Local controllability of the N-dimensional Boussinesq system with N-1 scalar controls in an arbitrary control domain

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Abstract

In this paper we deal with the local exact controllability to a particular class of trajectories of the $N-$dimensional Boussinesq system with internal controls having 2 vanishing components. The main novelty of this work is that no condition is imposed on the control domain.

Subject Classification: 34B15, 35Q30, 93C10, 93B05

Keywords: Navier-Stokes system, Boussinesq system, exact controllability, Carleman inequalities

1 Introduction

Let Ω be a nonempty bounded connected open subset of $\mathbf{R}^N$ ($N = 2$ or 3) of class C^∞. Let $T > 0$ and let $\omega \subset \Omega$ be a (small) nonempty open subset which is the control domain. We will use the notation $Q = \Omega \times (0, T)$ and $\Sigma = \partial\Omega \times (0, T)$.

We will be concerned with the following controlled Boussinesq system:

$$\begin{cases} y_t - \Delta y + (y \cdot \nabla)y + \nabla p = v\mathbb{1}_\omega + \theta\, e_N & \text{in } Q, \\ \theta_t - \Delta\theta + y \cdot \nabla\theta = v_0\mathbb{1}_\omega & \text{in } Q, \\ \nabla \cdot y = 0 & \text{in } Q, \\ y = 0,\ \theta = 0 & \text{on } \Sigma, \\ y(0) = y^0,\ \theta(0) = \theta^0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

where

$$e_N = \begin{cases} (0,1) & \text{if } N = 2, \\ (0,0,1) & \text{if } N = 3 \end{cases}$$

stands for the gravity vector field, $y = y(x, t)$ represents the velocity of the particules of the fluid, $\theta = \theta(x, t)$ their temperature and $(v_0, v) = (v_0, v_1, \ldots, v_N)$ stands for the control which acts over the set ω.

Let us recall the definition of some usual spaces in the context of incompressible fluids:

$$V = \{y \in H_0^1(\Omega)^N : \nabla \cdot y = 0 \text{ in } \Omega\}$$

and

$$H = \{y \in L^2(\Omega)^N : \nabla \cdot y = 0 \text{ in } \Omega,\ y \cdot n = 0 \text{ on } \partial\Omega\}.$$

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This paper concerns the local exact controllability to the trajectories of system (1.1) at time $t = T$ with a reduced number of controls. To introduce this concept, let us consider $(\bar{y}, \bar{\theta})$ (together with some pressure $\bar{p}$) a trajectory of the following uncontrolled Boussinesq system:

$$\begin{cases} \bar{y}_t - \Delta\bar{y} + (\bar{y}\cdot\nabla)\bar{y} + \nabla\bar{p} = \bar{\theta}\, e_N & \text{in } Q, \\ \bar{\theta}_t - \Delta\bar{\theta} + \bar{y}\cdot\nabla\bar{\theta} = 0 & \text{in } Q, \\ \nabla\cdot\bar{y} = 0 & \text{in } Q, \\ \bar{y} = 0,\ \bar{\theta} = 0 & \text{on } \Sigma, \\ \bar{y}(0) = \bar{y}^0,\ \bar{\theta}(0) = \bar{\theta}^0 & \text{in } \Omega. \end{cases} \tag{1.2}$$

We say that the local exact controllability to the trajectories $(\bar{y}, \bar{\theta})$ holds if there exists a number $\delta > 0$ such that if $\|(y^0, \theta^0) - (\bar{y}^0, \bar{\theta}^0)\|_X \le \delta$ (X is an appropriate Banach space), there exist controls $(v_0, v) \in L^2(\omega\times(0,T))^{N+1}$ such that the corresponding solution (y, θ) to system (1.1) matches $(\bar{y}, \bar{\theta})$ at time $t = T$, i.e.,

$$y(T) = \bar{y}(T) \text{ and } \theta(T) = \bar{\theta}(T) \text{ in } \Omega. \tag{1.3}$$

The first results concerning this problem were obtained in [7] and [8], with $N+1$ scalar controls acting in the whole boundary of Ω and with $N+1$ scalar controls acting in ω when Ω is a torus, respectively. Later, in [9], the author proved the local exact controllability for less regular trajectories $(\bar{y}, \bar{\theta})$ in an open bounded set and for an arbitrary control domain. Namely, the trajectories were supposed to satisfy

$$(\bar{y}, \bar{\theta}) \in L^\infty(Q)^{N+1},\ (\bar{y}_t, \bar{\theta}_t) \in L^2(0,T; L^r(\Omega))^{N+1}, \tag{1.4}$$

with $r > 1$ if $N = 2$ and $r > 6/5$ if $N = 3$.

In [5], the authors proved that local exact controllability can be achieved with $N-1$ scalar controls acting in ω when $\overline{\omega}$ intersects the boundary of Ω and (1.4) is satisfied. More precisely, we can find controls v_0 and v, with $v_N \equiv 0$ and $v_k \equiv 0$ for some $k < N$ (k is determined by some geometric assumption on ω, see [5] for more details), such that the corresponding solution to (1.1) satisfies (1.3).

In this work, we remove this geometric assumption on ω and consider a target trajectory of the form $(0, \bar{p}, \bar{\theta})$, i.e.,

$$\begin{cases} \nabla\bar{p} = \bar{\theta}\, e_N & \text{in } Q, \\ \bar{\theta}_t - \Delta\bar{\theta} = 0 & \text{in } Q, \\ \bar{\theta} = 0 & \text{on } \Sigma, \\ \bar{\theta}(0) = \bar{\theta}^0 & \text{in } \Omega, \end{cases} \tag{1.5}$$

where we assume

$$\bar{\theta} \in L^\infty(0,T; W^{3,\infty}(\Omega)) \text{ and } \nabla\bar{\theta}_t \in L^\infty(Q)^N. \tag{1.6}$$

The main result of this paper is given in the following theorem.

Theorem 1.1. *Let $i < N$ be a positive integer and $(\bar{p}, \bar{\theta})$ a solution to* (1.5) *satisfying* (1.6). *Then, for every $T > 0$ and $\omega \subset \Omega$, there exists $\delta > 0$ such that for every $(y^0, \theta^0) \in V \times H_0^1(\Omega)$ satisfying*

$$\|(y^0, \theta^0) - (0, \bar{\theta}^0)\|_{V\times H_0^1} \le \delta,$$

we can find controls $v^0 \in L^2(\omega\times(0,T))$ and $v \in L^2(\omega\times(0,T))^N$, with $v_i \equiv 0$ and $v_N \equiv 0$, such that the corresponding solution to (1.1) *satisfies* (1.3), *i.e.,*

$$y(T) = 0 \textit{ and } \theta(T) = \bar{\theta}(T) \textit{ in } \Omega. \tag{1.7}$$

Remark 1. Notice that when $N = 2$ we only need to control the temperature equation.

Remark 2. It would be interesting to know if the local controllability to the trajectories with $N-1$ scalar controls holds for $\bar{y} \neq 0$ and ω as in Theorem 1.1. However, up to our knowledge, this is an open problem even for the case of the Navier-Stokes system.

Remark 3. One could also try to just control the movement equation, that is, $v_0 \equiv 0$ in (1.1). However, this system does not seem to be controllable. To justify this, let us consider the control problem

$$\begin{cases} y_t - \Delta y + (y \cdot \nabla) y + \nabla p = v \mathbb{1}_\omega + \theta \, e_N & \text{in } Q, \\ \theta_t - \Delta \theta + y \cdot \nabla \theta = 0 & \text{in } Q, \\ \nabla \cdot y = 0 & \text{in } Q, \\ y = 0, \, \nabla \theta \cdot n = 0 & \text{on } \Sigma, \\ y(0) = y^0, \, \theta(0) = \theta^0 & \text{in } \Omega; \end{cases}$$

where we have homogeneous Neumann boundary conditions for the temperature. Integrating in Q, integration by parts gives

$$\int_\Omega \theta(T) \, dx = \int_\Omega \theta_0 \, dx,$$

so we can not expect in general null controllability.

Some recent works have been developed in the controllability problem with reduced number of controls. For instance, in [3] the authors proved the null controllability for the Stokes system with $N-1$ scalar controls, and in [2] the local null controllability was proved for the Navier-Stokes system with the same number of controls.

The present work can be viewed as an extension of [2]. To prove Theorem 1.1 we follow a standard approach introduced in [6] and [10] (see also [4]). We first deduce a null controllability result for the linear system

$$\begin{cases} y_t - \Delta y + \nabla p = f + v \mathbb{1}_\omega + \theta \, e_N & \text{in } Q, \\ \theta_t - \Delta \theta + y \cdot \nabla \bar{\theta} = f_0 + v_0 \mathbb{1}_\omega & \text{in } Q, \\ \nabla \cdot y = 0 & \text{in } Q, \\ y = 0, \, \theta = 0 & \text{on } \Sigma, \\ y(0) = y^0, \, \theta(0) = \theta^0 & \text{in } \Omega, \end{cases} \tag{1.8}$$

where f and f_0 will be taken to decrease exponentially to zero in $t = T$.

The main tool to prove this null controllability result for system (1.8) is a suitable Carleman estimate for the solutions of its adjoint system, namely,

$$\begin{cases} -\varphi_t - \Delta \varphi + \nabla \pi = g - \psi \nabla \bar{\theta} & \text{in } Q, \\ -\psi_t - \Delta \psi = g_0 + \varphi_N & \text{in } Q, \\ \nabla \cdot \varphi = 0 & \text{in } Q, \\ \varphi = 0, \, \psi = 0 & \text{on } \Sigma, \\ \varphi(T) = \varphi^T, \, \psi(T) = \psi^T & \text{in } \Omega, \end{cases} \tag{1.9}$$

where $g \in L^2(Q)^N$, $g_0 \in L^2(Q)$, $\varphi^T \in H$ and $\psi^T \in L^2(\Omega)$. In fact, this inequality is of the form

$$\iint_Q \widetilde{\rho}_1(t)(|\varphi|^2 + |\psi|^2) dx \, dt$$

$$\leq C \left(\iint_Q \widetilde{\rho}_2(t)(|g|^2 + |g_0|^2) dx \, dt + \int_0^T \int_\omega \widetilde{\rho}_3(t) |\varphi_j|^2 dx \, dt + \int_0^T \int_\omega \widetilde{\rho}_4(t) |\psi|^2 dx \, dt \right), \tag{1.10}$$

if $N=3$, and of the form

$$\iint\limits_Q \widetilde{\rho}_1(t)(|\varphi|^2 + |\psi|^2)dxdt \le C\left(\iint\limits_Q \widetilde{\rho}_2(t)(|g|^2+|g_0|^2)dxdt + \int_0^T\int_\omega \widetilde{\rho}_4(t)|\psi|^2dxdt\right),$$

if $N=2$, where $j=1$ or 2 and $\widetilde{\rho}_k(t)$ are positive smooth weight functions (see inequalities (2.4) and (2.5) below). From these estimates, we can find a solution (y,θ,v,v_0) of (1.8) with the same decreasing properties as f and f_0. In particular, $(y(T),\theta(T))=(0,0)$ and $v_i=v_N=0$.

We conclude the controllability result for the nonlinear system by means of an inverse mapping theorem.

This paper is organized as follows. In section 2, we prove a Carleman inequality of the form (1.10) for system (1.9). In section 3, we deal with the null controllability of the linear system (1.8). Finally, in section 4 we give the proof of Theorem 1.1.

2 Carleman estimate for the adjoint system

In this section we will prove a Carleman estimate for the adjoint system (1.9). In order to do so, we are going to introduce some weight functions. Let ω_0 be a nonempty open subset of $\mathbf{R}^N$ such that $\overline{\omega_0}\subset\omega$ and $\eta\in C^2(\overline{\Omega})$ such that

$$|\nabla\eta|>0 \text{ in } \overline{\Omega}\setminus\omega_0,\ \eta>0 \text{ in } \Omega \text{ and } \eta\equiv 0 \text{ on } \partial\Omega. \tag{2.1}$$

The existence of such a function η is given in [6]. Let also $\ell\in C^\infty([0,T])$ be a positive function satisfying

$$\begin{aligned}&\ell(t)=t \quad \forall t\in[0,T/4],\ \ell(t)=T-t \quad \forall t\in[3T/4,T],\\ &\ell(t)\le\ell(T/2),\ \forall t\in[0,T].\end{aligned}\tag{2.2}$$

Then, for all $\lambda\ge 1$ we consider the following weight functions:

$$\begin{aligned}&\alpha(x,t)=\frac{e^{2\lambda\|\eta\|_\infty}-e^{\lambda\eta(x)}}{\ell^8(t)},\ \xi(x,t)=\frac{e^{\lambda\eta(x)}}{\ell^8(t)},\\ &\alpha^*(t)=\max_{x\in\overline{\Omega}}\alpha(x,t),\ \xi^*(t)=\min_{x\in\overline{\Omega}}\xi(x,t),\\ &\widehat{\alpha}(t)=\min_{x\in\overline{\Omega}}\alpha(x,t),\ \widehat{\xi}(t)=\max_{x\in\overline{\Omega}}\xi(x,t).\end{aligned}\tag{2.3}$$

Our Carleman estimate is given in the following proposition.

Proposition 1. *Assume $N=3$, $\omega\subset\Omega$ and $(\bar p,\bar\theta)$ satisfies (1.6). There exists a constant λ_0, such that for any $\lambda\ge\lambda_0$ there exist two constants $C(\lambda)>0$ and $s_0(\lambda)>0$ such that for any $j\in\{1,2\}$, any $g\in L^2(Q)^3$, any $g_0\in L^2(Q)$, any $\varphi^T\in H$ and any $\psi^T\in L^2(\Omega)$, the solution of (1.9) satisfies*

$$\begin{aligned}&s^4\iint\limits_Q e^{-5s\alpha^*}(\xi^*)^4|\varphi|^2dx\,dt+s^5\iint\limits_Q e^{-5s\alpha^*}(\xi^*)^5|\psi|^2dx\,dt\\ &\le C\left(\iint\limits_Q e^{-3s\alpha^*}(|g|^2+|g_0|^2)dx\,dt+s^7\int_0^T\int_\omega e^{-2s\widehat{\alpha}-3s\alpha^*}(\widehat{\xi})^7|\varphi_j|^2dx\,dt\right.\\ &\left.\qquad+s^{12}\int_0^T\int_\omega e^{-4s\widehat{\alpha}-s\alpha^*}(\widehat{\xi})^{49/4}|\psi|^2dx\,dt\right)\end{aligned}\tag{2.4}$$

for every $s \geq s_0$.

For the sake of completeness, let us also state this result for the 2-dimensional case.

Proposition 2. *Assume* $N = 2$, $\omega \subset \Omega$ *and* $(\bar{p}, \bar{\theta})$ *satisfies* (1.6). *There exists a constant* λ_0, *such that for any* $\lambda \geq \lambda_0$ *there exist two constants* $C(\lambda) > 0$ *and* $s_0(\lambda) > 0$ *such that for any* $g \in L^2(Q)^2$, *any* $g_0 \in L^2(Q)$, *any* $\varphi^T \in H$ *and any* $\psi^T \in L^2(\Omega)$, *the solution of* (1.9) *satisfies*

$$s^4 \iint_Q e^{-5s\alpha^*}(\xi^*)^4|\varphi|^2 dx\,dt + s^5 \iint_Q e^{-5s\alpha^*}(\xi^*)^5|\psi|^2 dx\,dt$$
$$\leq C\left(\iint_Q e^{-3s\alpha^*}(|g|^2 + |g_0|^2)dx\,dt + s^{12}\int_0^T\int_\omega e^{-4s\widehat{\alpha}-s\alpha^*}(\widehat{\xi})^{49/4}|\psi|^2 dx\,dt\right) \tag{2.5}$$

for every $s \geq s_0$.

To prove Proposition 1 we will follow the ideas of [3] and [5] (see also [2]). An important point in the proof of the Carleman inequality established in [3] is that the laplacian of the pressure in the adjoint system is zero. In [2], a decomposition of the solution was made, so that we can essentially concentrate in a solution where the laplacian of the pressure is zero. For system (1.9) this will not be possible because of the coupling term $\psi\nabla\bar{\theta}$. However, under hypothesis (1.6) we can follow the same ideas to obtain (2.4). All the details are given below.

2.1 Technical results

Let us present now the technical results needed to prove Carleman inequalities (2.4) and (2.5). The first of these results is a Carleman inequality for parabolic equations with non-homogeneous boundary conditions proved in [11]. Consider the equation

$$u_t - \Delta u = F_0 + \sum_{j=1}^N \partial_j F_j \text{ in } Q, \tag{2.6}$$

where $F_0, F_1, \ldots, F_N \in L^2(Q)$. We have the following result.

Lemma 2.1. *There exists a constant* $\widehat{\lambda}_0$ *only depending on* Ω, ω_0, η *and* ℓ *such that for any* $\lambda > \widehat{\lambda}_0$ *there exist two constants* $C(\lambda) > 0$ *and* $\widehat{s}(\lambda)$, *such that for every* $s \geq \widehat{s}$ *and every* $u \in L^2(0,T;H^1(\Omega)) \cap H^1(0,T;H^{-1}(\Omega))$ *satisfying* (2.6), *we have*

$$\frac{1}{s}\iint_Q e^{-2s\alpha}\frac{1}{\xi}|\nabla u|^2 dx\,dt + s\iint_Q e^{-2s\alpha}\xi|u|^2 dx\,dt \leq C\left(s\int_0^T\int_{\omega_0} e^{-2s\alpha}\xi|u|^2 dx\,dt\right.$$
$$+ s^{-1/2}\left\|e^{-s\alpha}\xi^{-1/4}u\right\|^2_{H^{\frac{1}{4},\frac{1}{2}}(\Sigma)} + s^{-1/2}\left\|e^{-s\alpha}\xi^{-1/8}u\right\|^2_{L^2(\Sigma)}$$
$$\left. + s^{-2}\iint_Q e^{-2s\alpha}\xi^{-2}|F_0|^2 dx\,dt + \sum_{j=1}^N \iint_Q e^{-2s\alpha}|F_j|^2 dx\,dt\right). \tag{2.7}$$

Recall that

$$\|u\|_{H^{\frac{1}{4},\frac{1}{2}}(\Sigma)} = \left(\|u\|^2_{H^{1/4}(0,T;L^2(\partial\Omega))} + \|u\|^2_{L^2(0,T;H^{1/2}(\partial\Omega))}\right)^{1/2}.$$

The next technical result is a particular case of Lemma 3 in [3].

Lemma 2.2. *There exists a constant $\widehat{\lambda}_1$ such that for any $\lambda \geq \widehat{\lambda}_1$ there exists $C > 0$ depending only on λ, Ω, ω_0, η and ℓ such that, for every $T > 0$ and every $u \in L^2(0,T;H^1(\Omega))$,*

$$s^3 \iint\limits_Q e^{-2s\alpha}\xi^3|u|^2 dx\, dt \leq C\left(s\iint\limits_Q e^{-2s\alpha}\xi|\nabla u|^2 dx\, dt + s^3 \int_0^T\int_{\omega_0} e^{-2s\alpha}\xi^3|u|^2 dx\, dt\right), \quad (2.8)$$

for every $s \geq C$.

The next lemma is an estimate concerning the Laplace operator:

Lemma 2.3. *There exists a constant $\widehat{\lambda}_2$ such that for any $\lambda \geq \widehat{\lambda}_2$ there exists $C > 0$ depending only on λ, Ω, ω_0, η and ℓ such that, for every $u \in L^2(0,T;H_0^1(\Omega))$,*

$$s^6 \iint\limits_Q e^{-2s\alpha}\xi^6|u|^2 dx\, dt + s^4\iint\limits_Q e^{-2s\alpha}\xi^4|\nabla u|^2 dx\, dt \leq C\left(s^3\iint\limits_Q e^{-2s\alpha}\xi^3|\Delta u|^2 dx\, dt + s^6 \int_0^T\int_{\omega_0} e^{-2s\alpha}\xi^6|u|^2 dx\, dt\right), \quad (2.9)$$

for every $s \geq C$.

Inequality (2.9) comes from the classical result in [6] for parabolic equations applied to the laplacian with parameter $s/\ell^8(t)$. Then, multiplying by $\exp(-2se^{2\lambda\|\eta\|_\infty}/\ell^8(t))$ and integrating in $(0,T)$ we obtain (2.9). Details can be found in [3] or [2].

The last technical result concerns the regularity of the solutions to the Stokes system that can be found in [12] (see also [13]).

Lemma 2.4. *For every $T > 0$ and every $F \in L^2(Q)^N$, there exists a unique solution*

$$u \in L^2(0,T;H^2(\Omega)^N) \cap H^1(0,T;H)$$

to the Stokes system

$$\begin{cases} u_t - \Delta u + \nabla p = F & \text{in } Q, \\ \nabla \cdot u = 0 & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ u(0) = 0 & \text{in } \Omega, \end{cases}$$

for some $p \in L^2(0,T;H^1(\Omega))$, and there exists a constant $C > 0$ depending only on Ω such that

$$\|u\|^2_{L^2(0,T;H^2(\Omega)^N)} + \|u\|^2_{H^1(0,T;L^2(\Omega)^N)} \leq C\|F\|^2_{L^2(Q)^N}. \quad (2.10)$$

Furthermore, if $F \in L^2(0,T;H^2(\Omega)^N) \cap H^1(0,T;L^2(\Omega)^N)$, then $u \in L^2(0,T;H^4(\Omega)^N) \cap H^1(0,T;H^2(\Omega)^N)$ and there exists a constant $C > 0$ depending only on Ω such that

$$\|u\|^2_{L^2(0,T;H^4(\Omega)^N)} + \|u\|^2_{H^1(0,T;H^2(\Omega)^N)} \leq C\left(\|F\|^2_{L^2(0,T;H^2(\Omega)^N)} + \|F\|^2_{H^1(0,T;L^2(\Omega)^N)}\right). \quad (2.11)$$

From now on, we set $N = 3$, $i = 2$ and $j = 1$, i.e., we consider a control for the movement equation in (1.1) (and (1.8)) of the form $v = (v_1, 0, 0)$. The arguments can be easily adapted to the general case by interchanging the roles of i and j.

2.2 Proof of Proposition 1

Let us introduce (w, π_w), (z, π_z) and $\widetilde{\psi}$, the solutions of the following systems:

$$\begin{cases} -w_t - \Delta w + \nabla \pi_w = \rho\, g & \text{in } Q, \\ \nabla \cdot w = 0 & \text{in } Q, \\ w = 0 & \text{on } \Sigma, \\ w(T) = 0 & \text{in } \Omega, \end{cases} \tag{2.12}$$

$$\begin{cases} -z_t - \Delta z + \nabla \pi_z = -\rho' \varphi - \widetilde{\psi} \nabla \bar{\theta} & \text{in } Q, \\ \nabla \cdot z = 0 & \text{in } Q, \\ z = 0 & \text{on } \Sigma, \\ z(T) = 0 & \text{in } \Omega, \end{cases} \tag{2.13}$$

and

$$\begin{cases} -\widetilde{\psi}_t - \Delta \widetilde{\psi} = \rho\, g_0 + \rho\, \varphi_3 - \rho' \psi & \text{in } Q, \\ \widetilde{\psi} = 0 & \text{on } \Sigma, \\ \widetilde{\psi}(T) = 0 & \text{in } \Omega, \end{cases} \tag{2.14}$$

where $\rho(t) = e^{-\frac{3}{2} s \alpha^*}$. Adding (2.12) and (2.13), we see that $(w + z, \pi_w + \pi_z, \widetilde{\psi})$ solves the same system as $(\rho\, \varphi, \rho\, \pi, \rho\, \psi)$, where (φ, π, ψ) is the solution to (1.9). By uniqueness of the Cauchy problem we have

$$\rho\, \varphi = w + z,\ \rho\, \pi = \pi_w + \pi_z \text{ and } \rho\, \psi = \widetilde{\psi}. \tag{2.15}$$

Applying the divergence operator to (2.13) we see that $\Delta \pi_z = -\nabla \cdot (\widetilde{\psi} \nabla \bar{\theta})$. We apply now the operator $\nabla \Delta = (\partial_1 \Delta, \partial_2 \Delta, \partial_3 \Delta)$ to the equations satisfied by z_1 and z_3. We then have

$$\begin{aligned} -(\nabla \Delta z_1)_t - \Delta(\nabla \Delta z_1) &= \nabla \left(\partial_1 \nabla \cdot (\widetilde{\psi} \nabla \bar{\theta}) - \Delta(\widetilde{\psi} \partial_1 \bar{\theta}) - \rho' \Delta \varphi_1 \right) \text{ in } Q, \\ -(\nabla \Delta z_3)_t - \Delta(\nabla \Delta z_3) &= \nabla \left(\partial_3 \nabla \cdot (\widetilde{\psi} \nabla \bar{\theta}) - \Delta(\widetilde{\psi} \partial_3 \bar{\theta}) - \rho' \Delta \varphi_3 \right) \text{ in } Q. \end{aligned} \tag{2.16}$$

To the equations in (2.16), we apply the Carleman inequality in Lemma 2.1 with $u = \nabla \Delta z_k$ for $k = 1, 3$ to obtain

$$\begin{aligned} &\sum_{k=1,3} \left[\frac{1}{s} \iint\limits_Q e^{-2s\alpha} \frac{1}{\xi} |\nabla \nabla \Delta z_k|^2 dx\, dt + s \iint\limits_Q e^{-2s\alpha} \xi |\nabla \Delta z_k|^2 dx\, dt \right] \\ &\leq C \left(\sum_{k=1,3} \left[s \int_0^T \int_{\omega_0} e^{-2s\alpha} \xi |\nabla \Delta z_k|^2 dx\, dt + s^{-1/2} \left\| e^{-s\alpha^*} (\xi^*)^{-1/8} \nabla \Delta z_k \right\|^2_{L^2(\Sigma)^3} \right. \right. \\ &\qquad \left. + s^{-1/2} \left\| e^{-s\alpha^*} (\xi^*)^{-1/4} \nabla \Delta z_k \right\|^2_{H^{\frac{1}{4},\frac{1}{2}}(\Sigma)^3} + \iint\limits_Q e^{-2s\alpha} |\rho'|^2 |\Delta \varphi_k|^2 dx\, dt \right] \\ &\qquad \left. + \iint\limits_Q e^{-2s\alpha} \Big(\sum_{k,l=1}^3 |\partial^2_{kl} \widetilde{\psi}|^2 + |\nabla \widetilde{\psi}|^2 + |\widetilde{\psi}|^2 \Big) dx\, dt \right), \end{aligned} \tag{2.17}$$

for every $s \geq C$, where C depends also on $\|\bar{\theta}\|_{L^\infty(0,T;W^{3,\infty}(\Omega))}$.

Now, by Lemma 2.2 with $u = \Delta z_k$ for $k = 1, 3$ we have

$$\sum_{k=1,3} s^3 \iint_Q e^{-2s\alpha}\xi^3|\Delta z_k|^2 dx\, dt \\ \leq C \sum_{k=1,3}\left(s\iint_Q e^{-2s\alpha}\xi|\nabla\Delta z_k|^2 dx\, dt + s^3\int_0^T\int_{\omega_0} e^{-2s\alpha}\xi^3|\Delta z_k|^2 dx\, dt\right), \quad (2.18)$$

for every $s \geq C$, and by Lemma 2.3 with $u = z_k$ for $k = 1, 3$:

$$\sum_{k=1,3}\left[s^4\iint_Q e^{-2s\alpha}\xi^4|\nabla z_k|^2 dx\, dt + s^6\iint_Q e^{-2s\alpha}\xi^6|z_k|^2 dx\, dt\right] \\ \leq C\sum_{k=1,3}\left[s^3\iint_Q e^{-2s\alpha}\xi^3|\Delta z_k|^2 dx\, dt + s^6\int_0^T\int_{\omega_0} e^{-2s\alpha}\xi^6|z_k|^2 dx\, dt\right], \quad (2.19)$$

for every $s \geq C$.

Combining (2.17), (2.18) and (2.19) and considering a nonempty open set ω_1 such that $\omega_0 \Subset \omega_1 \Subset \omega$ we obtain after some integration by parts

$$\sum_{k=1,3}\left[\frac{1}{s}\iint_Q e^{-2s\alpha}\frac{1}{\xi}|\nabla\nabla\Delta z_k|^2 dx\, dt + s\iint_Q e^{-2s\alpha}\xi|\nabla\Delta z_k|^2 dx\, dt\right. \\ \left. + s^3\iint_Q e^{-2s\alpha}\xi^3|\Delta z_k|^2 dx\, dt + s^4\iint_Q e^{-2s\alpha}\xi^4|\nabla z_k|^2 dx\, dt + s^6\iint_Q e^{-2s\alpha}\xi^6|z_k|^2 dx\, dt\right] \\ \leq C\left(\sum_{k=1,3}\left[s^7\int_0^T\int_{\omega_1} e^{-2s\alpha}\xi^7|z_k|^2 dx\, dt + s^{-1/2}\left\| e^{-s\alpha^*}(\xi^*)^{-1/8}\nabla\Delta z_k\right\|^2_{L^2(\Sigma)^3}\right.\right. \\ \left. + s^{-1/2}\left\| e^{-s\alpha^*}(\xi^*)^{-1/4}\nabla\Delta z_k\right\|^2_{H^{\frac14,\frac12}(\Sigma)^3} + \iint_Q e^{-2s\alpha}|\rho'|^2|\Delta\varphi_k|^2 dx\, dt\right] \\ \left. + \iint_Q e^{-2s\alpha}\Big(\sum_{k,l=1}^3 |\partial^2_{kl}\widetilde{\psi}|^2 + |\nabla\widetilde{\psi}|^2 + |\widetilde{\psi}|^2\Big) dx\, dt\right), \quad (2.20)$$

for every $s \geq C$.

Notice that from the identities in (2.15), the regularity estimate (2.10) for w and $|\rho'|^2 \leq Cs^2\rho^2(\xi)^{9/4}$ we obtain for $k = 1, 3$

$$\iint_Q e^{-2s\alpha}|\rho'|^2|\Delta\varphi_k|^2 dx\, dt = \iint_Q e^{-2s\alpha}|\rho'|^2\rho^{-2}|\Delta(\rho\varphi_k)|^2 dx\, dt \\ \leq Cs^2\iint_Q e^{-2s\alpha}\xi^{9/4}|\Delta z_k|^2 dx\, dt + Cs^2\iint_Q e^{-2s\alpha}\xi^{9/4}|\Delta w|^2 dx\, dt \\ \leq Cs^2\iint_Q e^{-2s\alpha}\xi^3|\Delta z_k|^2 dx\, dt + C\|\rho\, g\|^2_{L^2(Q)^3},$$

where we have also used the fact that $s^2 e^{-2s\alpha}\xi^{9/4}$ is bounded and $1 \le C\xi^{3/4}$ in Q.

Now, from $z|_\Sigma = 0$ and the divergence free condition we readily have (notice that α^* and ξ^* do not depend on x)

$$s^4 \iint\limits_Q e^{-2s\alpha^*}(\xi^*)^4|z_2|^2 dx\,dt \le Cs^4 \iint\limits_Q e^{-2s\alpha^*}(\xi^*)^4|\partial_2 z_2|^2 dx\,dt$$
$$\le Cs^4 \iint\limits_Q e^{-2s\alpha}\xi^4(|\nabla z_1|^2 + |\nabla z_3|^2)dx\,dt.$$

Using these two last estimates in (2.20), we get

$$\begin{aligned} I(s,z) := \sum_{k=1,3} \Bigg[&\frac{1}{s}\iint\limits_Q e^{-2s\alpha}\frac{1}{\xi}|\nabla\nabla\Delta z_k|^2 dx\,dt + s\iint\limits_Q e^{-2s\alpha}\xi|\nabla\Delta z_k|^2 dx\,dt \\ &+ s^3\iint\limits_Q e^{-2s\alpha}\xi^3|\Delta z_k|^2 dx\,dt + s^4\iint\limits_Q e^{-2s\alpha}\xi^4|\nabla z_k|^2 dx\,dt \\ &+ s^6\iint\limits_Q e^{-2s\alpha}\xi^6|z_k|^2 dx\,dt\Bigg] + s^4\iint\limits_Q e^{-2s\alpha^*}(\xi^*)^4|z_2|^2 dx\,dt \\ \le C\Bigg(\sum_{k=1,3}\Bigg[&s^7\int_0^T\!\!\int_{\omega_1} e^{-2s\alpha}\xi^7|z_k|^2 dx\,dt + s^{-1/2}\left\|e^{-s\alpha^*}(\xi^*)^{-1/8}\nabla\Delta z_k\right\|^2_{L^2(\Sigma)^3} \\ &+ s^{-1/2}\left\|e^{-s\alpha^*}(\xi^*)^{-1/4}\nabla\Delta z_k\right\|^2_{H^{\frac14,\frac12}(\Sigma)^3}\Bigg] + \|\rho\, g\|^2_{L^2(Q)^3} \\ &+ \iint\limits_Q e^{-2s\alpha}(\sum_{k,l=1}^3|\partial^2_{kl}\widetilde{\psi}|^2 + |\nabla\widetilde{\psi}|^2 + |\widetilde{\psi}|^2)dx\,dt\Bigg), \end{aligned} \tag{2.21}$$

for every $s \ge C$.

For equation (2.14), we use the classical Carleman inequality for the heat equation (see for example [6]): there exists $\widehat{\lambda}_3 > 0$ such that for any $\lambda > \widehat{\lambda}_3$ there exists $C(\lambda, \Omega, \omega_1, \|\overline{\theta}\|_{L^\infty(0,T;W^{3,\infty}(\Omega))}) > 0$ such that

$$\begin{aligned} J(s,\widetilde{\psi}) := s &\iint\limits_Q e^{-2s\alpha}\xi(|\widetilde{\psi}_t|^2 + \sum_{k,l=1}^3|\partial^2_{kl}\widetilde{\psi}|^2)dx\,dt + s^3\iint\limits_Q e^{-2s\alpha}\xi^3|\nabla\widetilde{\psi}|^2 dx\,dt \\ &+ s^5\iint\limits_Q e^{-2s\alpha}\xi^5|\widetilde{\psi}|^2 dx\,dt \le C\Bigg(s^2\iint\limits_Q e^{-2s\alpha}\xi^2\rho^2(|g_0|^2 + |\varphi_3|^2)dx\,dt \\ &+ s^2\iint\limits_Q e^{-2s\alpha}\xi^2|\rho'|^2|\rho|^{-2}|\widetilde{\psi}|^2 dx\,dt + s^5\int_0^T\!\!\int_{\omega_1} e^{-2s\alpha}\xi^5|\widetilde{\psi}|^2 dx\,dt\Bigg), \end{aligned} \tag{2.22}$$

for every $s \ge C$.

We choose λ_0 in Proposition 1 (and Proposition 2) to be $\lambda_0 := \max\{\widehat{\lambda}_0, \widehat{\lambda}_1, \widehat{\lambda}_2, \widehat{\lambda}_3\}$ and we fix $\lambda \ge \lambda_0$.

Combining inequalities (2.21) and (2.22), and taking into account that $s^2 e^{-2s\alpha}\xi^2\rho^2$ is bounded, the identities in (2.15), estimate (2.10) for w and $|\rho'| \le Cs(\xi^*)^{9/8}\rho$ we have

$$I(s,z) + J(s,\widetilde{\psi}) \le C\left(\|\rho\, g\|^2_{L^2(Q)^3} + \|\rho\, g_0\|^2_{L^2(Q)} + s^5 \int_0^T\int_{\omega_1} e^{-2s\alpha}\xi^5|\widetilde{\psi}|^2 dx\, dt \right.$$
$$+ \sum_{k=1,3}\left[s^{-1/2}\left\| e^{-s\alpha^*}(\xi^*)^{-1/8}\nabla\Delta z_k\right\|^2_{L^2(\Sigma)^3} + s^{-1/2}\left\| e^{-s\alpha^*}(\xi^*)^{-1/4}\nabla\Delta z_k\right\|^2_{H^{\frac14,\frac12}(\Sigma)^3}\right.$$
$$\left.\left. + s^7\int_0^T\int_{\omega_1} e^{-2s\alpha}\xi^7|z_k|^2 dx\, dt\right]\right), \quad (2.23)$$

for every $s \ge C$.

It remains to treat the boundary terms of this inequality and to eliminate the local term in z_3.

Estimate of the boundary terms. First, we treat the first boundary term in (2.23). Notice that, since α^* and ξ^* do not depend on x, we can readily get by integration by parts, for $k = 1, 3$,

$$\begin{aligned}
&\left\| e^{-s\alpha^*}\nabla\Delta z_k\right\|^2_{L^2(\Sigma)^3} \\
&\le C\left\| s^{1/2}e^{-s\alpha^*}(\xi^*)^{1/2}\nabla\Delta z_k\right\|_{L^2(Q)^3}\left\| s^{-1/2}e^{-s\alpha^*}(\xi^*)^{-1/2}\nabla\nabla\Delta z_k\right\|_{L^2(Q)^3} \\
&\le C\left(s\iint_Q e^{-2s\alpha^*}\xi^*|\nabla\Delta z_k|^2 dx\, dt + \frac{1}{s}\iint_Q e^{-2s\alpha^*}\frac{1}{\xi^*}|\nabla\nabla\Delta z_k|^2 dx\, dt\right),
\end{aligned}$$

so $\|e^{-s\alpha^*}\nabla\Delta z_k\|^2_{L^2(\Sigma)^3}$ is bounded by $I(s,z)$. On the other hand, we can bound the first boundary term as follows:

$$s^{-1/2}\left\| e^{-s\alpha^*}(\xi^*)^{-1/8}\nabla\Delta z_k\right\|^2_{L^2(\Sigma)^3} \le Cs^{-1/2}\left\| e^{-s\alpha^*}\nabla\Delta z_k\right\|^2_{L^2(\Sigma)^3}.$$

Therefore, the first boundary terms can be absorbed by taking s large enough.

Now we treat the second boundary term in the right-hand side of (2.23). We will use regularity estimates to prove that z_1 and z_3 multiplied by a certain weight function are regular enough. First, let us observe that from (2.15) and the regularity estimate (2.10) for w we readily have

$$\|s^2e^{-s\alpha^*}(\xi^*)^2\rho\,\varphi\|^2_{L^2(Q)^3} \le C\left(I(s,z) + \|\rho\, g\|^2_{L^2(Q)^3}\right). \quad (2.24)$$

We define now

$$\widetilde{z} := se^{-s\alpha^*}(\xi^*)^{7/8}z,\ \widetilde{\pi}_z := se^{-s\alpha^*}(\xi^*)^{7/8}\pi_z.$$

From (2.13) we see that $(\widetilde{z}, \widetilde{\pi}_z)$ is the solution of the Stokes system:

$$\begin{cases}
-\widetilde{z}_t - \Delta\widetilde{z} + \nabla\widetilde{\pi}_z = R_1 & \text{in } Q, \\
\nabla\cdot\widetilde{z} = 0 & \text{in } Q, \\
\widetilde{z} = 0 & \text{on } \Sigma, \\
\widetilde{z}(T) = 0 & \text{in } \Omega,
\end{cases} \quad (2.25)$$

where $R_1 := -se^{-s\alpha^*}(\xi^*)^{7/8}\rho'\varphi - se^{-s\alpha^*}(\xi^*)^{7/8}\widetilde{\psi}\nabla\bar{\theta} - (se^{-s\alpha^*}(\xi^*)^{7/8})_t z$. Taking into account that $|\alpha_t^*| \le C(\xi^*)^{9/8}$, $|\rho'| \le Cs(\xi^*)^{9/8}\,\rho$, (1.6) and (2.24) we have

$$\|R_1\|^2_{L^2(Q)^3} \le C\left(I(s,z) + J(s,\widetilde{\psi}) + \|\rho g\|^2_{L^2(Q)^3}\right),$$

and therefore, by the regularity estimate (2.10) applied to (2.25), we obtain

$$\|\widetilde{z}\|^2_{L^2(0,T;H^2(\Omega)^3)\cap H^1(0,T;L^2(\Omega)^3)} \le C\left(I(s,z) + J(s,\widetilde{\psi}) + \|\rho\, g\|^2_{L^2(Q)^3}\right). \tag{2.26}$$

Next, let

$$\widehat{z} := e^{-s\alpha^*}(\xi^*)^{-1/4} z,\ \widehat{\pi}_z := e^{-s\alpha^*}(\xi^*)^{-1/4}\pi_z.$$

From (2.13), $(\widehat{z}, \widehat{\pi}_z)$ is the solution of the Stokes system:

$$\begin{cases} -\widehat{z}_t - \Delta\widehat{z} + \nabla\widehat{\pi}_z = R_2 & \text{in } Q,\\ \nabla\cdot\widehat{z} = 0 & \text{in } Q,\\ \widehat{z} = 0 & \text{on } \Sigma,\\ \widehat{z}(T) = 0 & \text{in } \Omega, \end{cases} \tag{2.27}$$

where $R_2 := -e^{-s\alpha^*}(\xi^*)^{-1/4}\rho'\varphi - e^{-s\alpha^*}(\xi^*)^{-1/4}\widetilde{\psi}\nabla\bar{\theta} - (e^{-s\alpha^*}(\xi^*)^{-1/4})_t z$. By the same arguments as before, and thanks to (2.26), we can easily prove that $R_2 \in L^2(0,T;H^2(\Omega)^3)\cap H^1(0,T;L^2(\Omega)^3)$ (for the first term in R_2, we use again (2.15) and (2.26)) and furthermore

$$\|R_2\|^2_{L^2(0,T;H^2(\Omega)^3)\cap H^1(0,T;L^2(\Omega)^3)} \le C\left(I(s,z) + J(s,\widetilde{\psi}) + \|\rho\, g\|^2_{L^2(Q)^3}\right).$$

By the regularity estimate (2.11) applied to (2.27), we have

$$\|\widehat{z}\|^2_{L^2(0,T;H^4(\Omega)^3)\cap H^1(0,T;H^2(\Omega)^3)} \le C\left(I(s,z) + J(s,\widetilde{\psi}) + \|\rho\, g\|^2_{L^2(Q)^3}\right).$$

In particular, $e^{-s\alpha^*}(\xi^*)^{-1/4}\nabla\Delta z_k \in L^2(0,T;H^1(\Omega)^3)\cap H^1(0,T;H^{-1}(\Omega)^3)$ for $k = 1,3$ and

$$\sum_{k=1,3} \|e^{-s\alpha^*}(\xi^*)^{-1/4}\nabla\Delta z_k\|^2_{L^2(0,T;H^1(\Omega)^3)} + \|e^{-s\alpha^*}(\xi^*)^{-1/4}\nabla\Delta z_k\|^2_{H^1(0,T;H^{-1}(\Omega)^3)}$$
$$\le C\left(I(s,z) + J(s,\widetilde{\psi}) + \|\rho\, g\|^2_{L^2(Q)^3}\right). \tag{2.28}$$

To end this part, we use a trace inequality to estimate the second boundary term in the right-hand side of (2.23):

$$\sum_{k=1,3} s^{-1/2}\left\|e^{-s\alpha^*}(\xi^*)^{-1/4}\nabla\Delta z_k\right\|^2_{H^{\frac14,\frac12}(\Sigma)^3}$$
$$\le C\, s^{-1/2}\sum_{k=1,3}\left[\left\|e^{-s\alpha^*}(\xi^*)^{-1/4}\nabla\Delta z_k\right\|^2_{L^2(0,T;H^1(\Omega)^3)} + \left\|e^{-s\alpha^*}(\xi^*)^{-1/4}\nabla\Delta z_k\right\|^2_{H^1(0,T;H^{-1}(\Omega)^3)}\right],$$

By taking s large enough in (2.23), the boundary terms $s^{-1/2}\|e^{-s\alpha}\xi^{-1/4}\nabla\Delta z_k\|^2_{H^{\frac14,\frac12}(\Sigma)^3}$ can be absorbed by the terms in the left-hand side of (2.28).

Thus, using (2.15) and (2.10) for w in the right-hand side of (2.23), we have for the moment

$$I(s,z)+J(s,\widetilde{\psi})\leq C\left(\|\rho\, g\|^2_{L^2(Q)^3}+\|\rho\, g_0\|^2_{L^2(Q)}+s^5\int_0^T\int_{\omega_1}e^{-2s\alpha}\xi^5|\widetilde{\psi}|^2dx\,dt\right.$$
$$\left.+s^7\int_0^T\int_{\omega_1}e^{-2s\alpha}\xi^7\rho^2|\varphi_1|^2dx\,dt+s^7\int_0^T\int_{\omega_1}e^{-2s\alpha}\xi^7\rho^2|\varphi_3|^2dx\,dt\right),$$

for every $s\geq C$. Furthermore, notice that using again (2.15), (2.10) for w and (2.26) we obtain from the previous inequality

$$s^2\iint_Q e^{-2s\alpha^*}(\xi^*)^{7/4}\rho^2|\varphi_{3,t}|^2dx\,dt+\widetilde{I}(s,\rho\,\varphi)+J(s,\widetilde{\psi})$$
$$\leq C\left(\|\rho\, g\|^2_{L^2(Q)^3}+\|\rho\, g_0\|^2_{L^2(Q)^3}+s^5\int_0^T\int_{\omega_1}e^{-2s\alpha}\xi^5|\widetilde{\psi}|^2dx\,dt\right.$$
$$\left.+s^7\int_0^T\int_{\omega_1}e^{-2s\alpha}\xi^7\rho^2|\varphi_1|^2dx\,dt+s^7\int_0^T\int_{\omega_1}e^{-2s\alpha}\xi^7\rho^2|\varphi_3|^2dx\,dt\right),\quad (2.29)$$

for every $s\geq C$, where

$$\widetilde{I}(s,\rho\,\varphi):=\sum_{k=1,3}\left[s^3\iint_Q e^{-2s\alpha}\xi^3\rho^2|\Delta\varphi_k|^2dx\,dt+s^4\iint_Q e^{-2s\alpha}\xi^4\rho^2|\nabla\varphi_k|^2dx\,dt\right.$$
$$\left.+s^6\iint_Q e^{-2s\alpha}\xi^6\rho^2|\varphi_k|^2dx\,dt\right]+s^4\iint_Q e^{-2s\alpha^*}(\xi^*)^4\rho^2|\varphi_2|^2dx\,dt.$$

Estimate of φ_3. We deal in this part with the last term in the right-hand side of (2.29). We introduce a function $\zeta_1\in C^2_0(\omega)$ such that $\zeta_1\geq 0$ and $\zeta_1=1$ in ω_1, and using equation (2.14) we have

$$C\,s^7\int_0^T\int_{\omega_1}e^{-2s\alpha}\xi^7\rho^2|\varphi_3|^2dx\,dt\leq Cs^7\int_0^T\int_{\omega}\zeta_1 e^{-2s\alpha}\xi^7\rho^2|\varphi_3|^2dx\,dt$$
$$=Cs^7\int_0^T\int_{\omega}\zeta_1 e^{-2s\alpha}\xi^7\rho\,\varphi_3(-\widetilde{\psi}_t-\Delta\widetilde{\psi}-\rho\, g_0+\rho'\psi)dx\,dt,$$

and we integrate by parts in this last term, in order to estimate it by local integrals of $\widetilde{\psi}$, g_0 and $\epsilon\, I(s,\rho\,\varphi)$. This approach was already introduced in [5].

We first integrate by parts in time taking into account that

$e^{-2s\alpha(0)}\xi^7(0) = e^{-2s\alpha(T)}\xi^7(T) = 0$:

$$-Cs^7 \int_0^T \int_\omega \zeta_1 e^{-2s\alpha} \xi^7 \rho\, \varphi_3 \widetilde{\psi}_t dx\, dt$$
$$= Cs^7 \int_0^T \int_\omega \zeta_1 e^{-2s\alpha} \xi^7 \rho\, \varphi_{3,t} \widetilde{\psi} dx\, dt + Cs^7 \int_0^T \int_\omega \zeta_1 (e^{-2s\alpha} \xi^7 \rho)_t\, \varphi_3 \widetilde{\psi} dx\, dt$$
$$\leq \epsilon \left(s^2 \int_0^T \int_\omega e^{-2s\alpha^*} (\xi^*)^{7/4} \rho^2 |\varphi_{3,t}|^2 dx\, dt + \widetilde{I}(s, \rho\, \varphi) \right)$$
$$+ C(\lambda, \epsilon) \left(s^{12} \int_0^T \int_\omega e^{-4s\alpha + 2s\alpha^*} \xi^{49/4} |\widetilde{\psi}|^2 dx\, dt + s^{10} \int_0^T \int_\omega e^{-2s\alpha} \xi^{41/4} |\widetilde{\psi}|^2 dx\, dt \right),$$

where we have used that
$$|(e^{-2s\alpha} \xi^7 \rho)_t| \leq Cse^{-2s\alpha} \xi^{65/8} \rho$$
and Young's inequality. Now we integrate by parts in space:

$$-Cs^7 \int_0^T \int_\omega \zeta_1 e^{-2s\alpha} \xi^7 \rho\, \varphi_3 \Delta \widetilde{\psi} dx\, dt = -Cs^7 \int_0^T \int_\omega \zeta_1 e^{-2s\alpha} \xi^7 \rho\, \Delta \varphi_3 \widetilde{\psi} dx\, dt$$
$$- 2Cs^7 \int_0^T \int_\omega \nabla(\zeta_1 e^{-2s\alpha} \xi^7) \cdot \rho\, \nabla \varphi_3 \widetilde{\psi} dx\, dt - C\, s^7 \int_0^T \int_\omega \Delta(\zeta_1 e^{-2s\alpha} \xi^7) \rho\, \varphi_3 \widetilde{\psi} dx\, dt$$
$$\leq \epsilon\, \widetilde{I}(s, \rho\, \varphi) + C(\epsilon)\, s^{12} \int_0^T \int_\omega e^{-2s\alpha} \xi^{12} |\widetilde{\psi}|^2 dx\, dt,$$

where we have used that

$$\nabla(\zeta_1 e^{-2s\alpha} \xi^7) \leq Cse^{-2s\alpha} \xi^8 \text{ and } \Delta(\zeta_1 e^{-2s\alpha} \xi^7) \leq Cs^2 e^{-2s\alpha} \xi^9,$$

and Young's inequality.

Finally,

$$Cs^7 \int_0^T \int_\omega \zeta_1 e^{-2s\alpha} \xi^7 \rho\, \varphi_3 (-\rho\, g_0 + \rho'\, \psi) dx\, dt$$
$$\leq Cs^7 \int_0^T \int_\omega \zeta_1 e^{-2s\alpha} \xi^7 \rho\, |\varphi_3| (\rho |g_0| + Cs\xi^{9/8}\, |\widetilde{\psi}|) dx\, dt$$
$$\leq \epsilon\, \widetilde{I}(s, \rho\, \varphi) + C(\epsilon) \left(s^8 \int_0^T \int_\omega e^{-2s\alpha} \xi^8 \rho^2\, |g_0|^2 dx\, dt + s^{10} \int_0^T \int_\omega e^{-2s\alpha} \xi^{41/4} |\widetilde{\psi}|^2 dx\, dt \right).$$

Setting $\epsilon = 1/2$ and noticing that

$$e^{-2s\alpha} \leq e^{-4s\alpha + 2s\alpha^*} \text{ in } Q,$$

(see (2.3)) we obtain (2.4) from (2.29). This completes the proof of Proposition 1.

3 Null controllability of the linear system

Here we are concerned with the null controllability of the system

$$\left\{\begin{array}{ll} Ly + \nabla p = f + (v_1, 0, 0)\mathbb{1}_\omega + \theta\, e_3 & \text{in } Q, \\ L\theta + y\cdot\nabla\bar\theta = f_0 + v_0\mathbb{1}_\omega & \text{in } Q, \\ \nabla\cdot y = 0 & \text{in } Q, \\ y = 0,\ \theta = 0 & \text{on } \Sigma, \\ y(0) = y^0,\ \theta(0) = \theta^0 & \text{in } \Omega, \end{array}\right. \tag{3.1}$$

where $y^0 \in V$, $\theta^0 \in H_0^1(\Omega)$, f and f_0 are in appropriate weighted spaces, the controls v_0 and v_1 are in $L^2(\omega\times(0,T))$ and

$$Lq = q_t - \Delta q.$$

Before dealing with the null controllability of (3.1), we will deduce a Carleman inequality with weights not vanishing at $t = 0$. To this end, let us introduce the following weight functions:

$$\begin{aligned} &\beta(x,t) = \frac{e^{2\lambda\|\eta\|_\infty} - e^{\lambda\eta(x)}}{\tilde\ell^8(t)},\ \gamma(x,t) = \frac{e^{\lambda\eta(x)}}{\tilde\ell^8(t)}, \\ &\beta^*(t) = \max_{x\in\overline\Omega}\beta(x,t),\ \gamma^*(t) = \min_{x\in\overline\Omega}\gamma(x,t), \\ &\widehat\beta(t) = \min_{x\in\overline\Omega}\beta(x,t),\ \widehat\gamma(t) = \max_{x\in\overline\Omega}\gamma(x,t), \end{aligned} \tag{3.2}$$

where

$$\tilde\ell(t) = \left\{\begin{array}{ll} \|\ell\|_\infty & 0\le t\le T/2, \\ \ell(t) & T/2 < t \le T. \end{array}\right.$$

Lemma 3.1. *Assume $N = 3$. Let s and λ be like in Proposition 1 and $(\bar p, \bar\theta)$ satisfy* (1.5)-(1.6)*. Then, there exists a constant $C > 0$ (depending on s, λ and $\bar\theta$) such that every solution (φ, π, ψ) of* (1.9) *satisfies:*

$$\begin{aligned} &\iint\limits_Q e^{-5s\beta^*}(\gamma^*)^4|\varphi|^2 dx\,dt + \iint\limits_Q e^{-5s\beta^*}(\gamma^*)^5|\psi|^2 dx\,dt + \|\varphi(0)\|^2_{L^2(\Omega)^3} + \|\psi(0)\|^2_{L^2(\Omega)} \\ &\le C\left(\iint\limits_Q e^{-3s\beta^*}(|g|^2 + |g_0|^2)dx\,dt + \int_0^T\!\!\int_\omega e^{-2s\widehat\beta - 3s\beta^*}\widehat\gamma^7|\varphi_1|^2 dx\,dt \right.\\ &\qquad\left. + \int_0^T\!\!\int_\omega e^{-4s\widehat\beta - s\beta^*}\widehat\gamma^{49/4}|\psi|^2 dx\,dt\right). \end{aligned} \tag{3.3}$$

Let us also state this result for $N = 2$.

Lemma 3.2. *Assume $N = 2$. Let s and λ be like in Proposition 2 and $(\bar p, \bar\theta)$ satisfy* (1.5)-(1.6)*. Then, there exists a constant $C > 0$ (depending on s, λ and $\bar\theta$) such that every solution (φ, π, ψ) of* (1.9) *satisfies:*

$$\begin{aligned} &\iint\limits_Q e^{-5s\beta^*}(\gamma^*)^4|\varphi|^2 dx\,dt + \iint\limits_Q e^{-5s\beta^*}(\gamma^*)^5|\psi|^2 dx\,dt + \|\varphi(0)\|^2_{L^2(\Omega)^2} + \|\psi(0)\|^2_{L^2(\Omega)} \\ &\le C\left(\iint\limits_Q e^{-3s\beta^*}(|g|^2 + |g_0|^2)dx\,dt + \int_0^T\!\!\int_\omega e^{-4s\widehat\beta - s\beta^*}\widehat\gamma^{49/4}|\psi|^2 dx\,dt\right). \end{aligned} \tag{3.4}$$

Proof of Lemma 3.1: We start by an a priori estimate for system (1.9). To do this, we introduce a function $\nu \in C^1([0,T])$ such that

$$\nu \equiv 1 \text{ in } [0,T/2],\ \nu \equiv 0 \text{ in } [3T/4,T].$$

We easily see that $(\nu\varphi, \nu\pi, \nu\psi)$ satisfies

$$\begin{cases} -(\nu\varphi)_t - \Delta(\nu\varphi) + \nabla(\nu\pi) = \nu\, g - (\nu\psi)\nabla\bar{\theta} - \nu'\varphi & \text{in } Q, \\ -(\nu\psi)_t - \Delta(\nu\psi) = \nu\, g_0 + \nu\,\varphi_3 - \nu'\psi & \text{in } Q, \\ \nabla\cdot(\nu\varphi) = 0 & \text{in } Q, \\ (\nu\varphi) = 0,\ (\nu\psi) = 0 & \text{on } \Sigma, \\ (\nu\varphi)(T) = 0,\ (\nu\psi)(T) = 0 & \text{in } \Omega, \end{cases}$$

thus we have the energy estimate

$$\begin{aligned} &\|\nu\varphi\|^2_{L^2(0,T;V)} + \|\nu\varphi\|^2_{L^\infty(0,T;H)} + \|\nu\psi\|^2_{L^2(0,T;H^1(\Omega))} + \|\nu\psi\|^2_{L^\infty(0,T;L^2(\Omega))} \\ &\qquad \le C(\|\nu g\|^2_{L^2(Q)^3} + \|\nu'\varphi\|^2_{L^2(Q)^3} + \|\nu g_0\|^2_{L^2(Q)} + \|\nu'\psi\|^2_{L^2(Q)}). \end{aligned}$$

Using the properties of the function ν, we readily obtain

$$\begin{aligned} &\|\varphi\|^2_{L^2(0,T/2;H)} + \|\varphi(0)\|^2_{L^2(\Omega)^3} + \|\psi\|^2_{L^2(0,T/2;L^2(\Omega))} + \|\psi(0)\|^2_{L^2(\Omega)} \\ &\qquad \le C\Big(\|g\|^2_{L^2(0,3T/4;L^2(\Omega)^3)} + \|\varphi\|^2_{L^2(T/2,3T/4;L^2(\Omega)^3)} \\ &\qquad\qquad + \|g_0\|^2_{L^2(0,3T/4;L^2(\Omega))} + \|\psi\|^2_{L^2(T/2,3T/4;L^2(\Omega))}\Big). \end{aligned}$$

From this last inequality, and the fact that

$$e^{-3s\beta^*} \ge C > 0,\ \forall t \in [0,3T/4] \text{ and } e^{-5s\alpha^*}(\xi^*)^4 \ge C > 0,\ \forall t \in [T/2,3T/4]$$

we have

$$\begin{aligned} &\int_0^{T/2}\int_\Omega e^{-5s\beta^*}(\gamma^*)^4|\varphi|^2 dx\, dt + \int_0^{T/2}\int_\Omega e^{-5s\beta^*}(\gamma^*)^5|\psi|^2 dx\, dt \\ &\quad + \|\varphi(0)\|^2_{L^2(\Omega)^3} + \|\psi(0)\|^2_{L^2(\Omega)} \le C\left(\int_0^{3T/4}\int_\Omega e^{-3s\beta^*}(|g|^2 + |g_0|^2) dx\, dt\right. \\ &\quad \left. + \int_{T/2}^{3T/4}\int_\Omega e^{-5s\alpha^*}(\xi^*)^4|\varphi|^2 dx\, dt + \int_{T/2}^{3T/4}\int_\Omega e^{-5s\alpha^*}(\xi^*)^5|\psi|^2 dx\, dt\right). \end{aligned} \tag{3.5}$$

Note that the last two terms in (3.5) are bounded by the left-hand side of the Carleman inequality (2.4). Since $\alpha = \beta$ in $\Omega \times (T/2,T)$, we have:

$$\begin{aligned} &\int_{T/2}^{T}\int_\Omega e^{-5s\beta^*}(\gamma^*)^4|\varphi|^2 dx\, dt + \int_{T/2}^{T}\int_\Omega e^{-5s\beta^*}(\gamma^*)^5|\psi|^2 dx\, dt \\ &= \int_{T/2}^{T}\int_\Omega e^{-5s\alpha^*}(\xi^*)^4|\varphi|^2 dx\, dt + \int_{T/2}^{T}\int_\Omega e^{-5s\alpha^*}(\xi^*)^5|\psi|^2 dx\, dt \\ &\quad \le \iint_Q e^{-5s\alpha^*}(\xi^*)^4|\varphi|^2 dx\, dt + \iint_Q e^{-5s\alpha^*}(\xi^*)^5|\psi|^2 dx\, dt. \end{aligned}$$

Combining this with the Carleman inequality (2.4), we deduce

$$\int_{T/2}^{T}\int_{\Omega} e^{-5s\beta^*}(\gamma^*)^4|\varphi|^2dx\,dt + \int_{T/2}^{T}\int_{\Omega} e^{-5s\beta^*}(\gamma^*)^5|\psi|^2dx\,dt$$
$$\leq C\left(\iint_Q e^{-3s\alpha^*}(|g|^2+|g_0|^2)dx\,dt + \int_0^T\int_\omega e^{-2s\widehat{\alpha}-3s\alpha^*}(\widehat{\xi})^7|\varphi_1|^2dx\,dt + \int_0^T\int_\omega e^{-4s\widehat{\alpha}-s\alpha^*}(\widehat{\xi})^{49/4}|\psi|^2dx\,dt\right).$$

Since

$$e^{-3s\beta^*}, e^{-2s\widehat{\beta}-3s\beta^*}\widehat{\gamma}^7, e^{-2s\widehat{\beta}-3s\beta^*}\widehat{\gamma}^7, e^{-4s\widehat{\beta}-s\beta^*}\widehat{\gamma}^{49/4} \geq C > 0,\ \forall t \in [0, T/2],$$

we can readily get

$$\int_{T/2}^{T}\int_{\Omega} e^{-5s\beta^*}(\gamma^*)^4|\varphi|^2dx\,dt + \int_{T/2}^{T}\int_{\Omega} e^{-5s\beta^*}(\gamma^*)^5|\psi|^2dx\,dt$$
$$\leq C\left(\iint_Q e^{-3s\beta^*}(|g|^2+|g_0|^2)dx\,dt + \int_0^T\int_\omega e^{-2s\widehat{\beta}-3s\beta^*}\widehat{\gamma}^7|\varphi_1|^2dx\,dt + \int_0^T\int_\omega e^{-4s\widehat{\beta}-s\beta^*}\widehat{\gamma}^{49/4}|\psi|^2dx\,dt\right),$$

which, together with (3.5), yields (3.3). □

Now we will prove the null controllability of (3.1). Actually, we will prove the existence of a solution for this problem in an appropriate weighted space. Let us introduce the space

$$\begin{aligned}
E = \{\,(y,p,v_1,\theta,v_0) : \ & e^{3/2s\beta^*}\,y,\ e^{s\widehat{\beta}+3/2s\beta^*}\widehat{\gamma}^{-7/2}\,(v_1,0,0)\mathbb{1}_\omega \in L^2(Q)^3,\\
& e^{3/2s\beta^*}\,\theta,\ e^{2s\widehat{\beta}+1/2s\beta^*}\widehat{\gamma}^{-49/8}\,v_0\mathbb{1}_\omega \in L^2(Q),\\
& e^{3/2s\beta^*}(\gamma^*)^{-9/8}y \in L^2(0,T;H^2(\Omega)^3)\cap L^\infty(0,T;V),\\
& e^{3/2s\beta^*}(\gamma^*)^{-9/8}\theta \in L^2(0,T;H^2(\Omega))\cap L^\infty(0,T;H_0^1(\Omega)),\\
& e^{5/2s\beta^*}(\gamma^*)^{-2}(Ly+\nabla p-\theta e_3-(v_1,0,0)\mathbb{1}_\omega) \in L^2(Q)^3,\\
& e^{5/2s\beta^*}(\gamma^*)^{-5/2}(L\theta + y\cdot\nabla\bar{\theta} - v_0\mathbb{1}_\omega) \in L^2(Q)\,\}.
\end{aligned}$$

It is clear that E is a Banach space for the following norm:

$$\begin{aligned}
\|(y,p,v_1,\theta,v_0)\|_E = \Big(&\|e^{3/2s\beta^*}\,y\|^2_{L^2(Q)^3} + \|e^{s\widehat{\beta}+3/2s\beta^*}\widehat{\gamma}^{-7/2}\,v_1\mathbb{1}_\omega\|^2_{L^2(Q)}\\
&+\|e^{3/2s\beta^*}\,\theta\|^2_{L^2(Q)} + \|e^{2s\widehat{\beta}+1/2s\beta^*}\widehat{\gamma}^{-49/8}\,v_0\mathbb{1}_\omega\|^2_{L^2(Q)}\\
&+\|e^{3/2s\beta^*}(\gamma^*)^{-9/8}\,y\|^2_{L^2(0,T;H^2(\Omega)^3)} + \|e^{3/2s\beta^*}(\gamma^*)^{-9/8}y\|^2_{L^\infty(0,T;V)}\\
&+\|e^{3/2s\beta^*}(\gamma^*)^{-9/8}\,\theta\|^2_{L^2(0,T;H^2(\Omega))} + \|e^{3/2s\beta^*}(\gamma^*)^{-9/8}\theta\|^2_{L^\infty(0,T;H_0^1)}\\
&+\|e^{5/2s\beta^*}(\gamma^*)^{-2}(Ly+\nabla p-\theta e_3-(v_1,0,0)\mathbb{1}_\omega)\|^2_{L^2(Q)^3}\\
&+\|e^{5/2s\beta^*}(\gamma^*)^{-5/2}(L\theta+y\cdot\nabla\bar{\theta}-v_0\mathbb{1}_\omega)\|^2_{L^2(Q)}\Big)^{1/2}
\end{aligned}$$

Remark 4. Observe in particular that $(y, p, v_1, \theta, v_0) \in E$ implies $y(T) = 0$ and $\theta(T) = 0$ in Ω. Moreover, the functions belonging to this space posses the interesting following property:

$$e^{5/2s\beta^*}(\gamma^*)^{-2}(y \cdot \nabla)y \in L^2(Q)^3 \text{ and } e^{5/2s\beta^*}(\gamma^*)^{-5/2}y \cdot \nabla\theta \in L^2(Q).$$

Proposition 3. *Assume $N = 3$, $(\bar{p}, \bar{\theta})$ satisfies* (1.5)-(1.6) *and*

$$y^0 \in V,\ \theta_0 \in H_0^1(\Omega),\ e^{5/2s\beta^*}(\gamma^*)^{-2}f \in L^2(Q)^3 \text{ and } e^{5/2s\beta^*}(\gamma^*)^{-5/2}f_0 \in L^2(Q).$$

Then, we can find controls v_1 and v_0 such that the associated solution (y, p, θ) to (3.1) *satisfies*
$(y, p, v_1, \theta, v_0) \in E$. In particular, $y(T) = 0$ and $\theta(T) = 0$.

Sketch of the proof: The proof of this proposition is very similar to the one of Proposition 2 in [9] (see also Proposition 2 in [4] and Proposition 3.3 in [2]), so we will just give the main ideas.

Following the arguments in [6] and [10], we introduce the space

$$P_0 = \{\, (\chi, \sigma, \kappa) \in C^2(\overline{Q})^5 : \nabla \cdot \chi = 0,\ \chi = 0 \text{ on } \Sigma,\ \kappa = 0 \text{ on } \Sigma\}$$

and we consider the following variational problem: find $(\widehat{\chi}, \widehat{\sigma}, \widehat{\kappa}) \in P_0$ such that

$$a((\widehat{\chi}, \widehat{\sigma}, \widehat{\kappa}), (\chi, \sigma, \kappa)) = \langle G, (\chi, \sigma, \kappa)\rangle \quad \forall(\chi, \sigma, \kappa) \in P_0, \tag{3.6}$$

where we have used the notations

$$a((\widehat{\chi}, \widehat{\sigma}, \widehat{\kappa}), (\chi, \sigma, \kappa)) = \iint_Q e^{-3s\beta^*}\, (L^*\widehat{\chi} + \nabla\widehat{\sigma} + \widehat{\kappa}\nabla\bar{\theta}) \cdot (L^*\chi + \nabla\sigma + \kappa\nabla\bar{\theta})\, dx\, dt$$

$$+ \iint_Q e^{-3s\beta^*}\, (L^*\widehat{\kappa} - \widehat{\chi}_3)(L^*\kappa - \chi_3)\, dx\, dt + \int_0^T\!\!\int_\omega e^{-2s\widehat{\beta} - 3s\beta^*}\,\widehat{\gamma}^7\, \widehat{\chi}_1\, \chi_1\, dx\, dt$$

$$+ \int_0^T\!\!\int_\omega e^{-4s\widehat{\beta} - s\beta^*}\,\widehat{\gamma}^{49/4}\, \widehat{\kappa}\, \kappa\, dx\, dt,$$

$$\langle G, (\chi, \sigma, \kappa)\rangle = \iint_Q f \cdot \chi\, dx\, dt + \iint_Q f_0\, \kappa\, dx\, dt + \int_\Omega y^0 \cdot \chi(0)\, dx + \int_\Omega \theta^0\, \kappa(0)\, dx$$

and L^* is the adjoint operator of L, i.e.

$$L^* q = -q_t - \Delta q.$$

It is clear that $a(\cdot\,, \cdot\,, \cdot) : P_0 \times P_0 \mapsto \mathbf{R}$ is a symmetric, definite positive bilinear form on P_0. We denote by P the completion of P_0 for the norm induced by $a(\cdot\,, \cdot\,, \cdot)$. Then $a(\cdot\,, \cdot\,, \cdot)$ is well-defined, continuous and again definite positive on P. Furthermore, in view of the Carleman estimate (3.3), the linear form $(\chi, \sigma, \kappa) \mapsto \langle G, (\chi, \sigma, \kappa)\rangle$ is well-defined and continuous on P. Hence, from Lax-Milgram's lemma, we deduce that the variational problem

$$\begin{cases} a((\widehat{\chi}, \widehat{\sigma}, \widehat{\kappa}), (\chi, \sigma, \kappa)) = \langle G, (\chi, \sigma, \kappa)\rangle \\ \forall(\chi, \sigma, \kappa) \in P, \quad (\widehat{\chi}, \widehat{\sigma}, \widehat{\kappa}) \in P, \end{cases}$$

possesses exactly one solution $(\widehat{\chi}, \widehat{\sigma}, \widehat{\kappa})$.

Let $\widehat{y}$, $\widehat{v}_1$, $\widehat{\theta}$ and $\widehat{v}_0$ be given by

$$\begin{cases} \widehat{y} = e^{-3s\beta^*}(L^*\widehat{\chi} + \nabla\widehat{\sigma} + \widehat{\kappa}\nabla\bar{\theta}), & \text{in } Q, \\ \widehat{v}_1 = -e^{-2s\widehat{\beta}-3s\beta^*}\widehat{\gamma}^7\,\widehat{\chi}_1, & \text{in } \omega\times(0,T), \\ \widehat{\theta} = e^{-3s\beta^*}(L^*\widehat{\kappa} - \widehat{\chi}_3), & \text{in } Q, \\ \widehat{v}_0 = -e^{-4s\widehat{\beta}-s\beta^*}\widehat{\gamma}^{49/4}\,\widehat{\kappa}, & \text{in } \omega\times(0,T). \end{cases}$$

Then, it is readily seen that they satisfy

$$\iint_Q e^{3s\beta^*}|\widehat{y}|^2dx\,dt + \iint_Q e^{3s\beta^*}|\widehat{\theta}|^2dx\,dt + \int_0^T\!\!\int_\omega e^{2s\widehat{\beta}+3s\beta^*}\widehat{\gamma}^{-7}|\widehat{v}_1|^2dx\,dt$$
$$+ \int_0^T\!\!\int_\omega e^{4s\widehat{\beta}+s\beta^*}\widehat{\gamma}^{-49/4}|\widehat{v}_0|^2dx\,dt = a((\widehat{\chi},\widehat{\sigma},\widehat{\kappa}),(\widehat{\chi},\widehat{\sigma},\kappa)) < +\infty$$

and also that $(\widehat{y},\widehat{\theta})$ is, together with some pressure $\widehat{p}$, the weak solution of the system (3.1) for $v_1 = \widehat{v}_1$ and $v_0 = \widehat{v}_0$.

It only remains to check that

$$e^{3/2s\beta^*}(\gamma^*)^{-9/8}\widehat{y} \in L^2(0,T;H^2(\Omega)^3) \cap L^\infty(0,T;V)$$

and

$$e^{3/2s\beta^*}(\gamma^*)^{-9/8}\widehat{\theta} \in L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;H_0^1(\Omega))$$

To this end, we define the functions

$$y^* = e^{3/2s\beta^*}(\gamma^*)^{-9/8}\,\widehat{y},\ p^* = e^{3/2s\beta^*}(\gamma^*)^{-9/8}\,\widehat{p},\ \theta^* = e^{3/2s\beta^*}(\gamma^*)^{-9/8}\,\widehat{\theta}$$

$$f^* = e^{3/2s\beta^*}(\gamma^*)^{-9/8}(f + (\widehat{v}_1,0,0)\mathbb{1}_\omega) \text{ and } f_0^* = e^{3/2s\beta^*}(\gamma^*)^{-9/8}(f_0 + \widehat{v}_0\mathbb{1}_\omega).$$

Then (y^*,p^*,θ^*) satisfies

$$\begin{cases} Ly^* + \nabla p^* = f^* + \theta^*\,e_3 + (e^{3/2s\beta^*}(\gamma^*)^{-9/8})_t\widehat{y} & \text{in } Q, \\ L\theta^* + y^*\cdot\nabla\bar{\theta} = f_0^* + (e^{3/2s\beta^*}(\gamma^*)^{-9/8})_t\widehat{\theta} & \text{in } Q, \\ \nabla\cdot y^* = 0 & \text{in } Q, \\ y^* = 0,\ \theta^* = 0 & \text{on } \Sigma, \\ y^*(0) = e^{3/2s\beta^*(0)}(\gamma^*(0))^{-9/8}y^0, & \text{in } \Omega, \\ \theta^*(0) = e^{3/2s\beta^*(0)}(\gamma^*(0))^{-9/8}\theta^0, & \text{in } \Omega. \end{cases}$$

From the fact that $f^* + (e^{3/2s\beta^*}(\gamma^*)^{-9/8})_t\,\widehat{y} \in L^2(Q)^3$, $f_0^* + (e^{3/2s\beta^*}(\gamma^*)^{-9/8})_t\widehat{\theta} \in L^2(Q)$, $y^0 \in V$ and $\theta^0 \in H_0^1(\Omega)$, we have indeed

$$y^* \in L^2(0,T;H^2(\Omega)^3) \cap L^\infty(0,T;V) \text{ and } \theta^* \in L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;H_0^1(\Omega))$$

(see (2.10)). This ends the sketch of the proof of Proposition 3. □

4 Proof of Theorem 1.1

In this section we give the proof of Theorem 1.1 using similar arguments to those in [10] (see also [4], [5], [9] and [2]). The result of null controllability for the linear system (3.1) given by Proposition 3 will allow us to apply an inverse mapping theorem. Namely, we will use the following theorem (see [1]).

Theorem 4.1. *Let B_1 and B_2 be two Banach spaces and let $\mathcal{A} : B_1 \to B_2$ satisfy $\mathcal{A} \in C^1(B_1; B_2)$. Assume that $b_1 \in B_1$, $\mathcal{A}(b_1) = b_2$ and that $\mathcal{A}'(b_1) : B_1 \to B_2$ is surjective. Then, there exists $\delta > 0$ such that, for every $b' \in B_2$ satisfying $\|b' - b_2\|_{B_2} < \delta$, there exists a solution of the equation*

$$\mathcal{A}(b) = b', \quad b \in B_1.$$

Let us set

$$y = \widetilde{y},\ p = \bar{p} + \widetilde{p} \text{ and } \theta = \bar{\theta} + \widetilde{\theta}.$$

Using (1.1) and (1.5) we obtain

$$\begin{cases} \widetilde{y}_t - \Delta \widetilde{y} + (\widetilde{y} \cdot \nabla)\widetilde{y} + \nabla \widetilde{p} = v \mathbb{1}_\omega + \widetilde{\theta}\, e_N & \text{in } Q, \\ \widetilde{\theta}_t - \Delta \widetilde{\theta} + \widetilde{y} \cdot \nabla \widetilde{\theta} + \widetilde{y} \cdot \nabla \bar{\theta} = v_0 \mathbb{1}_\omega & \text{in } Q, \\ \nabla \cdot \widetilde{y} = 0 & \text{in } Q, \\ \widetilde{y} = 0,\ \widetilde{\theta} = 0 & \text{on } \Sigma, \\ \widetilde{y}(0) = y^0,\ \widetilde{\theta}(0) = \theta^0 - \bar{\theta}^0 & \text{in } \Omega. \end{cases} \tag{4.1}$$

Thus, we have reduced our problem to the local null controllability of the nonlinear system (4.1).

We apply Theorem 4.1 setting

$$B_1 = E,$$

$$B_2 = L^2(e^{5/2s\beta^*}(\gamma^*)^{-2}(0,T); L^2(\Omega)^3) \times V \times L^2(e^{5/2s\beta^*}(\gamma^*)^{-5/2}(0,T); L^2(\Omega)) \times H_0^1(\Omega)$$

and the operator

$$\begin{aligned} \mathcal{A}(\widetilde{y}, \widetilde{p}, v_1, \widetilde{\theta}, v_0) = (L\widetilde{y} + (\widetilde{y} \cdot \nabla)\widetilde{y} + \nabla \widetilde{p} - \widetilde{\theta}\, e_3 - (v_1, 0, 0)\mathbb{1}_\omega,\ \widetilde{y}(0), \\ L\widetilde{\theta} + \widetilde{y} \cdot \nabla \widetilde{\theta} + \widetilde{y} \cdot \nabla \bar{\theta} - v_0 \mathbb{1}_\omega,\ \widetilde{\theta}(0)) \end{aligned}$$

for $(\widetilde{y}, \widetilde{p}, v_1, \widetilde{\theta}, v_0) \in E$.

In order to apply Theorem 4.1, it remains to check that the operator $\mathcal{A}$ is of class $C^1(B_1; B_2)$. Indeed, notice that all the terms in $\mathcal{A}$ are linear, except for $(\widetilde{y} \cdot \nabla)\widetilde{y}$ and $\widetilde{y} \cdot \nabla \widetilde{\theta}$. We will prove that the bilinear operator

$$((y^1, p^1, v_1^1, \theta^1, v_0^1), (y^2, p^2, v_1^2, \theta^2, v_0^2)) \to (y^1 \cdot \nabla) y^2$$

is continuous from $B_1 \times B_1$ to $L^2(e^{5/2s\beta^*}(\gamma^*)^{-2}(0,T); L^2(\Omega)^3)$. To do this, notice that

$$e^{3/2s\beta^*}(\gamma^*)^{-9/8} y \in L^2(0,T; H^2(\Omega)^3) \cap L^\infty(0,T; V)$$

for any $(y, p, v_1, \theta, v_0) \in B_1$, so we have

$$e^{3/2s\beta^*}(\gamma^*)^{-9/8} y \in L^2(0,T; L^\infty(\Omega)^3)$$

and

$$\nabla(e^{3/2s\beta^*}(\gamma^*)^{-9/8} y) \in L^\infty(0,T; L^2(\Omega)^3).$$

Consequently, we obtain

$$\begin{aligned} &\|e^{5/2s\beta^*}(\gamma^*)^{-2}(y^1 \cdot \nabla) y^2\|_{L^2(Q)^3} \\ &\le C\|(e^{3/2s\beta^*}(\gamma^*)^{-9/8}\, y^1 \cdot \nabla) e^{3/2s\beta^*}(\gamma^*)^{-9/8}\, y^2\|_{L^2(Q)^3} \\ &\le C\|e^{3/2s\beta^*}(\gamma^*)^{-9/8} y^1\|_{L^2(0,T;L^\infty(\Omega)^3)}\, \|e^{3/2s\beta^*}(\gamma^*)^{-9/8} y^2\|_{L^\infty(0,T;V)}. \end{aligned}$$

In the same way, we can prove that the bilinear operator

$$((y^1, p^1, v_1^1, \theta^1, v_0^1), (y^2, p^2, v_1^2, \theta^2, v_0^2)) \to y^1 \cdot \nabla \theta^2$$

is continuous from $B_1 \times B_1$ to $L^2(e^{5/2s\beta^*}(\gamma^*)^{-5/2}(0,T;L^2(\Omega))$ just by taking into account that

$$e^{3/2s\beta^*}(\gamma^*)^{-9/8}\theta \in L^\infty(0,T;H_0^1(\Omega)),$$

for any $(y,p,v_1,\theta,v_0) \in B_1$.

Notice that $\mathcal{A}'(0,0,0,0,0) : B_1 \to B_2$ is given by

$$\begin{aligned}\mathcal{A}'(0,0,0,0,0)(\widetilde{y},\widetilde{p},v_1,\widetilde{\theta},v_0) = (L\widetilde{y} + \nabla\widetilde{p} - \widetilde{\theta}\,e_3 - (v_1,0,0)\mathbb{1}_\omega,\, \widetilde{y}(0),\\ L\widetilde{\theta} + \widetilde{y}\cdot\nabla\bar{\theta} - v_0\mathbb{1}_\omega,\, \widetilde{\theta}(0)),\end{aligned}$$

for all $(\widetilde{y},\widetilde{p},v_1,\widetilde{\theta},v_0) \in B_1$, so this functional is surjective in view of the null controllability result for the linear system (3.1) given by Proposition 3.

We are now able to apply Theorem 4.1 for $b_1 = (0,0,0,0,0)$ and $b_2 = (0,0,0,0)$. In particular, this gives the existence of a positive number $\delta > 0$ such that, if $\|\widetilde{y}(0),\widetilde{\theta}(0)\|_{V\times H_0^1(\Omega)} \le \delta$, then we can find controls v_1 and v_0 such that the associated solution $(\widetilde{y},\widetilde{p},\widetilde{\theta})$ to (4.1) satisfies $\widetilde{y}(T) = 0$ and $\widetilde{\theta}(T) = 0$ in Ω.

This concludes the proof of Theorem 1.1.

Acknowledgments

The author would like to thank the “Agence Nationale de la Recherche” (ANR), Project CISIFS, grant ANR-09-BLAN-0213-02, for partially supporting this work.

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