Local controllability of the N-dimensional Boussinesq system with N-1 scalar controls in an arbitrary control domain

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Abstract

In this paper we deal with the local exact controllability to a particular class of trajectories of the N-dimensional Boussinesq system with internal controls having 2 vanishing components. The main novelty of this work is that no condition is imposed on the control domain.

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1 Introduction

Let Ω be a nonempty bounded connected open subset of $\mathbb{R}^N$ ($N = 2$ or $3$) of class $C^\infty$. Let $T > 0$ and let $\omega \subset \Omega$ be a (small) nonempty open subset which is the control domain. We will use the notation $Q = \Omega \times (0,T)$ and $\Sigma = \partial \Omega \times (0,T)$.

We will be concerned with the following controlled Boussinesq system:

\[
\begin{aligned}
&y_t - \Delta y + (y \cdot \nabla)y + \nabla p = v_1 \mathbb{1}_\omega + \theta e_N & \text{in } Q, \\
&\theta_t - \Delta \theta + y \cdot \nabla \theta = v_0 \mathbb{1}_\omega & \text{in } Q, \\
&\nabla \cdot y = 0 & \text{in } Q, \\
&y = 0, \theta = 0 & \text{on } \Sigma, \\
&y(0) = y^0, \theta(0) = \theta^0 & \text{in } \Omega,
\end{aligned}
\]

(1.1)

where

\[
e_N = \begin{cases}
(0,1) & \text{if } N = 2, \\
(0,0,1) & \text{if } N = 3
\end{cases}
\]

stands for the gravity vector field, $y = y(x,t)$ represents the velocity of the particles of the fluid, $\theta = \theta(x,t)$ their temperature and $(v_0,v) = (v_0,v_1,\ldots,v_N)$ stands for the control which acts over the set $\omega$.

Let us recall the definition of some usual spaces in the context of incompressible fluids:

\[
V = \{ y \in H^1_0(\Omega)^N : \nabla \cdot y = 0 \text{ in } \Omega \}
\]

and

\[
H = \{ y \in L^2(\Omega)^N : \nabla \cdot y = 0 \text{ in } \Omega, \ y \cdot n = 0 \text{ on } \partial \Omega \}.
\]

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This paper concerns the local exact controllability to the trajectories of system (1.1) at time $t = T$ with a reduced number of controls. To introduce this concept, let us consider $(\bar{y}, \bar{\theta})$ (together with some pressure $\bar{p}$) a trajectory of the following uncontrolled Boussinesq system:

$$
\begin{align*}
\bar{y}_t &= -\Delta \bar{y} + (\bar{y} \cdot \nabla) \bar{y} + \nabla \bar{p} = \bar{\theta} e_N \quad \text{in } Q, \\
\bar{\theta}_t &= -\Delta \bar{\theta} + \bar{y} \cdot \nabla \bar{\theta} = 0 \quad \text{in } Q, \\
\nabla \cdot \bar{y} &= 0 \quad \text{in } Q, \\
\bar{y} &= 0, \quad \bar{\theta} = 0 \quad \text{on } \Sigma, \\
\bar{y}(0) &= \bar{y}^0, \quad \bar{\theta}(0) = \bar{\theta}^0 \quad \text{in } \Omega.
\end{align*}
$$

We say that the local exact controllability to the trajectories $(\bar{y}, \bar{\theta})$ holds if there exists a number $\delta > 0$ such that if $\|(y^0, \theta^0) - (\bar{y}^0, \bar{\theta}^0)\|_X \leq \delta$ ($X$ is an appropriate Banach space), there exist controls $(\bar{v}_0, v) \in L^2(\omega \times (0, T))^{N+1}$ such that the corresponding solution $(y, \theta)$ to system (1.1) satisfies (1.3).

The first results concerning this problem were obtained in [7] and [8], with $N + 1$ scalar controls acting in the whole boundary of $\Omega$ and with $N + 1$ scalar controls acting in $\omega$ when $\Omega$ is a torus, respectively. Later, in [9], the author proved the local exact controllability for less regular trajectories $(\bar{y}, \bar{\theta})$ in an open bounded set and for an arbitrary control domain. Namely, the trajectories were supposed to satisfy

$$(\bar{y}, \bar{\theta}) \in L^\infty(Q)^{N+1}, \quad (\bar{y}_t, \bar{\theta}_t) \in L^2(0, T; L^r(\Omega))^{N+1},$$

with $r > 1$ if $N = 2$ and $r > 6/5$ if $N = 3$.

In [5], the authors proved that local exact controllability can be achieved with $N - 1$ scalar controls acting in $\omega$ when $\omega$ intersects the boundary of $\Omega$ and (1.4) is satisfied. More precisely, we can find controls $v_0$ and $v$, with $v_N \equiv 0$ and $v_k \equiv 0$ for some $k < N$ ($k$ is determined by some geometric assumption on $\omega$, see [5] for more details), such that the corresponding solution to (1.1) satisfies (1.3).

In this work, we remove this geometric assumption on $\omega$ and consider a target trajectory of the form $(0, \bar{p}, \bar{\theta})$, i.e.,

$$
\begin{align*}
\nabla \bar{p} &= \bar{\theta} e_N \quad \text{in } Q, \\
\bar{\theta}_t &= -\Delta \bar{\theta} = 0 \quad \text{in } Q, \\
\bar{\theta} &= 0 \quad \text{on } \Sigma, \\
\bar{\theta}(0) &= \bar{\theta}^0 \quad \text{in } \Omega,
\end{align*}
$$

where we assume

$$
\bar{\theta} \in L^\infty(0, T; W^{3, \infty}(\Omega)) \text{ and } \nabla \bar{\theta}_t \in L^\infty(\Omega)^N.
$$

The main result of this paper is given in the following theorem.

**Theorem 1.1.** Let $i < N$ be a positive integer and $(\bar{p}, \bar{\theta})$ a solution to (1.5) satisfying (1.6). Then, for every $T > 0$ and $\omega \subset \Omega$, there exists $\delta > 0$ such that for every $(y^0, \theta^0) \in V \times H^1_0(\Omega)$ satisfying

$$
\|(y^0, \theta^0) - (0, \bar{\theta}^0)\|_{V \times H^1_0} \leq \delta,
$$

we can find controls $v^0 \in L^2(\omega \times (0, T))$ and $v \in L^2(\omega \times (0, T))^N$, with $v_i \equiv 0$ and $v_N \equiv 0$, such that the corresponding solution to (1.1) satisfies (1.3), i.e.,

$$
y(T) = 0 \text{ and } \theta(T) = \bar{\theta}(T) \text{ in } \Omega.
$$

**Remark 1.** Notice that when $N = 2$ we only need to control the temperature equation.
Remark 2. It would be interesting to know if the local controllability to the trajectories with \( N - 1 \) scalar controls holds for \( \bar{y} \neq 0 \) and \( \omega \) as in Theorem 1.1. However, up to our knowledge, this is an open problem even for the case of the Navier-Stokes system.

Remark 3. One could also try to just control the movement equation, that is, \( v_0 \equiv 0 \) in (1.1). However, this system does not seem to be controllable. To justify this, let us consider the control problem

\[
\begin{aligned}
&\begin{cases}
  y_t - \Delta y + (y \cdot \nabla) y + \nabla p = v \mathbb{I}_\omega + \theta e_N \\
  \theta_t - \Delta \theta + y \cdot \nabla \theta = 0 \\
  \nabla \cdot y = 0 \\
  y = 0, \ n = 0
\end{cases} \quad \text{in } \Omega, \\
&\begin{cases}
  \theta_t - \Delta \theta + y \cdot \nabla \theta = f_0 + v_0 \mathbb{I}_\omega \\
  \nabla \cdot y = 0 \\
  y = 0, \ n = 0
\end{cases} \quad \text{on } \Sigma, \\
&y(0) = y^0, \ \theta(0) = \theta^0
\end{aligned}
\]

where we have homogeneous Neumann boundary conditions for the temperature. Integrating in \( Q \), integration by parts gives

\[\int_\Omega \theta(T) \, dx = \int_\Omega \theta_0 \, dx,\]

so we can not expect in general null controllability.

Some recent works have been developed in the controllability problem with reduced number of controls. For instance, in [3] the authors proved the null controllability for the Stokes system with \( N - 1 \) scalar controls, and in [2] the local null controllability was proved for the Navier-Stokes system with the same number of controls.

The present work can be viewed as an extension of [2]. To prove Theorem 1.1 we follow a standard approach introduced in [6] and [10] (see also [4]). We first deduce a null controllability result for the linear system

\[
\begin{aligned}
&\begin{cases}
  y_t - \Delta y + \nabla p = f + v_1 \mathbb{I}_\omega + \theta e_N \\
  \theta_t - \Delta \theta + y \cdot \nabla \theta = f_0 + v_0 \mathbb{I}_\omega \\
  \nabla \cdot y = 0 \\
  y = 0, \ \theta = 0
\end{cases} \quad \text{in } \Omega, \\
&\begin{cases}
  \nabla \cdot \phi = 0 \\
  \phi = 0
\end{cases} \quad \text{on } \Sigma, \\
&\phi(T) = \phi^T, \ \psi(T) = \psi^T
\end{aligned}
\]

where \( f \) and \( f_0 \) will be taken to decrease exponentially to zero in \( t = T \).

The main tool to prove this null controllability result for system (1.8) is a suitable Carleman estimate for the solutions of its adjoint system, namely,

\[
\begin{aligned}
&\begin{cases}
  -\varphi_t - \Delta \varphi + \nabla \pi = g - \psi \nabla \theta \\
  -\psi_t - \Delta \psi = g_0 + \varphi_N \\
  \nabla \cdot \varphi = 0 \\
  \varphi = 0, \ \psi = 0
\end{cases} \quad \text{in } \Omega, \\
&\begin{cases}
  \varphi(T) = \varphi^T, \ \psi(T) = \psi^T
\end{cases} \quad \text{in } \Omega,
\end{aligned}
\]

where \( g \in L^2(Q)^N \), \( g_0 \in L^2(Q) \), \( \varphi^T \in H \) and \( \psi^T \in L^2(\Omega) \). In fact, this inequality is of the form

\[
\iint_Q \tilde{\rho}_1(t)(|\varphi|^2 + |\psi|^2) \, dx \, dt \\
\leq C \left( \iint_Q \tilde{\rho}_2(t)(|g|^2 + |g_0|^2) \, dx \, dt + \int_0^T \iint_\omega \tilde{\rho}_3(t)|\varphi_j|^2 \, dx \, dt + \int_0^T \iint_\omega \tilde{\rho}_4(t)|\psi|^2 \, dx \, dt \right),
\]

(1.10)
if \( N = 3 \), and of the form
\[
\int_Q \tilde{\rho}_1(t)(|\varphi|^2 + |\psi|^2) \, dx \, dt \leq C \left( \int_Q \tilde{\rho}_2(t)(|g|^2 + |g_0|^2) \, dx \, dt + \int_0^T \int_\omega \tilde{\rho}_4(t)|\psi|^2 \, dx \, dt \right),
\]
if \( N = 2 \), where \( j = 1 \) or \( 2 \) and \( \tilde{\rho}_k(t) \) are positive smooth weight functions (see inequalities (2.4) and (2.5) below). From these estimates, we can find a solution \( (y, \theta, v) \) of (1.8) with the same decreasing properties as \( f \) and \( f_0 \). In particular, \( (y(T), \theta(T)) = (0, 0) \) and \( v_i = v_N = 0 \).

We conclude the controllability result for the nonlinear system by means of an inverse mapping theorem.

This paper is organized as follows. In section 2, we prove a Carleman inequality of the form (1.10) for system (1.9). In section 3, we deal with the null controllability of the linear system (1.8). Finally, in section 4 we give the proof of Theorem 1.1.

## 2 Carleman estimate for the adjoint system

In this section we will prove a Carleman estimate for the adjoint system (1.9). In order to do so, we are going to introduce some weight functions. Let \( \omega_0 \) be a nonempty open subset of \( \mathbb{R}^N \) such that \( \overlinet{\omega_0} \subset \omega \) and \( \eta \in C^2(\overlinet{\Omega}) \) such that
\[
|\nabla \eta| > 0 \text{ in } \overlinet{\Omega} \setminus \omega_0, \quad \eta > 0 \text{ in } \Omega \quad \text{and} \quad \eta \equiv 0 \text{ on } \partial \Omega. \tag{2.1}
\]
The existence of such a function \( \eta \) is given in [6]. Let also \( \ell \in C^\infty([0, T]) \) be a positive function satisfying
\[
\ell(t) = t \quad \forall t \in [0, T/4], \quad \ell(t) = T - t \quad \forall t \in [3T/4, T],
\]
\[
\ell(t) \leq \ell(T/2), \quad \forall t \in [0, T]. \tag{2.2}
\]
Then, for all \( \lambda \geq 1 \) we consider the following weight functions:
\[
\alpha(x, t) = \frac{e^{2\lambda \eta(x)}}{\ell^8(t)}, \quad \xi(x, t) = \frac{e^{\lambda \eta(x)}}{\ell^8(t)},
\]
\[
\alpha^*(t) = \max_{x \in \overlinet{\Omega}} \alpha(x, t), \quad \xi^*(t) = \min_{x \in \overlinet{\Omega}} \xi(x, t), \tag{2.3}
\]
\[
\tilde{\alpha}(t) = \min_{x \in \overlinet{\Omega}} \alpha(x, t), \quad \tilde{\xi}(t) = \max_{x \in \overlinet{\Omega}} \xi(x, t).
\]

Our Carleman estimate is given in the following proposition.

**Proposition 1.** Assume \( N = 3 \), \( \omega \subset \Omega \) and \( (\rho, \tilde{\theta}) \) satisfies (1.6). There exists a constant \( \lambda_0 \), such that for any \( \lambda \geq \lambda_0 \) there exist two constants \( C(\lambda) > 0 \) and \( s_0(\lambda) > 0 \) such that for any \( j \in \{1, 2\} \), any \( y \in L^2(Q)^3 \), any \( g_0 \in L^2(Q) \), any \( \varphi^T \in H \) and any \( \psi^T \in L^2(\Omega) \), the solution of (1.9) satisfies
\[
s^4 \int_Q \int e^{-5s\alpha^*}(\xi^*)^4|\varphi|^2 \, dx \, dt + s^5 \int_Q \int e^{-5s\alpha^*}(\xi^*)^5|\psi|^2 \, dx \, dt
\]
\[
\leq C \left( \int_Q \int e^{-3s\alpha^*}(|g|^2 + |g_0|^2) \, dx \, dt + s^7 \int_0^T \int_\omega e^{-2s\tilde{\alpha} - 3s\alpha^*}(\tilde{\xi})^7|\varphi|^2 \, dx \, dt \right)
\]
\[
+ s^{12} \int_0^T \int_\omega e^{-4s\tilde{\alpha} - s\alpha^*}(\tilde{\xi})^{10/4}|\psi|^2 \, dx \, dt \right) \tag{2.4}
\]
for every $s \geq s_0$.

For the sake of completeness, let us also state this result for the 2-dimensional case.

**Proposition 2.** Assume $N = 2$, $\omega \subset \Omega$ and $(\bar{p}, \bar{\theta})$ satisfies (1.6). There exists a constant $\lambda_0$, such that for any $\lambda \geq \lambda_0$ there exist two constants $C(\lambda) > 0$ and $s_0(\lambda) > 0$ such that for any $g \in L^2(\Omega)^2$, any $g_0 \in L^2(\Omega)^2$, any $\psi^T \in H$ and any $\psi^T \in L^2(\Omega)$, the solution of (1.9) satisfies

$$
\int_{\Omega} e^{-5s\alpha^*}(\xi^*)^4|\psi|^2\,dx\,dt + s^5 \int_{\Omega} e^{-5s\alpha^*}(\xi^*)^5|\psi|^2\,dx\,dt
\leq C \left( \int_{\Omega} e^{-3s\alpha^*}(|g|^2 + |g_0|^2)\,dx\,dt + s^{12} \int_{\omega} e^{-4s\alpha^*}(\bar{\xi})^{49/4}|\psi|^2\,dx\,dt \right) (2.5)
$$

for every $s \geq s_0$.

To prove Proposition 1 we will follow the ideas of [3] and [5] (see also [2]). An important point in the proof of the Carleman inequality established in [3] is that the laplacian of the pressure in the adjoint system is zero. In [2], a decomposition of the solution was made, so that we can essentially concentrate in a solution where the laplacian of the pressure is zero. For system (1.9) this will not be possible because of the coupling term $\psi \nabla \bar{\theta}$. However, under hypothesis (1.6) we can follow the same ideas to obtain (2.4). All the details are given below.

### 2.1 Technical results

Let us present now the technical results needed to prove Carleman inequalities (2.4) and (2.5). The first of these results is a Carleman inequality for parabolic equations with non-homogeneous boundary conditions proved in [11]. Consider the equation

$$
u_t - \Delta u = F_0 + \sum_{j=1}^N \partial_j F_j \text{ in } Q, \tag{2.6}$$

where $F_0, F_1, \ldots, F_N \in L^2(\Omega)$. We have the following result.

**Lemma 2.1.** There exists a constant $\tilde{\lambda}_0$ only depending on $\Omega$, $\omega_0$, $\eta$ and $\ell$ such that for any $\lambda > \tilde{\lambda}_0$ there exist two constants $C(\lambda) > 0$ and $\tilde{s}(\lambda)$, such that for every $s \geq \tilde{s}$ and every $u \in L^2((0,T;H^1(\Omega)) \cap H^1((0,T;H^{-1}(\Omega))$ satisfying (2.6), we have

$$
\frac{1}{s} \int_{\Omega} e^{-2s\alpha} \|\nabla u\|^2\,dx\,dt + s \int_{\Omega} e^{-2s\alpha} |u|^2\,dx\,dt \leq C \left( \int_0^T \int_{\omega_0} e^{-2s\alpha} |\xi|^2\,dx\,dt \right) \left( \int_0^T \int_{\omega_0} e^{-2s\alpha} |\xi|^2\,dx\,dt \right)
+ s^{-1/2} \left\| e^{-s\alpha \xi^{-1/4}} u \right\|^2_{H^{1/2}(\Sigma)} + s^{-1/2} \left\| e^{-s\alpha \xi^{-1/8}} u \right\|^2_{L^2(\Sigma)}
+ s^{-2} \int_{\Omega} e^{-2s\alpha} \xi^{-2} |F_0|^2\,dx\,dt + \sum_{j=1}^N \int_{\Omega} e^{-2s\alpha} |F_j|^2\,dx\,dt \right). \tag{2.7}
$$

Recall that

$$
\|u\|_{H^{1/2}(\Sigma)} = \left( \|u\|^2_{H^{1/4}(0,T;L^2(\partial\Omega))} + \|u\|^2_{L^2((0,T;H^{1/2}(\partial\Omega)))} \right)^{1/2}.
$$

The next technical result is a particular case of Lemma 3 in [3].
Lemma 2.4. There exists a constant $\hat{\lambda}_1$ such that for any $\lambda \geq \hat{\lambda}_1$, there exists $C > 0$ depending only on $\lambda$, $\Omega$, $\omega_0$, $\eta$ and $\ell$ such that, for every $T > 0$ and every $u \in L^2(0, T; H^1(\Omega))$, 
\[
s^3 \int_Q e^{-2s\alpha \xi^3}|u|^2 \, dx \, dt 
\leq C \left( s \int_Q e^{-2s\alpha \xi^2} |\nabla u|^2 \, dx \, dt + s^3 \int_0^T \int_{\omega_0} e^{-2s\alpha \xi^3} |u|^2 \, dx \, dt \right), \tag{2.8}
\]
for every $s \geq C$.

The next lemma is an estimate concerning the Laplace operator:

Lemma 2.3. There exists a constant $\hat{\lambda}_2$ such that for any $\lambda \geq \hat{\lambda}_2$, there exists $C > 0$ depending only on $\lambda$, $\Omega$, $\omega_0$, $\eta$ and $\ell$ such that, for every $u \in L^2(0, T; H^1_0(\Omega))$, 
\[
s^6 \int_Q e^{-2s\alpha \xi^6}|u|^2 \, dx \, dt + s^4 \int_Q e^{-2s\alpha \xi^4} |\nabla u|^2 \, dx \, dt 
\leq C \left( s^3 \int_Q e^{-2s\alpha \xi^3} |\Delta u|^2 \, dx \, dt + s^6 \int_0^T \int_{\omega_0} e^{-2s\alpha \xi^6} |u|^2 \, dx \, dt \right), \tag{2.9}
\]
for every $s \geq C$.

Inequality (2.9) comes from the classical result in [6] for parabolic equations applied to the laplacian with parameter $s/\ell^8(t)$. Then, multiplying by $\exp(-2s\alpha^2 H^3(\Omega))/\ell^8(t)$ and integrating in $(0, T)$ we obtain (2.9). Details can be found in [3] or [2].

The last technical result concerns the regularity of the solutions to the Stokes system that can be found in [12] (see also [13]).

Lemma 2.4. For every $T > 0$ and every $F \in L^2(Q)^N$, there exists a unique solution 
\[
u \in L^2(0, T; H^2(\Omega)^N) \cap H^1(0, T; H)
\]
to the Stokes system 
\[
\begin{aligned}
& u_t - \Delta u + \nabla p = F & \text{in } Q, \\
& \nabla \cdot u = 0 & \text{in } Q, \\
& u = 0 & \text{on } \Sigma, \\
& u(0) = 0 & \text{in } \Omega,
\end{aligned}
\]
for some $p \in L^2(0, T; H^1(\Omega))$, and there exists a constant $C > 0$ depending only on $\Omega$ such that 
\[
\|u\|_{L^2(0,T; H^2(\Omega)^N)} + \|u\|_{H^1(0,T; L^2(\Omega)^N)} \leq C \|F\|_{L^2(Q)^N}. \tag{2.10}
\]
Furthermore, if $F \in L^2(0, T; H^2(\Omega)^N) \cap \hat{\mathcal{H}}^1(0, T; L^2(\Omega)^N)$, then $u \in L^2(0, T; H^4(\Omega)^N) \cap H^1(0, T; H^2(\Omega)^N)$ and there exists a constant $C > 0$ depending only on $\Omega$ such that
\[
\|u\|_{L^2(0,T; H^4(\Omega)^N)} + \|u\|_{H^1(0,T; L^2(\Omega)^N)} 
\leq C \left( \|F\|_{L^2(0,T; H^2(\Omega)^N)} + \|F\|_{H^1(0,T; L^2(\Omega)^N)} \right). \tag{2.11}
\]

From now on, we set $N = 3$, $i = 2$ and $j = 1$, i.e., we consider a control for the movement equation in (1.1) (and (1.8)) of the form $v = (v_1, 0, 0)$. The arguments can be easily adapted to the general case by interchanging the roles of $i$ and $j$. 
2.2 Proof of Proposition 1

Let us introduce \((w, \pi_w), (z, \pi_z)\) and \(\tilde{\psi}\), the solutions of the following systems:

\[
\begin{align*}
-w_t - \Delta w + \nabla \pi_w &= \rho g & \text{in } Q, \\
\nabla \cdot w &= 0 & \text{in } Q, \\
w &= 0 & \text{on } \Sigma, \\
w(T) &= 0 & \text{in } \Omega,
\end{align*}
\]

and

\[
\begin{align*}
-w_t - \Delta z + \nabla \pi_z &= -\rho' \varphi - \tilde{\psi} \nabla \theta & \text{in } Q, \\
\nabla \cdot z &= 0 & \text{in } Q, \\
z &= 0 & \text{on } \Sigma, \\
z(T) &= 0 & \text{in } \Omega,
\end{align*}
\]

where \(\rho(t) = e^{-\frac{2}{3} s t^s}\). Adding (2.12) and (2.13), we see that \((w + z, \pi_w + \pi_z, \tilde{\psi})\) solves the same system as \((\varphi, \pi, \rho, \rho' \psi)\), where \((\varphi, \pi, \psi)\) is the solution to (1.9). By uniqueness of the Cauchy problem we have

\[
\rho \varphi = w + z, \quad \rho \pi = \pi_w + \pi_z \quad \text{and} \quad \rho \psi = \tilde{\psi}. \tag{2.15}
\]

Applying the divergence operator to (2.13) we see that \(\Delta \pi_z = -\nabla \cdot (\tilde{\psi} \nabla \theta)\). We apply now the operator \(\nabla \Delta = (\partial_1 \Delta, \partial_2 \Delta, \partial_3 \Delta)\) to the equations satisfied by \(z_1\) and \(z_3\). We then have

\[
\begin{align*}
-(\nabla \Delta z_1)_t - \Delta(\nabla \Delta z_1) &= \nabla \left( \partial_1 \nabla \cdot (\tilde{\psi} \nabla \theta) - \Delta(\tilde{\psi} \partial_1 \theta) - \rho' \Delta \varphi_1 \right) & \text{in } Q, \\
-(\nabla \Delta z_3)_t - \Delta(\nabla \Delta z_3) &= \nabla \left( \partial_3 \nabla \cdot (\tilde{\psi} \nabla \theta) - \Delta(\tilde{\psi} \partial_3 \theta) - \rho' \Delta \varphi_3 \right) & \text{in } Q.
\end{align*}
\]

(2.16)

To the equations in (2.16), we apply the Carleman inequality in Lemma 2.1 with \(u = \nabla \Delta z_k\) for \(k = 1, 3\) to obtain

\[
\sum_{k=1,3} \left[ \frac{1}{s} \int_Q e^{-2s o} \frac{1}{\xi} |\nabla \Delta z_k|^2 dx dt + s \int_Q e^{-2s o} |\nabla \Delta z_k|^2 dx dt \right]
\leq C \left( \sum_{k=1,3} \left[ \int_0^T s \int_Q e^{-2s o} |\nabla \Delta z_k|^2 dx dt + s^{-1/2} \left\| e^{-s o} (\xi^*)^{-1/8} \nabla \Delta z_k \right\|_{L^2(\Sigma)^3}^2 \\
+ s^{-1/2} \left\| e^{-s o} (\xi^*)^{-1/4} \nabla \Delta z_k \right\|_{H^{1/2}(\Sigma)^3}^2 + \int_Q e^{-2s o} |\rho'|^2 |\Delta \varphi_k|^2 dx dt \right]
\right)
\]

\[
+ \int_Q e^{-2s o} \left( \sum_{k,l=1}^3 |\partial_{kl} \tilde{\psi}|^2 + |\nabla \tilde{\psi}|^2 + |\tilde{\psi}|^2 \right) dx dt, \tag{2.17}
\]

for every \(s \geq C\), where \(C\) depends also on \(\|\theta\|_{L^\infty(0,T;W^{3,\infty}(\Omega))}\).
Now, by Lemma 2.2 with \( u = \Delta z_k \) for \( k = 1, 3 \) we have

\[
\sum_{k=1,3} s^3 \int_0^T \int_Q e^{-2s} \xi^3 |\Delta z_k|^2 \, dx \, dt
\]

\[
\leq C \sum_{k=1,3} \left( s \int_0^T \int_Q e^{-2s} \xi |\nabla \Delta z_k|^2 \, dx \, dt + s^3 \int_0^T \int_0^{\omega_0} e^{-2s} \xi^3 |\Delta z_k|^2 \, dx \, dt \right),
\]

for every \( s \geq C \), and by Lemma 2.3 with \( u = z_k \) for \( k = 1, 3 \):

\[
\sum_{k=1,3} s^4 \int_0^T \int_Q e^{-2s} \xi^4 |\nabla z_k|^2 \, dx \, dt + s^4 \int_0^T \int_Q e^{-2s} \xi^4 |\nabla z_k|^2 \, dx \, dt
\]

\[
\leq C \sum_{k=1,3} \left[ s^3 \int_0^T \int_Q e^{-2s} \xi^3 |\Delta z_k|^2 \, dx \, dt + s^6 \int_0^T \int_\omega e^{-2s} \xi^6 |z_k|^2 \, dx \, dt \right],
\]

for every \( s \geq C \).

Combining (2.17), (2.18) and (2.19) and considering a nonempty open set \( \omega_1 \) such that \( \omega_0 \subset \omega_1 \subset \omega \) we obtain after some integration by parts

\[
\sum_{k=1,3} \left[ \frac{1}{s} \int_0^T \int_Q e^{-2s} \xi^3 |\nabla \Delta z_k|^2 \, dx \, dt + s \int_0^T \int_Q e^{-2s} \xi |\nabla \Delta z_k|^2 \, dx \, dt \right]
\]

\[
+ s^3 \int_0^T \int_Q e^{-2s} \xi^3 |\Delta z_k|^2 \, dx \, dt + s \int_0^T \int_Q e^{-2s} \xi^4 |\nabla z_k|^2 \, dx \, dt + s^6 \int_0^T \int_\omega e^{-2s} \xi^6 |z_k|^2 \, dx \, dt
\]

\[
\leq C \sum_{k=1,3} \left[ s^7 \int_0^T \int_\omega e^{-2s} \xi^7 |z_k|^2 \, dx \, dt + s^{-1/2} \left\| e^{-s} \xi^{-1/2} \nabla \Delta z_k \right\|_{L^2(\Sigma)}^2
\]

\[
+ \sum_{k=1,3} \left( \frac{1}{s} \int_0^T \int_Q e^{-2s} |\varphi_k|^2 \, dx \, dt \right)
\]

\[
\leq C s^2 \left( \sum_{k=1,3} \left| \frac{\partial^2 \tilde{\psi}}{\partial x_i} \right|^2 + |\nabla \tilde{\psi}|^2 + |\tilde{\psi}|^2 \right) \, dx \, dt, \tag{2.20}
\]

for every \( s \geq C \).

Notice that from the identities in (2.15), the regularity estimate (2.10) for \( w \) and \( |\rho'|^2 \leq C s^2 \rho^2 (\xi)^{9/4} \) we obtain for \( k = 1, 3 \)

\[
\int_0^T \int_Q e^{-2s} |\rho'|^2 |\Delta \varphi_k|^2 \, dx \, dt = \int_0^T \int_Q e^{-2s} |\rho'|^2 |\Delta (\rho \varphi_k)|^2 \, dx \, dt
\]

\[
\leq C s^2 \int_0^T \int_Q e^{-2s} \xi^{9/4} |\Delta z_k|^2 \, dx \, dt + C s^2 \int_0^T \int_Q e^{-2s} \xi^{9/4} |\Delta w|^2 \, dx \, dt
\]

\[
\leq C s^2 \int_0^T \int_Q e^{-2s} \xi^3 |\Delta z_k|^2 \, dx \, dt + C \| \rho g \|_{L^2(\Sigma)}^2,
\]
where we have also used the fact that \( s^2 e^{-2s\alpha} \xi^{5/4} \) is bounded and \( 1 \leq C \xi^{3/4} \) in \( Q \).

Now, from \( z|_\Sigma = 0 \) and the divergence free condition we readily have (notice that \( \alpha^* \) and \( \xi^* \) do not depend on \( x \))

\[
\begin{align*}
ts^4 \int_Q e^{-2s\alpha^*} (\xi^*)^4 |z_2|^2 dx dt & \leq C s^4 \int_Q e^{-2s\alpha^*} (\xi^*)^4 |\partial_2 z_2|^2 dx dt \\
& \leq C s^4 \int_Q e^{-2s\alpha^*} (|\nabla z_1|^2 + |\nabla z_3|^2) dx dt.
\end{align*}
\]

Using these two last estimates in (2.20), we get

\[
I(s, z) := \sum_{k=1} s \left[ \frac{1}{s} \int_T e^{-2s\alpha^*} \xi |\nabla \Delta z_2|^2 dx dt + s \int_T e^{-2s\alpha^*} \xi |\Delta z_2|^2 dx dt + s^3 \int_T e^{-2s\alpha^*} \xi |\partial_2 z_2|^2 dx dt \right]
\]

\[
\leq C \left[ \sum_{k=1} s \left[ \frac{1}{s} \int_0^T e^{-2s\alpha^*} \xi |z_2|^2 dx dt + s^{-1/2} \left\| e^{-2s\alpha^*} (\xi^*)^{-1/4} \nabla \Delta z_k \right\|_{L^2(\Sigma)}^2 + \| \mu g \|^2_{L^2(Q)} \\
+ s^{1/2} \left\| e^{-2s\alpha^*} (\xi^*)^{-1/4} \nabla \Delta z_k \right\|_{H^{1/2}(\Sigma)}^2 \right] + \| \mu g \|^2_{L^2(Q)} + \int_T e^{-2s\alpha^*} \left( \sum_{k,l=1}^3 |\partial_{kl} \tilde{\psi}|^2 + |\nabla \tilde{\psi}|^2 + |\tilde{\psi}|^2 \right) dx dt \right),
\]

for every \( s \geq C \).

For equation (2.14), we use the classical Carleman inequality for the heat equation (see for example [6]): there exists \( \tilde{\lambda}_1 > 0 \) such that for any \( \lambda > \tilde{\lambda}_3 \) there exists

\[
C(\lambda, \Omega, \omega_1, ||\theta||_{L^\infty(0,T; W^{3,\infty}(\Omega)))}) > 0 \text{ such that}
\]

\[
J(s, \tilde{\psi}) := s \int_Q e^{-2s\alpha^*} \xi (|\tilde{\psi}|^2 + \sum_{k,l=1}^3 |\partial_{kl} \tilde{\psi}|^2) dx dt + s^3 \int_Q e^{-2s\alpha^*} \xi^3 |\nabla \tilde{\psi}|^2 dx dt
\]

\[
+ s^3 \int_Q e^{-2s\alpha^*} \xi^3 |\tilde{\psi}|^2 dx dt \leq C \left( s^2 \int_Q e^{-2s\alpha^*} \xi^2 \rho^2 (|g_0|^2 + |\varphi_3|^2) dx dt + s^2 \int_0^T e^{-2s\alpha^*} \xi \rho^2 (|g_0|^2 + |\varphi_3|^2) dx dt \right),
\]

for every \( s \geq C \).

We choose \( \lambda_0 \) in Proposition 1 (and Proposition 2) to be \( \lambda_0 := \max \{ \tilde{\lambda}_0, \tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3 \} \) and we fix \( \lambda \geq \lambda_0 \).
Combining inequalities (2.21) and (2.22), and taking into account that \( s^2e^{-2s\alpha}\xi^2 \rho^2 \) is bounded, the identities in (2.15), estimate (2.10) for \( w \) and \(|\rho| \leq Cs(\xi^*)^{9/8} \rho \) we have

\[
I(s, z) + J(s, \tilde{w}) \leq C \left( \|\rho g\|_{L^2(Q)^3}^2 + \|\rho g_0\|_{L^2(Q)}^2 + s^5 \int_0^T \int_{\omega_1} e^{-2s\alpha}\xi^2|\tilde{\psi}|^2 \, dx \, dt \right.
\]

\[
+ \sum_{k=1,3} \left[ s^{-1/2} \left\| e^{-s\alpha^*}(\xi^*)^{-1/8} \nabla \Delta z_k \right\|_{L^2(\Sigma)^3}^2 + s^{-1/2} \left\| e^{-s\alpha^*}(\xi^*)^{-1/4} \nabla \Delta z_k \right\|_{H^{1/4}(\Sigma)^3}^2 \right.
\]

\[
+ s^7 \int_0^T \int_{\omega_1} e^{-2s\alpha}\xi^7|z_k|^2 \, dx \, dt \bigg) \right), \tag{2.23}
\]

for every \( s \geq C \).

It remains to treat the boundary terms of this inequality and to eliminate the local term in \( z_3 \).

**Estimate of the boundary terms.** First, we treat the first boundary term in (2.23). Notice that, since \( \alpha^* \) and \( \xi^* \) do not depend on \( x \), we can readily get by integration by parts, for \( k = 1, 3 \),

\[
\left\| e^{-s\alpha^*} \nabla \Delta z_k \right\|_{L^2(\Sigma)^3}^2 
\leq C \left( s^{1/2} \left\| e^{-s\alpha^*}(\xi^*)^{1/2} \nabla \Delta z_k \right\|_{L^2(Q)} \right. 
\left\| s^{-1/2} e^{-s\alpha^*}(\xi^*)^{-1/2} \nabla \Delta z_k \right\|_{L^2(Q)^3} 
\leq C \left( s \int_Q e^{-2s\alpha^*} \xi^* \left| \nabla \Delta z_k \right|^2 \, dx \, dt + \frac{1}{s} \int_Q e^{-2s\alpha^*} \frac{1}{\xi^*} \left| \nabla \nabla \Delta z_k \right|^2 \, dx \, dt \right),
\]

so \( e^{-s\alpha^*} \nabla \Delta z_k \right\|_{L^2(\Sigma)^3}^2 \) is bounded by \( I(s, z) \). On the other hand, we can bound the first boundary term as follows:

\[
s^{-1/2} \left\| e^{-s\alpha^*}(\xi^*)^{-1/8} \nabla \Delta z_k \right\|_{L^2(\Sigma)^3}^2 \leq Cs^{-1/2} \left\| e^{-s\alpha^*} \nabla \Delta z_k \right\|_{L^2(\Sigma)^3}^2.
\]

Therefore, the first boundary terms can be absorbed by taking \( s \) large enough.

Now we treat the second boundary term in the right-hand side of (2.23). We will use regularity estimates to prove that \( z_1 \) and \( z_3 \) multiplied by a certain weight function are regular enough. First, let us observe that from (2.15) and the regularity estimate (2.10) for \( w \) we readily have

\[
\left\| s^2 e^{-s\alpha^*}(\xi^*)^{7/8} \right\|_{L^2(Q)^3}^2 \leq C \left( I(s, z) + \|\rho g\|_{L^2(Q)^3}^2 \right), \tag{2.24}
\]

We define now

\[
\tilde{z} := se^{-s\alpha^*}(\xi^*)^{7/8}z, \quad \pi \tilde{z} := se^{-s\alpha^*}(\xi^*)^{7/8}\pi z.
\]

From (2.13) we see that \( (\tilde{z}, \pi \tilde{z}) \) is the solution of the Stokes system:

\[
\begin{align*}
-\tilde{z}_t - \Delta \tilde{z} + \nabla \pi \tilde{z} &= R_1 & & \text{in } Q, \\
\nabla \cdot \tilde{z} &= 0 & & \text{in } Q, \\
\tilde{z} &= 0 & & \text{on } \Sigma, \\
\tilde{z}(T) &= 0 & & \text{in } \Omega, \\
\end{align*}
\tag{2.25}
\]
where \( R_1 := -se^{-sa^*}(\xi^*)^{7/8}\rho'\varphi - se^{-sa^*}(\xi^*)^{7/8}e^{\hat{\theta}} - (se^{-sa^*}(\xi^*)^{7/8})_t z \). Taking into account that \( |a^*_t| \leq C(\xi^*)^{9/8} \), \( |\rho'| \leq Cs(\xi^*)^{9/8} \rho \), (1.6) and (2.24) we have

\[
\| R_1 \|_{L^2(Q)^3} \leq C \left( I(s, z) + J(s, \hat{\psi}) + \| \rho g \|_{L^2(Q)^3}^2 \right),
\]

and therefore, by the regularity estimate (2.10) applied to (2.27), we have

\[
\| \hat{z} \|_{L^2(0,T;H^1(\Omega)^3) \cap H^1(0,T;L^2(\Omega)^3)} \leq C \left( I(s, z) + J(s, \hat{\psi}) + \| \rho g \|_{L^2(Q)^3}^2 \right). \tag{2.26}
\]

Next, let \( \hat{z} := e^{-sa^*}(\xi^*)^{-1/4} z, \hat{\pi}_z := e^{-sa^*}(\xi^*)^{-1/4}\pi_z \). From (2.13), \((\hat{z}, \hat{\pi}_z)\) is the solution of the Stokes system:

\[
\begin{cases}
-\hat{\zeta} - \Delta \hat{z} + \nabla \hat{\pi}_z = R_2 & \text{in } Q, \\
\nabla \cdot \hat{z} = 0 & \text{in } Q, \\
\hat{z} = 0 & \text{on } \Sigma, \\
\hat{z}(T) = 0 & \text{in } \Omega,
\end{cases} \tag{2.27}
\]

where \( R_2 := -e^{-sa^*}(\xi^*)^{-1/4}\rho'\varphi - e^{-sa^*}(\xi^*)^{-1/4}e^{\hat{\theta}} - (e^{-sa^*}(\xi^*)^{-1/4})_t z \). By the same arguments as before, and thanks to (2.26), we can easily prove that \( R_2 \in L^2(0,T;H^1(\Omega)^3) \) (for the first term in \( R_2 \)), we use again (2.15) and (2.26) and furthermore

\[
\| R_2 \|_{L^2(0,T;H^1(\Omega)^3) \cap H^1(0,T;L^2(\Omega)^3)} \leq C \left( I(s, z) + J(s, \hat{\psi}) + \| \rho g \|_{L^2(Q)^3}^2 \right).
\]

By the regularity estimate (2.11) applied to (2.27), we have

\[
\| \hat{z} \|_{L^2(0,T;H^1(\Omega)^3) \cap H^1(0,T;H^2(\Omega)^3)} \leq C \left( I(s, z) + J(s, \hat{\psi}) + \| \rho g \|_{L^2(Q)^3}^2 \right). \tag{2.28}
\]

In particular, \( e^{-sa^*}(\xi^*)^{-1/4}\nabla \Delta z_k \in L^2(0,T;H^1(\Omega)^3) \) \( \cap \ H^1(0,T;H^{-1}(\Omega)^3) \) for \( k = 1, 3 \) and

\[
\sum_{k=1}^{3} \| e^{-sa^*}(\xi^*)^{-1/4}\nabla \Delta z_k \|_{L^2(0,T;H^1(\Omega)^3)}^2 + \| e^{-sa^*}(\xi^*)^{-1/4}\nabla \Delta z_k \|_{H^1(0,T;H^{-1}(\Omega)^3)}^2 \leq C \left( I(s, z) + J(s, \hat{\psi}) + \| \rho g \|_{L^2(Q)^3}^2 \right). \tag{2.28}
\]

To end this part, we use a trace inequality to estimate the second boundary term in the right-hand side of (2.23):

\[
\sum_{k=1}^{3} s^{-1/2} \| e^{-sa^*}(\xi^*)^{-1/4}\nabla \Delta z_k \|_{H^{1/2}(\Sigma)^3}^2 \leq C s^{-1/2} \sum_{k=1}^{3} \left[ \| e^{-sa^*}(\xi^*)^{-1/4}\nabla \Delta z_k \|_{L^2(0,T;H^1(\Omega)^3)}^2 + \| e^{-sa^*}(\xi^*)^{-1/4}\nabla \Delta z_k \|_{H^1(0,T;H^{-1}(\Omega)^3)}^2 \right].
\]

By taking \( s \) large enough in (2.23), the boundary terms \( s^{-1/2} \| e^{-sa^*}(\xi^*)^{-1/4}\nabla \Delta z_k \|_{H^{1/2}(\Sigma)^3}^2 \) can be absorbed by the terms in the left-hand side of (2.28).
Thus, using (2.15) and (2.10) for \( w \) in the right-hand side of (2.23), we have for the moment

\[
I(s, z) + J(s, \tilde{\psi}) \leq C \left( \| \rho g \|_{L^2(Q)}^2 + \| \rho g_0 \|_{L^2(Q)}^2 + s^5 \int_0^T \int_{\omega_1} e^{-2s\alpha \xi^5} |\tilde{\psi}|^2 \, dx \, dt \\
+ s^7 \int_0^T \int_{\omega_1} e^{-2s\alpha \xi^7 \rho_2^2} |\varphi_1|^2 \, dx \, dt + s^7 \int_0^T \int_{\omega_1} e^{-2s\alpha \xi^7 \rho_3^2} |\varphi_3|^2 \, dx \, dt \right),
\]

for every \( s \geq C \). Furthermore, notice that using again (2.15), (2.10) for \( w \) and (2.26) we obtain from the previous inequality

\[
s^2 \int_Q e^{-2s\alpha^* (\xi^*)^{7/4} \rho_2^2} |\varphi_{3,t}|^2 \, dx \, dt + \tilde{I}(s, \rho \varphi) + J(s, \tilde{\psi}) \leq C \left( \| \rho g \|_{L^2(Q)}^2 + \| \rho g_0 \|_{L^2(Q)}^2 + s^5 \int_0^T \int_{\omega_1} e^{-2s\alpha \xi^5} |\tilde{\psi}|^2 \, dx \, dt \\
+ s^7 \int_0^T \int_{\omega_1} e^{-2s\alpha \xi^7 \rho_2^2} |\varphi_1|^2 \, dx \, dt + s^7 \int_0^T \int_{\omega_1} e^{-2s\alpha \xi^7 \rho_3^2} |\varphi_3|^2 \, dx \, dt \right), \quad (2.29)
\]

for every \( s \geq C \), where

\[
\tilde{I}(s, \rho \varphi) := \sum_{k=1,3} \left[ s^3 \int_Q e^{-2s\alpha \xi^3 \rho_2^2} |\Delta \varphi_k|^2 \, dx \, dt + s^4 \int_Q e^{-2s\alpha \xi^4 \rho_2^2} |\nabla \varphi_k|^2 \, dx \, dt \\
+ s^6 \int_Q e^{-2s\alpha \xi^6 \rho_2^2} |\varphi_k|^2 \, dx \, dt \right] + s^4 \int_Q e^{-2s\alpha^* (\xi^*)^{4} \rho_2^2} |\varphi_2|^2 \, dx \, dt.
\]

**Estimate of \( \varphi_3 \).** We deal in this part with the last term in the right-hand side of (2.29). We introduce a function \( \zeta_1 \in C_0^2(\omega) \) such that \( \zeta_1 \geq 0 \) and \( \zeta_1 = 1 \) in \( \omega_1 \), and using equation (2.14) we have

\[
Cs^7 \int_0^T \int_{\omega_1} e^{-2s\alpha \xi^7 \rho_3^2} |\varphi_3|^2 \, dx \, dt \leq Cs^7 \int_0^T \int_\omega \zeta_1 e^{-2s\alpha \xi^7 \rho_3^2} |\varphi_3|^2 \, dx \, dt \\
= Cs^7 \int_\omega \zeta_1 e^{-2s\alpha \xi^7 \rho} \varphi_3 (-\tilde{\psi}_t - \Delta \tilde{\psi} - \rho g_0 + \rho' \psi) \, dx \, dt,
\]

and we integrate by parts in this last term, in order to estimate it by local integrals of \( \tilde{\psi} \), \( g_0 \) and \( \epsilon I(s, \rho \varphi) \). This approach was already introduced in [5].

We first integrate by parts in time taking into account that
\[ e^{-2\alpha (0)} \xi^7 (0) = e^{-2\alpha (T)} \xi^7 (T) = 0: \]

\[-Cs^7 \int_0^T \int_\omega \zeta_1 e^{-2\alpha \xi^7} \rho \varphi_3 \tilde{\psi} dx \, dt \]

\[= Cs^7 \int_0^T \int_\omega \zeta_1 e^{-2\alpha \xi^7} \rho \varphi_3 \tilde{\psi} dx \, dt + Cs^7 \int_0^T \int_\omega \zeta_1 (e^{-2\alpha \xi^7} \rho) \varphi_3 \tilde{\psi} dx \, dt \]

\[\leq \epsilon \left( s^2 \int_0^T \int_\omega e^{-2\alpha \xi} (\xi^*)^{7/4} |\varphi_3|^2 dx \, dt + \tilde{I}(s, \rho \varphi) \right) + C(\lambda, \epsilon) \left( s^{12} \int_0^T \int_\omega e^{-4\alpha + 2\alpha^*} \xi^{19/4} |\tilde{\psi}|^2 dx \, dt + s^{10} \int_0^T \int_\omega e^{-2\alpha \xi^{11/4}} |\tilde{\psi}|^2 dx \, dt \right), \]

where we have used that

\[ |(e^{-2\alpha \xi^7} \rho)\xi| \leq Cs e^{-2\alpha \xi^6/8} \rho \]

and Young's inequality. Now we integrate by parts in space:

\[-Cs^7 \int_0^T \int_\omega \zeta_1 e^{-2\alpha \xi^7} \rho \varphi_3 \Delta \tilde{\psi} dx \, dt = -Cs^7 \int_0^T \int_\omega \zeta_1 e^{-2\alpha \xi^7} \rho \Delta \varphi_3 \tilde{\psi} dx \, dt \]

\[\quad - 2Cs^7 \int_0^T \int_\omega \nabla (\zeta_1 e^{-2\alpha \xi^7}) \cdot \rho \nabla \varphi_3 \tilde{\psi} dx \, dt - Cs^7 \int_0^T \int_\omega \Delta (\zeta_1 e^{-2\alpha \xi^7}) \rho \varphi_3 \tilde{\psi} dx \, dt \]

\[\leq \epsilon \tilde{I}(s, \rho \varphi) + C(\epsilon) s^{12} \int_0^T \int_\omega e^{-2\alpha \xi^{12}} |\tilde{\psi}|^2 dx \, dt, \]

where we have used that

\[ \nabla (\zeta_1 e^{-2\alpha \xi^7}) \leq Cs e^{-2\alpha \xi^8} \text{ and } \Delta (\zeta_1 e^{-2\alpha \xi^7}) \leq Cs^2 e^{-2\alpha \xi^9}, \]

and Young's inequality.

Finally,

\[Cs^7 \int_0^T \int_\omega \zeta_1 e^{-2\alpha \xi^7} \rho \varphi_3 (-\rho \varphi_3 + \rho' \tilde{\psi}) dx \, dt \]

\[\leq Cs^7 \int_0^T \int_\omega \zeta_1 e^{-2\alpha \xi^7} \rho |\varphi_3|((\rho) |\varphi_3| + Cs^{9/8} |\tilde{\psi}|) dx \, dt \]

\[\leq \epsilon \tilde{I}(s, \rho \varphi) + C(\epsilon) \left( s^8 \int_0^T \int_\omega e^{-2\alpha \xi^8} \rho^2 |\varphi_3|^2 dx \, dt + s^4 \int_0^T \int_\omega e^{-2\alpha \xi^{11/4}} |\tilde{\psi}|^2 dx \, dt \right). \]

Setting \( \epsilon = 1/2 \) and noticing that

\[ e^{-2\alpha} \leq e^{-4\alpha + 2\alpha^*} \text{ in } Q, \]

(see (2.3)) we obtain (2.4) from (2.29). This completes the proof of Proposition 1.
3 Null controllability of the linear system

Here we are concerned with the null controllability of the system

\[
\begin{aligned}
Ly + \nabla p &= f + (v_1, 0, 0) \mathbb{1}_\omega + \theta \varepsilon_3 \quad \text{in } Q, \\
L\theta + y \cdot \nabla \theta &= f_0 + v_0 \mathbb{1}_\omega \quad \text{in } Q, \\
\nabla \cdot y &= 0 \\
y &= 0, \quad \theta = 0 \quad \text{on } \Sigma, \\
y(0) = y^0, \quad \theta(0) = \theta^0 \\
\end{aligned}
\]

(3.1)

where \( y^0 \in V, \theta^0 \in H^1_0(\Omega) \), \( f \) and \( f_0 \) are in appropriate weighted spaces, the controls \( v_0 \) and \( v_1 \) are in \( L^2(\omega \times (0, T)) \) and

\[ Lq = q_t - \Delta q. \]

Before dealing with the null controllability of (3.1), we will deduce a Carleman inequality. Then, there exists a constant \( C > 0 \) such that every solution \( (\varphi, \pi, \psi) \) of (1.9) satisfies:

\[
\int_Q e^{-5s\beta} (\gamma^*)^4 |\varphi|^2 \, dx \, dt + \int_Q e^{-5s\beta} (\gamma^*)^5 |\psi|^2 \, dx \, dt + \|\varphi(0)\|^2_{L^2(\Omega)} + \|\psi(0)\|^2_{L^2(\Omega)}
\]

\[
\leq C \left( \int_Q e^{-3s\beta}|g|^2 \, dx \, dt + \int_0^T \int_\Omega e^{-2s\beta - 3s\beta} \bar{\gamma} T |\varphi|^2 \, dx \, dt + \int_0^T \int_\omega e^{-4s\beta - s\beta} \bar{\gamma}^{3/4} |\psi|^2 \, dx \, dt \right). \quad (3.3)
\]

Let us also state this result for \( N = 2 \).

**Lemma 3.2.** Assume \( N = 2 \). Let \( s \) and \( \lambda \) be like in Proposition 2 and \((\bar{p}, \bar{\theta})\) satisfy (1.5)-(1.6). Then, there exists a constant \( C > 0 \) (depending on\( s, \lambda \) and \( \bar{\theta} \)) such that every solution \((\varphi, \pi, \psi)\) of (1.9) satisfies:

\[
\int_Q e^{-5s\beta} (\gamma^*)^4 |\varphi|^2 \, dx \, dt + \int_Q e^{-5s\beta} (\gamma^*)^5 |\psi|^2 \, dx \, dt + \|\varphi(0)\|^2_{L^2(\Omega)} + \|\psi(0)\|^2_{L^2(\Omega)}
\]

\[
\leq C \left( \int_Q e^{-3s\beta}|g|^2 \, dx \, dt + \int_0^T \int_\Omega e^{-2s\beta - 3s\beta} \bar{\gamma} T |\varphi|^2 \, dx \, dt + \int_0^T \int_\omega e^{-4s\beta - s\beta} \bar{\gamma}^{3/4} |\psi|^2 \, dx \, dt \right). \quad (3.4)
\]
Proof of Lemma 3.1: We start by an a priori estimate for system (1.9). To do this, we introduce a function \( \nu \in C^1([0,T]) \) such that

\[
\nu \equiv 1 \text{ in } [0,T/2], \quad \nu \equiv 0 \text{ in } [3T/4, T].
\]

We easily see that \((\nu \varphi, \nu \pi, \nu \psi)\) satisfies

\[
\begin{aligned}
-(\nu \varphi)_t - \Delta (\nu \varphi) + \nabla (\nu \pi) &= \nu g - (\nu \psi) \nabla \theta - \nu' \varphi, & \text{ in } Q, \\
-(\nu \psi)_t - \Delta (\nu \psi) &= \nu g_0 + \nu \varphi_3 - \nu' \psi, & \text{ in } Q, \\
\nabla \cdot (\nu \varphi) &= 0, & \text{ in } Q, \\
(\nu \varphi)_t = 0, (\nu \psi)_t = 0 & \text{ on } \Sigma, \\
(\nu \varphi)(T) = 0, (\nu \psi)(T) = 0 & \text{ in } \Omega,
\end{aligned}
\]

thus we have the energy estimate

\[
\|\nu \varphi\|^2_{L^2(0,T;V)} + \|\nu \psi\|^2_{L^2(0,T;H^1(\Omega))} + \|\nu \psi\|^2_{L^2(0,T;L^2(\Omega))} \\
\leq C(\|g\|^2_{L^2(Q)} + \|\varphi_3\|^2_{L^2(Q)} + \|g_0\|^2_{L^2(Q)} + \|\psi\|^2_{L^2(Q)}).
\]

Using the properties of the function \( \nu \), we readily obtain

\[
\|\varphi\|^2_{L^2(0,T/2;H)} + \|\varphi(0)\|^2_{L^2(\Omega)} + \|\psi\|^2_{L^2(0,T/2;L^2(\Omega))} + \|\psi(0)\|^2_{L^2(\Omega)} \\
\leq C \left( \|g\|^2_{L^2(0,T/2;L^2(\Omega))} + \|\varphi_3\|^2_{L^2(T/2,T/4;L^2(\Omega))} \\
+ \|g_0\|^2_{L^2(0,T/2;L^2(\Omega))} + \|\psi\|^2_{L^2(T/2,T/4;L^2(\Omega))} \right).
\]

From this last inequality, and the fact that

\[e^{-3s\beta^*} \geq C > 0, \forall t \in [0,3T/4] \text{ and } e^{-5s\alpha^*}(\xi^*)^4 \geq C > 0, \forall t \in [T/2,3T/4]\]

we have

\[
\int_0^{T/2} \int_\Omega e^{-3s\beta^*} (\gamma^*)^4 |\varphi|^2 \, dx \, dt + \int_0^{T/2} \int_\Omega e^{-5s\beta^*} (\gamma^*)^5 |\psi|^2 \, dx \, dt \\
+ \|\varphi(0)\|^2_{L^2(\Omega)} + \|\psi(0)\|^2_{L^2(\Omega)} \leq C \left( \int_0^{3T/4} \int_\Omega e^{-3s\beta^*} |g|^2 + |g_0|^2 \, dx \, dt \\
+ \int_{T/2}^{3T/4} \int_\Omega e^{-5s\alpha^*} (\xi^*)^4 |\varphi|^2 \, dx \, dt + \int_{T/2}^{3T/4} \int_\Omega e^{-5s\alpha^*} (\xi^*)^5 |\psi|^2 \, dx \, dt \right). \tag{3.5}
\]

Note that the last two terms in (3.5) are bounded by the left-hand side of the Carleman inequality (2.4). Since \( \alpha = \beta \) in \( \Omega \times (T/2, T) \), we have:

\[
\int_{T/2}^T \int_\Omega e^{-5s\beta^*} (\gamma^*)^4 |\varphi|^2 \, dx \, dt + \int_{T/2}^T \int_\Omega e^{-5s\beta^*} (\gamma^*)^5 |\psi|^2 \, dx \, dt \\
= \int_{T/2}^T \int_\Omega e^{-5s\alpha^*} (\xi^*)^4 |\varphi|^2 \, dx \, dt + \int_{T/2}^T \int_\Omega e^{-5s\alpha^*} (\xi^*)^5 |\psi|^2 \, dx \, dt \\
\leq \int_Q e^{-5s\alpha^*} (\xi^*)^4 |\varphi|^2 \, dx \, dt + \int_Q e^{-5s\alpha^*} (\xi^*)^5 |\psi|^2 \, dx \, dt.
\]
Combining this with the Carleman inequality (2.4), we deduce
\[
\int_{T/2}^{T} \int_{\Omega} e^{-5s\beta^*}(\gamma^*)^4 |\varphi|^2 \, dx \, dt + \int_{T/2}^{T} \int_{\Omega} e^{-5s\beta^*}(\gamma^*)^5 |\psi|^2 \, dx \, dt \\
\leq C \left( \int_{Q} e^{-3s\alpha^*}(|y|^2 + |g_0|^2) \, dx \, dt + \int_{0}^{T} \int_{\omega} e^{-2s\bar{\alpha} - 3s\alpha^*}(\xi^7)|\varphi_1|^2 \, dx \, dt \\
+ \frac{T}{0} \int_{\omega} e^{-4s\bar{\alpha} - s\alpha^*}(\xi^7)^{49/4}|\psi|^2 \, dx \, dt \right).
\]
Since
\[
e^{-3s\beta^*}, e^{-2s\bar{\beta} - 3s\beta^*} \gamma_7, e^{-2s\bar{\beta} - 3s\beta^*} \gamma_7, e^{-4s\bar{\beta} - s\beta^*} \gamma^{49/4} \geq C > 0, \forall t \in [0,T/2],
\]
we can readily get
\[
\int_{T/2}^{T} \int_{\Omega} e^{-5s\beta^*}(\gamma^*)^4 |\varphi|^2 \, dx \, dt + \int_{T/2}^{T} \int_{\Omega} e^{-5s\beta^*}(\gamma^*)^5 |\psi|^2 \, dx \, dt \\
\leq C \left( \int_{Q} e^{-3s\beta^*}(|y|^2 + |g_0|^2) \, dx \, dt + \int_{0}^{T} \int_{\omega} e^{-2s\bar{\beta} - 3s\beta^*} \gamma_7 |\varphi_1|^2 \, dx \, dt \\
+ \frac{T}{0} \int_{\omega} e^{-4s\bar{\beta} - s\beta^*} \gamma^{49/4} |\psi|^2 \, dx \, dt \right),
\]
which, together with (3.5), yields (3.3).

Now we will prove the null controllability of (3.1). Actually, we will prove the existence of a solution for this problem in an appropriate weighted space. Let us introduce the space
\[
E = \{ (y,p,v_1,\theta,v_0) : e^{3/2s\beta^*} y, e^{2s\beta^*} \theta, e^{5/2s\beta^*} \gamma, e^{3/2s\beta^*} \gamma, e^{3/2s\beta^*} \gamma, e^{5/2s\beta^*} \gamma \in L^2(Q), \}
\]
It is clear that \( E \) is a Banach space for the following norm:
\[
\|(y,p,v_1,\theta,v_0)\|_E = \left( \|e^{3/2s\beta^*} y\|_{L^2(Q)}^2 + \|e^{3/2s\beta^*} \theta\|_{L^2(Q)}^2 + \|e^{2s\beta^*} \gamma, e^{3/2s\beta^*} \gamma, e^{5/2s\beta^*} \gamma\|_{L^\infty(0,T;V)}^2 \right).
Remark 4. Observe in particular that \((y, p, v_1, \theta, \nu_0) \in E\) implies \(y(T) = 0\) and \(\theta(T) = 0\) in \(\Omega\). Moreover, the functions belonging to this space possess the interesting following property:

\[
e^{5/2s^*}(\gamma^*)^{-2}(y \cdot \nabla)y \in L^2(Q)^3 \quad \text{and} \quad e^{5/2s^*}(\gamma^*)^{-5/2}y \cdot \nabla \theta \in L^2(Q).
\]

Proposition 3. Assume \(N = 3\), \((p, \theta)\) satisfies (1.5)-(1.6) and

\[
y^0 \in V, \theta_0 \in H^1_0(\Omega), e^{5/2s^*}(\gamma^*)^{-2}f \in L^2(Q)^3 \quad \text{and} \quad e^{5/2s^*}(\gamma^*)^{-5/2}f_0 \in L^2(Q).
\]

Then, we can find controls \(v_1\) and \(\nu_0\) such that the associated solution \((y, p, \theta)\) to (3.1) satisfies

\[
(y, p, v_1, \theta, \nu_0) \in E. \quad \text{In particular,} \quad y(T) = 0 \quad \text{and} \quad \theta(T) = 0.
\]

Sketch of the proof: The proof of this proposition is very similar to the one of Proposition 2 in [9] (see also Proposition 2 in [4] and Proposition 3.3 in [2]), so we will just give the main ideas.

Following the arguments in [6] and [10], we introduce the space

\[P_0 = \{ (\chi, \sigma, \kappa) \in C^2(\Omega)^5 : \nabla \cdot \chi = 0, \chi = 0 \text{ on } \Sigma, \kappa = 0 \text{ on } \Sigma \}\]

and we consider the following variational problem: find \((\hat{\chi}, \hat{\sigma}, \hat{\kappa}) \in P_0\) such that

\[a((\hat{\chi}, \hat{\sigma}, \hat{\kappa}), (\chi, \sigma, \kappa)) = \langle G, (\chi, \sigma, \kappa) \rangle \quad \forall (\chi, \sigma, \kappa) \in P_0, \tag{3.6}\]

where we have used the notations

\[
a((\hat{\chi}, \hat{\sigma}, \hat{\kappa}), (\chi, \sigma, \kappa)) = \iint_Q e^{-3s^*}(L^* \chi + \nabla \sigma + \kappa \nabla \theta) \cdot (L^* \chi + \nabla \sigma + \kappa \nabla \theta) \, dx \, dt
\]

\[+ \iint_Q e^{-3s^*}(L^* \kappa - \hat{\kappa} \lambda)(L^* \kappa - \chi \lambda) \, dx \, dt + \iint_0^T \iint_\Omega e^{-2s^*} \gamma^{-1} \chi_1 \chi_1 \, dx \, dt \]

\[+ \iint_0^T \iint_\Omega e^{-4s^*} \gamma^{-1} \kappa \kappa \, dx \, dt, \]

\[
\langle G, (\chi, \sigma, \kappa) \rangle = \iint_Q f \cdot \chi \, dx \, dt + \iint_Q f_0 \kappa \, dx \, dt + \iint_\Omega y^0 \cdot \chi(0) \, dx + \iint_\Omega \theta^0 \kappa(0) \, dx
\]

and \(L^*\) is the adjoint operator of \(L\), i.e.

\[
L^* q = -q_t - \Delta q.
\]

It is clear that \(a(\cdot, \cdot, \cdot) : P_0 \times P_0 \to \mathbf{R}\) is a symmetric, definite positive bilinear form on \(P_0\). We denote by \(P\) the completion of \(P_0\) for the norm induced by \(a(\cdot, \cdot, \cdot)\). Then \(a(\cdot, \cdot, \cdot)\) is well-defined, continuous and again definite positive on \(P\). Furthermore, in view of the Carleman estimate (3.3), the linear form \((\chi, \sigma, \kappa) \mapsto \langle G, (\chi, \sigma, \kappa) \rangle\) is well-defined and continuous on \(P\). Hence, from Lax-Milgram’s lemma, we deduce that the variational problem

\[
\begin{cases}
a((\hat{\chi}, \hat{\sigma}, \hat{\kappa}), (\chi, \sigma, \kappa)) = \langle G, (\chi, \sigma, \kappa) \rangle \\
\forall (\chi, \sigma, \kappa) \in P, \quad (\hat{\chi}, \hat{\sigma}, \hat{\kappa}) \in P,
\end{cases}
\]

possesses exactly one solution \((\hat{\chi}, \hat{\sigma}, \hat{\kappa})\).
Let $\hat{y}, \hat{v}_1, \hat{\theta}$ and $\hat{v}_0$ be given by
\[
\begin{align*}
\hat{y} &= e^{-3 \beta \gamma} (L^* \hat{x} + \nabla \hat{\sigma} + \kappa \nabla \hat{\theta}), \quad \text{in } Q, \\
\hat{v}_1 &= e^{-2 \beta - 3 \beta \gamma} \chi_1, \quad \text{in } \omega \times (0, T), \\
\hat{\theta} &= e^{-3 \beta \gamma} (L^* \hat{\kappa} - \hat{\chi}_3), \quad \text{in } Q, \\
\hat{v}_0 &= e^{-4 \beta - 3 \beta \gamma} \kappa_0, \quad \text{in } \omega \times (0, T).
\end{align*}
\]
Then, it is readily seen that they satisfy
\[
\begin{align*}
\iint_Q e^{3 \beta \gamma} |\hat{y}|^2 \, dx \, dt + \iint_Q e^{3 \beta \gamma} |\hat{\theta}|^2 \, dx \, dt + \int_0^T \iint_\omega e^{2 \beta + 3 \beta \gamma} \hat{\gamma}^{-2} |\hat{v}_1|^2 \, dx \, dt \\
+ \int_0^T \iint_\omega e^{4 \beta + s \gamma} \hat{\gamma}^{-4/4} |\hat{v}_0|^2 \, dx \, dt = a((\hat{x}, \hat{\sigma}, \hat{\kappa}), (\hat{x}, \hat{\sigma}, \kappa)) < +\infty
\end{align*}
\]
and also that $(\hat{y}, \hat{\theta})$ is, together with some pressure $\hat{p}$, the weak solution of the system (3.1) for $v_1 = \hat{v}_1$ and $v_0 = \hat{v}_0$.

It only remains to check that
\[
e^{3/2 \beta} (\gamma^*)^{-9/8} \hat{y} \in L^2(0, T; H^2(\Omega)^3) \cap L^\infty(0, T; V)
\]
and
\[
e^{3/2 \beta} (\gamma^*)^{-9/8} \hat{\theta} \in L^2(0, T; H^2(\Omega)) \cap L^\infty(0, T; H^0(\Omega))
\]
To this end, we define the functions
\[
y^* = e^{3/2 \beta} (\gamma^*)^{-9/8} \hat{y}, p^* = e^{3/2 \beta} (\gamma^*)^{-9/8} \hat{p}, \theta^* = e^{3/2 \beta} (\gamma^*)^{-9/8} \hat{\theta},
\]
\[
f^* = e^{3/2 \beta} (\gamma^*)^{-9/8} (f + (\hat{v}_1, 0, 0) \mathbb{1}_\omega) \text{ and } f_0^* = e^{3/2 \beta} (\gamma^*)^{-9/8} (f_0 + \hat{v}_0 \mathbb{1}_\omega).
\]
Then $(y^*, p^*, \theta^*)$ satisfies
\[
\begin{align*}
Ly^* + \nabla p^* &= f^* + \theta^* e_3 + (e^{3/2 \beta} (\gamma^*)^{-9/8}) \hat{y}, \quad \text{in } Q, \\
L\theta^* + y^* \cdot \nabla \theta &= f_0^* + (e^{3/2 \beta} (\gamma^*)^{-9/8}) \hat{\theta}, \quad \text{in } Q, \\
\nabla \cdot y^* &= 0, \quad \text{in } Q, \\
y^* = 0, \quad \text{on } \Sigma, \\
y^*(0) &= e^{3/2 \beta} (\gamma^*(0))^{-9/8} y_0, \quad \text{in } \Omega, \\
\theta^*(0) &= e^{3/2 \beta} (\gamma^*(0))^{-9/8} \theta_0, \quad \text{in } \Omega.
\end{align*}
\]
From the fact that $f^* + (e^{3/2 \beta} (\gamma^*)^{-9/8}) \hat{y} \in L^2(\Omega)^3$, $f_0^* + (e^{3/2 \beta} (\gamma^*)^{-9/8}) \hat{\theta} \in L^2(\Omega)$, $y_0 \in V$ and $\theta_0 \in H^0_0(\Omega)$, we have indeed
\[
y^* \in L^2(0, T; H^2(\Omega)^3) \cap L^\infty(0, T; V) \text{ and } \theta^* \in L^2(0, T; H^2(\Omega)) \cap L^\infty(0, T; H^0_0(\Omega)) \quad (\text{see (2.10)}).
\]
This ends the sketch of the proof of Proposition 3. \hfill \Box

4 Proof of Theorem 1.1

In this section we give the proof of Theorem 1.1 using similar arguments to those in [10] (see also [4], [5], [9] and [2]). The result of null controllability for the linear system (3.1) given by Proposition 3 will allow us to apply an inverse mapping theorem. Namely, we will use the following theorem (see [1]).
Thus, we have reduced our problem to the local null controllability of the nonlinear system and the operator $\tilde{C}$.

Let us set 

$$y = \tilde{y}, \quad p = \tilde{p} + \bar{p} \quad \text{and} \quad \theta = \tilde{\theta} + \bar{\theta}.$$ 

Using (1.1) and (1.5) we obtain 

$$\tilde{C}(y) = (\tilde{y}, \tilde{\theta}, \tilde{v}_0) \in B.$$ 

Theorem 4.1. 

We will prove that the bilinear operator $\tilde{C}$ is of class $C^1(B_1; B_2)$.

We apply Theorem 4.1 setting $b_1 = E$.

$$B_2 = L^2(e^{5/2s\beta^*} (\gamma^*)^{-2}(0, T); L^2(\Omega)^3) \times V \times L^2(e^{5/2s\beta^*} (\gamma^*)^{-5/2}(0, T); L^2(\Omega)) \times H_0^1(\Omega)$$ 

and the operator 

$$\mathcal{A}(\tilde{y}, \tilde{p}, v_1, \tilde{\theta}, v_0) = (L\tilde{y} + (\tilde{y} \cdot \nabla)\tilde{y} + \nabla\tilde{p} - \tilde{\theta} e_3 - (v_1, 0, 0) I_{\omega}, \tilde{y}, 0), \quad \tilde{\theta}(0) = 0.$$ 

for $(\tilde{y}, \tilde{p}, v_1, \tilde{\theta}, v_0) \in E$.

In order to apply Theorem 4.1, it remains to check that the operator $\mathcal{A}$ is of class $C^1(B_1; B_2)$. Indeed, notice that all the terms in $\mathcal{A}$ are linear, except for $(\tilde{y} \cdot \nabla)\tilde{y}$ and $\tilde{y} \cdot \nabla\tilde{\theta}$.

We will prove that the bilinear operator 

$$(y^1, p^1, v_1^1, \theta^1, v_0^1), (y^2, p^2, v_1^2, \theta^2, v_0^2) \rightarrow (y^1 \cdot \nabla) y^2$$ 

is continuous from $B_1 \times B_1$ to $L^2(e^{5/2s\beta^*} (\gamma^*)^{-2}(0, T); L^2(\Omega)^3)$.

To do this, notice that 

$$e^{3/2s\beta^*} (\gamma^*)^{-9/8} y \in L^2(0, T; H^2(\Omega)^3) \cap L^\infty(0, T; V)$$ 

for any $(y, p, v_1, \theta, v_0) \in B_1$, so we have 

$$e^{3/2s\beta^*} (\gamma^*)^{-9/8} y \in L^2(0, T; L^\infty(\Omega)^3)$$ 

and 

$$\nabla(e^{3/2s\beta^*} (\gamma^*)^{-9/8} y) \in L^\infty(0, T; L^2(\Omega)^3).$$ 

Consequently, we obtain 

$$\|e^{5/2s\beta^*} (\gamma^*)^{-2}(y^1 \cdot \nabla) y^2\|_{L^2(\Omega)^3} \leq C \|e^{3/2s\beta^*} (\gamma^*)^{-9/8} y^1 \cdot \nabla e^{3/2s\beta^*} (\gamma^*)^{-9/8} y^2\|_{L^2(\Omega)^3}$$ 

and 

$$\nabla(e^{3/2s\beta^*} (\gamma^*)^{-9/8} y) \|_{L^2(0, T; L^\infty(\Omega)^3)} \|e^{3/2s\beta^*} (\gamma^*)^{-9/8} y^2\|_{L^\infty(0, T; V)}.$$ 

In the same way, we can prove that the bilinear operator 

$$(y^1, p^1, v_1^1, \theta^1, v_0^1), (y^2, p^2, v_1^2, \theta^2, v_0^2) \rightarrow y^1 \cdot \nabla \theta^2$$
is continuous from $B_1 \times B_1$ to $L^2(\mathbb{R}^{5/2s\beta^*}(\gamma^*); L^2(\Omega))$ just by taking into account that

$$e^{5/2s\beta^*}(\gamma^*)^{-5/2}(0,T; L^2(\Omega)), $$

for any $(y, p, v_1, \theta, v_0) \in B_1$.

Notice that $A'(0,0,0,0) : B_1 \rightarrow B_2$ is given by

$$A'(0,0,0,0)(\tilde{y}, \tilde{p}, v_1, \tilde{\theta}, v_0) = (L\tilde{\theta} + \nabla \tilde{p} - \tilde{\theta} e_3 - (v_1,0,0) \mathbb{1}_\omega, \tilde{y}(0),$$

$$L\tilde{\theta} + \tilde{y} \cdot \nabla \tilde{\theta} - v_0 \mathbb{1}_\omega, \tilde{\theta}(0)), $$

for all $(\tilde{y}, \tilde{p}, v_1, \tilde{\theta}, v_0) \in B_1$, so this functional is surjective in view of the null controllability result for the linear system (3.1) given by Proposition 3.

We are now able to apply Theorem 4.1 for $b_1 = (0,0,0,0,0)$ and $b_2 = (0,0,0,0,0)$. In particular, this gives the existence of a positive number $\delta > 0$ such that, if $\|\tilde{y}(0), \tilde{\theta}(0)\|_{V \times H_0^1(\Omega)} \leq \delta$, then we can find controls $v_1$ and $v_0$ such that the associated solution $(\tilde{y}, \tilde{p}, \tilde{\theta})$ to (4.1) satisfies $\tilde{y}(T) = 0$ and $\tilde{\theta}(T) = 0$ in $\Omega$.

This concludes the proof of Theorem 1.1.

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References


