Compactness of trajectories to some nonlinear second order evolution equations and applications.

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Abstract

Under suitable growth and coercivity conditions on the nonlinear damping operator g, we establish boundedness or compactness properties of trajectories to the equation

$$\ddot{u}(t) + g(\dot{u}(t)) + Au(t) = h(t), \quad t \in \mathbb{R}_+,$$

where A is a positive selfadjoint operator and g is a nonlinear damping operator. The compactness results are used to prove the existence of almost periodic solutions when h is almost periodic, and to generalize some recent results of Chergui and Ben Hassen-Chergui concerning convergence to equilibrium when a nonlinear term depending on u is added and h dies off sufficiently fast for t large.

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1 Introduction.

In this paper we investigate the asymptotic behavior of solutions to the equation

$$\ddot{u}(t) + g(\dot{u}(t)) + Au(t) + f(u(t)) = h(t), \quad t \in \mathbb{R}_+, \tag{1.1}$$

where V is a real hilbert space, $A \in L(V, V')$ is a symmetric, positive, coercive operator , $g \in C(V, V')$ is monotone, f is a gradient operator satisfying some appropriate conditions and h is a forcing term. We are especially interested in the two following cases

1) $f = 0$ and h is almost periodic.

2) h tends to 0 sufficiently fast at infinity in t and f is the gradient of an analytic functional or more generally the gradient of a potential satisfying the Łojasiewicz gradient in equality in a sense which will be specified later.

Case 1) has been intensively studied in the Literature, covering the following topics: existence of almost periodic solutions, asymptotic behavior of the general solution, rate of decay to 0 of the difference of two solutions in the energy space in the best cases, cf.e.g.[1, 5, 6, 12, 15, 14, 24, 19, 25]. Until now, although boundedness of trajectories for $t \geq 0$ is sufficient in the linear case and when h is time-periodic, the treatment of the general case has always required the existence of a precompact trajectory, and the problem is that it is difficult to distinguish between different trajectories at this level: if we were able to exhibit a precompact orbit without knowing anything about the others, it would mean that we can (by going to the positive ω-limit set) localize an almost periodic orbit and this is precisely what becomes impossible in the nonlinear case. Therefore we are condemned to prove compactness of all trajectories or nothing (note that for g tangent to 0 at the origine, extra regularity of the initial state or even of the forcing term will not help when t becomes large). Another difficulty is the following: if the existence of bounded trajectories can be proved by combining a coerciveness property of g “at infinity” with some growth restrictions, compactness requires a global, uniform kind of coerciveness. In the past more and more general compactness results have been obtained, but only when g is a Nemytskii type operator. When g is a non-local operator or involves differential operators in space, the theory remained to be done: this is the main object of the present paper.

For case 2), compactness is the vital starting point. In the past several significative advances have been done, cf.e.g. [21, 16, 17, 18, 20, 11, 9, 10, 2, 3, 4], the other tool here being the Łojasiewicz gradient inequality [22, 23]. But here even the case $h = 0$ is non-trivial since the set of equilibria needs not have any particular structure except for the restrictions induced by the existence of a Łojasiewicz inequality : we know for instance in advance that the potential energy is constant on continua inside the set of equilibria, a property which can fail for C^∞ and even Gevrey potentials. The fact that precompactness of trajectories had been proved only for Nemytskii type damping operators limited until now the convergence results with non-linear damping to those damping operators. Therefore the second innovation of this paper is to contain the first convergence results in case 2) in presence of a non-local damping term.

The plan of the paper is as follows: In Section 2, we introduce the basic tools used in the statements and proofs of the main results. Section 3 is devoted to the initial value problem for (1.1). Sections 4 and 5 contain the statement and proof of the boundedness result and the compactness result, respectively. Sections 6 and 7 contain respectively the statement and proof of the asymptotic almost periodicity and the semilinear convergence result, respectively. Finally Section 8 is devoted to the application to PDE models with non-local damping terms.

2 Some useful tools.

In this section, we collect quite a few results of general interest which will reveal essential for the proofs of our main results. We also need to recall the definitions of some well known mathematical objects as well as their basic properties in the exact functional framework that shall be used in the main sections containing our new results.

2.1 Monotonicity theory

Let $\mathcal{H}$ be a real Hilbert space endowed with an inner product $(.,.)_{\mathcal{H}}$. We recall that a map $\mathcal{A}$ defined on a part $\mathcal{D} = \mathcal{D}(\mathcal{A})$ with values in $\mathcal{H}$ is monotone if

$$\forall (U, \hat{U}) \in \mathcal{D} \times \mathcal{D}, \quad (\mathcal{A}U - \mathcal{A}\hat{U}, U - \hat{U})_{\mathcal{H}} \geq 0.$$

In addition $\mathcal{A}$ is called maximal monotone if

$$\forall F \in \mathcal{H}, \quad \exists U \in D(\mathcal{A}) \quad \mathcal{A}\mathcal{U} + \mathcal{U} = \mathcal{F}.$$

The following result is well-known (cf. H. Brezis [7]) .

Proposition 2.1. *if $\mathcal{A}$ is maximal monotone, for each $T > 0$, each $U_0 \in D(\mathcal{A})$ and $F = F(t) \in W^{1,1}(0, T; \mathcal{H})$ there is a unique function $U \in W^{1,1}(0, T; \mathcal{H})$ with $U(t) \in D(\mathcal{A}))$ for almost all $t \in (0, T)$, $U(0) = U_0$ and such that for almost all $t \in (0, T)$*

$$U'(t) + \mathcal{A}U(t) = F(t). \tag{2.1}$$

In addition if for some $\hat{U}_0 \in D(\mathcal{A})$ and $\hat{H} \in W^{1,1}(0, T; \mathcal{H})$ we consider the solution $\hat{U} \in W^{1,1}(0, T; \mathcal{H})$ with $\hat{U}(t) \in D(\mathcal{A}))$ for almost all $t \in (0, T)$, $\hat{U}(0) = \hat{U}_0$ of

$$\hat{U}'(t) + \mathcal{A}\hat{U}(t) = \hat{F}(t)$$

then the difference satisfies the inequality

$$\forall t \in [0, T], \quad |U(t) - \hat{U}(t)| \leq \quad |U_0 - \hat{U}_0| + \int_0^t |F(s) - \hat{F}(s)| ds$$

This proposition allows to define by density, for any $U_0 \in \overline{D(\mathcal{A})}$ and $F = F(t) \in L^1(0, T; \mathcal{H})$ a weak solution of (2.1) such that $U(0) = U_0$, cf. H. Brezis [7].

2.2 A class of nonlinear operators

In the applications to non-local dissipations we shall use the following simple inequalities. Let X be any Hilbert space with norm denoted by $|.|$ and inner product by $\langle , \rangle$. Then for any $\alpha > 0$ we have

Lemma 2.2.

$$\forall (v,w) \in X \times X,\ \ ||v|^{\alpha} v - |w|^{\alpha} w| \leq (\alpha+1) \max\{|v|,|w|\}^{\alpha} |v-w|. \tag{2.2}$$

Proof. The inequality is trivial if v or w vanishes. Assuming $|v| \geq |w| > 0$, we can write

$$||v|^{\alpha} v - |w|^{\alpha} w| \leq (|v|^{\alpha} - |w|^{\alpha})|w| + |v|^{\alpha}|v-w|$$

Then we have 2 cases:

- If $\alpha \geq 1$, then

$$|v|^{\alpha} - |w|^{\alpha} \leq \alpha |v|^{\alpha-1}(|v|-|w|) \leq \alpha |v|^{\alpha-1}|v-w|$$

and then

$$(|v|^{\alpha} - |w|^{\alpha})|w| \leq \alpha |v|^{\alpha-1}|w|||v-w| \leq \alpha |v|^{\alpha}|v-w|$$

hence

$$||v|^{\alpha} v - |w|^{\alpha} w| \leq (\alpha+1)|v|^{\alpha}|v-w|$$

and the result is proved.

- If $\alpha < 1$, then

$$|v|^{\alpha} - |w|^{\alpha} \leq \alpha |w|^{\alpha-1}(|v|-|w|) \leq \alpha |w|^{\alpha-1}|v-w|$$

and then

$$(|v|^{\alpha} - |w|^{\alpha})|w| \leq \alpha |w|^{\alpha-1}|w|||v-w| = \alpha |w|^{\alpha}|v-w| \leq \alpha |v|^{\alpha}|v-w|$$

leading to the same conclusion.

□

Lemma 2.3.

$$\forall (v,w) \in X \times X,\ \ \langle |v|^{\alpha} v - |w|^{\alpha} w, v-w \rangle \geq 2^{-\alpha}|v-w|^{\alpha+2} \tag{2.3}$$

with equality if and only if $w = -v$.

Proof. We set $u = -w$ and $P = \langle |v|^\alpha v - |w|^\alpha w, v - w\rangle = \langle |u|^\alpha u + |v|^\alpha v, u + v\rangle$ so that we are left to prove $P \geq 2^{-\alpha}|u + v|^{\alpha+2}$ with equality if and only if $u = v$. We expand

$$P = |u|^{\alpha+2} + |v|^{\alpha+2} + (|u|^\alpha + |v|^\alpha)\langle u, v\rangle$$

$$= |u|^{\alpha+2} + |v|^{\alpha+2} + \frac{1}{2}(|u|^\alpha + |v|^\alpha)(|u + v|^2 - |u|^2 - |v|^2)$$

$$= \frac{1}{2}(|u|^\alpha + |v|^\alpha)|u + v|^2 + \frac{1}{2}(|u|^\alpha - |v|^\alpha)(|u|^2 - |v|^2).$$

If $u + v = 0$ there is nothing to prove, otherwise

$$\frac{2P}{|u + v|^{\alpha+2}} = \frac{|u|^\alpha + |v|^\alpha}{|u + v|^\alpha} + \frac{(|u|^\alpha - |v|^\alpha)(|u|^2 - |v|^2)}{|u + v|^{\alpha+2}}$$

hence

$$\frac{2P}{|u + v|^{\alpha+2}} \geq \frac{|u|^\alpha + |v|^\alpha}{(|u| + |v|)^\alpha} + \frac{(|u|^\alpha - |v|^\alpha)(|u|^2 - |v|^2)}{(|u| + |v|)^{\alpha+2}}$$

with equality if and only if u and v are proportional with nonnegative ratio. But we have

$$\frac{|u|^\alpha + |v|^\alpha}{(|u| + |v|)^\alpha} + \frac{(|u|^\alpha - |v|^\alpha)(|u|^2 - |v|^2)}{(|u| + |v|)^{\alpha+2}}$$

$$= \frac{(|u|^\alpha + |v|^\alpha)(|u| + |v|)^2) + (|u|^\alpha - |v|^\alpha)(|u|^2 - |v|^2)}{(|u| + |v|)^{\alpha+2}}$$

$$= \frac{(|u|^\alpha + |v|^\alpha)(|u| + |v|)) + (|u|^\alpha - |v|^\alpha)(|u| - |v|)}{(|u| + |v|)^{\alpha+1}} = 2\frac{|u|^{\alpha+1} + |v|^{\alpha+1}}{(|u| + |v|)^{\alpha+1}}.$$

Assuming for instance $|u| \geq |v|$ (in particular $u \neq 0$) and setting $t = \frac{|v|}{|u|} \leq 1$, we therefore obtain

$$\frac{P}{|u + v|^{\alpha+2}} \geq \frac{|u|^{\alpha+1} + |v|^{\alpha+1}}{(|u| + |v|)^{\alpha+1}} = \frac{1 + t^{\alpha+1}}{(1 + t)^{\alpha+1}}$$

with equality if and only if $u = tv$. To conclude it is sufficient to observe that $f(t) = \frac{1+t^{\alpha+1}}{(1+t)^{\alpha+1}}$ is decreasing on $(0, 1)$ with $f(1) = 2^{-\alpha}$. Indeed setting $h(t) = -\ln f(t)$ we have

$$\forall t \in (0, 1), \quad h'(t) = \frac{(\alpha + 1)(1 - t^\alpha)}{(1 + t)(1 + t^{\alpha+1})} > 0$$

$\square$

2.3 Almost periodic functions

A typical almost periodic numerical function is the sum of two periodic functions with incommensurable periods. Such objects often appear in the mechanics of vibrating systems, and sometimes infinite sums naturally impose their presence, for instance when studying the energy conservative vibrations of continuous media .

There are several equivalent definitions of almost periodicity, but in the theory of differential equations, the most convenient criterion is Bochner's functional definition:

Definition 2.4. Given a complete metric space X , a function $f : \mathbb{R} \to X$ is almost periodic iff the set of translates

$$\bigcup_{\alpha\in\mathbb{R}} \{f(. + \alpha\} = \mathcal{T}(f)$$

is precompact in the space $C_b(\mathbb{R}, X)$ endowed with the topology of uniform convergence $\mathbb{R} \to X$.

It follows clearly from the definition that

i) For any $T > 0$, any continuous T-periodic function: $f : \mathbb{R} \to X$ is almost periodic with $\mathcal{T}(f)$ compact.

ii) For 3 complete metric spaces X, Y, Z, if $f : \mathbb{R} \to X$ and $g : \mathbb{R} \to Y$ are- almost periodic and $C : X \times Y \to Z$ is uniformly continuous, the function $h(t) = C(f(t), g(t))$ is almost periodic $\mathbb{R} \to Z$ In particular if X is a Banach space, any finite sum of almost periodic functions: $\mathbb{R} \to X$ is almost periodic: $\mathbb{R} \to X$.

iii) Any uniform limit of almost periodic functions: $\mathbb{R} \to X$ is almost periodic: $\mathbb{R} \to X$.

iv) Any almost periodic function: $\mathbb{R} \to X$ is uniformly continuous with precompact range.

In the theory of abstract differential equations an important problem is the following: given a nonlinear operator $\mathcal{A} : D \to X$ and an exterior almost periodic force $F : \mathbb{R} \to X$ when can we garantee that the 'response ", i.e. the general solution of the evolution equation

$$U'(t) + \mathcal{A}U(t) = F(t)$$

adapts to F in the sense that it asymptotes an almost periodic function for t large? This problem is difficult even for ODEs and has only received a partial answer even in the case of equation (2.1), cf.e.g.[1, 5, 6, 12, 15, 14].

2.4 Stepanov spaces and generalized almost periodicity

Definition 2.5. Given a real Banach space X with norm $\|.\|$ and an infinite interval $J = [t_0, +\infty)$ we set

$$S^1(J,X) = \{f \in L^1_{loc}(J,X), \quad \sup_{t\in J}\int_t^{t+1} \|f(s)\|ds < \infty\}$$

where "S" stands for Stepanov. It is immediate to check that $S^1(J,X)$ endowed with the semi-norm

$$\|f\|_{S^1(J,X)} = \sup_{t\in J}\int_t^{t+1} \|f(s)\|ds$$

is a Banach space containing $L^\infty(J,X)$. One similarly defines for any $p \geq 1$

$$S^p(J,X) = \{f \in L^p_{loc}(J,X), \quad \sup_{t\in J}\int_t^{t+1} \|f(s)\|^p ds < \infty\}$$

and we may complete by setting $S^\infty(J,X) = L^\infty(J,X)$. When $X = \mathbb{R}$ we simplify the notation:

$$\forall p \in [1,\infty], \ S^p(J,\mathbb{R}) =: S^p(J)$$

For the theory of almost periodic functions the most important case is when $t_0 = -\infty$ i.e. $J = \mathbb{R}$ and $p = 1$. Indeed when we have a good adaptation result saying that the response to an almost periodic forcing asymptotes an almost periodic function for t large, most of the time we can afford discontinuous locally integrable forcings thanks to the smoothing effect of integration.

Definition 2.6. We say that $f \in S^p(\mathbb{R},X)$ is S^p- almost periodic : $\mathbb{R} \to X$ if the function $F(\alpha) : f(\ + \alpha) := T_\alpha f$ is almost periodic : $\mathbb{R} \to S^p(\mathbb{R},X)$

It is clear that usual (continuous) almost periodic functions are S^p- almost periodic for all p.

2.5 Boundedness via differential inequalities

Lemma 2.7. *Let $\alpha \in S^1(\mathbb{R}_+)$ and $F \in S^1(\mathbb{R}_+)$ be such that $F \geq 0$ and*

$$\forall t \in \mathbb{R}_+, \quad \int_t^{t+1} \alpha(r)dr \geq \alpha_0 > 0 \tag{2.4}$$

Let $\Phi \in L^1_{loc}(\mathbb{R}_+), \Phi \geq 0$ satisfy the differential inequality

$$\Phi'(t) + \alpha(t)\Phi(t) \leq F(t), \quad a.e. \ on\ \mathbb{R}_+ \tag{2.5}$$

Then Φ is bounded and we have

$$\forall t \in \mathbb{R}_+, \quad \Phi(t) \le e^{A_0}\{e^{-\alpha_0 t}\Phi(0) + (1+\frac{1}{\alpha_0})\|F\|_{S^1}\} \tag{2.6}$$

with

$$A_0 = \alpha_0 + \|\alpha\|_{S^1}$$

Proof. We observe that for any $(s,t) \in \mathbb{R}^2$ with $0 \le s \le t$ we have

$$\int_s^t \alpha(r)dr \ge \alpha_0(t-s) - A_0$$

Indeed setting $t = s+n+\rho$ with $n \in \mathbb{N}, 0 \le \rho < 1$ we have

$$\int_s^t \alpha(r)dr = \int_s^{s+n} \alpha(r)dr + \int_{s+n}^t \alpha(r)dr \ge n\alpha_0 - \|\alpha\|_{S^1} \ge (t-s-1)\alpha_0 - \|\alpha\|_{S^1}$$

The result then relies on the following more general Lemma

□

Lemma 2.8. *Let $\alpha \in L^1_{loc}(\mathbb{R}_+)$ and $F \in S^1(\mathbb{R}_+)$ be such that $F \ge 0$ and*

$$\forall t \in \mathbb{R}_+, \forall s \in [0,t], \quad \int_s^t \alpha(r)dr \ge \alpha_0(t-s) - A_0 \tag{2.7}$$

for some $\alpha_0 > 0, A_0 \ge 0$. Let $\Phi \in L^1_{loc}(\mathbb{R}_+)$ satisfy the differential inequality

$$\Phi'(t) + \alpha(t)\Phi(t) \le F(t), \quad \textit{a.e. on } \mathbb{R}_+ \tag{2.8}$$

Then Φ is bounded and we have

$$\forall t \in \mathbb{R}_+, \quad \Phi(t) \le e^{A_0}\{e^{-\alpha_0 t}\Phi(0) + (1+\frac{1}{\alpha_0})\|F\|_{S^1}\} \tag{2.9}$$

Proof. Introducing $A(t) = \int_0^t \alpha(r)dr$ we have almost-everywhere in t

$$\frac{d}{dt}(e^{A(t)}\Phi(t)) \le e^{A(t)}F(t)$$

which by integration provides

$$e^{A(t)}\Phi(t) \le \Phi(0) + \int_0^t e^{A(s)}F(s)ds$$

hence

$$\Phi(t) \le e^{-A(t)}\Phi(0) + \int_0^t e^{A(s)-A(t)}F(s)ds$$

and by using (2.7), we find

$$\Phi(t) \leq e^{A_0}\{e^{-\alpha_0 t}\Phi(0) + \int_0^t e^{\alpha_0(s-t)}F(s)ds\}$$

The conclusion follows easily from the next simple lemma.

$\square$

Lemma 2.9. *Let*

$$I(t) := \int_0^t e^{\alpha_0(s-t)}F(s)ds$$

Then I is bounded and we have

$$\forall t \in \mathbb{R}_+, \quad I(t) \leq (1 + \frac{1-e^{-\alpha_0 t}}{\alpha_0})\|F\|_{S^1} \tag{2.10}$$

In addition this inequality is optimal for $t \in \mathbb{N}$.

Proof. To simplify the formulas we write α instead of α_0 and we set, for t fixed

$$g(t) := \int_s^t F(r)dr$$

Then

$$\forall t \in \mathbb{R}_+, \forall s \in [0,t], \quad g(s) \leq (t-s+1)\|F\|_{S^1} \tag{2.11}$$

and

$$e^{\alpha t}I(t) = -\int_0^t e^{\alpha s}g'(s)ds = -[e^{\alpha s}g(s)]_0^t + \int_0^t \alpha e^{\alpha s}g(s)ds = g(o) + \int_0^t \alpha e^{\alpha s}g(s)ds$$

and by (2.11) we obtain

$$e^{\alpha t}I(t) \leq \|F\|_{S^1}\Big\{t+1+\int_0^t \alpha(t-s+1)e^{\alpha s}ds\Big\}$$

Finally, let

$$J(t) := t+1+\int_0^t \alpha(t-s+1)e^{\alpha s}ds$$

We find

$$J(t) = t+1+e^{\alpha t}-1+\int_0^t \alpha(t-s)e^{\alpha s}ds = t+e^{\alpha t}+\int_0^t \alpha(t-s)e^{\alpha s}ds$$

But integrating by parts again we obtain

$$\int_0^t \alpha(t-s)e^{\alpha s}ds = [(t-s)e^{\alpha s}]_0^t - \int_0^t (-1)\times e^{\alpha s}ds = -t + \int_0^t e^{\alpha s}ds = -t + \frac{e^{\alpha t}-1}{\alpha}$$

Finally

$$e^{\alpha t}I(t) \le \|F\|_{S^1}\Big[e^{\alpha t} + \frac{e^{\alpha t}-1}{\alpha}\Big]$$

and the result follows. $\square$

The following more nonlinear Lemma will be useful for the proof of our main boundedness result (cf. Theorem 4.1).

Lemma 2.10. *Let $\Psi \ge 0$ be a function in $W^{1,1}_{loc}(\mathbb{R}^+)$ which satisfies*

$$\frac{d}{dt}\Psi \le -\mu(t)\Psi^{\frac{1}{2}} + F(t) \tag{2.12}$$

with $\mu \in S^1(\mathbb{R}^+)$ and

$$\forall t \ge 0, \qquad \int_t^{t+1} \mu(s)ds \ge \rho > 0$$

Then $\Psi(t) : \mathbb{R}^+ \to \mathbb{R}^+$ is bounded.

Proof. We set

$$F^* = \|F\|_{S^1(\mathbb{R}^+)}; \quad \mu^* = \|\mu\|_{S^1(\mathbb{R}^+)}$$

As a consequence of Gronwall's inequality, $\Psi(t)$ is bounded on $[0,1]$ in terms of $\Psi(0)$, μ^* *and* F^*. To show the boundedness of $\Psi(t)$ on $[0,+\infty[$, we select $t \ge 0$ and we distinguish two cases.

case 1: Assume that $\exists s \in [t, t+1]$ such that:

$$\Psi(s) \le \left(\frac{F^*}{\rho}\right)^2 = K_1.$$

Since

$$\frac{d}{dt}\Psi \le \mu^-(\tau)\Psi(\tau) + F(\tau) + \frac{1}{4}\mu^-(\tau).$$

By Gronwall's inequality, we obtain:

$$\Psi(t+1) \le K_1 \exp(\mu^*) + \exp(\mu^*)(F^* + \frac{1}{4}\mu^*) = K_2. \tag{2.13}$$

case 2: Assume that we have:

$$\inf_{s\in[t,t+1]} \Psi(s) \ge \left(\frac{F^*}{\rho}\right)^2. \tag{2.14}$$

Then, for all $s \in [t, t+1]$, we have:

$$\frac{1}{\Psi(s)^{\frac{1}{2}}}\frac{d\Psi(s)}{ds} \le -\mu(s) + \frac{F(s)}{\Psi(s)^{\frac{1}{2}}}$$

Consequently

$$\frac{d}{ds}(\Psi(s)^{\frac{1}{2}}) \le -\frac{\mu(s)}{2} + \frac{\rho}{F^*}\frac{F(s)}{2}.$$

By integrating over $[t, t+1]$ we obtain

$$\Psi(t+1) \le \Psi(t). \tag{2.15}$$

Therefore in both cases 1 and 2 we have:

$$\forall t \ge 0, \quad \Psi(t+1) \le \max\big(K_2,\, \Psi(t)\big).$$

Finally, we conclude that:

$$\forall t \ge 0, \quad \Psi(t) \le \max\big(K_2,\, \sup_{0\le\varepsilon\le 1} \Psi(\varepsilon)\big). \tag{2.16}$$

□

3 Existence and regularity of solutions.

3.1 Functional setting

Throughout this article we let H and V be two Hilbert spaces with norms respectively denoted by $\|.\|$ and $|.|$. We assume that V is densely and continuously embedded into H. Identifying H with its dual H', we obtain $V \hookrightarrow H = H' \hookrightarrow V'$. We denote inner products by $(.,.)$ and duality products by $\langle\cdot,\cdot\rangle$; the spaces in question will be specified by subscripts. The notation $\langle f, u\rangle$ without any subscript will be used sometimes to denote $\langle f, u\rangle_{V',V}$. The duality map: $V \to V'$ will be denoted by A. We recall that A is characterized by the property

$$\forall (u, v) \in V \times V, \quad \langle Au, v\rangle_{V',V} = (u, v)_V$$

3.2 Weak solutions in the purely dissipative case.

We consider the dissipative evolution equation:

$$\ddot{u} + Au + g(\dot{u}) = h(t) \tag{3.1}$$

where $g \in C(V, V')$ satisfies

$$\forall (v, w) \in V \times V, \quad \langle g(v) - g(w), v - w\rangle \ge 0 \tag{3.2}$$

We consider the (generally unbounded) operator $\mathcal{A}$ defined on the Hilbert space $\mathcal{H} = V \times H$ by

$$D(\mathcal{A}) = \{(u, v) \in V \times V,\ Au + g(v) \in H\}$$

and

$$\forall (u, v) \in D(\mathcal{A}), \quad \mathcal{A}(u, v) = (-v, Au + g(v))$$

Lemma 3.1. *The operator $\mathcal{A}$ is maximal monotone.*

Proof. Let $U = (u, v)$ and $\hat{U} = (\hat{u}, \hat{v})$ be two elements of $D(\mathcal{A})$. We have

$$(\mathcal{A}U - \mathcal{A}\hat{U}, U - \hat{U})_{\mathcal{H}} = -(u - \hat{u}, v - \hat{v})_V + (Au + g(v) - A\hat{u} - g(\hat{v}), v - \hat{v})_H$$

$$-(u - \hat{u}, v - \hat{v})_V + \langle Au + g(v) - A\hat{u} - g(\hat{v}), v - \hat{v}\rangle_{V',V}$$

since $Au + g(v) \in H$ and $A\hat{u} + g(\hat{v}) \in H$ while $v, \hat{v}$ are in V . This reduces to

$$(\mathcal{A}U - \mathcal{A}\hat{U}, U - \hat{U})_{\mathcal{H}} = \langle g(v) - g(\hat{v}), v - \hat{v}\rangle_{V',V} \geq 0$$

Hence $\mathcal{A}$ is monotone. To prove that $\mathcal{A}$ is maximal monotone we are left to show that for any $F = (\varphi, \psi) \in \mathcal{H}$ the following equation

$$u - v = \varphi, \quad Au + g(v) + v = \psi$$

has a solution $U = (u, v) \in D(\mathcal{A})$. This is equivalent to finding a solution $v \in V$ of

$$Av + g(v) + v = \psi - A\varphi \in V'$$

But now the operator $\mathcal{C} \in C(V, V')$ defined by

$$\forall v \in V, \quad \mathcal{C}v = Av + g(v) + v$$

is continuous and coercive: $V \to V'$. Therefore by Corollary 14 p. 126 from H. Brezis [8], $\mathcal{C}$ is surjective. Finally $\mathcal{A}$ is maximal monotone as claimed. $\square$

As a consequence of Proposition 2.1, for any $h \in L^1_{loc}(\mathbb{R}^+, H)$ and for each $(u_0, u_1) \in V \times H$ there is a unique weak solution

$$u \in C(\mathbb{R}^+, V) \cap C^1(\mathbb{R}^+, H)$$

of (3.1) such that $u(0) = u_0$ and $\dot{u}(0) = u_1$. This solution is can be recovered on each compact interval $[0, T]$ by approximating the initial data by elements of the domain, the forcing term h by C^1 functions and passing to the limit: the limit is independent of the approximating elements so chosen. The next result shows that in fact the approximation can even be made uniform on $\mathbb{R}^+$.

3.3 Regularity properties and density of strong solutions.

Lemma 3.2. *For any $h \in L^1_{loc}(\mathbb{R}^+, H)$ and for each $(u_0, u_1) \in V \times H$ and for each $\delta > 0$ there exists $(w_0, w_1) \in D(\mathcal{A})$ and $k \in C^1(\mathbb{R}^+, H)$ for which the solution $w \in W^{1,1}_{loc}(\mathbb{R}^+, V) \cap W^{2,1}_{loc}(\mathbb{R}^+, H)$ of*

$$\ddot{w} + Aw + g(\dot{w}) = k(t); \quad w(0) = w_0, \ \dot{w}(0) = w_1$$

satisfies

$$\forall t \geq 0, \quad \|u(t) - w(t)\| + |\dot{u}(t) - \dot{w}(t)| \leq \delta$$

Proof. It suffices to use the last result of Proposition 2.1 by observing that for any $h \in L^1_{loc}(\mathbb{R}^+, H)$ we can find $k \in C^1(\mathbb{R}^+, H)$ such that

$$\forall n \in \mathbb{N}, \quad \int_n^{n+1} |k(s) - h(s)| ds \leq \delta 2^{-n-1}$$

Choosing $(w_0, w_1) \in D(\mathcal{A})$ such that $\|w_0 - u_0\| + |w_1 - v_1| \leq \delta 2^{-1}$ the result follows immediately $\square$

4 A Boundedness result.

We consider the dissipative evolution equation (3.1). We say that $h \in S^1(\mathbb{R}^+, H)$ if

$$h \in L^1_{loc}(\mathbb{R}^+, H) \quad \textit{and} \quad \sup_{t \geq 0} \int_t^{t+1} |h(s)| ds = h^* < +\infty. \tag{4.1}$$

Then we can state the following result which generalizes Theorem IV.2.1.1 from [14] to the case of possibly non-local damping terms.

Theorem 4.1. *Assume that* (4.1) *is satisfied and* $g \in C(V, V')$ *satisfies the conditions*

$$\exists \alpha > 0, \ \exists C_1 \geq 0 \quad \forall v \in V, \quad \langle g(v), v \rangle \geq \alpha |v|^2 - C_1. \tag{4.2}$$

$$\exists \tau > 0, \ \exists C_2 \geq 0 \quad \forall v \in V, \quad \|g(v)\|_* \leq C_2 + \tau \langle g(v), v \rangle. \tag{4.3}$$

Assume, also, that the following condition is fulfilled:

$$2\tau h^* < 1 \tag{4.4}$$

Then any solution $u \in C(\mathbb{R}^+, V) \cap C^1(\mathbb{R}^+, H)$ *of* (3.1) *is bounded on* $\mathbb{R}^+$ *in the sense that* u *has bounded range in* V *and* $\dot{u}$ *has bounded range in* H.

Proof of Theorem: We start by an estimate in the case of a strong solutions, i.e. we assume $u \in W^{1,1}_{loc}(\mathbb{R}^+, V) \cap W^{2,1}_{loc}(\mathbb{R}^+, H)$. The general case will follow by density. Let

$$E(t) = \frac{1}{2}(|\dot{u}|^2 + \|u\|^2)$$

Under the regularity conditions $[u_0, v_0] \in V \times V$, $\gamma(0, v_0) \in H$ and $h \in W^{1,1}_{loc}(\mathbb{R}^+, H)$, $t \to E(t)$ is absolutely continuous and we have $\forall t \in \mathbb{R}^+$:

$$\frac{d}{dt} E(t) = (h, \dot{u}) - \langle g(\dot{u}), \dot{u} \rangle. \tag{4.5}$$

In addition $t \to (u(t), \dot{u}(t))$ is absolutely continuous and

$$\frac{d}{dt}(u(t), \dot{u}(t)) = |\dot{u}|^2 - \|u\|^2 - \langle g(\dot{u}), u\rangle + (h, u).$$

By using (4.3), we obtain:

$$\frac{d}{dt}(u(t), \dot{u}(t)) \leq |\dot{u}|^2 - \|u\|^2 + \|u\| \left\{P|h| + C_2 + \tau\langle g(\dot{u}), \dot{u}\rangle\right\}$$

with $P = \sup\{|u|,\ u \in V,\ \|u\| = 1\}$. Introducing $\Phi(t) = 2E(t), \quad \forall t \geq 0$, we find:

$$\begin{aligned} \frac{d}{dt}\left\{(1+\Phi)^{-\frac{1}{2}}(u, \dot{u})\right\} \leq \frac{|\dot{u}|^2 - \|u\|^2}{(1+\Phi)^{\frac{1}{2}}} + \tau\langle g(\dot{u}), \dot{u}\rangle \\ + P|h| + C_2 + \frac{P|\dot{u}|\|u\|}{(1+\Phi)^{\frac{3}{2}}}\left|\frac{dE}{dt}\right| \end{aligned} \tag{4.6}$$

We have:

$$\begin{aligned} \frac{|\dot{u}|^2 - \|u\|^2}{(1+\Phi)^{\frac{1}{2}}} &= \frac{2|\dot{u}|^2 + 1}{(1+\Phi)^{\frac{1}{2}}} - \frac{1 + |\dot{u}|^2 + \|u\|^2}{(1+\Phi)^{\frac{1}{2}}} \\ &\leq 1 + 2|\dot{u}| - (1+\Phi)^{\frac{1}{2}} \end{aligned} \tag{4.7}$$

Then from (4.5), we obtain by Cauchy- Schwarz:

$$\begin{aligned} \frac{|\dot{u}|\|u\|}{(1+\Phi)^{\frac{3}{2}}}\left|\frac{dE}{dt}\right| &\leq \frac{1}{2(1+\Phi)^{\frac{1}{2}}}\left|\frac{dE}{dt}\right| \\ &\leq |h| + \frac{1}{2(1+\Phi)^{\frac{1}{2}}}\langle g(\dot{u}), \dot{u}\rangle - (h, \dot{u}) \\ &= |h| - \frac{1}{2(1+\Phi)^{\frac{1}{2}}}\frac{1}{2}\frac{d}{dt}(1+\Phi) \\ &= |h| - \frac{1}{2}\frac{d}{dt}[(1+\Phi)^{\frac{1}{2}}] \end{aligned} \tag{4.8}$$

Therefore, from (4.7) and (4.8) we deduce:

$$\begin{aligned} \frac{d}{dt}\left\{(1+\Phi)^{-\frac{1}{2}}(u, \dot{u}) + \frac{P}{2}(1+\Phi)^{\frac{1}{2}}\right\} \leq -(1+\Phi)^{\frac{1}{2}} + 2|\dot{u}| + 2P|h| \\ + \tau < g(\dot{u}), \dot{u} > +1 + C_2. \end{aligned} \tag{4.9}$$

By using (4.5) , (4.9) and (4.2) we obtain:

$$\begin{aligned} \frac{d}{dt}\{\tau\Phi + (1+\Phi)^{-\frac{1}{2}}(u, \dot{u}) + \frac{P}{2}(1+\Phi)^{\frac{1}{2}}\} \leq -(1+\Phi)^{\frac{1}{2}} + 2(1+\tau|h|)|\dot{u}| \\ - \alpha\tau|\dot{u}|^2 + 2P|h| + 1 + C_2 + C_1\tau. \end{aligned} \tag{4.10}$$

For $K > 0$ large enough, we set:

$$\Psi = \tau\Phi + (1+\Phi)^{-\frac{1}{2}}(u,\dot{u}) + \frac{P}{2}(1+\Phi)^{\frac{1}{2}} + K.$$

Therefore, we obtain:

$$\begin{aligned}
\frac{d}{dt}\Psi &\leq -(1+\Phi)^{\frac{1}{2}} + 2\tau|h|\Phi^{\frac{1}{2}} + 2|\dot{u}| - \alpha\tau|\dot{u}|^2 + 2P|h| + 1 + C_2 + C_1\tau \\
&\leq -(1-2\tau|h|)(1+\Phi)^{\frac{1}{2}} + 2|\dot{u}| - \alpha\tau|\dot{u}|^2 + 2P|h| + 1 + C_2 + C_1\tau.
\end{aligned}$$

This differential inequality is verified when $2\tau h^* < 1$. In addition for $K > 0$ large enough, Ψ is positive on $\mathbb{R}^+$ and we have:

$$\begin{aligned}
\Psi &\leq \tau\Phi + (1+\Phi)^{-\frac{1}{2}}|u||\dot{u}| + \frac{P}{2}(1+\Phi)^{\frac{1}{2}} + K \\
&\leq \tau\Phi + P(1+\Phi)^{-\frac{1}{2}}\|u\||\dot{u}| + \frac{P}{2}(1+\Phi)^{\frac{1}{2}} + K \\
&\leq \tau\Phi + \frac{P}{2}(1+\Phi)^{-\frac{1}{2}}\Phi + \frac{P}{2}(1+\Phi)^{\frac{1}{2}} + K \\
&\leq \tau\Phi + P(1+\Phi)^{\frac{1}{2}} + K
\end{aligned}$$

We notice that for any $\eta > 0$

$$P(1+\Phi)^{\frac{1}{2}} \leq \eta\Phi + c(\eta). \tag{4.11}$$

Then

$$\Psi \quad \leq (\tau+\eta)\Phi + c(\eta) + K$$

where $\eta > 0$ can be taken artibitrarily small. Setting $c(\eta) + K =: Q$, we obtain

$$\Psi \leq (\tau+\eta)\Phi + Q \tag{4.12}$$

Also, we have:

$$\begin{aligned}
\Psi &\geq \tau\Phi - (1+\Phi)^{-\frac{1}{2}}|u||\dot{u}| + \frac{P}{2}(1+\Phi)^{\frac{1}{2}} + K \\
&\geq \tau\Phi - P(1+\Phi)^{-\frac{1}{2}}\|u\||\dot{u}| + \frac{P}{2}(1+\Phi)^{\frac{1}{2}} + K \\
&\geq \tau\Phi - \frac{P}{2}(1+\Phi)^{-\frac{1}{2}}\Phi + \frac{P}{2}(1+\Phi)^{\frac{1}{2}} + K \\
&\geq \tau\Phi - P(1+\Phi)^{-\frac{1}{2}}\Phi + \frac{P}{2}(1+\Phi)^{\frac{1}{2}} + K \\
&\geq \tau\Phi - P(1+\Phi)^{\frac{1}{2}} + K
\end{aligned}$$

By using (4.11) we obtain

$$\Psi \geq (\tau-\eta)\Phi + K - c(\eta)$$

Assuming that $K - c(\eta) \geq 0$ and $\eta < \tau$, we deduce

$$\Psi \geq (\tau - \eta)\Phi \tag{4.13}$$

Hence by (4.12) and (4.13), we obtain for $Q > 0$ large enough:

$$(\tau - \eta)\Phi \leq \Psi \leq (\tau + \eta)\Phi + Q. \tag{4.14}$$

Then:

$$\frac{d}{dt}\Psi \leq -\frac{1}{\tau+\eta}\Psi^{\frac{1}{2}} + \frac{2\tau|h|}{\tau-\eta}\Psi^{\frac{1}{2}} + 2|\dot{u}| - \alpha\tau|\dot{u}|^2 + 2P|h| + 1 + C_2 + C_1\tau.$$

Since, by Cauchy-Schwarz , $2|\dot{u}| - \alpha\tau|\dot{u}|^2$ is less than a constant, we obtain:

$$\frac{d}{dt}\Psi \leq -\frac{1}{\tau+\eta}\Psi^{\frac{1}{2}} + \frac{2\tau|h|}{\tau-\eta}\Psi^{\frac{1}{2}} + 2P|h| + 1 + C_2 + C_1\tau + C_3. \tag{4.15}$$

We set

$$\sigma = \frac{1}{\tau+\eta}$$

$$m(t) = \frac{2\tau|h(t)|}{\tau-\eta}$$

and

$$F(t) = 2P|h(t)| + 1 + C_2 + C_1\tau + C_3.$$

We have

$$\frac{d}{dt}\Psi \leq -\sigma\Psi^{\frac{1}{2}} + m(t)\Psi^{\frac{1}{2}} + F(t). \tag{4.16}$$

As a consequence of (4.4), for η small enough, we have $\frac{2\tau}{\tau-\eta}h^* < \frac{1}{\tau+\eta} = \sigma$ hence

$$m^* = \sup_{t\geq 0}\int_t^{t+1} m(s)ds < \sigma \tag{4.17}$$

and

$$\begin{aligned} F^* &= \sup_{t\geq 0}\int_t^{t+1} F(s)ds \\ &= \sup_{t\geq 0}\int_t^{t+1} (2P|h(s)| + 1 + C_2 + C_1\tau + C_3)ds < +\infty \end{aligned} \tag{4.18}$$

For the rest of proof, we apply lemma 2.10. We obtain that $\Psi(t)$ is bounded on $\mathbb{R}^+$. Hence by (4.14), $\Phi(t)$ and $E(t)$ are bounded for $t \geq 0$. The general case of weak solutions follows easily by density by using Lemma 3.2.

□

5 Precompactness of bounded orbits.

The main result of this section is

Theorem 5.1. *Let $u \in C(\mathbb{R}^+, V) \cap C^1(\mathbb{R}^+, H)$ be a solution of*

$$\ddot{u}(t) + g(\dot{u}(t)) + Au(t) = h(t), \quad t \in \mathbb{R}_+,$$

with $h \in S^1(\mathbb{R}^+, H)$ such that u has bounded range in V and $\dot{u}$ has bounded range in H. Assume the following

i) h is S^1-uniformly continuous with values in H in the sense that

$$\limsup_{\epsilon \to 0} \sup_{t \geq 0} \int_t^{t+1} |h(s+\epsilon) - h(s)| ds = 0. \tag{5.1}$$

ii) $g \in C(V, V')$ satisfies $g(0) = 0$ and the following conditions

$$\forall (v,w) \in V \times V, \ \langle g(v) - g(w), v - w \rangle_{V',V} \geq 0 \tag{5.2}$$

$$\forall \delta > 0, \exists C(\delta) > 0, \quad \forall (v,w) \in V \times V, \ |v - w|^2 \leq \delta + C(\delta) \langle g(v) - g(w), v - w \rangle_{V',V} \tag{5.3}$$

$$\exists C > 0, \quad \forall v \in V, \ \|g(v)\|_{Z'} \leq C(1 + \langle g(v), v \rangle_{V',V}) \tag{5.4}$$

where Z' is the dual of a reflexive Banach space such that

$$Z \subset H \quad \text{and the imbedding: } Z \to H \ \text{ is continuous} \tag{5.5}$$

$$V \subset Z \quad \text{and the imbedding: } V \to Z \ \text{ is compact} \tag{5.6}$$

$$V \text{ is everywhere dense in } Z. \tag{5.7}$$

Then u has precompact range in V and $\dot{u}$ has precompact range in H.

Proof. Let us denote the norm in Z by $||| \ |||$ and the norm in Z' by $||| \ |||_*$.
We observe that $Z' \subset V'$ with continuous imbedding. From (5.5) and (5.6) it follows that $\forall \alpha > 0, \quad \exists c(\alpha) \geq 0$ such that

$$\forall u \in V, \quad |||u||| \leq \alpha \|u\| + c(\alpha)|u| \tag{5.8}$$

We prove first the result under the additional assumption $u \in W^{1,1}_{loc}(\mathbb{R}^+, V) \cap W^{2,1}_{loc}(\mathbb{R}^+, H)$.
First $w_\epsilon(t) = u(t+\epsilon) - u(t)$ satisfies the equation:

$$\ddot{w}_\epsilon + Lw_\epsilon + g(\dot{u}(t+\epsilon)) - g(\dot{u}(t)) = h(t+\epsilon) - h(t) \tag{5.9}$$

where $u(t)$ is the solution of equation (3.1).
We set

$$f_\epsilon(t) = h(t+\epsilon) - h(t) \tag{5.10}$$

and

$$\forall t \in \mathbb{R}_+, \forall w \in V, \quad g(\dot{u}(t)+w) - g(\dot{u}(t)) = \gamma(t,w). \tag{5.11}$$

Then (5.9) becomes

$$\ddot{w}_\epsilon + Lw_\epsilon + \gamma(t,\dot{w}_\epsilon) = f_\epsilon(t), \ \ \forall t \geq 0 \tag{5.12}$$

As a consequence of section 2, we know that $u(t)$ and $\dot{u}(t)$ are bounded on $\mathbb{R}^+$. We denote by

$$\begin{aligned} E_\epsilon(t) &= \frac{1}{2}\{|\dot{u}(t+\epsilon) - \dot{u}(t)|^2 + \|u(t+\varepsilon) - u(t)\|^2\} \\ &= \frac{1}{2}\{|\dot{w}_\epsilon(t)|^2 + \|w_\epsilon(t)\|^2\} \end{aligned}$$

the energy of the solution to (5.12).
Let us introduce, for some $\beta > 0$ to be chosen later

$$\Phi = \frac{1}{2}\{|\dot{w}_\epsilon|^2 + \|w_\epsilon\|^2\} + \beta(w_\epsilon, \dot{w}_\epsilon).$$

Then, we have:

$$\Phi' \leq -\langle \dot{w}_\epsilon, \gamma(t,\dot{w}_\epsilon)\rangle + \beta|\dot{w}_\epsilon|^2 - \beta\|w_\epsilon\|^2 - \beta\langle\gamma(t,\dot{w}_\epsilon), w_\epsilon\rangle + \langle f_\epsilon, \dot{w}_\epsilon + \beta w_\epsilon\rangle$$

we obtain

$$\Phi' \leq -\beta E_\epsilon - \langle \dot{w}_\epsilon, \gamma(t,\dot{w}_\epsilon)\rangle + \frac{3\beta}{2}|\dot{w}_\epsilon|^2 - \frac{\beta}{2}\|w_\epsilon\|^2 + \beta|||w_\epsilon||| \ \ |||\gamma(t,\dot{w}_\epsilon)|||_* + |f_\epsilon|\Big(|\dot{w}_\epsilon| + \beta|w_\epsilon|\Big)$$

If we impose $\beta \leq \sqrt{\lambda_1}$, then $\Phi \geq 0$ and from (5.8) we deduce

$$\begin{aligned} \Phi' \leq& -\beta E_\epsilon - \langle \dot{w}_\epsilon, \gamma(t,\dot{w}_\epsilon)\rangle + \frac{3\beta}{2}|\dot{w}_\epsilon|^2 - \frac{\beta}{2}\|w_\epsilon\|^2 + \beta\Big(\alpha\|w_\epsilon\| + c(\alpha)|w_\epsilon|\Big)|||\gamma(t,\dot{w}_\epsilon)|||_* \\ &+ |f_\epsilon|\Big(|\dot{w}_\epsilon| + \|w_\epsilon\|\Big) \end{aligned}$$

From (5.3) and $|\dot{w}_\epsilon| + \|w_\epsilon\| \leq 2\sqrt{E_\epsilon}$, we have:

$$\begin{aligned} \Phi' \leq& -\beta E_\epsilon - \langle \dot{w}_\epsilon, \gamma(t,\dot{w}_\epsilon)\rangle + \frac{3\beta}{2}\delta + \frac{3\beta}{2}K(\delta)\langle \dot{w}_\epsilon, \gamma(t,\dot{w}_\epsilon)\rangle - \frac{\beta}{2}\|w_\epsilon\|^2 \\ &+ \beta\Big(\alpha\|w_\epsilon\| + c(\alpha)|w_\epsilon|\Big)|||\gamma(t,\dot{w}_\epsilon)|||_* + 2\sqrt{E_\epsilon}|f_\epsilon| \\ \leq& -\beta E_\epsilon + (\frac{3\beta}{2}K(\delta) - 1)\langle \dot{w}_\epsilon, \gamma(t,\dot{w}_\epsilon)\rangle - \frac{\beta}{2}\|w_\epsilon\|^2 + \beta\Big(\alpha\|w_\epsilon\| + c(\alpha)|w_\epsilon|\Big)|||\gamma(t,\dot{w}_\epsilon)|||_* \\ &+ 2\sqrt{E_\epsilon}|f_\epsilon| + \frac{3\beta}{2}\delta \end{aligned}$$

From now on we fix $\delta > 0$ small enough and we choose β such that

$$\frac{3\beta}{2}K(\delta) - 1 \le 0$$

which is equivalent to

$$\beta \le \frac{2}{3K(\delta)}.$$

Then

$$\Phi' \le -\beta E_\epsilon + \beta\Big(\alpha\|w_\epsilon\| + c(\alpha)|w_\epsilon|\Big)|||\gamma(t,\dot{w}_\epsilon)|||_* + 2\sqrt{E_\epsilon}|f_\epsilon| + \frac{3\beta}{2}\delta.$$

We have by Cauchy-Schwarz:

$$\begin{aligned}\Phi &= \frac{1}{2}\{|\dot{w}_\epsilon|^2 + \|w_\epsilon\|^2\} + \beta(w_\epsilon, \dot{w}_\epsilon)\\ &\le E_\epsilon + \frac{\beta^2}{2}|w_\epsilon|^2 + \frac{1}{2}|\dot{w}_\epsilon|^2.\end{aligned}$$

Then, for $\beta \le \sqrt{\lambda_1}$, we have

$$\Phi \le 2E_\epsilon$$

Also, if we assume the stronger condition $\beta \le \frac{1}{2}\sqrt{\lambda_1}$, then

$$\Phi \ge E_\epsilon - \beta^2|w_\epsilon|^2 - \frac{1}{4}|\dot{w}_\epsilon|^2 \ge \frac{1}{2}E_\epsilon$$

Therefore , we have

$$\frac{1}{2}E_\epsilon(t) \le \Phi(t) \le 2E_\epsilon(t), \quad \forall t \in \mathbb{R} \tag{5.13}$$

From (5.13), we obtain

$$\Phi' \quad \le -\beta E_\epsilon + 2\sqrt{E_\epsilon}|f_\epsilon| + \beta\Big(\alpha\|w_\epsilon\| + c(\alpha)|w_\epsilon|\Big)|||\gamma(t,\dot{w}_\epsilon)|||_* + \tfrac{3\beta\delta}{2} \tag{5.14}$$

Then, we set

$$F(t) = \beta\{\Big(\alpha\|w_\epsilon\| + c(\alpha)|w_\epsilon|\Big)|||\gamma(t,\dot{w}_\epsilon)|||_* + \frac{3\delta}{2}\}$$

Then

$$\Phi' \quad \le -\tfrac{\beta}{2}\Phi + 2\sqrt{E_\epsilon}|f_\epsilon| + F(t)$$

By using (5.13), we find:

$$\Phi' \quad \leq -\tfrac{\beta}{2}\Phi + 2\sqrt{2}\sqrt{\Phi}|f_\epsilon| + F(t)$$

By using the inequality $2\sqrt{\Phi} \leq 1 + \Phi$ in the second term of the RHS and setting

$$m(t) = \sqrt{2}|f_\epsilon(t)|$$

we find

$$\Phi' \quad \leq -\tfrac{\beta}{2}\Phi + m(t)\Phi + F(t) + m(t)$$

Introducing

$$H(t) := F(t) + m(t)$$

we have:

$$\Phi'(t) \quad \leq -(\tfrac{\beta}{2} - m(t))\Phi(t) + H(t), \quad \forall t \in \mathbb{R}^+ \tag{5.15}$$

On the other hand:

$$\int_t^{t+1} m(s)ds = \sqrt{2}\int_t^{t+1} f_\epsilon(s)ds = \sqrt{2}\int_t^{t+1} |h(s+\epsilon) - h(s)|ds.$$

with

$$\limsup_{\epsilon \to 0} \sup_{t \geq 0} \int_t^{t+1} |h(s+\epsilon) - h(s)|ds = 0.$$

In addition

$$\begin{aligned}
\int_t^{t+1} F(s)ds &= \beta \int_t^{t+1} \{\Big(\alpha\|w_\epsilon(s)\| + c(\alpha)|w_\epsilon(s)|\Big)|||\gamma(s, \dot{w}_\epsilon(s))|||_* + \frac{3\delta}{2}\}ds \\
&\leq \beta(2\alpha \sup_{t \geq 0}\|u(t)\| + c(\alpha) \sup_{t \leq s \leq t+1} |w_\epsilon(s)|) \int_t^{t+1} |||\gamma(t, \dot{w}_\epsilon(s))|||_* ds + \frac{3\beta\delta}{2}
\end{aligned}$$

Now

$$\gamma(s, \dot{w}_\epsilon(s)) = g(\dot{u}(s+\epsilon)) - g(\dot{u}(s))$$

From boundedness of the energy $E(t)$ for $t \geq 0$, $\exists C > 0$ such that

$$\int_t^{t+1} \langle g(\dot{u}(s)), \dot{u}(s)\rangle \, dx\, ds \leq C, \;\; \forall t \geq 0.$$

As a consequence of (5.4) we have

$$g(\dot{u}(s)) \in L^1(t, t+1, Z')$$

and

$$\int_t^{t+1} |||g(\dot{u}(s))|||_* ds \leq N(C+1) \tag{5.16}$$

Therefore

$$\int_t^{t+1} F(s)ds \leq 2N(C+1)\beta(2\alpha \sup_{t\geq 0} \|u(t)\| + c(\alpha) \sup_{t\leq s\leq t+1} |w_\epsilon(s)|) + \frac{3\beta\delta}{2}$$

We have

$$\sup_{t\leq s\leq t+1} |w_\epsilon(s)| \leq \epsilon \sup_{s\geq 0} |\dot{u}(s)|.$$

By choosing α small enough, we obtain

$$\int_t^{t+1} F(s)ds \leq \beta\delta + P\epsilon \sup_{s\geq 0} |\dot{u}(s)| + \frac{3\beta\delta}{2}.$$

Then for $\epsilon \leq \epsilon_2$ small enough, we obtain:

$$\sup_{t\geq 0} \int_t^{t+1} F(s)ds \leq 3\beta\delta. \tag{5.17}$$

Then by (5.17), we obtain:

$$\sup_{t\geq 0} \int_t^{t+1} H(s)ds \leq 3\beta\delta + \sqrt{2} \int_t^{t+1} |h(s+\epsilon) - h(s)|ds$$

Now we may fix

$$\beta = \beta_0 \leq \min\{\frac{1}{2}\sqrt{\lambda_1}, \frac{2}{3K(\delta)}\}$$

Then by imposing that ε is small enough to ensure

$$\sqrt{2} \sup_{t\geq 0} \int_t^{t+1} |h(s+\epsilon) - h(s)|ds \leq \min\{\beta\delta, \frac{\beta}{4}\} \tag{5.18}$$

we obtain

$$\sup_{t\geq 0} \int_t^{t+1} H(s)ds \leq 4\beta\delta \tag{5.19}$$

and (5.15) becomes

$$\Phi'(t) \quad \leq -\alpha(t)\Phi(t) + H(t), \quad \forall t \in \mathbb{R}^+ \tag{5.20}$$

where

$$\int_t^{t+1} \alpha(s)ds \geq \frac{\beta}{4}$$

Lemma 2.9 then provides

$$\forall t \geq 0, \quad \Phi(t) \leq C[\Phi(0) + (1 + \frac{4}{\beta})4\beta\delta]$$

where C is bounded in terms of the S^1 norm of h. The uniform continuity follows easily and since this property is robust with respect to uniform convergence in the energy norm on $\mathbb{R}^+$, we obtain it for general weak solutions by using Lemma 3.2 . The compactness result now follows from the same argument as in [14], p.167-168: the main idea is that the average $\frac{1}{\varepsilon}\int_t^{t+\varepsilon} \dot{u}(s)ds$ of $\dot{u}$ on a small time interval remains bounded (by a large constant) in V while the equation shows that the average $\frac{1}{\varepsilon}\int_t^{t+\varepsilon} u(s)ds$ of u remains bounded (by a large constant) in $A^{-1}(Z')$ which imbeds compactly in V. The uniform continuity of the vector $(u(t), \dot{u}(t))$ shows that this vector is arbitrary close to its average on a small time interval. Finally precompactness follows from the total boundedness criterion.

□

Corollary 5.2. *Assume that the conditions of Theorem 4.1 are all satisfied except* (5.3) *which is replaced by*

$$\exists p \geq 2, \exists \eta > 0, \quad \forall (v, w) \in V \times V, \ \langle g(v) - g(w), v - w \rangle_{V',V} \geq \eta |v - w|^p \tag{5.21}$$

Then the conclusion of Theorem 4.1 holds true.

Proof. Given $\delta > 0$, we have

$$\forall (v, w) \in V \times V, \ |v - w|^2 \leq \delta + C_1(\delta)|v - w|^p \leq \delta + \frac{C_1(\delta)}{\eta}\langle g(v) - g(w), v - w \rangle_{V',V}$$

In particular (5.3) is fulfilled. □

In the applications to non-local dissipations we shall use Lemma 2.3.

6 The purely dissipative almost periodic case.

Compactness of trajectories is a basic tool to prove the existence of almost periodic (weak) solutions to the equation

$$U'(t) + \mathcal{A}U(t) = F(t)$$

Indeed if $\mathcal{A}$ is maximal monotone on $\mathcal{H}$ and $F : \mathbb{R} \to \mathcal{H}$ is almost periodic, it follows from [12] or ISHII that the existence of a precompact trajectory is equivalent to the existence of an almost periodic solution. This property has been used in [14] to prove the existence of almost-periodic solutions of

$$\begin{cases} u_{tt} + g(u_t) - \Delta u = h(t, x), \quad \text{in } \mathbb{R}_+ \times \Omega, \\ u = 0 \quad \text{on } \mathbb{R}_+ \times \partial\Omega. \end{cases} \tag{6.1}$$

where Ω is a bounded domain, g is the Nemytskii operator generated by an increasing function $\gamma \in C(\mathbb{R})$ such that γ^{-1} is uniformly continuous and satisfying some dimension dependant growth conditions and h is S^1-almost periodic :$\mathbb{R} \to L^2(\Omega)$.

6.1 A general result

We are now in a position to state and prove a more general result valid also for non-local dissipation terms. More precisely we have

Theorem 6.1. *Assume that g satisfies the hypotheses of Theorem 5.1 and* (4.3) *with τ arbitrary. Then for any h which is S^1-almost periodic: $\mathbb{R} \to H$ the equation*

$$\ddot{u}(t) + g(\dot{u}(t)) + Au(t) = h(t), \quad t \in \mathbb{R}_+, \tag{6.2}$$

has at least one solution ω such that the vector $(\omega, \dot{\omega})$ is almost periodic $\mathbb{R} \to V \times H$. In addition for any other solution u we have for some constant vector $a \in V$

$$\lim_{t\to\infty} (\|u(t) - \omega(t) - a\| + |\dot{u}(t) - \dot{\omega}(t)|) = 0$$

In addition if $g \in C(H, V')$ the almost periodic solution is unique and the previous convergence result is satisfied with $a = 0$ for any solution u.

Proof. The existence follows from [14], Theorem IV.3.3.3 p.173. The uniqueness result up to a constant vector will be a consequence of the second part of the theorem since an almost-periodic vector whose norm tends to 0 is identically 0 (c.f. e.g. [1, 15]). For the last result, let us first select a common sequence a_n tending to $+\infty$ for which $h(. + a_n)$ converges to h in $S^1(\mathbb{R}, H)$ and $\omega(. + a_n)$ converges to ω in $S^1(\mathbb{R}, V)$ (The Vector $(h, \omega, \dot{\omega})$ being S^1-almost periodic with values in$H \times V \times H$), hence $\omega(. + a_n)$ converges to ω also in $C_b(\mathbb{R}, V)$ since ω is almost periodic in the usual sense. if u is any solution, precompactness of the range of u implies that a subsequence of $u(. + a_n)$ converges uniformly on all compact subintervals with values in V to some limit z, while the same subsequence of $\dot{u}(. + a_n)$ converges to $\dot{z}$ uniformly on all compact subintervals with values in H. Since the energy norm of the difference $u - \omega$ is non-increasing, it converges to some limit $l \geq 0$. The energy norm of the difference $z - \omega$ is equal to l for all t. We finally show that $z - \omega$ is constant (and equal to 0 if $g \in C(H, V')$) by using the following Lemma

Lemma 6.2. *Let $J = [a, b]$ be any compact interval of $\mathbb{R}$ with $a < b$ and let u, v be two weak solutions of*

$$\ddot{u}(t) + g(\dot{u}(t)) + Au(t) = h(t), \quad t \in J$$

such as $\|u(t) - v(t)\|^2 + |\dot{u}(t) - \dot{v}(t)|^2$ is constant on J. Then $\dot{u} = \dot{v}$. In addition if $g \in C(H, V')$ then $u = v$.

Proof. The result is almost trivial if u and v are strong solutions. To prove it in the general case we approximate h and $(u(a), \dot{u}(a)), (v(a), \dot{v}(a))$ by $h_n \in C^1(J, H)$ and $(u_n^0, u_n^1), (v_n^0, v_n^1) \in D(\mathcal{A})^2$. Let $\eta > 0$ be an arbitrary small fixed number. We choose n in such a way that

$$\sup\{\|u_n^0 - u(a)\|^2 + |u_n^1 - \dot{u}(a)|^2, \|v_n^0 - v(a)\|^2 + |v_n^1 - \dot{v}(a)|^2\} \leq \frac{\eta^2}{16}$$

and

$$\|h_n - h\|_{L^1(J)} \leq \frac{\eta}{4}$$

which implies

$$\sup_J\{[\|u_n(t) - u(t)\|^2 + |\dot{u_n}(t) - \dot{u}(t)|^2]^{1/2} \leq \frac{\eta}{2}$$

where u_n is the strong solution of

$$\ddot{u_n}(t) + g(\dot{u_n}(t)) + Au_n(t) = h_n(t), \quad t \in J; \quad u_n(a) = u_n^0, \ \dot{u_n}(a) = u_n^1. \tag{6.3}$$

Defining similarly the solution v_n of

$$\ddot{v_n}(t) + g(\dot{v_n}(t)) + Av_n(t) = h_n(t), \quad t \in J; \quad v_n(a) = v_n^0, \ \dot{v_n}(a) = v_n^1 \tag{6.4}$$

we find by the triangular inequality

$$[\|u_n(a) - v_n(a)\|^2 + |\dot{u_n}(a) - \dot{v_n}(a)|^2]^{1/2} \leq [\|u(a) - v(a)\|^2 + |\dot{u}(a) - \dot{v}(a)|^2]^{1/2} + \frac{\eta}{2}$$

$$= [\|u(b) - v(b)\|^2 + |\dot{u}(b) - \dot{v}(b)|^2]^{1/2} + \frac{\eta}{2} \leq [\|u_n(b) - v_n(b)\|^2 + |\dot{u_n}(b) - \dot{v_n}(b)|^2]^{1/2} + \eta.$$

Hence by squaring and using boundedness of the sequence $\|u_n(b) - v_n(b)\|^2 + |\dot{u_n}(b) - \dot{v_n}(b)|^2$ we obtain

$$\|u_n(a) - v_n(a)\|^2 + |\dot{u_n}(a) - \dot{v_n}(a)|^2 - [\|u_n(b) - v_n(b)\|^2 + |\dot{u_n}(b) - \dot{v_n}(b)|^2] \leq \eta^2 + C\eta$$

which can be rewritten, assuming $\eta < 1$, as

$$\int_J \langle g(\dot{u_n}(t)) - g(\dot{v_n}(t)), \dot{u_n}(t) - \dot{v_n}(t)\rangle dt \leq C'\eta.$$

By using the hypothesis (5.3) on g we deduce

$$\int_J |\dot{u_n}(t) - \dot{v_n}(t)|^2 dt \leq |J|\delta + C(\delta) \int_J \langle g(\dot{u_n}(t)) - g(\dot{v_n}(t)), \dot{u_n}(t)) - \dot{v_n}(t)\rangle dt \leq |J|\delta + C'C(\delta)\eta$$

so that for η small enough we find

$$\int_J |\dot{u_n}(t) - \dot{v_n}(t)|^2 dt \leq 2|J|\delta$$

As a first consequence $\dot{u} = \dot{v}$ on J. In addition if $g \in C(H, V')$, since $\dot{u_n}$ converges uniformly to $\dot{u}$ on J and $\dot{v_n}$ converges uniformly to $\dot{v}$ on J with values in H, by considering the equation

$$\ddot{u_n}(t) - \ddot{v_n}(t) + g(\dot{u_n}(t)) - g(\dot{v_n}(t)) + A(u_n(t) - v_n(t)) = 0$$

After integration on J we find that in the sense of V'

$$\int_J A(u_n(t) - v_n(t))dt \to 0$$

Hence if $u - v \equiv a$ we end up with $|J|Aa = 0$ and finally $a = 0$

□

The end of proof of Theorem 6.1 follows very easily by considering any interval J as in the Lemma applied with u replaced by z and v replaced by ω. □

Remark 6.3. 1) It does not seem easy to construct a counterexample in which $a \neq 0$.

2) In order to have uniqueness of ω it is sufficient to assume a much weaker property, it suffices for instance that g be continuous from H to X weak where X is a reflexive Banach space such that $V \subset X$ with continuous imbedding.

3) In the next subsection, we derive a better result for special kinds of damping terms.

6.2 A more precise result for a special class of damping operators.

In this section we consider a reflexive Banach space Z satisfying the conditions (5.5), (5.6) and (5.7).

Definition 6.4. Given $\alpha > 0$, we say that $g \in C(Z, Z')$ is (Z, α)-admissible if $g(0) = 0$ and for some positive constants c, C we have

$$\forall (v, w) \in V \times V, \quad \langle g(v) - g(w), v - w \rangle_{V',V} \geq c\|v - w\|_Z{}^{\alpha+2} \tag{6.5}$$

$$\forall (v, w) \in V \times V, \quad \|g(v) - g(w)\|_{Z'} \leq C(\|v\|_Z^\alpha + \|w\|_Z^\alpha)\|v - w\|_Z \tag{6.6}$$

The class of (Z, α)-admissible functions will be denoted by $G(Z, \alpha)$.

The following Lemma shows that the functions of $G(Z, \alpha)$ satifies all the properties used in our main boundedness and compactness results.

Lemma 6.5. *Let $g \in G(Z, \alpha)$. Then g satisfies* (5.21) *with $p = \alpha + 2$ hence* (5.2), (5.3) *and* (4.2), (4.3) *for any $\tau > 0$ and* (5.4).

Proof. It is clear that (6.5) implies (5.21) with $p = \alpha + 2$ since the norm in Z dominates the norm in H. (5.2) and (5.3) are immediate consequences. (4.2) follows by taking $w = 0$. Finally (4.3) for any $\tau > 0$ and (5.4) follow easily from the combination of (6.5) and (6.6) applied with $w = 0$.

□

The next proposition clarifies the regularity of weak solutions when g is (Z, α)-admissible.

Proposition 6.6. *Let $g \in G(Z, \alpha)$. Then for any $J = [a, b]$ compact interval of $\mathbb{R}$ with $a < b$, any $h \in L^1(J, H)$ and u any weak solution of*

$$\ddot{u}(t) + g(\dot{u}(t)) + Au(t) = h(t), \quad t \in J \tag{6.7}$$

we have

$$\dot{u} \in L^{\alpha+2}(J, Z); \quad g(\dot{u}) \in L^{\frac{\alpha+2}{\alpha+1}}(J, Z')$$

$$\ddot{u}(t) + g(\dot{u}(t)) + Au(t) = h(t) \quad in \;\; L^1(J, V')$$

Proof. The result is obvious for a strong solution u since then $\dot{u} \in C(J, V)$ and therefore $g(\dot{u}) \in C(J, V')$. Now let u be a weak solution and u_n be a sequence converging to u in $C(J, V) \cap C^1(J, H)$ with u_n a strong solution of

$$\ddot{u_n}(t) + g(\dot{u_n}(t)) + Au_n(t) = h_n(t)$$

and

$$\lim_{n \to \infty} \|h_n - h\|_{L^1(J,H)} = 0$$

It is an immediate consequence of (6.5) that $\dot{u}_n$ is a Cauchy sequence in $L^{\alpha+2}(J, Z)$, then (6.6) shows that $g(\dot{u}_n)$ is Cauchy in $L^{\frac{\alpha+2}{\alpha+1}}(J, Z')$. The result follows easily.

□

Corollary 6.7. *In this case the almost periodic solution is unique*

Proof. Indeed if ω_1, ω_2 are two such solutions, then $\dot{\omega_1} = \dot{\omega_2}$ and $\ddot{\omega_1} = \ddot{\omega_2}$ in the sense of distributions from $\mathrm{Int}(J)$ with values in H, then also in $L^1(J, V')$ on any bounded interval J. Then the equation gives $A\omega_1 = A\omega_2$ in the sense of $L^1(J, V')$ and the conclusion follows easily □

Finally the following result improves our main asymptotic theorem by giving a rate of convergence:

Theorem 6.8. *Assume that g satisfies the hypotheses of Proposition 6.6. Then for any h which is S^1-almost periodic: $\mathbb{R} \to H$ the equation*

$$\ddot{u}(t) + g(\dot{u}(t)) + Au(t) = h(t), \quad t \in \mathbb{R}_+,$$

has a unique solution ω such that the vector $(\omega, \dot{\omega})$ is almost periodic $\mathbb{R} \to V \times H$. In addition for any other solution u we have for some $M \geq 0$

$$\forall t \geq 0, \quad \|u(t) - \omega(t)\| + |\dot{u}(t) - \dot{\omega}(t)| \leq M(1+t)^{-\frac{1}{\alpha}} \tag{6.8}$$

Proof. We extend the method of proof of [19], theorem 3.1 p. 200 (cf. also [15], theorem 7.5.1 p. 98) to the case of a non-local damping satisfying (6.5)- (6.6). Since the proof is quite similar we just sketch out the main steps for completeness. We start from 2 strong solutions u and v of the same equation

$$\ddot{u}(t) + g(\dot{u}(t)) + Au(t) = h(t), \quad t \in J$$

and we try to derive the estimate

$$\forall t \geq 0, \quad \|u(t) - v(t)\| + |\dot{u}(t) - \dot{v}(t)| \leq M(1+t)^{-\frac{1}{\alpha}} \tag{6.9}$$

with M bounded in terms of the initial data and the S^1 norm of h . To this end we introduce $z := u - v$ and the function

$$E(t) := \frac{1}{2}(|\dot{z}(t)|^2 + \|w(t)\|^2)$$

In particular, E is non-increasing and therefore bounded. Then we define

$$\Psi(t) := E(t)^{\frac{\alpha}{2}}(z(t), \dot{z}(t))$$

First we have

$$E'(t) = -\langle g(\dot{u}(t)) - g(\dot{v}(t)), \dot{z}(t)\rangle_{V',V} \leq -c\|\dot{z}(t)\|_Z{}^{\alpha+2}$$

Then we find

$$\Psi'(t) = \frac{\alpha}{2}E(t)^{\frac{\alpha}{2}-1}E'(t)(z(t), z'(t)) + E(t)^{\frac{\alpha}{2}}\langle -Az(t) - g(\dot{u}(t)) + g(\dot{v}(t)), z(t)\rangle_{V',V}$$

$$\leq -C_1E'(t) - E(t)^{\frac{\alpha}{2}}\|z(t)\|^2 + CE(t)^{\frac{\alpha}{2}}(\|\dot{u}(t)\|_Z^{\alpha} + \|\dot{v}(t)\|_Z^{\alpha})\|\dot{z}(t)\|_Z Vert z(t)\|$$

It follows from the properties of g that $r(t) := \|\dot{u}(t)\|_Z^{\alpha+2} + \|\dot{v}(t)\|_Z^{\alpha+2} \in S^1(\mathbb{R}_+)$ Introducing $k(t) := \|\dot{u}(t)\|_Z^{\alpha} + \|\dot{v}(t)\|_Z^{\alpha}$ a rather straightforward calculation yields for any $\delta > 0$

$$\Psi'(t) \leq -C_1E'(t) - E(t)^{\frac{\alpha}{2}}\|z(t)\|^2 + \delta kE(t)^{\frac{\alpha}{2}}\|z(t)\|^2 + \delta k^{\frac{\alpha+2}{\alpha}}E(t)^{\frac{\alpha+2}{2}} + C_2\delta^{-(\alpha+1)}\|\dot{z}(t)\|_Z^{\alpha+2}$$

Then setting

$$F_\varepsilon := (1 + C_1\varepsilon)E + \varepsilon\Psi$$

we have for all ε small enough

$$\frac{1}{2}E \leq F_\varepsilon \leq 2E$$

and then we find

$$F'_\varepsilon \leq -c\|\dot{z}(t)\|_Z{}^{\alpha+2} - \varepsilon E(t)^{\frac{\alpha}{2}}\|z(t)\|^2 + \varepsilon\delta k^{\frac{\alpha+2}{\alpha}}E(t)^{\frac{\alpha+2}{2}} + C_2\varepsilon\delta^{-(\alpha+1)}\|\dot{z}(t)\|_Z^{\alpha+2} + \varepsilon\delta k E(t)^{\frac{\alpha}{2}}\|z(t)\|^2$$

By choosing $\delta = (\frac{2C_2\varepsilon}{c})^{\frac{1}{\alpha+1}}$ we derive

$$F'_\varepsilon \leq -\frac{c}{2}\|\dot{z}(t)\|_Z{}^{\alpha+2} - \varepsilon E(t)^{\frac{\alpha}{2}}\|z(t)\|^2 + \varepsilon\delta k^{\frac{\alpha+2}{\alpha}}E(t)^{\frac{\alpha+2}{2}} + \varepsilon\delta k E(t)^{\frac{\alpha}{2}}\|z(t)\|^2$$

which, for ε sufficiently small, gives

$$F'_\varepsilon \leq -\varepsilon(\|\dot{z}(t)\|_Z{}^{\alpha+2} + E(t)^{\frac{\alpha}{2}}\|z(t)\|^2) + K\varepsilon^{1+\frac{1}{\alpha+1}}(1 + k^{\frac{\alpha+2}{\alpha}})E(t)^{\frac{\alpha+2}{2}}$$

This inequality reduces to

$$F'_\varepsilon \leq -\eta\varepsilon F_\varepsilon^{\frac{\alpha+2}{2}} + K'\varepsilon^{1+\frac{1}{\alpha+1}}(1 + k^{\frac{\alpha+2}{\alpha}})F_\varepsilon^{\frac{\alpha+2}{2}}$$

Because $h_1(t) = 1 + k^{\frac{\alpha+2}{\alpha}}$ is in $S^1(\mathbb{R}_+)$ we conclude by a direct application of Lemma 1.7 from [19].

□

7 Semilinear perturbations with rapidly decaying source terms.

7.1 An abstract convergence theorem

In this section we consider the equation

$$\ddot{u}(t) + g(\dot{u}(t)) + \mathcal{M}(u(t)) = h(t), \quad t \in \mathbb{R}_+, \tag{7.1}$$

where $\mathcal{M} = \nabla E$ is the gradient operator of a C^2 functional E on V and $g \in C(V, V')$ is a nonlinear damping operator such that there exists $\alpha \in (0,1)$, $\rho_1 > 0$ and $\rho_2 > 0$ for which

$$\forall v \in V,\ \langle g(v), v\rangle_{V',V} \geq \rho_1\|v\|_H^{\alpha+2} \tag{7.2}$$

$$\forall v \in V,\ \|g(v)\|_{V'} \leq \rho_2\langle g(v), v\rangle_{V',V}^{\frac{\alpha+1}{\alpha+2}}. \tag{7.3}$$

We assume that E satisfies a uniform Łojasiewicz gradient inequality near the set $\mathcal{E} = \mathcal{M}^{-1}\{0\}$ of equilibria with exponent $\theta \in \left(\frac{\alpha}{\alpha+1}, \frac{1}{2}\right]$ which means that for some $\rho > 0$ we have for some constant C

$$\forall u \in V, \quad \mathrm{dist}_V(u, \mathcal{E}) \le \rho \Longrightarrow \forall a \in \mathcal{E}, \quad |E(u) - E(a)|^{1-\theta} \le C\|\mathcal{M}u\|_{V'} \tag{7.4}$$

We also need an additional technical assumption as follows . Since $E \in C^2(V)$ we have for each $u \in V$

$$E''(u) = \mathcal{M}'(u) \in \mathcal{L}(V, V')$$

and therefore

$$A^{-1}\mathcal{M}'(u) \in \mathcal{L}(V, V)$$

we require the slightly different condition

$$\forall u \in V, \ A^{-1}\mathcal{M}'(u) \in \mathcal{L}(H, H)$$

and moreover

$$\forall u \in V, \ \|A^{-1}\mathcal{M}'(u)\|_{\mathcal{L}(H,H)} \le C(\|u\|) \tag{7.5}$$

where $C(R)$ is bounded on bounded subsets of $\mathbb{R}_+$. We obtain the following generalization of a result due to Ben Hassen and Chergui [3].

Theorem 7.1. *Let $u \in W^{1,1}_{loc}(\mathbb{R}^+, V) \cap W^{2,1}_{loc}(\mathbb{R}^+, H)$ be a solution of (7.1))such that u has precompact range in V and $\dot{u}$ has precompact range in H. Assume in addition that*

$$\exists C \ge 0, \ \exists \delta > 0, \ \|h(t, \cdot)\|_H \le \frac{C}{(1+t)^{1+\delta+\alpha}}, \text{ for all } t \in \mathbb{R}_+. \tag{7.6}$$

Then we have for some $a \in \mathcal{E}$

$$\lim_{t\to\infty}(\|u(t) - a\| + |\dot{u}(t)|) = 0$$

Proof. First we prove that $\|\dot{u}(t)\|_H$ tends to 0 at infinity. Indeed introducing

$$\mathcal{E}_0(t) = \frac{1}{2}\|\dot{u}(t)\|_H^2 + E(u(t))$$

we have, thanks to assumption (7.2)

$$\mathcal{E'}_0(t) = -\langle \dot{u}, g(\dot{u})\rangle_{V,V'} + \langle h, \dot{u}\rangle_H \le -\rho_1\|\dot{u}\|_H^{\alpha+2} + \|h\|_H\|\dot{u}\|_H.$$

$$\mathcal{E}_0'(t) \le -\frac{\rho_1}{2}\|\dot{u}\|_H^{\alpha+2} + C_0\|h\|_H^{\frac{\alpha+2}{\alpha+1}}.$$

and therefore the bounded function

$$\mathcal{F}(t) := \mathcal{E}_0(t) + C_0\int_t^\infty \|h(s)\|_H^{\frac{\alpha+2}{\alpha+1}} ds$$

is non-increasing and consequently convergent at infinity. Since $\mathcal{F}(t) - \mathcal{E}_0$ tends to 0 at infinity, it follows that $\mathcal{E}_0(t)$ itself has a limit $\bar{E}$. In addition we have

$$\|\dot{u}\|_H^{\alpha+2} \leq -\frac{2}{\rho_1}\mathcal{E}_0'(t) + D_0\|h\|_H^{\frac{\alpha+2}{\alpha+1}}.$$

hence $\|\dot{u}(t)\|_H \in L^{\alpha+2}(\mathbb{R}_+)$ and in particular

$$\dot{u} \in L^{\alpha+2}(\mathbb{R}_+, V')$$

But by the equation we also have

$$\ddot{u} \in L^{\infty}(\mathbb{R}_+, V')$$

therefore $\lim\limits_{t\to+\infty} \dot{u}(t) = 0$ in V'. By compactness,

$$\lim_{t\to+\infty} \|\dot{u}(t)\|_H = 0 \tag{7.7}$$

From the definition of $\mathcal{E}_0$ we deduce

$$\lim_{t\to+\infty} E(u(t)) = \bar{E} \tag{7.8}$$

It is then rather straightforward to deduce the following property

$$\lim_{t\to+\infty} \mathrm{dist}_V(u(t), \mathcal{E}_*) = 0 \tag{7.9}$$

where

$$\mathcal{E}_* = \{\varphi \in \mathcal{E}, E(\varphi) = \bar{E}\}$$

Indeed by compactness, u, being Lipschitz continuous with values in H, is uniformy continuous with values in V. In particular we have

$$\lim_{\varepsilon\to 0}\sup_{t\geq 0} \|\frac{1}{\varepsilon}\int_t^{t+\varepsilon} \mathcal{M}(u(s))ds - \mathcal{M}(u(t))\|_{V'} = 0$$

From

$$\mathcal{E}'{}_0(t) = -\langle \dot{u}, g(\dot{u})\rangle_{V,V'} + \langle h, \dot{u}\rangle_H \leq -\frac{1}{2}\langle \dot{u}, g(\dot{u})\rangle_{V,V'} - \frac{\rho_1}{2}\|\dot{u}\|_H^{\alpha+2} + \|h\|_H\|\dot{u}\|_H$$

it follows that $\langle \dot{u}, g(\dot{u})\rangle_{V,V'} \in L^1(\mathbb{R}_+)$ Then we observe that (7.3) implies the inequality

$$\forall \delta > 0, \forall v \in V, \quad \|g(v)\|_{V'} \leq \delta + C(\delta)\langle g(v), v\rangle_{V',V}$$

Therefore for each ε fixed

$$\lim_{t\to+\infty} \|\frac{1}{\varepsilon}\int_t^{t+\varepsilon} g(\dot{u}(s))ds\|_{V'} = 0$$

By integrating the equation over $[t, t+\varepsilon]$ and dividing through by ε we now obtain

$$\frac{1}{\varepsilon}\int_t^{t+\varepsilon} \mathcal{M}(u(s))ds = \frac{1}{\varepsilon}\int_t^{t+\varepsilon} h(s)ds - \frac{1}{\varepsilon}\int_t^{t+\varepsilon} g(\dot{u}(s))ds - \frac{1}{\varepsilon}(\dot{u}(t+\varepsilon) - \dot{u}(t))$$

The right-hand side tends to 0 in V' as $t \to \infty$ for any fixed $\varepsilon > 0$. By choosing $\varepsilon > 0$ small enough first and then letting $t \to \infty$ we finally see that

$$\lim_{t\to+\infty} \|\mathcal{M}(u(t))\|_{V'} = 0$$

and the result follow easily since for any limiting point φ of $u(t)$ as $t \to \infty$ we have $\mathcal{M}(\varphi) = 0$ and $E(\varphi) = \bar{E}$.

Now let $0 < \varepsilon \leq 1$ be a real constant. We define the function :

$$H(t) = \frac{1}{2}\|\dot{u}\|_H^2 + E(u) - \bar{E} + \varepsilon\|\dot{u}\|_{V'}^{\alpha}\langle \mathcal{M}(u), \dot{u}\rangle_{V'} + \int_t^{+\infty} \langle h(s), \dot{u}(s)\rangle_H$$
$$+\frac{\varepsilon}{2}\frac{(1+\alpha)^2}{1-\alpha}\int_t^{+\infty} \|\dot{u}\|_{V'}^{\alpha}\|h(s)\|_{V'}^2\, ds,$$

It is clear that

$$\lim_{t\to+\infty} H(t) = 0$$

On the other hand H is differentiable at any point where $\dot{u}$ does not vanish and at those points we have

$$\begin{gathered}H'(t) = -\langle \dot{u}, g(\dot{u})\rangle_{V,V'} + \varepsilon\|\dot{u}\|_{V'}^{\alpha}\langle \mathcal{M}'(u)\dot{u}, \dot{u}\rangle_{V'} + \\ +\varepsilon\|\dot{u}\|_{V'}^{\alpha}\langle \mathcal{M}(u), h(t)\rangle_{V,V'} - \varepsilon\|\dot{u}\|_{V'}^{\alpha}\langle \mathcal{M}(u), g(\dot{u})\rangle_{V'} - \varepsilon\|\dot{u}\|_{V'}^{\alpha}\|\mathcal{M}(u)\|_{V'}^2 + \\ \varepsilon\alpha\|\dot{u}\|_{V'}^{\alpha-2}\langle \mathcal{M}(u), \dot{u}\rangle_{V'}\langle \dot{u}, h\rangle_{V,V'} - \varepsilon\alpha\|\dot{u}\|^{\alpha-2}{}_{V'}\langle \mathcal{M}(u), \dot{u}\rangle^2{}_{V'} - \\ \varepsilon\alpha\|\dot{u}\|_{V'}^{\alpha-2}\langle \mathcal{M}(u), \dot{u}\rangle_{V'}\langle \dot{u}, g(\dot{u})\rangle_{V,V'} - \frac{\varepsilon}{2}\frac{(1+\alpha)^2}{1-\alpha}\|\dot{u}\|_{V'}^{\alpha}\|h(s)\|_{V'}^2.\end{gathered}$$

By using Cauchy-Schwarz inequality, together with assumption (7.2) we obtain

$$H'(t) \leq -\frac{1}{2}\langle \dot{u}, g(\dot{u})\rangle_{V,V'} - \frac{\rho_1}{2}\|\dot{u}\|_H^{\alpha+2} - \varepsilon(1-\alpha)\|\dot{u}\|_{V'}^{\alpha}\|\mathcal{M}(u)\|_{V'}^2 + \varepsilon(1+\alpha)\|\dot{u}\|_{V'}^{\alpha}\|\mathcal{M}(u)\|_{V'}\|h\|_{V'} +$$
$$\varepsilon\|\dot{u}\|_{V'}^{\alpha}|\langle \mathcal{M}'(u)\dot{u}, \dot{u}\rangle_{V'}| + \varepsilon(1+\alpha)\|\dot{u}\|_{V'}^{\alpha}\|\mathcal{M}(u)\|_{V'}\|g(\dot{u})\|_{V'} - \frac{\varepsilon}{2}\frac{(1+\alpha)^2}{(1-\alpha)}\|\dot{u}\|_{V'}^{\alpha}\|h\|_{V'}^2.$$

It is not difficult to check that H is differentiable with derivative equal to 0 any point where $\dot{u} = 0$, therefore the above inequality is in fact valid everywhere. Since $\|\mathcal{M}(u)\|_{V'}$ is bounded and, by using Young's inequality together with assumption (7.3) we get

$$\|g(\dot{u})\|_{V'}\|\mathcal{M}(u)\|_{V'} \leq \rho_2\langle \dot{u}, g(\dot{u})\rangle_{V,V'}^{\frac{\alpha+1}{\alpha+2}}\|\mathcal{M}(u)\|_{V'} \leq C_1\|\mathcal{M}(u)\|_{V'}^{\alpha+2} + C_2\langle \dot{u}, g(\dot{u})\rangle_{V,V'},$$

where C_1 and C_2 are two positive constants.
It is easy to see that

$$\|\mathcal{M}(u)\|_{V'}\|h\|_{V'} \leq \frac{1}{2}\left\{\frac{1-\alpha}{1+\alpha}\|\mathcal{M}(u)\|_{V'}^2 + \frac{1+\alpha}{1-\alpha}\|h\|_{V'}^2\right\}.$$

Since $\lim_{t\to+\infty}\|\dot{u}(t)\|_H = 0$, there exists T such that for all $t \geq T$ we have $\|\dot{u}\|_H \leq 1$. Then we get for all $t \geq T$

$$H'(t) \leq (-\frac{\rho_1}{2} + \varepsilon(1+\alpha)C_3)\|\dot{u}\|_H^{\alpha+2} - \varepsilon\frac{1-\alpha}{2}\|\dot{u}\|_{V'}^{\alpha}\|\mathcal{M}(u)\|_{V'}^2 +$$

$$\varepsilon\|\dot{u}\|_{V'}^{\alpha}|\langle\mathcal{M}'(u)\dot{u},\dot{u}\rangle_{V'}| + \varepsilon(1+\alpha)C_4\|\dot{u}\|_{V'}^{\alpha}\|\mathcal{M}(u)\|_{V'}^{\alpha+2}.$$

By assumption (7.5), and by choosing ε small enough we have for all $t \geq T$

$$H'(t) \leq -\varepsilon C_5\|\dot{u}\|_{V'}^{\alpha}(\|\dot{u}\|_H^2 + \|\mathcal{M}(u)\|_{V'}^2).$$

which also implies for all $t \geq T$

$$H'(t) \leq -\varepsilon C_6\|\dot{u}\|_{V'}^{\alpha}(\|\dot{u}\|_H + \|\mathcal{M}(u)\|_{V'})^2. \tag{7.10}$$

From (7.10) we deduce that H is nonincreasing on $[T,+\infty[$ and in particular $H(t) \geq 0$ on $\mathbb{R}_+$. If for some $t_0 \geq T$ it happens that $H(t_0) = 0$, then $H(t)$ vanishes on $[t_0,\infty)$ and by (7.10) we conclude that $\|\dot{u}\|_{V'}^{\alpha}\|\dot{u}\|_H^2 = 0$, hence $u(t)$ is constant for t large and there is nothing to prove.

From now on we assume that for all $t \geq T$ we have $H(t) > 0$. Let $\delta > 0$ be as in (7.6). Let $\theta \in]\frac{\alpha}{\alpha+1}, \frac{1}{2}]$ be the Łojasiewicz exponent and let $\beta = \theta - \alpha(1-\theta)$. We have

$$-\frac{1}{\beta}\frac{d}{dt}(H(t)^{\beta}) = \frac{-H'(t)}{\{H(t)^{1-\theta}\}^{1+\alpha}}. \tag{7.11}$$

By applying Cauchy-Schwarz inequality we obtain

$$\begin{aligned} H(t)^{1-\theta} \quad \leq \quad & C_7\Big\{\|\dot{u}\|_H^{2(1-\theta)} + |E(u)-\bar{E}|^{1-\theta} + \|\dot{u}\|_{V'}^{(\alpha+1)(1-\theta)}\|\mathcal{M}(u)\|_{V'}^{1-\theta} + \\ & + \left(\int_t^{+\infty}|\langle h,\dot{u}\rangle_H|\,ds\right)^{1-\theta} + \left(\int_t^{+\infty}\|h(s)\|_H^2 ds\right)^{1-\theta}\Big\}. \end{aligned}$$

and we observe that, assuming T large enough to insure $\mathrm{dist}_V(u,\mathcal{E}_*) \leq \rho$ for all $t \geq T$, we have $|E(u)-\bar{E}|^{1-\theta} \leq C\|\mathcal{M}(u)\|_{V'}$. Moreover by Young's inequality and since $\|\dot{u}\|_H \leq 1$ for all $t \geq T$ we get

$$\|\dot{u}\|_{V'}^{(\alpha+1)(1-\theta)}\|\mathcal{M}(u)\|_{V'}^{1-\theta} \leq C_7(\|\dot{u}\|_{V'} + \|\mathcal{M}(u)\|_{V'}).$$

Now, let

$$\mathcal{E}(t) = \frac{1}{2}\|\dot{u}\|_H^2 + E(u) - \bar{E}.$$

We have thanks to assumption (7.2)

$$\mathcal{E}'(t) = -\langle \dot{u}, g(\dot{u})\rangle_{V,V'} + \langle h, \dot{u}\rangle_H \leq -\rho_1 \|\dot{u}\|_H^{\alpha+2} + \|h\|_H \|\dot{u}\|_H.$$

By Young's inequality we obtain

$$\mathcal{E}'(t) \leq -\frac{\rho_1}{2} \|\dot{u}\|_H^{\alpha+2} + C_8 \|h\|_H^{\frac{\alpha+2}{\alpha+1}}.$$

By integrating we have

$$\int_t^{+\infty} \|\dot{u}\|_H^{\alpha+2}\, ds \leq C_9 \mathcal{E}(t) + C_{10} \int_t^{+\infty} \|h\|_H^{\frac{\alpha+2}{\alpha+1}}\, ds.$$

Then we have

$$(\int_t^{+\infty} |\langle \dot{u}, h\rangle_H|\, ds)^{1-\theta} \leq C_{11} \mathcal{E}(t)^{1-\theta} + C_{12} (\int_t^{+\infty} \|h\|_H^{\frac{\alpha+2}{\alpha+1}}\, ds)^{1-\theta}.$$

Now, as a consequence of (7.9) for t large enough we can use the Łojasiewicz gradient inequality. Combining with the last calculations and assumption (7.6), we obtain

$$H(t)^{1-\theta} \leq C_{13} (\|\dot{u}\|_H + \|\mathcal{M}(u)\|_{V'} + \frac{1}{(1+t)^{\xi(1-\theta)}}) \tag{7.12}$$

where $\xi = \alpha + 1 + \delta \left(\frac{\alpha+2}{\alpha+1}\right)$. As in [3] we can replace, if necessary, θ by a smaller number still greater than $\frac{\alpha}{\alpha+1}$, for which $\xi(1-\theta) > 1$, that we still call θ from now on. By combining (7.10), (7.11) and (7.12) we find

$$-C_{13} \frac{d}{dt}(H(t)^\beta) + \frac{1}{(1+t)^{\xi(1-\theta)}} \geq \frac{\|\dot{u}\|_{V'}^\alpha (\|\dot{u}\|_H + \|\mathcal{M}(u)\|_{V'})^2}{(\|\dot{u}\|_H + \|\mathcal{M}(u)\|_{V'} + \frac{1}{(1+t)^{\xi(1-\theta)}})^{1+\alpha}} + \frac{1}{(1+t)^{\xi(1-\theta)}}.$$

By using lemma 4.2 in [3] this implies

$$-C_{13} \frac{d}{dt}(H(t)^\beta) + \frac{1}{(1+t)^{\xi(1-\theta)}} \geq C_{14} \|\dot{u}\|_{V'}^\alpha \|\dot{u}\|_H^{1-\alpha}.$$

By using the embedding of H into V' we obtain

$$\|\dot{u}\|_{V'} \leq -C_{15} \frac{d}{dt}(H(t)^\beta) + \frac{C_{16}}{(1+t)^{\xi(1-\theta)}}.$$

Hence by integrating, we get for all $t \geq T$

$$\int_T^t \|\dot{u}(s)\|_{V'}\, ds \leq C_{15}(H(T)^\beta) + \frac{C_{17}}{(1+T)^{\xi(1-\theta)-1}}.$$

This implies that $\lim_{t\to+\infty} u(t)$ exists in V'. By compactness, $\lim_{t\to+\infty} u(t)$ exists in V.

□

Remark 7.2. It is easy to check that any $g \in G(Z, \alpha)$ for some Z satisfying the conditions (5.5), (5.6) and (5.7) verifies (7.2)and (7.3). Indeed (7.2) follows from (5.21) with $p = \alpha + 2$ by taking $w = 0$ and (7.3) follows easily from the combination of (6.5) and (6.6) applied with $w = 0$.

7.2 A semilinear convergence theorem

In this section we consider the semilinear equation

$$\ddot{u}(t) + g(\dot{u}(t)) + Au(t) + f(u(t)) = h(t), \quad t \in \mathbb{R}_+, \tag{7.13}$$

where A is as in the previous sections, g satisfies (7.2)-(7.3) and $f = \nabla F \in C(V, V')$ is the gradient operator of $F \in C^2(V)$. We assume

i) There is a Banach space $W \subset H$ such that $V \subset W$ with compact imbedding for which $f : W \longrightarrow H$ is Lipschitz continuous on bounded sets of W.

ii) The functional $E(u) := \frac{1}{2}\|u\|^2 + F(u)$ satisfies a uniform Łojasiewicz gradient inequality (7.4) with exponent $\theta \in \left(\frac{\alpha}{\alpha+1}, \frac{1}{2}\right]$ near the set of equilibria

$$\mathcal{E} = \mathcal{M}^{-1}\{0\} = \{u \in V, Au + f(u) = 0\}.$$

iii) For all $u \in V$ we have $A^{-1}f'(u) \in \mathcal{L}(H, H)$ with

$$\forall u \in V, \ \|A^{-1}f'(u)\|_{\mathcal{L}(H,H)} \le C(\|u\|) \tag{7.14}$$

where $C(R)$ is bounded on bounded subsets of $\mathbb{R}_+$.

We obtain

Theorem 7.3. *Let $u \in W^{1,1}_{loc}(\mathbb{R}^+, V) \cap W^{2,1}_{loc}(\mathbb{R}^+, H)$ be a solution of* (7.13) *such that u has bounded range in V and $\dot{u}$ has bounded range in H. Assume in addition that g satisfies* (5.2), (5.3) *and* (5.4), *and that h satisfies* (7.6). *Then we have for some $a \in \mathcal{E}$*

$$\lim_{t \to \infty} (\|u(t) - a\| + |\dot{u}(t)|) = 0$$

Proof. First we observe that under condition i) the function $p(t) := f(u(t)$ is uniformly continuous: $\mathbb{R}_+ \to H$. Indeed since u is bounded in W and Lipschitz continuous : $\mathbb{R}_+ \to H$

$$|f(u(t+a) - f(u(t)| \le C\|u(t+a) - u(t)\|_W \le \varepsilon\|u(t+a) - u(t)\|_V + K(\varepsilon)\|u(t+a) - u(t)\|_H$$

$$\le \varepsilon\|u(t+a) - u(t)\|_V + MK(\varepsilon)|a| \le 2\varepsilon \sup_{t\ge 0} \|u(t)\|_V + MK(\varepsilon)|a|$$

and the result follows immediately. Then since h tends to 0 it is clear that $k := h - p$ is S^1-uniformly continuous. As a conserquence of Theorem 5.1, we deduce that u has precompact range in V and $\dot{u}$ has precompact range in H. Then Theorem 7.3 becomes an immediate consequence of Theorem 7.1. $\square$

8 Applications.

8.1 Some classes of admissible damping terms

When $H = L^2(\Omega, d\mu)$ with Ω, μ some finitely measured space, an important class of damping terms satisfying the hypotheses of Theorems 4.1- 5.1 is the class of Nemytsckii operators associated to a numerical non-decreasing function $\gamma \in C(\mathbb{R}, \mathbb{R})$ which means that

$$\forall v \in V, \quad g(v)(x) = \gamma(v(x)), \quad \mu - \text{a.e in } \Omega$$

Assuming that γ satisfies

$$\exists c_0 > 0, \exists c_1 \geq 0 \quad \forall s \in \mathbb{R} \quad \gamma(s)s \ \geq c_0 s^2 - c_1$$

the coerciveness condition will be satisfied whatever be the space V as in subsection 3.1 with $C_1 = c_1|\Omega|$. In this section we consider different kinds of damping operators of non local or semi-local type. The first class that we consider corresponds to the gradient of the convex functional

$$\Phi_1(v) := \frac{c}{\alpha + 2}|A^{\frac{\beta}{2}}v|^{\alpha+2}$$

Proposition 8.1. *Let $\alpha > 0, \beta \geq 0, c > 0$ Then the operator defined by*

$$g(v) := c|A^{\frac{\beta}{2}}v|^{\alpha}A^{\beta}v$$

with $\beta < 1$ is (Z, α)-admissible with $Z = D(A^{\frac{\beta}{2}})$.

Proof. The result will in fact be a consequence of the following more general property. □

Proposition 8.2. *let Z be a reflexive banach space satisfying the conditions* (5.5), (5.6) *and* (5.7)*. Let $C \in \mathcal{L}(Z, H)$ be one-to one. Then the operator defined by*

$$g(v) := |Cv|^{\alpha}C^*Cv$$

is (Z, α)-admissible.

Proof. First we show that (6.5) is an immediate consequence of Lemma 2.3. Indeed we have

$$\forall (v, w) \in V \times V, \ \langle g(v) - g(w), v - w\rangle = \langle |Cv|^{\alpha}C^*Cv - |Cw|^{\alpha}C^*Cw, v - w\rangle$$

$$= \langle C^*(|Cv|^{\alpha}Cv - |Cw|^{\alpha}Cw), v - w\rangle = \langle |Cv|^{\alpha}Cv - |Cw|^{\alpha}Cw, Cv - Cw\rangle$$
$$\geq c(\alpha)|Cv - Cw|^{\alpha+2}$$

by Lemma 2.3. By Banach's theorem we have $C^{-1} \in \mathcal{L}(H, Z)$ and therefore

$$|Cv - Cw| \geq \delta\|v - w\|_Z$$

hence the result follows immediately. Next we show that (6.6) is a consequence of Lemma 2.2. Indeed we have

$$\forall (v,w)\in V\times V,\ \ \|g(v)-g(w)\|_{Z'}\le \|C^*\|_{\mathcal{L}(H,Z')}||Cv|^\alpha Cv-|Cw|^\alpha Cw|$$
$$\le (\alpha+1)\|C^*\|_{\mathcal{L}(H,Z')}(|Cv|^\alpha+|Cw|^\alpha)|Cv-Cw|$$

and the result follows immediately since $C\in\mathcal{L}(Z,H)$.

□

Another interesting class is the following class of "semi-local" damping terms corresponding to to the gradient of the convex functional

$$\Phi_2(v):=\frac{c}{\alpha+2}\int_\Omega |(Dv)(x)|_X^{\alpha+2}dx$$

where X is a finite dimensional Hilbert space with norm denoted by $|.|_X$ and $D\in\mathcal{L}(V,L^{\alpha+2}(\Omega,X))$, cf also [15] for results of weak convergence involving this kind of damping term in presence of almost periodic forcing.

Proposition 8.3. *Let X be a Hilbert space with norm denoted by $|.|_X$, $\mathcal{Y}=L^{\alpha+2}(\Omega,X)$, Z be as in the previous result and and $D\in\mathcal{L}(Z,\mathcal{Y})$ be be one-to one. Then the operator defined by*

$$g(v):=cD^*(|Dv|_X^\alpha Dv)$$

is (Z,α)-admissible.

Proof. We have, applying Lemma 2.3 in the Hilbert space X and using $D^{-1}\in\mathcal{L}(\mathcal{Y},Z)$

$$\forall (v,w)\in V\times V,\ \ \langle g(v)-g(w),v-w\rangle=\langle cD^*(|Dv|_X^\alpha Dv)-cD^*(|Dw|_X^\alpha Dw),v-w\rangle$$
$$=\langle |Dv|_X^\alpha Dv-|Dw|_X^\alpha Dw,Dv-Dw\rangle_{\mathcal{Y}',\mathcal{Y}}=c\int_\Omega (|Dv|_X^\alpha Dv-|Dw|_X^\alpha Dw),Dv-Dw)_X dx$$
$$\ge c'\int_\Omega |Dv-Dw|_X^{\alpha+2}dx\ge c''\|v-w\|_Z^{\alpha+2}$$

This proves (6.5). In order to check (6.6) we write, using Lemma 2.2 in the Hilbert space X

$$\forall (v,w)\in V\times V,\ \ \|g(v)-g(w)\|_{Z'}\le \|D^*\|_{\mathcal{L}(\mathcal{Y}',Z')}\||Dv|_X^\alpha Dv-|Dw|_X^\alpha Dw\|_{\mathcal{Y}'}$$
$$\le K\left\{\int_\Omega ||Dv|_X^\alpha Dv-|Dw|_X^\alpha Dw|_X^{\frac{\alpha+2}{\alpha+1}}dx\right\}^{\frac{\alpha+1}{\alpha+2}}\le CK\left\{\int_\Omega (|Dv|_X^\alpha+|Dw|_X^\alpha)^{\frac{\alpha+2}{\alpha+1}}|Dv-Dw|_X^{\frac{\alpha+2}{\alpha+1}}dx\right\}^{\frac{\alpha+1}{\alpha+2}}$$
$$\le K'\left\{\int_\Omega (|Dv|_X^{\alpha+2}+|Dw|_X^{\alpha+2})dx\right\}^{\frac{\alpha}{\alpha+2}}\left\{\int_\Omega |Dv-Dw|_X^{\alpha+2})dx\right\}^{\frac{1}{\alpha+2}}$$

and the result follows easily since $D\in\mathcal{L}(Z,\mathcal{Y})$.

□

8.2 Almost periodic forcing in presence of a non-local damping

In this Section, Ω denotes a bounded open domain of $\mathbb{R}^N$ with C^2 boundary and $\alpha \geq 0,\ c > 0$. We consider the 5 following special cases of (6.2)

Example 1: the wave equation with nonlinear averaged damping

$$\begin{cases} u_{tt} + c[\int_\Omega {u_t}^2(t,x)dx]^{\frac{\alpha}{2}} u_t - \Delta u = h(t,x), \quad \text{in } \mathbb{R}_+ \times \Omega, \\ u = 0 \ \text{ on } \mathbb{R}_+ \times \partial\Omega. \end{cases} \tag{8.1}$$

Here $V = H_0^1(\Omega)$ and $H = L^2(\Omega) = Z$.

Example 2: A clamped plate equation with nonlinear structural averaged damping

$$\begin{cases} u_{tt} - c[\int_\Omega |\nabla u_t|^2 dx]^{\frac{\alpha}{2}} \Delta u_t + \Delta^2 u = h(t,x), \quad \text{in } \mathbb{R}_+ \times \Omega, \\ u = |\nabla u| = 0 \ \text{ on } \mathbb{R}_+ \times \partial\Omega, \end{cases} \tag{8.2}$$

Here $V = H_0^2(\Omega)$, $H = L^2(\Omega)$ and $Z = H_0^1(\Omega)$.

Example 3: A simply supported plate equation with nonlinear structural averaged damping

$$\begin{cases} u_{tt} - c[\int_\Omega |\nabla u_t|^2 dx]^{\frac{\alpha}{2}} \Delta u_t + \Delta^2 u = h(t,x), \quad \text{in } \mathbb{R}_+ \times \Omega, \\ u = \Delta u = 0, \quad \text{on } \mathbb{R}_+ \times \partial\Omega, \end{cases} \tag{8.3}$$

Here $V = H^2 \cap H_0^1(\Omega)$, $H = L^2(\Omega)$and $Z = H_0^1(\Omega)$.

Example 4: A clamped plate equation with a semi-local non linear damping

$$\begin{cases} u_{tt} - c\,\mathrm{div}(|\nabla u_t|^\alpha \nabla u_t) + \Delta^2 u = h(t,x), \quad \text{in } \mathbb{R}_+ \times \Omega, \\ u(t,x) = |\nabla u| = 0, \quad \text{on } \mathbb{R}_+ \times \partial\Omega, \end{cases} \tag{8.4}$$

Here $V = H_0^2(\Omega)$ and $H = L^2(\Omega)$.

Example 5: A simply supported plate equation with a semi-local non linear damping

$$\begin{cases} u_{tt} - c\,\mathrm{div}(|\nabla u_t|^\alpha \nabla u_t) + \Delta^2 u = h(t,x), \quad \text{in } \mathbb{R}_+ \times \Omega, \\ u(t,x) = \Delta u = 0, \quad \text{on } \mathbb{R}_+ \times \partial\Omega, \end{cases} \tag{8.5}$$

Here $V = H^2 \cap H_0^1(\Omega)$ and $H = L^2(\Omega)$.

The following result is an immediate consequence of Theorem 6.8 and the properties established in Section 8.1.

Theorem 8.4. *Let $\alpha \geq 0$ be arbitrary with the restriction $(N-2)\alpha < 4$ in examples 4-5. Then for any h which is S^1-almost periodic: $\mathbb{R} \to H = L^2(\Omega)$ each equation above has a unique solution ω such that the vector $(\omega, \dot{\omega})$ is almost periodic $\mathbb{R} \to V \times H$. In addition for any other solution u we have for some $M > 0$*

$$\forall t \geq 0, \quad \|u(t) - \omega(t)\| + |\dot{u}(t) - \dot{\omega}(t)| \leq M(1+t)^{-\frac{1}{\alpha}}$$

Proof. The only thing to check is that in all the examples, the operator g is (Z, α)- admissible for some relevant choice of Z. In examples 1,2, 3 we apply Proposition 8.2 and in examples 4-5 we use Proposition 8.3 with $\mathcal{Y} = L^{\alpha+2}(\Omega, \mathbb{R}^N)$ and Z the closure of V in $H^{2-\varepsilon}$ for some $\varepsilon > 0$. We skip the details. □

8.3 Convergence in presence of a non-local damping

We now give some generalizations of the main infinite-dimensional result from [3]. The spaces V and H are the same as in the corresponding examples in the previous subsection

Example 6: the wave equation with averaged damping

Under the following assumptions on f and h :

$$f : \mathbb{R} \to \mathbb{R} \text{ is analytic,} \tag{8.6}$$

there exists $C \geq 0$ and $\eta > 0$ with $(N-2)\eta < 2$ such that :

$$|f'(s)| \leq C(1 + |s|^\eta) \text{ on } \mathbb{R}, \tag{8.7}$$

we consider the equation

$$\begin{cases} u_{tt} + c[\int_\Omega {u_t}^2(t,x)dx]^{\frac{\alpha}{2}} u_t - \Delta u + f(u) = h(t,x), \quad \text{in } \mathbb{R}_+ \times \Omega, \\ u = 0 \quad \text{on } \mathbb{R}_+ \times \partial\Omega. \end{cases} \tag{8.8}$$

Example 7: A clamped plate equation with nonlinear structural averaged damping

Under the following assumptions on f :

$$f : \mathbb{R} \to \mathbb{R} \text{ is analytic,} \tag{8.9}$$

there exists $C \geq 0$ and $\eta > 0$ with $(N-4)\eta < 4$ such that :

$$|f'(s)| \leq C(1 + |s|^\eta) \text{ on } \mathbb{R}, \tag{8.10}$$

we consider the equation

$$\begin{cases} u_{tt} - c[\int_\Omega |\nabla u_t|^2 dx]^{\frac{\alpha}{2}} \Delta u_t + \Delta^2 u + f(u) = h(t,x), & \text{in } \mathbb{R}_+ \times \Omega, \\ u = |\nabla u| = 0, \quad \text{on } \mathbb{R}_+ \times \partial\Omega, \end{cases} \tag{8.11}$$

Example 8: A simply supported plate equation with with nonlinear structural averaged damping

Assuming that f satisfies (8.9) and (8.10) we consider the equation

$$\begin{cases} u_{tt} - c[\int_\Omega |\nabla u_t|^2 dx]^{\frac{\alpha}{2}} \Delta u_t + \Delta^2 u + f(u) = h(t,x), & \text{in } \mathbb{R}_+ \times \Omega, \\ u = \Delta u = 0, \quad \text{on } \mathbb{R}_+ \times \partial\Omega, \end{cases} \tag{8.12}$$

Example 9: A clamped plate equation with a semi-local non linear damping

Assuming that f satisfies (8.9) and (8.10), assuming in addition $(N-2)\alpha < 4$ we consider the equation

$$\begin{cases} u_{tt} - c\,\mathrm{div}(|\nabla u_t|^\alpha \nabla u_t) + \Delta^2 u + f(u) = h(t,x), & \text{in } \mathbb{R}_+ \times \Omega, \\ u = |\nabla u| = 0, \quad \text{on } \mathbb{R}_+ \times \partial\Omega, \end{cases} \tag{8.13}$$

Example 10: A simply supported plate equation with a semi-local non linear damping

Assuming that f satisfies (8.9) and (8.10), assuming in addition $(N-2)\alpha < 4$ we consider the equation

$$\begin{cases} u_{tt} - c\,\mathrm{div}(|\nabla u_t|^\alpha \nabla u_t) + \Delta^2 u + f(u) = h(t,x), & \text{in } \mathbb{R}_+ \times \Omega, \\ u = \Delta u = 0, \quad \text{on } \mathbb{R}_+ \times \partial\Omega, \end{cases} \tag{8.14}$$

The following result is an immediate consequence of Theorem and the properties established in Section 8.1.

Theorem 8.5. *Define V and H as in the examples of Section 8.2. Let $u \in W^{1,1}_{loc}(\mathbb{R}^+, V) \cap W^{2,1}_{loc}(\mathbb{R}^+, H)$ be a solution of one of the above equations. Assume in addition that $F(s) = \int_0^s f(s)ds$ is bounded from below, that α is small enough (depending on f) and that h satisfies*

$$\exists K \geq 0,\ \exists \delta > 0,\ \|h(t,\cdot)\|_{L^2(\Omega)} \leq \frac{K}{(1+t)^{1+\delta+\alpha}},\ \textit{for all } t \in \mathbb{R}_+. \tag{8.15}$$

Then we have for some $\varphi \in V$ solution of the corresponding elliptic problem

$$-\Delta\varphi + f(\varphi) = 0 \quad \textit{(example 6)}$$

or

$$\Delta^2\varphi + f(\varphi) = 0 \qquad \textit{(examples 7-10)}$$

with the relevant boundary conditions

$$\lim_{t\to\infty}(\|u(t)-\varphi\| + |\dot{u}(t)|) = 0.$$

Proof. From the hypothesis on f it is easy to check that all solutions have a bounded energy and the set of stationary solutions is compact, hence the potential energy E satisfies (7.4). Then we apply Theorem 7.3. We skip the details which are rather classical. □

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References

[1] L. Amerio & G. Prouse, *Uniqueness and almost-periodicity theorems for a non linear wave equation*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. **8**, 46 (1969) 1–8.

[2] I. Ben Hassen, *Decay estimates to equilibrium for some asymptotically autonomous semi-linear evolution equations*, .Asymptot. Anal. **69** (2010), no. 1-2, 31–44.

[3] I. Ben Hassen & L. Chergui, *Convergence of global and bounded solutions of some nonautonomous second order evolution equations with nonlinear dissipation*, J. Dynam. Differential Equations **23**(2011), no. 2, 315-332.

[4] I. Ben Hassen & A. Haraux, *Convergence and decay estimates for a class of second order dissipative equations involving a non-negative potential energy*, J. Funct. Anal.**260** (2011), no. 10, 2933–2963.

[5] M. Biroli, Bounded or almost periodic solution of the non linear vibrating membrane equation, Ricerche Mat. **22** (1973), 190-202.

[6] M. Biroli, & A. Haraux, Asymptotic behavior for an almost periodic, strongly dissipative wave equation. J. Differential Equations **38** (1980), no. 3, 422-440.

[7] H. Brezis, *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert* North-Holland Mathematics Studies, No. 5, Amsterdam-London (1973)183 pp.

[8] H. Brezis, *Equations et inéquations non linéaires dans les espaces vectoriels en dualité* Ann. Inst. Fourier (Grenoble) **18** (1968) fasc. 1, 115-175.

[9] L. Chergui, *Convergence of global and bounded solutions of a second order gradient like system with nonlinear dissipation and analytic nonlinearity*, J. Dynam. Diff. Equa. **20** (2008), 643-652.

[10] L.Chergui, *Convergence of global and bounded solutions of the wave equation with nonlinear dissipation and analytic nonlinearity*, J. Evol. Equ. **9** (2009), 405-418.

[11] R.Chill & M.A. Jendoubi, *Convergence to steady states in asymptotically autonomous semilinear evolution equations,*, Nonlinear Anal. **53** (2000), no 7-8, 1017–1039.

[12] A. Haraux, *Nonlinear evolution equations- Global behavior of solutions*, Lecture Notes in Mathematics **841**, Springer-Verlag, Berlin-New York, (1981). 313 pp.

[13] A. Haraux, *Damping out of transient states for some semi-linear, quasi-autonomous systems of hyperbolic type*, Rc. Accad. Naz. Sci. dei 40 (Memorie di Matematica) **101** 7, fasc.7 (1983), 89–136.

[14] A. Haraux, *Semi-linear hyperbolic problems in bounded domains*, Mathematical reports Vol 3, Part 1 (1987), J. Dieudonné Editor, Harwood Academic Publishers, Gordon & Breach.

[15] A. Haraux, *Systèmes Dynamiques Dissipatifs et Applications*, R.M.A.**17** Masson, Paris (1991).

[16] A. Haraux & M.A. Jendoubi, *Convergence of bounded weak solutions of the wave equation with dissipation and analytic nonlinearity.* Calc. Var. Partial Differential Equations **9** (1999), no. 2, 95–124.

[17] A. Haraux & M.A. Jendoubi, *Decay estimates to equilibrium for some evolution equations with an analytic nonlinearity.* Asymptotic Analysis **26** (2001), no. 1, 21–36.

[18] A. Haraux, M.A. Jendoubi & O. Kavian, *Rate of decay to equilibrium in some semilinear parabolic equations*, J. Evol. Equ. **3** (2003), no. 3, 463–484.

[19] A. Haraux, E. Zuazua, *Decay estimates for some semilinear damped hyperbolic problems*, Arch. Rational Mech. Anal.**100** (1988), 191–206

[20] S.Z. Huang & P. Takac, *Convergence in gradient-like systems which are asymptotically autonomous and analytic*, Nonlinear Anal. **46** (2001), no. 5, Ser. A: Theory Methods, 675-698.

[21] M.A. Jendoubi, *Convergence of global and bounded solutions of the wave equation with linear dissipation and analytic nonlinearity.* J. Differential Equations **144** (1998), no. 2, 302–312.

[22] S. Łojasiewicz, Une propriété topologique des sous-ensembles analytiques réels, Colloques internationaux du C.N.R.S.: Les équations aux dérivées partielles, Paris (1962), Editions du C.N.R.S., Paris, 1963, pp. 87–89.

[23] S. Łojasiewicz, Ensembles semi-analytiques, Preprint, I.H.E.S. Bures-sur-Yvette, 1965.

[24] G. Prouse, Soluzioni quasi-periodiche dell'equazione non omogenea delle onde, con termine dissipativo non lineare. I.-IV Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. **8**, 38-39 (1965)

[25] E. Zuazua, *Stability and decay for a class of nonlinear hyperbolic problems*, Asymptotic Anal. **1** (1988), no. 2, 161–185.