Sharp ultimate bounds of solutions to a class of second order linear evolution equations with bounded forcing term

Faouzia ALOUI
UPMC Univ Paris 06, UMR 7598, Laboratoire Jacques-Louis Lions,
F-75005, Paris, France.
aloui@ann.jussieu.fr

Alain HARAUX (1, 2)
1. UPMC Univ Paris 06, UMR 7598, Laboratoire Jacques-Louis Lions,
F-75005, Paris, France.
2- CNRS, UMR 7598, Laboratoire Jacques-Louis Lions,
Boîte courrier 187, 75252 Paris Cedex 05, France.
haraux@ann.jussieu.fr

Abstract

We establish a precise estimate of the ultimate bound of solutions to some second order evolution equations with possibly unbounded linear damping and bounded forcing term.

Introduction

Let H be a real Hilbert space. In the sequel we denote by (u, v) the inner product of two vectors u, v in H and by $|u|$ the $H-$ norm of u. Given $f \in L^\infty(\mathbb{R}, H)$, we consider the second order evolution equation with possibly unbounded and time-dependent damping operator B:

$$u'' + Au + Bu' = f(t) \tag{0.1}$$

where A is a fixed linear, self-adjoint and positive operator in H. We assume that the domain of A is dense in H and A is coercive, in other terms:

$$\exists \lambda > 0, \quad \forall u \in D(A), \quad (Au, u) \geq \lambda |u|^2. \tag{0.2}$$

Obviously the set of λ satisfying (0.2) is closed. For our purpose the best possible is the largest one, ie.

$$\lambda = \inf_{u \in D(A), |u|=1} (Au, u) =: \lambda_1(A).$$

We introduce $V = D(A^{\frac{1}{2}})$ endowed with the norm given by

$$\forall u \in V, \quad \|u\| = |A^{\frac{1}{2}} u|.$$

This norm defined on V is equivalent to the graph norm of $A^{\frac{1}{2}}$ as a result of the coerciveness hypothesis on A.

In the sequel, $B : V \to V'$ may be a time-dependent continuous operator. When B is linear and time-independent, we write (0.1) in the following form:

$$U' + LU = F(t) \tag{0.3}$$

with $U = (u, u')$, $L = \begin{pmatrix} 0 & -I \\ A & B \end{pmatrix}$ and $F = (0, f)$. If $B \in L(V, V')$ satisfies

$$\langle Bv, v \rangle \geq 0 \quad \forall v \in V$$

then it is not difficult to check (cf.e.g. [1, 3, 4]) that L is a maximal monotone operator with dense domain $D(L) = \{(u, v) \in V \times V,\ Au + Bv \in H\}$ in $V \times H$. Then, by Hille -Yosida's Theorem (cf.e.g. [3, 13]), L generates a C^0 contraction semi-group $S(t)$ that insures the existence and uniqueness of a mild solution $u \in C(\mathbb{R}^+, V) \cap C^1(\mathbb{R}^+, H)$ to (0.1) on $\mathbb{R}^+$ for any pair of initial data $u_0 = u(0) \in V; u_1 = u'(0) \in H$. Moreover, the two following properties are equivalent cf [9]:

1) $S(t)$ is exponentially damped on $V \times H$ which means that for some constants $M \geq 1, \delta > 0$

$$\forall t \geq 0, \quad \|S(t)\|_{L(H)} \leq M \exp(-\delta t)$$

2) $\forall F \in L^\infty(\mathbb{R}^+, H)$, any solution of (0.3) is bounded in $V \times H$ for $t \geq 0$.

In addition in this case we have

$$\overline{\lim_{t\to\infty}} \|U(t)\| \leq \frac{M}{\delta} \overline{\lim_{t\to\infty}} \|F(t)\|_H$$

In applications to infinite or even finite dimensional second order equations, this method does not give the best possible estimate because it is not easy to optimize on M and δ. This was already observed in [11] and [12] where precise estimates of $\overline{\lim}_{t\to\infty} \|U(t)\|$ were given in the case of (0.1) with $B = cI$ or $B = cA^{\frac{1}{2}}$.

The main objective of this paper is to generalize the results of [11, 12] for B time independent and improve some of the results in the specific cases $B = cI$ and$B = cA^{\frac{1}{2}}$. We shall consider also the case $B = cA$ which was not studied before.

The plan of the paper is the following: section 1 contains an improvement of the main result from [11] in the general case $B = \beta(t)$. Section 2 is devoted to the case where $B = B(t)$ is linear and self-adjoint. Section 3 gives the precise statements when $B = cA^\alpha$ with a special treatment in the case $B = B_0 = cA^{\frac{1}{2}}$ and Section 4 is devoted to the main concrete applications of Theorem 2.1. Finally Section 5 is devoted to slightly different examples and some additional remarks.

1 An ultimate bound valid for general time-dependent damping terms

We consider the equation:

$$u'' + \beta(t)u' + Au = f(t) \tag{1.1}$$

where $t \in \mathbb{R}^+$. For this equation, we improve some general estimates obtained in [11] when $\beta(t) : \mathbb{R}^+ \to C(V, V')$ is a measurable family of possibly nonlinear continuous operators which satisfies the two hypotheses:

$$\exists c > 0, \quad \forall t \in \mathbb{R}^+, \quad \forall v \in V, \quad \langle \beta(t)v, v\rangle \geq c|v|^2. \tag{1.2}$$

$$\exists C > 0, \quad \forall t \in \mathbb{R}^+, \quad \forall v \in V, \quad \|\beta(t)v\|_*^2 \leq C\langle \beta(t)v, v\rangle. \tag{1.3}$$

It is immediate (cf. e.g. [11]) that $c \leq C\lambda_1$ where $\lambda_1 = \lambda_1(A)$. Our main result is the following

Theorem 1.1. *For any solution* $u \in W^{1,\infty}_{loc}(\mathbb{R}^+, V) \cap W^{2,\infty}_{loc}(\mathbb{R}^+, H)$ *of* (1.1) *we have the estimate :*

$$\max(\overline{\lim_{t\to\infty}} \|u(t)\|, \overline{\lim_{t\to\infty}} |u'(t)|) \leq \max(\sqrt{12}\sqrt{\frac{C}{c}}, \frac{3}{c}) \overline{\lim_{t\to\infty}} |f(t)| \tag{1.4}$$

Proof. For simplicity of the formulas, we drop the variable t whenever possible and we denote by z' the time derivative of a (scalar or vector) time-dependent function z. We consider for some $\alpha > 0$ to be chosen later the following modified energy functional:

$$\Phi = |u'|^2 + \|u\|^2 + \alpha(u, u') - \frac{\alpha^2}{4}|u|^2.$$

Then

$$\begin{aligned}\Phi' &= -2\langle \beta u', u'\rangle + \alpha|u'|^2 - \alpha\|u\|^2 - \alpha\langle \beta u', u\rangle + \langle f, 2u' + \alpha u\rangle - \frac{\alpha^2}{2}(u, u')\\ &= -\frac{\alpha}{2}(|u'|^2 + \|u\|^2 + \alpha(u, u')) - 2\langle \beta u', u'\rangle + \frac{3\alpha}{2}|u'|^2 - \frac{\alpha}{2}\|u\|^2 - \alpha\langle \beta u', u\rangle + \langle f, 2u' + \alpha u\rangle\end{aligned}$$

we set $\Psi = |u'|^2 + \|u\|^2 + \alpha(u, u') \geq \Phi$.

Then, by using (1.2), we have:

$$\Phi' \leq -\frac{\alpha}{2}\Phi - \frac{1}{2}\langle \beta u', u'\rangle - (\frac{3c}{2} - \frac{3\alpha}{2})|u'|^2 - \frac{\alpha}{2}\|u\|^2 - \alpha\langle \beta u', u\rangle + \langle f, 2u' + \alpha u\rangle$$

we have, from (1.3):

$$|\langle \beta u', u\rangle| \leq \sqrt{C}\langle \beta u', u'\rangle^{\frac{1}{2}}\|u\|$$

By using Young's inequality we deduce :

$$|\alpha\langle \beta u', u\rangle| \leq \alpha C\langle \beta u', u'\rangle + \alpha\frac{\|u\|^2}{4}$$

Assuming $\alpha C \leq \frac{1}{2}$, then:

$$\Phi' + \frac{\alpha}{2}\Phi \leq -\frac{3}{2}(c - \alpha)|u'|^2 + 2\langle f, u'\rangle - \frac{\alpha}{4}\|u\|^2 + \alpha\langle f, u\rangle$$

Assuming $\frac{3}{2}(c - \alpha) \geq \frac{1}{2}c$, then $\alpha \leq \frac{2}{3}c$.

We have, by using Young's inequality:

$$\begin{aligned}-\frac{3}{2}(c - \alpha)|u'|^2 + 2\langle f, u'\rangle &\leq -\frac{c}{2}|u'|^2 + 2\langle f, u'\rangle\\ &\leq \frac{2}{c}|f|^2\end{aligned}$$

Moreover

$$\alpha\langle f,u\rangle \le \frac{\alpha}{\sqrt{\lambda_1}}|f|\|u\|$$

Therefore, by Young's inequality:

$$\begin{aligned} -\frac{\alpha}{4}\|u\|^2+\alpha\langle f,u\rangle &\le \alpha(-\frac{\|u\|^2}{4}+\frac{1}{\sqrt{\lambda_1}}|f|\|u\|) \\ &\le \frac{\alpha}{\lambda_1}|f|^2 \\ &\le \frac{\alpha C}{c}|f|^2 \\ &\le \frac{1}{2c}|f|^2 \end{aligned}$$

Then

$$\Phi' + \frac{\alpha}{2}\Phi \le \frac{5}{2c}|f|^2$$

Then, we find that Φ is bounded with:

$$\overline{\lim_{t\to\infty}}\,\Phi(t) \le \frac{5}{c\alpha}\overline{\lim_{t\to\infty}}\,|f(t)|^2.$$

Moreover, we have:

$$-\alpha(u,u') \le |u'|^2 + \frac{\alpha^2}{4}|u|^2$$

We set $F = \overline{\lim}_{t\to\infty}|f(t)|^2$.

In particular for any $\epsilon > 0$ we have for t large enough

$$(1-\frac{\alpha^2}{2\lambda_1})\|u(t)\|^2 \le \|u(t)\|^2 - \frac{\alpha^2}{2}|u(t)|^2 \le \Phi(t) \le \frac{5}{c\alpha}F + \frac{\epsilon}{2}.$$

Now since $\alpha \le \frac{2}{3}c$ and $\alpha \le \frac{1}{2C}$, we have

$$\frac{\alpha^2}{2\lambda_1} \le \frac{c}{6\lambda_1 C} \le \frac{1}{6}$$

Then we find

$$\overline{\lim_{t\to\infty}}\,\|u(t)\|^2 \le \frac{6}{c\alpha}F + 2\epsilon$$

Finally, by choosing $\alpha = \inf(\frac{2}{3}c, \quad \frac{1}{2C})$, we obtain by letting $\epsilon \to 0$:

$$\overline{\lim_{t\to\infty}}\,\|u(t)\| \le \max(\sqrt{\frac{12C}{c}}, \frac{3}{c})\,\overline{\lim_{t\to\infty}}\,|f(t)|.$$

In order to estimate u', observe that for t large enough:

$$|u'(t)|^2 + \lambda_1 |u(t)|^2 + \alpha(u, u') - \frac{\alpha^2}{4}|u(t)|^2 \leq \frac{5}{c\alpha}F + \frac{\epsilon}{2}$$

Since $\alpha \leq \frac{2}{3}c \leq c$ and $\alpha \leq \frac{1}{2C} \leq \frac{\lambda_1}{2c}$, then $\alpha^2 \leq \alpha c \leq \frac{\lambda_1}{2}$.

Consequently for t large enough

$$\frac{5}{6}|u'(t)|^2 + 2\alpha^2 |u(t)|^2 + \frac{1}{6}|u'(t)|^2 + \alpha(u, u') - \frac{\alpha^2}{4}|u(t)|^2 \leq \frac{5}{c\alpha}F + \frac{\epsilon}{2}$$

In other terms

$$\frac{5}{6}|u'(t)|^2 + \frac{\alpha^2}{4}|u(t)|^2 + |\frac{1}{\sqrt{6}}u' + \frac{\sqrt{3}}{\sqrt{2}}\alpha u|^2 \leq \frac{5}{c\alpha}F + \frac{\epsilon}{2}$$

Then:

$$\overline{\lim_{t\to\infty}}\, |u'(t)|^2 \leq \frac{6}{c\alpha}F + 2\epsilon$$

Also assuming $\alpha = \inf(\frac{1}{2C}, \frac{2}{3}c)$ and letting $\epsilon \to 0$, we have:

$$\overline{\lim_{t\to\infty}}\, |u'(t)| \leq \max(\sqrt{\frac{12C}{c}}, \frac{3}{c})\, \overline{\lim_{t\to\infty}}\, |f(t)|.$$

□

Remark 1.2. If $\beta(t) = B_0 \in L(V, V')$, it is well known that the conditions $(u_0, u_1) \in D(A) \times V$ and $f \in C^1(\mathbb{R}^+, V)$ imply $u \in C^1(\mathbb{R}^+, V) \cap C^2(\mathbb{R}^+, H)$. By density on (u_0, u_1, f) we obtain easily the following

Corollary 1.3. *Let $\beta(t) = B_0 \in L(V, V')$. In this case any mild solution $u \in C(\mathbb{R}^+, V) \cap C^1(\mathbb{R}^+, H)$ of* (1.1) *satisfies* (1.4).

Remark 1.4. In [11], the following estimate was established

$$\sup\{\overline{\lim_{t\to\infty}}\, \|u(t)\|, \overline{\lim_{t\to\infty}}\, |u'(t)|\} \leq \sqrt{3}(C + \frac{4}{c})\, \overline{\lim_{t\to\infty}}\, |f(t)| \qquad (1.5)$$

Since

$$\sqrt{\frac{12C}{c}} \leq \frac{\sqrt{12}}{4}(C + \frac{4}{c}) = \frac{\sqrt{3}}{2}(C + \frac{4}{c})$$

and

$$\frac{3}{c} \leq \frac{3}{4}(C + \frac{4}{c}) \leq \frac{\sqrt{3}}{2}(C + \frac{4}{c}),$$

we can see that Theorem 1.1 improves the estimate (1.5) by a factor 2 for all values of c and C. Moreover if $C \to \infty$ with $\frac{C}{c}$ bounded, $\max(\sqrt{\frac{12C}{c}}, \frac{3}{c})$ remains bounded and $(C + \frac{4}{c})$ tends to infinity, therefore (1.4) improves (1.5) by an arbitrarily large amount. A typical case is : $\beta = cB_0$ with $c \to \infty$ since then $\frac{C}{c}$ is fixed and $C \to \infty$.

2 The case of a linear self-adjoint damping operator

In this section, we study the equation (0.1) where $B : \mathbb{R}^+ \longrightarrow L(V, V')$ is a self-adjoint and possibly unbounded operator and satisfies the following hypotheses:

$$\exists c > 0, \quad \forall t \in \mathbb{R}^+, \quad \forall v \in V, \quad \langle B(t)v, v\rangle \geq c|v|^2 \tag{2.1}$$

$$\exists C > 0, \quad \forall t \in \mathbb{R}^+, \quad \forall v \in V, \quad \langle B(t)v, v\rangle \leq C\langle Av, v\rangle \tag{2.2}$$

The following result, will give close to optimal estimates even when B is independent of time.

Theorem 2.1. *Any solution $u \in W^{1,\infty}_{loc}(\mathbb{R}^+, V) \cap W^{2,\infty}_{loc}(\mathbb{R}^+, H)$ of* (0.1) *satisfies the following estimate:*

$$\max(\overline{\lim_{t\to\infty}}\,\|u(t)\|, \overline{\lim_{t\to\infty}}\,|u'(t)|) \leq \max(\sqrt{\frac{3C}{c}}, \frac{3}{\sqrt{2c}})\,\overline{\lim_{t\to\infty}}\,|f(t)| \tag{2.3}$$

Proof. Considering again the energy functional $\Phi = |u'|^2 + \|u\|^2 + \alpha(u, u') - \frac{\alpha^2}{4}|u|^2$ we find:

$$\begin{aligned}
\Phi' &= -2|B^{\frac{1}{2}}u'|^2 + \alpha|u'|^2 - \alpha\|u\|^2 - \alpha(Bu', u) + (f, 2u' + \alpha u) - \frac{\alpha^2}{2}(u, u') \\
&= -\frac{\alpha}{2}\Psi - (2 - \frac{3\alpha}{2c})|B^{\frac{1}{2}}u'|^2 - \frac{\alpha}{2}\|u\|^2 - \alpha(Bu', u) + (f, 2u' + \alpha u) \\
&\leq -\frac{\alpha}{2}\Phi - (2 - \frac{3\alpha}{2c})|B^{\frac{1}{2}}u'|^2 - \frac{\alpha}{2}\|u\|^2 - \alpha(Bu', u) + (f, u' + \alpha u) + (f, u')
\end{aligned}$$

where $\Psi = |u'|^2 + \|u\|^2 + \alpha(u, u') \geq \Phi$. By (2.1) and Young's inequality, we have

$$\begin{aligned}
(f, u') &\leq \frac{1}{2c}|f|^2 + \frac{c}{2}|u'|^2 \\
&\leq \frac{1}{2c}|f|^2 + \frac{1}{2}|B^{\frac{1}{2}}u'|^2
\end{aligned}$$

Therefore by using (2.2), we obtain

$$\Phi' \leq -\frac{\alpha}{2}\Phi - (\frac{3}{2} - \frac{3\alpha}{2c})|B^{\frac{1}{2}}u'|^2 - \frac{\alpha}{2C}|B^{\frac{1}{2}}u|^2 - \alpha(Bu', u) + \frac{1}{2c}|f|^2 + (f, u' + \alpha u)$$

Assuming

$$\frac{3}{2} - \frac{3\alpha}{2c} \geq \frac{1}{2} \quad \text{and} \quad \alpha^2 \leq \frac{\alpha}{C}$$

which means

$$\alpha \leq \frac{2}{3}c \quad \text{and} \quad \alpha \leq \frac{1}{C}$$

we deduce

$$\begin{aligned}
\Phi' &\leq -\frac{\alpha}{2}\Phi - \frac{1}{2}|B^{\frac{1}{2}}u'|^2 - \frac{\alpha^2}{2}|B^{\frac{1}{2}}u|^2 - \alpha(Bu', u) + \frac{1}{2c}|f|^2 + (f, u' + \alpha u) \\
&\leq -\frac{\alpha}{2}\Phi - \frac{1}{2}|B^{\frac{1}{2}}(u' + \alpha u)|^2 + \frac{1}{2c}|f|^2 + (f, u' + \alpha u)
\end{aligned}$$

By using (2.1), we find

$$\Phi' \leq -\frac{\alpha}{2}\Phi - \frac{c}{2}|u' + \alpha u|^2 + \frac{1}{2c}|f|^2 + (f, u' + \alpha u)$$

By using Young's inequality in the last term, we have

$$(f, u' + \alpha u) \leq \frac{1}{2c}|f|^2 + \frac{c}{2}|u' + \alpha u|^2$$

Then

$$\Phi' \leq -\frac{\alpha}{2}\Phi + \frac{1}{c}|f|^2$$

Then we find that Φ is bounded with

$$\overline{\lim_{t\to\infty}}\, \Phi(t) \leq \frac{2}{c\alpha} \overline{\lim_{t\to\infty}}\, |f(t)|^2$$

By setting $F = \overline{\lim}_{t\to\infty} |f(t)|^2$ we see that for t large enough and any $\epsilon > 0$

$$|u'(t)|^2 + \|u(t)\|^2 + \alpha(u(t), u'(t)) - \frac{\alpha^2}{4}|u(t)|^2 \leq \frac{2}{c\alpha}F + \frac{\epsilon}{2}$$

In other terms

$$\|u(t)\|^2 + |u'(t) + \frac{\alpha}{2}u(t)|^2 - \frac{\alpha^2}{2}|u(t)|^2 \leq \frac{2}{c\alpha}F + \frac{\epsilon}{2}$$

By using $\alpha \leq \frac{2}{3}c$ and (2.1), we obtain for t large enough:

$$\|u(t)\|^2 - \frac{\alpha}{3}|B^{\frac{1}{2}}u(t)|^2 \leq \frac{2}{c\alpha}F + \frac{\epsilon}{2}$$

now using $\alpha \leq \frac{1}{C}$ and (2.2), for t large enough we obtain :

$$\|u(t)\|^2 \leq \frac{3}{\alpha c}F + 2\epsilon$$

Finally by selecting $\alpha = \inf(\frac{2}{3}c, \frac{1}{C})$and letting $\epsilon \to 0$ we find :

$$\overline{\lim_{t\to\infty}}\, \|u(t)\| \leq \max(\sqrt{\frac{3C}{c}}, \frac{3}{\sqrt{2}c}) \overline{\lim_{t\to\infty}}\, |f(t)|$$

In order to estimate u', for t large enough by using (0.2)

$$\frac{2}{3}|u'(t)|^2 + \lambda_1|u(t)|^2 + \alpha(u(t), u'(t)) + \frac{1}{3}|u'(t)|^2 - \frac{\alpha^2}{4}|u(t)|^2 \le \frac{2}{c\alpha}F + \frac{\epsilon}{2}$$

Since $\alpha \le \frac{2}{3}c \le c$ and $\alpha \le \frac{1}{C} \le \frac{\lambda_1}{c}$, we have $\alpha^2 \le \alpha c \le \lambda_1$.

Therefore, for t large enough:

$$\frac{2}{3}|u'(t)|^2 + \alpha^2|u(t)|^2 + \alpha(u(t), u'(t)) + \frac{1}{3}|u'(t)|^2 - \frac{\alpha^2}{4}|u(t)|^2 \le \frac{2}{c\alpha}F + \frac{\epsilon}{2}$$

Then, for t large enough

$$\frac{2}{3}|u'(t)|^2 + \frac{3\alpha^2}{4}|u(t)|^2 + \alpha(u(t), u'(t)) + \frac{1}{3}|u'(t)|^2 \le \frac{2}{c\alpha}F + \frac{\epsilon}{2}$$

In other terms

$$\frac{2}{3}|u'(t)|^2 + |\frac{\sqrt{3}}{2}\alpha u(t) + \frac{1}{\sqrt{3}}u'(t)|^2 \le \frac{2}{c\alpha}F + \frac{\epsilon}{2}$$

Hence, for t large enough

$$|u'(t)|^2 \le \frac{3}{\alpha c}F + 2\epsilon$$

Finally by letting $\epsilon \to 0$

$$\overline{\lim_{t\to\infty}}\, |u'(t)| \le \max(\sqrt{\frac{3C}{c}}, \frac{3}{\sqrt{2c}})\, \overline{\lim_{t\to\infty}}\, |f(t)|$$

□

By using Remark 1.2 we obtain

Corollary 2.2. *Let* $\beta(t) = B_0 \in L(V, V')$. *In this case any mild solution* $u \in C(\mathbb{R}^+, V) \cap C^1(\mathbb{R}^+, H)$ *of* (0.1) *satisfies* (2.3).

Remark 2.3. When B is linear and self-adjoint, Theorem 2.1 improves the result (1.4) with $\beta(t) = B(t)$ by a factor $\in [\sqrt{2}, 2]$ depending on the values of C and c. Indeed in this case (but not in general) the two inequalities (1.3) and (2.2) are equivalent, see Section 5 below.

3 Applications when $B = \gamma A^\alpha$, $0 \le \alpha \le 1$

In this section we consider the case of a time independent self-adjoint B proportional to some positive power of A. In order to garantee exponential damping of the associated semi-group the power will be taken ≤ 1.

3.1 The ODE case

We consider the equation:

$$u'' + \gamma u' + \omega^2 u = f(t) \tag{3.1}$$

We apply theorem 2.1 to (3.1) with $c = \gamma$ and $C = \frac{\gamma}{\omega^2}$, we find

$$\forall t \in \mathbb{R}, |u(t)| \leq \max(\frac{\sqrt{3}}{\omega^2}, \frac{3}{\sqrt{2}\gamma\omega}) \overline{\lim_{t\to\infty}} |f(t)| \tag{3.2}$$

By comparaison with the estimates in [10], we find that the result of theorem 2.1 is optimal up to a factor $K(\omega,\gamma) = \frac{3\pi}{4\sqrt{2}}$, if $\gamma < 2\omega$ and $\sqrt{3}$ if $\gamma \geq 2\omega$. More precisely, in [10] the exact minimum global bound for solutions bounded on the whole line is given, and the minimum turns out to be achieved on some periodic solutions (corresponding to a periodic source term) for which the ultimate bound of course coincides with the global bound on $\mathbb{R}$.

3.2 The case $B = \gamma A^\alpha$, $0 \leq \alpha \leq 1$

We consider the equation

$$u'' + \gamma A^\alpha u' + Au = f(t) \tag{3.3}$$

In this case (cf. Proposition 5.4)we have $c = \gamma\lambda_1^\alpha$ and $C = \frac{\gamma}{\lambda_1^{1-\alpha}}$, then, by Theorem 2.1, we have the following estimates

$$\max(\overline{\lim_{t\to\infty}} \|u(t)\|, \overline{\lim_{t\to\infty}} |u'(t))| \leq \max(\sqrt{\frac{3}{\lambda_1}}, \frac{3}{\sqrt{2}\gamma\lambda_1^\alpha}) \overline{\lim_{t\to\infty}} |f(t)| \tag{3.4}$$

Considering the special case $H = \mathbb{R}$, $A = \omega^2 I$ we conclude that this result is always sharp up to a factor $\sqrt{3}$.

3.3 The case $B = \gamma I$:

we consider the equation:

$$u'' + \gamma u' + Au = f(t) \tag{3.5}$$

Applying Theorem 2.1 to (3.5) with $C = \frac{\gamma}{\lambda_1}$ and $c = \gamma$ we find :

$$\max(\overline{\lim_{t\to\infty}} \|u(t)\|, \overline{\lim_{t\to\infty}} |u'(t))| \leq \max(\sqrt{\frac{3}{\lambda_1}}, \frac{3}{\sqrt{2}\gamma}) \overline{\lim_{t\to\infty}} |f(t)| \tag{3.6}$$

Remark 3.1. Let us compare our result on (3.6) with the estimates from [8].
In [8] it was shown that

$$\overline{\lim_{t\to\infty}} \|u(t)\| \le \sqrt{\frac{4}{\gamma^2} + \frac{1}{\lambda_1}}\, \overline{\lim_{t\to\infty}} |f(t)|. \tag{3.7}$$

If γ is fixed and $\lambda_1 \to \infty$ we have:

$$\max(\sqrt{\frac{3}{\lambda_1}}, \frac{3}{\sqrt{2}\gamma}) = \frac{3}{\sqrt{2}\gamma}$$

and

$$\sqrt{\frac{4}{\gamma^2} + \frac{1}{\lambda_1}} \backsimeq \frac{2}{\gamma}$$

therefore we find that (3.6) is worse than (3.7), hence Theorem 2.1 is weaker than the result of [8] in this case.

If λ_1 is fixed and $\gamma \to \infty$ we have:

$$\max(\sqrt{\frac{3}{\lambda_1}}, \frac{3}{\sqrt{2}\gamma}) = \sqrt{\frac{3}{\lambda_1}}$$

and

$$\sqrt{\frac{4}{\gamma^2} + \frac{1}{\lambda_1}} \backsimeq \sqrt{\frac{1}{\lambda_1}}$$

therefore in this case Theorem 2.1 is also weaker than [8].

Let us determine the values of γ and λ_1 for which condition (3.6) is better than (3.7). To this end we can study the condition:

$$\frac{\sqrt{\frac{4}{\gamma^2} + \frac{1}{\lambda_1}}}{\max(\sqrt{\frac{3}{\lambda_1}}, \frac{3}{\sqrt{2}\gamma})} > 1$$

Therefore, we introduce:

$$g(\gamma, \lambda_1) = \frac{\sqrt{4 + \frac{\gamma^2}{\lambda_1}}}{\max(\sqrt{\frac{3\gamma^2}{\lambda_1}}, \frac{3}{\sqrt{2}})}$$

By setting $r = \frac{\gamma}{\sqrt{\lambda_1}}$, we obtain:

$$g(\gamma, \lambda_1) = p(r) = \frac{\sqrt{4 + r^2}}{\max(\sqrt{3r^2}, \frac{3}{\sqrt{2}})}$$

Introducing $\tau = r^2$, we have:

$$P(\tau) = \frac{4+\tau}{\max(\frac{9}{2}, 3\tau)}$$

A simple calculation shows that

$$P(\tau) > 1 \quad \Longleftrightarrow \quad \tau \in]\frac{1}{2}, 2[\quad \Longleftrightarrow \quad r^2 \in]\frac{1}{2}, 2[\quad \Longleftrightarrow \quad r \in]\frac{1}{\sqrt{2}}, \sqrt{2}[.$$

Finally, we obtain that if $\gamma \in]\sqrt{\frac{\lambda_1}{2}}, \sqrt{2\lambda_1}[$, Theorem 2.1 improves the result of [8].

3.4 The case $B = \gamma A$:

Let us consider the equation:

$$u'' + \gamma A u' + Au = f(t) \tag{3.8}$$

with $\gamma > 0$.

When we apply Theorem 2.1 to the equation (3.8) with $C = \gamma$ and $c = \gamma\lambda_1$, we obtain immediately:

Corollary 3.2. *Any solution of* (3.8) *satisfies the following hypotheses:*

$$\max(\overline{\lim_{t\to\infty}} \|u(t)\|, \overline{\lim_{t\to\infty}} |u'(t)|) \le \max(\sqrt{\frac{3}{\lambda_1}}, \frac{3}{\sqrt{2\gamma\lambda_1}}) \overline{\lim_{t\to\infty}} |f(t)| \tag{3.9}$$

Remark 3.3. This result is new and was not obtained in [11].

3.5 The case $B = \gamma A^{\frac{1}{2}}$

In this subsection we consider the so-called structural damping (cf [5, 6, 7] for the terminology and main properties). Therefore we consider as in [12] the equation:

$$u'' + \gamma A^{\frac{1}{2}} u' + Au = f(t) \tag{3.10}$$

with $\gamma > 0$.
If we apply theorem (2.1) with $c = \gamma\sqrt{\lambda_1}$ and $C = \frac{\gamma}{\sqrt{\lambda_1}}$, we obtain

$$\max(\overline{\lim_{t\to\infty}} \|u(t)\|, \overline{\lim_{t\to\infty}} |u'(t)|) \le \max(\sqrt{\frac{3}{\lambda_1}}, \frac{3}{\sqrt{2\gamma\sqrt{\lambda_1}}}) \overline{\lim_{t\to\infty}} |f(t)| \tag{3.11}$$

By comparaison with [12], we remark that (2.1) gives a weaker result. We shall now recover the estimate on u from [12] in the case of large damping by a method introduced by C. Fitouri (cf. [8]) which is less complicated than the method of [12].
We recall the main result from [12].

Theorem 3.4. *The bounded solution of* (3.10) *satisfies the estimate*

$$\forall t \in \mathbb{R}, \quad \|u(t)\| \leq \frac{1}{\sqrt{\lambda_1}} \max(1, \frac{2}{\gamma})\|f(t)\|_{L^\infty(\mathbb{R},H)}.$$

Proof. In the case of a small damping we refer to [2]. We now prove (3.4) when

$$\gamma \geq 2 \tag{3.12}$$

We choose the energy functional

$$\Phi = |A^{\frac{1}{4}}u'|^2 + |A^{\frac{3}{4}}u|^2 + \alpha(A^{\frac{1}{2}}u', A^{\frac{1}{2}}u)$$

Then, we have:

$$\begin{aligned}
\Phi' &= (2A^{\frac{1}{2}}u', u'' + Au) + \alpha|A^{\frac{1}{2}}u'|^2 + \alpha(Au, u'') \\
&= -2\gamma|A^{\frac{1}{2}}u'|^2 + \alpha|A^{\frac{1}{2}}u'|^2 - \gamma\alpha(Au, A^{\frac{1}{2}}u') - \alpha|Au|^2 + (f, 2A^{\frac{1}{2}}u' + \alpha Au) \\
&= -\frac{\alpha}{2}(|A^{\frac{1}{2}}u'|^2 + \alpha(Au, A^{\frac{1}{2}}u') + |Au|^2) + (\frac{3\alpha}{2} - 2\gamma)|A^{\frac{1}{2}}u'|^2 + (\frac{\alpha^2}{2} - \gamma\alpha)(Au, A^{\frac{1}{2}}u') \\
&\quad - \frac{\alpha}{2}|Au|^2 + (f, 2A^{\frac{1}{2}}u' + \alpha Au)
\end{aligned}$$

we set

$$\Psi = |A^{\frac{1}{2}}u'|^2 + \alpha(Au, A^{\frac{1}{2}}u') + |Au|^2$$

Then:

$$\Phi' = -\frac{\alpha}{2}\Psi + (\frac{3\alpha}{2} - 2\gamma)|A^{\frac{1}{2}}u'|^2 + (\frac{\alpha^2}{2} - \gamma\alpha)(Au, A^{\frac{1}{2}}u') - \frac{\alpha}{2}|Au|^2 + (f, 2A^{\frac{1}{2}}u' + \alpha Au)$$

by using Young's inequality, we obtain:

$(f, 2A^{\frac{1}{2}}u' + \alpha Au) \leq \frac{\alpha}{2}|f|^2 + \frac{1}{2\alpha}(4|A^{\frac{1}{2}}u'|^2 + 4\alpha(Au, A^{\frac{1}{2}}u') + \alpha^2|Au|^2)$

Therefore

$\Phi' \leq -\frac{\alpha}{2}\Psi + (\frac{3\alpha}{2} + \frac{2}{\alpha} - 2\gamma)|A^{\frac{1}{2}}u'|^2 + (\frac{\alpha^2}{2} - \gamma\alpha + 2)(Au, A^{\frac{1}{2}}u') + \frac{\alpha}{2}|f|^2$

we remark that $\alpha = \gamma - \sqrt{\gamma^2 - 4}$ is a solution of the equation:

$x^2 - 2\gamma x + 4 = 0$, then:

$$\frac{\alpha^2}{2} - \gamma\alpha + 2 = 0$$

we have also

$$2\gamma - \frac{3\alpha}{2} - 2\gamma + \frac{2}{\alpha} = \alpha - \gamma < 0$$

then

$$\Phi' \leq -\frac{\alpha}{2}\Psi + \frac{\alpha}{2}|f|^2$$

We have:

$$\alpha = \gamma - \sqrt{\gamma^2 - 4} = \frac{4}{\gamma + \sqrt{\gamma^2 - 4}} \leq \frac{4}{\gamma}$$

then, from (3.12)

$$0 < \frac{\alpha^2}{4} \leq \frac{4}{\gamma^2} < 1$$

We have

$$\begin{aligned}
\Psi &= |A^{\frac{1}{2}}u'|^2 + \alpha(Au, A^{\frac{1}{2}}u') + |Au|^2 \\
&= |A^{\frac{1}{4}}(A^{\frac{1}{4}}u' + \frac{\alpha}{2}A^{\frac{3}{4}}u)|^2 + (1 - \frac{\alpha^2}{4})|Au|^2 \\
&\geq \sqrt{\lambda_1}|A^{\frac{1}{4}}u' + \frac{\alpha}{2}A^{\frac{3}{4}}u)|^2 + (1 - \frac{\alpha^2}{4})|A^{\frac{3}{4}}u|^2 \\
&= \sqrt{\lambda_1}\Phi
\end{aligned}$$

Hence

$$\Phi' \leq -\frac{\alpha\sqrt{\lambda_1}}{2}\Phi + \frac{\alpha}{2}|f|^2$$

since Φ is bounded, we have

$$\forall t \in \mathbb{R}, \quad \Phi(t) \leq \frac{1}{\sqrt{\lambda_1}}\|f(t)\|_\infty^2$$

which means

$$\forall t \in \mathbb{R}, \quad |A^{\frac{1}{4}}u'(t)|^2 + |A^{\frac{3}{4}}u(t)|^2 + \alpha(A^{\frac{1}{2}}u(t), A^{\frac{1}{2}}u'(t)) \leq \frac{1}{\sqrt{\lambda_1}}\|f(t)\|_\infty^2$$

Then

$$\forall t \in \mathbb{R}, \quad \sqrt{\lambda_1}|A^{\frac{1}{2}}u(t)|^2 + \frac{\alpha}{2}\frac{d}{dt}|A^{\frac{1}{2}}u(t)|^2 \leq \frac{1}{\sqrt{\lambda_1}}\|f(t)\|_\infty^2$$

Finally, since u is bounded in V on $\mathbb{R}$, we obtain

$$\forall t \in \mathbb{R}, \quad \|u(t)\| \leq \frac{1}{\sqrt{\lambda_1}}\|f(t)\|_\infty \tag{3.13}$$

□

Remark 3.5. By this method, we do not recover the estimate of u' from [12] in the strongly damped case $\gamma > 2$.

4 Main examples

Let Ω be a bounded domain in $\mathbb{R}^{\mathbb{N}}$ and $\gamma > 0$.

Example 4.1. We consider the following equation

$$\left\{ \begin{array}{l} u_{tt} - \Delta u + \gamma u_t = f \\ u_{/\partial\Omega} = 0 \end{array} \right. \tag{4.1}$$

Then, as a consequence of (3.5) we have the following result valid for all mild solutions

$$\overline{\lim_{t\to\infty}} \left\{ \int_\Omega \|\nabla u\|^2 \, dx \right\}^{\frac{1}{2}} \leq \max\left(\sqrt{\frac{3}{\lambda_1(\Omega)}}, \frac{3}{\sqrt{2}\gamma} \right) \overline{\lim_{t\to\infty}} |f(t)|$$

This result improves on [11] when $\sqrt{\frac{\lambda_1(\Omega)}{2}} < \gamma < \sqrt{2\lambda_1(\Omega)}$.

Example 4.2. We consider the equation

$$\left\{ \begin{array}{l} u_{tt} - \Delta u - \gamma \Delta u_t = f \\ u_{/\partial\Omega} = 0 \end{array} \right. \tag{4.2}$$

We have the following result valid for all mild solutions

$$\overline{\lim_{t\to\infty}} \left\{ \int_\Omega \|\nabla u\|^2 \, dx \right\}^{\frac{1}{2}} \leq \max\left(\sqrt{\frac{3}{\lambda_1(\Omega)}}, \frac{3}{\sqrt{2}\gamma\lambda_1(\Omega)} \right) \overline{\lim_{t\to\infty}} |f(t)|$$

Example 4.3. We consider the equation

$$\left\{ \begin{array}{l} u_{tt} + \Delta^2 u - \gamma \Delta u_t = f \\ u = \Delta u = 0 \quad \text{on} \quad \partial\Omega \end{array} \right. \tag{4.3}$$

Then, we have for all mild solutions

$$\overline{\lim_{t\to\infty}} \left\{ \int_\Omega |\Delta u|^2 \, dx \right\}^{\frac{1}{2}} \leq \frac{1}{\lambda_1(\Omega)} \max\left(1, \ \frac{2}{\gamma} \right) \overline{\lim_{t\to\infty}} |f(t)|$$

This follows from Theorem 3.4 since here $\lambda_1(A) = \lambda_1(\Omega)^2$

Example 4.4. We consider the equation

$$\left\{ \begin{array}{l} u_{tt} + \Delta^2 u - \gamma \Delta u_t = f \\ u = |\nabla u| = 0 \quad \text{on} \quad \partial\Omega \end{array} \right. \tag{4.4}$$

Then, we shall establish

$$\overline{\lim_{t\to\infty}} \left\{ \int_\Omega |\Delta u|^2 \, dx \right\}^{\frac{1}{2}} \leq \max \left(\sqrt{\frac{3}{\lambda_1(\Omega)\lambda_1(A)}}, \frac{3}{\sqrt{2}\gamma\lambda_1(\Omega)} \right) \overline{\lim_{t\to\infty}} |f(t)|$$

Indeed, in this example, we have

$$B = -\gamma\Delta : H_0^1 \to H^{-1}; \quad A = \Delta^2$$

with domain

$$D(A) = \{u \in H^2(\Omega) | \quad u = |\nabla u| = 0 \quad \text{on} \quad \partial\Omega\}$$

and

$$cI \leq B \leq CA,$$

with

$$c = \gamma\lambda_1(\Omega)$$

To get an estimate for C we observe that

$$\begin{aligned}
(Bv, v) &= \gamma \int_\Omega \|\nabla u\|^2 \, dx \\
&= -\gamma \int_\Omega \Delta v.v \, dx \\
&= \gamma (\int_\Omega |\Delta v|^2 \, dx)^{\frac{1}{2}} (\int_\Omega |v|^2 \, ds)^{\frac{1}{2}} \\
&= \frac{\gamma}{\lambda_1(A)} (\int_\Omega |\Delta v|^2 \, dx) \\
&= \frac{\gamma}{\lambda_1(A)} (Av, v)
\end{aligned}$$

Therefore, we can take $C \leq \frac{\gamma}{\lambda_1(A)}$ and this shows the claim.

Remark 4.5. Actually, since we used a Cauchy-Schwarz inequality for two linearly independent functions it is clear that the optimal value of C is stricly less than $\frac{\gamma}{\lambda_1(A)}$. More precisely to obtain the optimum we need to evaluate

$$\mu = \inf \left\{ \frac{\int_\Omega |\Delta v|^2 \, dx}{\int_\Omega |\nabla v|^2 \, dx}, \ v \in H_0^2(\Omega), \ v \neq 0 \right\} = \inf \left\{ \int_\Omega |\Delta v|^2 \, dx, \ v \in H_0^2(\Omega), \quad \int_\Omega |\nabla v|^2 = 1 \right\}$$

By the Lagrange multiplier theory, there is $v \neq 0$ such that

$$\begin{cases} \Delta^2 v = -\mu\Delta v \\ v \in H_0^2(\Omega) \end{cases}$$

with

$$\Delta v \in L^2(\Omega)$$

and

$$-(\Delta + \mu)(\Delta v) = 0$$

Then we have $C = \frac{1}{\mu}$. To illustrate this we consider the one dimensional case.

Proposition 4.6. *If $N = 1$, $\Omega = (0, \pi)$ then $C = \frac{1}{4}$.*

Proof. In order to compute C we need to find the minimal value of μ when

$$u^{(4)} = -\mu u'', \quad u \in H_0^2(0, \pi)$$

Then, setting $\lambda = \sqrt{\mu}$, we have

$$u = c_1 x + c_2 + c_3 \cos(\lambda x) + c_4 \sin(\lambda x)$$

$$u' = c_1 - \lambda c_3 \sin(\lambda x) + \lambda c_4 \cos(\lambda x)$$

$$0 = c_2 + c_3$$

$$0 = c_1 \pi + c_2 + c_3 \cos(\pi\lambda) + c_4 \sin(\pi\lambda)$$

$$0 = c_1 + \lambda c_4$$

$$0 = c_1 - \lambda c_3 \sin(\lambda\pi) + \lambda c_4 \cos(\lambda\pi)$$

$$c_4 = -c_3 \sin(\pi\lambda) + c_4 \cos(\pi\lambda)$$

$$c_4(1 - \cos(\pi\lambda) = -c_3 \sin(\pi\lambda)$$

We distinguish 3 possibilities.

case 1: If $\sin(\pi\lambda) = 0$ and $\cos(\pi\lambda) \neq 1$ $(= -1)$ then

$$c_4 = 0 \Longrightarrow c_1 = 0, \quad c_2 = -c_3 \cos(\pi\lambda) = c_3 \Longrightarrow c_2 = c_3 = 0$$

then $u \equiv 0$ and this case is excluded.

case 2: If $\sin(\pi\lambda) = 0$ and $\cos(\pi\lambda) = 1 \Longrightarrow \lambda = 2k, \quad k \in \mathbb{N}$
then

$$0 = c_1 \pi + c_2 + c_3 = c_2 + c_3 \Longrightarrow c_1 = 0$$

and

$$c_4 = -\frac{1}{\lambda} c_1 = 0.$$

Therefore $u = c_2(1 - \cos(2kx)) = 2c_2 \sin^2 kx$. In this case $\mu = 4k^2$ and therefore $\mu \geq 4$.

case 3: If $\sin(\pi\lambda) \neq 0$, then

$$2c_4 \sin^2(\frac{\pi\lambda}{2}) = -2c_3 \sin(\frac{\pi\lambda}{2}) \cos(\frac{\pi\lambda}{2})$$

hence

$$c_3 = -c_4 \tan(\frac{\pi\lambda}{2}) \; c_2 = -c_3, \; c_1 = -\lambda c_4.$$

and

$$-\lambda\pi c_4 + c_4 \tan(\frac{\pi\lambda}{2}) - c_4 \cos(\pi\lambda) \tan(\frac{\pi\lambda}{2}) + c_4 \sin(\pi\lambda) = 0.$$

If $c_4 = 0$, then $u = 0$.

If $c_4 \neq 0$, we can reduce to $c_4 = 1$, then we find

$$\tan(\frac{\pi\lambda}{2})(1 - \cos(\pi\lambda) + 2\cos^2(\frac{\pi\lambda}{2})) = \lambda\pi \iff 2\tan(\frac{\pi\lambda}{2}) = \lambda\pi \iff \tan(\frac{\pi\lambda}{2}) = \frac{\pi\lambda}{2}$$

Therefore

$$\frac{\pi\lambda}{2} > \pi \quad \Longrightarrow \lambda > 2$$

and

$$\mu = \lambda^2 > 4.$$

Summarizing the 3 cases we conclude that the minimal possible value of μ is 4. □

Corollary 4.7. *Any mild solution u of*

$$\begin{cases} u_{tt} + u_{xxxx} - \gamma u_{xxt} = f \\ u(t,0) = u(t,\pi) = u_x(t,0) = u_x(t,\pi) = 0 \end{cases} \tag{4.5}$$

satisfies the asymptotic bound:

$$\overline{\lim_{t\to\infty}} \left\{ \int_\Omega |u_{xx}|^2 \, dx \right\}^{\frac{1}{2}} \leq \max\left(\frac{\sqrt{3}}{2}, \frac{3}{\sqrt{2}\gamma}\right) \overline{\lim_{t\to\infty}} |f(t)| \tag{4.6}$$

5 Additional results

5.1 The first eigenvalue of a square root.

At several places in this paper we used implicitely the property

$$\lambda_1(A^{\frac{1}{2}}) = (\lambda_1(A))^{\frac{1}{2}}$$

where A is a self-adjoint coercive operator. This property is obvious when A has compact inverse, but it is natural to ask what happens in general. In the next subsection we shall derive a similar property for any positive power of A, but in the case of square roots an easier proof can be given. The result is as follows

Proposition 5.1. *Let A be as the introduction. Then $A^{\frac{1}{2}}$ is also coercive and $\lambda_1(A^{\frac{1}{2}}) = (\lambda_1(A))^{\frac{1}{2}}$.*

The proof of this proposition relies on 2 simple lemmas :

Lemma 5.2. *Let $B \in L(H)$ be symmetric and nonnegative. Then we have*

$$\|B^2\| = \|B\|^2$$

$$\forall v \in H, \quad |Bv|^2 \leq \|B\|(Bv, v)$$

Proof. First we have $B^2 \in L(H)$ and $\|B^2\| \leq \|B\|^2$. The reverse inequality is also immediate since

$$|Bu|^2 = (B^2u, u) \leq \|B^2\||u|^2$$

Finally we have for any $v \in H$

$$|Bv|^2 = |B^{\frac{1}{2}}(B^{\frac{1}{2}}v)|^2 \leq \|B^{\frac{1}{2}}\|^2|B^{\frac{1}{2}}v|^2 = \|B\|(Bv, v)$$

□

Lemma 5.3. *Let A be a self-adjoint, positive, coercive operator. Then*

$$\lambda_1(A) = \frac{1}{\|A^{-1}\|}$$

Proof. By definition it is clear that

$$\lambda_1(A) = \frac{1}{\|A^{-\frac{1}{2}}\|^2}$$

Then the result follows from the previous Lemma.

□

Proof of Proposition 5.1. We first show that $A^{\frac{1}{2}}$ is coercive. Actually $A^{\frac{1}{2}} \in L(V,H)$ is clearly injective. Moreover for any $h \in H$, there is $u \in D(A)$ with $Au = h$. But then $v = A^{\frac{1}{2}}u \in V$ and $A^{\frac{1}{2}}v = h$. Hence $A^{\frac{1}{2}} \in L(V,H)$ is onto and by Banach Theorem, $A^{-\frac{1}{2}} \in L(H,V)$. By Lemma 5.2 we find that that $A^{\frac{1}{2}}$ is coercive. Then $\lambda_1(A^{\frac{1}{2}}) = \frac{1}{\|A^{-\frac{1}{2}}\|}$ and the result follows from a last application of Lemma 5.2 □

5.2 The first eigenvalue of a fractional power.

Let A be a self-adjoint coercive operator. The fractional power A^α with $\alpha \in (0,1)$ is defined as the inverse of the operator

$$A^{-\alpha} = \frac{\sin(\pi\alpha)}{\pi}\int_0^\infty t^{-\alpha}(tI + A)^{-1}dt$$

with domain equal to the range of $A^{-\alpha}$. $D(A^\alpha)$ is also the closure of $D(A)$ under the semi-norm $p_\alpha(u) := |A^\alpha(u)|$ (cf. e.g. [2, 13]).
We now generalize Proposition 5.1 to any positive power by relying on the above formula.

Proposition 5.4. *For any $\alpha \in (0,1)$, A^α is also coercive and $\lambda_1(A^\alpha) = (\lambda_1(A))^\alpha$.*

Proof. By homogeneity it is clearly sufficient to establish the result when $\lambda_1(A) = 1$ Then applying the result to $A_1 = \lambda_1(A)^{-1}A$ gives the general case. First we show that

$$\lambda_1(A) = 1 \Longrightarrow \lambda_1((A^\alpha) \geq 1$$

Indeed we have

$$\|A^{-\alpha}\| \leq \frac{\sin(\pi\alpha)}{\pi}\int_0^\infty t^{-\alpha}\|(tI + A)^{-1}\|dt \leq \frac{\sin(\pi\alpha)}{\pi}\int_0^\infty t^{-\alpha}(t+1)^{-1}dt = 1$$

and then Lemma 5.3 gives the result. Now we have for any $u \in D(A)$

$$(Au, u) = (A^\alpha A^{1-\alpha}u, u) = (A^\alpha A^\beta u, A^\beta u)$$

with $\beta = \frac{1-\alpha}{2}$. Hence

$$(Au, u) \geq \lambda_1(A^\alpha)|A^\beta u, A^\beta u|^2 = \lambda_1(A^\alpha)(A^{1-\alpha}u, u) \geq \lambda_1(A^\alpha)\lambda_1(A^{1-\alpha})|u|^2$$

Then

$$1 = \lambda_1(A) \geq \lambda_1(A^\alpha)\lambda_1(A^{1-\alpha})$$

Finally

$$\lambda_1(A^\alpha) = \lambda_1(A^{1-\alpha}) = 1$$

□

5.3 The relationship between the two main results.

In Remark 2.3 we said that when B is linear and self-adjoint, the two inequalities (1.3) and (2.2) are equivalent. This is a consequence of the following

Proposition 5.5. *Let A be as the introduction and $\in L(V, V')$ be symmetric and nonnegative. Then the 3 following conditions are equivalent*

$$\|B\|_{L(V,V')} \leq C \tag{5.1}$$

$$B \leq CA \tag{5.2}$$

$$\forall u \in V, \quad \|Bu\|_*^2 \leq C\langle Bu, u\rangle \tag{5.3}$$

Proof. We proceed in 3 steps

1) Proof of (5.1) $\Longrightarrow$ (5.2). Assuming (5.1) we have

$$\forall u \in V, \quad \langle Bu, u\rangle \leq \|B\|_{L(V,V')}\|u\| \leq C\|u\|^2 = C\langle Au, u\rangle$$

Hence $B \leq CA$.

2) Proof of (5.3) $\Longrightarrow$ (5.1). Assuming (5.3) we have

$$\forall u \in V, \quad \|Bu\|_*^2 \leq C\langle Bu, u\rangle \leq C\|Bu\|_*\|u\|$$

Hence, either $Bu = 0$ or $\|Bu\|_* \leq C\|u\|$ and we have (5.1).

3) Proof of (5.2) $\Longrightarrow$ (5.3). Since $B \geq 0$ we have

$$\forall (u, v) \in V \times V, \quad \langle Bu, v\rangle^2 \leq \langle Bu, u\rangle\langle Bv, v\rangle$$

In this formula we choose $v = A^{-1}(Bu)$ Then

$$\langle Bu, v\rangle = \langle Bu, A^{-1}(Bu)\rangle = \|A^{-1}Bu\|^2 = \|Bu\|_*^2$$

so that we find

$$\|Bu\|_*^4 \leq \langle Bu, u\rangle\langle BA^{-1}(Bu), A^{-1}(Bu)\rangle \leq C\langle Bu, u\rangle\langle AA^{-1}(Bu), A^{-1}(Bu)\rangle$$

by using (5.2). Now

$$\langle Bu, u\rangle\langle AA^{-1}(Bu), A^{-1}(Bu) = \langle Bu, u\rangle\|A^{-1}(Bu)\|^2 = \|Bu\|_*^2\langle Bu, u\rangle$$

and if $Bu \neq 0$ we obtain (5.1) on dividing through by $\|Bu\|_*^2$. □

Remark 5.6. For a general positive operator the conditions are not equivalent . For instance take $V = H = \mathbb{C}$ and for some $\alpha > 0, \beta \in \mathbb{R}$

$$\forall v \in V, \quad Bv = (\alpha + i\beta)v$$

In this case we have

$$\|B\| = (\alpha^2 + \beta^2)^{\frac{1}{2}}$$

$$\forall v \in V, \quad (Bv, v) = \alpha |v|^2$$

so that the optimal value of C in (5.2) is α. The optimal value of C in (5.3) is $\frac{\alpha^2+\beta^2}{\alpha}$. As soon as $\beta \neq 0$ we have

$$\alpha < (\alpha^2 + \beta^2)^{\frac{1}{2}} < \frac{\alpha^2 + \beta^2}{\alpha}$$

and therefore the three constants are all different.

5.4 Some more examples.

Sometimes Theorem 2.1 can be applied to equations in unbounded domains. For brevity we give only 2 typical examples

Example 5.7. Let Ω be a possibly unbounded domain in $\mathbb{R}^{\mathbb{N}}$ and $m > 0, \gamma > 0$. We consider the following equation

$$\begin{cases} u_{tt} - \Delta u + mu + \gamma u_t = f \\ u_{/\partial\Omega} = 0 \end{cases} \tag{5.4}$$

Then, as a consequence of (3.5) we have the following result valid for all mild solutions

$$\overline{\lim_{t\to\infty}} \left\{ \int_\Omega m|u|^2 + \|\nabla u\|^2 \, dx \right\}^{\frac{1}{2}} \leq \max \left(\sqrt{\frac{3}{m + \lambda_1(\Omega)}}, \frac{3}{\sqrt{2}\gamma} \right) \overline{\lim_{t\to\infty}} |f(t)|$$

Example 5.8. Let Ω be a bounded domain in $\mathbb{R}^{\mathbb{N}}$ and $\gamma > 0$. We consider the cylinder $\mathcal{C} = \Omega \times \mathbb{R}$ and the following equation in $\mathbb{R}^+ \times \mathcal{C}$

$$\begin{cases} u_{tt} - \Delta u + \gamma u_t = f \\ u_{/\partial\mathcal{C}} = 0 \end{cases} \tag{5.5}$$

Then, as a consequence of (3.5), since $A = -\Delta$ is coercive in $\mathcal{C}$ with $\lambda_1(\mathcal{C}) = \lambda_1(\Omega)$we have the following result valid for all mild solutions

$$\overline{\lim_{t\to\infty}} \left\{ \int_\Omega \|\nabla u\|^2 \, dx \right\}^{\frac{1}{2}} \leq \max \left(\sqrt{\frac{3}{\lambda_1(\Omega)}}, \frac{3}{\sqrt{2}\gamma} \right) \overline{\lim_{t\to\infty}} |f(t)|$$

We conclude this section by giving 2 examples of application for Theorem 1.1 and Theorem 2.1: a case where B is not selfadjoint and a case where B is non linear.

Example 5.9. Let $V = H = \mathbb{C}$. Then any solution u of the ODE

$$u'' + u + (\alpha + i\beta)u' = f \tag{5.6}$$

satisfies

$$\max(\overline{\lim_{t\to\infty}} |u(t)|, \overline{\lim_{t\to\infty}} |u'(t)|) \le \max(\sqrt{12}\sqrt{1+\frac{\beta^2}{\alpha^2}}, \frac{3}{\alpha}) \overline{\lim_{t\to\infty}} |f(t)|$$

We did not investigate how close from optimality this estimate is.

Example 5.10. Let Ω be a bounded domain in $\mathbb{R}^{\mathbb{N}}$ and $\gamma > 0$. We consider the following equation

$$\begin{cases} u_{tt} - \Delta u + \alpha(t,x)u_t^+ - \beta(t,x)u_t^- = f \\ u_{/\partial\Omega} = 0 \end{cases} \tag{5.7}$$

where $\alpha, \beta \in C^1(\mathbb{R}^+, C^0(\overline{\Omega})$ are nonnegative functions with

$$0 < a \le \min(\alpha(t,x), \beta(t,x)) \le \max(\alpha(t,x), \beta(t,x)) \le A.$$

It is tempting to apply Theorem 1.1 in this situation. However it is better to use Theorem 2.1 as follows. First we can approach the solutions by strong solutions with f replaced by a smooth function with a smaller or equal $L^2(\Omega)$-ultimate bound. For such a solution we can write

$$\alpha(t,x)u_t^+ - \beta(t,x)u_t^- = B(t,x)u_t$$

where

$$B(t,x) = \alpha(t,x)\chi(u_t > 0) - \beta(t,x)\chi(u_t \le 0)$$

is a multiplication operator. Then, as a consquence of Theorem 2.1 we find

$$\overline{\lim_{t\to\infty}} \left\{ \int_\Omega \|\nabla u\|^2 \, dx \right\}^{\frac{1}{2}} \le \max\left(\sqrt{\frac{3A}{a\lambda_1(\Omega)}}, \frac{3}{a\sqrt{2}} \right) \overline{\lim_{t\to\infty}} |f(t)|$$

We skip the details.

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