replacing  $\rho_k$  by the translated  $\rho_k^-$ , and using the property established in Corollary 5.5. We do not reproduce the proof for this case.  $\square$

**Corollary 5.7.** *The function  $u_\varepsilon \in AC^2([0, +\infty); L^2(\Omega))$  solves*

$$\begin{cases} u_\varepsilon' = \operatorname{div}((\eta_\varepsilon + \rho_\varepsilon^2) \nabla u_\varepsilon) - \beta(u_\varepsilon - g) & \text{in } L^2(0, +\infty; L^2(\Omega)), \\ \frac{\partial u_\varepsilon}{\partial \nu} = 0 & \text{in } L^2(0, +\infty; H^{1/2}(\partial\Omega)), \\ u_\varepsilon(0) = u_0. \end{cases} \quad (5.8)$$

**Proof.** It suffices to combine (4.15) (replacing  $\rho_k(t)$  by  $\rho_k^-(t)$  for the case  $(u_\varepsilon, \rho_\varepsilon) \in GU AMM(u_0, \rho_0^\varepsilon)$ ) with the convergence results of Lemma 4.11 and Proposition 5.6.  $\square$

We are now in position to prove the decrease monotonicity of the bulk energy  $\mathfrak{B}_\varepsilon$ .

**Proposition 5.8.** *Let  $t_0 > 0$  and set  $\rho_\varepsilon^{t_0}(t) := \rho_\varepsilon(t + t_0)$ . For any  $w_0 \in H^1(\Omega) \cap L^\infty(\Omega)$  there exists a unique solution  $w_\varepsilon \in AC^2([0, +\infty); L^2(\Omega)) \cap L^\infty(0, +\infty; H^1(\Omega))$  of*

$$\begin{cases} w'_\varepsilon = \operatorname{div}((\eta_\varepsilon + (\rho_\varepsilon^{t_0})^2) \nabla w_\varepsilon) - \beta(w_\varepsilon - g) & \text{in } L^2_{\text{loc}}([0, +\infty); H^{-1}(\Omega)), \\ (\eta_\varepsilon + (\rho_\varepsilon^{t_0})^2) \nabla w_\varepsilon \cdot \nu = 0 & \text{in } L^2_{\text{loc}}([0, +\infty); H^{-1/2}(\partial\Omega)), \\ w_\varepsilon(0) = w_0, \end{cases} \quad (5.9)$$

and  $w_\varepsilon$  satisfies the following energy inequality for every  $t \geq 0$ ,

$$\frac{1}{2} \int_{\Omega} (\eta_\varepsilon + (\rho_\varepsilon^{t_0}(t))^2) |\nabla w_\varepsilon(t)|^2 dx + \frac{\beta}{2} \int_{\Omega} (w_\varepsilon(t) - g)^2 dx \leq \frac{1}{2} \int_{\Omega} (\eta_\varepsilon + \rho_\varepsilon^2(t_0)) |\nabla w_0|^2 dx + \frac{\beta}{2} \int_{\Omega} (w_0 - g)^2 dx. \quad (5.10)$$

In particular, for any  $t_0 > 0$  the function  $u_\varepsilon(\cdot + t_0)$  is the unique solution of (5.9) with initial datum  $w_0 := u_\varepsilon(t_0)$ . As a consequence, the bulk energy  $\mathfrak{B}_\varepsilon$  defined in (5.3) is non increasing on  $[0, +\infty)$ , and thus continuous outside an (at most) countable subset  $\mathcal{B}_\varepsilon$  of  $[0, +\infty)$ .

**Proof.** *Step 1, Uniqueness.* Let  $w_{\varepsilon,1}$  and  $w_{\varepsilon,2}$  be two solutions of (5.9), and set  $z_\varepsilon := w_{\varepsilon,1} - w_{\varepsilon,2}$ . Then  $z_\varepsilon(0) = 0$ . The variational formulation of (5.9) implies that for any  $T > 0$  and any test function  $\phi \in L^2(0, T; H^1(\Omega))$ ,

$$\int_0^T \int_{\Omega} (z'_\varepsilon \phi + (\eta_\varepsilon + (\rho_\varepsilon^{t_0})^2) \nabla z_\varepsilon \cdot \nabla \phi + \beta z_\varepsilon \phi) dx dt = 0.$$

Choosing  $\phi(t) := z_\varepsilon(t) \chi_{[0, T]}(t)$  as test function above yields

$$\int_0^T \int_{\Omega} z'_\varepsilon z_\varepsilon dx dt \leq 0 \quad \text{for every } T > 0.$$

On the other hand, since  $z_\varepsilon \in AC^2([0, +\infty); L^2(\Omega))$ , we have  $\|z_\varepsilon(\cdot)\|_{L^2(\Omega)}^2 \in AC([0, +\infty))$  and

$$\frac{d}{dt} \|z_\varepsilon(t)\|_{L^2(\Omega)}^2 = 2 \int_{\Omega} z'_\varepsilon(t) z_\varepsilon(t) dx \quad \text{for a.e. } t \in (0, +\infty).$$

Therefore,

$$0 \geq \int_0^T \int_{\Omega} z'_\varepsilon z_\varepsilon dx dt = \frac{1}{2} \|z_\varepsilon(T)\|_{L^2(\Omega)}^2 \quad \text{for every } T > 0,$$

which shows that  $z_\varepsilon \equiv 0$ , i.e.,  $w_{\varepsilon,1} = w_{\varepsilon,2}$ .

*Step 2, Existence.* For what concerns existence, we reproduce a minimizing movement scheme as before. More precisely, given a sequence  $\tau_k \downarrow 0$ , we set  $\tau_k^i := i\tau_k$  for  $i \in \mathbb{N}$ . Taking  $w_k^0 := w_0$ , we define recursively for all integer  $i \geq 1$ ,  $w_k^i \in H^1(\Omega)$  as the unique solution of the minimization problem

$$\min_{v \in H^1(\Omega)} \left\{ \frac{1}{2} \int_{\Omega} (\eta_\varepsilon + (\rho_\varepsilon^{t_0}(\tau_k^{i-1}))^2) |\nabla v|^2 dx + \frac{\beta}{2} \int_{\Omega} (v - g)^2 dx + \frac{1}{2\tau_k} \int_{\Omega} (v - w_k^{i-1})^2 dx \right\}.$$

Using the minimality of  $w_k^i$  at each step and the fact that  $0 \leq \rho_\varepsilon^{t_0}(\tau_k^i) \leq \rho_\varepsilon^{t_0}(\tau_k^{i-1})$ , we obtain that for every integer  $i \geq 1$ ,

$$\begin{aligned} \frac{1}{2} \int_{\Omega} (\eta_\varepsilon + (\rho_\varepsilon^{t_0}(\tau_k^{i-1}))^2) |\nabla w_k^i|^2 dx + \frac{\beta}{2} \int_{\Omega} (w_k^i - g)^2 dx + \sum_{j=1}^i \frac{1}{2\tau_k} \int_{\Omega} (w_k^j - w_k^{j-1})^2 dx \\ \leq \frac{1}{2} \int_{\Omega} (\eta_\varepsilon + \rho_\varepsilon^2(t_0)) |\nabla w_0|^2 dx + \frac{\beta}{2} \int_{\Omega} (w_0 - g)^2 dx. \end{aligned} \quad (5.11)$$

Let us now define the following piecewise constant and piecewise affine interpolations. Set  $w_k(0) = \hat{w}_k(0) = w_0$ , and for  $t \in (\tau_k^{i-1}, \tau_k^i]$ ,

$$\begin{cases} w_k(t) := w_k^i, \\ \varrho_k^{t_0}(t) := \rho_\varepsilon^{t_0}(\tau_k^{i-1}), \\ \hat{w}_k(t) := w_k^{i-1} + \tau_k^{-1}(t - \tau_k^{i-1})(w_k^i - w_k^{i-1}). \end{cases}$$

By Lemma 5.4, we have  $\varrho_k^{t_0}(t) \rightarrow \rho_\varepsilon^{t_0}(t)$  strongly in  $W^{1,p}(\Omega)$  for all  $t \in [0, +\infty) \setminus (-t_0 + \mathcal{R}_\varepsilon)$ . Arguing exactly as in the proof of Lemma 4.11 we prove that (for a suitable subsequence)  $w_k(t) \rightharpoonup w_\varepsilon(t)$  weakly in  $H^1(\Omega)$  for every

$t \geq 0$  and  $\hat{w}'_k \rightharpoonup w'_\varepsilon$  weakly in  $L^2(0, +\infty; L^2(\Omega))$ , for some  $w_\varepsilon \in AC^2([0, +\infty); L^2(\Omega)) \cap L^\infty(0, +\infty; H^1(\Omega)) \cap L^2_{\text{loc}}([0, +\infty); H^2(\Omega))$ . Then we can reproduce with minor modifications the proof of Proposition 5.6 and Corollary 5.7 to show that  $w_\varepsilon$  is a solution of (5.9).

Since  $0 \leq \rho_\varepsilon^{t_0}(t) \leq \rho_k^{t_0}(t)$  and  $w_k(t) \rightarrow w_\varepsilon(t)$  strongly in  $L^2(\Omega)$  for every  $t \geq 0$ , we infer from (5.11) that for every  $t \geq 0$ ,

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} (\eta_\varepsilon + \rho_\varepsilon^2(t_0)) |\nabla w_0|^2 dx + \frac{\beta}{2} \int_{\Omega} (w_0 - g)^2 dx \\ & \geq \liminf_{k \rightarrow \infty} \left( \frac{1}{2} \int_{\Omega} (\eta_\varepsilon + (\rho_k^{t_0}(t))^2) |\nabla w_k(t)|^2 dx + \frac{\beta}{2} \int_{\Omega} (w_k(t) - g)^2 dx \right) \\ & \geq \liminf_{k \rightarrow \infty} \frac{1}{2} \int_{\Omega} (\eta_\varepsilon + (\rho_\varepsilon^{t_0}(t))^2) |\nabla w_k(t)|^2 dx + \frac{\beta}{2} \int_{\Omega} (w_\varepsilon(t) - g)^2 dx \\ & \geq \frac{1}{2} \int_{\Omega} (\eta_\varepsilon + (\rho_\varepsilon^{t_0}(t))^2) |\nabla w_\varepsilon(t)|^2 dx + \frac{\beta}{2} \int_{\Omega} (w_\varepsilon(t) - g)^2 dx, \end{aligned}$$

and (5.10) is proved.  $\square$

**Remark 5.9.** We notice that the proof of Lemma 5.4 together with Remark 4.12 show that the function  $\mathfrak{B}_\varepsilon$  is actually continuous at time  $t = 0$ , i.e.,  $0 \notin \mathcal{B}_\varepsilon$ .

As a consequence of Lemma 5.4 and Proposition 5.8, we obtain the strong continuity in  $H^1(\Omega)$  of the mapping  $t \mapsto u_\varepsilon(t)$  outside a countable subset of  $(0, +\infty)$  containing the discontinuity points of  $\mathfrak{S}_\varepsilon$  and  $\mathfrak{B}_\varepsilon$ .

**Corollary 5.10.** *The mapping  $u_\varepsilon : [0, +\infty) \rightarrow H^1(\Omega)$  is strongly continuous on  $[0, +\infty) \setminus (\mathcal{R}_\varepsilon \cup \mathcal{B}_\varepsilon)$ .*

**Proof.** Let us consider  $t_0 \in [0, +\infty) \setminus (\mathcal{R}_\varepsilon \cup \mathcal{B}_\varepsilon)$  and  $\{t_n\} \subset [0, +\infty)$  an arbitrary sequence such that  $t_n \rightarrow t_0$ . Since  $t_0 \notin \mathcal{R}_\varepsilon \cup \mathcal{B}_\varepsilon$  we have  $\mathfrak{B}_\varepsilon(t_n) \rightarrow \mathfrak{B}_\varepsilon(t_0)$  and  $\rho_\varepsilon(t_n) \rightarrow \rho_\varepsilon(t_0)$  strongly in  $W^{1,p}(\Omega)$ . Therefore  $\mathcal{E}_\varepsilon(u_\varepsilon(t_n), \rho_\varepsilon(t_n)) \rightarrow \mathcal{E}_\varepsilon(u_\varepsilon(t_0), \rho_\varepsilon(t_0))$ . On the other hand  $u_\varepsilon(t_n) \rightharpoonup u_\varepsilon(t_0)$  weakly in  $H^1(\Omega)$  by Remark 4.12, and the conclusion follows from Lemma 3.1.  $\square$

### 5.3. Strong convergences and limiting minimality

Thanks to the equation solved by  $u_\varepsilon$ , we are now able to improve the weak  $H^1(\Omega)$ -convergence of the sequence  $\{u_k\}_{k \in \mathbb{N}}$  into a strong convergence. We start by proving that the bulk energy converges in time average.

**Lemma 5.11.** *If  $(u_\varepsilon, \rho_\varepsilon) \in GUMM(u_0, \rho_0^\varepsilon)$  we have*

$$\lim_{k \rightarrow \infty} \int_s^t \int_{\Omega} (\eta_\varepsilon + \rho_k^2(r)) |\nabla u_k(r)|^2 dx dr = \int_s^t \int_{\Omega} (\eta_\varepsilon + \rho_\varepsilon^2(r)) |\nabla u_\varepsilon(r)|^2 dx dr \quad \text{for every } t > s \geq 0, \quad (5.12)$$

while

$$\lim_{k \rightarrow \infty} \int_s^t \int_{\Omega} (\eta_\varepsilon + (\rho_k^-(r))^2) |\nabla u_k(r)|^2 dx dr = \int_s^t \int_{\Omega} (\eta_\varepsilon + \rho_\varepsilon^2(r)) |\nabla u_\varepsilon(r)|^2 dx dr \quad \text{for every } t > s \geq 0,$$

if  $(u_\varepsilon, \rho_\varepsilon) \in GUAMM(u_0, \rho_0^\varepsilon)$ .

**Proof.** We only consider the case  $(u_\varepsilon, \rho_\varepsilon) \in GUMM(u_0, \rho_0^\varepsilon)$ . In the case  $(u_\varepsilon, \rho_\varepsilon) \in GUAMM(u_0, \rho_0^\varepsilon)$ , it suffices to reproduce the argument below with  $\rho_k^-$  instead of  $\rho_k$ .

Taking  $u_k(r)$  as test function in the variational formulation (4.15) and integrating in time between  $s$  and  $t$  leads to

$$\int_s^t \int_{\Omega} (\eta_\varepsilon + \rho_k^2(r)) |\nabla u_k(r)|^2 dx dr = - \int_s^t \int_{\Omega} v'_k(r) u_k(r) dx dr - \beta \int_s^t \int_{\Omega} (u_k(r) - g) u_k(r) dx dr.$$

From Lemma 4.11 we have  $u_k \rightarrow u_\varepsilon$  strongly in  $L^2_{\text{loc}}([0, +\infty); L^2(\Omega))$  and  $v'_k \rightharpoonup u'_\varepsilon$  weakly in  $L^2(0, +\infty; L^2(\Omega))$ . Therefore,

$$\lim_{k \rightarrow \infty} \int_s^t \int_{\Omega} (\eta_\varepsilon + \rho_k^2(r)) |\nabla u_k(r)|^2 dx dt = - \int_s^t \int_{\Omega} u'_\varepsilon(r) u_\varepsilon(r) dx dr - \beta \int_s^t \int_{\Omega} (u_\varepsilon(r) - g) u_\varepsilon(r) dx dr.$$

On the other hand, according to equation (5.8) solved by  $u_\varepsilon$ , we have

$$-\int_s^t \int_\Omega u'_\varepsilon(r) u_\varepsilon(r) dx dr - \beta \int_s^t \int_\Omega (u_\varepsilon(r) - g) u_\varepsilon(r) dx dr = \int_s^t \int_\Omega (\eta_\varepsilon + \rho_\varepsilon^2(r)) |\nabla u_\varepsilon(r)|^2 dx dr,$$

which leads to (5.12).  $\square$

From Lemma 5.11, we deduce the announced strong convergence of the sequence  $\{u_k\}_{k \in \mathbb{N}}$ .

**Lemma 5.12.** *For every  $t \in [0, +\infty) \setminus \mathcal{B}_\varepsilon$ ,  $u_k(t) \rightarrow u_\varepsilon(t)$  strongly in  $H^1(\Omega)$ .*

**Proof.** *Step 1.* We first assume that  $(u_\varepsilon, \rho_\varepsilon) \in GUMM(u_0, \rho_0^\varepsilon)$ . Let  $t_0 \in [0, +\infty) \setminus \mathcal{B}_\varepsilon$ . Since  $\mathfrak{B}_\varepsilon$  is continuous at  $t_0$ , for every  $\alpha > 0$  there exists  $\delta_\alpha > 0$  such that  $\mathfrak{B}_\varepsilon(t) \leq \mathfrak{B}_\varepsilon(t_0) + \alpha$  for all  $t \in [t_0 - \delta_\alpha, t_0]$ .

Let us fix  $\alpha > 0$  arbitrary. Since  $\mathcal{E}_\varepsilon(u_k^i, \rho_k^i) \leq \mathcal{E}_\varepsilon(u_k^{i-1}, \rho_k^i)$  and  $\rho_k^i \leq \rho_k^{i-1}$  in  $\Omega$  for each integers  $k$  and  $i \geq 1$ , we infer that the function

$$t \mapsto \frac{1}{2} \int_\Omega (\eta_\varepsilon + \rho_k^2(t)) |\nabla u_k(t)|^2 dx + \frac{\beta}{2} \int_\Omega (u_k(t) - g)^2 dx$$

is non increasing on  $[0, +\infty)$ , and thus

$$\begin{aligned} \delta_\alpha \left( \frac{1}{2} \int_\Omega (\eta_\varepsilon + \rho_k^2(t_0)) |\nabla u_k(t_0)|^2 dx + \frac{\beta}{2} \int_\Omega (u_k(t_0) - g)^2 dx \right) \\ \leq \int_{t_0 - \delta_\alpha}^{t_0} \left( \frac{1}{2} \int_\Omega (\eta_\varepsilon + \rho_k^2(t)) |\nabla u_k(t)|^2 dx + \frac{\beta}{2} \int_\Omega (u_k(t) - g)^2 dx \right) dt. \end{aligned}$$

By Lemma 5.11 and the strong convergence of  $u_k$  to  $u_\varepsilon$  in  $L^2_{\text{loc}}([0, +\infty); L^2(\Omega))$ , we infer that

$$\begin{aligned} \delta_\alpha \limsup_{k \rightarrow \infty} \left( \frac{1}{2} \int_\Omega (\eta_\varepsilon + \rho_k^2(t_0)) |\nabla u_k(t_0)|^2 dx + \frac{\beta}{2} \int_\Omega (u_k(t_0) - g)^2 dx \right) \\ \leq \lim_{k \rightarrow \infty} \int_{t_0 - \delta_\alpha}^{t_0} \left( \frac{1}{2} \int_\Omega (\eta_\varepsilon + \rho_k^2(t)) |\nabla u_k(t)|^2 dx + \frac{\beta}{2} \int_\Omega (u_k(t) - g)^2 dx \right) dt \\ = \int_{t_0 - \delta_\alpha}^{t_0} \mathfrak{B}_\varepsilon(t) dt \leq (\mathfrak{B}_\varepsilon(t_0) + \alpha) \delta_\alpha. \end{aligned}$$

Dividing the previous inequality by  $\delta_\alpha$  and using the strong convergence of  $u_k(t_0)$  in  $L^2(\Omega)$ , we derive in view of the arbitrariness of  $\alpha$  that

$$\limsup_{k \rightarrow \infty} \int_\Omega (\eta_\varepsilon + \rho_k^2(t_0)) |\nabla u_k(t_0)|^2 dx \leq \int_\Omega (\eta_\varepsilon + \rho_\varepsilon^2(t_0)) |\nabla u_\varepsilon(t_0)|^2 dx.$$

As in the proof of Lemma 3.1 we obtain

$$\int_\Omega (\eta_\varepsilon + \rho_\varepsilon^2(t_0)) |\nabla u_\varepsilon(t_0)|^2 dx \leq \liminf_{k \rightarrow \infty} \int_\Omega (\eta_\varepsilon + \rho_k^2(t_0)) |\nabla u_k(t_0)|^2 dx.$$

Combining the last two inequalities we conclude

$$\lim_{k \rightarrow \infty} \int_\Omega (\eta_\varepsilon + \rho_k^2(t_0)) |\nabla u_k(t_0)|^2 dx = \int_\Omega (\eta_\varepsilon + \rho_\varepsilon^2(t_0)) |\nabla u_\varepsilon(t_0)|^2 dx. \quad (5.13)$$

Finally, using (5.13), the weak convergence of  $\rho_k(t_0)$  to  $\rho_\varepsilon(t_0)$  in  $W^{1,p}(\Omega)$ , and the weak convergence of  $u_k(t_0)$  to  $u_\varepsilon(t_0)$  in  $H^1(\Omega)$ , we can argue as in the proof of Lemma 3.1 to show that  $u_k(t_0) \rightarrow u_\varepsilon(t_0)$  strongly in  $H^1(\Omega)$ .

*Step 2.* If  $(u_\varepsilon, \rho_\varepsilon) \in GUAMM(u_0, \rho_0^\varepsilon)$ , we essentially proceed as in Step 1 up to the modification indicated below. First notice that inequality (4.13) shows that for each integer  $k$  the function

$$t \mapsto \frac{1}{2} \int_\Omega (\eta_\varepsilon + (\rho_k^-(t))^2) |\nabla u_k(t)|^2 dx + \frac{\beta}{2} \int_\Omega (u_k(t) - g)^2 dx$$

is non increasing on  $[0, +\infty)$ . Then we conclude as in Step 1 that

$$\begin{aligned} \limsup_{k \rightarrow \infty} \int_\Omega (\eta_\varepsilon + \rho_k^2(t_0)) |\nabla u_k(t_0)|^2 dx &\leq \limsup_{k \rightarrow \infty} \int_\Omega (\eta_\varepsilon + (\rho_k^-(t_0))^2) |\nabla u_k(t_0)|^2 dx \\ &\leq \int_\Omega (\eta_\varepsilon + \rho_\varepsilon^2(t_0)) |\nabla u_\varepsilon(t_0)|^2 dx. \end{aligned}$$

At this stage, it suffices to continue the argument of Step 1 to reach the conclusion.  $\square$

We now derive the (strong) minimality property for  $\rho_\varepsilon(t)$  at all times, as well as the strong  $W^{1,p}(\Omega)$ -convergence of  $\{\rho_k(t)\}_{k \in \mathbb{N}}$  at all continuity points of the bulk energy.

**Proposition 5.13.** *For every  $t \geq 0$  the function  $\rho_\varepsilon(t)$  satisfies*

$$\mathcal{E}_\varepsilon(u_\varepsilon(t), \rho_\varepsilon(t)) \leq \mathcal{E}_\varepsilon(u_\varepsilon(t), \rho) \quad \text{for all } \rho \in W^{1,p}(\Omega) \text{ such that } \rho \leq \rho_\varepsilon(t) \text{ in } \Omega. \quad (5.14)$$

In addition, if  $t \in [0, +\infty) \setminus \mathcal{B}_\varepsilon$  then

$$\rho_\varepsilon(t) = \operatorname{argmin} \{ \mathcal{E}_\varepsilon(u_\varepsilon(t), \rho) : \rho \in W^{1,p}(\Omega) \text{ such that } \rho \leq \rho_\varepsilon^-(t) \text{ in } \Omega \}, \quad (5.15)$$

and  $\rho_k(t) \rightarrow \rho_\varepsilon(t)$  strongly in  $W^{1,p}(\Omega)$ .

**Proof.** Let us fix an arbitrary  $t \geq 0$ . Since  $u_k(t) \rightharpoonup u_\varepsilon(t)$  weakly in  $H^1(\Omega)$ , we can find a (not relabeled) subsequence and a nonnegative Radon measure  $\mu \in \mathcal{M}(\mathbb{R}^N)$  supported in  $\bar{\Omega}$  such that

$$|\nabla u_k(t)|^2 \mathcal{L}^N \llcorner \Omega \rightharpoonup |\nabla u_\varepsilon(t)|^2 \mathcal{L}^N \llcorner \Omega + \mu$$

weakly\* in  $\mathcal{M}(\mathbb{R}^N)$ . Then we consider the functionals  $\mathcal{F}_k$  and  $\mathcal{F}$  defined on  $W^{1,p}(\Omega)$  by

$$\mathcal{F}_k(\rho) := \begin{cases} \mathcal{E}_\varepsilon(u_k(t), \rho) & \text{if } \rho \leq \rho_k^-(t), \\ +\infty & \text{otherwise,} \end{cases} \quad \text{and} \quad \mathcal{F}(\rho) := \begin{cases} \mathcal{E}_\varepsilon(u_\varepsilon(t), \rho) + \frac{1}{2} \int_{\bar{\Omega}} (\eta_\varepsilon + \rho^2) d\mu & \text{if } \rho \leq \rho_\varepsilon^-(t), \\ +\infty & \text{otherwise.} \end{cases}$$

Since  $\rho_k^-(t) \rightharpoonup \rho_\varepsilon^-(t)$  weakly in  $W^{1,p}(\Omega)$ , we may argue as in the proofs of Proposition 3.9 (Step 1) and Proposition 3.11 to show that  $\mathcal{F}_k$   $\Gamma$ -converges to  $\mathcal{F}$  with respect to the sequential weak topology in  $W^{1,p}(\Omega)$ . Since

$$\rho_k(t) = \operatorname{argmin}_{\rho \in W^{1,p}(\Omega)} \mathcal{F}_k(\rho),$$

and  $\rho_k(t) \rightharpoonup \rho_\varepsilon(t)$  weakly in  $W^{1,p}(\Omega)$ , we infer that

$$\rho_\varepsilon(t) = \operatorname{argmin}_{\rho \in W^{1,p}(\Omega)} \mathcal{F}(\rho). \quad (5.16)$$

Let us now fix an arbitrary  $\rho \in W^{1,p}(\Omega)$  such that  $\rho \leq \rho_\varepsilon(t)$  in  $\Omega$ , and set  $\rho^+ := \rho \wedge 0$ . Then  $\rho^+ \in W^{1,p}(\Omega)$ ,  $0 \leq \rho^+ \leq \rho_\varepsilon(t)$  in  $\Omega$ , and  $\mathcal{E}_\varepsilon(u_\varepsilon(t), \rho^+) \leq \mathcal{E}_\varepsilon(u_\varepsilon(t), \rho)$ . Since  $\rho^+ \leq \rho_\varepsilon(t) \leq \rho_\varepsilon^-(t)$  in  $\Omega$ , we have  $\mathcal{F}(\rho_\varepsilon(t)) \leq \mathcal{F}(\rho^+)$  which leads to

$$\mathcal{E}_\varepsilon(u_\varepsilon(t), \rho_\varepsilon(t)) \leq \mathcal{E}_\varepsilon(u_\varepsilon(t), \rho_\varepsilon(t)) + \frac{1}{2} \int_{\bar{\Omega}} (\rho_\varepsilon^2(t) - (\rho^+)^2) d\mu \leq \mathcal{E}_\varepsilon(u_\varepsilon(t), \rho^+) \leq \mathcal{E}_\varepsilon(u_\varepsilon(t), \rho),$$

and (5.14) is proved.

Next we observe that if  $t \in [0, +\infty) \setminus \mathcal{B}_\varepsilon$ , then  $\mu = 0$  by Lemma 5.12. Hence  $\mathcal{F}(\rho) = \mathcal{E}_\varepsilon(u_\varepsilon(t), \rho)$  for every  $\rho \in W^{1,p}(\Omega)$  such that  $\rho \leq \rho_\varepsilon^-(t)$  in  $\Omega$ , and (5.15) is a consequence of (5.16). From the  $\Gamma$ -convergence of  $\mathcal{F}_k$  to  $\mathcal{F}$  we also have  $\min \mathcal{F}_k \rightarrow \min \mathcal{F}$ , and thus

$$\mathcal{E}_\varepsilon(u_k(t), \rho_k(t)) \xrightarrow[k \rightarrow \infty]{} \mathcal{E}_\varepsilon(u_\varepsilon(t), \rho_\varepsilon(t)),$$

and the strong convergence in  $W^{1,p}(\Omega)$  of  $\rho_k(t)$  follows from Lemma 3.1.  $\square$

#### 5.4. Energy inequalities

**Proposition 5.14.** *The mapping  $(u_\varepsilon, \rho_\varepsilon) : [0, +\infty) \rightarrow L^2(\Omega) \times L^p(\Omega)$  is a curve of maximal unilateral slope for  $\mathcal{E}_\varepsilon$ .*

**Proof.** Let us define for each  $k \in \mathbb{N}$  and  $t \geq 0$ ,  $\lambda_k(t) := \mathcal{E}_\varepsilon(u_k(t), \rho_k(t))$ . By Lemmas 4.6 & 4.7 the function  $\lambda_k : [0, +\infty) \rightarrow [0, +\infty)$  is non-increasing and bounded uniformly with respect to  $k$ . By Helly's Theorem for monotone functions we can find a (not relabeled) subsequence such that

$$\lambda_k(t) \xrightarrow[k \rightarrow \infty]{} \lambda(t) \quad \text{for every } t \geq 0,$$

for some non-increasing function  $\lambda : [0, +\infty) \rightarrow [0, +\infty)$ . Then we infer from Lemma 5.12 and Proposition 5.13 that

$$\lambda(t) = \mathcal{E}_\varepsilon(u_\varepsilon(t), \rho_\varepsilon(t)) \quad \text{for every } t \in [0, +\infty) \setminus \mathcal{B}_\varepsilon. \quad (5.17)$$

We shall now distinguish both cases  $(u_\varepsilon, \rho_\varepsilon) \in GUMM(u_0, \rho_0^\varepsilon)$  and  $(u_\varepsilon, \rho_\varepsilon) \in GUAMM(u_0, \rho_0^\varepsilon)$ .

*Case 1.* We first assume that  $(u_\varepsilon, \rho_\varepsilon) \in GUMM(u_0, \rho_0^\varepsilon)$ . Let us fix  $t \geq s > 0$ , and write  $s \in (t_k^{i_s-1}, t_k^{i_s}]$  and  $t \in (t_k^{i_t-1}, t_k^{i_t}]$ . As in the proof of Lemma 4.6 we deduce from (4.8) that

$$\lambda_k(t) + \frac{1}{2} \int_{t_k^{i_s}}^{t_k^{i_t}} \|v'_k(r)\|_{L^2(\Omega)}^2 dr + \frac{1}{2} \int_{t_k^{i_s}}^{t_k^{i_t}} |G_k(r)|^2 dr \leq \lambda_k(s). \quad (5.18)$$

Applying (vi) in Lemma 4.11 and Fatou's lemma, we let  $k \rightarrow \infty$  in (5.18) to obtain

$$\lambda(t) + \frac{1}{2} \int_s^t \|u'_\varepsilon(r)\|_{L^2(\Omega)}^2 dr + \frac{1}{2} \int_s^t \liminf_{k \rightarrow \infty} |G_k(r)|^2 dr \leq \lambda(s).$$

We now claim that there exists an  $\mathcal{L}^1$ -negligible set  $\mathcal{L}_\varepsilon \subset (0, +\infty)$  such that

$$\liminf_{k \rightarrow \infty} |G_k(t)|^2 \geq |\partial \mathcal{E}_\varepsilon|^2(u_\varepsilon(t), \rho_\varepsilon(t)) \quad \text{for every } t \in (0, +\infty) \setminus \mathcal{L}_\varepsilon. \quad (5.19)$$

Before proving this claim we complete the argument. Thanks to (5.19) we have

$$\lambda(t) + \frac{1}{2} \int_s^t \|u'_\varepsilon(r)\|_{L^2(\Omega)}^2 dr + \frac{1}{2} \int_s^t |\partial \mathcal{E}_\varepsilon|^2(u_\varepsilon(r), \rho_\varepsilon(r)) dr \leq \lambda(s). \quad (5.20)$$

Notice that the second integral term above is well defined by the explicit formula (3.10), Proposition 5.13, and items (iii) & (iv) in Lemma 4.11. Since  $s$  and  $t$  are arbitrary, we deduce from (5.20) that

$$\lambda'(t) \leq -\frac{1}{2} \|u'_\varepsilon(t)\|_{L^2(\Omega)}^2 - \frac{1}{2} |\partial \mathcal{E}_\varepsilon|^2(u_\varepsilon(t), \rho_\varepsilon(t)) \quad \text{for a.e. } t \in (0, +\infty).$$

In view of (5.17), we have thus proved that  $(u_\varepsilon, \rho_\varepsilon)$  is a curve of maximal unilateral slope for  $\mathcal{E}_\varepsilon$ .

It now remains to prove (5.19). First notice that Lemma 4.6 implies through Fatou's lemma that

$$\int_0^{+\infty} \liminf_{k \rightarrow \infty} |G_k(r)|^2 dr < +\infty.$$

Hence we can find an  $\mathcal{L}^1$ -negligible set  $\tilde{\mathcal{L}}_\varepsilon \subset (0, +\infty)$  such that

$$C(t) := \liminf_{k \rightarrow \infty} |G_k(t)|^2 < +\infty \quad \text{for every } t \in (0, +\infty) \setminus \tilde{\mathcal{L}}_\varepsilon.$$

We set  $\mathcal{L}_\varepsilon := \tilde{\mathcal{L}}_\varepsilon \cup \mathcal{B}_\varepsilon$ , so that  $\mathcal{L}^1(\mathcal{L}_\varepsilon) = 0$ . Let us now fix an arbitrary  $t \in (0, +\infty) \setminus \mathcal{L}_\varepsilon$ . We then extract a subsequence  $\{k_n\}$  (depending on  $t$ ) such that

$$\lim_{n \rightarrow \infty} |G_{k_n}(t)|^2 = C(t).$$

In view of estimate (4.9) we deduce that

$$\sup_{n \in \mathbb{N}} |\partial \mathcal{E}_\varepsilon|(\tilde{u}_{k_n}(t), \tilde{\rho}_{k_n}(t)) < +\infty. \quad (5.21)$$

On the other hand, we infer from (4.7) and (4.11) that

$$\sup_{n \in \mathbb{N}} \mathcal{E}_\varepsilon(\tilde{u}_{k_n}(t), \tilde{\rho}_{k_n}(t)) < +\infty. \quad (5.22)$$

Then, (5.21) together with (5.22) implies that the sequence  $\{\tilde{u}_{k_n}(t)\}$  is bounded in  $H^2(\Omega)$  by Proposition 3.9, and thus  $\tilde{u}_{k_n}(t) \rightarrow u_\varepsilon(t)$  strongly in  $H^1(\Omega)$  by Lemma 4.13. By (5.22),  $\{\tilde{\rho}_{k_n}(t)\}$  is bounded in  $W^{1,p}(\Omega)$ , and up to the extraction of a further subsequence we may assume that  $\tilde{\rho}_{k_n}(t) \rightharpoonup \rho_*$  weakly in  $W^{1,p}(\Omega)$  for some  $\rho_* \in W^{1,p}(\Omega)$ .

Now we consider once again the functionals  $\tilde{\mathcal{F}}_k$  and  $\tilde{\mathcal{F}}$  defined on  $W^{1,p}(\Omega)$  by

$$\tilde{\mathcal{F}}_k(\rho) := \begin{cases} \mathcal{E}_\varepsilon(\tilde{u}_k(t), \rho) & \text{if } \rho \leq \rho_k^-(t), \\ +\infty & \text{otherwise,} \end{cases} \quad \text{and} \quad \tilde{\mathcal{F}}(\rho) := \begin{cases} \mathcal{E}_\varepsilon(u_\varepsilon(t), \rho) & \text{if } \rho \leq \rho_\varepsilon^-(t), \\ +\infty & \text{otherwise.} \end{cases}$$

Arguing as in the proof of Proposition 5.13 we obtain that  $\tilde{\mathcal{F}}_k$   $\Gamma$ -converges as  $k \rightarrow \infty$  to  $\tilde{\mathcal{F}}$  with respect to the sequential weak topology of  $W^{1,p}(\Omega)$ . By the very definition of  $\tilde{\rho}_{k_n}(t)$  we have

$$\tilde{\rho}_{k_n}(t) = \operatorname{argmin}_{\rho \in W^{1,p}(\Omega)} \tilde{\mathcal{F}}_{k_n}(\rho),$$

and since  $\tilde{\rho}_{k_n}(t) \rightharpoonup \rho_*$  weakly in  $W^{1,p}(\Omega)$ , we deduce that

$$\rho_* = \operatorname{argmin}_{\rho \in W^{1,p}(\Omega)} \tilde{\mathcal{F}}(\rho).$$

Since  $t \notin \mathcal{B}_\varepsilon$  we conclude from Proposition 5.13 that  $\rho_* = \rho_\varepsilon(t)$  by uniqueness of the minimizer (again due to the strict convexity of  $\mathcal{E}_\varepsilon(u_\varepsilon(t), \cdot)$ ).

To summarize, we have established that  $\tilde{u}_{k_n}(t) \rightarrow u_\varepsilon(t)$  strongly in  $H^1(\Omega)$ , and that  $\tilde{\rho}_{k_n}(t) \rightharpoonup \rho_\varepsilon(t)$  weakly in  $W^{1,p}(\Omega)$ . By (5.21) we can now apply Proposition 3.11 and use (4.9) to derive that

$$|\partial \mathcal{E}_\varepsilon|^2(u_\varepsilon(t), \rho_\varepsilon(t)) \leq \liminf_{n \rightarrow \infty} |\partial \mathcal{E}_\varepsilon|^2(\tilde{u}_{k_n}(t), \tilde{\rho}_{k_n}(t)) \leq \lim_{n \rightarrow \infty} |G_{k_n}(t)|^2,$$

and (5.19) is proved.

*Case 2.* We now assume that  $(u_\varepsilon, \rho_\varepsilon) \in GUAMM(u_0, \rho_0^\varepsilon)$ . Let us fix again  $t \geq s > 0$ , and write  $s \in (t_k^{i_s-1}, t_k^{i_s}]$  and  $t \in (t_k^{i_t-1}, t_k^{i_t}]$ . From Lemma 4.7 we deduce that

$$\lambda_k(t) + \int_{t_k^{i_s}}^{t_k^{i_t-1}} \|v'_k(r)\|_{L^2(\Omega)}^2 dr \leq \lambda_k(s). \quad (5.23)$$

Still applying (vi) in Lemma 4.11 and Fatou's lemma, we let  $k \rightarrow \infty$  in (5.23) to obtain

$$\lambda(t) + \int_s^t \|u'_\varepsilon(r)\|_{L^2(\Omega)}^2 dr \leq \lambda(s). \quad (5.24)$$

On the other hand, in view of Lemma 4.11 (iii)-(iv), the minimality (5.14) satisfied by  $\rho_\varepsilon$ , and the equation (5.8) satisfied by  $u_\varepsilon$ , we infer from Proposition 3.9 that

$$\|u'_\varepsilon(r)\|_{L^2(\Omega)} = |\partial \mathcal{E}_\varepsilon|(u_\varepsilon(r), \rho_\varepsilon(r)) \quad \text{for a.e. } r \in (0, +\infty). \quad (5.25)$$

Then (5.24) yields (5.20), and the conclusion follows as in Case 1.  $\square$

**Corollary 5.15.** *For every  $s \in [0, +\infty) \setminus \mathcal{B}_\varepsilon$  and every  $t \geq s$ ,*

$$\mathcal{E}_\varepsilon(u_\varepsilon(t), \rho_\varepsilon(t)) + \int_s^t \|u'_\varepsilon(r)\|_{L^2(\Omega)}^2 dr \leq \mathcal{E}_\varepsilon(u_\varepsilon(s), \rho_\varepsilon(s)).$$

**Proof.** *Step 1.* We assume in this step that  $(u_\varepsilon, \rho_\varepsilon) \in GUMM(u_0, \rho_0^\varepsilon)$ . Let us fix  $s \in (0, +\infty) \setminus \mathcal{B}_\varepsilon$  and  $t \geq s$ . Using the notation of the previous proof, we first notice that  $\mathcal{E}_\varepsilon(u_\varepsilon(t), \rho_\varepsilon(t)) \leq \lambda(t)$  for every  $t \geq 0$  by Lemma 3.1. Then, combining (5.17) with (5.20) yields

$$\mathcal{E}_\varepsilon(u_\varepsilon(t), \rho_\varepsilon(t)) + \frac{1}{2} \int_s^t \|u'_\varepsilon(r)\|_{L^2(\Omega)}^2 dr + \frac{1}{2} \int_s^t |\partial \mathcal{E}_\varepsilon|^2(u_\varepsilon(r), \rho_\varepsilon(r)) dr \leq \mathcal{E}_\varepsilon(u_\varepsilon(s), \rho_\varepsilon(s)).$$

As in the previous proof, Lemma 4.11 (iii)-(iv), (5.14), (5.8), and Proposition 3.9 imply that (5.25) holds, and the result follows.

*Step 2.* If  $(u_\varepsilon, \rho_\varepsilon) \in GUAMM(u_0, \rho_0^\varepsilon)$ , we still have  $\mathcal{E}_\varepsilon(u_\varepsilon(t), \rho_\varepsilon(t)) \leq \lambda(t)$  for every  $t \geq 0$  by Lemma 3.1. Then the result is a direct consequence of (5.17) and (5.24).  $\square$

## 6. Asymptotic behavior of unilateral minimizing movements in the Mumford-Shah limit

The main goal of this section is to analyse the behavior of a unilateral (alternate) minimizing movement as  $\varepsilon$  tends to zero. We prove that in the limit  $\varepsilon \rightarrow 0$ , we recover a parabolic type evolution for the Mumford-Shah functional under the irreversible growth constraint on the crack set similar to [12]. The result rests on the approximation of the Mumford-Shah functional by the Ambrosio-Tortorelli functional by means of  $\Gamma$ -convergence proved in [5,6,28]. The main result of this section is the following theorem.



**Theorem 6.1.** *Assume that  $\Omega$  has a  $\mathcal{C}^{1,1}$ -boundary. Let  $\varepsilon_n \downarrow 0$  be an arbitrary sequence,  $u_0 \in H^1(\Omega) \cap L^\infty(\Omega)$ , and  $\rho_0^{\varepsilon_n}$  determined by (3.1). Let  $\{(u_{\varepsilon_n}, \rho_{\varepsilon_n})\}_{n \in \mathbb{N}}$  be a sequence in either  $GUMM(u_0, \rho_0^{\varepsilon_n})$  or  $GUAMM(u_0, \rho_0^{\varepsilon_n})$ . In the case  $\{(u_{\varepsilon_n}, \rho_{\varepsilon_n})\}_{n \in \mathbb{N}} \subset GUAMM(u_0, \rho_0^{\varepsilon_n})$ , assume in addition that assumption (5.1) in Theorem 5.2 holds. Then there exist a (not relabeled) subsequence and  $u_* \in AC^2([0, +\infty); L^2(\Omega))$  such that*

$$\begin{cases} \rho_{\varepsilon_n}(t) \rightarrow 1 \text{ strongly in } L^p(\Omega) \text{ for every } t \geq 0, \\ u_{\varepsilon_n}(t) \rightarrow u_*(t) \text{ strongly in } L^2(\Omega) \text{ for every } t \geq 0, \\ u'_{\varepsilon_n} \rightharpoonup u'_* \text{ weakly in } L^2(0, +\infty; L^2(\Omega)). \end{cases} \quad (6.1)$$

For every  $t \geq 0$  the function  $u_*(t)$  belongs to  $SBV^2(\Omega) \cap L^\infty(\Omega)$  with

$$\|u_*(t)\|_{L^\infty(\Omega)} \leq \max\{\|u_0\|_{L^\infty(\Omega)}, \|g\|_{L^\infty(\Omega)}\}, \quad (6.2)$$

and  $\nabla u_* \in L^\infty(0, +\infty; L^2(\Omega; \mathbb{R}^N))$ . Moreover  $u_*$  solves

$$\begin{cases} u'_* = \operatorname{div}(\nabla u_*) - \beta(u_* - g) & \text{in } L^2(0, +\infty; L^2(\Omega)), \\ \nabla u_* \cdot \nu = 0 & \text{in } L^2(0, +\infty; H^{-1/2}(\partial\Omega)), \\ u_*(0) = u_0, \end{cases}$$

and there exists a family of countably  $\mathcal{H}^{N-1}$ -rectifiable subsets  $\{\Gamma(t)\}_{t \geq 0}$  of  $\overline{\Omega}$  such that

- (i)  $\Gamma(s) \subset \Gamma(t)$  for every  $0 \leq s \leq t$ ;
- (ii)  $J_{u_*(t)} \tilde{\subset} \Gamma(t)$  for every  $t \geq 0$ ;
- (iii) for every  $t \geq 0$ ,

$$\begin{aligned} \frac{1}{2} \int_{\Omega} |\nabla u_*(t)|^2 dx + \mathcal{H}^{N-1}(\Gamma(t)) + \frac{\beta}{2} \int_{\Omega} (u_*(t) - g)^2 dx + \int_0^t \|u'_*(s)\|_{L^2(\Omega)}^2 ds \\ \leq \frac{1}{2} \int_{\Omega} |\nabla u_0|^2 dx + \frac{\beta}{2} \int_{\Omega} (u_0 - g)^2 dx. \end{aligned}$$

This section is now essentially devoted to the proof of this theorem. To this purpose, we consider for the rest of the section an open set  $\Omega$  with  $\mathcal{C}^{1,1}$ -boundary, a sequence  $\varepsilon_n \downarrow 0$ , and an arbitrary sequence  $\{(u_{\varepsilon_n}, \rho_{\varepsilon_n})\}_{n \in \mathbb{N}}$  in  $GUMM(u_0, \rho_0^{\varepsilon_n})$  or  $GUAMM(u_0, \rho_0^{\varepsilon_n})$  (and we assume that the assumptions of Theorem 5.2 hold in this later case).

### 6.1. Compactness and the limiting heat equation

We start by proving compactness properties for the sequence  $\{(u_{\varepsilon_n}, \rho_{\varepsilon_n})\}_{n \in \mathbb{N}}$ .

**Proposition 6.2.** *There exist a (not relabeled) subsequence  $\{u_{\varepsilon_n}\}_{n \in \mathbb{N}}$  and a function  $u_* \in AC^2([0, +\infty); L^2(\Omega))$  such that (6.1) holds. In addition,  $u_*(t) \in SBV^2(\Omega) \cap L^\infty(\Omega)$  with (6.2) for every  $t \geq 0$ , and the mapping  $t \mapsto \nabla u_*(t) \in L^2(\Omega; \mathbb{R}^N)$  is strongly measurable with  $\nabla u_* \in L^\infty(0, +\infty; L^2(\Omega; \mathbb{R}^N))$ . Moreover, for every  $t \geq 0$  and any  $0 < \delta_1 < \delta_2 < 1$ , there exists  $s_n = s_n(t, \delta_1, \delta_2) \in (\delta_1, \delta_2)$  such that  $E_n := \{\rho_{\varepsilon_n}(t) < s_n\}$  has finite perimeter in  $\Omega$ ,  $\tilde{u}_{\varepsilon_n}(t) := (1 - \chi_{E_n})u_{\varepsilon_n}(t) \in SBV^2(\Omega) \cap L^\infty(\Omega)$ , and*

$$\begin{cases} \tilde{u}_{\varepsilon_n}(t) \rightarrow u_*(t) \text{ strongly in } L^2(\Omega), \\ \tilde{u}_{\varepsilon_n}(t) \rightharpoonup u_*(t) \text{ weakly}^* \text{ in } L^\infty(\Omega), \\ \nabla \tilde{u}_{\varepsilon_n}(t) \rightharpoonup \nabla u_*(t) \text{ weakly } L^2(\Omega; \mathbb{R}^N). \end{cases}$$

Finally, for any open subset  $A \subset \Omega$ ,

$$\begin{cases} \mathcal{H}^{N-1}(J_{u_*(t)} \cap A) \leq \liminf_{n \rightarrow \infty} \frac{p}{2} \int_A (1 - \rho_{\varepsilon_n}(t))^{p-1} |\nabla \rho_{\varepsilon_n}(t)| dx, \\ \int_A |\nabla u_*(t)|^2 dx \leq \liminf_{n \rightarrow \infty} \int_A (\eta_{\varepsilon_n} + \rho_{\varepsilon_n}^2(t)) |\nabla u_{\varepsilon_n}(t)|^2 dx. \end{cases} \quad (6.3)$$

**Proof.** *Step 1.* We first derive *a priori* estimates from the energy inequality obtained in Corollary 5.15. Indeed according to that result together with the minimality property (3.1) of  $\rho_0^{\varepsilon_n}$ , we infer that for every  $t \geq 0$ ,

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} (\eta_{\varepsilon_n} + \rho_{\varepsilon_n}^2(t)) |\nabla u_{\varepsilon_n}(t)|^2 dx + \frac{\varepsilon_n^{p-1}}{p} \int_{\Omega} |\nabla \rho_{\varepsilon_n}(t)|^p dx + \frac{\alpha}{p' \varepsilon_n} \int_{\Omega} (1 - \rho_{\varepsilon_n}(t))^p dx \\ & + \int_0^t \|u'_{\varepsilon_n}(s)\|_{L^2(\Omega)}^2 ds \leq \mathcal{E}_{\varepsilon_n}(u_0, \rho_0^{\varepsilon_n}) \leq \mathcal{E}_{\varepsilon_n}(u_0, 1) \leq \int_{\Omega} |\nabla u_0|^2 dx + \frac{\beta}{2} \int_{\Omega} (u_0 - g)^2 dx. \end{aligned} \quad (6.4)$$

Then, applying Young's inequality and using (2.2), we obtain

$$\frac{\varepsilon_n^{p-1}}{p} \int_{\Omega} |\nabla \rho_{\varepsilon_n}(t)|^p dx + \frac{\alpha}{p' \varepsilon_n} \int_{\Omega} (1 - \rho_{\varepsilon_n}(t))^p dx \geq \frac{p}{2} \int_{\Omega} (1 - \rho_{\varepsilon_n}(t))^{p-1} |\nabla \rho_{\varepsilon_n}(t)| dx, \quad (6.5)$$

from which we deduce the following uniform bound

$$\|u'_{\varepsilon_n}\|_{L^2(0, +\infty; L^2(\Omega))}^2 + \int_{\Omega} (\eta_{\varepsilon_n} + \rho_{\varepsilon_n}^2(t)) |\nabla u_{\varepsilon_n}(t)|^2 dx + \int_{\Omega} (1 - \rho_{\varepsilon_n}(t))^{p-1} |\nabla \rho_{\varepsilon_n}(t)| dx \leq C_0, \quad (6.6)$$

for some constant  $C_0 > 0$  independent of  $\varepsilon_n$  and  $t$ .

*Step 2.* We now establish the weak convergence of  $\{u_{\varepsilon_n}\}$  and the bound (6.2). Recalling that  $0 \leq \rho_{\varepsilon_n} \leq 1$ , the fact that

$$\rho_{\varepsilon}(t) \rightarrow 1 \quad \text{strongly in } L^p(\Omega) \text{ for every } t \geq 0,$$

is a direct consequence of (6.4). According to items (v) and (ii) in Lemma 4.11, the sequence  $\{u_{\varepsilon_n}\}$  in uniformly equi-continuous in  $L^2(\Omega)$ , and for each  $t \in [0, +\infty)$ , the sequence  $\{u_{\varepsilon_n}(t)\}$  is sequentially weakly relatively compact in  $L^2(\Omega)$ . Therefore, according to Ascoli-Arzelà Theorem, we can find a (not relabeled) subsequence and  $u_* \in AC^2([0, +\infty); L^2(\Omega))$  such that  $u_{\varepsilon_n}(t) \rightharpoonup u_*(t)$  weakly in  $L^2(\Omega)$  (and also weakly\* in  $L^\infty(\Omega)$ ) for every  $t \geq 0$ , and  $u'_{\varepsilon_n} \rightharpoonup u'_*$  weakly in  $L^2(0, +\infty; L^2(\Omega))$ . In particular, (6.2) follows from item (ii) in Lemma 4.11.

*Step 3.* We now examine more accurately the asymptotic behavior of the sequence  $\{u_{\varepsilon_n}\}$  as in [5,6,28], and prove (6.3). Let us fix  $t \geq 0$ ,  $0 < \delta_1 < \delta_2 < 1$  and an arbitrary open subset  $A$  of  $\Omega$ . According to the BV-coarea formula (see [3, Theorem 3.40]),

$$\begin{aligned} \int_A (1 - \rho_{\varepsilon_n}(t))^{p-1} |\nabla \rho_{\varepsilon_n}(t)| dx &= \int_0^1 (1-s)^{p-1} \mathcal{H}^{N-1}(\partial^* \{\rho_{\varepsilon_n}(t) < s\} \cap A) ds \\ &\geq \int_{\delta_1}^{\delta_2} (1-s)^{p-1} \mathcal{H}^{N-1}(\partial^* \{\rho_{\varepsilon_n}(t) < s\} \cap A) ds. \end{aligned} \quad (6.7)$$

Consequently, by the mean value theorem there exists some  $s_n = s_n(t, \delta_1, \delta_2, A) \in (\delta_1, \delta_2)$  such that

$$\int_A (1 - \rho_{\varepsilon_n}(t))^{p-1} |\nabla \rho_{\varepsilon_n}(t)| dx \geq \frac{\delta_2^p - \delta_1^p}{p} \mathcal{H}^{N-1}(\partial^* E_n \cap A), \quad (6.8)$$

where  $E_n := \{\rho_{\varepsilon_n}(t) < s_n\} \cap A$ . Note that from (6.4) we have

$$\mathcal{L}^N(E_n) \leq \frac{1}{(1-s_n)^p} \int_{\Omega} (1 - \rho_{\varepsilon_n}(t))^p dx \leq \frac{C \varepsilon_n}{(1-\delta_2)^p} \rightarrow 0 \text{ as } n \rightarrow \infty, \quad (6.9)$$

for some constant  $C > 0$  independent of  $n$ .

Let us define the new sequence

$$\tilde{u}_{\varepsilon_n}(t) := (1 - \chi_{E_n}) u_{\varepsilon_n}(t). \quad (6.10)$$

By (6.9) we have

$$\|u_{\varepsilon_n}(t) - \tilde{u}_{\varepsilon_n}(t)\|_{L^2(A)} \leq \|u_{\varepsilon_n}\|_{L^\infty(\Omega)} \sqrt{\mathcal{L}^N(E_n)} \rightarrow 0 \quad (6.11)$$

as  $n \rightarrow \infty$ , from which we deduce that  $\tilde{u}_{\varepsilon_n}(t) \rightharpoonup u_*(t)$  weakly in  $L^2(A)$ . On the other hand, according to [3, Theorem 3.84] we have  $\tilde{u}_{\varepsilon_n}(t) \in SBV^2(A) \cap L^\infty(A)$  with

$$\begin{cases} J_{\tilde{u}_{\varepsilon_n}(t)} \tilde{\subset} \partial^* E_n, \\ \nabla \tilde{u}_{\varepsilon_n}(t) = (1 - \chi_{E_n}) \nabla u_{\varepsilon_n}(t), \\ \|\tilde{u}_{\varepsilon_n}(t)\|_{L^\infty(A)} \leq \max\{\|u_0\|_{L^\infty(\Omega)}, \|g\|_{L^\infty(\Omega)}\}. \end{cases}$$

By the energy estimate (6.6) together with (6.7) and (6.8),

$$\|\nabla \tilde{u}_{\varepsilon_n}(t)\|_{L^2(A; \mathbb{R}^N)}^2 \leq \frac{1}{s_n^2} \int_{\Omega} \rho_{\varepsilon_n}^2(t) |\nabla u_{\varepsilon_n}(t)|^2 dx \leq \frac{C_0}{\delta_1^2},$$

and

$$\mathcal{H}^{N-1}(J_{\tilde{u}_{\varepsilon_n}(t)} \cap A) \leq \mathcal{H}^{N-1}(\partial^* E_n \cap A) \leq \frac{C_0 p}{\delta_2^p - \delta_1^p}.$$

We are now in position to apply Ambrosio's compactness Theorem in *SBV* (see Theorems 4.7 and 4.8 in [3]) to deduce that  $u_*(t) \in SBV^2(\Omega)$  (by arbitrariness of  $A$ ), and that

$$\begin{cases} \tilde{u}_{\varepsilon_n}(t) \rightarrow u_*(t) \text{ strongly in } L^2(A), \\ \tilde{u}_{\varepsilon_n}(t) \rightarrow u_*(t) \text{ weakly}^* \text{ in } L^\infty(A), \\ \nabla \tilde{u}_{\varepsilon_n}(t) \rightharpoonup \nabla u_*(t) \text{ weakly } L^2(A; \mathbb{R}^N). \end{cases}$$

In view of (6.11) we deduce that  $u_{\varepsilon_n}(t) \rightarrow u_*(t)$  strongly in  $L^2(\Omega)$  for each  $t \geq 0$  (again by arbitrariness of  $A$ ). Next Proposition 2.1 yields

$$2\mathcal{H}^{N-1}(J_{u_*(t)} \cap A) \leq \liminf_{n \rightarrow \infty} \mathcal{H}^{N-1}(\partial^* E_n \cap A).$$

Combining this inequality with (6.8) we get that

$$(\delta_2^p - \delta_1^p) \mathcal{H}^{N-1}(J_{u_*(t)} \cap A) \leq \liminf_{n \rightarrow \infty} \frac{p}{2} \int_A (1 - \rho_{\varepsilon_n}(t))^{p-1} |\nabla \rho_{\varepsilon_n}(t)| dx,$$

and the first inequality of (6.3) follows by letting  $\delta_1 \rightarrow 0$  and  $\delta_2 \rightarrow 1$ .

For what concerns the bulk energy, we have

$$\int_A (\eta_{\varepsilon_n} + \rho_{\varepsilon_n}^2(t)) |\nabla u_{\varepsilon_n}(t)|^2 dx \geq s_n^2 \int_{A \setminus E_n} |\nabla u_{\varepsilon_n}(t)|^2 dx \geq \delta_1^2 \int_{A \setminus E_n} |\nabla u_{\varepsilon_n}(t)|^2 dx.$$

Since  $\tilde{u}_{\varepsilon_n}(t) = (1 - \chi_{E_n})u_{\varepsilon_n}(t)$ , we have  $\nabla \tilde{u}_{\varepsilon_n}(t) = (1 - \chi_{E_n})\nabla u_{\varepsilon_n}(t) \rightharpoonup \nabla u_*(t)$  weakly in  $L^2(A; \mathbb{R}^N)$ , and thus

$$\liminf_{n \rightarrow \infty} \int_A (\eta_{\varepsilon_n} + \rho_{\varepsilon_n}^2(t)) |\nabla u_{\varepsilon_n}(t)|^2 dx \geq \delta_1^2 \liminf_{n \rightarrow \infty} \int_A |\nabla \tilde{u}_{\varepsilon_n}(t)|^2 dx \geq \delta_1^2 \int_A |\nabla u_*(t)|^2 dx,$$

and the second inequality of (6.3) follows by letting  $\delta_1 \rightarrow 1$ .

*Step 4.* It now remains to prove the strong measurability in  $L^2(\Omega; \mathbb{R}^N)$  of  $t \mapsto \nabla u_*(t)$ , and that  $\nabla u_* \in L^\infty(0, +\infty; L^2(\Omega; \mathbb{R}^N))$ . Given  $t \geq 0$  and  $0 < \delta_1 < \delta_2 < 1$  arbitrary, let us consider as in Step 3 the set  $E_n$  and the function  $\tilde{u}_{\varepsilon_n}(t) \in SBV^2(\Omega)$  given by (6.10) with  $A = \Omega$ . Then,

$$(\eta_{\varepsilon_n} + \rho_{\varepsilon_n}^2(t))(1 - \chi_{E_n})\nabla u_{\varepsilon_n}(t) = (\eta_{\varepsilon_n} + \rho_{\varepsilon_n}^2(t))\nabla \tilde{u}_{\varepsilon_n}(t).$$

Note that this last sequence is bounded in  $L^2(\Omega; \mathbb{R}^N)$ . Since  $\rho_{\varepsilon_n}(t) \rightarrow 1$  strongly in  $L^p(\Omega)$  with  $0 \leq \rho_{\varepsilon_n} \leq 1$ , and  $\nabla \tilde{u}_{\varepsilon_n}(t) \rightharpoonup \nabla u_*(t)$  weakly in  $L^2(\Omega; \mathbb{R}^N)$ , we deduce that

$$(\eta_{\varepsilon_n} + \rho_{\varepsilon_n}^2(t))(1 - \chi_{E_n})\nabla u_{\varepsilon_n}(t) \rightharpoonup \nabla u_*(t) \text{ weakly in } L^2(\Omega; \mathbb{R}^N). \quad (6.12)$$

On the other hand, from the a priori estimate (6.6), the Cauchy-Schwarz inequality, and (6.9), we infer that for every  $\Phi \in \mathcal{C}_c^\infty(\Omega; \mathbb{R}^N)$ ,

$$\begin{aligned} & \int_{E_n} (\eta_{\varepsilon_n} + \rho_{\varepsilon_n}^2(t)) \nabla u_{\varepsilon_n}(t) \cdot \Phi dx \\ & \leq \|\Phi\|_{L^\infty(\Omega; \mathbb{R}^N)} \left( \int_{\Omega} (\eta_{\varepsilon_n} + \rho_{\varepsilon_n}^2(t)) |\nabla u_{\varepsilon_n}(t)|^2 dx \right)^{1/2} \left( \int_{E_n} (\eta_{\varepsilon_n} + \rho_{\varepsilon_n}^2(t)) dx \right)^{1/2} \\ & \leq \|\Phi\|_{L^\infty(\Omega; \mathbb{R}^N)} \sqrt{C_0(1 + \eta_{\varepsilon_n})} \mathcal{L}^N(E_n) \rightarrow 0. \end{aligned} \quad (6.13)$$

By (6.6) and the boundedness of  $\rho_{\varepsilon_n}$ , the sequence  $\{(\eta_{\varepsilon_n} + \rho_{\varepsilon_n}(t)^2)\nabla u_{\varepsilon_n}(t)\}$  is thus bounded in  $L^2(\Omega; \mathbb{R}^N)$ , so that (6.12) and (6.13) yield

$$(\eta_{\varepsilon_n} + \rho_{\varepsilon_n}^2(t))\nabla u_{\varepsilon_n}(t) \rightharpoonup \nabla u_*(t) \text{ weakly in } L^2(\Omega; \mathbb{R}^N). \quad (6.14)$$

Finally, Lemmas 4.9 and 4.11 ensure that, for each  $n \in \mathbb{N}$ , the mappings  $t \mapsto (\eta_{\varepsilon_n} + \rho_{\varepsilon_n}(t)^2)\nabla u_{\varepsilon_n}(t)$  are strongly measurable in  $L^2(\Omega; \mathbb{R}^N)$ . Hence  $t \mapsto \nabla u_*(t)$  is weakly measurable in  $L^2(\Omega; \mathbb{R}^N)$ , and thus strongly measurable owing to Pettis Theorem. The fact that  $\nabla u_* \in L^\infty(0, +\infty; L^2(\Omega; \mathbb{R}^N))$  is a consequence of the second relation in (6.3) together with the uniform bound (6.6).  $\square$

Our next goal is to pass to the limit as  $\varepsilon_n \rightarrow 0$  in the inhomogeneous heat equation solved by  $u_{\varepsilon_n}$ .

**Proposition 6.3.** *The function  $u_*$  solves the generalized heat equation*

$$\begin{cases} u'_* = \operatorname{div}(\nabla u_*) - \beta(u_* - g) & \text{in } L^2(0, +\infty; L^2(\Omega)), \\ \nabla u_* \cdot \nu = 0 & \text{in } L^2(0, +\infty; H^{-1/2}(\partial\Omega)), \\ u_*(0) = u_0. \end{cases}$$

**Proof.** By Corollary 5.7  $u_{\varepsilon_n}$  is the solution of the equation

$$\begin{cases} u'_{\varepsilon_n} = \operatorname{div}((\eta_{\varepsilon_n} + \rho_{\varepsilon_n}^2)\nabla u_{\varepsilon_n}) - \beta(u_{\varepsilon_n} - g) & \text{in } L^2(0, +\infty; L^2(\Omega)) \\ \frac{\partial u_{\varepsilon_n}}{\partial \nu} = 0 & \text{in } L^2(0, +\infty; H^{1/2}(\partial\Omega)) \\ u_{\varepsilon_n}(0) = u_0. \end{cases}$$

According to Proposition 6.2,  $u'_{\varepsilon_n} + \beta(u_{\varepsilon_n} - g) \rightharpoonup u'_* + \beta(u_* - g)$  weakly in  $L^2_{\text{loc}}([0, +\infty); L^2(\Omega))$ , so that it remains to pass to the limit in the divergence term. Let us fix  $\Phi \in L^2(0, +\infty; L^2(\Omega; \mathbb{R}^N))$  arbitrary. Thanks to (6.14), for a.e.  $t \geq 0$  we have

$$\int_{\Omega} (\eta_{\varepsilon_n} + \rho_{\varepsilon_n}^2(t))\nabla u_{\varepsilon_n}(t) \cdot \Phi(t) \, dx \xrightarrow{n \rightarrow \infty} \int_{\Omega} \nabla u_*(t) \cdot \Phi(t) \, dx.$$

By the dominated convergence theorem, we deduce that for every  $T > 0$ ,

$$\int_0^T \int_{\Omega} (\eta_{\varepsilon_n} + \rho_{\varepsilon_n}^2)\nabla u_{\varepsilon_n} \cdot \Phi \, dx \, dt \xrightarrow{n \rightarrow \infty} \int_0^T \int_{\Omega} \nabla u_* \cdot \Phi \, dx \, dt.$$

Hence  $(\eta_{\varepsilon_n} + \rho_{\varepsilon_n}^2)\nabla u_{\varepsilon_n} \rightharpoonup \nabla u_*$  weakly in  $L^2_{\text{loc}}([0, +\infty); L^2(\Omega))$ . On the other hand, it follows from the equation that the sequence  $\{\operatorname{div}((\eta_{\varepsilon_n} + \rho_{\varepsilon_n}^2)\nabla u_{\varepsilon_n})\}$  is uniformly bounded in  $L^2_{\text{loc}}([0, +\infty); L^2(\Omega))$ . Up to a subsequence, it is therefore weakly convergent in  $L^2_{\text{loc}}([0, +\infty); L^2(\Omega))$ . Identifying the limits in the distributional sense, we infer that  $\operatorname{div}((\eta_{\varepsilon_n} + \rho_{\varepsilon_n}^2)\nabla u_{\varepsilon_n}) \rightharpoonup \operatorname{div}(\nabla u_*)$  weakly in  $L^2_{\text{loc}}([0, +\infty); L^2(\Omega))$  which yields

$$u'_* = \operatorname{div}(\nabla u_*) - \beta(u_* - g) \quad \text{in } L^2(0, +\infty; L^2(\Omega)). \quad (6.15)$$

Concerning the Neumann boundary condition, we first notice that  $\nabla u_* \in L^\infty(0, +\infty; L^2(\Omega; \mathbb{R}^N))$  and  $\operatorname{div}(\nabla u_*) \in L^2_{\text{loc}}([0, +\infty); L^2(\Omega))$  imply that  $\nabla u_*(t)$  has a well defined weak normal trace for a.e.  $t \geq 0$  on  $\partial\Omega$  (denoted by  $\nabla u_*(t) \cdot \nu$ ), and that  $t \mapsto \nabla u_*(t) \cdot \nu$  belongs to  $L^2_{\text{loc}}([0, +\infty); H^{-1/2}(\partial\Omega))$ . Then, given  $T > 0$ , we infer from the equation solved by  $u_{\varepsilon_n}$  that

$$\int_0^T \int_{\Omega} (u'_{\varepsilon_n} \phi + (\eta_{\varepsilon_n} + \rho_{\varepsilon_n}^2)\nabla u_{\varepsilon_n} \cdot \nabla \phi + \beta(u_{\varepsilon_n} - g)\phi) \, dx \, dt = 0 \quad \text{for all } \phi \in L^2(0, T; H^1(\Omega)).$$

Passing to the limit as  $\varepsilon_n \rightarrow 0$  in this variational equality yields

$$\int_0^T \int_{\Omega} (u'_* \phi + \nabla u_* \cdot \nabla \phi + \beta(u_* - g)\phi) \, dx \, dt = 0 \quad \text{for all } \phi \in L^2(0, T; H^1(\Omega)).$$

Together with (6.15) this last property ensures that  $\nabla u_* \cdot \nu = 0$  in  $L^2(0, T; H^{-1/2}(\partial\Omega))$  for every  $T > 0$ .

Finally, the initial condition  $u_*(0) = u_0$  is a consequence of the fact that  $u_{\varepsilon_n}(0) = u_0$  together with the strong convergence in  $L^2(\Omega)$  of  $u_{\varepsilon_n}(0)$  to  $u_*(0)$ .  $\square$

## 6.2. Limiting crack set and the energy inequality

Our main goal is now to pass to the limit as  $\varepsilon_n \rightarrow 0$  in the energy inequality established in Corollary 5.15. We first notice that Theorem 2.2 and Proposition 6.2 (with  $A = \Omega$ ) immediately imply that for every  $t \geq 0$ ,

$$\mathcal{E}(u_*(t)) = \frac{1}{2} \int_{\Omega} |\nabla u_*(t)|^2 dx + \mathcal{H}^{N-1}(J_{u_*(t)}) + \frac{\beta}{2} \int_{\Omega} (u_*(t) - g)^2 dx \leq \liminf_{n \rightarrow \infty} \mathcal{E}_{\varepsilon_n}(u_{\varepsilon_n}(t), \rho_{\varepsilon_n}(t)).$$

We emphasize that this lower bound only involves the measure of the jump set of  $u_*(t)$ . It will be later improved in Proposition 6.7 by replacing  $J_{u_*(t)}$  by a countably  $\mathcal{H}^{N-1}$ -rectifiable set  $\Gamma(t)$  containing  $J_{u_*(t)}$  and increasing with respect to  $t$ . This monotonicity property of the crack acts as a memory of the irreversibility of the process characterized by the non-increasing property of  $t \mapsto \rho_{\varepsilon_n}(t)$  together with the non-decreasing property of the diffuse surface energy  $\mathfrak{S}_{\varepsilon_n}$  established in Proposition 5.3.

To prove the assertion above, we fix an arbitrary countable dense subset  $D$  of  $[0, +\infty)$ , and we consider for each  $t \in D$  and  $n \in \mathbb{N}$  the bounded Radon measure

$$\mu_n(t) := \left( \frac{\varepsilon_n^{p-1}}{p} |\nabla \rho_{\varepsilon_n}(t)|^p + \frac{\alpha}{p' \varepsilon_n} (1 - \rho_{\varepsilon_n}(t))^p \right) \mathcal{L}^N \llcorner \Omega.$$

By the energy inequality (6.4), we infer that the sequences  $\{\mu_n(t)\}_{n \in \mathbb{N}}$  are uniformly bounded with respect to  $t \in D$ . Then, a standard diagonalization procedure together with the metrizable of bounded subsets of  $\mathcal{M}(\mathbb{R}^N)$  yields the existence of a subsequence (not relabeled) and a family of bounded non negative Radon measures  $\{\mu(t)\}_{t \in D}$  (supported in  $\overline{\Omega}$ ) such that

$$\mu_n(t) \rightharpoonup \mu(t) \quad \text{weakly* in } \mathcal{M}(\mathbb{R}^N) \text{ for every } t \in D.$$

We first claim that the mapping  $t \in D \mapsto \mu(t)$  inherits the increase monotonicity of the diffuse surface energy.

**Lemma 6.4.** *For every  $s$  and  $t \in D$  with  $0 \leq s \leq t$  we have*

$$\mu(s) \leq \mu(t).$$

**Proof.** Let us fix  $s$  and  $t \in D$  with  $0 \leq s \leq t$ . Let  $B \subset \mathbb{R}^N$  be an arbitrary Borel set, and  $K \subset B \subset A$  where  $A$  is open and  $K$  is compact. Let us consider a cut-off function  $\zeta \in \mathcal{C}_c^\infty(\mathbb{R}^N; [0, 1])$  such that  $\zeta = 1$  on  $K$  and  $\zeta = 0$  on  $\mathbb{R}^N \setminus A$ , and let us define

$$\hat{\rho}_n := \zeta \rho_{\varepsilon_n}(t) + (1 - \zeta) \rho_{\varepsilon_n}(s).$$

Note that  $\hat{\rho}_n \in W^{1,p}(\Omega)$ , and since  $t \geq s$ , we have  $\hat{\rho}_n \leq \rho_{\varepsilon_n}(s)$  in  $\Omega$ . As a consequence of the minimality property established in Proposition 5.3, we have

$$\int_{\Omega} \left( \frac{\varepsilon_n^{p-1}}{p} |\nabla \rho_{\varepsilon_n}(s)|^p + \frac{\alpha}{p' \varepsilon_n} (1 - \rho_{\varepsilon_n}(s))^p \right) dx \leq \int_{\Omega} \left( \frac{\varepsilon_n^{p-1}}{p} |\nabla \hat{\rho}_n|^p + \frac{\alpha}{p' \varepsilon_n} (1 - \hat{\rho}_n)^p \right) dx.$$

Since  $\nabla \hat{\rho}_n = \zeta \nabla \rho_{\varepsilon_n}(t) + (1 - \zeta) \nabla \rho_{\varepsilon_n}(s) + (\rho_{\varepsilon_n}(t) - \rho_{\varepsilon_n}(s)) \nabla \zeta$ , there exists a constant  $C > 0$  (independent of  $n$ ) such that

$$\begin{aligned} \int_{\Omega} |\nabla \hat{\rho}_n|^p dx &\leq \int_{\Omega} |\zeta \nabla \rho_{\varepsilon_n}(t) + (1 - \zeta) \nabla \rho_{\varepsilon_n}(s)|^p \\ &+ C \int_{\Omega} |\nabla \zeta| (\rho_{\varepsilon_n}(s) - \rho_{\varepsilon_n}(t)) (1 + |\nabla \rho_{\varepsilon_n}(t)|^{p-1} + |\nabla \rho_{\varepsilon_n}(s)|^{p-1} + |\nabla \zeta|^{p-1} |\rho_{\varepsilon_n}(t) - \rho_{\varepsilon_n}(s)|^{p-1}) dx \\ &\leq \int_{\Omega} (\zeta |\nabla \rho_{\varepsilon_n}(t)|^p + (1 - \zeta) |\nabla \rho_{\varepsilon_n}(s)|^p) dx + C (1 + \|\nabla \rho_{\varepsilon_n}(t)\|_{L^p(\Omega; \mathbb{R}^N)}^{p-1} + \|\nabla \rho_{\varepsilon_n}(s)\|_{L^p(\Omega; \mathbb{R}^N)}^{p-1}), \end{aligned}$$

where we used Hölder's inequality and the fact that  $0 \leq \rho_{\varepsilon_n} \leq 1$ . Hence,

$$\begin{aligned} \mu_n(s)(\mathbb{R}^N) &\leq \int_{\mathbb{R}^N} \zeta d\mu_n(t) + \int_{\mathbb{R}^N} (1 - \zeta) d\mu_n(s) + C \varepsilon_n^{p-1} \\ &\quad + C \varepsilon_n^{(p-1)/p} (\|\varepsilon_n^{(p-1)/p} \nabla \rho_{\varepsilon_n}(t)\|_{L^p(\Omega; \mathbb{R}^N)}^{p-1} + \|\varepsilon_n^{(p-1)/p} \nabla \rho_{\varepsilon_n}(s)\|_{L^p(\Omega; \mathbb{R}^N)}^{p-1}), \end{aligned} \quad (6.16)$$

and passing to the limit as  $n \rightarrow \infty$  yields

$$\mu(s)(\mathbb{R}^N) \leq \int_{\mathbb{R}^N} \zeta d\mu(t) - \int_{\mathbb{R}^N} \zeta d\mu(s) + \mu(s)(\mathbb{R}^N).$$

From this inequality we deduce that

$$\mu(s)(K) \leq \int_{\mathbb{R}^N} \zeta d\mu(s) \leq \int_{\mathbb{R}^N} \zeta d\mu(t) \leq \mu(t)(A).$$

Taking the supremum among all compact sets  $K \subset B$ , the infimum among all open sets  $A \supset B$ , and using the outer-inner regularity of the measures  $\mu(s)$  and  $\mu(t)$  leads to  $\mu(s)(B) \leq \mu(t)(B)$ .  $\square$

We can now define a family of increasing cracks for times in the countable dense set  $D$ .

**Lemma 6.5.** *There exists a family of countably  $\mathcal{H}^{N-1}$ -rectifiable subsets  $\{\hat{\Gamma}(t)\}_{t \in D}$  of  $\bar{\Omega}$  such that*

- $\hat{\Gamma}(s) \tilde{\subset} \hat{\Gamma}(t)$  for every  $s \leq t$  with  $s, t \in D$ ;
- $J_{u(s)} \tilde{\subset} \hat{\Gamma}(t)$  for every  $t \in D$  and  $0 \leq s \leq t$ ;
- $\mu(t) \geq \mathcal{H}^{N-1} \llcorner \hat{\Gamma}(t)$  for every  $t \in D$ .

**Proof.** For  $t \in D$ , let us define the upper density of  $\mu(t)$  at  $x$  by

$$\Theta^*(t, x) := \limsup_{r \rightarrow 0} \frac{\mu(t)(B_r(x))}{\omega_{N-1} r^{N-1}} = \limsup_{r \rightarrow 0} \frac{\mu(t)(\bar{B}_r(x))}{\omega_{N-1} r^{N-1}}$$

for all  $x \in \mathbb{R}^N$ , and the Borel set

$$K(t) := \{x \in \mathbb{R}^N : \Theta^*(t, x) \geq 1\} \subset \bar{\Omega}.$$

Note that the monotonicity property established in Lemma 6.4 ensures that  $D \ni t \mapsto \Theta^*(t, x)$  is non-decreasing for every  $x \in \bar{\Omega}$ . Consequently,

$$K(s) \subset K(t) \text{ for every } 0 \leq s \leq t \text{ with } s, t \in D.$$

Moreover, from standard properties of densities (see [3, Theorem 2.56]) we infer that for every  $t \in D$ ,

$$\mathcal{H}^{N-1} \llcorner K(t) \leq \mu(t). \quad (6.17)$$

Let us now fix  $t \in D$  and  $s \in [0, t]$  (not necessarily in  $D$ ), and let  $A$  and  $A' \subset \mathbb{R}^N$  be open sets such that  $\bar{A} \subset A'$ . We consider a cut-off function  $\zeta \in \mathcal{C}_c^\infty(\mathbb{R}^N; [0, 1])$  such that  $\zeta = 1$  on  $A \cap \Omega$  and  $\zeta = 0$  on  $\mathbb{R}^N \setminus A'$ . Arguing exactly as in the proof of Lemma 6.4, we obtain inequality (6.16) from which we deduce that

$$\mu_n(s)(A) \leq \int_{\mathbb{R}^N} \zeta d\mu_n(s) \leq \int_{\mathbb{R}^N} \zeta d\mu_n(t) + C\varepsilon_n^{(p-1)/p}, \quad (6.18)$$

for some constant  $C > 0$  independent of  $n$ . By (6.5) and (6.3), we infer that

$$\liminf_{n \rightarrow \infty} \mu_n(s)(A) \geq \mathcal{H}^{N-1}(J_{u_*(s)} \cap A).$$

Passing to the limit in (6.18) then leads to

$$\mathcal{H}^{N-1}(J_{u_*(s)} \cap A) \leq \int_{\mathbb{R}^N} \zeta d\mu(t) \leq \mu(t)(A').$$

Taking the infimum with respect to all open sets  $A'$  containing  $\bar{A}$  yields

$$\mu(t)(\bar{A}) \geq \mathcal{H}^{N-1}(J_{u_*(s)} \cap A) \quad \text{for every open set } A.$$

In particular, since  $J_{u_*(s)}$  is countably  $\mathcal{H}^{N-1}$ -rectifiable, we infer from the Besicovitch-Mastrand-Mattila Theorem (see [3, Theorem 2.63]) that  $\Theta^*(t, x) \geq 1$  for  $\mathcal{H}^{N-1}$ -a.e.  $x \in J_{u_*(s)}$ , and hence

$$J_{u_*(s)} \tilde{\subset} K(t) \quad \text{for every } s \in [0, t]. \quad (6.19)$$

The Borel sets  $\{K(t)\}_{t \in D}$  have all the required properties, except that they might not be countably  $\mathcal{H}^{N-1}$ -rectifiable. However, since  $\mathcal{H}^{N-1}(K(t)) < +\infty$  by (6.17), it is possible to decompose each  $K(t)$  into the union of a countably  $\mathcal{H}^{N-1}$ -rectifiable set  $\hat{\Gamma}(t)$ , and a purely  $\mathcal{H}^{N-1}$ -unrectifiable set  $K(t) \setminus \hat{\Gamma}(t)$  (see *e.g.* [3, page 83]). This decomposition being unique up to  $\mathcal{H}^{N-1}$ -negligible sets, and  $J_{u_*(s)}$  being countably  $\mathcal{H}^{N-1}$ -rectifiable, we deduce from (6.19) that

$$J_{u_*(s)} \tilde{\subset} \hat{\Gamma}(t) \quad \text{for every } t \in D \text{ and } s \in [0, t].$$

Moreover, for  $s, t \in D$  with  $s \leq t$  we have  $\hat{\Gamma}(s) \subset K(s) \subset K(t)$ , and since  $\hat{\Gamma}(s)$  is countably  $\mathcal{H}^{N-1}$ -rectifiable we finally conclude that  $\hat{\Gamma}(s) \tilde{\subset} \hat{\Gamma}(t)$ .  $\square$

We now extend our definition of crack set for arbitrary times. We set for each  $t \geq 0$ ,

$$\Gamma(t) := \bigcap_{\tau > t, \tau \in D} \hat{\Gamma}(\tau).$$

**Lemma 6.6.** *For every  $t \geq 0$ , the set  $\Gamma(t)$  is countably  $\mathcal{H}^{N-1}$ -rectifiable, and it satisfies*

- $\Gamma(s) \subset \Gamma(t)$  for every  $0 \leq s \leq t$ ;
- $J_{u_*(t)} \tilde{\subset} \Gamma(t)$  for every  $t \geq 0$ .

**Proof.** Clearly  $\{\Gamma(t)\}_{t \geq 0}$  is a family of countably  $\mathcal{H}^{N-1}$ -rectifiable sets satisfying  $\Gamma(s) \subset \Gamma(t)$  for every  $0 \leq s \leq t$ . Moreover, for  $t \geq 0$  we have

$$\begin{aligned} \mathcal{H}^{N-1}(J_{u_*(t)} \setminus \Gamma(t)) &= \mathcal{H}^{N-1}\left(J_{u_*(t)} \setminus \bigcap_{\tau > t, \tau \in D} \hat{\Gamma}(\tau)\right) \\ &= \mathcal{H}^{N-1}\left(\bigcup_{\tau > t, \tau \in D} (J_{u_*(t)} \setminus \hat{\Gamma}(\tau))\right) \leq \sum_{\tau > t, \tau \in D} \mathcal{H}^{N-1}(J_{u_*(t)} \setminus \hat{\Gamma}(\tau)) = 0, \end{aligned}$$

since  $J_{u_*(t)} \tilde{\subset} \hat{\Gamma}(\tau)$  for all  $\tau \in D$  such that  $\tau > t$  by Lemma 6.5. Consequently,  $J_{u_*(t)} \tilde{\subset} \Gamma(t)$ .  $\square$

We are now in position to improve the energy inequality by replacing the jump set of  $u_*(t)$  by the increasing family of cracks  $\Gamma(t)$  constructed before.

**Proposition 6.7.** *For every  $t \geq 0$ ,*

$$\begin{aligned} \frac{1}{2} \int_{\Omega} |\nabla u_*(t)|^2 dx + \mathcal{H}^{N-1}(\Gamma(t)) + \frac{\beta}{2} \int_{\Omega} (u_*(t) - g)^2 dx + \int_0^t \|u'_*(s)\|_{L^2(\Omega)}^2 ds \\ \leq \frac{1}{2} \int_{\Omega} |\nabla u_0|^2 dx + \frac{\beta}{2} \int_{\Omega} (u_0 - g)^2 dx. \end{aligned}$$

**Proof.** *Step 1.* We first consider the case  $t \in D$ . According to the energy inequality in Corollary 5.15, we have

$$\frac{1}{2} \int_{\Omega} (\eta_{\varepsilon_n} + \rho_{\varepsilon_n}^2(t)) |\nabla u_{\varepsilon_n}(t)|^2 dx + \mu_n(t)(\mathbb{R}^N) + \frac{\beta}{2} \int_{\Omega} (u_{\varepsilon_n}(t) - g)^2 dx + \int_0^t \|u'_{\varepsilon_n}(s)\|_{L^2(\Omega)}^2 ds \leq \mathcal{E}_{\varepsilon_n}(u_0, \rho_0^{\varepsilon_n}).$$

Since  $\mu_n(t) \rightharpoonup \mu(t)$  weakly\* in  $\mathcal{M}(\mathbb{R}^N)$  and  $\mu(t) \geq \mathcal{H}^{N-1} \llcorner \hat{\Gamma}(t)$  by Lemma 6.5, we have

$$\liminf_{n \rightarrow \infty} \mu_n(t)(\mathbb{R}^N) \geq \mu(t)(\mathbb{R}^N) \geq \mathcal{H}^{N-1}(\hat{\Gamma}(t)).$$

On the other hand the second inequality in (6.3) with  $A = \Omega$  yields

$$\liminf_{n \rightarrow \infty} \int_{\Omega} (\eta_{\varepsilon_n} + \rho_{\varepsilon_n}(t)^2) |\nabla u_{\varepsilon_n}(t)|^2 dx \geq \int_{\Omega} |\nabla u_*(t)|^2 dx.$$

Therefore,

$$\begin{aligned} \frac{1}{2} \int_{\Omega} |\nabla u_*(t)|^2 dx + \mathcal{H}^{N-1}(\hat{\Gamma}(t)) + \frac{\beta}{2} \int_{\Omega} (u_*(t) - g)^2 dx + \int_0^t \|u'_*(s)\|_{L^2(\Omega)}^2 ds \\ \leq \liminf_{n \rightarrow \infty} \left\{ \frac{1}{2} \int_{\Omega} (\eta_{\varepsilon_n} + \rho_{\varepsilon_n}(t)^2) |\nabla u_{\varepsilon_n}(t)|^2 dx + \mu_n(t)(\mathbb{R}^N) + \frac{\beta}{2} \int_{\Omega} (u_{\varepsilon_n}(t) - g)^2 dx + \int_0^t \|u'_{\varepsilon_n}(s)\|_{L^2(\Omega)}^2 ds \right\}. \end{aligned}$$

Then, by the minimality property (3.1) of  $\rho_0^{\varepsilon_n}$ , we have

$$\mathcal{E}_{\varepsilon_n}(u_0, \rho_0^{\varepsilon_n}) \leq \mathcal{E}_{\varepsilon_n}(u_0, 1) = \frac{1 + \eta_{\varepsilon_n}}{2} \int_{\Omega} |\nabla u_0|^2 dx + \frac{\beta}{2} \int_{\Omega} (u_0 - g)^2 dx \rightarrow \frac{1}{2} \int_{\Omega} |\nabla u_0|^2 dx + \frac{\beta}{2} \int_{\Omega} (u_0 - g)^2 dx,$$

which leads to

$$\begin{aligned} \frac{1}{2} \int_{\Omega} |\nabla u_*(t)|^2 dx + \mathcal{H}^{N-1}(\hat{\Gamma}(t)) + \frac{\beta}{2} \int_{\Omega} (u_*(t) - g)^2 dx + \int_0^t \|u'_*(s)\|_{L^2(\Omega)}^2 ds \\ \leq \frac{1}{2} \int_{\Omega} |\nabla u_0|^2 dx + \frac{\beta}{2} \int_{\Omega} (u_0 - g)^2 dx. \end{aligned} \quad (6.20)$$

*Step 2.* We now extend the inequality above to the case where  $t \geq 0$  is arbitrary. In that case, there exists a sequence  $\{t_j\} \subset D$  such that  $t_j \rightarrow t$  with  $t_j > t$ . By (6.20) we have

$$\sup_{j \in \mathbb{N}} \{ \|u_*(t_j)\|_{L^\infty(\Omega)} + \|\nabla u_*(t_j)\|_{L^2(\Omega; \mathbb{R}^N)} + \mathcal{H}^{N-1}(J_{u_*(t_j)}) \} < \infty,$$

since  $J_{u_*(t_j)} \tilde{\subset} \hat{\Gamma}(t_j)$  by Lemma 6.5. On the other hand,  $u_* \in AC^2(0, +\infty; L^2(\Omega))$  by Proposition 6.2, and thus  $u_*(t_j) \rightarrow u_*(t)$  strongly in  $L^2(\Omega)$ . Applying Ambrosio's compactness Theorem (Theorems 4.7 and 4.8 in [3]), we deduce that  $\nabla u(t_j) \rightharpoonup \nabla u(t)$  weakly in  $L^2(\Omega; \mathbb{R}^N)$ . Since  $\Gamma(t) \subset \hat{\Gamma}(t_j)$  for all  $j \in \mathbb{N}$ , we finally conclude that

$$\begin{aligned} \frac{1}{2} \int_{\Omega} |\nabla u_*(t)|^2 dx + \mathcal{H}^{N-1}(\Gamma(t)) + \frac{\beta}{2} \int_{\Omega} (u_*(t) - g)^2 dx + \int_0^t \|u'_*(s)\|_{L^2(\Omega)}^2 ds \\ \leq \liminf_{j \rightarrow \infty} \left\{ \frac{1}{2} \int_{\Omega} |\nabla u_*(t_j)|^2 dx + \mathcal{H}^{N-1}(\hat{\Gamma}(t_j)) + \frac{\beta}{2} \int_{\Omega} (u_*(t_j) - g)^2 dx + \int_0^{t_j} \|u'_*(s)\|_{L^2(\Omega)}^2 ds \right\}, \end{aligned}$$

which, in view of (6.20), completes the proof of the energy inequality.  $\square$

### 6.3. Relation with the unilateral slope of the Mumford-Shah functional

In [19] the authors have introduced a new notion of unilateral slope for the Mumford-Shah functional (in dimension  $N = 2$  and for  $\beta = 0$ ). In this paper, the Mumford-Shah functional is slightly different from the one we consider here (see (2.3)). It is rather defined on pairs  $(u, K)$  by

$$\mathcal{E}_*(u, K) := \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx + \mathcal{H}^{N-1}(K) + \frac{\beta}{2} \int_{\Omega} (u - g)^2 dx,$$

where  $u \in SBV^2(\Omega)$  and  $K$  is a subset of  $\Omega$  satisfying  $\mathcal{H}^{N-1}(K) < \infty$  and  $J_u \tilde{\subset} K$ . The related unilateral slope of  $\mathcal{E}_*$  is then given by

$$|\partial \mathcal{E}_*(u, K)| := \limsup_{v \rightarrow u} \frac{(\mathcal{E}_*(u, K) - \mathcal{E}_*(v, K \cup J_v))^+}{\|v - u\|_{L^2(\Omega)}},$$

where  $v \rightarrow u$  in  $L^2(\Omega)$ . In [19], the authors proved that if  $|\partial \mathcal{E}_*(u, K)| < \infty$ , then  $\operatorname{div}(\nabla u) \in L^2(\Omega)$ , and that a weak form of

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } K$$

holds, where  $\nu$  denotes a unit normal vector on  $K$ . They also obtained the inequality  $|\partial \mathcal{E}_*(u, K)| \geq \|\operatorname{div}(\nabla u) - \beta(u - g)\|_{L^2(\Omega)}$ , and that equality holds if  $u$  and  $K$  are smooth enough. By means of an explicit counterexample, they have shown that  $|\partial \mathcal{E}_*|$  is not lower semicontinuous for any reasonable notion of convergence. In view of this result, they have introduced a notion of relaxed slope corresponding to a lower semicontinuous envelope of  $|\partial \mathcal{E}_*|$  with respect to a suitable sequential topology. More precisely, the relaxed slope  $|\bar{\partial} \mathcal{E}_*|$  is defined for a pair  $(u, K)$  in the domain of  $\mathcal{E}_*$  by

$$|\bar{\partial} \mathcal{E}_*(u, K)| := \inf \left\{ \liminf_{n \rightarrow \infty} |\partial \mathcal{E}_*(u_n, K_n)| \right\}$$

where the infimum is taken over all sequences  $\{(u_n, K_n)\}_{n \in \mathbb{N}}$  such that  $u_n \rightarrow u$  strongly in  $L^2(\Omega)$ ,  $\nabla u_n \rightharpoonup \nabla u$  weakly in  $L^2(\Omega; \mathbb{R}^N)$ , and  $K_n$   $\sigma^2$ -converges to  $K$  (see [14, Definition 4.1] for a precise definition). They established that if  $|\bar{\partial} \mathcal{E}_*(u, K)| < \infty$ , then there exists  $f \in L^2(\Omega)$  such that

$$\begin{cases} -\operatorname{div}(\nabla u) = f & \text{in } L^2(\Omega), \\ |\nabla u|^2 - \operatorname{div}(u \nabla u) \leq f u & \text{in } \mathcal{D}'(\Omega), \\ \nabla u \cdot \nu = 0 & \text{in } H^{-1/2}(\partial \Omega). \end{cases} \quad (6.21)$$



Again, there is an inequality  $|\overline{\partial\mathcal{E}_*}|(u, K) \geq \|\operatorname{div}(\nabla u) - \beta(u - g)\|_{L^2(\Omega)}$ , and equality holds in some particular cases. Note that, in the case where  $u$  and  $K$  are smooth enough, the first line in (6.21) implies the continuity of  $\frac{\partial u}{\partial \nu}$  across  $K$ , and the second one is then a weak reformulation of

$$(u^+ - u^-) \frac{\partial u}{\partial \nu} \geq 0 \quad \text{on } K,$$

where  $u^\pm$  are the one-sided traces of  $u$  on  $K$  according to the orientation  $\nu$ .

In our context, the analogy between the definitions of the unilateral slopes  $|\partial\mathcal{E}_\varepsilon|$  and  $|\partial\mathcal{E}_*|$  is quite clear, and it was actually one of the motivations to introduce  $|\partial\mathcal{E}_\varepsilon|$ . In view of the relation between the Ambrosio-Tortorelli functional and the Mumford-Shah functional in terms of  $\Gamma$ -convergence, a very interesting issue would be to find a precise relation between  $|\overline{\partial\mathcal{E}_*}|$  and the asymptotic behavior as  $\varepsilon \downarrow 0$  of  $|\partial\mathcal{E}_\varepsilon|$ . Even if we do not pursue this issue here, we prove for completeness that a conclusion similar to [19, Proposition 1.3] holds for  $|\partial\mathcal{E}_\varepsilon|$ . For clarity reasons we only state the result in terms of the asymptotic limit  $u_*$  previously obtained.

**Proposition 6.8.** *For a.e.  $t \geq 0$ , we have*

$$\|\operatorname{div}(\nabla u_*(t)) - \beta(u_*(t) - g)\|_{L^2(\Omega)} \leq \liminf_{n \rightarrow \infty} |\partial\mathcal{E}_{\varepsilon_n}|(u_{\varepsilon_n}(t), \rho_{\varepsilon_n}(t)) < \infty, \quad (6.22)$$

and

$$|\nabla u_*(t)|^2 - \operatorname{div}(u_*(t) \nabla u_*(t)) \leq -u_*(t) \operatorname{div}(\nabla u_*(t)) \quad \text{in } \mathcal{D}'(\Omega).$$

**Proof.** From Propositions 3.8 & 5.14 together with (6.4) and Fatou's lemma, we first deduce that

$$\int_0^{+\infty} \liminf_{n \rightarrow \infty} |\partial\mathcal{E}_{\varepsilon_n}|^2(u_{\varepsilon_n}(t), \rho_{\varepsilon_n}(t)) dt \leq \liminf_{n \rightarrow \infty} \int_0^{+\infty} |\partial\mathcal{E}_{\varepsilon_n}|^2(u_{\varepsilon_n}(t), \rho_{\varepsilon_n}(t)) dt \leq C,$$

for a constant  $C > 0$  independent of  $n$ . Hence there exists an  $\mathcal{L}^1$ -negligible set  $\mathcal{L} \subset (0, +\infty)$  such that

$$\liminf_{n \rightarrow \infty} |\partial\mathcal{E}_{\varepsilon_n}|(u_{\varepsilon_n}(t), \rho_{\varepsilon_n}(t)) < \infty \quad \text{for } t \in (0, +\infty) \setminus \mathcal{L}.$$

Let us now fix  $t \in (0, +\infty) \setminus \mathcal{L}$  and extract a subsequence (depending on  $t$ ) such that

$$\lim_{j \rightarrow \infty} |\partial\mathcal{E}_{\varepsilon_{n_j}}|(u_{\varepsilon_{n_j}}(t), \rho_{\varepsilon_{n_j}}(t)) = \liminf_{n \rightarrow \infty} |\partial\mathcal{E}_{\varepsilon_n}|(u_{\varepsilon_n}(t), \rho_{\varepsilon_n}(t)).$$

By Proposition 3.9, the sequence  $\{\operatorname{div}((\eta_{\varepsilon_{n_j}} + \rho_{\varepsilon_{n_j}}^2(t)) \nabla u_{\varepsilon_{n_j}}(t))\}$  is thus bounded in  $L^2(\Omega)$ , and in view of (6.14) we deduce that

$$\operatorname{div}((\eta_{\varepsilon_{n_j}} + \rho_{\varepsilon_{n_j}}^2(t)) \nabla u_{\varepsilon_{n_j}}(t)) \rightharpoonup \operatorname{div}(\nabla u_*(t)) \quad \text{weakly in } L^2(\Omega). \quad (6.23)$$

Then (6.22) follows from the convergences in (6.1), and the lower semicontinuity of the  $L^2(\Omega)$ -norm.

Using again Proposition 3.9, we next notice that

$$\begin{aligned} \int_{\Omega} (\eta_{\varepsilon_n} + \rho_{\varepsilon_n}^2(t)) |\nabla u_{\varepsilon_n}(t)|^2 \varphi dx + \int_{\Omega} u_{\varepsilon_n}(t) (\eta_{\varepsilon_n} + \rho_{\varepsilon_n}^2(t)) \nabla u_{\varepsilon_n}(t) \cdot \nabla \varphi dx \\ = - \int_{\Omega} u_{\varepsilon_n}(t) \operatorname{div}((\eta_{\varepsilon_n} + \rho_{\varepsilon_n}^2(t)) \nabla u_{\varepsilon_n}(t)) \varphi dx \end{aligned}$$

for any nonnegative function  $\varphi \in \mathcal{D}(\Omega)$ , and the conclusion follows from (6.14) and (6.23).  $\square$

### Acknowledgements

The authors wish to thank Franois Murat for many helpful discussions and suggestions at different stages of the preparation of this work. They have been supported by the *Agence Nationale de la Recherche* under Grant No. ANR 10-JCJC 0106.

**References**

- [1] L. AMBROSIO: Minimizing Movements, *Rend. Accad. Naz. Sci. XL Mem. Mat. Appl.* **19** (1995) 191–246.
- [2] L. AMBROSIO, A. BRAIDES: Energies in *SBV* and variational models in fracture mechanics, in *Homogenization and applications to material sciences* (Nice, 1995), GAKUTO Internat. Ser. Math. Sci. Appl., Vol. 9 (Gakkotosho, 1995) 1–22.
- [3] L. AMBROSIO, N. FUSCO, D. PALLARA: *Functions of bounded variation and free discontinuity problems*, Oxford University Press (2000).
- [4] L. AMBROSIO, N. GIGLI, G. SAVARÉ: *Gradient flows in metric spaces and in the space of probability measures*, Second edition. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel (2008).
- [5] L. AMBROSIO, V. M. TORTORELLI: On the approximation of free discontinuity problems, *Bollettino U. M. I* **7** (1992) 105–123.
- [6] L. AMBROSIO, V. M. TORTORELLI: Approximation of functionals depending on jumps by elliptic functionals by  $\Gamma$ -convergence, *Comm. Pure Appl. Math.* **43** (1990) 999–1036.
- [7] B. BOURDIN, G. A. FRANCFORT, J.-J. MARIGO: Numerical experiments in revisited brittle fracture, *J. Mech. Phys. Solids* **48** (2000) 797–826.
- [8] B. BOURDIN, G. A. FRANCFORT, J.-J. MARIGO: The variational approach to fracture, *J. Elasticity* **9** (2008) 5–148.
- [9] A. BRAIDES: *A handbook of Gamma-convergence*, Handbook of Differential Equations. Stationary Partial Differential Equations, Volume 3 (M. Chipot and P. Quittner, eds.) Elsevier 2006.
- [10] A. BRAIDES, A. CHAMBOLLE, M. SOLCI: A relaxation result for energies defined on pairs set-function and applications, *ESAIM Control Optim. Calc. Var.* **13** (2007) 717–734.
- [11] H. BRÉZIS: *Operateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*, North-Holland, Amsterdam-London; American Elsevier, New York (1973).
- [12] A. CHAMBOLLE, F. DOVERI: Minimizing movements of the Mumford-Shah functional, *Discrete Contin. Dyn. Syst. A* **3** (1997) 153–174.
- [13] D. CIORANESCU, F. MURAT: Un terme étrange venu d’ailleurs, *Nonlinear partial differential equations and their applications. Collège de France Seminar, Vol. III (Paris, 1980/1981), Res. Notes in Math.* **70**, Pitman: Boston, Mass., 1982, 154–178, 425–426.
- [14] G. DAL MASO, G. FRANCFORT, R. TOADER: Quasistatic crack growth in nonlinear elasticity, *Arch. Rational Mech. Anal.* **176** (2005) 165–225.
- [15] G. DAL MASO, C. J. LARSEN: Existence for wave equations on domains with arbitrary growing cracks, *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei*, (9) Mat. Appl. **22** (2011) 387–408.
- [16] G. DAL MASO, P. LONGO:  $\Gamma$ -limits of obstacles, *Ann. Mat. Pura Appl.* **128** (1981) 1–50.
- [17] G. DAL MASO, R. TOADER: A model for the quasi-static growth of brittle fractures: existence and approximation results, *Arch. Rational Mech. Anal.* **162** (2002) 101–135.
- [18] G. DAL MASO, R. TOADER: A model for the quasi-static growth of brittle fractures based on local minimization, *Math. Models Methods Appl. Sci.* **12** (2002) 1773–1800.
- [19] G. DAL MASO, R. TOADER: On a notion of unilateral slope for the Mumford-Shah functional, *NoDEA* **13** (2007) 713–734.
- [20] E. DE GIORGI: New problems on minimizing movements, In: C. Baiocchi - J.-L. Lions (eds.), *Boundary Value Problems for PDE and Applications*, Masson, Paris (1993) 81–98
- [21] E. DE GIORGI, M. CARRIERO, A. LEACI: Existence theorem for a minimum problem with free discontinuity set, *Arch. Rational Mech. Anal.* **108**, (1989) 195–218.
- [22] E. DE GIORGI, A. MARINO, M. TOSQUES Problemi di evoluzione in spazi metrici e curve di massima pendenza, *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.* **68** (1980) 180–187.
- [23] J. DIESTEL, J. J. UHL: *Vector measures*, Mathematical Surveys, No. 15. American Mathematical Society, Providence, R.I. (1977).
- [24] G. A. FRANCFORT, C. J. LARSEN: Existence and convergence for quasi-static evolution in brittle fracture, *Comm. Pure Appl. Math.* **56** (2003) 1465–1500.
- [25] G. A. FRANCFORT, N. Q. LE, S. SERFATY: Critical Points of Ambrosio-Tortorelli converge to critical points of Mumford-Shah in the one-dimensional Dirichlet case, *ESAIM Control Optim. Calc. Var.* **15** (2009) 576–598.
- [26] G. A. FRANCFORT, J.-J. MARIGO: Revisiting brittle fracture as an energy minimization problem, *J. Mech. Phys. Solids* **46** (1998) 1319–1342.
- [27] X. FENG, A. PROHL: Analysis of gradient flow of a regularized Mumford-Shah functional for image segmentation and image inpainting, *ESAIM: M2AN* **38** (2004) 291–320.
- [28] M. FOCARDI: On the variational approximation of free discontinuity problems in the vectorial case, *Math Models Methods Appl. Sci.* **11** (2001) 663–684.
- [29] A. GIACOMINI: Ambrosio-Tortorelli approximation of quasi-static evolution of brittle fractures, *Calc. Var. and PDEs* **22** (2005) 129–172.
- [30] M. GOBBINO: Gradient flow for the one-dimensional Mumford-Shah functional, *Ann. Sc. Norm. Sup. Pisa, Cl. Sc.* **4** **27** (1998) 145–193.
- [31] P. GRISVARD: *Elliptic problems in nonsmooth domains*, Monographs and Studies in Mathematics, 24. Pitman (Advanced Publishing Program), Boston, MA (1985).
- [32] C. J. LARSEN: Epsilon-stable quasi-static brittle fracture evolution, *Comm. Pure Appl. Math.* **63** (2010) 630–654.

- [33] C. J. LARSEN, C. ORTNER, E. SÜLI: Existence of solutions to a regularized model of dynamic fracture, *Math Models Methods Appl. Sci.* **20** (2010) 1021–1048.
- [34] N. Q. LE: Convergence results for critical points of the one-dimensional Ambrosio-Tortorelli functional with fidelity term, *Adv. Differential Equations* **15** (2010), 255–282.
- [35] A. MAINIK, A. MIELKE: Existence results for energetic models for rate-independent systems, *Calc. Var. and PDEs* **22** (2005) 73–99.
- [36] D. MUMFORD, J. SHAH: Optimal approximation by piecewise smooth functions and associated variational problems, *Comm. Pure Appl. Math.* **17** (1989) 577–685.
- [37] R. ROSSI, G. SAVARÉ: Gradient flows of non convex functionals in Hilbert spaces and applications, *ESAIM Control Optim. Calc. Var.* **12** (2006) 564–614.
- [38] E. SANDIER, S. SERFATY: Gamma-convergence of gradient flows with applications to Ginzburg-Landau, *Comm. Pure Appl. Math* **57** (2004) 1627–1672.
- [39] S. SERFATY: Gamma-convergence of gradient flows on Hilbert and metric spaces and applications, *Discrete Contin. Dyn. Syst. A* **31** (2011) 1427–1451.