From the Ginzburg-Landau Model to Vortex Lattice Problems

Etienne Sandier and Sylvia Serfaty

Second version, January 2012

Abstract

We introduce a "Coulombian renormalized energy" W which is a logarithmic type of interaction between points in the plane, computed by a "renormalization." We prove various of its properties, such as the existence of minimizers, and show in particular, using results from number theory, that among lattice configurations the triangular lattice is the unique minimizer. Its minimization in general remains open.

Our motivation is the study of minimizers of the two-dimensional Ginzburg-Landau energy with applied magnetic field, between the first and second critical fields H_{c_1} and H_{c_2}. In that regime, minimizing configurations exhibit densely packed triangular vortex lattices, called Abrikosov lattices. We derive, in some asymptotic regime, W as a Γ-limit of the Ginzburg-Landau energy. More precisely we show that the vortices of minimizers of Ginzburg-Landau, blown-up at a suitable scale, converge to minimizers of W, thus providing a first rigorous hint at the Abrikosov lattice. This is a next order effect compared to the mean-field type results we previously established.

The derivation of W uses energy methods: the framework of Γ-convergence, and an abstract scheme for obtaining lower bounds for "2-scale energies" via the ergodic theorem that we introduce.

keywords: Ginzburg-Landau, vortices, Abrikosov lattice, triangular lattice, renormalized energy, Gamma-convergence.
MSC classification: 35B25, 82D55, 35Q99, 35J20, 52C17.

1 Introduction

In this paper, we are interested in deriving a "Coulombian renormalized energy" from the Ginzburg-Landau model of superconductivity. We will start by defining and presenting the renormalized energy in Section 1.1, then state some results about it in Section 1.2. In Section 1.3, we then present an abstract method for lower bounds for two-scale energies using ergodic theory. In Sections 1.5–1.10 we turn to the Ginzburg-Landau model, and give our main results about it as well as ingredients for the proof.

1.1 The Coulombian renormalized energy W

The interaction energy W that we wish to define just below is a natural energy for the Coulombian interaction of charged particles in the plane screened by a uniform background: it could be called a "screened Coulombian renormalized energy". It can be seen in our context as the analogue for an infinite number of points in $\mathbb{R}^2$ of the renormalized energy W introduced in Bethuel-Brezis-Hélein [BBH] for a finite number of points in a bounded domain, or of the

Kirchhof-Onsager function. We believe that this energy is quite ubiquitous in all problems that have an underlying Coulomb interaction: it already arises in the study of weighted Fekete sets and of the statistical mechanics of Coulomb gases and random matrices [SS6], as well as a limit in some parameter regime for the Ohta-Kawasaki model [GMS2]. In [SS7] we introduce a one-dimensional analogue (a renormalized logarithmic interaction for points on the line) which we also connect to one-dimensional Fekete sets as well as "log gases" and random matrices.

We will discuss more at the end of this subsection and in the next, but let us first give the precise definition.

In all the paper, B_R denotes the ball centered at 0 and of radius R, and $|\cdot|$ denotes the area of a set.

Definition 1.1. *Let m be a positive number. Let j be a vector field in $\mathbb{R}^2$. We say j belongs to the admissible class $\mathcal{A}_m$ if*

$$\operatorname{curl} j = \nu - m, \qquad \operatorname{div} j = 0, \tag{1.1}$$

where ν has the form

$$\nu = 2\pi \sum_{p\in\Lambda} \delta_p \quad \textit{for some discrete set } \Lambda \subset \mathbb{R}^2,$$

and

$$\frac{\nu(B_R)}{|B_R|} \quad \textit{is bounded by a constant independent of } R > 1. \tag{1.2}$$

For any family of sets $\{\mathbf{U}_R\}_{R>0}$ in $\mathbb{R}^2$ we use the notation $\chi_{\mathbf{U}_R}$ for positive cutoff functions satisfying, for some constant C independent of R,

$$|\nabla \chi_{\mathbf{U}_R}| \le C, \quad \operatorname{Supp}(\chi_{\mathbf{U}_R}) \subset \mathbf{U}_R, \quad \chi_{\mathbf{U}_R}(x) = 1 \text{ if } d(x, {\mathbf{U}_R}^c) \ge 1. \tag{1.3}$$

We will always implicitly assume that $\{\mathbf{U}_R\}_{R>0}$ is an increasing family of bounded open sets, and we will use the following set of additional assumptions:

- $$\{\mathbf{U}_R\} \text{ is a Vitali family and } \lim_{R\to+\infty} \frac{|(\lambda + \mathbf{U}_R) \triangle \mathbf{U}_R|}{|\mathbf{U}_R|} = 0. \tag{1.4}$$

  for any $\lambda \in \mathbb{R}^2$. Here, a Vitali family (see [Ri]) means that the intersection of the closures is $\{0\}$, that $R \mapsto |\mathbf{U}_R|$ is left continuous, and that $|\mathbf{U}_R - \mathbf{U}_R| \le C|\mathbf{U}_R|$ for some constant $C > 0$ independent of R.

- There exists $\theta < 2$ such that for any $R > 0$

  $$\mathbf{U}_R + B(0,1) \subset \mathbf{U}_{R+C}, \quad \mathbf{U}_{R+1} \subset \mathbf{U}_R + B(0,C), \quad |\mathbf{U}_{R+1} \setminus \mathbf{U}_R| = O(R^\theta). \tag{1.5}$$

Definition 1.2. *The Coulombian renormalized energy W is defined, for $j \in \mathcal{A}_m$, by*

$$W(j) = \limsup_{R\to\infty} \frac{W(j, \chi_{B_R})}{|B_R|}, \tag{1.6}$$

where for any function χ we denote

$$W(j,\chi) = \lim_{\eta\to 0}\left(\frac{1}{2}\int_{\mathbb{R}^2\setminus\cup_{p\in\Lambda}B(p,\eta)} \chi|j|^2 + \pi\log\eta\sum_{p\in\Lambda}\chi(p)\right). \tag{1.7}$$

We similarly define the renormalized energy relative to the family $\{\mathbf{U}_R\}_{R>0}$ by

$$W_U(j) = \limsup_{R\to\infty}\frac{W(j,\chi_{\mathbf{U}_R})}{|\mathbf{U}_R|}. \tag{1.8}$$

Let us make several remarks about the definition.

1. We will see in Theorem 1 that the value of W does not depend on $\{\chi_{B_R}\}_R$ as long as it satisfies (1.3). The corresponding statement holds for W_U under the assumptions (1.4)–(1.5).
2. Since in the neighborhood of $p\in\Lambda$ we have $\operatorname{curl} j = 2\pi\delta_p - 1$, $\operatorname{div} j = 0$, we have near p the decomposition $j(x) = \nabla^\perp \log|x-p| + f(x)$ where f is smooth, and it easily follows that the limit (1.7) exists. It also follows that j belongs to L^p_{loc} for any $p<2$.
3. From (1.1) we have $j = -\nabla^\perp H$ for some H, and then
$$-\Delta H = 2\pi\sum_{p\in\Lambda}\delta_p - m.$$
Then the energy in (1.7) can be seen as the (renormalized) interaction energy between the "charged particles" at $p\in\Lambda$ and between them and a constant background $-m$. We prefer to take $j=-\nabla^\perp H$ as the unknown, though, because it is related to the superconducting current j_ε.
4. We will see in Theorem 1 that the minimizers and the value of the minimum of W_U are independent of U, provided (1.4) and (1.5) hold. However there are examples of admissible j's (nonminimizers) for which $W_U(j)$ depends on the family of shapes $\{\mathbf{U}_R\}_{R>0}$ which is used.
5. Because the number of points is infinite, the interaction over large balls needs to be normalized by the volume and thus W does not feel compact perturbations of the configuration of points. Even though the interactions are long-range, this is not difficult to justify rigorously.
6. The cut-off function χ_R cannot simply be replaced by the characteristic function of B_R because for every $p\in\Lambda$
$$\lim_{\substack{R\to|p|\\ R<|p|}} W(j,\mathbf{1}_{B_R}) = +\infty, \quad \lim_{\substack{R\to|p|\\ R>|p|}} W(j,\mathbf{1}_{B_R}) = -\infty.$$
7. It is easy to check that if j belongs to $\mathcal{A}_m$ then $j' = \frac{1}{\sqrt{m}}j(\cdot/\sqrt{m})$ belongs to $\mathcal{A}_1$ and
$$W(j) = m\left(W(j') - \frac{1}{4}\log m\right). \tag{1.9}$$
so we may reduce to the study of W over $\mathcal{A}_1$.

When the set of points Λ is periodic with respect to some lattice $\mathbb{Z}\vec{u}+\mathbb{Z}\vec{v}$ then it can be viewed as a set of n points $a_1, \cdots, a_n$ over the torus $\mathbb{T}_{(\vec{u},\vec{v})} = \mathbb{R}^2/(\mathbb{Z}\vec{u}+\mathbb{Z}\vec{v})$. There also exists a unique periodic (with same period) $j_{\{a_i\}}$ with mean zero and satisfying (1.1) for some m which from (1.1) and the periodicity of $j_{\{a_i\}}$ must be equal to $2\pi n$ divided by the surface of the periodicity cell. Moreover $j_{\{a_i\}}$ minimizes W among $(\vec{u},\vec{v})$-periodic solutions of (1.1) (see Proposition 3.1). The computation of W in this setting where both Λ and j are periodic is quite simpler (the need for the limit $R \to \infty$ and the cutoff function disappear). By the scaling formula (1.9), we may reduce to working in $\mathcal{A}_n$ in a situation where the volume of the torus is 2π. Then we will see in Section 3.1 the following

Lemma 1.3. *With the above notation, we have*

$$W(j_{\{a_i\}}) = \frac{1}{2}\sum_{i\neq j} G(a_i - a_j) + nc_{(\vec{u},\vec{v})} \tag{1.10}$$

where $c_{(\vec{u},\vec{v})}$ is a constant depending only on $(\vec{u},\vec{v})$ and G is the Green function of the torus with respect to its volume form, i.e. the solution to

$$-\Delta G(x) = 2\pi\delta_0 - 1 \quad \text{in} \quad \mathbb{T}_{(\vec{u},\vec{v})}.$$

Moreover, $j_{\{a_i\}}$ is the minimizer of $W(j)$ among all $\mathbb{T}_{(\vec{u},\vec{v})}$-periodic j's satisfying (1.1)*.*

Remark 1.4. *The Green function of the torus admits an explicit Fourier series expansion, through this we can obtain a more explicit formula for the right-hand side of* (1.10)*:*

$$W(j_{\{a_i\}}) = \frac{1}{2}\sum_{i\neq j}\sum_{p\in(\mathbb{Z}\vec{u}+\mathbb{Z}\vec{v})^*\setminus\{0\}} \frac{e^{2i\pi p\cdot(a_i-a_j)}}{4\pi^2|p|^2} + \frac{n}{2}\lim_{x\to 0}\left(\sum_{p\in(\mathbb{Z}\vec{u}+\mathbb{Z}\vec{v})^*\setminus\{0\}} \frac{e^{2i\pi p\cdot x}}{4\pi^2|p|^2} + \log|x|\right) \tag{1.11}$$

where * refers to the dual of a lattice.

The function $\sum_{i\neq j} G(a_i - a_j)$ is the sum of pairwise Coulombian interactions between particles on a torus. It arises for example in number theory (Arakelov theory), see [La2] p. 150, where a result attributed to Elkies is stated: $\sum_{i\neq j} G(a_i - a_j) \geq -\frac{n}{4}\log n + O(n)$ (on any Riemann surface of genus ≥ 1). Note that we can retrieve this estimate in the case of the torus by using the fact that $\min_{\mathcal{A}_1} W$ is finite and formula (1.12) with $m = n$.

So conversely, another way of looking at our energy W is that it provides a way of computing an analogue of $\sum_{i\neq j} G(a_i - a_j)$ in an infinite-size domain.

1.2 Results and conjecture on the renormalized energy

The following theorem summarizes the basic results about the minimization of W. Note that by the scaling relation (1.9) we may reduce to the case of $\mathcal{A}_1$, and we have

$$\min_{\mathcal{A}_m} W = m\left(\min_{\mathcal{A}_1} W - \frac{1}{4}\log m\right). \tag{1.12}$$

Theorem 1. *Let W be as in Definition 1.2.*

1. *Let $\{\mathbf{U}_R\}_{R>0}$ be a family of sets satisfying* (1.4)–(1.5)*, then for any $j \in \mathcal{A}_1$, the value of $W_U(j)$ is independent of the choice of $\chi_{\mathbf{U}_R}$ in its definition as long as it satisfies* (1.3)*.*

2. W is Borel measurable on $L^p_{loc}(\mathbb{R}^2, \mathbb{R}^2)$, $p < 2$.

3. $\min_{\mathcal{A}_1} W_U$ is achieved and finite, and it is independent of the choice of U, as long as $\{\mathbf{U}_R\}$ satisfies (1.4)–(1.5).

4. There exists a minimizing sequence $\{j_n\}_{n\in\mathbb{N}}$ for $\min_{\mathcal{A}_1} W$ consisting of vector-fields which are periodic (with respect to a square lattice of sidelength $\sqrt{2\pi n}$).

The question of identifying the minimum and minimizers of W seems very difficult. In fact it is natural to expect that the triangular lattice (of appropriate volume) minimizes W over any $\mathcal{A}_m$, we will come back to this below. We show here a weaker but nontrivial result: the triangular lattice is the unique minimizer among lattice configurations.

When the set of points Λ itself is a lattice, i.e. of the form $\mathbb{Z}\vec{u} \oplus \mathbb{Z}\vec{v}$, denoting by j_Λ the j which is as in Lemma 1.3, that lemma shows that $W(j_\Lambda)$ is equal to $c_{(\vec{u},\vec{v})}$ and only depends on the lattice Λ. We will denote it in this case by $W(\Lambda)$. For the sake of generality, we state the result for any volume normalization:

Theorem 2. *Let $\mathcal{L} = \{\Lambda \mid \Lambda$ is a lattice and $j_\Lambda \in \mathcal{A}_m\}$. Then the minimum of $\Lambda \mapsto W(\Lambda)$ over $\mathcal{L}$ is achieved uniquely, modulo rotation, by the triangular lattice*

$$\Lambda_m = \sqrt{\frac{4\pi}{m\sqrt{3}}} \left((1,0)\mathbb{Z} \oplus \left(\frac{1}{2}, \frac{\sqrt{3}}{2} \right) \mathbb{Z} \right).$$

The normalizing factor ensures that the periodicity cell has area $2\pi/m$, or equivalently that $\Lambda_m \in \mathcal{A}_m$.

Remark 1.5. *The value of W for the triangular lattice with $m = 1$ estimated numerically from formula (3.6) or (3.10) is $\simeq -0.2011$. For the square lattice it is $\simeq -0.1958$.*

Theorem 2 is proven by expressing j_Λ using Fourier series. Then minimizing W over $\mathcal{L}$ is a limit case of minimizing the Epstein ζ function

$$\Lambda \mapsto \sum_{p \in \Lambda \setminus \{0\}} \frac{1}{|p|^{2+x}}$$

over $\mathcal{L}$ when $x \to 0$. This question was answered by Cassels, Rankin, Ennola, Diananda, [Cas, Ran, En1, En2, Di]. In a later self-contained paper [Mont], Montgomery shows that the ζ-function is actually the Mellin transform of another classical function from number theory: the Theta function

$$\theta_\Lambda(\alpha) = \sum_{p\in\Lambda} e^{-\pi\alpha|p|^2}. \tag{1.13}$$

He then deduces the minimality of the ζ function at the triangular lattice from the corresponding result for the θ function (we will give more details in Section 3).

The main open question is naturally to show that the triangular lattice is a minimizer among all configurations:

Conjecture 1. *The lattice Λ_m being defined as in Theorem 2, we have $W(\Lambda_m) = \min_{\mathcal{A}_m} W$.*

Note that a minimizer of W cannot be unique since compact perturbations do not affect the value of W, as seen in the fifth remark in Section 1.1. However, it could be that the triangular lattice is the only minimizer which is also a local minimizer in the following sense: if Λ' is any set of points differing from Λ by a finite number of points, and $j_{\Lambda'}$ a corresponding perturbation of j_Λ, then

$$\lim_{R\to\infty} W(j_{\Lambda'}, \chi_{B_R}) - W(j_\Lambda, \chi_{B_R}) \geq 0.$$

(Note that here we do not normalize by $|B_R|$.)

A first motivation for this conjecture comes from the physics: in the experiments on superconductors, triangular lattices of vortices, called Abrikosov lattices, are observed, as predicted by the physicist Abrikosov from Ginzburg-Landau theory. But we shall prove here, cf. Theorem 4, that vortices of minimizers of the Ginzburg-Landau energy functional (or rather, their associated "currents") converge in some asymptotic limit to minimizers of W, so if one believes experiments show ground states, then it can be expected that the triangular lattice corresponds to a minimizer of W. Another motivation for this conjecture is that, returning to the expression (1.6)–(1.7), W can be seen as a renormalized way of computing $\|2\pi\sum_p \delta_p - 1\|_{H^{-1}}$, thus minimizing W over $\mathcal{A}_1$ is heuristically like trying to minimize $\|2\pi\sum_p \delta_p - 1\|_{H^{-1}}$ over points in the plane, or trying to allocate points in the plane in the most uniform manner. By analogy with packing problems and other crystallisation problems, it seems natural, although far out of reach, that this could be accomplished by the triangular lattice. Positive answers are found in [Rad] for packing problems, [Th] for some very short-range pairwise interaction potential, and one also finds the same conjecture and some supporting arguments in [CK] Section 9, for a certain (but different) class of interaction potentials. But we note however again that here the interaction between the points is logarithmic hence long range, in contrast with these known results. Finally, in dimension 1, the situation is much easier, since we can prove in [SS7] that the minimum of the one-dimensional analogue of W is indeed achieved by the perfect lattice $\mathbb{Z}$ (suitably rescaled).

We have seen in Theorem 1 that W has a minimizer which is a limit of periodic configurations with large period. This will be used crucially for the energy upper bound on Ginzburg-Landau in Section 7. It also connects the question of minimizing W in all generality to the simpler one of minimizing it in the periodic setting. By the formula (1.10) the problem in the periodic setting reduces to minimizing $\sum_{i\neq j} G(a_i - a_j)$ or (1.11) over the torus. The points that achieve such minima (on all types of surfaces) are called Fekete points (or weighted Fekete sets) and are important in potential theory, random matrices, approximation, see [ST]. Their average distribution is well-known (see [ST]), in our setting it is uniform, but their precise location is more delicate to study (in [SS6] we make progress in that direction and connect them to W). As already mentioned in the last remark of the previous subsection, estimates on the minimum value of W in this setting are also used in number theory.

The energy W also bears some ressemblance with a nonlocal interaction energy related to diblock copolymers, sometimes called "Ohta-Kawasaki model" and studied in particular in a recent paper of Alberti, Choksi and Otto [ACO], see also [Mu, GMS1] (and previous references therein): there, one also has a logarithmic interaction, but the Dirac masses are replaced by nonsingular charges, and so no renormalization is needed. More precisely the interaction energy is $\|u - m\|_{H^{-1}}$ where m is a fixed constant in $[-1, 1]$ and u takes values in $\{-1, 1\}$, whose BV norm is also penalized. There, triangular lattice configurations are also observed, it can even be shown [GMS2] that the energy W can be derived as a limit in the regime where $m \to 1$ (in this regime, the problem becomes singular again). Also, the analogue result to our Theorem 2 is proven for that model in [CO] using also modular functions. In [ACO] it is proven that for all $m \in [-1, 1]$ the energy for minimizers is uniformly distributed. Our study of W in Section 4 is similar in spirit. Note that results of equidistribution of energy analogous to [ACO] could also most likely be proven with our method.

1.3 Lower bounds for two-scale energies via the ergodic theorem

In this subsection we present an abstract framework for proving lower bounds on energies which contain two scales (one much smaller than the other). This framework will then be crucially used in this paper, both for proving the results of W in Theorem 1 and for obtaining the lower bounds for Ginzburg-Landau in Theorem 4 and 5. We believe it is of independent interest as well.

The question is to deduce from a Γ-convergence (in the sense of De Giorgi) result at a certain scale a statement at a larger scale. The framework can thus be seen as a type of Γ-convergence result for 2-scale energies. The lower bound is expressed in terms of a probability measure, which can be seen as a Young measure on profiles (i.e. limits of the configuration functions viewed in the small scale). Following the suggestion of Varadhan, this is achieved by using Wiener's multiparameter ergodic theorem, as stated in Becker [Be]. Alberti and Müller introduced in [AM] a different framework for a somewhat similar goal, with a similar notion of Young measure, that they called "Young measures on micropatterns". In contrast with Young measures, these measures (just like ours) are not a probability measure on values taken by the functions, but rather probability measures on the whole limiting profile. The spirit of both frameworks is the same, however Alberti and Müller's method did not use the ergodic theorem. It was also a bit more general since it dealt with problems that are not homogeneous at the larger scale but admit slowly varying parameters (however we generalize to such dependence in forthcoming work), and it adressed the Γ-limsup aspect as well. Our method is more rudimentary, but maybe also more flexible. In forthcoming work, we refine it and include a version with dependence on the "slow (larger scale) variable" in [SS6], as well as an application to random homogenization in [BSS].

Let X denote a Polish metric space (for reference see [Du]). When we speak of measurable functions on X we will always mean Borel-measurable. We assume that there exists an n-parameter group of transformations θ_λ acting continuously on X. More precisely we require that

- For all $u \in X$ and $\lambda, \mu \in \mathbb{R}^n$, $\theta_\lambda(\theta_\mu u) = \theta_{\lambda+\mu} u$, $\theta_0 u = u$.
- The map $(\lambda, u) \mapsto \theta_\lambda u$ is measurable on $\mathbb{R}^n \times X$.
- The map $(\lambda, u) \mapsto \theta_\lambda u$ is continuous with respect to each variable (hence measurable with respect to both).

Typically we think of X as a space of functions defined on $\mathbb{R}^n$ and θ as the action of translations, i.e. $\theta_\lambda u(x) = u(x + \lambda)$.

We also consider a family $\{\omega_\varepsilon\}_\varepsilon$ of domains of $\mathbb{R}^n$ such that for any $R > 0$ and letting $\omega_{\varepsilon,R} = \{x \in \omega_\varepsilon \mid \text{dist } (x, \partial\omega_\varepsilon) > R\}$, we have

$$|\omega_\varepsilon| \sim |\omega_{\varepsilon,R}| \text{ as } \varepsilon \to 0. \tag{1.14}$$

In particular the diameter of ω_ε tends to $+\infty$ as $\varepsilon \to 0$.

Finally we let $\{f_\varepsilon\}_\varepsilon$ and f be measurable nonnegative functions on X, and assume that for any family $\{u_\varepsilon\}_\varepsilon$ such that

$$\forall R > 0, \quad \limsup_{\varepsilon\to 0} \int_{B_R} f_\varepsilon(\theta_\lambda u_\varepsilon)\, d\lambda < +\infty \tag{1.15}$$

the following holds:

1. (Coercivity) $\{u_\varepsilon\}_\varepsilon$ admits a convergent subsequence.
2. (Γ-liminf) If $\{u_\varepsilon\}_\varepsilon$ is a convergent subsequence (not relabeled) and u is its limit, then

$$\liminf_{\varepsilon\to 0} f_\varepsilon(u_\varepsilon) \geq f(u). \tag{1.16}$$

The abstract result is

Theorem 3. *Let $\{\theta_\lambda\}_\lambda$, $\{\omega_\varepsilon\}_\varepsilon$ and $\{f_\varepsilon\}_\varepsilon$, f be as above. Let $\{\mathbf{U}_R\}_{R>0}$ be a family satisfying* (1.4)*. Let*

$$F_\varepsilon(u) = \fint_{\omega_\varepsilon} f_\varepsilon(\theta_\lambda u)\, d\lambda. \tag{1.17}$$

Assume that $\{F_\varepsilon(u_\varepsilon)\}_\varepsilon$ is bounded. Let P_ε be the image of the normalized Lebesgue measure on ω_ε under the map $\lambda \mapsto \theta_\lambda u_\varepsilon$. Then P_ε converges along a subsequence to a Borel probability measure P on X invariant under the action θ and such that

$$\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) \geq \int f(u)\, dP(u). \tag{1.18}$$

Moreover, for any family $\{\mathbf{U}_R\}_{R>0}$ satisfying (1.4)*, we have*

$$\int f(u)\, dP(u) = \int f^*(u)\, dP(u), \tag{1.19}$$

with f^* given by

$$f^*(u) := \lim_{R\to+\infty} \fint_{\mathbf{U}_R} f(\theta_\lambda u)\, d\lambda. \tag{1.20}$$

In particular, the right-hand side of (1.19) *is independent of the choice of $\{\mathbf{U}_R\}_{R>0}$.*

Remark 1.6. *The result* (1.19) *is simply the ergodic theorem in multiparameter form. Part of the result is that the limit in* (1.19) *exists for P-almost every u.*

The probability measure P is the “Young measure” on limiting profiles we were referring to before, indeed it encodes the limit of all translates of u_ε. Note that $P_\varepsilon \to P$ implies that P almost every u is of the form $\lim_{\varepsilon\to 0} \theta_{\lambda_\varepsilon} u_\varepsilon$. The limit defining f^* can be viewed as a “cell problem”, using the terminology of homogenization, providing the limiting small scale functional.

To illustrate this result, we shall give two examples, in order of increasing generality, both models for what we will use here.

Example 1. *Consider $X = \mathcal{M}(\mathbb{R}^n)$ the set of positive bounded measures on $\mathbb{R}^n$, and θ_λ the action of translations of $\mathbb{R}^n$. Let χ be a given nonnegative smooth function with support in the unit ball of $\mathbb{R}^n$, and define $f_\varepsilon(\mu) = f(\mu) = \int_{\mathbb{R}^n} \chi \, d\mu$. Then one can check that F_ε, as defined in* (1.17)*, is*

$$F_\varepsilon(\mu) = \frac{1}{|\omega_\varepsilon|} \int_{\mathbb{R}^n} \chi * \mathbf{1}_{\omega_\varepsilon} \, d\mu.$$

The result of the theorem (choosing balls for example) is the the assertion that

$$\liminf_{\varepsilon \to 0} F_\varepsilon(\mu_\varepsilon) \geq \int f(\mu) \, dP(\mu) = \mathbb{E}^P \left(\lim_{R \to +\infty} \int \chi * \mathbf{1}_{B_R} \, d\mu \right).$$

The probability P gives a measure over all limiting profiles of μ_ε (depending on the centering point), and the result says that the quantity we are computing, here the average of μ_ε over the large sets ω_ε, can be bounded below by an average, this time over P, of a similar quantity for the limiting profiles. This implies in particular that there is a μ_0 in the support of P, hence of the form $\lim_{\varepsilon \to 0} \mu_\varepsilon(\lambda_\varepsilon + \cdot)$ such that

$$\liminf_{\varepsilon \to 0} \frac{1}{|\omega_\varepsilon|} \int_{\mathbb{R}^n} \chi * \mathbf{1}_{\omega_\varepsilon} \, d\mu_\varepsilon \geq \lim_{R \to +\infty} \int \chi * \mathbf{1}_{B_R} \, d\mu_0.$$

In other words we can find a good centering sequence λ_ε such that the average of μ_ε over the large sets ω_ε can be bounded from below by the average over large balls of the limit after centering, μ_0. The averages over ω_ε or B_R, and the average with respect to P are not of the same nature (in standard ergodic settings, the first ones are usually time averages, while the latter is a space average).

Example 2. *Let us assume we want to bound from below an energy which is the average over large (as $\varepsilon \to 0$) domains ω_ε of some nonnegative energy density $e_\varepsilon(u)$, defined on a space of functions X (functions over $\mathbb{R}^n$), $\fint_{\omega_\varepsilon} e_\varepsilon(u(x)) \, dx$, and we know the Γ-liminf behavior of $e_\varepsilon(u)$ on small (i.e. here, bounded) scales, say we know how to prove that $\liminf_{\varepsilon \to 0} \int_{B_R} e_\varepsilon(u) \, dx \geq \int_{B_R} e(u) \, dx$. We cannot always directly apply such a knowledge to obtain a lower bound on the average over large domains, this may be due to a loss of boundary information, or due to the difficulty to reverse limits $R \to \infty$ and $\varepsilon \to 0$. (These are two obstructions that we encounter specifically both for Ginzburg-Landau and W, as illustrated by the 6th remark in Section 1.1) What we can do is let χ be a smooth cutoff function as above, and define the functions f_ε by*

$$f_\varepsilon(u) = \int e_\varepsilon(u(x)) \, \chi(x) \, dx,$$

that is f_ε can be seen as the small scale local functional. Since we know the Γ-liminf behavior of the energy density e_ε on small scales, let us assume we can prove that (1.16) *holds for some function(al) f, a function on u (we may expect that f will also be of the form $\int e(u(x)) \, \chi(x) \, dx$.)*

Defining F_ε as in (1.17) *and using Fubini's theorem, we see that*

$$F_\varepsilon(u) = \fint_{\omega_\varepsilon} \int_{\mathbb{R}^n} e_\varepsilon(u(x+y)) \chi(x) \, dx \, dy = \frac{1}{|\omega_\varepsilon|} \int_{z \in \omega_\varepsilon - x} \int_x e_\varepsilon(u(z)) \chi(x) \, dx \, dz.$$

Since χ is supported in B_1, $\int \chi = 1$, and ω_ε satisfies (1.14)*, we check that*

$$F_\varepsilon(u) \sim_{\varepsilon \to 0} \fint_{\omega_\varepsilon} e_\varepsilon(u(y) \, dy,$$

hence F_ε is asymptotically equal to the average we wanted to bound from below. We may thus apply the theorem and it yields a lower bound for the desired quantity:

$$\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) \geq \int f^*(u)\, dP(u)$$

with

$$f^*(u) = \lim_{R\to\infty} \fint_{B_R} f(u(x+\cdot))\, dx \tag{1.21}$$

The "cell-function" f^* is simply an average over large balls of the local Γ-liminf f. If typically f is of the form $\int e(u(x))\,\chi(x)\,dx$ then we can compute by Fubini that

$$f^*(u) = \lim_{R\to\infty} \frac{1}{|B_R|} \int e(u(x))(\chi * \mathbf{1}_{B_R})(x)\, dx.$$

1.4 The Ginzburg-Landau model

Our original motivation in this article is to analyze the behaviour of minimizers of the Ginzburg-Landau energy, given by

$$G_\varepsilon(u,A) = \frac{1}{2}\int_\Omega |\nabla_A u|^2 + |\mathrm{curl}\, A - h_{\mathrm{ex}}|^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2}. \tag{1.22}$$

This is a celebrated model in physics, introduced by Ginzburg and Landau in the 1950's as a model for superconductivity. Here Ω is a two dimensional bounded and simply connected domain, u is a complex-valued function, "order parameter" in physics, describing the local state of the superconductor, $A:\Omega\mapsto\mathbb{R}^2$ is the vector potential of the magnetic field $h = \mathrm{curl}\, A = \nabla\times A$, and ∇_A denotes the operator $\nabla - iA$. Finally the parameter h_{ex} denotes the intensity of the applied magnetic field and ε is a constant corresponding to a characteristic lengthscale (of the material). It is the inverse of κ, the Ginzburg-Landau parameter in physics. We are interested in the $\varepsilon\to 0$ asymptotics. The quantity $|u|^2$ measures the local density of superconducting electron pairs ($|u|\leq 1$). The material is in the superconducting phase wherever $|u|\simeq 1$ and in the normal phase where $|u|\simeq 0$. We focus our attention on the zeroes of u with nonzero topological degree (recall that u is complex-valued), also known as the *vortices* of u. Here the typical lengthscale of the set where $|u|$ is small, hence of the vortex "cores", is ε.

The Euler-Lagrange equations associated to this energy with natural boundary conditions are the Ginzburg-Landau equations

$$\begin{cases} -(\nabla_A)^2 u = \frac{u}{\varepsilon^2}(1-|u|^2) & \text{in } \Omega \\ -\nabla^\perp h = (iu, \nabla_A u) & \text{in } \Omega \\ h = h_{\mathrm{ex}} & \text{on } \partial\Omega \\ \nu\cdot\nabla_A u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.23}$$

where $\nabla^\perp$ denotes the operator $(-\partial_2,\partial_1)$, ν the outer unit normal to $\partial\Omega$ and $(\cdot,\cdot)$ the canonical scalar product in $\mathbb{C}$ obtained by identifying $\mathbb{C}$ with $\mathbb{R}^2$.

This model is also famous as the simplest "Abelian gauge theory". Indeed it admits the $\mathbb{U}(1)$ gauge-invariance : the energy (1.22), the equation (1.23), and all the physical quantities are invariant under the transformation

$$\begin{cases} u \to ue^{i\Phi} \\ A \to A + \nabla\Phi. \end{cases}$$

For a more detailed mathematical presentation of the functional, one may refer to [SS4] and the references therein.

1.5 Physical behaviour: critical fields and vortex lattices

Here and in all the paper $a \ll b$ means $\lim a/b = 0$.

For ε small, the minimizers of (1.22) depend on the intensity $h_{\rm ex}$ of the applied field as follows.

- If $h_{\rm ex}$ is below a critical value called the *first critical field* and denoted by H_{c_1}, the superconductor is in the so-called *Meissner state* characterized by the expulsion of the magnetic field and the fact that $|u| \simeq 1$ everywhere. Vortices are absent in this phase.
- If $h_{\rm ex}$ is above H_{c_1}, which is equivalent to $\lambda_\Omega|\log\varepsilon|$ as $\varepsilon \to 0$, then energy minimizers have one vortex, then two,... The number of vortices increases with $h_{\rm ex}$, so as to become equal to leading order to $h_{\rm ex}/2\pi$ for fields much larger than $|\log\varepsilon|$ (see [SS4]). In this case, according to the picture we owe to A. Abrikosov and dating back to the late 1950's, vortices repell each other and organize themselves in *triangular lattices* named *Abrikosov lattices*. The theoretical predictions of Abrikosov have received ample experimental confirmation and there are numerous and striking observations of the lattices.
- If $h_{\rm ex}$ increases beyond a second critical value $H_{c_2} = 1/\varepsilon^2$, another phase transition occurs where superconductivity disappears from the material, except for a boundary layer. Even this boundary layer completely disappears when $h_{\rm ex}$ is above a third critical field H_{c_3}.

Since the works of Ginzburg, Landau and Abrikosov, this model has been largely studied in the physics literature. We refer to the classic monographs and textbooks by De Gennes [DeG], Saint-James Sarma - Thomas [SST], Tinkham [Ti].

The above picture describes the phenomenology of the model for small values of ε (or high values of κ) in a casual way, but by now many rigorous mathematical results support this picture. Except for the third critical field however, whose existence was proven in a strong sense by Fournais and Helffer [FH1, FH2], mathematical results really prove the existence of intervals where the transition between the different types of behaviour occur, rather than critical values, with estimates on these intervals as $\varepsilon \to 0$.

Our goal is the study and description of the vortices in the whole range $H_{c_1} < h_{\rm ex} \ll H_{c_2}$, where minimizers of the Ginzburg-Landau energy have a large number of vortices, expected to form Abrikosov lattices.

1.6 Connection to earlier mathematical works

There is an abundant mathematical literature related to the study of (1.22) and to the justification of the physics picture above. A relatively extensive bibliography is given in [SS4], Chap. 14.

The techniques most relevant to us for the description of vortices originate in the pioneering book of Bethuel-Brezis-Hélein [BBH] and have been further expanded by several authors including in particular Jerrard, Soner [Je, JS], ourselves, etc... For a more detailed description of these techniques we refer to [SS4]. In that book we describe how these techniques allow to derive the asymptotic values of the critical field H_{c_1} as $\varepsilon \to 0$ as well as the mean-field description of minimizers and their vortices in the regimes $h_{\rm ex} \ll H_{c_2}$.

In order to describe the vortices of minimizers, one introduces the *vorticity* associated to a configuration (u, A), defined by

$$\mu(u,A) = \operatorname{curl} j(u,A) + \operatorname{curl} A, \quad \text{where } j(u,A) := (iu, \nabla_A u) \tag{1.24}$$

is the superconducting current. Here (a, b) denotes the Euclidean scalar product in $\mathbb{C}$ identified with $\mathbb{R}^2$, so $j(u, A)$ may also be written as $\frac{i}{2}\left(u\overline{\nabla_A u} - \bar{u}\nabla_A u\right)$, or as $\rho^2(\nabla\varphi - A)$ if $u = \rho e^{i\varphi}$, at least where $\rho \neq 0$.

This vorticity is the appropriate quantity to consider in this context, rather than the quantity $\operatorname{curl}(iu, \nabla_A u)$ which could come to mind first. It may be seen as a gauge-invariant version of $\operatorname{curl}(iu, \nabla u)$ which is also (twice) the Jacobian determinant of u. Indeed if $A = 0$, then $\mu(u, A) = 2\partial_x u \times \partial_y u$. One can prove (this is the so-called Jacobian estimate, see [JS, SS4]) that assuming a suitable bound on $G_\varepsilon(u, A)$, the vorticity $\mu(u, A)$ is well approximated, in some weak sense, as $\varepsilon \to 0$ by a measure of the form $2\pi \sum_i d_i \delta_{a_i}$. As points of concentration of the vorticity, the points $\{a_i\}_i$ are naturally called vortices of u and d_i, which is an integer, is called the degree of a_i. The vorticity $\mu(u, A)$ may either describe individual vortices or, after normalization by the number of vortices, their density.

In previous work (summarized in [SS4], Chap. 7) we showed that

$$H_{c_1} \sim \lambda_\Omega |\log \varepsilon|, \tag{1.25}$$

for a constant $\lambda_\Omega > \frac{1}{2}$ depending only on Ω (and such that $\lambda_\Omega \to \frac{1}{2}$ as $\Omega \to \mathbb{R}^2$). More precisely, letting $h_{\rm ex} = \lambda|\log \varepsilon|$, we established by a Γ-convergence approach that minimizers $(u_\varepsilon, A_\varepsilon)$ of G_ε satisfy

$$\frac{\operatorname{curl} A_\varepsilon}{h_{\rm ex}} \text{ converges to } h_\lambda, \quad \frac{\mu(u_\varepsilon, A_\varepsilon)}{h_{\rm ex}} \text{ converges to } \mu_\lambda, \text{ as } \varepsilon \to 0, \tag{1.26}$$

where $\mu_\lambda = -\Delta h_\lambda + h_\lambda$, and h_λ is the solution of the following minimization problem

$$\min_{h-1 \in H_0^1(\Omega)} \frac{1}{2\lambda} \int_\Omega |-\Delta h + h| + \frac{1}{2}\int_\Omega |\nabla h|^2 + |h-1|^2. \tag{1.27}$$

This problem is in turn equivalent to an obstacle problem, and as a consequence there exists a subdomain ω_λ such that

$$\mu_\lambda = m_\lambda \mathbf{1}_{\omega_\lambda}, \quad \text{where } m_\lambda = 1 - \frac{1}{2\lambda}, \tag{1.28}$$

and where $\mathbf{1}_A$ denotes the characteristic function of A. In other words the optimal limiting vortex density μ_λ is uniform and equal to m_λ over a subregion of Ω, completely determined by λ i.e. by the applied field $h_{\rm ex}$. The constant λ_Ω introduced in (1.25) is characterized by the fact that $\omega_\lambda = \varnothing$ if and only if $\lambda < \lambda_\Omega$. Since $\lambda_\Omega > \frac{1}{2}$, if $\omega_\lambda \neq \varnothing$ we have

$$1 \ge m_\lambda \ge 1 - \frac{1}{2\lambda_\Omega} > 0. \tag{1.29}$$

Since the number of vortices is proportional to $h_{\rm ex} = \lambda |\log \varepsilon|$, it tends to $+\infty$ as $\varepsilon \to 0$. It is also established in [SS4] that as $\varepsilon \to 0$, the minimal energy has the following expansion,

$$\min G_\varepsilon = \frac{1}{2} h_{\rm ex} \log \frac{1}{\varepsilon \sqrt{h_{\rm ex}}} \int_\Omega \mu_\lambda + \frac{{h_{\rm ex}}^2}{2} \int_\Omega |\nabla h_\lambda|^2 + |h_\lambda - 1|^2 + o\left(h_{\rm ex} \log \frac{1}{\varepsilon\sqrt{h_{\rm ex}}} \right). \tag{1.30}$$

When the applied field is much larger than $|\log \varepsilon|$, but much less than $1/\varepsilon^2$, (1.26) and (1.30) still hold, replacing λ by $+\infty$. In this case $h_\lambda = 1$ and $\omega_\lambda = \Omega$.

There are other cases where the distribution of vortices for minimizers of the Ginzburg-Landau functional is understood. First in a periodic setting: Minimizing the Ginzburg-Landau energy among configurations (u, A) which are periodic (modulo gauge equivalence) with respect to a certain lattice independent of ε, one obtains (see [Ay, AyS]) as above a limiting vortex density $\mu_\lambda = -\Delta h_\lambda + h_\lambda$ where h_λ minimizes the energy in (1.27) among periodic functions. The minimizer in this case is clearly a constant, which is easily found to be $\max\left(1 - \frac{1}{2\lambda}, 0\right)$.

Second, in the regime of applied fields $h_{\rm ex}$ where the number of vortices tends to $+\infty$, but is negligible compared to $|\log \varepsilon|$, (this corresponds to applied fields such that $\log |\log \varepsilon| \ll h_{\rm ex} - \lambda_\Omega |\log \varepsilon| \ll |\log \varepsilon|$ as $\varepsilon \to 0$), it is shown in [SS4] that for simply connected domains satisfying a certain generic property (see (1.31) below) — including convex domains — vortices concentrate around a single point (a finite number of points for general simply connected domains). Then, blowing up at the suitable scale and normalizing the vorticity, one obtains in the limit a probability measure μ which describes the distribution of vortices around the point, and μ is the unique minimizer among probability measures in $\mathbb{R}^2$ of

$$I(\mu) = -\pi \iint \log |x - y| \, d\mu(x) \, d\mu(y) + \pi \int Q(x) \, d\mu(x).$$

Note that here Q is a positive definite quadratic form which depends on the domain Ω. It is the Hessian of a certain function at the point of concentration of the vortices. Note that the precise regime of applied field modifies the number of vortices, i.e. the normalizing factor of the vorticity, and the scaling, but does not influence the limit distribution μ, which is a characteristic of Q.

In our previous work, the treatment for the "intermediate" regime $\log |\log \varepsilon| \ll h_{\rm ex} - \lambda_\Omega |\log \varepsilon| \ll |\log \varepsilon|$ and for the regime $h_{\rm ex} = \lambda |\log \varepsilon|$, $\lambda > \lambda_\Omega$ were different, the former one being more delicate. Here we provide (and this is part of the technical difficulties) a unified approach for both, and treat all regimes $h_{\rm ex} \ll H_{c_2}$ where the number of vortices is blowing up.

1.7 Main result on Ginzburg-Landau

The mean field description above tells us that the vortices tend to be distributed uniformly in ω_λ but is insensitive to the pattern formed by vortices. This pattern is in fact, as we shall see, selected by the minimization of the next term in the asymptotic expansion of the energy as $\varepsilon \to 0$. The proof of this is achieved in this paper by a splitting of the energy that separates the leading order term found in (1.30) from a remainder term, and then by studying the remainder term after blow up at the scale of the expected intervortex distance, which from the above considerations is of the order of $1/\sqrt{h_{\rm ex}}$; this remainder term is then shown to Γ-converge to W (as introduced in Section 1.1), hence allows to distinguish among vortex configurations.

As before, we use the current $j(u, A) = (iu, \nabla_A u)$ (cf. (1.24)) to describe the vortex locations, we study through the abstract framework of Section 1.3 the probability measure carried by all possible limiting profiles of blow ups of j at the scale $1/\sqrt{h_{\rm ex}}$ centered at all possible blow-up points in ω_λ, and show that this probability is concentrated on minimizers of W.

Before stating the simplest form of our main result, let us explain the main assumptions we need to make. First, we make the simplifying assumption that the domain Ω is convex. This is only used at one point in the proof (the upper bound construction) and avoids the possibility that ω_λ may have cusps. We believe our results still hold without this restriction. Then letting h_0 be the solution to $-\Delta h_0 + h_0 = 0$ with $h_0 = 1$ on $\partial\Omega$ — which is consistent with the definition of h_λ in (1.27) — it is known (see Caffarelli-Friedman [CF], the result in dimensions greater than two can be found in [KL]) that in the case where Ω is convex, h_0 is strictly convex hence achieves its minimum at a unique point $x_0 \in \Omega$. Moreover this minimum is nondegenerate:

$$\min_\Omega h_0 \text{ is achieved at a unique point } x_0 \text{ and } \ Q := D^2 h_0(x_0) \text{ is positive definite.} \tag{1.31}$$

For a family $(u_\varepsilon, A_\varepsilon)$, we denote

$$\tilde{\jmath}_{\varepsilon,x}(\cdot) := \frac{1}{\sqrt{h_{\rm ex}}} j(u_\varepsilon, A_\varepsilon)\left(x + \frac{\cdot}{\sqrt{h_{\rm ex}}}\right) \tag{1.32}$$

their blown-up current, where in the right-hand side $j(u_\varepsilon, A_\varepsilon)$ is implicitly extended by 0 outside the domain Ω.

We have

Theorem 4. *Assume that Ω is convex, so that (1.31) is satisfied, and that*

$$h_{\rm ex} = \lambda|\log\varepsilon| \textit{ with } \lambda > \lambda_\Omega\ .$$

Let $(u_\varepsilon, A_\varepsilon)$ be a minimizer of G_ε, and let $\tilde{\jmath}_{\varepsilon,x}$ be as in (1.32) for $x \in \omega_\lambda$. Then, given $1 < p < 2$, there exists a probability measure P on $L^p_{loc}(\mathbb{R}^2, \mathbb{R}^2)$ such that the following hold:

1. *Up to extraction, for any bounded continuous function Φ on $L^p_{loc}(\mathbb{R}^2, \mathbb{R}^2)$, we have*

$$\lim_{\varepsilon\to 0} \frac{1}{|\omega_\lambda|}\int_{\omega_\lambda} \Phi(\tilde{\jmath}_{\varepsilon,x})\, dx = \int \Phi(j)\, dP(j). \tag{1.33}$$

2. *P-almost every j minimizes W over $\mathcal{A}_{m_\lambda}$ and*

$$G_\varepsilon(u_\varepsilon, A_\varepsilon) = G_\varepsilon^{N_0} + h_{\rm ex}|\omega_\lambda|(\min W + cst) + o(h_{\rm ex}) \quad \textit{as } \varepsilon \to 0, \tag{1.34}$$

where $G_\varepsilon^{N_0} \in \mathbb{R}$ is explicited in (1.39) below.

We may informally describe $G_\varepsilon^{N_0}$: It is the minimum value for an obstacle problem similar to (1.27). But whereas (1.27) is derived assuming the vortex energy is $\pi|\log\varepsilon|$, to obtain $G_\varepsilon^{N_0}$ we must use instead the more precise value $\pi|\log\varepsilon'|$, where

$$\varepsilon' = \varepsilon\sqrt{h_{\rm ex}}.$$

As $\varepsilon \to 0$ we have $|\log\varepsilon| \approx |\log\varepsilon'|$, but using $|\log\varepsilon'|$ induces a correction which *is not* $o(h_{\rm ex})$, hence is important to us.

As we have seen in Section 1.2, W allows to distinguish between configurations of points since it distinguishes between lattices, and it is expected to favor the triangular lattice. The result of Theorem 4 can be informally understood as follows: if one chooses a point at random in ω_λ and blows up at the scale $1/\sqrt{h_{\rm ex}}$, then in the limit $\varepsilon \to 0$, almost surely (with respect to the blow up center), one sees a minimizer of W. This derivation of this limiting energy W is, to our knowledge, the first rigorous justification of the Abrikosov triangular lattice in this regime: at least the triangular lattice is the best among lattice configurations, and it is conjectured to be a global minimizer (see Section 1.2).

In Theorem 5 below, we will give a more precise and a full Γ-convergence version of Theorem 4, valid for the other regimes of applied field where vortex lattices are expected to arise. The latter result encompasses the regimes of Theorems 1.3, 1.4 and 1.5 in [SS4] and allows to reprove these results.

Returning to the reference to the Theta function (1.13) in Section 1.2, it is striking to observe that the problem of minimizing the Theta function also arises in the context of Ginzburg-Landau, but in a very different regime: when $h_{\rm ex} \sim H_{c_2}$ or more precisely when $h_{\rm ex} = \frac{b}{\varepsilon^2}$ with $b \nearrow 1$. As seen in [SS2, AS] this is a regime which is essentially linear (contrarily to the one we study here) and the energy minimization can be reduced to the minimization of a function on a finite dimensional space (the "lowest Landau level"- this is essentially the result of Abrikosov's original calculation). This function can be viewed as the linear analogue of W and reduces, in the case where the points are on a lattice, to the θ function θ_Λ (and so again the optimal lattice is the triangular one). In that sense the limiting lattice energies for Ginzburg-Landau in the regime $H_{c_1} \ll h_{\rm ex} \ll H_{c_2}$ and in the regime $h_{\rm ex} \sim H_{c_2}$ can be viewed as Mellin transforms of each other.

We now go into more detail on the method of the proof of Theorem 4. It follows from a result of Γ-convergence, i.e. by showing a general lower bound for the energy and a matching upper bound via an explicit construction. Thus the minimality of $(u_\varepsilon, A_\varepsilon)$ and the Euler-Lagrange equation it solves is not used per se. The proof of the lower bound involves three ingredients: an energy splitting, a blow-up, and the abstract method of Theorem 3.

1.8 The energy splitting

The first ingredient of the proof, detailed in Section 5, is a new algebraic splitting of the energy, which allows to isolate the constant leading order part from the next-order.

First we define a mean field $h_{0,\varepsilon}$ similar to $h_{\rm ex}h_\lambda$, except that when computing it we take more precisely into account the cost of a vortex which is $\pi|\log\varepsilon'|$, where we recall $\varepsilon' = \varepsilon\sqrt{h_{\rm ex}}$. Accordingly, assuming $h_{\rm ex} < 1/\varepsilon^2$, we let $h_{0,\varepsilon}$ be the minimizer of

$$\min_{h - h_{\rm ex}\in H_0^1(\Omega)} \frac{1}{2}|\log\varepsilon'| \int_\Omega |-\Delta h + h| + \frac{1}{2}\int_\Omega |\nabla h|^2 + |h - h_{\rm ex}|^2. \tag{1.35}$$

Thus $h_{0,\varepsilon}/h_{\rm ex}$ solves (1.27), with λ replaced by $\lambda_\varepsilon = \frac{h_{\rm ex}}{|\log \varepsilon'|}$.

This is equivalent (see [Br, BS]) to saying that $h_{0,\varepsilon}/h_{\rm ex}$ minimizes the H^1 norm subject to the constraints $h_{0,\varepsilon}/h_{\rm ex} = 1$ on $\partial\Omega$ and

$$\frac{h_{0,\varepsilon}}{h_{\rm ex}} \geq m_{0,\varepsilon} := 1 - \frac{|\log \varepsilon'|}{2h_{\rm ex}} \quad \text{in } \Omega. \tag{1.36}$$

In other words $h_{0,\varepsilon}$ is the solution to an obstacle problem with constant obstacle. Letting then $\mu_{0,\varepsilon} = -\Delta h_{0,\varepsilon} + h_{0,\varepsilon}$, we have

$$\mu_{0,\varepsilon} = m_{0,\varepsilon} h_{\rm ex} \mathbf{1}_{\omega_{0,\varepsilon}}, \quad \text{where} \quad \omega_{0,\varepsilon} = \{x \mid h_{0,\varepsilon}(x) = h_{\rm ex} m_{0,\varepsilon}\}. \tag{1.37}$$

We define

$$N_0 = \frac{1}{2\pi} \int_\Omega \mu_{0,\varepsilon}. \tag{1.38}$$

The splitting function could be taken to be $h_{0,\varepsilon}$ in most regimes of applied field. Then one should define $G_\varepsilon^{N_0}$ in Theorem 4 as

$$G_\varepsilon^{N_0} := \pi N_0 |\log \varepsilon'| + \frac{1}{2} \|h_{0,\varepsilon} - h_{\rm ex}\|^2_{H^1(\Omega)}, \tag{1.39}$$

where $\|\cdot\|_{H^1}$ denotes the Sobolev space norm $\left(\|\cdot\|^2_{L^2} + \|\nabla \cdot\|^2_{L^2}\right)^{\frac{1}{2}}$.

However, when $h_{\rm ex} - \lambda_\Omega |\log \varepsilon| \ll |\log \varepsilon|$ — i.e. when the number of vortices is small compared to $h_{\rm ex}$ though divergent as $\varepsilon \to 0$ — then (1.35), which is a refinement of (1.27), must itself be refined to take into account the constraint that the vorticity is quantized. For the other regimes, the error made by ignoring this constraint is negligible in our analysis.

More precisely, given N such that $0 \leq N \leq \frac{1}{2\pi} h_{\rm ex}|\Omega|$, we consider $h_{\varepsilon,N}$ the minimizer of

$$\min_{\substack{h - h_{\rm ex} \in H^1_0(\Omega) \\ \int_\Omega |\Delta h + h| = 2\pi N}} \frac{1}{2} \int_\Omega |\nabla h|^2 + |h - h_{\rm ex}|^2, \tag{1.40}$$

and we define

$$G_\varepsilon^N := \pi N |\log \varepsilon'| + \frac{1}{2} \|h_{\varepsilon,N} - h_{\rm ex}\|^2_{H^1(\Omega)}. \tag{1.41}$$

Since $h_{\rm ex}$ will be a given function of ε, we denote the dependence of $h_{\varepsilon,N}$ as of ε instead of $h_{\rm ex}$. G_ε^N is minimal at $N = N_0$ and we will see that $h_{\varepsilon,N_0} = h_{0,\varepsilon}$.

The refinement with respect to (1.35) consists in taking N to be an integer, when N_0 is not necessarily one. More precisely, N will be taken to be either N_0^-, the largest integer $\leq N_0$, or N_0^+, the smallest integer $\geq N_0$. With that choice, we will sometimes call $h_{\varepsilon,N}$ the "splitting function".

The leading order term in the energy is not exactly $G_\varepsilon^{N_0}$ but rather $\min_{N \in \{N_0^-, N_0^+\}} G_\varepsilon^N$. We may immediately check however that for $N \in \{N_0^-, N_0^+\}$, as $\varepsilon \to 0$,

$$G_\varepsilon^N \sim G_\varepsilon^{N_0} \sim \frac{1}{2} h_{\rm ex} |\log \varepsilon'| \int_\Omega \mu_\lambda + \frac{{h_{\rm ex}}^2}{2} \int_\Omega |\nabla h_\lambda|^2 + |h_\lambda - 1|^2$$

when $h_{\rm ex} \sim \lambda|\log \varepsilon|$ in the notation of Section 1.6, hence it recovers the leading order term of the minimal energy (1.30). The difference between $\min_{N\in\{N_0^-,N_0^+\}} G_\varepsilon^N$ and $G_\varepsilon^{N_0}$ is $o(h_{\rm ex})$ hence it is negligible for the precision of $o(N)$ we want to achieve as soon as $\lambda > \lambda_\Omega$, but not always when $\lambda = \lambda_\Omega$.

$h_{\varepsilon,N}$ is the solution to an obstacle problem and we have $h_{\varepsilon,N} \ge m_{\varepsilon,N} h_{\rm ex}$ and $-\Delta h_{\varepsilon,N} + h_{\varepsilon,N} = h_{\rm ex} m_{\varepsilon,N} \mathbf{1}_{\omega_{\varepsilon,N}}$ where $\omega_{\varepsilon,N} = \{h_{\varepsilon,N} = m_{\varepsilon,N} h_{\rm ex}\}$ is called the coincidence set, for some constant $m_{\varepsilon,N}$ such that $m_{\varepsilon,N} \to m_\lambda$ as $\varepsilon \to 0$. Note that $\omega_{\varepsilon,N}$ will depend on the choice of $N = N_0^-$ or N_0^+ but sometimes we will forget it and simply write ω_ε. We let $\mu_{\varepsilon,N} = -\Delta h_{\varepsilon,N} + h_{\varepsilon,N} = h_{\rm ex} m_{\varepsilon,N} \mathbf{1}_{\omega_{\varepsilon,N}}$. It is a perturbation of μ_λ and $\omega_{\varepsilon,N}$ a perturbation of ω_λ, defined in Section 1.6. We have the relation

$$|\omega_{\varepsilon,N}| m_{\varepsilon,N} h_{\rm ex} = \int_\Omega \mu_{\varepsilon,N} = 2\pi N. \tag{1.42}$$

Then we (temporarily) introduce $A_{0,\varepsilon} = \nabla^\perp h_{\varepsilon,N}$. Letting (u, A) be a configuration of finite energy, we will write

$$A_{1,\varepsilon} = A - A_{0,\varepsilon} = A - \nabla^\perp h_{\varepsilon,N},$$

where $A_{0,\varepsilon} = \nabla^\perp h_{\varepsilon,N}$ is understood as the leading order term, and $A_{1,\varepsilon}$ as a remainder term. The energy-splitting is the observation of the following identity (valid even if N is not an integer):

(1.43)
$$G_\varepsilon(u, A) = G_\varepsilon^N + \int_\Omega (h_{\varepsilon,N} - h_{\rm ex})\mu(u, A_{1,\varepsilon}) + \frac{1}{2}\int_\Omega |\nabla_{A_{1,\varepsilon}} u|^2 + |\mathrm{curl}\, A_{1,\varepsilon} - \mu_{\varepsilon,N}|^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2} - c_{\varepsilon,N}\int_\Omega \mu_{\varepsilon,N} - \frac{1}{2}\int_\Omega (1-|u|^2)|\nabla h_{\varepsilon,N}|^2,$$

where we recall (1.24), and $c_{\varepsilon,N}$ is a constant explicited in (5.3). The last term in (1.43) can be shown to be negligible if $G_\varepsilon(u, A)$ is not too large. Thus the study of the energy near its minimum reduces to that of the remainder

$$F_\varepsilon(u, A_{1,\varepsilon}) := \frac{1}{2}\int_\Omega |\nabla_{A_{1,\varepsilon}} u|^2 + |\mathrm{curl}\, A_{1,\varepsilon} - \mu_{\varepsilon,N}|^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2} + \int_\Omega (h_{\varepsilon,N} - h_{\rm ex})\mu(u, A_{1,\varepsilon}) - c_{\varepsilon,N}\int_\Omega \mu_{\varepsilon,N}. \tag{1.44}$$

It turns out that when we make the right specific choice $N = N_0^-$ or $N = N_0^+$ (depending on $(u_\varepsilon, A_\varepsilon)$), this expression simplifies and one has

$$F_\varepsilon(u, A_{1,\varepsilon}) \ge \frac{1}{2}\int_\Omega |\nabla_{A_{1,\varepsilon}} u|^2 + |\mathrm{curl}\, A_{1,\varepsilon} - \mu_{\varepsilon,N}|^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2} - \int_\Omega \zeta_\varepsilon \mu(u, A_{1,\varepsilon}) + o(1) \tag{1.45}$$

where ζ_ε is a positive function, equal to its maximum $\frac{1}{2}|\log \varepsilon'|$ on $\omega_{\varepsilon,N} = \mathrm{Supp}(\mu_{\varepsilon,N})$. It is this remainder F_ε (with this choice of N) whose Γ-convergence we study. We will see through the upper bound construction that

$$\min F_\varepsilon \le C h_{\rm ex} |\omega_{\varepsilon,N}|$$

(this is equivalent to $\min F_\varepsilon \le CN$ or $\le CN_0$ from (1.42)), and we will work in that class $F_\varepsilon(u, A_{1,\varepsilon}) \le CN_0$, thus reducing to a relatively narrow class of "almost minimizers" i.e. configurations whose leading order energy is the minimal one $G_\varepsilon^{N_0}$. Note that once we know that this remainder F_ε is of lower order, this shows that the leading order component in $A = \nabla^\perp h_{\varepsilon,N} + A_{1,\varepsilon}$ is $\nabla^\perp h_{\varepsilon,N}$ (in other words $A_{1,\varepsilon} \ll A_{0,\varepsilon}$) so also at leading order $\mu(u_\varepsilon, A_\varepsilon) \sim \mu_{\varepsilon,N} \sim \mu_\lambda$ and this allows to recover in essence the results of [SS4] Theorems 1.3, 1.4, 1.5 for all configurations in this almost minimizing class.

The next step is to make the change of scales $x' = \sqrt{h_{\rm ex}} x$ in order to study (1.44). Under this rescaling, the inter-vortex distance becomes of order 1 (recall that the average vortex density is precisely $m_{\varepsilon,N} h_{\rm ex} \mathbf{1}_{\omega_\varepsilon}$ with $m_{\varepsilon,N} \to m_\lambda$ and (1.28)). After this change of scales the right-hand side of (1.45) becomes, in terms of $u'(x') = u(x)$ and $A'(x') = A(x)/\sqrt{h_{\rm ex}}$, and $\zeta'_\varepsilon(x') = \zeta(x)$,

(1.46)
$$F'_\varepsilon(u', A') = \frac{1}{2} \int_{\Omega'_\varepsilon} |\nabla_{A'} u'|^2 + h_{\rm ex} |\mathrm{curl}\, A' - m_\varepsilon \mathbf{1}_{\omega'_\varepsilon}|^2 + \frac{(1 - |u'|^2)^2}{2\varepsilon'^2} - \int_{\Omega'_\varepsilon} \zeta'_\varepsilon(x') \mu(u', A') + o(1)$$

where ω'_ε is the rescaled domain $\sqrt{h_{\rm ex}}\omega_{\varepsilon,N}$ and we recall $\varepsilon' = \varepsilon\sqrt{h_{\rm ex}}$.

Combining all these elements, the conclusion of the splitting procedure, found in Section 5 is

Proposition 1.7. *For any (u, A), there exists $N \in \{N_0^+, N_0^-\}$ such that*

$$G_\varepsilon(u, A) \ge G_\varepsilon^N + F'_\varepsilon(u', A') + o(1)$$

where F'_ε is as in (1.46), $u'(x') = u(x'/\sqrt{h_{\rm ex}})$, $A'(x') = A(x'/\sqrt{h_{\rm ex}})$ and ζ' is a positive function, equal to its maximum $\frac{1}{2}|\log \varepsilon'|$ on ω'_ε.

1.9 Full version of the main result

We may now give the more complete version of Theorem 4 and the stronger statement of Γ-convergence of $\frac{1}{N}(G_\varepsilon(u, A) - G_\varepsilon^N)$. In all the paper, the weak convergence of probabilities will mean convergence against bounded continuous test-functions, see [Bi]. We will say that a probability measure is concentrated on a set if that set has probability 1.

We consider configurations $(u_\varepsilon, A_\varepsilon)$ and assume that $|u_\varepsilon| \le 1$ everywhere together with the second Ginzburg-Landau equation, i.e. we assume

$$|u| \le 1 \text{ in } \Omega, \qquad -\nabla^\perp \mathrm{curl}\, A = (iu, \nabla_A u) \text{ in } \Omega \tag{1.47}$$

is satisfied. This is obviously true for minimizers and critical points of (1.22). Moreover given (u, A), replacing u by $u/|u|$ wherever $|u| \ge 1$ and replacing A by the minimizer of $A \mapsto G_\varepsilon(u, A)$, with u remaining fixed, decreases the energy without displacing the vortices, and the modified (u, A) verify (1.47). Since the change can only decrease the energy, our results allow to bound from below $G_\varepsilon(u_\varepsilon, A_\varepsilon)$ for the original arbitrary (u, A). Thus, assuming (1.47) is no loss of generality. Note that the only consequence of (1.47) that we will really use is that $\mathrm{div}\, j(u, A) = 0$.

Note that when $h_{\rm ex} - \lambda_\Omega |\log \varepsilon| = O(\log |\log \varepsilon|)$ we already know from [SS4], Chap. 12, that the number of vortices remains bounded as $\varepsilon \to 0$ and we already characterized their limiting location. For $h_{\rm ex} \ge \frac{\beta}{\varepsilon^2}$ the study of vortices involves an analysis quite different from that used in the present paper, see [SS2, AS, FH2]. So we focus on the remaining regimes, namely

$$\log|\log \varepsilon| \ll h_{\rm ex} - \lambda_\Omega|\log \varepsilon| \quad \text{and} \quad h_{\rm ex} \ll 1/\varepsilon^2, \tag{1.48}$$

and we have

Theorem 5. *Assume Ω is convex, hence satisfies (1.31). Assume (1.48) and $\frac{h_{\rm ex}}{|\log\,\varepsilon|} \to \lambda \in [\lambda_\Omega, +\infty]$ as $\varepsilon \to 0$. Let any $1 < p < 2$ be chosen.*

1. *Let $(u_\varepsilon, A_\varepsilon)$ satisfy (1.47), and*

$$G_\varepsilon(u_\varepsilon, A_\varepsilon) \le \min_{N\in\{N_0^-,N_0^+\}} G_\varepsilon^N + CN_0$$

for some C independent of ε. Then there is a choice of $N \in \{N_0^-, N_0^+\}$ such that, letting P_ε be the probability measure on $L^p_{\rm loc}(\mathbb{R}^2, \mathbb{R}^2)$ defined as the push-forward of the normalized uniform measure on $\omega_{\varepsilon,N}$ by the map $x \mapsto \tilde{\jmath}_{\varepsilon,x}$ (cf. (1.32)), as $\varepsilon \to 0$ the measures $\{P_\varepsilon\}_\varepsilon$ converge up to extraction to a probability measure P on $L^p_{\rm loc}(\mathbb{R}^2, \mathbb{R}^2)$ which is invariant under the action of translations, concentrated on $\mathcal{A}_{m_\lambda}$ (m_λ as in (1.28)) and

$$G_\varepsilon(u_\varepsilon, A_\varepsilon) \ge \min_{N\in\{N_0^-,N_0^+\}} G_\varepsilon^N + N\left(\frac{2\pi}{m_\lambda}\int W_U(j)\,dP(j) + \gamma + o(1)\right), \tag{1.49}$$

where γ is defined in (1.52), and W_U is computed according to (1.8) relatively to any family of sets satisfying (1.4)–(1.5).

In the case $h_{\rm ex} \le \frac{1}{\varepsilon^\beta}$ for some $\beta > 0$ small enough, then

$$\|\mu(u_\varepsilon, A_\varepsilon) - \mu_{\varepsilon,N}\|_{W^{-1,p}(\Omega)} \le C_p\sqrt{N}. \tag{1.50}$$

2. *For any probability P on $L^p_{\rm loc}(\mathbb{R}^2, \mathbb{R}^2)$ which is invariant under the action of translations and concentrated on $\mathcal{A}_{m_\lambda}$ and for every $N \in \{N_0^-, N_0^+\}$, there exists $(u_\varepsilon, A_\varepsilon)$ such that, letting P_ε be the push-forward of the normalized Lebesgue measure on $\omega_{\varepsilon,N}$ by the map $x \mapsto \frac{1}{\sqrt{h_{\rm ex}}} j(u_\varepsilon, A_\varepsilon)\left(x + \frac{\cdot}{\sqrt{h_{\rm ex}}}\right)$, we have as $\varepsilon \to 0$, $P_\varepsilon \to P$ and*

$$G_\varepsilon(u_\varepsilon, A_\varepsilon) \le G_\varepsilon^N + N\left(\frac{2\pi}{m_\lambda}\int W_K(j)\,dP(j) + \gamma + o(1)\right), \tag{1.51}$$

where W_K is the renormalized energy relative to the family of squares $\{K_R\}_R$, where $K_R = [-R/2, R/2]^2$, as defined in (1.8).

3. *If we assume that $(u_\varepsilon, A_\varepsilon)$ minimizes G_ε then it satisfies all the assumptions of item 1), P-almost every j minimizes W over $\mathcal{A}_{m_\lambda}$, and there is equality in (1.49).*

Remark 1.8. - *The constant γ in (1.49) and (1.51) was introduced in [BBH] and may be defined by*

$$\gamma = \lim_{R\to\infty}\left(\frac{1}{2}\int_{B_R} |\nabla u_0|^2 + \frac{(1-|u_0|^2)^2}{2} - \pi \log R\right), \tag{1.52}$$

where $u_0(r,\theta) = f(r)e^{i\theta}$ is the unique (up to translation and rotation) radially symmetric degree-one vortex (see [BBH, Mi]).

- *There exists* $\{(u_\varepsilon, A_\varepsilon)\}_\varepsilon$ *satisfying* (1.51) *for any* N *such that* $1 \ll N \le \frac{h_{\rm ex}|\Omega|}{2\pi}$, *see Theorem 7 in Section 7.*

- *As we already mentioned in Section 1.8, this theorem allows to retrieve the results of [SS4], but it gives a stronger result: in [SS4] we establish the leading order behaviour of the vorticity for minimizers:* $\mu(u_\varepsilon, A_\varepsilon) \sim h_{\rm ex}\mu_\lambda \sim \mu_{\varepsilon,N}$, *while here* (1.50) *gives (at least for small enough applied fields) for the whole class of "almost minimizers" the order of the fluctuations of the vorticity around the constant density* $\mu_{\varepsilon,N}$*: it is of order* $\sqrt{N}$, *hence in particular the number of vortices is* N *with an error of order* $\sqrt{N}$.

- *For energy minimizers it is possible to deduce from this result that* $\bar{N}$, *now being defined as the total degree of the vortices, satisfies* $G_\varepsilon^{\bar{N}} = \min_{N\in\mathbb{N}} G_\varepsilon^N + o(\bar{N})$. *On the other hand* $\min_{N\in\mathbb{N}} G_\varepsilon^N$ *is achieved at* N_0^- *or* N_0^+. *From examining carefully the variations of* $N \mapsto G_\varepsilon^N$ *one should be able to deduce that* $\bar{N} = N_0^-$ *or* N_0^+ *with a smaller error than previously (at least for* $h_{\rm ex} \le H_{c_1} + O(\sqrt{|\log \varepsilon|})$ *we expect this error to be* 0*), see Remark 6.7 for more details. So we expect, for small enough fields, to be able to estimate exactly the total degree of the vortices of a minimizer, and for larger fields, to estimate it with an error which is at least better than* $\sqrt{N}$.

1.10 Use of the ergodic theorem for Ginzburg-Landau

As announced, the method consists in applying the framework of Section 1.3 to the Ginzburg-Landau energy.

We sketch the method in the case $h_{\rm ex} = \lambda|\log \varepsilon|$, $\lambda_\Omega < \lambda < +\infty$. The case of higher fields $h_{\rm ex} \gg |\log \varepsilon|$ will reduce to this one by scaling.

Let $(u_\varepsilon, A_\varepsilon)$ — or $(u'_\varepsilon, A'_\varepsilon)$ in rescaled coordinates — denote a minimizer of G_ε and let $\mu'_\varepsilon = \mu(u'_\varepsilon, A'_\varepsilon)$. The splitting result of Section 1.8 combined with the blow up procedure reduces us (cf. Proposition 1.7) to bounding from below $F'_\varepsilon(u'_\varepsilon, A'_\varepsilon)$.

Thus we are in the setting of Example 2 in Section 1.3, i.e. the case where we want to bound from below the average over large domains ω'_ε of some energy density e_ε. Here the energy density is of the form

$$e_\varepsilon(u, A) = \frac{1}{2}|\nabla_A u|^2 + h_{\rm ex}|\mathrm{curl}\, A - m\mathbf{1}_{\omega'_\varepsilon}|^+ \frac{(1-|u|^2)^2}{2\varepsilon^2} - \int \zeta'\, d\mu(u, A).$$

From the Jacobian estimate μ'_ε is well approximated by $\nu_\varepsilon = 2\pi\sum_i d_i\delta_{a_i}$ (where a_i denotes the vortex center and d_i its degree) and we should have $a_i \in \omega'_\varepsilon$. Using in addition that $\zeta' = \frac{1}{2}|\log\varepsilon'|$ there, we may thus formally replace the energy-density above by

$$e_\varepsilon(u, A) = \frac{1}{2}|\nabla_A u|^2 + h_{\rm ex}|\mathrm{curl}\, A - m\mathbf{1}_{\omega'_\varepsilon}|^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2} - \pi \sum |\log \varepsilon| \sum_i d_i\delta_{a_i}.$$

To apply the framework of Section 1.3 we need to check the coercivity and Γ-liminf properties of $f_\varepsilon(u, A) = \int e_\varepsilon(u, A)\,\chi$ for a cut-off function χ as in the example of Section 1.3. But the framework also requires that f_ε be nonnegative (or bounded below by a constant will do) but this is obviously not the case, and one of the major difficulties of our analysis consists in getting around this problem. This part of the analysis was carried out in our companion paper [SS3], where we introduced a method that consists in displacing the negative part of e_ε so as to absorb it into the positive part and to obtain a density bounded below, called g_ε. We show there that e_ε can be replaced by $g_\varepsilon \geq -C$ with making only a small error (in average at the small scale). Note that since we are dealing with cancelling leading order terms, we need very precise estimates of the free-energy f_ε, in fact we need to make errors which are at most $o(1)$ per vortex (the total acceptable error is $o(N)$). Hence in [SS3] we need refined estimates in the "ball construction methods" which are devised to obtain general lower bounds for the energy of the vortices even when their number is unbounded. We also show in [SS3] the crucial fact that g_ε controls the number of vortices.

Returning to the question of proving the coercivity of f_ε, we may now heuristically replace e_ε by g_ε, and the coercivity requires proving 1.15 which in our case becomes

$$\forall R > 0,\ \limsup_{\varepsilon \to 0} \int g_\varepsilon(\chi * \mathbf{1}_{B_R}) < \infty \Longrightarrow (u_\varepsilon, A_\varepsilon) \text{ compact.}$$

This is satisfied thanks to the fact that g_ε controls the number of vortices, in other words we have roughly

$$\forall R > 0,\ \limsup_{\varepsilon \to 0} \int g_\varepsilon(\chi * \mathbf{1}_{B_R}) < \infty \Longrightarrow \sum_{a_i \in B_R} |d_i| \leq C_R.$$

To prove the Γ-liminf relation on f_ε, we may reduce to that setting, i.e. that where the number of vortices is bounded independently of ε on the compact support of χ. In that setting, it is now standard to retrieve very precise estimates for the Ginzburg-Landau energy, using an analysis of the type of [BBH]. This way, we obtain the precise Γ-liminf for f_ε and can show it can be expressed as a function of j (limit of $j(u'_\varepsilon, A'_\varepsilon)$ and is equal (up to a constant) to $f(j) = W(j, \chi)$. But then, using the definition (1.20), the "cell-energy" is

$$f^*(j) = \lim_{R \to +\infty} \fint_{\mathbf{U}_R} W(j(\lambda + \cdot), \chi)\, d\lambda.$$

One may immediately check, using Fubini, that this is equal to $\lim_{R \to +\infty} \frac{1}{|\mathbf{U}_R|} W(j, \chi * \mathbf{1}_{\mathbf{U}_R})$, which is in turn clearly equal to $W_U(j)$ (cf. (1.8) and the first item in Theorem 1). Combining all these elements, the result of Theorem 3 yields

$$\liminf_{\varepsilon \to 0} \fint_{\omega'_\varepsilon} \frac{1}{2} |\nabla_{A'_\varepsilon} u'_\varepsilon|^2 + h_{\text{ex}} |\operatorname{curl} A'_\varepsilon - m \mathbf{1}_{\omega'_\varepsilon}|^+ \frac{(1 - |u'_\varepsilon|^2)^2}{2\varepsilon^2} - \int \zeta'\, d\mu(u'_\varepsilon, A'_\varepsilon) \geq \int W(j)\, dP(j),$$

which is essentially the desired lower bound.

The rigorous proof of this is detailed in Section 6.

1.11 Plan of the paper

The paper contains two parts. The first part is completely independent of the Ginzburg-Landau energy and can thus be read independently. It starts in Section 2 with the proof of Theorem 3. In Section 3, we prove Theorem 2 i.e. that W is minimized among lattice configurations by the triangular lattice. In Section 4, we study W more generally, and prove Theorem 1.

The second part is about the application of the tools of the first part to the Ginzburg-Landau energy and the derivation of W as its Γ-limit. In Section 5, we prove the new energy-splitting formula, in the same section we recall or prove some results on the splitting function that will be needed in the sequel and we also derive simple a priori bounds for energy-minimizers. In Section 6, we show how to apply the abstract framework of Section 1.3 in the specific case of the Ginzburg-Landau energy, and this way we obtain the main energy lower bound, and prove Theorems 4 and 5 assuming the upper bound, which is proven in Section 7. This requires the improved lower bounds for the energy of vortices borrowed from [SS3]. In Section 7, we prove the matching upper bound for the energy via an explicit construction, using the periodic minimizing sequence found in Theorem 1.

In the Appendix, we prove some additional qualitative results on the solutions to the obstacle problem which can be of independent interest, in particular estimates when the size of the coincidence set is small, which are needed in the regime $h_{\rm ex} - H_{c_1} \ll |\log \varepsilon|$.

Acknowledgements: We would like to thank warmly S.R.S. Varadhan for his suggestion to use the ergodic theorem. Many thanks also to our colleagues A. Venkatesh, C.S. Güntürk, L. Caffarelli, A. Chambert-Loir, N. Fournier, L. Mazet, M. Traizet and E. Lesigne for providing information and/or help on various steps. We also thank the referee, as well as our students and post-docs H. Ben Moussa, D. Goldman and A. Contreras, for their careful and critical reading. This work would also not have been possible without several invitations of E.S to New York University and S.S to Paris XII. We thank these two institutions, and acknowledge the financial support of the National Science Foundation, an EURYI award (SS), and the Institut Universitaire de France (ES).

Part I
The renormalized energy

2 Proof of Theorem 3

Before giving the proof of the theorem, we state a preliminary lemma.

Lemma 2.1. *(E. Lesigne) Assume P_n are Borel probability measures on a Polish metric space X and that for any $\delta > 0$ there exists $\{K_n\}_n$ such that $P_n(K_n) \ge 1-\delta$ for every n and such that if $\{x_n\}_n$ satisfies for every n that $x_n \in K_n$, then any subsequence of $\{x_n\}_n$ admits a convergent subsequence (note that we do not assume K_n to be compact).*

Then $\{P_n\}_n$ admits a subsequence which converges tightly, i.e. converges weakly to a probability *measure P.*

Proof. From Prohorov's Theorem, it suffices to show that the sequence of measures is tight. As a finite Borel measure on a Polish space, the measure P_n is regular [Co], thus there is a compact subset $K_n' \subset K_n$ such that $P_n(K_n') \ge 1 - 2\delta$. Then, letting

$$K = \overline{\bigcup_n K_n'},$$

we have $P_n(K) \ge 1 - 2\delta$ for every n and the assumption made on $\{K_n\}_n$ implies that K is compact. Indeed, a sequence in K is either included in a finite union of the compact sets K_n (then is compact) or has a subsequence which can be relabelled (x_n) and satisfied $x_n' \in K_n'$ along a subsequence n', hence compact by assumption. Therefore $\{P_n\}_n$ is tight. $\square$

We now start proving the theorem. First we choose a sequence $\{\varepsilon_n\}_n$ tending to 0 such that

$$\lim_{n\to+\infty} F_{\varepsilon_n}(u_{\varepsilon_n}) = \liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon).$$

In this proof, ε will always be assumed to belong to this sequence, and $\lim_{\varepsilon\to 0}$ will mean the limit along this sequence.

Recall that P_ε is the image of the normalized Lebesgue measure restricted to ω_ε under the map $\lambda \mapsto \theta_\lambda u_\varepsilon$. In particular for any positive measurable function Φ on X

$$\int \Phi(u)\, dP_\varepsilon(u) = \fint_{\omega_\varepsilon} \Phi(\theta_\lambda u_\varepsilon)\, d\lambda. \tag{2.1}$$

Step 1: $\{P_\varepsilon\}_\varepsilon$ is tight. Letting $\omega_{\varepsilon,R} = \{x \in \omega_\varepsilon \mid \text{dist }(x, \partial\omega_\varepsilon) > R\}$ we have

$$\begin{aligned}\int_{\omega_{\varepsilon,R}} \int_{B_R} f_\varepsilon(\theta_{\lambda+\mu} u_\varepsilon)\, d\lambda\, d\mu &= \iint_{\mathbb{R}^2\times\mathbb{R}^2} \mathbf{1}_{\omega_{\varepsilon,R}}(\lambda)\mathbf{1}_{B_R}(\mu) f_\varepsilon(\theta_{\lambda+\mu} u_\varepsilon)\, d\lambda\, d\mu \\ &= \int_{\mathbb{R}^2} \mathbf{1}_{\omega_{\varepsilon,R}} * \mathbf{1}_{B_R}(\lambda) f_\varepsilon(\theta_\lambda u_\varepsilon)\, d\lambda \\ &\le |B_R| \int_{\omega_\varepsilon} f_\varepsilon(\theta_\lambda u_\varepsilon)\, d\lambda \le C|B_R||\omega_\varepsilon|.\end{aligned} \tag{2.2}$$

Let us denote $Y_{\varepsilon,R}$ the image of $\omega_{\varepsilon,R}$ by $\lambda \mapsto \theta_\lambda u_\varepsilon$, and

$$X^\varepsilon_{R,K} = \left\{ u \in X \mid \fint_{B_R} f_\varepsilon(\theta_\lambda u)\, d\lambda > K \right\}.$$

The left-hand side in (2.2) is larger than $K|B_R||\omega_\varepsilon| P_\varepsilon\left(X^\varepsilon_{R,K} \cap Y_{\varepsilon,R}\right)$. In addition $P_\varepsilon(X^\varepsilon_{R,K}) \ge P_\varepsilon\left(X^\varepsilon_{R,K} \cap Y_{\varepsilon,R}\right) - P_\varepsilon(Y^c_{\varepsilon,R}) \ge P_\varepsilon\left(X^\varepsilon_{R,K} \cap Y_{\varepsilon,R}\right) - \frac{|\omega_\varepsilon \setminus \omega_{\varepsilon,R}|}{|\omega_\varepsilon|}$, so we deduce from (2.2) that

$$P_\varepsilon(X^\varepsilon_{R,K}) \le \frac{C}{K} + \frac{|\omega_\varepsilon \setminus \omega_{\varepsilon,R}|}{|\omega_\varepsilon|}.$$

From (1.14) and for any $\delta > 0$, there exists a subsequence $\{\varepsilon_n\}_n$ such that

$$\frac{|\omega_{\varepsilon_n} \setminus \omega_{\varepsilon_n,R}|}{|\omega_{\varepsilon_n}|} < \delta 2^{-n}$$

and then

$$P_{\varepsilon_n}\left(\cup_{k=1}^n X^{\varepsilon_n}_{k,2^k/\delta}\right) \le C\delta.$$

Now we have that the hypotheses of Lemma 2.1 are satisfied. Indeed, letting K_n be the complement of $\cup_{k=1}^n X^{\varepsilon_n}_{k,2^k/\delta}$, we have $P_{\varepsilon_n}(K_n) \ge 1 - C\delta$. Moreover, if $u_n \in K_n$ for every n then $\forall R > 0$, $\forall n > R$ we have $u_n \notin X^{\varepsilon_n}_{R,2^R/\delta}$, i.e.

$$\fint_{B_R} f_{\varepsilon_n}(\theta_\lambda u_n)\, d\lambda \le \frac{2^R}{\delta}.$$

Then, from the coercivity assumption, a subsequence of $\{u_n\}$ converges.

Applying Lemma 2.1 we can conclude that $\{P_{\varepsilon_n}\}$ is tight and then that a subsequence converges weakly to a probability measure P. □

Step 2: P is θ-invariant. Let Φ be bounded continuous on X. Then from the definition of P_ε,

$$\int \Phi(u)\, dP(u) = \lim_{\varepsilon\to 0} \int \Phi(u)\, dP_\varepsilon(u) = \lim_{\varepsilon\to 0} \fint_{\omega_\varepsilon} \Phi(\theta_\lambda u_\varepsilon)\, d\lambda.$$

Moreover,

$$\int_{\omega_\varepsilon} \Phi(\theta_\lambda u_\varepsilon)\, d\lambda - \int_{\omega_\varepsilon} \Phi(\theta_{\lambda+\lambda_0} u_\varepsilon)\, d\lambda = \int_{\omega_\varepsilon} \Phi(\theta_\lambda u_\varepsilon)\, d\lambda - \int_{\omega_\varepsilon+\lambda_0} \Phi(\theta_\lambda u_\varepsilon)\, d\lambda.$$

Thus,

$$\left| \fint_{\omega_\varepsilon} \Phi(\theta_\lambda u_\varepsilon)\, d\lambda - \fint_{\omega_\varepsilon} \Phi(\theta_{\lambda+\lambda_0} u_\varepsilon)\, d\lambda \right| \le \frac{|\omega_\varepsilon \triangle (\omega_\varepsilon + \lambda_0)|}{|\omega_\varepsilon|} \|\Phi\|_{L^\infty},$$

where $\triangle$ denotes the symmetric difference between sets, and follows from (1.14) that $|\omega_\varepsilon \triangle (\omega_\varepsilon + \lambda_0)| = o(|\omega_\varepsilon|)$ as $\varepsilon \to 0$. We deduce that

$$\int \Phi(u)\, dP(u) = \lim_{\varepsilon\to 0} \fint_{\omega_\varepsilon} \Phi(\theta_{\lambda+\lambda_0} u_\varepsilon)\, d\lambda = \int \Phi(\theta_{\lambda_0} u)\, dP(u),$$

hence P is invariant under the action θ. □

We state the proof of (1.18) as a lemma.

Lemma 2.2. *Assume that X is a Polish metric space, that $\{P_\varepsilon\}_{\varepsilon>0}$, P are Borel probability measures on X such that $P_\varepsilon \to P$ as $\varepsilon \to 0$, and that $\{f_\varepsilon\}_{\varepsilon>0}$ and f are positive measurable functions on X such that $\liminf_{\varepsilon\to 0} f_\varepsilon(x_\varepsilon) \ge f(x)$ whenever $x_\varepsilon \to x$.*

Then,

$$\liminf_{\varepsilon\to 0} \int f_\varepsilon\, dP_\varepsilon \ge \int f\, dP. \tag{2.3}$$

This is (1.18) since $\int f_\varepsilon\, dP_\varepsilon = \fint_{\omega_\varepsilon} f_\varepsilon(\theta_\lambda u)\, d\lambda$.

Proof of the Lemma. It suffices to show that for any $\lambda, \delta > 0$, we have

$$\liminf_{\varepsilon\to 0} P_\varepsilon(\{f_\varepsilon > \lambda - \delta\}) \ge P(\{f > \lambda\}). \tag{2.4}$$

Indeed, using the standard expression for the integral of a positive function f

$$\int f(x)\, d\mu(x) = \int_0^\infty \mu(\{f > \lambda\})\, d\lambda,$$

we find by applying it to f_ε and P_ε, and then to f and P, in view of (2.4) that

$$\begin{aligned}
\liminf_{\varepsilon\to 0} \int f_\varepsilon(u)\, dP_\varepsilon(u) &= \liminf_{\varepsilon\to 0} \int_0^\infty P_\varepsilon(\{f_\varepsilon > \lambda\})\, d\lambda \\
&\ge \int_0^\infty \liminf_{\varepsilon\to 0} P_\varepsilon(\{f_\varepsilon > \lambda\})\, d\lambda \\
&\ge \int_0^\infty P(\{f > \lambda + \delta\})\, d\lambda \\
&\ge \int_0^\infty P(\{f > \lambda\})\, d\lambda - \delta \\
&= \int f(u)\, dP(u) - \delta,
\end{aligned}$$

where we have used Fatou's lemma, and the fact that P is a probability measure. Since this is true for any $\delta > 0$, we have (2.3).

We prove (2.4). For any $\delta > 0$ and $u \in X$ we claim that there exists an open neighborhood V_u of u and $\eta > 0$ such that,

$$\forall \varepsilon < \eta,\ \forall v \in V_u,\ f_\varepsilon(v) > F(u) - \delta. \tag{2.5}$$

Indeed, assume this were wrong, then there would exist $\delta > 0$ and $u \in X$ together with a sequence $\{\varepsilon\}$ tending to 0 and a corresponding sequence $\{u_\varepsilon\}_\varepsilon$ tending to u such that $f_\varepsilon(u_\varepsilon) \le f(u) - \delta$, thus contradicting the Γ-liminf assumption. Hence (2.5) holds.

We denote by $\mathcal{V}_\eta$ the set of u's such that $f_\varepsilon(v) > F(u) - \delta$ holds on V_u for every $\varepsilon < \eta$. Clearly, $\{\mathcal{V}_\eta\}_\eta$ is decreasing and from the above $\cup_{\eta>0}\mathcal{V}_\eta = X$. Thus, if we let for some $\lambda > 0$

$$E = \{u \in X \mid f(u) > \lambda\}, \quad E_\eta = E \cap \mathcal{V}_\eta,$$

then $\{E_\eta\}_\eta$ is decreasing and $E = \cup_\eta E_\eta$. Moreover, from the definition of $\mathcal{V}_\eta$, we have $f_\varepsilon > \lambda - \delta$ on the open set $O_\eta = \cup_{u\in E_\eta} V_u$ for every $\varepsilon < \eta$.

It follows that for any $\varepsilon < \eta$,

$$P_\varepsilon(\{f_\varepsilon > \lambda - \delta\}) \ge P_\varepsilon(O_\eta). \tag{2.6}$$

Then, since $P_\varepsilon \to P$ and since O_η is open and contains E_η, we have

$$\liminf_{\varepsilon\to 0} P_\varepsilon(O_\eta) \ge P(O_\eta) \ge P(E_\eta).$$

It then follows from (2.6) that

$$\liminf_{\varepsilon\to 0} P_\varepsilon(\{f_\varepsilon > \lambda - \delta\}) \ge P(E_\eta).$$

Since $E_\eta \nearrow E$ as $\eta \searrow 0$ we deduce by monotone convergence that (2.4) holds. □

Step 4: Proof of (1.19). Since P is invariant w.r.t. the action θ, we may apply the ergodic theorem as stated in [Be] to obtain that

$$\mathbf{E}^P(f(u)) = \mathbf{E}^P(f^*(u)),$$

where f^* is θ-invariant and P-a.e. equal to

$$\lim_{R\to\infty} \fint_{B_R} f(\theta_\lambda u)\, d\lambda. \tag{2.7}$$

It is also true (see [Be]) that this limit exists for P-a.e. u, and that balls may be replaced by any Vitali family satisfying (1.4). This proves (1.19).

□

Remark 2.3.

1. *The lower bound of Theorem 3 implies in particular that* $\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) \ge \min f^*$ *where* f^* *is given by* (2.7)*. If we assume in addition that for some family there is equality i.e. that* $\limsup_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) \le \min f^*$*, then comparing with the lower bound obtained in Theorem 3 we deduce that we have*

$$f^*(u) = \min f^* \quad \textit{for } P-a.e.u.$$

2. We may apply the same reasoning as Theorem 3 in $B(x_\varepsilon, R)$ instead of ω_ε, with R large, to the functional $\fint_{B(x_\varepsilon,R)} f_\varepsilon(\theta_\lambda u)\, d\lambda$, and we will obtain

$$\liminf_{\varepsilon\to 0} \fint_{B(x_\varepsilon,R)} f_\varepsilon(\theta_\lambda u_\varepsilon)\, d\lambda \ge \min f^* + o_R(1)$$

where $o_R(1) \to 0$ as $R \to \infty$ (the $o_R(1)$ is due to near boundary errors). If in addition we know that for some x_ε,

$$\limsup_{\varepsilon\to 0} \fint_{B(x_\varepsilon,R)} f_\varepsilon(\theta_\lambda u)\, d\lambda \le \min f^* + o_R(1) \tag{2.8}$$

we deduce

$$\lim_{R\to\infty} \lim_{\varepsilon\to 0} \fint_{B(x_\varepsilon,R)} f_\varepsilon(\theta_\lambda u_\varepsilon)\, d\lambda = \min f^*. \tag{2.9}$$

But if we assume $\limsup_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) \le \min f^*$, by a Fubini argument (2.8) holds for most $x_\varepsilon \in \omega_\varepsilon$, so the local estimate (2.9) too. This says that when the upper and lower bounds match, the energy density is essentially uniformly distributed, at any scale $\gg 1$.

3 Minimization of W in the periodic case : optimality of the triangular lattice

3.1 Calculation of W in the periodic case

In this section we study the minimization of the renormalized energy $W(j)$ among lattices in the following sense: We assume that $\Lambda = \mathbb{Z}\vec{u} + \mathbb{Z}\vec{v}$, where the vectors $(\vec{u}, \vec{v})$ form a basis of $\mathbb{R}^2$, that j is invariant w.r.t. translations by the vectors $\vec{u}$, $\vec{v}$ and

$$\operatorname{curl} j = 2\pi \sum_{p\in\Lambda} \delta_p - 1, \qquad \operatorname{div} j = 0.$$

It is not difficult to check that there exists such a current j if and only if $\det(\vec{u}, \vec{v}) = 2\pi$. We will denote $\mathbb{T}_\Lambda$ the torus $\mathbb{T}_\Lambda = \mathbb{R}^2/\Lambda$. We have

Proposition 3.1. *Assume $\Lambda = \mathbb{Z}\vec{u} + \mathbb{Z}\vec{v}$ and j are as above. Then, letting $\{\mathbf{U}_R\}$ be any family satisfying (1.4), (1.5), defining $W_U(j)$ as in (1.8) we have*

$$W_U(j) = \lim_{\eta\to 0} \frac{1}{2\pi} \left(\frac{1}{2} \int_{\mathbb{T}_\Lambda \setminus B(0,\eta)} |j|^2 + \pi \log \eta \right). \tag{3.1}$$

Moreover, letting H_Λ be the unique solution (with mean zero) on $\mathbb{T}_\Lambda$ to

$$-\Delta H_\Lambda = 2\pi\delta_0 - 1, \tag{3.2}$$

and denoting

$$j_\Lambda = -\nabla^\perp H_\Lambda \tag{3.3}$$

we have that $j - j_\Lambda$ is a constant and

$$W_U(j) \ge W_U(j_\Lambda)$$

with equality if and only if $j = j_\Lambda$.

We will denote $W_U(j_\Lambda)$ simply $W(\Lambda)$: it is the minimum of $W_U(j)$ among Λ-periodic configurations, and using (3.1) it is given by

$$W(\Lambda) = \lim_{\eta\to 0} \frac{1}{2\pi}\left(\frac{1}{2}\int_{\mathbb{T}_\Lambda\setminus B(0,\eta)} |\nabla H_\Lambda|^2 + \pi \log \eta\right).$$

Proof. Since $\operatorname{curl}(j - j_\Lambda) = \operatorname{div}\,(j - j_\Lambda) = 0$ and j, j_Λ are periodic, we have that $j - j_\Lambda$ is constant. We now prove (3.1). Let us denote by $\mathcal{K}$ the set of cells K of the form $\{t\vec{u} + s\vec{v}, t \in (l - \frac{1}{2}, l + \frac{1}{2}), s \in (m - \frac{1}{2}, m + \frac{1}{2})\}$, where $l, m \in \mathbb{Z}$. For $K \in \mathcal{K}$, let c_K be the center of the cell. The cells of $\mathcal{K}$ tile $\mathbb{R}^2$, hence for every $R > 0$, writing χ_R for $\chi_{\mathbf{U}_R}$,

$$W(j, \chi_R) = \lim_{\eta\to 0} \sum_{K\in\mathcal{K}} \frac{1}{2}\int_{K\setminus B(c_K,\eta)} |j|^2\chi_R + \pi\chi_R(c_K)\log\eta.$$

There are only finitely many K's on which χ_R is not identically zero, thus we may write

$$W(j, \chi_R) = \sum_{K\in\mathcal{K}} w_K(j, \chi_R)$$

where

$$w_K(j, \chi_R) = \lim_{\eta\to 0} \frac{1}{2}\int_{K\setminus B(c_K,\eta)} |j|^2\chi_R + \pi\chi_R(c_K)\log\eta.$$

Then, if $\chi_R \equiv 1$ on K, we have (by periodicity of j)

$$w_K(j, \chi_R) = \lim_{\eta\to 0} \int_{\mathbb{T}_\Lambda\setminus B(0,\eta)} |j|^2 + \pi \log \eta. \tag{3.4}$$

On the other hand, there exists $C > 0$ such that,

$$\forall R > 0, \forall K \in \mathcal{K}, \quad |w_K(j, \chi_R)| \le C. \tag{3.5}$$

Indeed, $j = cst - \nabla^\perp H_\Lambda$ with $H_\Lambda(x) = -\log|x| + U(x)$ where U is a C^2 function, so for any $r > 0$, we have

$$E(r) := \frac{1}{2}\int_{\mathbb{T}_\Lambda\setminus B(0,r)} |j|^2 \le C + \pi\log\frac{1}{r}.$$

From this we deduce first, by letting $r = \eta$, that

$$\left|\frac{1}{2}\int_{\mathbb{T}_\Lambda\setminus B(0,\eta)} |j|^2\chi_R(0) + \pi\chi_R(0)\log\eta\right| \le C,$$

and second, that, since $|\nabla\chi_R| \le C$, for any $0 < \eta < r_0$,

$$\left|\int_{B(0,r_0)\setminus B(0,\eta)} |j|^2(\chi_R - \chi_R(0))\right| \le C\int_{B(0,r_0)\setminus B(0,\eta)} |j|^2|x| = -C\int_\eta^{r_0} E'(t)t\,dt$$
$$= -C\left(E(r_0)r_0 - E(\eta)\eta + \int_\eta^{r_0} E(r)\,dr\right) \le C.$$

Adding the two proves (3.5). To conclude, we note that by definition of χ_R and (1.5)

$$\#\{K \in \mathcal{K} | \chi_R \equiv 1 \text{ on } K\} \sim_{R\to\infty} \tfrac{|\mathbf{U}_R|}{2\pi}$$
$$\#\{K \in \mathcal{K} | \chi_R \not\equiv 0 \text{ and } \chi_R \not\equiv 1 \text{ on } K\} = o(|\mathbf{U}_R|) \text{ as } R \to +\infty.$$

Together with (3.4)–(3.5) this yields

$$W(j, \chi_R) = \frac{|\mathbf{U}_R|}{2\pi} \lim_{\eta\to 0} \int_{\mathbb{T}_\Lambda \setminus B(0,\eta)} |j|^2 + \pi \log \eta + o(|\mathbf{U}_R|).$$

Combining with (1.8), we deduce (3.1). It remains to show that $W_U(j) \geq W_U(j_\Lambda)$ with equality iff $j = j_\Lambda$. Since $j = j_\Lambda + c$, using (3.1), we have

$$4\pi W_U(j) = \lim_{\eta\to 0} \int_{\mathbb{T}_\Lambda \setminus B(0,\eta)} |j_\Lambda|^2 + |c|^2 + 2c \cdot j_\Lambda + \pi \log \eta = 4\pi W_U(j_\Lambda) + |\mathbb{T}_\Lambda| c^2$$

since $\int_{\mathbb{T}_\Lambda} c \cdot j_\Lambda = -c \cdot \int_{\mathbb{T}_\Lambda} \nabla^\perp H_\Lambda = 0$. The result follows. □

We next express $W(\Lambda)$ as a series using the Fourier decomposition of H_Λ.

Lemma 3.2. *For all $\Lambda \in \mathcal{L}$ we have*

$$W(\Lambda) = \frac{1}{2} \lim_{x\to 0} \left(\sum_{p \in \Lambda^* \setminus \{0\}} \frac{e^{2i\pi p \cdot x}}{4\pi^2 |p|^2} + \log |x| \right), \tag{3.6}$$

where Λ^* denotes the lattice dual to Λ.

Proof. Integrating by parts and using (3.2) we find

$$\frac{1}{2} \int_{\mathbb{T}_\Lambda \setminus B(0,\eta)} |\nabla H_\Lambda|^2 + \pi \log \eta = \frac{1}{2} \left(\int_{\partial(B(0,\eta))} H_\Lambda \frac{\partial H_\Lambda}{\partial \nu} - \int_{\mathbb{T}_\Lambda \setminus B(0,\eta)} H_\Lambda \right) + \pi \log \eta. \tag{3.7}$$

But $\int_{\mathbb{T}_\Lambda} H_\Lambda = 0$ and $H_\Lambda(x) + \log |x|$ is a C^1 function in a neighbourhood of 0, thus passing to the limit $\eta \to 0$ above we find

$$W(\Lambda) = \frac{1}{2} \lim_{x\to 0} (H_\Lambda(x) + \log |x|). \tag{3.8}$$

Using the following normalisation of the Fourier transform

$$\hat{f}(y) = \int_{\mathbb{R}^2} f(x) e^{-2i\pi x \cdot y} \, dx,$$

since $-\Delta H_\Lambda = 2\pi \sum_{p \in \Lambda} \delta_p - 1$ in $\mathbb{R}^2$ and H_Λ has zero mean we have

$$\hat{H_\Lambda}(y) = \frac{1}{4\pi^2 |y|^2} \sum_{p \in \Lambda^* \setminus \{0\}} \delta_p(y),$$

where Λ^* is the dual lattice of Λ, i.e. the set of vectors q such that $p \cdot q \in \mathbb{Z}$ for every $p \in \Lambda$. By Fourier inversion formula, we obtain the expression of H_Λ in Fourier series:

$$H_\Lambda(x) = \sum_{p \in \Lambda^* \setminus \{0\}} \frac{e^{2i\pi p \cdot x}}{4\pi^2 |p|^2}, \tag{3.9}$$

and the result follows from (3.8). □

We may now prove Lemma 1.3. For the more general periodic situation of Lemma 1.3 where Λ is assumed to be merely $(\vec{u},\vec{v})$ periodic instead of being itself a lattice, and j is also $(\vec{u},\vec{v})$ periodic with $\operatorname{curl} j = 2\pi\sum_{p\in\Lambda}\delta_p - n$, we observe that Proposition 3.1 can be adapted with identical proofs. In this case, denoting by $\mathbb{T}_{(\vec{u},\vec{v})}$ the quotient $\mathbb{R}^2/(\mathbb{Z}\vec{u}+\mathbb{Z}\vec{v})$, the configuration Λ may be seen as a finite family of points $(a_1,\dots,a_n)$ in $\mathbb{T}_{(\vec{u},\vec{v})}$, and the statements of Proposition 3.1 remain true, replacing $\mathbb{T}_\Lambda$ by $\mathbb{T}_{(\vec{u},\vec{v})}$, replacing $\mathbb{T}_\Lambda\setminus B(0,\eta)$ by $\mathbb{T}_{(\vec{u},\vec{v})}\setminus\cup_i B(a_i,\eta)$ and replacing $-\Delta H_\Lambda = 2\pi\delta_0 - 1$ by $-\Delta H_{\{a_i\}} = 2\pi\sum_{i=1}^n \delta_{a_i} - n$, in particular $W_U(j_{\{a_i\}}) = \frac{1}{2\pi}W(j_{\{a_i\}}, \mathbf{1}_{\mathbb{T}_{(\vec{u},\vec{v})}})$. Moreover, writing $j_{\{a_i\}}$ as $-\nabla^\perp H_{\{a_i\}}$ and using the translation invariance of the equation, we have

$$H_{\{a_i\}}(x) = \sum_{i=1}^n G(x-a_i)$$

where G is the solution to $-\Delta G(x) = 2\pi\delta_0 - 1$ on the torus $\mathbb{T}_{(\vec{u},\vec{v})}$. We may also define $R(x) = G(x) + \log|x|$, which is known to be a continuous function. Next, integrating by parts exactly as in (3.7)–(3.8), we easily find that

$$W(j_{\{a_i\}}) = \frac{1}{2}\left(\sum_{i\neq j} G(a_i-a_j) + \sum_{i=1}^n R(0)\right),$$

i.e. we deduce the result of Lemma 1.3.

Note that we may also compute W in Fourier series just as above and we find (1.11) i.e.

$$W(j_{\{a_i\}}) = \frac{1}{2}\sum_{i\neq j}\sum_{p\in(\mathbb{Z}\vec{u}+\mathbb{Z}\vec{v})^*\setminus\{0\}} \frac{e^{2i\pi p\cdot(a_i-a_j)}}{4\pi^2|p|^2} + \frac{n}{2}\lim_{x\to 0}\left(\sum_{p\in(\mathbb{Z}\vec{u}+\mathbb{Z}\vec{v})^*\setminus\{0\}} \frac{e^{2i\pi p\cdot x}}{4\pi^2|p|^2} + \log|x|\right),$$

which is formula (1.11).

This may also be rewritten in the form

$$W(j_{\{a_i\}}) = \frac{1}{2}\sum_{i\neq j} H_{\mathbb{Z}\vec{u}+\mathbb{Z}\vec{v}}(a_i-a_j) + nW(\mathbb{Z}\vec{u}+\mathbb{Z}\vec{v})$$

where $H_{\mathbb{Z}\vec{u}+\mathbb{Z}\vec{v}}$ is as in (3.2) or (3.9) and $W(\mathbb{Z}\vec{u}+\mathbb{Z}\vec{v})$ as in (3.8). This way W is expressed as a sum of the form $\sum_{i\neq j} f(a_i-a_j)$. Note that the series defining $H_{\mathbb{Z}\vec{u}+\mathbb{Z}\vec{v}}$ is an Eisenstein series and can thus be expressed in terms of the Dedekind eta function via the "second Kronecker limit formula" as in the proof of Lemma 3.3 below, see also [BoS] for more computations.

3.2 Proof of Theorem 2

First, from (1.12) we may consider only the case $m = 1$. Thus the lattice Λ is in $\mathcal{L}$ iff its fundamental cell has area 2π. We return to the expression (3.6) for $W(\Lambda)$, and, using standard functions and formulas from number theory, give a closed form for it in terms of the Dedekind eta function. One can also view the series expression of $W(\Lambda)$ as a regularization, or renormalization, of the divergent series

$$\sum_{p\in\Lambda^*\setminus\{0\}} \frac{1}{8\pi^2|p|^2}.$$

We also show that modulo a constant independent of Λ, this particular regularization is equal to the regularization which uses the Zeta functions

$$\zeta_{\Lambda^*}(x) = \sum_{p\in\Lambda^*\setminus\{0\}} \frac{1}{8\pi^2|p|^{2+x}},$$

for which the minimizers w.r.t. Λ are known. Both results are the object of the following lemmas:

Lemma 3.3. *We have*

$$W(\Lambda) = -\frac{1}{2}\log(\sqrt{2\pi b}|\eta(\tau)|^2) \tag{3.10}$$

with η the Dedekind eta function, where the dual lattice Λ^* to Λ has, up to rotation, a fundamental cell given by the vectors $\frac{1}{\sqrt{2\pi b}}(1,0)$ and $\frac{1}{\sqrt{2\pi b}}(a,b)$, and τ denotes $a+ib$.

Proof. We may parametrize Λ^* as $\{\frac{1}{\sqrt{2\pi b}}(m\tau+n), (m,n)\in\mathbb{Z}^2\}$. Then for $x=(x_1,x_2)$ we have

$$\sum_{p\in\Lambda^*\setminus\{0\}} \frac{e^{2i\pi p\cdot x}}{4\pi^2|p|^2} = \frac{1}{2\pi}\sum_{(m,n)\in\mathbb{Z}^2\setminus\{0\}} e^{\frac{2i\pi}{\sqrt{2\pi b}}(m(ax_1+bx_2)+nx_1)}\frac{b}{|m\tau+n|^2}. \tag{3.11}$$

One can recognize that this is an Eisenstein series i.e. of the form

$$E_{u,v}(\tau) = \sum_{(m,n)\in\mathbb{Z}^2\setminus\{0\}} e^{2i\pi(mu+nv)}\frac{b}{|m\tau+n|^2}$$

with $\tau=a+ib$, $u=\frac{1}{\sqrt{2\pi b}}(ax_1+bx_2)$ and $v=\frac{1}{\sqrt{2\pi b}}x_1$. But the "second Kronecker limit formula" (see [La1]) states that

$$E_{u,v}(\tau) = -2\pi\log|f(u-v\tau,\tau)q^{v^2/2}|$$

where

$$f(z,\tau) := q^{1/12}(p^{1/2}-p^{-1/2})\prod_{n\geq 1}(1-q^np)(1-q^n/p)$$

with $q=e^{2i\pi\tau}$, $p=e^{2i\pi z}$. Here $z=u-v\tau=\sqrt{\frac{b}{2\pi}}(x_2-ix_1)$. As $x\to 0$, we have $p\to 1$, and then the expression above can be expressed in terms of the Dedekind eta function

$$\eta(\tau) := q^{1/24}\prod_{n\geq 1}(1-q^n).$$

Inserting all the above in (3.11) we obtain that as $|x|\to 0$,

$$\sum_{p\in\Lambda^*\setminus\{0\}} \frac{e^{2i\pi p\cdot x}}{4\pi^2|p|^2} \sim -\log(\sqrt{2\pi b}|\eta(\tau)|^2|x|)$$

and combining with (3.6) we find the result (3.10). □

Lemma 3.4. *There exists $C \in \mathbb{R}$ such that for any $\Lambda \in \mathcal{L}$ we have*

$$W(\Lambda) = C + \lim_{x\to 0}\left(\zeta_{\Lambda^*}(x) - \int_{\mathbb{R}^2} \frac{\pi}{1+4\pi^2|y|^{2+x}}\,dy\right).$$

Two proofs can be given for this : one, which we give in the following subsection below, uses standard analysis methods. The other uses the closed form (3.10) and the fact that the Dedekind eta function is in turn related to the Epstein zeta function

$$Z(\tau,s) = \sum_{(m,n)\in\mathbb{Z}^2\backslash 0} \frac{b^s}{|m\tau+n|^{2s}} = \frac{1}{(2\pi)^s} \sum_{p\in\frac{1}{\sqrt{2\pi b}}(\mathbb{Z}+\tau\mathbb{Z})\backslash 0} \frac{1}{|p|^{2s}}$$

via the "first Kronecker limit formula"

$$Z(\tau,s) = \frac{\pi}{s-1} + 2\pi(\gamma_0 - \log 2 - \log(\sqrt{b}|\eta(\tau)|^2)) + O(s-1) \quad \text{as } s\to 1,$$

where γ_0 is this time the Euler constant. Putting these pieces together correctly yields a proof of Lemma 3.4.

Then, in order to minimize $W(\Lambda)$ over $\mathcal{L}$, i.e. over lattices under the constraint $|\mathbb{T}_\Lambda| = 2\pi$, one is reduced to the question of minimizing $\zeta_{\Lambda^*}(x)$. It is proven in [Mont] that

$$\frac{2^{1+\frac{x}{2}}8\pi^2\Gamma(1+x/2)}{(2\pi)^{1+x/2}}\zeta_{\Lambda^*}(x) = \frac{2}{x} - \frac{1}{1+\frac{x}{2}} + \int_1^\infty (\theta_{\Lambda^*}(a)-1)(a^{-x/2}+a^{1+x/2})\frac{da}{a}$$

where

$$\theta_\Lambda(a) = \sum_{p\in\Lambda} e^{-\pi a|p|^2} \tag{3.12}$$

is the Jacobi Theta function and Γ is the Gamma function. Moreover, from [Cas, Ran, En1, Di, Mont] and modulo rotation, the minimum of θ over $\mathcal{L}^*$ is uniquely achieved by $(\Lambda_0)^*$, where $\Lambda_0 = \alpha\left(\mathbb{Z}(1,0)+\mathbb{Z}\left(1/2,\sqrt{3}/2\right)\right)$ and the factor α is chosen such that $|\mathbb{T}_{\Lambda_0}| = 2\pi$. But, from the above formula,

$$\frac{2^{1+\frac{x}{2}}8\pi^2\Gamma(1+x/2)}{(2\pi)^{1+x/2}}(\zeta_{\Lambda^*}(x)-\zeta_{\Lambda_0^*}(x)) = \int_1^\infty (\theta_{\Lambda^*}(a)-\theta_{\Lambda_0^*}(a))(a^{-x/2}+a^{1+x/2})\frac{da}{a}. \tag{3.13}$$

In view of (3.12) the integrand in (3.13) is dominated by an integrable function independently of x. Hence, letting x tend to 0, Lebesgue's dominated convergence theorem yields

$$8\pi\Gamma(1)\lim_{x\to 0}\zeta_{\Lambda^*}(x) - \zeta_{\Lambda_0^*}(x) = \int_1^\infty (\theta_{\Lambda^*}(a)-\theta_{\Lambda_0^*}(a))(1+a)\,\frac{da}{a}.$$

Hence, using Lemma 3.4, we have

$$8\pi\Gamma(1)(W(\Lambda)-W(\Lambda_0)) = \int_1^\infty (\theta_{\Lambda^*}(a)-\theta_{\Lambda_0^*}(a))(1+a)\,\frac{da}{a}.$$

We deduce that $W(\Lambda) \geq W(\Lambda_0)$ for any Λ, with equality if and only if $\theta_{\Lambda^*}(a) = \theta_{\Lambda_0^*}(a)$ for almost every a, i.e. if, modulo rotations, $\Lambda = \Lambda_0$. This proves Theorem 2.

Analytic proof of Lemma 3.4

Denoting by G_0 the unique solution of $-\Delta G_0 + G_0 = 2\pi\delta_0$ in $\mathbb{R}^2$, we may write

$$H_\Lambda(x) + \log|x| = U_\Lambda + (G_0(x) + \log|x|)\,,$$

where $U_\Lambda = H_\Lambda - G_0$ and $G_0(x) + \log|x|$ are C^1 near the origin. Taking limits, we obtain from (3.8)

$$W(\Lambda) = \gamma_0 + \frac{1}{2}U_\Lambda(0),$$

where $\gamma_0 = \frac{1}{2}\lim_{x\to 0}(G_0(x) + \log|x|)$. Denote by $\varphi(x) = (2\pi)^{-1}e^{-|x|^2/2}$ the Gaussian distribution in $\mathbb{R}^2$ and for any $n \in \mathbb{N}$ let $\varphi_n(x) = n^2\varphi(nx)$, so that $\{\varphi_n\}_n$ is an approximate identity. We have $\hat{\varphi}_n(y) = e^{-|y|^2/2n^2}$. Then, since U_Λ is continuous at 0 and bounded in $\mathbb{R}^2$, we have

$$U_\Lambda(0) = \lim_{n\to+\infty} w(n,\Lambda),\ \text{ where } w(n,\Lambda) = \int_{\mathbb{R}^2} \varphi_n(x)U_\Lambda(x)dx = \int_{\mathbb{R}^2} \hat{\varphi}_n(y)\hat{U_\Lambda}(y)\,dy.$$

Also, it is standard that $\hat{G}_0 = 2\pi/(1+4\pi^2|y|^2)$. Then, since $\hat{U_\Lambda} = \hat{H}_\Lambda - \hat{G}_0$, we get

$$w(n,\Lambda) = \sum_{p\in\Lambda^*\setminus\{0\}} \frac{e^{-|p|^2/2n^2}}{4\pi^2|p|^2} - 2\pi\int_{\mathbb{R}^2} \frac{e^{-|y|^2/2n^2}}{1+4\pi^2|y|^2}\,dy.$$

Note that $|\mathbb{T}_{\Lambda^*}| = 1/|\mathbb{T}_\Lambda|^{-1} = 1/2\pi$, which accounts for the factor 2π in front of the integral.

The second step is to show that

(3.14)
$$\lim_{n\to+\infty} w(n,\Lambda) = \lim_{x\to 0} v(x,\Lambda),\ \text{ where } v(x,\Lambda) = \sum_{p\in\Lambda^*\setminus\{0\}} \frac{1}{4\pi^2|p|^{2+x}} - \int_{\mathbb{R}^2} \frac{2\pi}{1+4\pi^2|y|^{2+x}}\,dy.$$

First we truncate $w(n,\Lambda)$ and $v(x,\Lambda)$, letting for each $N \in \mathbb{N}$

$$w^N(n,\Lambda) = \sum_{\substack{p\in\Lambda^*\setminus\{0\}\\ |p|<N}} \frac{e^{-|p|^2/2n^2}}{4\pi^2|p|^2} - 2\pi\int_{B(0,N)} \frac{e^{-|y|^2/2n^2}}{1+4\pi^2|y|^2}\,dy, \quad \tilde{w}^N(n,\Lambda) = w(n,\Lambda) - w^N(n,\Lambda),$$

and defining similarly $v^N(x,\Lambda)$, $\tilde{v}^N(x,\Lambda)$. It is clear that for any N

$$\lim_{n\to+\infty} w^N(n,\Lambda) = \lim_{x\to 0} v^N(x,\Lambda) = \frac{1}{4\pi^2}\sum_{\substack{p\in\Lambda^*\setminus\{0\}\\ |p|<N}} \frac{1}{|p|^2} - 2\pi\int_{B(0,N)} \frac{1}{1+4\pi^2|y|^2}\,dy. \tag{3.15}$$

Then we claim that

$$\lim_{N\to+\infty}\lim_{n\to+\infty} \tilde{w}^N(n,\Lambda) = \lim_{N\to+\infty}\lim_{x\to 0} \tilde{v}^N(x,\Lambda) = 0, \tag{3.16}$$

which together with (3.15) clearly implies (3.14). To prove (3.16), for any $p \in \Lambda^*$ we let K_p be the Voronoi cell centered at p, i.e. the set of points in $\mathbb{R}^2$ which are closer to p than to any other point of Λ^*. Then

$$\tilde{w}^N(n,\Lambda) = \sum_{\substack{p\in\Lambda^*\setminus\{0\}\\ |p|\geq N}} \fint_{K_p} \frac{e^{-|p|^2/2n^2}}{4\pi^2|p|^2} - \frac{e^{-|y|^2/2n^2}}{1+4\pi^2|y|^2}\,dy + \delta(N,n,\Lambda), \tag{3.17}$$

where

$$\delta(N,n,\Lambda) = 2\pi \left(\sum_{p\notin B(0,N)} \int_{K_p\cap B(0,N)} \frac{e^{-|y|^2/2n^2}}{1+4\pi^2|y|^2}\,dy - \sum_{p\in B(0,N)} \int_{K_p\setminus B(0,N)} \frac{e^{-|y|^2/2n^2}}{1+4\pi^2|y|^2}\,dy \right).$$

We can bound $|\delta|$ by the integral of $(1+4\pi^2|y|^2)^{-1}$ over those cells which intersect both $B(0,N)$ and its complement, the union of which is included in $B(0,N+C_\Lambda)\setminus B(0,N-C_\Lambda)$. Thus we have for every Λ

$$\lim_{N\to+\infty}\lim_{n\to+\infty} \delta(N,n,\Lambda) = 0. \tag{3.18}$$

Then we consider the sum in (3.17). If $|p| > n^2$ it is straightforward to bound the contribution of K_p, which we denote M_p, by $C_\Lambda e^{-|p|/2}/|p|^2$, and from there to deduce

$$\lim_{N\to+\infty}\lim_{n\to+\infty} \sum_{\substack{p\in\Lambda^*\setminus\{0\}\\ |p|\ge n^2}} |M_p| = 0. \tag{3.19}$$

If $|p|\le n^2$, we use the fact that the gradient of $y\to e^{-|y|^2/2n^2}$ is bounded by C/n on $\mathbb{R}^2$, and that $e^{-|p|^2/2n^2}\le 1$ to write

$$|M_p| \le \left| \fint_{K_p} \frac{1}{4\pi^2|p|^2} - \frac{1}{1+4\pi^2|y|^2}\,dy \right| + \frac{C}{n}\fint_{K_p} \frac{dy}{1+4\pi^2|y|^2}.$$

The sum w.r.t $p\in\Lambda^*$ of the first term on the right-hand side is convergent, therefore the sum w.r.t $p\in\Lambda^*$, $n^2\ge|p|\ge N$ tends to 0 as $N\to+\infty$, uniformly in n. For the second term, the corresponding sum may be compared with the integral

$$\frac{C}{n}2\pi \int_{N\le|y|\le n^2} \frac{dy}{1+4\pi^2|y|^2},$$

which is $O(\log n/n)$ as $n\to+\infty$. Therefore

$$\lim_{N\to+\infty}\lim_{n\to+\infty} \sum_{\substack{p\in\Lambda^*\setminus\{0\}\\ n^2\ge|p|\ge N}} |M_p| = 0.$$

Together with (3.19) and (3.18), this proves that $\lim_{N\to+\infty}\lim_{n\to+\infty}\tilde{w}^N(n,\Lambda) = 0$. The proof that $\lim_{N\to+\infty}\lim_{x\to 0}\tilde{v}^N(x,\Lambda) = 0$ is similar, we omit it. Then (3.16) holds, which proves (3.14) and the lemma.

4 The renormalized energy in the general case: proof of Theorem 1

In this section we prove Theorem 1. We recall that we can reduce by scaling to studying the case of $\mathcal{A}_m = \mathcal{A}_1$ i.e. where (see Definition 1.1)

$$\operatorname{curl} j = \nu - 1 \qquad \operatorname{div} j = 0, \tag{4.1}$$

where $\nu = 2\pi \sum_{p\in\Lambda} \delta_p$ is such that $\{\frac{\nu(B_R)}{|B_R|}\}_R$ is bounded. For convenience, once the class $\mathcal{A}_1$ has been chosen, if $p < 2$ we may extend the definition of W_U to all $j \in L^p_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ by letting

$$W_U(j) = +\infty \quad \text{if } j \notin \mathcal{A}_1.$$

We now assume W has been likewise extended for a certain $1 < p < 2$. In what follows we will show that the minimizers and the minimum of W_U do not depend on U. However W_U itself does. When a statement is independent of U, we will sometimes write W instead of W_U.

One of the difficulties about W is that it is not lower semi-continuous. Here is a hint as to why: consider a set of points Λ_n which is equal to the square lattice (of density 1) in the ball B_n and to the triangular lattice (of density 1) outside B_n. Let j_n be corresponding vector fields. Then, since W is insensitive to compact perturbations, we have for every n, in informal notation, $W(j_n) = W(\text{triangular})$. On the other hand, as $n \to \infty$, $j_n \to j_{\text{square}}$ by construction. So if W was lower semi-continuous, we should have $W(\text{square}) \le W(\text{triangular})$, which is false by Theorem 2.

However, we will show that W is lower semi-continuous "up to translations".

We split the results into several propositions:

Proposition 4.1. *Let U refer to any family $\{\mathbf{U}_R\}$ satisfying* (1.4), (1.5). *Let $1 < p < 2$. We have:*

- *The value of $W_U(j)$ is independent of the choice of cutoff functions $\{\chi_{\mathbf{U}_R}\}_R$ satisfying* (1.3).
- *$W_U : L^p_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2) \to \mathbb{R} \cup \{+\infty\}$ is a Borel function.*
- *The sublevel sets $\{j, W_U(j) \le \alpha\}$ are "compact in $L^p_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ up to translation", more precisely: for every j_n such that $\limsup_{n\to\infty} W(j_n) < +\infty$, after extraction of a subsequence, there exists a sequence $\lambda_n \in \mathbb{R}^2$ and $j \in L^p_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ such that $j_n(\lambda_n + \cdot) \to j$ in $L^p_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ and*

  $$W_U(j) \le \liminf_{n\to\infty} W_U(j_n). \tag{4.2}$$

  In particular $\inf_{L^p_{\text{loc}}(\mathbb{R}^2,\mathbb{R}^2)} W_U = inf_{\mathcal{A}_1} W_U$ is achieved and is finite.
- *The minimizers and the value of the minimum of W_U are independent of U.*

We will also prove a result of uniform approximation by construction, which implies in particular that the minimum of W may be approximated by suitable periodic configurations. This construction will be crucial in particular in constructing test functions. In what follows K_R denotes the square $[-R, R]^2$, and W_K denotes the renormalized energy relative to $\{K_R\}_R$. In the following results, squares could easily be replaced by arbitrary parallelograms.

Proposition 4.2. *Let $G \subset \mathcal{A}_1$ be such that there exist $C > 0$ such that for any $j \in G$, we have*

$$\forall R > 1, \quad \frac{\nu(K_R)}{|K_R|} < C \tag{4.3}$$

for the associated ν's and such that, uniformly with respect to $j \in G$,

$$\lim_{R\to+\infty} \frac{W(j,\chi_{K_R})}{|K_R|} = W_K(j) \le C. \tag{4.4}$$

Then for any $\varepsilon > 0$, there exists $R_0 > 0$ such that if $R > R_0$ and $|K_R| \in 2\pi\mathbb{N}$, for any $j \in G$ there exists j_R such that

$$\begin{cases} \operatorname{curl} j_R = 2\pi \sum_{p\in\Lambda_R} \delta_p - 1 & \text{in } K_R, \\ j_R \cdot \tau = 0 & \text{on } \partial K_R, \end{cases} \tag{4.5}$$

where Λ_R is a discrete subset of the interior of K_R, and

- *if $x \in K_{R-2R^{3/4}}$ then $j_R(x) = j(x)$,*
- $\dfrac{W(j_R, \mathbf{1}_{K_R})}{|K_R|} \le W_K(j) + \varepsilon.$

Remark 4.3. *An inspection of the proof allows to see that the construction can alternatively be made in any rectangle which is a small perturbation of the square K_R, i.e whose sidelengths are in $[2R, 2R(1+\eta)]$, where η depends on ε.*

Note that this result is close to establishing that $\mathcal{A}_1$ is "uniformly W-approximable" in the sense of [AM], Definition 4.14.

The following corollaries are proved at the end of this section.

Corollary 4.4. *Given any j such that $W_K(j) < \infty$, there exists a sequence $\{j_R\}_{R^2\in 2\pi\mathbb{N}}$ in $\mathcal{A}_1$ such that each j_R is K_R periodic (i.e. $j_R(x + 2Rke_1 + 2Rle_2) = j_R(x)$ for $k, l \in \mathbb{Z}$ where (e_1, e_2) is the canonical basis of $\mathbb{R}^2$) and*

$$\limsup_{R\to\infty} W_K(j_R) = \limsup_{R\to\infty} \frac{W(j_R, \mathbf{1}_{K_R})}{|K_R|} \le W_K(j). \tag{4.6}$$

In particular there exists a minimizing sequence for $\min_{\mathcal{A}_1} W$ consisting of periodic vector-fields.

Corollary 4.5. *Let $p < 2$ and let P be a probability measure on $L^p_{loc}(\mathbb{R}^2, \mathbb{R}^2)$ which is invariant under the action of translations and concentrated in $\mathcal{A}_1$. Then there exists a sequence $R \to +\infty$ and a sequence $\{j_R\}_R$ of vector fields defined over K_R such that*

- *There exists a finite subset Λ_R of the interior of K_R such that*

$$\begin{cases} \operatorname{curl} j_R = 2\pi \sum_{p\in\Lambda_R} \delta_p - 1 & \text{in } K_R \\ j_R \cdot \tau = 0 & \text{on } \partial K_R. \end{cases} \tag{4.7}$$

- *Letting P_R be the probability measure on $L^p_{loc}(\mathbb{R}^2, \mathbb{R}^2)$ which is defined as the image of the normalized Lebesgue measure on K_R by $x \mapsto j_R(x + \cdot)$ where j_R is extended periodically to the whole $\mathbb{R}^2$, we have $P_R \to P$ weakly as $R \to \infty$.*
- $\displaystyle \limsup_{R\to\infty} \frac{W(j_R, \mathbf{1}_{K_R})}{|K_R|} \le \int W_K(j)\, dP(j).$

4.1 A mass displacement result

In this subsection, we show that even though the integrand in the definition of W is not bounded below, we can somehow reduce to that case by a "mass displacement" method, which is an adaptation of that of [SS3]. The situation is much simpler here due to the fact that all degrees are +1. It follows the same steps however, consisting in ball construction combined with mass displacement.

We begin with a ball construction argument à la Sandier and Jerrard [Sa, Je]. For the sake of generality, we prove the result for an average density m which may depend on x.

Proposition 4.6. *Assume* $\operatorname{curl} j = \nu - m$ *in the sense of distributions in some open set* U, *where* $m \in L^\infty(U)$ *and* $\nu = 2\pi \sum_{p\in\Lambda} \delta_p$ *for some finite subset* Λ *of* U, *and that* $j \in L^2_{\mathrm{loc}}(U\setminus\Lambda)$. *Let* $n = \#\Lambda$ *and* $\eta_0 > 0$ *be the minimal distance between points of* Λ. *Then there exists for any* $r \in (0, 1]$ *a family of disjoint closed balls* $\mathcal{B}_r$ *of total radius* r *covering* Λ *such that:*

- *If* $\Lambda = \{p\}$ *then* $\mathcal{B}_r = \{B(p, r)\}$.
- *If* $r/n \le \eta_0$, *then* $\mathcal{B}_r = \{B(p, \frac{r}{n})\}_{p\in\Lambda}$.
- *The set* $\cup_{B\in\mathcal{B}_r} B$ *is increasing as a function of* r. *Moreover, for any* $\eta \le r/n$, *and every* $B \in \mathcal{B}_r$ *such that* $B \subset U$ *we have, letting* $d_B = \#(\Lambda \cap B)$,

$$\frac{1}{2}\int_{B\setminus\cup_{p\in\Lambda}B(p,\eta)} |j|^2 \ge \pi d_B \left(\log\frac{r}{n\eta} - \|m\|_\infty \frac{r^2}{2}\right).$$

- *If* $B \in \mathcal{B}_r$ *and* χ *is a positive function with support in* U, *then*

$$\int_{B\setminus\cup_{p\in\Lambda}B(p,\eta)} \chi|j|^2 - 2\pi\left(\log\frac{r}{n\eta} - \|m\|_\infty\frac{r^2}{2}\right)\sum_{p\in B\cap\Lambda}\chi(p) \ge -2r\nu(B)\|\nabla\chi\|_\infty.$$

Proof of the first three items. We let $M = \|m\|_\infty$. The first items are obtained by a standard ball construction argument à la Jerrard/Sandier. Since $\operatorname{curl} j = \nu - m$ we have for any circle $C = \partial B$ of radius $r(B)$ not intersecting Λ, and letting $d_B = \#(\Lambda \cap B)$,

$$\int_C j\cdot\tau \ge \nu(B) - M\pi {r_B}^2 = 2\pi d_B - M\pi {r_B}^2.$$

Using the Cauchy-Schwarz inequality and the fact that d_B is a nonnegative integer, we deduce that

$$\int_C |j|^2 \ge \frac{2\pi}{r_B}(d_B)^2 - 2\pi M d_B r_B \ge 2\pi d_B\left(\frac{1}{r_B} - Mr_B\right). \tag{4.8}$$

Define $\mathcal{F}(x, r) = \int_{B(x,r)} |j|^2$. The above yields

$$\frac{\partial\mathcal{F}}{\partial r} \ge 2\pi d_B\left(\frac{1}{r_B} - Mr_B\right).$$

In order to define $\mathcal{B}_r$, we first fix a reference family of balls produced via a ball-growth. Set $\eta_1 = \min\{\eta_0/3, 1/(n+1)\}$ and let $\mathcal{B}_0 = \{\bar{B}(p, \eta_1)\}_{p\in\Lambda}$. According to the definition of η_0, we have that $\mathcal{B}_0$ is a finite, disjoint collection of closed balls of total radius $n\eta_1 < 1$. We then apply Theorem 4.2 of [SS4] to $\mathcal{B}_0$ to produce a family of collections $\{\mathcal{B}(s)\}_{s\in[0,\log\frac{1}{n\eta_1}]}$, such that the total radius of the balls in $\mathcal{B}(s)$ is $r = n\eta_1 e^s$. Then, given $n\eta_1 < r < 1$ we may choose $s = \log\frac{r}{n\eta_1}$ and write $\mathcal{B}_r$ for $\mathcal{B}(s)$. We then extend this reference family "backward" to radii smaller than $n\eta_1$ by letting $\mathcal{B}_r = \{\bar{B}(p, r/n)\}_{p\in\Lambda}$ for any $0 < r \le n\eta_1$.

Since the balls in these collections never become tangent when $r < n\eta_1$, for any $\eta > 0$ we may trivially view $\{\mathcal{B}(s)\}_s$, where $\mathcal{B}(s) = \mathcal{B}_r$, with $r = n\eta e^s$ as having been generated by a ball-growth from $\mathcal{B}(0) = \{\bar{B}(p,\eta)\}_{p\in\Lambda}$, i.e. satisfying all the results of Theorem 4.2 in [SS4]. Then each $\mathcal{B}_r$ has total radius r and covers Λ and from Proposition 4.1 in [SS4] for every $B \in \mathcal{B}_r = \mathcal{B}(s)$, with $r = n\eta e^s$,

$$\int_{B\setminus\cup_{p\in\Lambda}B(p,\eta)} |j|^2 \ge \int_0^s \sum_{\substack{B'\in\mathcal{B}(t)\\ B'\subset B}} 2\pi d_{B'}(1 - M{r_{B'}}^2)\, dt.$$

Since the sum of the $d_{B'}$'s is $\#(B\cap\Lambda)$ and since the sum of the $r_{B'}$'s is less than $n\eta e^t$, we find

$$\int_{B\setminus\cup_{p\in\Lambda}B(p,\eta)} |j|^2 \ge 2\pi d_B \int_0^s (1 - M(n\eta e^t)^2)\, dt = 2\pi d_B \left(\log\frac{r}{n\eta} - M\frac{r^2 - n^2\eta^2}{2}\right).$$

Since this is true for any $0 < \eta < r/n$, the first three items are satisfied. □

Proof of the last item. Let $B_\eta = B \setminus \cup_{p\in\Lambda}B(p,\eta)$. Then by the "layer-cake" theorem

$$\int_{B_\eta} \chi|j|^2 = \int_0^{+\infty}\left(\int_{B_\eta\cap\{\chi>t\}} |j|^2\right) dt. \tag{4.9}$$

Now if $a \in \Lambda\cap B$, then for any $s \in (0, r]$ there exists a closed ball $B_{a,s} \in \mathcal{B}_s$ containing a. For $t > 0$ we call

$$s(a,t) = \sup\{s \in (0,1], B_{a,s} \subset \{\chi > t\}\}$$

if this set is nonempty, and let $s(a,t) = 0$ otherwise, i.e. if $\chi(a) \le t$. Then we let $B_a^t = B_{a,s(a,t)}$. Note that a is not necessarily the center of B_a^t, note also that $s(a,t)$ bounds from above the radius of B_a^t, but is not necessarily equal to it.

As noted above $s(a,t) = 0$ iff $\chi(a) \le t$ while if $s(a,t) \in (0,r)$ then $B_a^t \not\subset \{\chi > t\}$, otherwise there would exist $s' > s(a,t)$ such that $B_{a,s'} \subset \{\chi > t\}$, contradicting the definition of $s(a,t)$. Thus, choosing y in $B_a^t \setminus \{\chi > t\}$, we have

$$\chi(a) - t \le \chi(a) - \chi(y) \le 2s(a,t)\|\nabla\chi\|_\infty. \tag{4.10}$$

Also, for any $t \ge 0$ the collection $\{B_a^t\}_a$, where $a \in \Lambda$ and the a's for which $s(a,t) = 0$ have been excluded, is disjoint. Indeed if $a, b \in \Lambda$ and $s(a,t) \ge s(b,t)$ then, since $\mathcal{B}_{s(a,t)}$ is disjoint, the balls $B_{a,s(a,t)}$ and $B_{b,s(a,t)}$ are either equal or disjoint. If they are disjoint we note that $s(a,t) \ge s(b,t)$ implies that $B_{b,s(b,t)} \subset B_{b,s(a,t)}$ and therefore $B_b^t = B_{b,s(b,t)}$ and $B_a^t = B_{a,s(a,t)}$ are disjoint. If they are equal, then $B_{b,s(a,t)} \subset E_t \cap B$ and therefore $s(b,t) \ge s(a,t)$, which implies $s(b,t) = s(a,t)$ and then $B_b^t = B_a^t$.

Now assume that $B' \in \{B_a^t\}_a$ and let s be the common value of $s(a,t)$ for a's in $B'\cap\Lambda$ and $n = \#\Lambda$. Then the previous item of the proposition yields for any $\eta < \min(\eta_0, r/n)$ (but the inequality is trivially true if $\eta > r/n$),

$$\int_{B'\setminus\cup_{p\in\Lambda}B(p,\eta)} |j|^2 \ge \nu(B')\left(\log\frac{s}{n\eta} - M\frac{s^2}{2}\right)_+.$$

We may rewrite the above as

$$\int_{B'\setminus\cup_{p\in\Lambda}B(p,\eta)}|j|^2 \geq 2\pi \sum_{a\in B'\cap\Lambda}\left(\log\frac{s(a,t)}{n\eta} - M\frac{s(a,t)^2}{2}\right)_+$$

and summing over $B' \in \{B_a^t\}_a$ we deduce, noting that the a's for which $s(a,t)=0$ do not contribute to the sum,

$$\int_{B_\eta\cap\{\chi>t\}}|j|^2 \geq 2\pi \sum_{a\in B\cap\Lambda}\left(\log\frac{s(a,t)}{n\eta} - M\frac{s(a,t)^2}{2}\right)_+.$$

Integrating the above in view of (4.9) yields, using (4.10) and the fact that $s(a,t)\leq r$,

$$\int_{B_\eta}\chi|j|^2 \geq 2\pi\sum_{a\in B\cap\Lambda}\int_0^{\chi(a)}\left(\log\frac{s(a,t)}{n\eta} - M\frac{s(a,t)^2}{2}\right)_+ dt \geq$$
$$2\pi\sum_{a\in B\cap\Lambda}\int_0^{\chi(a)}\left(\log\frac{r}{n\eta} + \log\left(\frac{\chi(a)-t}{2r\|\nabla\chi\|_\infty}\wedge 1\right) - M\frac{r^2}{2}\right) dt.$$

The right-hand side is greater than

$$2\pi\sum_{a\in B\cap\Lambda}\left(\chi(a)\left(\log\frac{r}{n\eta} - M\frac{r^2}{2}\right) - 2r\|\nabla\chi\|_\infty\right),$$

which proves the result. □

We deduce a control of j by W, which we point out is substantially improved in [ST] into a control in a critical Lorentz space.

Lemma 4.7. *Let χ be a positive function compactly supported in an open set U and assume that* $\operatorname{curl} j = \nu - m$ *in*

$$\widehat{U} := \{x \mid d(x,U) < 1\}$$

where $\nu = 2\pi\sum_{p\in\Lambda}\delta_p$ for some finite subset Λ of $\widehat{U}$. Then, there exists $C>0$ universal and for any $p\in[1,2)$, $C_p>0$ depending only on p, such that

$$\int_U \chi^{p/2}|j|^p \leq C(|U|+C_p)^{1-p/2}\left(W(j,\chi) + n(\log n + \|m\|_\infty)\|\chi\|_\infty + n\|\nabla\chi\|_\infty\right)^{p/2}.$$

where $n = \nu(\widehat{U})/2\pi = \#\Lambda$.

Proof. We use an argument of M. Struwe [St]. Let $M = \|m\|_\infty$. We construct as above balls $\mathcal{B}_r$ in $\widehat{U}$ and define for $k\geq 1$ the set $U_k = \mathcal{B}_{2^{-k}}\setminus\mathcal{B}_{2^{-(k+1)}}$ and $U_0 = \widehat{U}\setminus\mathcal{B}_{1/2}$. Then since χ is supported in U, we have by Hölder's inequality,

$$\int_U \chi^{p/2}|j|^p = \int_{\widehat{U}}\chi^{p/2}|j|^p = \sum_{k=0}^\infty\int_{U_k}\chi^{p/2}|j|^p \leq \sum_{k=0}^\infty |U_k|^{1-p/2}\left(\int_{U_k}\chi|j|^2\right)^{p/2}. \tag{4.11}$$

For any given $k \in \mathbb{N}$ and if η is small enough we have $U_k \subset U_\eta \setminus (\mathcal{B}_{2^{-(k+1)}})_\eta$, where we have written $A_\eta = A \setminus \cup_{p\in\Lambda} B(p,\eta)$. For any ball $B \in \mathcal{B}_{2^{-(k+1)}}$, if B intersects the support of χ then $B \subset \widehat{U}$ and thus the previous proposition yields

$$\int_{B_\eta} \chi |j|^2 \geq 2\pi \left(\log \frac{2^{-(k+1)}}{n\eta} - CM \right) \sum_{p\in B\cap\Lambda} \chi(p) - C\nu(B)\|\nabla\chi\|_\infty.$$

Summing over balls in $\mathcal{B}_{2^{-(k+1)}}$ and subtracting from $\int_{U_\eta} \chi|j|^2$ we find

$$\int_{U_k} \chi|j|^2 \leq \int_{U_\eta} \chi|j|^2 - 2\pi \left(\log \frac{2^{-(k+1)}}{n\eta} - CM \right) \sum_{p\in U\cap\Lambda} \chi(p) + C\nu(U)\|\nabla\chi\|_\infty.$$

Taking the limit $\eta \to 0$ and plugging into (4.11) we find, recalling that $n = \#\Lambda$,

$$\int_U \chi^{p/2}|j|^p \leq \sum_{k=0}^{\infty} |U_k|^{1-p/2} \left(W(j,\chi) + Cn\left(M + k + \log n\right)\|\chi\|_\infty + Cn\|\nabla\chi\|_\infty \right)^{p/2}.$$

We have $|U_0| \leq |U|$ while for $k \geq 1$ we have $|U_k| \leq C2^{-k}$. It follows easily that for some $C_p, C > 0$,

$$\int_U \chi^{p/2}|j|^p \leq C(|U| + C_p)^{1-p/2} \left(2W(j,\chi) + n(\log n + M)\|\chi\|_\infty + n\|\nabla\chi\|_\infty \right)^{p/2}.$$

□

Lemma 4.8. *Assume that $m \in L^\infty(B_R)$ and $\{(j_n,\nu_n)\}_n$ is a sequence in $L^p_{\text{loc}}(\mathbb{R}^2,\mathbb{R}^2) \times \mathcal{M}$ such that ν_n restricted to B_R is of the form $2\pi\sum_i \delta_{a_i}$ for every n, with* div $j_n = 0$ *and* curl $j_n = \nu_n - m$, *and which converges in the distributional sense to (j,ν) on B_R.*

Then div $j = 0$ *and* curl $j = \nu - m$ *on B_R, where ν is a locally finite sum of the form $2\pi\sum_{p\in\Lambda} d_p\delta_p$, where Λ is a discrete subset of B_R and $d_p \in \mathbb{N}^*$. Moreover, if $\chi \in C_c^\infty(B_R)$ is positive and $\sup_n W(j_n,\chi) < +\infty$, then $d_p = 1$ for every p such that $\chi(p) \neq 0$. In addition, for any smooth function ξ compactly supported in $\{\chi > 0\}$ we have*

$$\liminf_{n\to+\infty} W(j_n,\xi) = W(j,\xi).$$

Proof. This is essentially a consequence of the type of analysis of [BBH].

First, since ν_n is positive for each n, the distributional convergence of $\{\nu_n\}_n$ is in fact a weak convergence of the measures, and $\{\nu_n\}_n$ is locally bounded on B_R, and we deduce that ν is of the form $2\pi\sum_{p\in\Lambda} d_p\delta_p$ with $d_p \in \mathbb{N}^*$. That div $j = 0$ and curl $j = \nu - m$ on B_R follows by passing to the limit in the corresponding equations satisfied by j_n, ν_n.

Now assume $\sup_n W(j_n,\chi) < +\infty$, let U be an open set compactly included in B_R and containing the support of χ, and choose $\eta > 0$. Let $U_\eta = U \setminus \cup_{p\in\Lambda} B(p,\eta)$. If n is large enough depending on η we have $\Lambda_n \cap U_\eta = \varnothing$ hence div $(j - j_n) =$ curl $(j - j_n) = 0$ in U_η. Elliptic regularity then implies that $j_n \to j$ in $C^k(U_{2\eta})$ for any $k \in \mathbb{N}$ and thus we have convergence in $C^k_{\text{loc}}(B_R \setminus \Lambda)$. In particular we have for any $\rho > 0$

$$\lim_{n\to\infty} \frac{1}{2}\int_{U_\rho} \chi|j_n|^2 = \frac{1}{2}\int_{U_\rho} \chi|j|^2.$$

We choose ρ small enough so that the balls $B(p,\rho)$ for $p \in \Lambda \cap U$ are disjoint. Then for each $p \in \Lambda \cap U$ and n large enough, there are exactly $d_p \geq 1$ points $p_n^1, \dots, p_n^{d_p}$ in $\Lambda_n \cap B(p,\rho)$, and these points converge to p. We apply Proposition 4.6 to (j_n, ν_n) in $B(p,\rho)$, with $r = \rho^2$. We deduce the existence of a family of balls $\mathcal{B}_n$ of total radius ρ^2 containing the points p_n^k and such that for $\eta > 0$ small enough

$$\int_{\mathcal{B}_n \setminus \cup_k B(p_n^k, \eta)} \chi |j_n|^2 - 2\pi \left(\log \frac{\rho^2}{d_p \eta} - CM \right) \sum_{k=1}^{d_p} \chi(p_n^k) \geq -C d_p \rho^2 \|\nabla \chi\|_\infty.$$

Moreover, since the p_n^k's converge to p as $n \to +\infty$, if n is chosen large enough then they are at distance less than ρ^2 from p hence $\mathcal{B}_n \subset B(p, 2\rho^2)$. Using (4.8) to bound from below $\int_{\partial B(p,t)} |j_n|^2$ for $2\rho^2 < t < \rho$, we find

$$\frac{1}{2} \int_{B(p,\rho) \setminus \mathcal{B}_n} \chi |j_n|^2 \geq \frac{1}{2} \left(\min_{B(p,\rho)} \chi \right) \int_{B(p,\rho) \setminus \mathcal{B}_n} |j_n|^2 \geq \pi \left(\min_{B(p,\rho)} \chi \right) {d_p}^2 \log \frac{\rho}{2\rho^2} - C d_p \rho.$$

Also, since on $B(p,\rho)$ we have $\min_{B(p,\rho)} \chi \geq \chi - 2\rho \|\nabla \chi\|_\infty$, we may write

$$\left(\min_{B(p,\rho)} \chi \right) {d_p}^2 \geq \sum_k \chi(p_n^k) + \chi(p) \left({d_p}^2 - d_p \right) - C \rho {d_p}^2 \|\nabla \chi\|_\infty.$$

Putting together the above lower bounds, replacing $\chi(p_n^k)$ by $\chi(p) - C\rho \|\nabla \chi\|_\infty$, and summing with respect to k and p, we deduce

$$\frac{1}{2} \int_{U \setminus \cup_{p \in \Lambda_n} B(p,\eta)} \chi |j_n|^2 \geq \frac{1}{2} \int_{U_\rho} \chi |j_n|^2 + \pi \log \frac{\rho^2}{\eta} \sum_{p \in \Lambda_n} \chi(p) + \pi \log \frac{\rho}{2\rho^2} \sum_{p \in \Lambda_n} \chi(p)$$
$$+ \pi \log \frac{\rho}{2\rho^2} \sum_{p \in \Lambda} ({d_p}^2 - d_p) \chi(p) - C \sum_{p \in \Lambda} d_p (\log d_p + 1) C(\chi) - C\rho \log \frac{1}{2\rho} \sum_{p \in \Lambda} {d_p}^2 C(\chi),$$

where $C(\chi)$ denotes a constant depending only on the function χ and possibly on M. Adding $\pi \log \eta \sum_p \chi(p)$ on both sides and passing to the limit $\eta \to 0$ we find

$$W(j_n, \chi) \geq \frac{1}{2} \int_{U_\rho} \chi |j_n|^2 + \pi \log \frac{\rho}{2} \sum_{p \in \Lambda_n} \chi(p) + \pi \log \frac{1}{2\rho} \sum_{p \in \Lambda} ({d_p}^2 - d_p) \chi(p) - R,$$

where the error term R is bounded independently of $\rho \in (0,1]$, n. This inequality is true for n large enough, but the left-hand side was assumed to be bounded above independently of n, hence the right-hand side is bounded above independently of ρ, n, which can only be true if $({d_p}^2 - d_p - 1)\chi(p) \leq 0$ for every $p \in \Lambda$, i.e. (since d_p is a nonzero integer) if $d_p = 1$ for every $p \in \Lambda$ such that $\chi(p) \neq 0$.

We now prove that $\liminf_{n \to +\infty} W(j_n, \xi) = W(j, \xi)$ if $\mathrm{Supp}(\xi) \subset \{\chi > 0\}$. For this we note that since $j_n \to j$ in $C^k_{\mathrm{loc}}(B_R \setminus \Lambda)$, then for any $\rho > 0$ we have

$$I(j_n, \rho) := \frac{1}{2} \int_{U_\rho} \xi |j_n|^2 + \pi \sum_{p \in \Lambda_n} \xi(p) \log \rho \xrightarrow{n \to +\infty} I(j, \rho).$$

Then, the convergence of $W(j_n,\xi)$ to $W(j,\xi)$ will follow if we may reverse the limits $n\to\infty$ and $\rho\to 0$, which is the case provided for instance that $I(j_n,\rho)$ tends to $W(j_n,\xi)$ as $\rho\to 0$ *uniformly with respect to n.*

We prove this fact. Denote by $\{p_1,\dots,p_k\}$ the set $\Lambda\cap\operatorname{Supp}(\xi)$. We know already that the points are distinct hence there exist a neighborhood $\mathcal{V}$ of $\operatorname{Supp}(\xi)$ and k sequences $\{p_i^n\}_n$, $1\le i\le k$ such that $\Lambda_n\cap\mathcal{V}=\{p_i^n\}_i$ and such that $p_i^n\to p_i$ as $n\to+\infty$. There exists N_0 such that if $n>N_0$ then $|p_i^n-p_i|<\rho_0$ when $i\neq j$, where $\rho_0=\frac{1}{4}min_{i\neq j}|p_i-p_j|$.

Now choose $\rho_1<\rho_2<\rho_0$ and $n>N_0$. We have $j_n=-\nabla^\perp H_n$, where $-\Delta H_n=2\pi\delta_{p_i^n}-m$ in $B(p_i^n,2\rho_0)$ with $\|m\|_\infty<M$. Then $H_n=\log|\cdot-p_i^n|+H_{i,n}$, where $H_{i,n}$ is bounded by a constant independent of n in $C^1(B(p_i^n,\rho))$. It follows straightforwardly by writing $|j_n|^2=|\nabla\log+\nabla H_{i,n}|^2$, expanding and estimating each term, that

$$\left|\frac{1}{2}\int_{B(p_i^n,\rho_2)\setminus B(p_i^n,\rho_1)}\xi|j_n|^2-\pi\xi(p_i^n)\log\frac{\rho_2}{\rho_1}\right|\le C\rho_2\|\nabla\xi\|_\infty+C\sqrt{\rho_2}\|\xi\|_\infty,$$

where C is independent of $n\le N_0$. The left-hand side is nothing but $|I(j_n,\rho_2)-I(j_n,\rho_1)|$ thus we have proved the uniform convergence w.r.t. n of $I(j_n,\rho)$ as $\rho\to 0$, and then it follows that $W(j_n,\xi)\to W(j,\xi)$. □

The energy density defining W, $|j|^2+\log\eta\sum_p\delta_p$ has no sign, which makes it impossible to apply directly Theorem 3 to it. We now show that it can be modified into a density bounded below by a constant, using again the mass displacement method introduced in [SS3] (but in a simpler setting), that is by absorbing the negative part into the positive part while making a controlled error.

The next proposition summarizes the properties of the modified density g.

Proposition 4.9. *Assume $U\subset\mathbb{R}^2$ is open and (j,ν) are such that $\nu=2\pi\sum_{p\in\Lambda}\delta_p$ for some finite subset Λ of $\widehat{U}$ and* $\operatorname{curl} j=\nu-m$, $\operatorname{div} j=0$ *in $\widehat{U}$, where $m\in L^\infty(\widehat{U})$. Then there exists a measure g supported on $\widehat{U}$ and such that*

- *$g\ge -C(\|m\|_\infty^2+1)$ on $\widehat{U}$, where C is a universal constant.*
- *For any function χ compactly supported in U we have*

$$\left|W(j,\chi)-\int\chi\,dg\right|\le Cn(\log n+\|m\|_\infty)\|\nabla\chi\|_\infty, \tag{4.12}$$

  where $n=\#\{p\in\Lambda\mid B(p,C)\cap \mathit{Supp}(\nabla\chi)\neq\varnothing\}$ and C is universal.
- *For any $E\subset U$,*

$$\#(\Lambda\cap E)\le C\left(1+\|m\|_\infty^2|\widehat{E}|+g(\widehat{E})\right), \tag{4.13}$$

  where C is universal.

Proof. The proof follows the method of [SS3]. Throughout $M=\|m\|_\infty$. We cover $\mathbb{R}^2$ by the balls of radius $1/4$ whose centers are in $\frac{\mathbb{Z}}{4}\times\frac{\mathbb{Z}}{4}$. We call this cover $\{U_\alpha\}_\alpha$ and $\{x_\alpha\}_\alpha$ the centers. In each $U_\alpha\cap\widehat{U}$ and for any $r\in(0,1/4)$ we construct disjoint balls $\mathcal{B}_r^\alpha$ using Proposition 4.6. Then choosing a small enough $\rho\in(0,1/4)$ to be specified below, we may extract from $\cup_\alpha\mathcal{B}_\rho^\alpha$

a disjoint family which still covers Λ as follows: Denoting by $\mathcal{C}$ a connected component of $\cup_\alpha \mathcal{B}_\rho^\alpha$, we claim that there exists α_0 such that $\mathcal{C} \subset U_{\alpha_0}$. Indeed if $x \in \mathcal{C}$ and letting ℓ be a Lebesgue number of the covering $\{U_\alpha\}_\alpha$ (in our case $\ell \le 1/4$), there exists α_0 such that $B(x,\ell) \subset U_{\alpha_0}$. If $\mathcal{C}$ intersected the complement of U_{α_0}, there would exist a chain of balls connecting x to $(U_{\alpha_0})^c$, each of which would intersect U_{α_0}. Each of the balls in the chain would belong to some $\mathcal{B}_\rho^\beta$ with β such that dist $(U_\beta, U_{\alpha_0}) \le 2\rho < 1/2$. Thus, calling k the maximum number of β's such that dist $(U_\beta, U_{\alpha_0}) < 1/2$, the length of the chain is at most $2k\rho$ and thus $\ell \le 2k\rho$. If we choose $\rho < \ell/2k$, this is impossible and the claim is proven. Let us then choose $\rho = \ell/4k$. By the above, each $\mathcal{C}$ is included in some U_α.

Then, to obtain a disjoint cover of Λ from $\cup_\alpha \mathcal{B}_\rho^\alpha$, we let $\mathcal{C}$ run over all the connected components of $\cup_\alpha \mathcal{B}_\rho^\alpha$ and for a given $\mathcal{C}$ such that $\mathcal{C} \subset U_{\alpha_0}$, we remove from $\mathcal{C}$ the balls which do not belong to $\mathcal{B}_\rho^{\alpha_0}$. We will still denote $\mathcal{B}_\rho^\alpha$ the family with deleted balls, and let $\mathcal{B}_\rho = \cup_\alpha \mathcal{B}_\rho^\alpha$. Then $\mathcal{B}_\rho$ covers Λ and is disjoint.

We then proceed to the mass displacement. Note that by construction every ball in $\mathcal{B}_\rho^\alpha$ is included in U_α.

From the last item of Proposition 4.6 applied to a ball $B \in \mathcal{B}_\rho^\alpha$, if η is small enough then, letting $B_\eta = B \setminus \cup_{p\in\Lambda} B(p,\eta)$, for any function χ vanishing outside $B \cap \widehat{U}$ we have

$$\int_{B_\eta} \chi |j|^2 - 2\pi \left(\log \frac{\rho}{n_\alpha \eta} - CM \right) \sum_{p \in B\cap\Lambda} \chi(p) \ge -C\nu(B) \|\nabla \chi\|_{L^\infty(B)},$$

where $n_\alpha = \nu(U_\alpha)$ and $M = \|m\|_\infty$. Then applying Lemma 3.1 of [SS3] to

$$f_{B,\eta} = \frac{1}{2} \left(|j|^2 \mathbf{1}_{B_\eta} - \left(\log \frac{\rho}{n_\alpha \eta} - CM \right) 2\pi \sum_{p\in B\cap\Lambda} \delta_p \right)$$

we deduce the existence of a positive measure $g_{B,\eta}$ such that $\|f_{B,\eta} - g_{B,\eta}\| \le C\nu(B)$, where the norm is that of the dual of the space Lip of Lipschitz functions in B which vanish outside $B \cap \widehat{U}$. Now we let $\eta \to 0$. Since $g_{B,\eta} \ge 0$, it subsequentially converges to a positive measure g_B and for any $\chi \in$ Lip,

$$\left| \int \chi \, dg_B - W_B(j,\chi) \right| \le C\nu(B) \|\nabla\chi\|_{L^\infty(B)}, \quad \text{where} \quad W_B(j,\chi) = \lim_{\eta\to 0} \int \chi \, df_{B,\eta}. \tag{4.14}$$

Next we note that, letting $W'(j,\chi) = W(j,\chi) - \sum_{B\in\mathcal{B}_\rho} W_B(j,\chi)$, we have

$$W'(j,\chi) = \int \chi \, df', \quad \text{where} \quad f' = \frac{1}{2}|j|^2 \mathbf{1}_{U\setminus\mathcal{B}_\rho} + \pi \sum_{p\in\Lambda} \left(\log \frac{\rho}{n_{\alpha_p}} - CM \right) \delta_p,$$

where, denoting Λ_α the set of $p \in \Lambda$ belonging to a ball of $\mathcal{B}_\rho^\alpha$, we define α_p as the index α such that p belongs to Λ_α.

We define a set C_α as follows: recall that ρ was assumed equal to $\ell/4k$, where $\ell \le 1/4$ and k bounds the number of β's such that dist $(U_\beta, U_\alpha) < 1/2$ for any given α. Therefore the total radius of the balls in $\mathcal{B}_\rho$ which are at distance less than $\frac{1}{2}$ from U_α is at most $k\rho < 1/16$. In particular, letting T_α denote the set of $t \in (\frac{1}{4}, \frac{1}{2})$ such that the circle of center x_α (where we recall x_α is the center of U_α) and radius t does not intersect $\mathcal{B}_\rho^\alpha$, we have $|T_\alpha| \ge 3/16$. We let $C_\alpha = \{x \mid |x - x_\alpha| \in T_\alpha\}$ and $D_\alpha = U_\alpha \cup C_\alpha$. If $U_\alpha \cap U \ne \varnothing$ then $d(x_\alpha, U) \le 1/4$ hence

$B(x_\alpha, \frac{1}{2}) \subset \widehat{U}$. In particular, we have $C_\alpha \subset D_\alpha \subset \widehat{U}$. Then there exists universal constants $c > 0$ and C such that

$$\int_{C_\alpha} |j|^2 \geq c{n_\alpha}^2 - CM^2. \tag{4.15}$$

To see this, apply the lower bound (4.8) on the circle $S_t = \{|x - x_\alpha| = t\}$, i.e. with $r_B = t$ and $d_B = \#(\Lambda \cap B(x_\alpha, t))$. Using $d_B \geq n_\alpha$ and $t \in (\frac{1}{8}, \frac{1}{2})$ we deduce that $\int_{S_t} |j|^2 \geq \pi(n_\alpha)^2 - \pi M n_\alpha$ and integrating with respect to $t \in T_\alpha$ yields (4.15).

The overlap number of the sets $\{C_\alpha\}_\alpha$, defined as the maximum number of sets to which a given x belongs is bounded by the overltap number of $\{B(x_\alpha, 3/4)\}_\alpha$, call it k'. Then, letting

$$f_\alpha = \frac{1}{2k'}|j|^2 \mathbf{1}_{C_\alpha} + \pi \sum_{p \in \Lambda_\alpha} \left(\log \frac{\rho}{n_\alpha} - CM \right) \delta_p,$$

we have

$$f' - \sum_\alpha f_\alpha \geq |j|^2 \mathbf{1}_{U \setminus \mathcal{B}_\rho} - \frac{1}{2k'}|j|^2 \mathbf{1}_{C_\alpha} \geq \frac{1}{2}|j|^2 \mathbf{1}_{U \setminus \mathcal{B}_\rho} \geq 0 \tag{4.16}$$

and, from (4.15),

$$f_\alpha(D_\alpha) = \frac{1}{2k'} \int_{C_\alpha} |j|^2 + \pi n_\alpha \left(\log \frac{\rho}{n_\alpha} - CM \right) \geq c{n_\alpha}^2 - C(M^2 + 1), \tag{4.17}$$

where the constants may have changed. We then apply [SS3] Lemma 3.2 over D_α to $f_\alpha + \frac{C(M^2+1)}{|D_\alpha|}$, where C is the constant in the right-hand side of (4.17). We deduce *for any α such that $U_\alpha \cap U \neq \varnothing$* the existence of a measure g_α supported in D_α such that $g_\alpha \geq -\frac{C(M^2+1)}{|D_\alpha|}$ and such that, for every Lipschitz function χ

$$\int \chi \, d(f_\alpha - g_\alpha) \leq C \|\nabla \chi\|_{L^\infty(D_\alpha)} (f_\alpha)_-(D_\alpha) \leq C n_\alpha (\log n_\alpha + C(M+1)) \|\nabla \chi\|_{L^\infty(D_\alpha)}. \tag{4.18}$$

In particular, taking $\chi = 1$,

$$g_\alpha(D_\alpha) = f_\alpha(D_\alpha) \geq c{n_\alpha}^2 - C(M^2 + 1). \tag{4.19}$$

Now we let

$$g = \sum_{B \in \mathcal{B}_\rho} g_B + \sum_{\substack{\alpha \text{ s.t.} \\ U_\alpha \cap U \neq \varnothing}} g_\alpha + \left(f' - \sum_\alpha f_\alpha \right).$$

The term g_B is positive for every B, $f' - \sum_\alpha f_\alpha$ is bounded below by (4.16), and $\sum_\alpha g_\alpha$ is bounded below by $-k'C(M^2 + 1)$ since $g_\alpha \geq -C(M^2 + 1)$. Thus g is bounded below by $\frac{1}{2}|j|^2 \mathbf{1}_{U \setminus \mathcal{B}_\rho} - C(M^2 + 1) \geq -C(M^2 + 1)$, and we have proved the first item. In addition, if χ has support in U then

$$\sum_\alpha \int \chi \, df_\alpha = \sum_{\substack{\alpha \text{ s.t.} \\ U_\alpha \cap U \neq \varnothing}} \int \chi \, df_\alpha$$

hence

$$\int \chi\, dg = \int \chi\, d\left(\sum_B g_B + \sum_{\substack{\alpha \text{ s.t.}\\ U_\alpha\cap U\neq\varnothing}} (g_\alpha - f_\alpha) + f'\right) = \\ \sum_B \int \chi\, dg_B + \sum_{\substack{\alpha \text{ s.t.}\\ U_\alpha\cap U\neq\varnothing}} \int \chi\, d(g_\alpha - f_\alpha) + W'(j,\chi)$$

hence in view of the definition of W', (4.14) and (4.18),

(4.20)

$$\int \chi\, dg - W(j,\chi) = \sum_{B\in\mathcal{B}_\rho}\left(\int \chi\, dg_B - W_B(j,\chi)\right) + \sum_{\substack{\alpha \text{ s.t.}\\ U_\alpha\cap U\neq\varnothing}} \int \chi\, d(g_\alpha - f_\alpha) \\ \leq C\left(\sum_{B\in\mathcal{B}_\rho} \nu(B)\|\nabla\chi\|_{L^\infty(B)} + \sum_\alpha n_\alpha(\log n_\alpha + C(M+1))\|\nabla\chi\|_{L^\infty(D_\alpha)}\right).$$

Then if we denote by A the set of α's such that $\|\nabla\chi\|_{L^\infty(D_\alpha)} \neq 0$, and k is the overlap number of the U_α's,

$$2\pi \sum_{\alpha\in A} n_\alpha \leq k\nu\left(\cup_{\alpha\in A} U_\alpha\right),$$

and $x \in U_\alpha$ with $\alpha \in A$ implies that $\nabla\chi(y) \neq 0$ for some $y \in D_\alpha$ hence $B(x, 5/4)\cap \mathrm{Supp}\nabla\chi \neq \varnothing$. It follows that $\sum_{\alpha\in A} n_\alpha \leq kn$, where n is defined after (4.12). Similarly, the sum of $\nu(B)$ for B's in $\mathcal{B}_\rho$ such that $\|\nabla\chi\|_{L^\infty(B)} \neq 0$ is less then $2\pi n$. Thus (4.20) may be rewritten as (4.12).

Finally, summing (4.19) for α's such that $U_\alpha \cap E \neq \varnothing$ and recalling that $g - \sum_\alpha g_\alpha \geq 0$, we easily deduce (4.13). □

Remark 4.10. *We have in fact proved the following stronger property on g: There exists $\rho > 0$ and a family $\mathcal{B}_\rho$ of disjoint closed balls covering Λ, such that the sum of the radii of the balls in $\mathcal{B}_\rho$ intersected with any ball of radius 1 is bounded by $C\rho < \frac{1}{2}$, and such that on $\hat{U}$*

$$g \geq -C(\|m\|_\infty^2 + 1) + \frac{1}{2}|j|^2 \mathbf{1}_{U\setminus\mathcal{B}_\rho}.$$

4.2 Application: proof of Proposition 4.1

We start with

Lemma 4.11. *Assume $j \in \mathcal{A}_1$. Then, for any family $\{\mathbf{U}_R\}_R$ satisfying* (1.5) *and $R > C$, for any $p > 1$,*

$$\max\left\{|\nu(\mathbf{U}_R) - |\mathbf{U}_R||\,,\ \nu(\widehat{\mathbf{U}_R}) - \nu(\mathbf{U}_R)\right\} \leq C\left(R^\theta + R^{\theta(1-\frac{1}{p})}\right)\left(\|j\|_{L^p(\mathbf{U}_{R+C})} + 1\right),$$

where $\theta < 2$ is the exponent in (1.5)*, and C only depends on the constants in* (1.5).

Proof. Let d_R denote the signed distance to $\partial\mathbf{U}_R$, i.e. $d_R(x) = -d(x, {\mathbf{U}_R}^c)$ if $x \in \mathbf{U}_R$ and $d_R(x) = d(x, \mathbf{U}_R)$ otherwise. Let ${\mathbf{U}_R}^t = \{x \mid d_R(x) \le t.\}$.

Integrating (4.1) over ${\mathbf{U}_R}^t$ and using Hölder's inequality we find

$$\left|\nu({\mathbf{U}_R}^t) - |{\mathbf{U}_R}^t|\right| = \left|\int_{\partial{\mathbf{U}_R}^t} j\cdot\tau\right| \le |\partial{\mathbf{U}_R}^t|^{1-\frac{1}{p}}\|j\|_{L^p(\partial{\mathbf{U}_R}^t)}. \tag{4.21}$$

Then, using the coarea formula,

$$\int_0^1 |\partial{\mathbf{U}_R}^t|\,dt = |{\mathbf{U}_R}^1\setminus\mathbf{U}_R|, \qquad \int_0^1 \|j\|^p_{L^p(\partial{\mathbf{U}_R}^t)}\,dt = \|j\|^p_{L^p({\mathbf{U}_R}^1\setminus\mathbf{U}_R)}$$

and from (1.5) there exists $C > 0$ such that ${\mathbf{U}_R}^1 \subset \mathbf{U}_{R+C}$. Using a mean-value argument and (1.5) again there exists $t \in (0,1)$ such that

$$|\partial{\mathbf{U}_R}^t| \le CR^\theta, \quad \|j\|^p_{L^p(\partial{\mathbf{U}_R}^t)} \le 2\|j\|^p_{L^p({\mathbf{U}_R}^1)} \le 2\|j\|^p_{L^p(\mathbf{U}_{R+C})},$$

and therefore

$$\left|\nu({\mathbf{U}_R}^t) - |{\mathbf{U}_R}^t|\right| \le CR^{\theta(1-\frac{1}{p})}\|j\|_{L^p(\mathbf{U}_{R+C})}. \tag{4.22}$$

A similar mean value argument yields the existence of $s \in (-1,0)$ such that

$$|\nu({\mathbf{U}_R}^s) - |{\mathbf{U}_R}^s|| \le CR^{\theta(1-\frac{1}{p})}\|j\|_{L^p(\mathbf{U}_R)}.$$

Since $t \mapsto \nu({\mathbf{U}_R}^t)$ is increasing we deduce, since $\mathbf{U}_{R-C} \subset {\mathbf{U}_R}^s$,

$$\nu({\mathbf{U}_R}^s) - |{\mathbf{U}_R}^s| \le \nu(\mathbf{U}_R) - |\mathbf{U}_R| + |\mathbf{U}_R\setminus\mathbf{U}_{R-C}| \le \nu({\mathbf{U}_R}^t) - |{\mathbf{U}_R}^t| + |\mathbf{U}_{R+C}\setminus\mathbf{U}_{R-C}|,$$

which in view of (4.21)-(4.22) and (1.5) proves the bound for $|\nu(\mathbf{U}_R) - |\mathbf{U}_R||$. The bound for $\nu(\widehat{\mathbf{U}_R}) - \nu(\mathbf{U}_R)$ follows easily since $\mathbf{U}_R \subset \widehat{\mathbf{U}_R} \subset \mathbf{U}_{R+C}$. □

Corollary 4.12. *Assume that $j \in \mathcal{A}_1$, that $\{\mathbf{U}_R\}_R$, $\{\mathbf{V}_R\}_R$ satisfy* (1.5) *and that $c > 0$ is such that $\mathbf{U}_{cR} \subset \mathbf{V}_R$ for any $R \ge 1$ (it is easy to show from* (1.5) *that such a c exists). We assume that either $W_U(j) < +\infty$, or $W_V(j) < +\infty$.*

Then, denoting $g_{\mathbf{U}_R}$ the result of applying Proposition 4.9 to (j,ν) in $\mathbf{U}_R$, we have

$$\lim_{R\to+\infty}\frac{1}{R^2}\left(\int_{\mathbf{U}_{cR}} \chi_{\mathbf{U}_{cR}}\,dg_{\mathbf{V}_R} - W(j,\chi_{\mathbf{U}_{cR}})\right) = 0. \tag{4.23}$$

In particular if $\{\mathbf{V}_R\}_R = \{\mathbf{U}_R\}_R$, we may take $c = 1$ to obtain

$$\lim_{R\to+\infty}\frac{1}{R^2}\left(\int_{\mathbf{U}_R} \chi_{\mathbf{U}_R}\,dg_{\mathbf{U}_R} - W(j,\chi_{\mathbf{U}_R})\right) = 0. \tag{4.24}$$

As a consequence, W_U does not depend on the particular choice of $\chi_{\mathbf{U}_R}$ satisfying (1.3).

Remark 4.13. *Let $G \subset \mathcal{A}_1$ satisfy the hypothesis of Proposition 4.2 and let $\{\mathbf{U}_R\}_R = \{K_R\}_R$. Then the convergence in* (4.24) *is uniform with respect to $j \in G$.*

Proof. Since $W_U(j) < +\infty$ and $j \in \mathcal{A}_1$, both $W(j, \chi_{\mathbf{U}_R})$ and $\nu(\mathbf{U}_R)$ are $O(R^2)$ as $R \to \infty$. Applying Lemma 4.7 in $\mathbf{U}_{R+C}$ we find, choosing some $p \in (1,2)$, that $\int_{\mathbf{U}_R} |j|^p = O(R^2 \log R)$. Then, using Lemma 4.11, we have $|\nu(\widehat{\mathbf{U}_R}) - \nu(\mathbf{U}_R)| = O(R^{\theta_p})$ for some exponent $\theta_p \in (0,2)$, and thus the same holds for $|\nu(\mathbf{U}_R) - \nu(\mathbf{U}_{R-C})|$. Inserting into (4.12) yields

$$\left| \frac{W(j, \chi_{\mathbf{U}_{cR}})}{|\mathbf{U}_{cR}|} - \fint_{\mathbf{U}_{cR}} \chi_{\mathbf{U}_{cR}}\, dg_{\mathbf{V}_R} \right| \xrightarrow{R \to +\infty} 0, \tag{4.25}$$

hence (4.23), and (4.24) easily follows.

In view of (4.24), proving that the definition of W_U is independent of the choice of $\{\chi_{\mathbf{U}_R}\}_R$ satisfying (1.3) now reduces to proving the same statement for

$$\limsup_{R \to +\infty} \fint_{\mathbf{U}_R} \chi_{\mathbf{U}_R}\, dg_{\mathbf{U}_R}.$$

But, since $g_{\mathbf{U}_R} \geq -C$ and since $\mathbf{U}_{R-C} \subset \{\chi_{\mathbf{U}_R} = 1\}$,

$$g_{\mathbf{U}_R}(\mathbf{U}_{R-C}) - C|\mathbf{U}_R \setminus \mathbf{U}_{R-C}| \leq \int_{\mathbf{U}_R} \chi_{\mathbf{U}_R}\, dg_{\mathbf{U}_R} \leq g_{\mathbf{U}_R}(\mathbf{U}_R) + C|\mathbf{U}_R \setminus \mathbf{U}_{R-C}|.$$

Dividing by $|\mathbf{U}_R|$ and in view of (1.5) we obtain that

$$\limsup_{R \to +\infty} \fint_{\mathbf{U}_R} \chi_{\mathbf{U}_R}\, dg_{\mathbf{U}_R} = \limsup_{R \to +\infty} \frac{g_{\mathbf{U}_R}(\mathbf{U}_R)}{|\mathbf{U}_R|},$$

which is clearly independent of $\chi_{\mathbf{U}_R}$ and finite thanks to (4.24).

The proof of Remark 4.13 follows from the fact that under the hypothesis of Proposition 4.2 we clearly have bounds $\nu(K_R) < CR^2$ and $W(j, \chi_{K_R}) < CR^2$ which are uniform with respect to $j \in G$ and thus that the convergence in (4.25) is uniform with respect to $j \in G$ as well, when $\{\mathbf{U}_R\}_R = \{K_R\}_R$. ☐

We may now give the proof of Proposition 4.1, in several steps.

W_U is measurable. First we show that $\mathcal{A}_1$ (recall Definition 1.1) is a Borel subset of $X := L^p_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$. For $R, \varepsilon > 0$ we let $A_{R,\varepsilon}$ be the set of $j \in L^p_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ such that first curl $j = \nu - 1$ and div $j = 0$, where the restriction of ν to B_{2R} is of the form $2\pi \sum_i \delta_{a_i}$ with $a_i \in B_R$ and $|a_i - a_j| \geq \varepsilon$ (in particular the sum is finite), and second $\|j\|_{L^p(B_{2R})} \leq 1/\varepsilon$. We also let

$$A_{R,\varepsilon,C} = \{j \in A_{R,\varepsilon} \mid \nu(B_R) \leq CR^2\}.$$

Clearly both $A_{R,\varepsilon}$, and $A_{R,\varepsilon,C}$ are closed. Noting that

$$\mathcal{A}_1 = \left(\cup_{C>1} \cap_{R>1} \cup_{\varepsilon>0} A_{R,\varepsilon,C}\right),$$

we find that $\mathcal{A}_1$ is a Borel subset of X.

For $j \in \mathcal{A}_1$ we have $W(j) = \limsup_R W(j, \chi_{\mathbf{U}_R})/|\mathbf{U}_R|$, hence proving that W is Borel reduces to proving that

$$W_\chi : j \mapsto \begin{cases} W(j, \chi) & \text{if } j \in \mathcal{A}_1 \\ +\infty & \text{otherwise} \end{cases}$$

is Borel for any smooth, positive χ with compact support.

This follows from Lemma 4.8. Choosing $R > 0$ such that $\mathrm{Supp}(\chi) \subset B_R$ the lemma implies that W_χ restricted to the closed set $A_{R,\varepsilon}$ is continuous, therefore $\{j \in A_{R,\varepsilon} \mid W_\chi \le t\}$ is closed for any t, and

$$\{j \in \mathcal{A}_1 \mid W_\chi(j) \le t\} = \mathcal{A}_1 \cap (\cap_{R>0} \cup_{\varepsilon>0} \{j \in A_{R,\varepsilon} \mid W_\chi \le t\})$$

is Borel. □

$\inf_{\mathcal{A}_1} W_U$ is finite. The results of Section 3 for example show that $\inf W_U < +\infty$. The fact that $\inf W_U > -\infty$ is a direct consequence of Corollary 4.12 and the fact that for any $R > 0$ we have $g_{\mathbf{U}_R} \ge -C$, where $g_{\mathbf{U}_R}$ is the result of applying Proposition 4.9 on $\mathbf{U}_R$. □

Sub-level sets are compact up to translation, $\min W_U$ is achieved. Consider a family $\{\mathbf{V}_R\}_R$ satisfying (1.4)–(1.5) and consider a sequence $\{\bar{\jmath}_n\}_n$ such that $W_V(\bar{\jmath}_n) \le \alpha$. In particular $\bar{\jmath}_n \in \mathcal{A}_1$. Let $\bar{\nu}_n$ be $\mathrm{curl}\, \bar{\jmath}_n + 1$. Then since $\bar{\nu}_n(B_R) = O(R^2)$ and from the definition of W_V as a lim sup, for any fixed arbitrary $C > 0$ and any $\theta > 1$ there exists a sequence $R_n \to +\infty$ such that

$$\liminf_{n\to+\infty} \left(W_V(\bar{\jmath}_n) - \frac{W(\bar{\jmath}_n, \chi_{V_{R_n}})}{|V_{R_n}|} \right) \ge 0, \quad \bar{\nu}_n \left(V_{R_n+C} \setminus V_{R_n-C}\right) \le {R_n}^\theta.$$

Indeed, given n, the second relation is satisfied by arbitrarily large R's, using a mean value argument.

Now, letting $V_n = V_{R_n}$, we apply Proposition 4.9 to $(\bar{\jmath}_n, \bar{\nu}_n)$ with $U = V_n$ and deduce the existence of a measure $\bar{g}_n \ge -C$ satisfying the properties described there. The choice of $\{R_n\}_n$ ensures that

$$\liminf_{n\to+\infty} W_V(\bar{\jmath}_n) \ge \liminf_{n\to+\infty} \frac{W(\bar{\jmath}_n, \chi_{V_n})}{|V_n|} = \liminf_{n\to+\infty} \fint_{V_n} \chi_{V_n}\, d\bar{g}_n, \tag{4.26}$$

using (4.12).

Now we apply the abstract scheme described in Theorem 3. Let $X = L^p_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2) \times \mathcal{M}_0$, where $\mathcal{M}_0$ is the space of Radon measures on $\mathbb{R}^2$ bounded below by twice the fixed constant given by item 1 of Proposition 4.9 (this means that we are considering measures μ such that $\mu + C$ is a positive Radon measure), and the topology is that of convergence in $L^p_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ and weak convergence on $\mathcal{M}_0$. X is a Polish space, and on it we have the natural continuous action $\theta_\lambda(j, g) = (j(\lambda + \cdot), g(\lambda + \cdot))$. We may check the hypotheses of Section 2 are satisfied. Then we choose a smooth positive χ with compact support in $B(0, 1)$ and integral 1, we let $V'_n = V_{R_n - C}$ where C is chosen (according to (1.5)) such that

$$\chi_{V_n} \ge \chi * \mathbf{1}_{V'_n} \text{ everywhere} \quad \text{and} \quad \chi_{V_n} = \chi * \mathbf{1}_{V'_n} = 1 \text{ in } \; V_{R_n - 2C} \tag{4.27}$$

and define

$$\mathbf{f}_n(j, g) = \begin{cases} \int \chi(y)\, dg(y) & \text{if there exists } \lambda \in V'_n \text{ such that } (j, g) = \theta_\lambda(\bar{\jmath}_n, \bar{g}_n), \\ +\infty & \text{otherwise.} \end{cases}$$

We also let

$$\mathbf{F}_n(j, g) = \fint_{V'_n} \mathbf{f}_n\left(\theta_\lambda(j, g)\right) d\lambda = \begin{cases} \frac{1}{|V'_n|} \int \chi * \mathbf{1}_{V'_n}\, d\bar{g}_n & \text{if } (j, g) = (\bar{\jmath}_n, \bar{g}_n), \\ +\infty & \text{otherwise.} \end{cases}$$

Since $\bar{g}_n \geq -C$, the property (4.27) implies that

$$\int \chi_{V_n}\, d\bar{g}_n \geq |V_n'|\mathbf{F}_n(\bar{\jmath}_n, \bar{g}_n) - O(R_n). \tag{4.28}$$

Then we check the coercivity and Γ-lim inf properties of $\{\mathbf{f}_n\}_n$ as in (1.15)–(1.16). The latter is the trivial observation that if $(j_n, g_n) \to (j, g)$ then for any subsequence (not relabeled) such that $\{\mathbf{f}_n(j_n, g_n)\}_n$ is bounded we have

$$\lim_{n\to\infty} \mathbf{f}_n(j_n, g_n) = \lim_{n\to+\infty} \int \chi\, dg_n = \mathbf{f}(j, g), \quad \textit{where} \quad \mathbf{f}(j, g) = \int \chi\, dg.$$

To prove coercivity, assume as in (1.15) that for every R

$$\limsup_{n\to+\infty} \int_{B_R} \mathbf{f}_n(\theta_\lambda(j_n, g_n))\, d\lambda < +\infty. \tag{4.29}$$

Then for every R, if n is large enough the integrand is finite a.e. hence there exists $\lambda_n \in V_n'$ such that $(j_n, g_n) = \theta_{\lambda_n}(\bar{\jmath}_n, \bar{g}_n)$ and $\lambda + \lambda_n \in V_n'$ for almost every $\lambda \in B_R$, i.e. $\lambda_n + B_R \subset V_n'$. Then (4.29) reads

$$\limsup_{n\to+\infty} \int_{B_R} \int \chi(x - \lambda_n - \lambda)\, d\bar{g}_n(x)\, d\lambda = \int \chi * \mathbf{1}_{\lambda_n + B_R}\, d\bar{g}_n < +\infty,$$

which is equivalent to saying that $\{g_n = \bar{g}_n(\lambda_n + \cdot)\}_n$ is bounded in $L^1(B_R)$ for every R. This implies that a subsequence converges in $\mathcal{M}_0$.

Then, in view of (4.13) this proves that $\{\bar{\nu}_n(\lambda_n + \cdot)\}_n$ is locally bounded. Inserting this information into (4.12) we find that $\{W(\bar{\jmath}_n(\lambda_n + \cdot), \chi_R)\}_n$ is bounded and then using Lemma 4.7 we deduce that $\{\bar{\jmath}_n(\lambda_n + \cdot)\}_n$ is bounded in $L^p(B_R)$ for any R. Thus going to a further subsequence $\{j_n = \bar{\jmath}_n(\lambda_n + \cdot)\}_n$ converges to j locally weakly in L^p. Moreover $\mathrm{div}\, j_n = 0$ and by the above $\mathrm{curl}\, j_n$ is locally bounded in the sense of measure, hence weakly compact in $W^{-1,p}_{loc}$. By elliptic regularity it follows that the convergence of j_n to j is strong in $L^p_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2)$. This proves coercivity.

We may now apply Theorem 3. Letting P_n be the image of the normalized Lebesgue measure on B_n' by the map $\lambda \mapsto \theta_\lambda(\bar{\jmath}_n, \bar{g}_n)$, there is a subsequence such that $P_n \to P$, where P is a probability measure on X and

$$\liminf_{n\to+\infty} \mathbf{F}_n(\bar{\jmath}_n, \bar{g}_n) \geq \int \mathbf{f}^*_U(j, g)\, dP(j, g), \quad \text{where} \quad \mathbf{f}^*_U(j, g) = \lim_{R\to+\infty} \fint_{\mathbf{U}_R} \int \chi(x - \lambda)\, dg(x)\, d\lambda,$$

for any family of open sets $\{\mathbf{U}_R\}_{R>0}$ satisfying (1.4).

We claim that if $(j, g) \in \mathrm{Supp}(P)$ and $\mathbf{f}^*_U(j, g) \leq +\infty$, then

$$j \in \mathcal{A}_1 \text{ and } \mathbf{f}^*_U(j, g) = W_U(j). \tag{4.30}$$

Assuming this, and choosing $(j, g) \in \mathrm{Supp}(P)$ such that $\mathbf{f}^*_U(j, g) \leq \liminf_n \mathbf{F}_n(\bar{\jmath}_n, \bar{g}_n)$ we obtain, using (4.26) and (4.28),

$$\liminf_n W_V(\bar{\jmath}_n) \geq \liminf_n \mathbf{F}_n(\bar{\jmath}_n, \bar{g}_n) \geq W_U(j). \tag{4.31}$$

Choosing $V = U$ shows that $\inf_{\mathcal{A}_1} W_U$ is achieved. Taking for $\bar{j}_n$ a minimizing sequence for W_V, it follows from (4.31) that $\min_{\mathcal{A}_1} W_V \geq \min_{\mathcal{A}_1} W_U$, hence the value of $\min_{\mathcal{A}_1} W_U$ is independent of U.

We prove the claim (4.30). Since $\mathbf{f}_U^*(j,g) \leq +\infty$, we have $g(\mathbf{U}_R) < C|\mathbf{U}_R| \leq CR^2$ and thus $\forall R > 1$ there exists $N_R \in \mathbb{N}$ such that $n \geq N_R$ implies $g_n(\mathbf{U}_R) \leq CR^2$. Using (4.13) this in turn implies that if $n \geq N_R$ then $\nu_n(\mathbf{U}_R) \leq CR^2$ and then, passing to the limit $n \to \infty$, that $\nu(\mathbf{U}_R) \leq CR^2$, so that in particular $j \in \mathcal{A}_1$.

Moreover, still if $n > N_R$, from $g_n(\mathbf{U}_R)$, $\nu_n(\mathbf{U}_R) \leq CR^2$ and using (4.12) we deduce that $W(j_n, \chi_{\mathbf{U}_R}) \leq CR^2 \log R$ and then, as in the proof of Corollary 4.12, that

$$\left| W(j_n, \chi_{\mathbf{U}_R}) - \int \chi_{\mathbf{U}_R}\, dg_n \right| \leq o(R^2),$$

for some $\theta < 2$. Passing to the liminf $n \to \infty$ we obtain in view of Lemma 4.8 the same relation for j, g. Dividing by $|\mathbf{U}_R|$ — which is bounded below by cR^2 for some $c > 0$ — and letting $R \to +\infty$ we find that $W_U(j) = \mathbf{f}_U^*(j,g)$, which finishes the proof of (4.30).

□

Independence w.r.t. the shape. We have just seen that if U and V refer to two families of sets, the infimum of W_V and W_U are both achieved and are equal. There remains to show that minimizers are also the same.

Consider j_V a minimizer of W_V. By Corollary 4.12 we have for any $\{\mathbf{U}_R\}_R$ satisfying (1.5),

$$\lim_{R\to\infty} \frac{W(j_V, \chi_{\mathbf{U}_R})}{|\mathbf{U}_R|} - \fint_{\mathbf{U}_R} \chi_{\mathbf{U}_R}\, dg_{\mathbf{U}_R} = 0, \tag{4.32}$$

where $g_{\mathbf{U}_R}$ is the result of applying Proposition 4.9 in $\mathbf{U}_R$. We deduce that

$$W_U(j_V) = \limsup_{R\to\infty} \fint_{\mathbf{U}_R} \chi_{\mathbf{U}_R}\, dg_{\mathbf{U}_R} = \limsup_{R\to\infty} \int \mathbf{f}(j,g)\, dP_{\mathbf{U}_R}(j,g) \geq \liminf_{R\to\infty} \int \mathbf{f}(j,g)\, dP_{\mathbf{U}_R}(j,g) \geq \int \mathbf{f}(j,g)\, dP_U(j,g), \tag{4.33}$$

where $\mathbf{f}(j,g) = \int \chi\, dg$, $\chi_{\mathbf{U}_R} = \mathbf{1}_{\mathbf{U}_{R-C}} * \chi$, and where $P_{\mathbf{U}_R}$ is the image of the normalized Lebesgue measure on $\mathbf{U}_{R-C}$ by $\lambda \mapsto \theta_\lambda(j_V, g_{\mathbf{U}_R})$. Moreover we have chosen a subsequence $\{R\}$ such that $\{P_{\mathbf{U}_R}\}_R$ converges to a probability measure P_U.

Since P_U is θ-invariant and using the ergodic theorem and (4.30) we get

$$\int \mathbf{f}(j,g)\, dP_U(j,g) = \int \mathbf{f}_U^*(j,g)\, dP_U(j,g) = \int W_U(j)\, dP_U(j,g) \geq \min_{\mathcal{A}_1} W,$$

where W can be defined using any family of sets satisfying (1.4), (1.5), *not necessarily the family* U. Together with (4.33) we get

$$W_U(j_V) = \limsup_{R\to\infty} \int \mathbf{f}(j,g)\, dP_{\mathbf{U}_R}(j,g) \geq \int \mathbf{f}(j,g)\, dP_U(j,g) = \int W_U(j)\, dP_U(j,g) \tag{4.34}$$

and this is bounded below by $\min_{\mathcal{A}_1} W$. From the minimality of j_V and since $\min W_U = \min W_V$, applying (4.34) to $\{\mathbf{U}_R\}_R = \{\mathbf{V}_R\}_R$ implies that there is equality everywhere and therefore

$$\text{(i)} \ \limsup_R \int \mathbf{f}\, dP_{\mathbf{V}_R} = \int \mathbf{f}\, dP_V, \quad \text{(ii)} \ P_V\text{-almost every } j \text{ minimizes } W.$$

Since $\mathbf{f}$ is continuous and bounded below on the support of $P_{\mathbf{V}_R}$ independently of R,

$$\limsup_R \int \mathbf{f}\, dP_{\mathbf{V}_R} = \int \mathbf{f}\, dP_V \iff \sup_R \int_{\{\mathbf{f}>M\}} \mathbf{f}\, dP_{\mathbf{V}_R} \xrightarrow{M\to+\infty} 0.$$

Now choose $c>0$ such that $\mathbf{U}_{cR} \subset \mathbf{V}_{R-C}$ for every $R\ge 1$ and let $P'_{\mathbf{V}_R}$ be the image of the normalized Lebesgue measure on $\mathbf{U}_{cR}$ by $\lambda \mapsto \theta_\lambda(j_V, g_{\mathbf{V}_R})$. Then

$$P_{\mathbf{V}_R} \ge \frac{|\mathbf{U}_{cR}|}{|\mathbf{V}_{R-C}|} P'_{\mathbf{V}_R} \ge \delta P'_{\mathbf{V}_R}, \tag{4.35}$$

for some $\delta>0$ independent of R (this follows from (1.5)). It follows that

$$\sup_R \int_{\{\mathbf{f}>M\}} \mathbf{f}\, dP'_{\mathbf{V}_R} \xrightarrow{M\to+\infty} 0$$

and then that, choosing a subsequence $\{R\}$ such that $P'_{\mathbf{V}_R} \to P'_V$, that $\limsup_R \int \mathbf{f}\, dP'_{\mathbf{V}_R} = \int \mathbf{f}\, dP'_V$ (cf. [Bi, Du]).

Now we claim that the support of P'_V is included in the support of P_V. Indeed if $\varphi\ge 0$ is continuous with compact support in $(\text{Supp } P_V)^c$, then $\int \varphi\, dP_{\mathbf{V}_R} \to \int \varphi\, dP_V = 0$ hence from (4.35)

$$\delta \int \varphi\, dP'_{\mathbf{V}_R} \xrightarrow{R\to\infty} 0$$

and thus $\int \varphi\, dP'_V = \lim_R \int \varphi\, dP'_{\mathbf{V}_R} = 0$.

It follows that P'_V-almost every j minimizes W_U and that

$$\int \mathbf{f}\, dP'_{\mathbf{V}_R} \xrightarrow{R\to\infty} \min W_U. \tag{4.36}$$

But $\int \mathbf{f}\, dP'_{\mathbf{V}_R} = \fint_{\mathbf{U}_{cR}} \chi_{\mathbf{U}_{cR}}\, dg_{\mathbf{V}_R}$ by definition of $\mathbf{f}$ and $P'_{\mathbf{V}_R}$ and using Corollary 4.12, we have that $\fint_{\mathbf{U}_{cR}} \chi_{\mathbf{U}_{cR}}\, d(g_{\mathbf{V}_R} - g_{\mathbf{U}_{cR}})$ tends to 0 as $R\to+\infty$. Therefore $\int \mathbf{f}\, d(P'_{\mathbf{V}_R} - P_{\mathbf{U}_{cR}})$ tends to 0 as well, which together with (4.36) and (4.34) yields

$$W_U(j_V) = \limsup_{R\to+\infty} \int \mathbf{f}\, dP_{\mathbf{U}_{cR}} = \min_{\mathcal{A}_1} W_U.$$

It follows that j_V minimizes W_U. $\square$

4.3 Proof of Proposition 4.2 and Corollaries 4.4, 4.5

The first lemma (whose proof is postponed to the end of the subsection) serves to extract a good boundary. We denote by W_K the renormalized energy relative to the family $\{K_R = [-R,R]^2\}_R$.

Lemma 4.14. *Let G satisfy the assumptions of Proposition 4.2. Then for any $\gamma\in(0,1)$, any R large enough depending on γ, and any $p\in[1,2)$ there exists, for any $j\in G$, some $t\in[R-2R^\gamma, R-R^\gamma]$ such that*

$$\int_{\partial K_t} |j|^p \le C_p R^{2-\gamma} \tag{4.37}$$

$$\lim_{R\to\infty} \frac{W(j,\mathbf{1}_{K_t})}{|K_t|} = W_K(j), \qquad |\nu(K_t) - |K_t|| \le CR^{2-\gamma}, \tag{4.38}$$

where C, C_p do not depend on $j \in G$, and where the convergence in (4.38) *is uniform with respect to $j \in G$.*

The next steps consist in modifying j in $K_R \backslash K_t$ so that $j \cdot \tau = 0$ on ∂K_R, and so that the ensuing modification of $W(j, \mathbf{1}_{K_R})$ is negligible compared to R^2 as $R \to +\infty$. This relies on the following two lemmas.

Lemma 4.15. *Let $\mathcal{R}$ be a rectangle with sidelengths in $[\frac{L}{2}, \frac{3L}{2}]$. Let $p \in (1,2)$. Let $g \in L^p(\partial\mathcal{R})$ be a function which is 0 except on one side of the rectangle $\mathcal{R}$. Let m be a constant such that $(m-1)|\mathcal{R}| = -\int_{\partial\mathcal{R}} g$. Then the mean zero solution to*

$$\begin{cases} -\Delta u = m-1 & \text{in } \mathcal{R} \\ \frac{\partial u}{\partial \nu} = g & \text{on } \partial\mathcal{R} \end{cases} \tag{4.39}$$

satisfies for every $q \in [1, 2p]$

$$\int_{\mathcal{R}} |\nabla u|^q \le C_{p,q} L^{2-\frac{q}{p}} \|g\|^q_{L^p(\partial\mathcal{R})}. \tag{4.40}$$

Proof. We write the solution u of (4.39) as $u = u_1 + u_2$ where

$$\begin{cases} -\Delta u_1 = m-1 & \text{in } \mathcal{R} \\ \frac{\partial u_1}{\partial \nu} = \bar{g} & \text{on } \partial\mathcal{R} \end{cases} \tag{4.41}$$

where $\bar{g}$ is equal to the average of g on the side where g is supported and is 0 on the other sides; and

$$\begin{cases} -\Delta u_2 = 0 & \text{in } \mathcal{R} \\ \frac{\partial u_2}{\partial \nu} = g - \bar{g} & \text{on } \partial\mathcal{R}. \end{cases}$$

Assume that $\mathcal{R} = [0,\ell_1] \times [0,\ell_2]$, with $\ell_i \in [\frac{L}{2}, \frac{3L}{2}]$, and that $\bar{g}$ is supported on the side $x_2 = 0$. Then, up a constant, the solution of (4.41) is $u_1(x_1,x_2) = \frac{m-1}{2}(x_2 - \ell_2)^2$. Therefore

$$\int_{\mathcal{R}} |\nabla u_1|^q = (m-1)^q \ell_1 \int_0^{\ell_2} (x_2-\ell_2)^q \, dx_2 = (m-1)^q \ell_1 \frac{\ell_2^{q+1}}{q+1} \le C(m-1)^q L^{2+q}.$$

Then, $m - 1 = -\frac{1}{\ell_1 \ell_2} \int_{\partial\mathcal{R}} g$ and using Hölder's inequality $|m-1| \le CL^{-2} \|g\|_{L^p(\partial\mathcal{R})} L^{1-\frac{1}{p}}$. Inserting above we are led to

$$\int_{\mathcal{R}} |\nabla u_1|^q \le C_{p,q} L^{2-\frac{q}{p}} \|g\|^q_{L^p(\partial\mathcal{R})}. \tag{4.42}$$

For u_2, note that the conjugate harmonic function u_2^* has trace φ which satisfies $\partial_\tau \varphi = g - \bar{g}$, hence $\|\nabla\varphi\|_{L^p(\partial\mathcal{R})} \le \|g - \bar{g}\|_{L^p(\partial\mathcal{R})}$. Then from the Sobolev imbedding $W^{1,p}(\partial\mathcal{R}) \hookrightarrow W^{1-\frac{1}{2p},2p}(\partial\mathcal{R})$, which is the trace space of $W^{1,2p}(\mathcal{R})$, and elliptic regularity it follows that

$$\|\nabla u_2\|_{L^{2p}(\mathcal{R})} = \|\nabla u_2^*\|_{L^{2p}(\mathcal{R})} \le C_{\mathcal{R}} \|g - \bar{g}\|_{L^p(\partial\mathcal{R})},$$

and it is easy to check that the constant $C_{\mathcal{R}}$ may be chosen to depend only on p, L and not on ℓ_i, as long as $\ell_i \in [\frac{L}{2}, \frac{3L}{2}]$. Scaling arguments then show that for some $C_{p,q}$ independent of L we have

$$\int_{\mathcal{R}} |\nabla u_2|^q \le C_{p,q} L^{2-\frac{q}{p}} \|g\|^q_{L^p(\partial\mathcal{R})}. \tag{4.43}$$

Combining (4.42) and (4.43), we obtain the result (4.40). □

Lemma 4.16. *Let $\mathcal{R}$ be a rectangle of barycenter 0, and sidelengths $\in \sqrt{2\pi}[\frac{1}{2}, \frac{3}{2}]$, and let m be a constant such that $m|\mathcal{R}| = 2\pi$. Then the solution to*

$$\begin{cases} -\Delta f = 2\pi\delta_0 - m & \text{in } \mathcal{R} \\ \frac{\partial f}{\partial \nu} = 0 & \text{on } \partial\mathcal{R} \end{cases}$$

satisfies

$$\lim_{\eta\to 0} \left| \int_{\mathcal{R}\setminus B(0,\eta)} |\nabla f|^2 + 2\pi \log \eta \right| \le C \tag{4.44}$$

where C is universal, and for every $1 \le q < 2$

$$\int_{\mathcal{R}} |\nabla f|^q \le C_q, \tag{4.45}$$

where C_q depends only on q.

Proof. This is a standard computation, of the type of [BBH], Chap. 1, observing that $f = -\log|x| + S(x)$ where S is a C^1 function. □

4.3.1 Proof of Proposition 4.2

Let $j \in G$. Apply Lemma 4.14 with $p \in (\frac{3}{2}, 2)$ and $\gamma = \frac{3}{4}$. For any R large enough it provides us with a square K_t, where t depends on $j \in G$ but satisfies

$$R^{\frac{3}{4}} \le R - t \le 2R^{\frac{3}{4}}. \tag{4.46}$$

We wish to extend j in $K_R\backslash K_t$, keeping j and Λ unchanged in $K_R\backslash K_t$, to obtain a j_R satisfying $j_R \cdot \tau = 0$ on ∂K_R and $\mathrm{curl}\, j_R = 2\pi \sum_{p\in\Lambda_R} \delta_p - 1$ — while the extension's contribution to the renormalized energy remains negligible compared to R^2, uniformly with respect to $j \in G$.

Below, the notations $o(\cdot)$, $\sim$ and $O(\cdot)$ are understood with respect to $R \to +\infty$, and *uniform with respect to $j \in G$.*

We divide each side of K_t into $[\sqrt{R}]$ intervals, so that there are a total of $4[\sqrt{R}]$ intervals, which we label $\{I_i\}_i$, of length $\frac{2t}{[\sqrt{R}]}$, which is equivalent to $2\sqrt{R}$ as $R \to +\infty$, uniformly with respect to t satisfying (4.46). For each i we consider the square $K_i \subset K_R \setminus K_t$ with one side equal to I_i. By perturbing the length of the other side of an amount which is $O(1/\sqrt{R})$ as $R \to \infty$ we may obtain a rectangle $\mathcal{R}_i$ whose aspect ratio tends to 1. and such that

$$|\mathcal{R}_i| - \int_{I_i} j \cdot \tau \in 2\pi\mathbb{N}. \tag{4.47}$$

We let g_i denote the restriction of $j \cdot \tau$ to I_i (and extend it by 0 on the rest of $\partial \mathcal{R}_i$), and let $m_i = 1 - \frac{\int_{\partial \mathcal{R}_i} g_i}{|\mathcal{R}_i|}$. Using Hölder's inequality, we have

$$|m_i - 1| \le \frac{C}{|\mathcal{R}_i|} \left(\int_{I_i} |j \cdot \tau|^p \right)^{\frac{1}{p}} R^{\frac{1}{2}(1-\frac{1}{p})} \le C R^{\frac{1}{2}(-1-\frac{1}{p})} \left(\int_{\partial K_t} |j|^p \right)^{\frac{1}{p}}.$$

In view of (4.37), we deduce

$$|m_i - 1| = O(R^{-\frac{1}{2}-\frac{1}{2p}+\frac{2}{p}-\frac{\gamma}{p}}) = o(1),$$

since $\gamma = \frac{3}{4}$ and $p > \frac{3}{2}$.

By (4.47) and by choice of m_i we also have $m_i |\mathcal{R}_i| \in 2\pi\mathbb{N}$. We may thus tile $\mathcal{R}_i$ by an integer number of rectangles $\mathcal{R}_{ik}$, whose sidelengths are in $\sqrt{2\pi}[\frac{1}{2}, \frac{3}{2}]$ and such that for each i, k, we have $m_i |\mathcal{R}_{ik}| = 2\pi$. Since $m_i \sim 1$, the number of rectangles inside each $\mathcal{R}_i$ is equivalent to $|\mathcal{R}_i|/2\pi \sim R/2\pi$ as $R \to +\infty$.

On each of these rectangles, we may apply Lemma 4.16, which yields a function f_{ik} satisfying (4.44)–(4.45). We then define the vector field j_1 in $\cup_i \mathcal{R}_i$ by $j_1 = -\nabla f_{ik}$ in each $\mathcal{R}_{ik}$. We can check that j_1 satisfies

$$\begin{cases} \operatorname{curl} j_1 = 2\pi \sum_{p \in \Gamma} \delta_p - \sum_i m_i \mathbf{1}_{\mathcal{R}_i} & \text{in } \cup_i \mathcal{R}_i \\ j_1 \cdot \tau = 0 & \text{on } \partial(\cup_i \mathcal{R}_i) \end{cases} \tag{4.48}$$

where Γ is the union over i, k, of the centers of the rectangles $\mathcal{R}_{ik}$. Indeed since $\frac{\partial f_{ij}}{\partial \nu} = 0$ no curl is created at the interfaces between the $\mathcal{R}_{ik}$'s, and no curl is created either at the interfaces between the $\mathcal{R}_i$'s.

Moreover, by (4.44)–(4.45), since the number of $\mathcal{R}_{ik}$ for each i is of order $|\mathcal{R}_i|$, and since the number of $\mathcal{R}_i$'s is $O(\sqrt{R})$, j_1 satisfies

$$\lim_{\eta \to 0} \left| \int_{\cup_i \mathcal{R}_i \setminus \cup B(p,\eta)} |j_1|^2 + \pi \#\Gamma \log \eta \right| \le C R^{\frac{3}{2}}, \tag{4.49}$$

and for $q < 2$

$$\int_{\cup_i \mathcal{R}_i} |j_1|^q \le C_q R^{\frac{3}{2}}. \tag{4.50}$$

Since $(m_i - 1)|\mathcal{R}_i| = -\int_{\partial \mathcal{R}_i} g_i$, we may also apply Lemma 4.15 in each $\mathcal{R}_i$ with $g = g_i$ for boundary data. It yields a function u_i satisfying (4.40). We then define the vector field j_2 as $-\nabla^\perp u_i$ in each $\mathcal{R}_i$. It satisfies

$$\begin{cases} \operatorname{curl} j_2 = \sum_i m_i \mathbf{1}_{\mathcal{R}_i} - 1 & \text{in } \cup_i \mathcal{R}_i \\ j_2 \cdot \tau = g & \text{on } \partial \cup_i \mathcal{R}_i \end{cases} \tag{4.51}$$

where $g = j \cdot \tau$ on ∂K_t and 0 on the rest of $\partial(\cup_i \mathcal{R}_i)$. Indeed, g_i is only supported on the I_i i.e. on the sides of the $\mathcal{R}_i$ which are in ∂K_t, so $j_2 \cdot \tau = 0$ on all the boundaries of the $\mathcal{R}_i$ which intersect, therefore again no curl is created there. For every $1 \le q \le 2p$ we have, since $|I_i| \sim \sqrt{R}$,

$$\int_{\mathcal{R}_i} |j_2|^q \le C_{p,q} R^{1-\frac{q}{2p}} \|g_i\|^q_{L^p(\partial \mathcal{R}_i)}.$$

Adding these relations, we obtain

$$\int_{\cup_i \mathcal{R}_i} |j_2|^q \le C_{p,q} R^{1-\frac{q}{2p}} \sum_i \left(\int_{\partial \mathcal{R}_i} |g_i|^p \right)^{\frac{q}{p}}.$$

But when $q/p > 1$ we have $\sum_i x_i^{q/p} \le (\sum_i \max(1, x_i))^{q/p}$ and number of $\mathcal{R}_i$'s is $O(\sqrt{R})$ hence,

$$\int_{\cup_i \mathcal{R}_i} |j_2|^q \le C_{p,q} R^{1-\frac{q}{2p}} \left(\int_{\partial K_t} |g|^p + \sqrt{R} \right)^{\frac{q}{p}}.$$

Using (4.37), we deduce, for all $1 \le q \le 2p$

$$\int_{\cup_i \mathcal{R}_i} |j_2|^q \le C_{p,q} R^{1-\frac{q}{2p}+\frac{q}{p}(2-\frac{3}{4})} \le C_{p,q} R^{1+\frac{3q}{4p}}. \tag{4.52}$$

From now on we choose $p \in (\frac{3}{2}, 2)$ and we have for every $q < \frac{4p}{3}$,

$$\int_{\cup_i \mathcal{R}_i} |j_2|^q \le C_q R^\sigma, \quad \text{for some } \sigma < 2. \tag{4.53}$$

We can now define j_R more precisely. In $\cup_i \mathcal{R}_i$ we let $j_R = j_1 + j_2$ and $\nu_R = 2\pi \sum_{p\in\Gamma} \delta_p$. By summing (4.48) and (4.51) we have

$$\begin{cases} \operatorname{curl} j_R = 2\pi \sum_{p\in\Gamma} \delta_p - 1 & \text{in} \quad \cup_i \mathcal{R}_i \\ j_R \cdot \tau = g & \text{on } \partial(\cup_i \mathcal{R}_i), \end{cases}$$

where $g = j \cdot \tau$ on $\partial(\cup_i \mathcal{R}_i) \cap \partial K_t$ and $g = 0$ elsewhere. Also,

$$\int_{\cup_i \mathcal{R}_i \setminus \cup B(p,\eta)} |j_R|^2 = \int_{\cup_i \mathcal{R}_i \setminus \cup B(p,\eta)} |j_1|^2 + |j_2|^2 + 2 j_1 \cdot j_2. \tag{4.54}$$

We have

$$\left| \int_{\cup_i \mathcal{R}_i \setminus \cup B(p,\eta)} j_1 \cdot j_2 \right| \le \left(\int_{\cup_i \mathcal{R}_i} |j_1|^{q'} \right)^{1/q'} \left(\int_{\cup_i \mathcal{R}_i} |j_2|^q \right)^{1/q}$$

where $q > 2$ and $\frac{1}{q'} = 1 - \frac{1}{q}$. Using (4.50) and (4.53) where we can choose $q > 2$ since $p > \frac{3}{2}$, we find

$$\left| \int_{\cup_i \mathcal{R}_i \setminus \cup B(p,\eta)} j_1 \cdot j_2 \right| \le C R^{\frac{3}{2q'} + \frac{\sigma}{q}} = o(R^2),$$

since $\sigma < 2$ and $\frac{1}{q'} + \frac{1}{q} = 1$. Inserting into (4.54) and combining with (4.49) and (4.52) we obtain

$$\lim_{\eta \to 0} \left| \int_{\cup_i \mathcal{R}_i \setminus \cup B(p,\eta)} |j_R|^2 + \pi \#\Gamma \log \eta \right| = O(R^{\frac{3}{2}}) + o(R^2) = o(R^2).$$

There remains to define j_R in $A := K_R \setminus (K_t \cup_i \mathcal{R}_i)$. First we note that $|A| \in 2\pi\mathbb{N}$. Indeed, from $\operatorname{curl} j = \nu - 1$ and (4.47), we have

$$2\pi \#(\Lambda \cap K_t) - |K_t| = \int_{\partial K_t} j \cdot \tau = \sum_i |\mathcal{R}_i| \pmod{2\pi},$$

thus $|A| = |K_R| - \sum_i |\mathcal{R}_i| - |K_t|$ is in $2\pi\mathbb{N}$ if $|K_R| \in 2\pi\mathbb{N}$.

The set A can be described as follows: It is the union of four squares of sidelength $R-t$ positioned at the corners of K_R, and a union $\cup_i \mathcal{R}'_i$, where $\mathcal{R}'_i$ is a rectangle having a side of length $|I_i|$ in common with $\mathcal{R}_i$, and such that their union is isometric to $I_i \times [0, R-t]$. Since both dimensions of these rectangles as well as those of the four squares tend to $+\infty$ as $R \to +\infty$, and since $|A| \in 2\pi\mathbb{N}$, it is possible to tile A by rectangles of area 2π and aspect ratio close to 1. Applying Lemma 4.16 in each of them yields a current j_A which satisfies $\operatorname{curl} j_A = 2\pi \sum_{p\in\Gamma'} \delta_p - 1$ in A and $j_A \cdot \tau = 0$ on ∂A — where Γ' is the set of centers of the rectangles tiling A.

The cardinal of Γ' is $|A|/2\pi$, which is $O(R^{1+\frac{3}{4}})$, and therefore from (4.44) we deduce

$$\lim_{\eta\to 0} \left| \int_{K_R\setminus(K_t\cup_i\mathcal{R}_i)} |j_A|^2 + \pi \#\Gamma' \log \eta \right| = O(R^{\frac{7}{4}}), \tag{4.55}$$

and from (4.45), for all $1 \le q < 2$,

$$\int_{K_R\setminus(K_t\cup_i\mathcal{R}_i)} |j_A|^q \le C_q R^{\frac{7}{4}}. \tag{4.56}$$

Letting $j_R = j_A$ in A and

$$\Lambda_R = (\Lambda \cap K_t) \cup \Gamma \cup \Gamma', \quad \nu_R = 2\pi \sum_{p\in\Lambda_R} \delta_p$$

we have $j = j_R$ in K_t, $\nu_R = \nu$ in K_t, $\operatorname{curl} j_R = \nu_R - 1$ in K_R, $j_R \cdot \tau = 0$ on ∂K_R, and combining (4.55), (4.54) and (4.38) we get

$$\frac{W(j_R, \mathbf{1}_{K_R})}{|K_R|} = W_K(j) + o(1) \quad \text{as } R \to +\infty, \tag{4.57}$$

where the $o(1)$ is uniform with respect to $j \in G$. This completes the proof of Proposition 4.2. We also note that from (4.50), (4.53) and (4.56), for every $1 \le q < 2$, we have

$$\int_{K_R\setminus K_t} |j_R|^q \le R^\sigma \quad \text{for some } \sigma < 2. \tag{4.58}$$

4.3.2 Proof of Lemma 4.14

Let G satisfy the assumptions of Proposition 4.2, and $R > 2$ with $|K_R| \in 2\pi\mathbb{N}$, $0 < \gamma < 1$ be given. Assume $j \in G$. In this proofs the constants and limits as $R \to +\infty$ are understood to be uniform with respect to $j \in G$.

Step 1: Denote by g_R the result of applying Proposition 4.9 in K_R to (j, ν). We apply (4.12) to functions of the form $\chi(x) = \rho(\|x\|_\infty)$, i.e. whose level sets are squares, with the additional assumption that $\rho'(t) = 0$ outside $[R-2, R-1]$ and $\rho = 0$ on $[R-1, +\infty]$. Since for any Radon measure μ on K_R we have

$$\int \chi \, d\mu = -\int_0^{R-1} \rho'(t)\mu(K_t)\, dt,$$

we deduce that

$$(4.59)\quad \int_{R-2}^{R-1} (W(j, \mathbf{1}_{K_t}) - g_R(K_t))\, \rho'(t)\, dt = -W(j,\chi) + \int \chi\, dg_R \le Cn(\log n + 1)\|\rho'\|_\infty,$$

where the last inequality is (4.12), and $n = \#\{p \in \Lambda, B(p,C) \cap \mathrm{Supp}\, \nabla\chi \neq \varnothing\}$, so that $2\pi n \le \nu(K_{R+C}) - \nu(K_{R-C})$. Here C denotes a universal constant, hence independant of $j \in G$. We deduce by duality that for some universal $C > 0$ we have

$$(4.60)\quad \int_{R-2}^{R-1} |W(j, \mathbf{1}_{K_t}) - g_R(K_t)|\; dt \le Cn(\log n + 1).$$

On the other hand, Lemma 4.7 yields that for any $p \in [1,2)$ and $R > 0$,

$$\|j\|_{L^p(K_R)} \le CR^{2/p} \log^{1/2} R,$$

where C depends only on p and on the constants in (4.3) and (4.4). Arguing as in the proof of Lemma 4.11 this implies that

$$(4.61)\quad n \le \frac{1}{2\pi} |\nu(K_{R+C}) - \nu(K_R)| \le CR^{\theta(1-\frac{1}{p})} \|j\|_{L^p(K_{R+2C})} \le CR^{\theta(1-\frac{1}{p}) + \frac{2}{p}} \log^{\frac{1}{2}} R \le CR^\beta$$

for some $\beta < 2$, where we have chosen $p < 2$ close enough to 2 and used $\theta < 2$ in (1.5). It then follows from (4.60), (4.61) that

$$(4.62)\quad \frac{1}{R^2} \int_{R-2}^{R-1} |W(j, \mathbf{1}_{K_t}) - g_R(K_t)|\; dt \le CR^{\beta-2} \log R.$$

Now denote by $\{\chi_R\}_R$ a family of functions satisfying (1.3) relative to the family $\{K_R\}_R$. We also assume $\chi_R \le 1$. Since $g_R \ge -C$ and since $\chi_R = 1$ on K_{R-1} we have for any $t \in [R-2, R-1]$ the inequalities

$$\int \chi_{R-2}\, dg_R - C|K_{R-1} \setminus K_{R-3}| \le g_R(K_t) \le \int \chi_R\, dg_R + C|K_R \setminus K_{R-2}| \le \int \chi_R\, dg_R + CR.$$

and thus

$$(4.63)\quad \int \chi_{R-2}\, dg_R - CR \le g_R(K_t) \le \int \chi_R\, dg_R + CR.$$

Step 2: For any integer $k \ge 1$ let $\xi_k = \chi_{k+1} - \chi_k$, and let $\xi_0 = \chi_1$. Then $\xi_k \ge 0$, since $\chi_{k+1} = 1$ on K_k and since $\chi_k \le 1$ and is supported in K_k. Moreover ξ_k is supported in $\mathcal{C}_k := K_{k+1} \setminus K_{k-1}$. Since (4.3) holds, the number of integers k in $[R - 2R^\gamma + 1, R - R^\gamma]$ such that $\nu(K_{k+2} \setminus K_{k-2}) \le CR^{2-\gamma}$ is greater than $R^\gamma/2$ if C is chosen large enough. Similarly,

$$\sum_{k=[R-2R^\gamma]}^{[R-R^\gamma]} \int \xi_k\, dg_R = \int (\chi_{[R-R^\gamma]+1} - \chi_{[R-2R^\gamma]})\, dg_R \le CR^2,$$

where we use Corollary 4.12, Remark 4.13 and (4.4). Since $g_R \ge -C$ we have $\int \xi_k\, dg_R \ge -CR$ and therefore the number of integer k's between $[R - 2R^\gamma] + 1$ and $[R - R^\gamma]$ such that

$\int \xi_k \, dg_R \le C R^{2-\gamma}$ is larger than $R^\gamma/2$ if C and R are chosen large enough, and thus such a k satisfying

$$\nu(K_{k+2} \setminus K_{k-2}) \le C R^{2-\gamma}, \qquad \int \xi_k \, dg_R \le C R^{2-\gamma},$$

for some C uniform with respect to $j \in G$.

Applying Proposition 4.9 in $\mathcal{C}_k$ to ξ_k, we have

$$\left| W(j, \xi_k) - \int \xi_k \, dg_R \right| \le C R^{2-\gamma} \log R$$

hence $W(j, \xi_k) \le C R^{2-\gamma} \log R$ and applying Lemma 4.7 we find that for $p < 2$,

$$\int_{\mathcal{C}_k} |\xi_k|^{\frac{p}{2}} |j|^p \le C R^{1-\frac{p}{2}} \left(R^{2-\gamma} \log R\right)^{\frac{p}{2}} \le C R^{2-\gamma}.$$

But $\xi_k(x) = 1$ if $\|x\|_\infty = k$ and thus from the gradient bound $\xi_k \ge 1/2$ if $k - 1/C \le \|x\|_\infty \le k + 1/C$. Therefore

$$\int_{K_{k-\frac{1}{C}} \setminus K_{k+\frac{1}{C}}} |j|^p \le C R^{2-\gamma}.$$

By a mean value argument on this integral as well as on (4.62) (applied with $R = k+1$), we deduce the existence $t \in [k-1, k]$ — hence $t \in [R - 2R^\gamma, R - R^\gamma]$ — such that, one the one hand

$$\int_{\partial K_t} |j|^p \le C R^{2-\gamma},$$

proving (4.37) since C is uniform with respect to $j \in G$ — and on the other hand

$$g_R(K_t) - W(j, \mathbf{1}_{K_t}) \le C R^\beta \log R. \tag{4.64}$$

Now, from (4.63) applied to $R = k+1$ and using (4.12) in Proposition 4.9 together with (4.61) we obtain

$$W(j, \chi_{R-2}) - C R^\beta \log R \le g_R(K_t) \le W(j, \chi_R) + C R^\beta \log R,$$

Which together with (4.64) and in view of (4.4) yields (4.38). Finally,

$$|\nu(K_t) - |K_t|| = \left| \int_{\partial K_t} j \cdot \tau \right| \le \|j\|_{L^p(\partial K_t)} |\partial K_t|^{1-\frac{1}{p}} \le C R^{2-\gamma},$$

using (4.37). Lemma 4.14 is proved.

4.3.3 Proof of Corollary 4.4

Let $j \in \mathcal{A}_1$ be such that $W_K(j) < +\infty$. Let R be such that $R^2 \in 8\pi\mathbb{N}$, and j_R be obtained by applying Proposition 4.2 over $K = [0, R] \times [0, R]$. We let $\tilde{j}_R = j_R - \nabla\zeta$, where $\Delta\zeta = \operatorname{div} j_R$ on K and $\zeta = 0$ on ∂K. Then $\tilde{j}_R = -\nabla^\perp H_R$ in K since $\operatorname{div} \tilde{j}_R = 0$ there, and we have $\partial_\nu H_R = 0$ on ∂K since $\tilde{j}_R \cdot \tau = j_R \cdot \tau = 0$ there. Thus defining H_R on $K_R = [-R, R] \times [-R, R]$ by letting $H_R(\pm x, \pm y) = H_R(x, y)$ we have $-\Delta H_R = \sum_{p \in \Lambda_R} \delta_p - 1$, where Λ_R is obtained from the restriction of $\operatorname{curl} j + 1$ to K by reflections across the coordinate axis. Moreover $H_R(-R, y) = H_R(R, y)$ and $H_R(x, -R) = H_R(x, R)$ so that we may periodize H_R to have it defined on $\mathbb{R}^2$. Then $j_R := -\nabla^\perp H_R$ belongs to $\mathcal{A}_1$ and since everything is periodic W can be computed through the results of Section 3.1:

$$W_K(j_R) = \frac{W(j_R, \mathbf{1}_{K_R})}{|K_R|} \le W_K(j) + o(1) \quad \text{as } R \to \infty.$$

The last assertion of the Corollary follows by taking j to minimize W_K over $\mathcal{A}_1$ (a minimizer exists by Proposition 4.1), and remembering that the minimum does not depend on the choice of shapes used.

4.3.4 Proof of Corollary 4.5

The proof of Corollary 4.5 consists in constructing a sequence from a Young measure on micropatterns, to use the terminology of [AM], while retaining an energy control. We thus assume P is a probability measure on $L^p_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ which is invariant under the action of translations and concentrated in $\mathcal{A}_1$.

First we choose distances which metrize the topologies of $L^p_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ and $\mathcal{B}(L^p_{\text{loc}})$, the set of finite Borel measures on $L^p_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$. For $j_1, j_2 \in L^p_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ we let

$$d_p(j_1, j_2) = \sum_{k=1}^{\infty} 2^{-k} \frac{\|j_1 - j_2\|_{L^p(B(0,k))}}{1 + \|j_1 - j_2\|_{L^p(B(0,k))}}.$$

On $\mathcal{B}(L^p_{\text{loc}})$ we define a distance by choosing a sequence of bounded continuous functions $\{\varphi_k\}_k$ which is dense in $C_b(L^p_{\text{loc}})$ and we let, for any $\mu_1, \mu_2 \in \mathcal{B}(L^p_{\text{loc}})$,

$$d_{\mathcal{B}}(\mu_1, \mu_2) = \sum_{k=1}^{\infty} 2^{-k} \frac{|\langle \varphi_k, \mu_1 - \mu_2 \rangle|}{1 + |\langle \varphi_k, \mu_1 - \mu_2 \rangle|},$$

where we have used the notation $\langle \varphi, \mu \rangle = \int \varphi \, d\mu$.

We have the following general facts.

Lemma 4.17. *For any $\varepsilon > 0$ there exists $\eta_0 > 0$ such that if $P, Q \in \mathcal{B}(L^p_{\text{loc}})$ and $\|P - Q\| < \eta_0$, then $d(P, Q) < \varepsilon$. Here $\|P - Q\|$ denotes the total variation of the signed measure $P - Q$, i.e. the supremum of $\langle \varphi, P - Q \rangle$ over measurable functions φ such that $|\varphi| \le 1$.*

In particular, if $P = \sum_{i=1}^{\infty} \alpha_i \delta_{x_i}$ and $Q = \sum_{i=1}^{\infty} \beta_i \delta_{x_i}$ with $\sum_i |\alpha_i - \beta_i| < \eta_0$, then $d_{\mathcal{B}}(P, Q) < \varepsilon$.

Lemma 4.18. *Let $K \subset L^p_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ be compact. For any $\varepsilon > 0$ there exists $\eta_1 > 0$ such that if $x \in K, y \in L^p_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ and $d_p(x, y) < \eta_1$ then $d_{\mathcal{B}}(\delta_x, \delta_y) < \varepsilon$.*

Lemma 4.19. *Let $0 < \varepsilon < 1$. If μ is a probability measure on a set A and $f, g : A \to L^p_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ are measurable and such that $d_{\mathcal{B}}(\delta_{f(x)}, \delta_{g(x)}) < \varepsilon$ for every $x \in A$, then*

$$d_{\mathcal{B}}(f^\# \mu, g^\# \mu) < C\varepsilon(|\log \varepsilon| + 1).$$

Proof. Take any bounded continuous function φ_k defining the distance on $\mathcal{B}(L^p_{\text{loc}})$. Then if $d_{\mathcal{B}}(\delta_{f(x)}, \delta_{g(x)}) < \varepsilon$ for any $x \in L^p_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ we have in particular

$$\frac{|\varphi_k(f(x)) - \varphi_k(g(x))|}{1 + |\varphi_k(f(x)) - \varphi_k(g(x))|} \le 2^k \varepsilon.$$

It follows that

$$d_{\mathcal{B}}(f^{\#}\mu, g^{\#}\mu) \le \sum_k 2^{-k}\min(\varepsilon 2^k, 1) \le \varepsilon\left([\log_2 \varepsilon] + 1\right) + \sum_{k=[\log_2 \varepsilon]+1}^{\infty} 2^{-k} \le C\varepsilon(|\log \varepsilon| + 1).$$

□

Selection of a good subset of L^p_{loc}. We must restrict to a compact subset of $L^p_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ in a suitable way. This is not surprising when constructing an approximation: note that the set of micropatterns in [AM] is assumed to be compact, and we need to reduce to this case. This is the aim of the following Lemma.

Lemma 4.20. *Given P as above and $\varepsilon, R > 0$ there exist subsets $H_\varepsilon \subset G_\varepsilon$ in $L^p_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ with G_ε compact and such that:*

i) η_0 being given by Lemma 4.17 we have

$$P({G_\varepsilon}^c) < \min({\eta_0}^2, \eta_0\varepsilon), \quad P(H_\varepsilon^c) < \min(\eta_0, \varepsilon). \tag{4.65}$$

ii) For every $j \in H_\varepsilon$ there is a subset $\Gamma(j) \subset K_R$ such that

$$|\Gamma(j)| < CR^2\eta_0 \text{ and } \lambda \notin \Gamma(j) \implies \theta_\lambda j \in G_\varepsilon. \tag{4.66}$$

iii)

$$W_K(j) \text{ and } \frac{\nu(K_t)}{t^2} \text{ are bounded uniformly with respect to } j \in G_\varepsilon \text{ and } t > 1, \tag{4.67}$$

where $\mathrm{curl}\, j = \nu - 1$, and the convergence in the definition of $W_K(j)$ is uniform.

iv) We have

$$d_{\mathcal{B}}(P, P'') < C\varepsilon(|\log \varepsilon| + 1), \quad \text{where} \quad P'' = \int_{H_\varepsilon} \frac{1}{|K_R|} \int_{K_R \setminus \Gamma(j)} \delta_{\theta_\lambda j}\, d\lambda\, dP(j). \tag{4.68}$$

Moreover, there exists a partition $H_\varepsilon = \cup_{i=1}^{N_\varepsilon} H_\varepsilon^i$ such that diam $(H_\varepsilon^i) < \eta_3$*, where η_3 is such that*

$$j \in H_\varepsilon,\ d_p(j, j') < \eta_3,\ \lambda \in K_R \setminus \Gamma(j) \implies d_{\mathcal{B}}(\delta_{\theta_\lambda j}, \delta_{\theta_\lambda j'}) < \varepsilon; \tag{4.69}$$

and for all $1 \le i \le N_\varepsilon$ there exists $J_i \in H_\varepsilon^i$ such that

$$W_K(J_i) < \inf_{H_\varepsilon^i} W_K + \varepsilon. \tag{4.70}$$

Proof. Choice of G_ε. Since $L^p_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ is Polish we can always find a compact set G_ε satisfying (4.65) and $P({G_\varepsilon}^c) < \eta_0$. Then from Lemma 4.17, $P \llcorner G_\varepsilon$ (the restriction of P to G_ε) satisfies $d_{\mathcal{B}}(P, P \llcorner G_\varepsilon) < \varepsilon$.

From the translation-invariance of P, we have for any λ that $P(\theta_\lambda G_\varepsilon) > 1 - \eta_0$ and therefore that $d_{\mathcal{B}}(P, P \llcorner \theta_\lambda G_\varepsilon) < \varepsilon$. It follows that for any $\lambda \in \mathbb{R}^2$ we have $\|P - P_\lambda\| < \eta_0$ and then $d_{\mathcal{B}}(P, P_\lambda) < \varepsilon$, where

$$P_\lambda = \int_{\theta_\lambda G_\varepsilon} \delta_j \, dP(j) = \int_{G_\varepsilon} \delta_{\theta_\lambda j} \, dP(j).$$

Then using Lemma 4.19 we deduce that if $A \subset \mathbb{R}^2$ is any measurable set of positive measure, then

$$d_{\mathcal{B}}(P, P') < C\varepsilon(|\log \varepsilon| + 1), \quad \text{where} \quad P' = \int_{G_\varepsilon} \fint_A \delta_{\theta_\lambda j} \, d\lambda \, dP(j). \tag{4.71}$$

Moreover, since P is invariant, choosing χ to be a smooth positive function with integral 1 supported in $B(0,1)$, the ergodic theorem (as in [Be]) ensures that for P-almost every j the limit

$$\lim_{t \to +\infty} \frac{1}{|K_t|} \int_{K_t} W(j(\lambda + \cdot), \chi(\lambda + \cdot)) \, d\lambda$$

exists. Then $\mathbf{1}_{K_t} * \chi$ is a family of functions which satisfies (1.3) with respect to the family of squares $\{K_t\}_t$, and from the definition of the renormalized energy relative to $\{K_t\}_t$ we may rewrite the limit above as

$$W_K(j) = \lim_{t \to +\infty} \frac{1}{|K_t|} W(j, \mathbf{1}_{K_t} * \chi). \tag{4.72}$$

By Egoroff's theorem we may choose the compact set G_ε above to be such that, in addition to (4.71), the convergence in (4.72) is uniform on G_ε. In fact, since $W_K(j) < +\infty$ and $\limsup_t \nu(K_t)/t^2 < +\infty$ for P-a.e. j, where $\operatorname{curl} j = \nu - 1$, we may choose G_ε such that (4.67) holds.

The difficulty we have to face next is that $\theta_\lambda j$ need not belong to G_ε if j does.

Choice of H_ε. For $j \in G_\varepsilon$, let $\Gamma(j)$ be the set of λ's in K_R such that $\theta_\lambda j \notin G_\varepsilon$. Since, from (4.65) and the translation-invariance of P, for any $\lambda \in \mathbb{R}^2$ we have $P(\theta_\lambda (G_\varepsilon)^c) < {\eta_0}^2$, it follows from Fubini's theorem that

$$\int_{G_\varepsilon} |\Gamma(j)| \, dP(j) = \int_{K_R} P((\theta_\lambda G_\varepsilon)^c) \, d\lambda < 4R^2 \min({\eta_0}^2, \eta_0 \varepsilon).$$

Therefore, letting

$$H_\varepsilon = \{j \in G_\varepsilon : |\Gamma(j)| < 4R^2 \eta_0\}, \tag{4.73}$$

we have that (4.65) holds. Combining (4.65) and (4.73) with Lemma 4.17, we deduce from (4.71) that (4.68) holds.

Then we use the fact that G_ε is compact and Lemma 4.18 to find that there exists $\eta_4 > 0$ such that

$$\forall j \in G_\varepsilon, \forall j' \in L^p_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2), \quad d_p(j, j') < \eta_4 \implies d_{\mathcal{B}}\left(\delta_j, \delta_{j'}\right) < \varepsilon. \tag{4.74}$$

Moreover, from the continuity of $(\lambda, j) \mapsto \theta_\lambda j$, there exists $\eta_3 > 0$ such that

$$\forall j \in G_\varepsilon, \forall \lambda \in K_R, \quad d_p(j, j') < \eta_3 \implies d_p\left(\theta_\lambda j, \theta_\lambda j'\right) < \eta_4.$$

Now if $j \in H_\varepsilon$ and $\lambda \in K_R$ this implies that if $d_p(j, j') < \eta_3$ then $d_p(\theta_\lambda j, \theta_\lambda j') < \eta_4$. But if $\lambda \notin \backslash\Gamma(j)$ we have $\theta_\lambda j \in G_\varepsilon$ hence applying (4.74) to $\theta_\lambda j$, $\theta_\lambda j'$, we get (4.69).

Choice of $J_1, \dots, J_{N_\varepsilon}$. Now we cover the relatively compact H_ε by a finite number of balls $B_1, \dots, B_{N_\varepsilon}$ of radius $\eta_3/2$ and derive from it a partition of H_ε by sets with diameter less than η_3 by letting $H_\varepsilon^1 = B_1 \cap H_\varepsilon$ and

$$H_\varepsilon^{i+1} = B_{i+1} \cap H_\varepsilon \setminus (B_1 \cup \cdots \cup B_i).$$

We then have

$$H_\varepsilon = \bigcup_{i=1}^{N_\varepsilon} H_\varepsilon^i, \quad \text{diam}\,(H_\varepsilon^i) < \eta_3(R), \tag{4.75}$$

where the union is disjoint. Then we may choose $J_i \in H_\varepsilon^i$ such that (4.70) holds. □

Completion of the construction. First we apply Proposition 4.2 with $G = G_\varepsilon$. The proposition yields $R_0 > 1$ such that for any $j \in G_\varepsilon$ and any $R > R_0$ such that $|K_R| \in 2\pi\mathbb{N}$ there exists j_R defined in K_R such that (4.5) is satisfied and such that, if $x \in K_{R-2R^{3/4}}$, then $j_R(x) = j(x)$. Moreover,

$$\frac{W(j_R, \mathbf{1}_{K_R})}{|K_R|} \le W_K(j) + \varepsilon. \tag{4.76}$$

We choose $R_\varepsilon > R_0$ such that $|K_{R_\varepsilon}| \in 2\pi\mathbb{N}$ and large enough so that

$$K_{R_\varepsilon(1-\eta_0)} \subset \{x : d(x, {K_{R_\varepsilon}}^c) > {R_\varepsilon}^{\frac{3}{4}}\}. \tag{4.77}$$

where η_0 is the constant in Lemma 4.17.

If $\lambda \in K_{R_\varepsilon(1-\eta_0)}$ and since $j(x) = j_{R_\varepsilon}(x)$ if $d(x, {K_{R_\varepsilon}}^c) > {R_\varepsilon}^{\frac{3}{4}}$, we deduce from (4.77) that $\theta_\lambda j_{R_\varepsilon} = \theta_\lambda j$ in $B(0, {R_\varepsilon}^{\frac{3}{4}})$ as soon as $\eta_0 R_\varepsilon > 2{R_\varepsilon}^{\frac{3}{4}}$, so that from the definition of d_p, taking R_ε larger if necessary,

$$\forall j' \in L^p_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2), \forall \lambda \in K_{R_\varepsilon(1-\eta_0)}, \quad j' = j_{R_\varepsilon} \text{ on } K_{R_\varepsilon} \implies d_p(\theta_\lambda j, \theta_\lambda j') < \eta_1, \tag{4.78}$$

where η_1 comes from Lemma 4.18 applied on G_ε, i.e. is such that

$$j \in G_\varepsilon, j' \in L^p_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2) \text{ and } d_p(j, j') < \eta_1 \implies d_{\mathcal{B}}(\delta_j, \delta_{j'}) < \varepsilon. \tag{4.79}$$

Having chosen R_ε, we get from Lemma 4.20 a set H_ε and a partition $H_\varepsilon = \cup_{i=1}^{N_\varepsilon} H_\varepsilon^i$ and in each H_ε^i a current J_i satisfying (4.70). We also choose an arbitrary $J_0 \in \mathcal{A}_1$ such that $W_K(J_0) < +\infty$.

Second we choose an integer q_ε large enough so that

$$\frac{N_\varepsilon}{{q_\varepsilon}^2} < \eta_0, \qquad \frac{N_\varepsilon}{{q_\varepsilon}^2} \times \max_{0 \le i \le N_\varepsilon} W_K(J_i) < \varepsilon. \tag{4.80}$$

Now if $j \in H_\varepsilon^i$ then $d_p(j, J_i) < \eta_3$ and we deduce from (4.69) that for every $\lambda \in K_{R_\varepsilon} \setminus \Gamma(j)$ we have

$$d_{\mathcal{B}}(\delta_{\theta_\lambda j}, \delta_{\theta_\lambda J_i}) < \varepsilon. \tag{4.81}$$

Using (4.81) together with Lemmas 4.18, 4.17, and the bound (4.66), we deduce from (4.68) that $d_{\mathcal{B}}(P, P''') < C\varepsilon(|\log \varepsilon| + 1)$, where

$$P''' = \sum_{1 \le i \le N_\varepsilon} p_i \fint_{K_{R_\varepsilon}} \delta_{\theta_\lambda J_i}\, d\lambda, \quad \text{where} \quad p_i = P\left(H_\varepsilon^i\right). \tag{4.82}$$

We now replace p_i in the definition (4.82) by

$$\frac{n_i}{{q_\varepsilon}^2}, \quad \text{where} \quad n_i = \left[{q_\varepsilon}^2 p_i\right]. \tag{4.83}$$

Then $\sum_{i=1}^{N_\varepsilon} n_i \le {q_\varepsilon}^2$ and

$$\sum_{1 \le i \le N_\varepsilon} \left| \frac{n_i}{{q_\varepsilon}^2} - p_i \right| < \frac{N_\varepsilon}{{q_\varepsilon}^2} < \eta_0. \tag{4.84}$$

Then Lemma 4.17 implies that $d_{\mathcal{B}}(P, P^{(4)}) < C\varepsilon(|\log \varepsilon| + 1)$ where

$$P^{(4)} = \sum_{1 \le i \le N_\varepsilon} \frac{n_i}{{q_\varepsilon}^2} \fint_{K_{R_\varepsilon}} \delta_{\theta_\lambda J_i}\, d\lambda. \tag{4.85}$$

Now we let $K_\varepsilon = [-q_\varepsilon R_\varepsilon, q_\varepsilon R_\varepsilon]^2$, and $n_0 := {q_\varepsilon}^2 - \sum_{i=1}^{N_\varepsilon} n_i$, so that $\sum_{i=0}^{N_\varepsilon} n_i = {q_\varepsilon}^2$. Then we divide K_ε in a collection $\mathcal{L}_\varepsilon$ of ${q_\varepsilon}^2$ identical subrectangles which are translates of K_{R_ε}, and for each $0 \le i \le N_\varepsilon$ we choose n_i subrectangles in an arbitrary way and call the collection of these subrectangles $\mathcal{L}_{\varepsilon,i}$, so that $\{\mathcal{L}_{\varepsilon,i}\}_{0 \le i \le N_\varepsilon}$ is a partition of $\mathcal{L}_\varepsilon$.

Let us call J_{i,R_ε} the currents obtained from J_i using Proposition 4.2. They satisfy (4.76) and (4.78). We claim that, as a consequence of the latter, we have for any $L \in \mathcal{L}_{\varepsilon,i}$ that

$$j' = J_{i,R_\varepsilon} \text{ on } K_{R_\varepsilon} \implies d_{\mathcal{B}}\left(\fint_{K_{R_\varepsilon}} \delta_{\theta_\lambda J_i}\, d\lambda, \fint_{K_{R_\varepsilon}} \delta_{\theta_\lambda j'}\, d\lambda \right) < C\varepsilon(|\log \varepsilon| + 1). \tag{4.86}$$

This goes as follows: (i) Using (4.66) and Lemma 4.17, integrating on $K_{(1-\eta_0)R_\varepsilon} \setminus \Gamma_(J_i)$ instead of K_{R_ε} induces an error of $C\varepsilon$. (ii) From (4.78), and (4.79) applied to $\theta_\lambda J_i$ by $\theta_\lambda j'$ we have $d_{\mathcal{B}}(\delta_{\theta_\lambda J_i}, \delta_{\theta_\lambda j'}) < \varepsilon$ and thus in view of Lemma 4.19 we may replace $\theta_\lambda J_i$ by $\theta_\lambda j'$ in the integral with an error of $C\varepsilon|\log \varepsilon|$ at most. (iii) Using (4.66) and Lemma 4.17 again, we may integrate back on K_{R_ε} rather than on $K_{(1-\eta_0)R_\varepsilon} \setminus \Gamma_(J_i)$, with an additional error of $C\varepsilon$. this proves (4.86).

Then combining (4.86) with (4.85) and $d_{\mathcal{B}}(P, P^{(4)}) < C\varepsilon(|\log \varepsilon| + 1)$, using Lemma 4.19 we find $d_{\mathcal{B}}(P, P^{(5)}) < C\varepsilon(|\log \varepsilon| + 1)$, where

$$P^{(5)} = \sum_{1 \le i \le N_\varepsilon} \frac{n_i}{{q_\varepsilon}^2} \fint_{K_{R_\varepsilon}} \delta_{\theta_\lambda \tilde{J}_{i,R_\varepsilon}}\, d\lambda, \tag{4.87}$$

and $\tilde{J}_{i,R_\varepsilon}$ is an arbitrary field in $L^p_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ such that $\tilde{J}_{i,R_\varepsilon} = J_{i,R_\varepsilon}$ on K_{R_ε}, the constant C being independent of the choice of $\tilde{J}_{i,R_\varepsilon}$.

We chose above an arbitrary J_0 in $\mathcal{A}_1$ such that $W_K(J_0) < +\infty$. Let the sum in (4.87) range over $0 \le i \le N_\varepsilon$ instead of $1 \le i \le N_\varepsilon$, this defines a measure $P^{(6)}$ such that, by (4.80),

$$\|P^{(5)} - P^{(6)}\| \le \frac{n_0}{{q_\varepsilon}^2} \le \eta_0, \tag{4.88}$$

where we have used (4.84) and the fact that $1-\sum_i p_i = P({H_\varepsilon}^c) < \eta_0$, from (4.82), (4.65). Hence using Lemma 4.17 we have $d_{\mathcal{B}}(P^{(5)},P^{(6)}) < \varepsilon$ and then $d_{\mathcal{B}}(P,P^{(6)}) < C\varepsilon(|\log\varepsilon|+1)$.

We now define the vector field $j_\varepsilon:\mathbb{R}^2\to\mathbb{R}^2$ by letting $j_\varepsilon(x) = J_{i,R_\varepsilon}(x-x_L)$ on every $L\in\mathcal{L}_{\varepsilon,i}$, $0\le i\le N_\varepsilon$ — where x_L is the center of L and thus $L = x_L+K_{R_\varepsilon}$ — and by requiring j_ε to be K_ε-periodic. For every $L\in\mathcal{L}_{\varepsilon,i}$ we have $j_\varepsilon(x_L+\cdot) = J_{i,R_\varepsilon}$ on K_{R_ε}, therefore we may choose $\tilde J_{i,R_\varepsilon} = j_\varepsilon(x_L+\cdot)$ in (4.87) and then

$$\int_{K_\varepsilon}\delta_{\theta_\lambda j_\varepsilon}\,d\lambda = \sum_{\substack{0\le i\le N_\varepsilon\\ L\in\mathcal{L}_{\varepsilon,i}}}\int_L \delta_{\theta_\lambda j_\varepsilon}\,d\lambda = \sum_{\substack{0\le i\le N_\varepsilon\\ L\in\mathcal{L}_{\varepsilon,i}}}\int_{K_{R_\varepsilon}}\delta_{\theta_\lambda j_\varepsilon(x_L+\cdot)}\,d\lambda = \sum_{0\le i\le N_\varepsilon} n_i\int_{K_{R_\varepsilon}}\delta_{\theta_\lambda \tilde J_{i,R_\varepsilon}}\,d\lambda.$$

Therefore we may summarize the discussion concerning $P^{(5)}$, $P^{(6)}$ by writing

$$d_{\mathcal{B}}(P,P^{(6)}) < C\varepsilon(|\log\varepsilon|+1),\quad P^{(6)} = \fint_{K_\varepsilon}\delta_{\theta_\lambda j_\varepsilon}\,d\lambda. \tag{4.89}$$

Note that since $J_{i,R_\varepsilon}=0$ outside K_{R_ε}, and $J_{i,R_\varepsilon}\cdot\tau = 0$ on $\partial K_{R_\varepsilon}$ we have, in K_ε,

$$j_\varepsilon = \sum_{\substack{1\le i\le N_\varepsilon\\ L\in\mathcal{L}_{\varepsilon,i}}} J_{i,R_\varepsilon}(\cdot - x_L),\quad \operatorname{curl} j_\varepsilon = 2\pi\sum_{p\in\Lambda_\varepsilon}\delta_p - 1,\quad j_\varepsilon\cdot\tau = 0 \text{ on } \partial K_\varepsilon. \tag{4.90}$$

where Λ_ε is a finite subset of the interior of K_ε. This completes the construction of j_ε.

Estimate of the energy. We have

$$W(j_\varepsilon,\mathbf{1}_{K_\varepsilon}) = \sum_{\substack{0\le i\le N_\varepsilon\\ L\in\mathcal{L}_{\varepsilon,i}}} W(J_{i,R_\varepsilon}(\cdot-x_L),\mathbf{1}_L) = \sum_{\substack{0\le i\le N_\varepsilon\\ L\in\mathcal{L}_{\varepsilon,i}}} W(J_{i,R_\varepsilon},\mathbf{1}_{K_{R_\varepsilon}}).$$

From (4.76) applied to the J_i's we deduce that

$$W(j_\varepsilon,\mathbf{1}_{K_\varepsilon}) \le |K_{R_\varepsilon}|\sum_{i=0}^{N_\varepsilon} n_i\left(W_K(J_i)+C\varepsilon\right). \tag{4.91}$$

Now from (4.84) and the fact that $1-\sum_i p_i = P({H_\varepsilon}^c) < \varepsilon$ we deduce that $n_0\le N_\varepsilon + {q_\varepsilon}^2\varepsilon$, and then from (4.80) that $n_0\le C\varepsilon {q_\varepsilon}^2$, where we have included $W_K(J_0)$ in the constant. This and (4.91), together with the estimates (4.84), (4.80) and the definition of p_i in (4.82), implies that

$$W(j_\varepsilon,\mathbf{1}_{K_\varepsilon}) \le {q_\varepsilon}^2|K_{R_\varepsilon}|\left(\sum_{i=1}^{N_\varepsilon} P(H_\varepsilon^i)W_K(J_i) + C\varepsilon\right),$$

and then from (4.70), and since $|K_\varepsilon| = {q_\varepsilon}^2|K_{R_\varepsilon}|$, that

$$W(j_\varepsilon,\mathbf{1}_{K_\varepsilon}) \le |K_\varepsilon|\left(\int_{H_\varepsilon} W_K(j)\,dP(j) + C\varepsilon\right) \le |K_\varepsilon|\left(\int W_K(j)\,dP(j)+C\varepsilon\right), \tag{4.92}$$

using (4.65) and the fact from Proposition 4.1 that W_K is bounded below.

Choosing a sequence $\{\varepsilon\}\to 0$ we thus obtain a sequence $\{R\}$ tending to $+\infty$, where $R = q_\varepsilon R_\varepsilon$ and a sequence of currents $\{j_R\}$, where $j_R = j_\varepsilon$ such that (4.7) is satisfied — this is

(4.90) — and $\limsup_{R\to\infty} \frac{W(j_R, \mathbf{1}_{K_R})}{|K_R|} \le \int W_K(j)\, dP(j)$ — this is (4.92). Moreover from (4.89) we have $P_R \to P$, using the notations of Corollary 4.5. Thus Corollary 4.5 is proved.

Part II
From Ginzburg-Landau to the renormalized energy

5 The energy-splitting formula and the blow up

In this section, we return to the Ginzburg-Landau energy and prove an algebraic splitting formula on it already discussed in Section 1.8, as well as results on the splitting function. We recall that if $\lim_{\varepsilon\to 0} h_{\rm ex}/|\log\varepsilon| > \lambda_\Omega$, then $h_{0,\varepsilon}$ may be used as the splitting function. Only when the limit is equal to λ_Ω does one need to use $h_{\varepsilon,N}$ with $N \neq N_0$ instead.

We recall $h_{0,\varepsilon}$ is the minimizer of (1.35) and $h_{\varepsilon,N}$ is given by (1.40). We also introduce the notation of the appendix: for $m \in (-\infty, 1]$, H_m denotes the minimizer of

$$\min_{h-1\in H^1_0(\Omega)} (1-m)\int_\Omega |-\Delta H + H| + \frac{1}{2}\int_\Omega |\nabla H|^2 + |H-1|^2. \tag{5.1}$$

H_m is the solution of an obstacle problem and its properties are studied in the appendix. In particular $H_m \ge m$ and $-\Delta H_m + H_m = m\mathbf{1}_{\omega_m}$, where $\omega_m = \{H_m = m\}$ is the so-called coincidence set. We have

Lemma 5.1. *For any $0 \le N \le \frac{|\Omega| h_{\rm ex}}{2\pi}$ there exists a unique $m \in [1 - \frac{1}{2\lambda_\Omega}, 1]$ (and conversely) such that $h_{\varepsilon,N} = h_{\rm ex} H_m$, and m and N are related by $2\pi N = h_{\rm ex} m|\omega_m|$. Moreover m and $|\omega_m|$ are continuous increasing functions of N.*

Proof. The minimization problem (1.40) has a unique minimizer $h_{\varepsilon,N}$ by convexity. On the other hand, by the theory of Lagrange multipliers, $h_{\varepsilon,N}$ is the minimizer of

$$\min_{h-h_{\rm ex}\in H^1_0(\Omega)} \frac{1}{2}\int_\Omega |\nabla h|^2 + |h-h_{\rm ex}|^2 + \lambda \int_\Omega |-\Delta h + h|$$

for some number λ characterized by the fact that the unique minimizer satisfies $\int_\Omega |-\Delta h + h| = 2\pi N$. But this minimizer is precisely $h_{\rm ex} H_m$ with m such that $2\pi N = h_{\rm ex} m|\omega_m|$ hence $h_{\varepsilon,N} = h_{\rm ex} H_m$. From Proposition A.1, $m \mapsto |\omega_m|$ is continuous increasing and one to one from $I = [1 - \frac{1}{2\lambda_\Omega}, 1]$ to $[0, |\Omega|]$, hence if $2\pi N$ is between 0 and $h_{\rm ex}|\Omega|$, then there exists a unique $m \in I$ such that $h_{\varepsilon,N} = h_{\rm ex} H_m$, and m is a continuous increasing function of $2\pi N/h_{\rm ex}$ — hence of N — characterized by $m|\omega_m| = 2\pi N h_{\rm ex}$, and obviously $|\omega_m|$ is too. ☐

We will denote by $m_{\varepsilon,N}$ the m corresponding to $h_{\varepsilon,N}$, and note that

$$0 < 1 - \frac{1}{2\lambda_\Omega} \le m_{\varepsilon,N} \le 1. \tag{5.2}$$

It follows from Lemma 5.1 that $h_{\varepsilon,N}$ is also the solution to an obstacle problem, hence $h_{\varepsilon,N} \in C^{1,1}$ (see [Fr]) and satisfies $h_{\rm ex} m_{\varepsilon,N} \le h_{\varepsilon,N} \le h_{\rm ex}$ and

$$\mu_{\varepsilon,N} := -\Delta h_{\varepsilon,N} + h_{\varepsilon,N} = m_{\varepsilon,N} h_{\rm ex} \mathbf{1}_{\omega_{\varepsilon,N}}, \quad \text{where} \quad \omega_{\varepsilon,N} = \{h_{\varepsilon,N} = h_{\rm ex} m_{\varepsilon,N}\}.$$

Note that $\mu_{\varepsilon,N}(\Omega) = 2\pi N$.

We will also let

$$c_{\varepsilon,N} := h_{\rm ex}(m_{\varepsilon,N} - m_{0,\varepsilon}) = h_{\rm ex} m_{\varepsilon,N} - h_{\rm ex} + \frac{1}{2}|\log \varepsilon'|. \tag{5.3}$$

It is immediate from (1.48), (5.2) that

$$c_{\varepsilon,N} = O(h_{\rm ex}).$$

The minimizer $h_{0,\varepsilon}$ of (1.35) is equal to h_{ε,N_0}, where N_0 is given by (1.38). Moreover, we recall (see (1.36)) that $h_{0,\varepsilon} = h_{\rm ex} - \frac{1}{2}|\log \varepsilon'|$ on its coincidence set, hence $h_{\rm ex} m_{\varepsilon,N_0} = h_{\rm ex} - \frac{1}{2}|\log \varepsilon'|$ and $c_{\varepsilon,N_0} = 0$.

On the other hand H_m is increasing with respect to m, see Proposition A.1, hence if $m_1 \le m_2$ then $H_{m_1} \le H_{m_2}$ (see Proposition A.1 in the appendix) so

$$c_{\varepsilon,N} \ge 0 \quad \text{if } N \ge N_0 \qquad c_{\varepsilon,N} \le 0 \quad \text{if } N \le N_0. \tag{5.4}$$

We will be most interested in the cases where N is one of the two integers closest to N_0.

5.1 Energy-splitting

Let h_μ, a "splitting function", be any function such that $\mu := -\Delta h_\mu + h_\mu \in L^2(\Omega)$ and $h_\mu = h_{\rm ex}$ on $\partial\Omega$. Let

$$A_1 = A - \nabla^\perp h_\mu. \tag{5.5}$$

Then,

Lemma 5.2. *For any (u, A) and h_μ as above we have*

$$G_\varepsilon(u, A) = \frac{1}{2}\|h_\mu - h_{\rm ex}\|^2_{H^1(\Omega)} + \frac{1}{2}\int_\Omega \left(|u|^2 - 1\right)|\nabla h_\mu|^2 + \tag{5.6}$$

$$+ \frac{1}{2}\int_\Omega |\nabla_{A_1} u|^2 + (\operatorname{curl} A_1 - \mu)^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2} + \int_\Omega (h_\mu - h_{\rm ex})\left(\mu(u, A_1) - \mu\right).$$

Proof. From (5.5), we have

$$|\nabla_A u|^2 = |\nabla_{A_1} u|^2 - 2\nabla^\perp h_\mu \cdot j(u, A_1) + |u|^2 |\nabla h_\mu|^2,$$

where we have used the notation $j(u, A) = (iu, \nabla_A u)$. Also, since $\operatorname{curl} A = \operatorname{curl} A_1 + \Delta h_\mu = \operatorname{curl} A_1 - \mu + h_\mu$, we may write

$$(\operatorname{curl} A - h_{\rm ex})^2 = (h_\mu - h_{\rm ex})^2 + 2(\operatorname{curl} A_1 - \mu)(h_\mu - h_{\rm ex}) + (\operatorname{curl} A_1 - \mu)^2.$$

Replacing in (1.22) and integrating by parts the term $\nabla^\perp h_\mu \cdot j(u,A_1)$, using the fact that $h_\mu = h_{\rm ex}$ on $\partial\Omega$, we find

$$G_\varepsilon(u,A) = \frac{1}{2}\|h_\mu - h_{\rm ex}\|^2_{H^1(\Omega)} + \frac{1}{2}\int_\Omega \left(|u|^2-1\right)|\nabla h_\mu|^2 +$$

$$+\frac{1}{2}\int_\Omega |\nabla_{A_1} u|^2 + (\operatorname{curl} A_1 - \mu)^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2} + \int_\Omega (h_\mu - h_{\rm ex})\left(\operatorname{curl} j(u,A_1) + \operatorname{curl} A_1 - \mu\right).$$

This yields (5.6), using the fact that $\mu(u,A_1) = \operatorname{curl} j(u,A_1) + \operatorname{curl} A_1$. □

Using the particular choice of splitting function $h_\mu = h_{\varepsilon,N}$ in Lemma 5.2, we obtain the following result:

Proposition 5.3. *Let $0 \le N \le \frac{h_{\rm ex}|\Omega|}{2\pi}$ and let $h_{\varepsilon,N}$ be the corresponding minimizer of* (1.40)*, then for any (u,A), denoting $A_{1,\varepsilon} = A - \nabla^\perp h_{\varepsilon,N}$, and using the notation* (1.41) *we have*

$$G_\varepsilon(u,A) = G_\varepsilon^N + F_\varepsilon(u,A_{1,\varepsilon}) - \frac{1}{2}\int_\Omega (1-|u|^2)|\nabla h_{\varepsilon,N}|^2, \tag{5.7}$$

where

$$F_\varepsilon(u,A) = \frac{1}{2}\int_\Omega |\nabla_A u|^2 + (\operatorname{curl} A - \mu_{\varepsilon,N})^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2} + \int_\Omega (h_{\varepsilon,N} - h_{\rm ex} - c_{\varepsilon,N})\mu(u,A) + c_{\varepsilon,N}\int_\Omega (\mu(u,A) - \mu_{\varepsilon,N}). \tag{5.8}$$

Proof. In (5.6), we replace $\int_\Omega (h_{\varepsilon,N} - h_{\rm ex})\left(\mu(u,A_{1,\varepsilon}) - \mu_{\varepsilon,N}\right)$ by using the fact that by definition of $h_{\varepsilon,N}$, on the support of $\mu_{\varepsilon,N}$, $h_{\varepsilon,N} - h_{\rm ex} = h_{\rm ex} m_{\varepsilon,N} - h_{\rm ex} = -\frac{1}{2}|\log\varepsilon'| + c_{\varepsilon,N}$ and $\int_\Omega \mu_{\varepsilon,N} = 2\pi N$. □

Note that the functional F_ε depends on N but for simplicity we will not denote that dependence.

5.2 Dependence on N

We have the following

Lemma 5.4. *For $N \in [0, \frac{|\Omega| h_{\rm ex}}{2\pi}]$ we have*

$$\frac{dG_\varepsilon^N}{dN} = 2\pi h_{\rm ex}(m - m_{0,\varepsilon}) \tag{5.9}$$

where m is the one-to-one function of N given by Lemma 5.1. Moreover, G_ε^N is minimized uniquely at N_0 and minimized among integers at N_0^- or N_0^+ (or both) where N_0^- is the largest integer below N_0 and N_0^+ the smallest integer above.

Proof. From (1.41) and Lemma 5.1 we have

$$G_\varepsilon^N = \pi N|\log\varepsilon'| + \frac{1}{2}{h_{\rm ex}}^2\|H_m - 1\|^2_{H^1(\Omega)} \tag{5.10}$$

where m and N are related by $2\pi N = h_{\rm ex} m|\omega_m|$. On the other hand, since H_m minimizes (5.1) we have

$$(5.11)\qquad (1-m)\frac{2\pi N}{h_{\rm ex}} + \frac{1}{2}\|H_m - 1\|^2_{H^1(\Omega)} \le (1-m)\frac{2\pi N'}{h_{\rm ex}} + \frac{1}{2}\|H_{m'} - 1\|^2_{H^1(\Omega)}$$

where $2\pi N' = h_{\rm ex} m'|\omega_{m'}|$. Reversing the roles of m and m' and taking $m' = m+\delta$ with $\delta \to 0$, we obtain

$$\frac{d}{dm}\|H_m - 1\|^2_{H^1(\Omega)} = \frac{4\pi}{h_{\rm ex}}(m-1)\frac{dN}{dm}$$

(in the sense of BV derivatives). Inserting into (5.10) we deduce

$$\frac{dG_\varepsilon^N}{dN} = \pi|\log\varepsilon'| + 2\pi h_{\rm ex}(m-1) = 2\pi h_{\rm ex}\left(m - 1 + \frac{|\log\varepsilon'|}{2h_{\rm ex}}\right)$$

hence the result (5.9) in view of (1.36). But $m_{0,\varepsilon}$ is the m that corresponds to N_0. It immediately follows with the monotonicity of $N \mapsto m$ that G_ε^N is decreasing in $[0, N_0]$, increasing in $[N_0, \frac{|\Omega|h_{\rm ex}}{2\pi}]$, hence minimized at N_0, and minimized among integers at N_0^- or N_0^+. □

For the rest of Section 5 and Section 6, $N \in \{N_0^-, N_0^+\}$**.** Since $2\pi N_0 = \int_\Omega \mu_{0,\varepsilon} \le |\Omega|\left(h_{\rm ex} - \frac{1}{2}|\log\varepsilon'|\right)$, see (1.37), it is clear that $N_0^- \le N_0^+ < \frac{|\Omega|h_{\rm ex}}{2\pi}$ so Lemma 5.1 applies and the corresponding $h_{\varepsilon,N}$ are well-defined. We also record the following

Lemma 5.5. *If $N_0 \gg 1$ and $N \in \{N_0^-, N_0^+\}$ then*

$$(5.12)\qquad \lim_{\varepsilon\to 0}\frac{c_{\varepsilon,N}}{|\log\varepsilon'|} = 0 \qquad \lim_{\varepsilon\to 0} m_{\varepsilon,N} = m_\lambda,$$

where $\lambda = \lim_{\varepsilon\to 0}\frac{h_{\rm ex}}{|\log\ \varepsilon|}$ and $m_\lambda = 1 - \frac{1}{2\lambda}$.

Proof. First we recall that $m_{0,\varepsilon} = 1 - \frac{|\log\varepsilon'|}{2h_{\rm ex}}$, hence by definition of λ, we have $\lim_{\varepsilon\to 0} m_{0,\varepsilon} = m_\lambda$.

Assume that $N = N_0^+$. We use the fact, seen in Lemma 5.1, that m and $|\omega_m|$ are increasing functions of N. Thus, since $N_0 \le N \le N_0 + 1$, we have $|\omega_{\varepsilon,N}| \ge |\omega_{0,\varepsilon}|$ and, using

$$0 \le m_{\varepsilon,N} - m_{0,\varepsilon} = \frac{2\pi N_0^+}{h_{\rm ex}|\omega_{\varepsilon,N}|} - \frac{2\pi N_0}{h_{\rm ex}|\omega_{0,\varepsilon}|} \le \frac{2\pi N_0^+}{h_{\rm ex}|\omega_{0,\varepsilon}|} - \frac{2\pi N_0}{\le |\omega_{0,\varepsilon}|} \le \frac{2\pi}{h_{\rm ex}|\omega_{0,\varepsilon}|} = \frac{m_{0,\varepsilon}}{N_0} \le \frac{1}{N_0},$$

because we always have $m \le 1$. Since we are in the regime $N_0 \gg 1$, we find that $m_{\varepsilon,N}$ and $m_{0,\varepsilon}$ have the same limit, that is m_λ. The case $N = N_0^-$ is treated analogously, and we deduce in both cases

$$(5.13)\qquad |m_{\varepsilon,N} - m_{0,\varepsilon}| \le O(\frac{1}{N_0}).$$

Assume then that $h_{\rm ex} \le O(|\log\ \varepsilon|)$, from (5.3) and (5.13), we deduce $c_{\varepsilon,N} = o(h_{\rm ex})$, which proves (5.12). If $h_{\rm ex} \gg |\log\ \varepsilon|$, then $m_{0,\varepsilon} \sim 1$ and $2\pi N_0 \sim h_{\rm ex}|\Omega|$ hence combining with (5.3) and (5.13), we find $c_{\varepsilon,N} = O(1)$ which also implies the result in this case. □

5.3 Blow-up procedure

We write for simplicity ω_ε instead $\omega_{\varepsilon,N}$ and m_ε instead of $m_{\varepsilon,N}$, when the precise value of $N \in \{N_0^-, N_0^+\}$ does not need to appear explicitly.

Assume (1.48) holds. Let

$$\ell_\varepsilon = \frac{1}{\sqrt{h_{\text{ex}}}},$$

and, assuming $0 \in \omega_\varepsilon \subset \Omega$, write $x = \ell_\varepsilon x'$. Under this change of coordinates the domain Ω becomes Ω'_ε and the subdomain ω_ε becomes ω'_ε, both becoming infinitely large as $\varepsilon \to 0$ — this is obvious for Ω'_ε and for ω'_ε, it is proven to be a consequence of (1.48) in Proposition 5.6 below. We call F'_ε the expression of $F_\varepsilon(u, A_{1,\varepsilon})$ (see (5.8)) in terms of the rescaled unknowns $u'(x') = u(x)$ and $A'(x') = \ell_\varepsilon A_{1,\varepsilon}(x)$. It is given by

$$(5.14)\quad F'_\varepsilon(u', A') = \frac{1}{2}\int_{\Omega'_\varepsilon} |\nabla_{A'} u'|^2 + \frac{1}{{\ell_\varepsilon}^2}|\text{curl}\, A' - m_\varepsilon \mathbf{1}_{\omega'_\varepsilon}|^2 + \frac{(1-|u'|^2)^2}{2(\varepsilon')^2} - \int_{\Omega'_\varepsilon} {\zeta_\varepsilon}' \mu(u', A') + c_{\varepsilon,N}\int_{\Omega'_\varepsilon} (\mu(u', A') - m_\varepsilon \mathbf{1}_{\omega'_\varepsilon}),$$

where ε', ${\zeta_\varepsilon}'$ are given by

$$(5.15)\qquad \varepsilon' = \frac{\varepsilon}{\ell_\varepsilon}, \quad {\zeta_\varepsilon}'(x') = h_{\text{ex}} - h_{\varepsilon,N}(x) + c_{\varepsilon,N}.$$

We also define the blown-up current in Ω'_ε

$$(5.16)\qquad j'_\varepsilon = \text{curl}\,(iu', \nabla_{A'} u')$$

and the blow-up measure $\mu'_\varepsilon = \mu(u'_\varepsilon, A'_\varepsilon)$ and extend them by 0 outside Ω'_ε. Note that $j'_\varepsilon(x') = \ell_\varepsilon j_{1,\varepsilon}(\ell_\varepsilon x')$ if $j_{1,\varepsilon}$ denotes the current $(iu_\varepsilon, \nabla_{A_{1,\varepsilon}} u_\varepsilon)$ in the original variables.

The function $h_{\varepsilon,N}$ is in $C^{1,1}(\Omega)$ and equal to h_{ex} on $\partial\Omega$. It attains its minimum $h_{\text{ex}} m_{\varepsilon,N}$ on $\omega_{\varepsilon,N}$. From (5.3), (5.15)

$$c_{\varepsilon,N} = \min_\Omega (h_{\varepsilon,N} - h_{\text{ex}}) + \frac{1}{2}|\log \varepsilon'|, \quad {\zeta_\varepsilon}'(x') = \frac{1}{2}|\log \varepsilon'| - h_{\varepsilon,N}(x) + \min_\Omega h_{\varepsilon,N},$$

thus ${\zeta_\varepsilon}' \in C^{1,1}(\Omega'_\varepsilon)$, ${\zeta_\varepsilon}' = c_{\varepsilon,N}$ on $\partial\Omega'_\varepsilon$ and ${\zeta_\varepsilon}'$ attains its maximum on ω'_ε. Moreover

$$(5.17)\qquad \max_{\Omega'_\varepsilon} {\zeta_\varepsilon}' = \frac{1}{2}|\log \varepsilon'|, \quad \|\nabla {\zeta_\varepsilon}'\|_\infty \le C\sqrt{|\log \varepsilon'|},$$

this last assertion following from (5.20) in Proposition 5.6 below.

5.4 Additional results on the splitting function

In this subsection, we adapt some results from the appendix that we will need below. In the appendix we introduce an ellipse E_Q of measure 1 and a function $U_Q \ge 0$ defined in $\mathbb{R}^2$ such that

$$\Delta U_Q = \frac{\Delta Q}{2}\mathbf{1}_{\mathbb{R}^2 \setminus E_Q}, \quad \{U_Q = 0\} = E_Q,$$

where Q is the quadratic form $D^2 h_0(x_0)$ as introduced in (1.31).

The following proposition can be applied to $N \in \{N_0, N_0^-, N_0^+\}$. In either case we denote by ω_ε the associated coincidence set, and by ω'_ε its blow-up.

Proposition 5.6. *Assume* (1.48) *holds. Let* $h_{\varepsilon,N}$ *be as above with* $|N - N_0| \le 1$.

1. *We have* $N_0 \gg 1$, *and if* $h_{\text{ex}} - H_{c_1} \ll |\log \varepsilon|$ *then* $N_0 \ll h_{\text{ex}}$ *and*

$$h_{\text{ex}} - H_{c_1} \sim \lambda_\Omega N_0 \log \frac{h_{\text{ex}}}{N_0}. \tag{5.18}$$

2. *For every* $x \in \Omega$

$$d(x, \omega_\varepsilon) \le C\sqrt{\frac{|\log \varepsilon'|}{h_{\text{ex}}}}, \quad |\Omega \backslash \omega_\varepsilon| \le C\sqrt{\frac{|\log \varepsilon'|}{h_{\text{ex}}}}. \tag{5.19}$$

Moreover

$$\|\nabla h_{\varepsilon,N}\|_{L^\infty(\Omega)} \le C\sqrt{h_{\text{ex}} |\log \varepsilon'|}. \tag{5.20}$$

3. *If* K *is any compact subset of* $(\lambda_\Omega, +\infty)$,

$$\lim_{\delta\to 0} \frac{|\{x \mid d(x, \partial\omega_\varepsilon) < \delta\}|}{|\omega_\varepsilon|} = 0, \quad \lim_{\delta\to 0} \frac{|\{h_{\text{ex}} m_{\varepsilon,N} < h_{\varepsilon,N} < h_{\text{ex}} m_{\varepsilon,N} + \delta h_{\text{ex}}\}|}{|\omega_\varepsilon|} = 0 \tag{5.21}$$

uniformly with respect to ε *such that* $\frac{h_{\text{ex}}}{|\log\ \varepsilon|} \in K$.

4. *If* $h_{\text{ex}}/|\log\ \varepsilon| \to \lambda_\Omega$ *then there exists* $\{L_\varepsilon\}_\varepsilon$ *such that for any* $\delta, M > 0$ *and if* ε *is small enough*

$$\begin{gathered} \left\{\frac{h_{\varepsilon,N} - \min_\Omega h_{\varepsilon,N}}{h_{\text{ex}} {L_\varepsilon}^2} \ge M\right\} \subset x_0 + L_\varepsilon\{U_Q \ge M - \delta\}, \\ \left\{\frac{h_{\varepsilon,N} - \min_\Omega h_{\varepsilon,N}}{h_{\text{ex}} {L_\varepsilon}^2} \le M\right\} \subset x_0 + L_\varepsilon\{U_Q \le M + \delta\}, \\ \{d(x, {\omega_Q}^c) > \delta L_\varepsilon\} \subset \omega_\varepsilon \subset \{d(x, \omega_Q) < \delta L_\varepsilon\}, \end{gathered} \tag{5.22}$$

where $\omega_Q = x_0 + L_\varepsilon E_Q$ *and* x_0 *is defined in* (1.31). *Moreover there is a constant* C_Ω *depending only on* Ω *such that*

$$L_\varepsilon^2 |\log L_\varepsilon| \sim C_\Omega \frac{h_{\text{ex}} - \lambda_\Omega |\log\ \varepsilon|}{h_{\text{ex}}}. \tag{5.23}$$

In particular $\frac{\log\ |\log\ \varepsilon|}{|\log\ \varepsilon|} \ll L_\varepsilon^2 |\log L_\varepsilon|$ *and* $\ell_\varepsilon \ll L_\varepsilon$, *and from* (5.22)

$$L_\varepsilon^2 |\omega_Q| \sim |\omega_\varepsilon|, \quad \{x \mid d(x, \partial\omega_\varepsilon) \le \delta\ell_\varepsilon\} \ll |\omega_\varepsilon|.$$

5. *For any* $R > 0$,

$$\big|\{x \mid d(x, (\omega'_\varepsilon)^c) > R\}\big| \sim |\omega'_\varepsilon| \tag{5.24}$$

as $\varepsilon \to 0$, *i.e.* ω'_ε *satisfies* (1.14).

6. *We have*

$$G_\varepsilon^N = G_\varepsilon^{N_0} + O\left(\frac{h_{\rm ex}}{N_0}\right) \le C h_{\rm ex} |\log\varepsilon'|. \tag{5.25}$$

If $|\log\,\varepsilon|^4 \ll h_{\rm ex} \ll \frac{1}{\varepsilon^2}$*,*

$$G_\varepsilon^N = \frac{1}{2} h_{\rm ex} |\Omega| |\log\varepsilon'| + o(h_{\rm ex}). \tag{5.26}$$

Proof. Proof of item 1: We apply Proposition A.1 to $m_{0,\varepsilon} = 1 - \frac{|\log\varepsilon'|}{2h_{\rm ex}}$. We denote by L_ε the L_m corresponding to $m_{0,\varepsilon}$ given by Proposition A.1. In particular we have $|\omega_{0,\varepsilon}| \sim L_\varepsilon^2$. In that proposition $\underline{h}_0$ denotes $\min h_0$ and is equal to $1 - \frac{1}{2\lambda_\Omega}$. Moreover $h_{\rm ex} - \lambda_\Omega |\log\,\varepsilon| \gg \log\,|\log\,\varepsilon|$ is equivalent to $\frac{1}{2(1-m_{0,\varepsilon})} - \frac{1}{2(1-\underline{h}_0)} \gg \frac{\log\,|\log\,\varepsilon|}{|\log\,\varepsilon|}$ which in turn is equivalent to $m_{0,\varepsilon} - \underline{h}_0 \gg \frac{\log\,|\log\,\varepsilon|}{|\log\,\varepsilon|}$. From (A.7)

$$L_\varepsilon^2 |\log L_\varepsilon| \sim 2\pi(m_{0,\varepsilon} - \underline{h}_0)/\underline{h}_0 = \frac{2\pi}{\underline{h}_0}\left(\frac{1}{2\lambda_\Omega} - \frac{|\log\varepsilon'|}{2h_{\rm ex}}\right). \tag{5.27}$$

Then, since $|\omega_{0,\varepsilon}| \sim L_\varepsilon^2$ and $N_0 = h_{\rm ex} m_{0,\varepsilon} |\omega_{0,\varepsilon}|$ we have $L_\varepsilon^2 |\log L_\varepsilon| \sim \frac{1}{2} |\omega_{0,\varepsilon}| |\log|\omega_{0,\varepsilon}|| \sim \frac{1}{2}\frac{N_0}{m_{0,\varepsilon} h_{\rm ex}} |\log \frac{N_0}{m_{0,\varepsilon} h_{\rm ex}}|$ and inserting into (5.27) we find

$$\frac{1}{2}\frac{N_0}{m_{0,\varepsilon} h_{\rm ex}} |\log \frac{N_0}{m_{0,\varepsilon} h_{\rm ex}}| \sim \frac{2\pi}{\underline{h}_0}\left(\frac{1}{2\lambda_\Omega} - \frac{|\log\varepsilon'|}{2h_{\rm ex}}\right) = \frac{\pi}{\lambda_\Omega \underline{h}_0 h_{\rm ex}}\left(h_{\rm ex} - \lambda_\Omega |\log\,\varepsilon| + \frac{1}{2}\log h_{\rm ex}\right). \tag{5.28}$$

It follows, since $m_{0,\varepsilon} \ge \underline{h}_0 > 0$, that if (1.48) holds, we must have $N_0 \gg 1$, for otherwise the left-hand side is $O(\log\,|\log\,\varepsilon|)$ while the right-hand side is $\gg \log\,|\log\,\varepsilon|$. Moreover, if $h_{\rm ex} - H_{c_1} \ll |\log\,\varepsilon|$ then $m_{0,\varepsilon} \sim 1 - \frac{1}{2\lambda_\Omega} = \underline{h}_0$ and inserting into (5.28) and rearranging terms, we obtain $N_0 \ll h_{\rm ex}$ and (5.18).

Proof of item 2. From (1.36), $m_{0,\varepsilon} = 1 - \frac{|\log\varepsilon'|}{2h_{\rm ex}}$ hence using Proposition A.1 we have $\|\nabla h_{0,\varepsilon}\|_\infty = h_{\rm ex} \|\nabla H_{m_{0,\varepsilon}}\|_\infty \le C\sqrt{h_{\rm ex}|\log\varepsilon'|}$. If $|N - N_0| \le 1$ then from (5.12) we have $|m_{0,\varepsilon} - m_{\varepsilon,N}| \ll |\log\varepsilon'|/h_{\rm ex}$ and therefore $m_{\varepsilon,N} \le 1 - \frac{|\log\varepsilon'|}{h_{\rm ex}}$ for ε small enough. Proposition A.1 applied to $m_{\varepsilon,N}$ yields the same bound for $\nabla h_{\varepsilon,N}$, and (5.19) follows similarly from Proposition A.1 using $m_{\varepsilon,N} \le 1 - \frac{|\log\varepsilon'|}{h_{\rm ex}}$.

Proof of item 3. First note that if $h_{\rm ex}/|\log\,\varepsilon|$ is bounded above as $\varepsilon \to 0$ then $|\log\,\varepsilon| \sim |\log\varepsilon'|$ hence $h_{\rm ex}/|\log\varepsilon'|$ is bounded as well. Moreover, since $|\log\varepsilon'| \le |\log\,\varepsilon|$, if $h_{\rm ex}/|\log\,\varepsilon| \ge \alpha > \lambda_\Omega$ then $h_{\rm ex}/|\log\varepsilon'| \ge \alpha$ hence if $h_{\rm ex}/|\log\,\varepsilon|$ belongs to a compact subset K of $(\lambda_\Omega, +\infty)$, then $h_{\rm ex}/|\log\varepsilon'|$ belong to a (different) compact subset K' of $(\lambda_\Omega, +\infty)$. But, from (1.36), if this is the case then $m_{0,\varepsilon}$ belongs to a compact subset $\tilde{K}$ of $(\underline{h}_0, 1)$. Then from (5.12) and if ε is small enough the same is true of $m_{\varepsilon,N}$, and the result follows from Proposition A.1 applied to $m_{\varepsilon,N}$.

Proof of items 4,5. This is again Proposition A.1. Indeed $h_{\rm ex}/|\log\,\varepsilon| \to \lambda_\Omega$ implies that $m_{0,\varepsilon} \to \underline{h}_0 = 1 - \frac{1}{2\lambda_\Omega}$ hence $\lim_{\varepsilon\to 0} m_{\varepsilon,N} = \underline{h}_0$ and (A.7) holds, and for any $\delta, M > 0$, (A.8) also if ε is small enough. It is easy to check using (5.12), (1.36) and (A.3) that (A.7) implies (5.23), and (5.22) is (A.8) since $h_{\varepsilon,N} - \min_\Omega h_{\varepsilon,N} = h_{\rm ex}(H_{m_{\varepsilon,N}} - m_{\varepsilon,N})$.

To prove (5.24) we distinguish the case $m_{\varepsilon,N} \to \underline{h}_0$ from the case where $m_{\varepsilon,N}$ is bounded away from $\underline{h}_0$. In the latter and using Proposition A.1, $|\omega_{\varepsilon,N}|$ is bounded away from 0 and

$$\frac{|\{x \mid d(x, (\omega'_\varepsilon)^c) > R\}|}{|\omega'_\varepsilon|} = \frac{|\omega_R|}{|\omega_\varepsilon|}, \quad \text{where } \omega_R = \{x \mid d(x, {\omega_\varepsilon}^c) > R\ell_\varepsilon\}.$$

Since $R\ell_\varepsilon \to 0$ we have $|\omega_R| - |\omega_\varepsilon| \to 0$, proving (5.24) in this case.

If $m_{\varepsilon,N} \to \underline{h}_0$ then $m_{0,\varepsilon} \to \underline{h}_0$, i.e. $h_{\text{ex}}/|\log \varepsilon'| \to \lambda_\Omega$, or equivalently $h_{\text{ex}}/|\log\, \varepsilon| \to \lambda_\Omega$. Then $L_\varepsilon \gg \ell_\varepsilon$ and therefore, using (5.22),

$$\{d(x, {\omega_Q}^c) > \delta L_\varepsilon\} \subset \omega_R \subset \{d(x, \omega_Q) < \delta L_\varepsilon\}$$

holds for any $R, \delta > 0$ if ε is small enough. It follows, since $|\omega_Q| = L_\varepsilon^2$, that

$$|\omega_R| \sim |\omega_Q|$$

as $\varepsilon \to 0$ for any $R > 0$, which implies in particular (5.24).

Proof of item 6. Combining (5.9) with (5.13) it follows that

$$|G_\varepsilon^N - G_\varepsilon^{N_0}| = \left| \int_{N_0}^N 2\pi h_{\text{ex}}(m - m_{0,\varepsilon}) \right| \le \frac{2\pi h_{\text{ex}}}{N_0}$$

where we have used the fact that $N \mapsto m$ is increasing and $|N - N_0| \le 1$.

The upper bound part of (5.26) follows from (5.25) by noting that

$$|\log \varepsilon'| \int \mu_{0,\varepsilon} + \|h_{0,\varepsilon} - h_{\text{ex}}\|_{H^1}^2 \le |\Omega| |\log \varepsilon'|,$$

which follows from the minimality of $H_{m_{0,\varepsilon}}$ in (5.1), using 1 as a test function. Finally, for the lower bound part of (5.26) we note that from (5.19)

$$\begin{aligned} |\log \varepsilon'| \int \mu_{0,\varepsilon} + \|h_{0,\varepsilon} - h_{\text{ex}}\|_{H^1}^2 &\ge |\log \varepsilon'| |\omega_{0,\varepsilon}| m_{0,\varepsilon} \\ &\ge |\log \varepsilon'| \left(|\Omega| - C\sqrt{\frac{|\log \varepsilon'|}{h_{\text{ex}}}} \right) \left(1 - \frac{|\log \varepsilon'|}{2h_{\text{ex}}} \right) \\ &\ge |\log \varepsilon'| |\Omega| + o(h_{\text{ex}}), \end{aligned}$$

if $h_{\text{ex}} \gg |\log\, \varepsilon|^4$. □

Remark 5.7. *In* (5.18)*, we have recovered the formula (9.88) from [SS4].*

5.5 A priori bounds

Lemma 5.8. *If* $(u_\varepsilon, A_\varepsilon)$ *minimizes* G_ε *then*

$$G_\varepsilon(u_\varepsilon, A_\varepsilon) \le C h_{\text{ex}} |\log \varepsilon'|. \tag{5.29}$$

Moreover, for any $(u_\varepsilon, A_\varepsilon)$ *satisfying (5.29) we have, if* $N \in \{N_0^-, N_0^+\}$*,*

$$G_\varepsilon(u_\varepsilon, A_\varepsilon) = G_\varepsilon^N + F_\varepsilon(u_\varepsilon, A_{1,\varepsilon}) + o(N), \tag{5.30}$$

where F_ε *is defined in* (5.8)*.*

Proof. The a priori bound (5.29) is a consequence of the upper bound in Theorem 7, see Corollary 7.1. For the other relation we observe that, since $\nabla h_{\varepsilon,N}$ is supported in $\Omega \setminus \omega_\varepsilon$, it follows from (5.20)–(5.19) that

$$\int_\Omega |\nabla h_{\varepsilon,N}|^4 \le C{h_{\rm ex}}^{3/2}|\log \varepsilon'|^{5/2}. \tag{5.31}$$

We then claim that

$$\int_\Omega (1-|u|^2)|\nabla h_{\varepsilon,N}|^2 = o(h_{\rm ex}|\omega_{\varepsilon,N}|). \tag{5.32}$$

Indeed we have $\int_\Omega (1-|u|^2)^2 \le \varepsilon^2 G_\varepsilon(u,A) \le C\varepsilon^2 h_{\rm ex}|\log\varepsilon'|$. Applying the Cauchy-Schwarz inequality, it follows from (5.31) that

$$\int_\Omega (1-|u|^2)|\nabla h_{\varepsilon,N}|^2 \le C\varepsilon\sqrt{h_{\rm ex}|\log\varepsilon'|}\|\nabla h_{\varepsilon,N}\|_{L^4}^2 \le C\varepsilon {h_{\rm ex}}^{5/4}|\log\varepsilon'|^{7/4}.$$

If $h_{\rm ex} \le C|\log \varepsilon|$ this is $o(1)$ hence $o(N)$. If $|\log\varepsilon| \ll h_{\rm ex} \ll 1/\varepsilon^2$ then $h_{\rm ex}|\omega_{\varepsilon,N}| \sim h_{\rm ex}|\Omega|$ and this is also $o(h_{\rm ex}|\omega_{\varepsilon,N}|)$. (5.32) is proven, since we recall that $2\pi N = m_{\varepsilon,N}h_{\rm ex}|\omega_{\varepsilon,N}|$ with (5.2). Then (5.30) follows from (5.7).

□

We also note some consequences of the second Ginzburg-Landau equation (1.47).

Lemma 5.9. *Assume that* $(u_\varepsilon, A_\varepsilon)$ *satisfies* (1.47)*. Then, letting* $A_{1,\varepsilon} = A_\varepsilon - \nabla^\perp h_{\varepsilon,N}$ *and* $j_{1,\varepsilon} = j(u_\varepsilon, A_{1,\varepsilon})$*, we have*

$$\|\text{div } j_{1,\varepsilon}\|_{H^{-1}(\Omega)} = o(N)$$

$$\|\text{curl}\, j_{1,\varepsilon} - \mu(u_\varepsilon, A_\varepsilon) + \text{curl}\, A_{1,\varepsilon}\|_{H^{-1}(\Omega)} = o(N). \tag{5.33}$$

Proof. By definition of $j_{1,\varepsilon}$ and $A_{1,\varepsilon}$ we have $j_{1,\varepsilon} = (iu_\varepsilon, \nabla_{A_\varepsilon} u_\varepsilon) - |u_\varepsilon|^2\nabla^\perp h_{\varepsilon,N}$. It follows that div $j_{1,\varepsilon} =$ div $(iu_\varepsilon, \nabla_{A_\varepsilon}u_\varepsilon) +$ div $((1-|u_\varepsilon|^2)\nabla^\perp h_{\varepsilon,N})$. If (1.47) is satisfied then div $(iu_\varepsilon, \nabla_{A_\varepsilon}u_\varepsilon) = 0$ and div $j_{1,\varepsilon} =$ div $((1-|u_\varepsilon|^2)\nabla^\perp h_{\varepsilon,N})$. On the other hand, combining (5.32) with $|u_\varepsilon| \le 1$ we find

$$\int_\Omega (1-|u_\varepsilon|^2)^2|\nabla h_{\varepsilon,N}|^2 = o(N). \tag{5.34}$$

It follows that

$$\|\text{div } j_{1,\varepsilon}\|_{H^{-1}(\Omega)} = o(\sqrt{N}).$$

By direct calculation we have $\mu(u_\varepsilon, A_{1,\varepsilon}) - \mu(u_\varepsilon, A_\varepsilon) = \text{curl}\,((|u_\varepsilon|^2-1)\nabla^\perp h_{\varepsilon,N})$ and since $\mu(u, A_{1,\varepsilon}) = \text{curl}\, j_{1,\varepsilon} + \text{curl}\, A_{1,\varepsilon}$, (5.33) follows again from (5.34).

□

6 Proof of Theorem 4, lower bound

In this section we state the key result from [SS3] which we need in order to apply the framework of Section 2 to the minimization of the Ginzburg-Landau functional as explained in Section 1.8. In paragraphs 2 (resp. paragraph 3) we use it to derive the lower bound part of Theorems 4, 5 in the case of moderate (resp. high) applied fields. In paragraph 4 we prove Theorems 4 and 5 assuming the upper bound of Theorem 7, proven in Section 7.

6.1 Mass displacement

There are two problems which arise when trying to apply the abstract scheme described in Section 2 to the Ginzburg-Landau energy. The first, and less problematic one, is that F_ε is not translation invariant. Indeed $\mathbf{1}_{\omega'_\varepsilon}$ and $\zeta_\varepsilon{}'$ are constant only in the subdomain ω'_ε. Therefore we need to reduce to integrating on ω'_ε rather than Ω'_ε.

The second, more delicate problem, is that the integrand in F_ε is not positive, and not even expected to be bounded below uniformly as $\varepsilon \to 0$: Each vortex creates in F_ε two terms which get infinitely large as $\varepsilon \to 0$ and which balance each other. To capture the difference in the limit we need to absorb the negative part in the positive part to obtain an essentially positive integrand (this will by the way solve in essence the first issue). Note that the cancellation is not a pointwise cancellation of the different terms: While the negative contribution is very concentrated near each vortex, the positive one is more spread out.

The method, introduced in [SS3] is the same that we used in Proposition 4.9 in a simpler setting, so the reader can refer to that section for an idea of it.

Let us denote the free energy functional

$$G^0_\varepsilon(u,A) = \int_{\Omega_\varepsilon} e_\varepsilon, \quad e_\varepsilon = \frac{1}{2}|\nabla_A u|^2 + \frac{1}{2}(\operatorname{curl} A)^2 + \frac{1}{4\varepsilon^2}\left(1-|u|^2\right)^2, \tag{6.1}$$

where Ω_ε is a smooth domain depending on ε and large as $\varepsilon \to 0$.

We now state the result of [SS3] for the sake of completeness. It contains (in a slightly different form) Theorems 1 and 2, as well as Corollary 1.1 of [SS3].

First, f_+ and f_- will denote the positive and negative parts of a function or measure, both being positive functions or measures. If f and g are two measures, we will write $f \le g$ in the sense of measures to mean that $g - f$ is a positive measure.

For any set E in the plane, $\widehat{E}$ will denote the 1-tubular neighborhood of E in Ω_ε i.e.

$$\widehat{E} = \{x \in \Omega_\varepsilon, \text{dist }(x,E) \le 1\}.$$

This way

$$\widehat{\partial\Omega_\varepsilon} = \{x \in \Omega_\varepsilon, \text{dist }(x,\partial\Omega_\varepsilon) \le 1\}.$$

For any function v on Ω_ε we denote (notice the absolute value)

$$\widehat{v}(x) = \sup_{y \in B(x,1)\cap\Omega_\varepsilon} |v(y)|.$$

Note that here the choice of the number 1 is arbitrary.

Theorem 6 ([SS3]). *Let $\{\Omega_\varepsilon\}_{\varepsilon>0}$ be a family of bounded open sets in $\mathbb{R}^2$. Assume that $\{(u_\varepsilon, A_\varepsilon)\}_\varepsilon$, where $(u_\varepsilon, A_\varepsilon)$ is defined over Ω_ε, satisfies for some $0 < \beta < 1$ small enough*

$$G^0_\varepsilon(u_\varepsilon, A_\varepsilon) \le \varepsilon^{-\beta}. \tag{6.2}$$

Then for any small enough ε, there exists a measure g_ε defined over Ω_ε and a measure ν_ε depending only on u_ε of the form $2\pi\sum_i d_i\delta_{a_i}$ for some points $a_i \in \Omega_\varepsilon$ and some integers d_i such that, C denoting a generic constant independent of ε:

1. *We have*

$$\|\mu(u_\varepsilon, A_\varepsilon) - \nu_\varepsilon\|_{(C^{0,1}_0(\Omega_\varepsilon))^*} \le C\sqrt{\varepsilon}G_\varepsilon(u_\varepsilon, A_\varepsilon), \tag{6.3}$$

2. *The following inequality holds*

$$-C \le g_\varepsilon \le e_\varepsilon + \frac{1}{2}|\log\,\varepsilon|(\nu_\varepsilon)_-.$$

3. *For any measurable set $E \subset \Omega_\varepsilon$,*

$$(g_\varepsilon)_-(E) \le C\frac{e_\varepsilon(\widehat{E})}{|\log\,\varepsilon|}, \quad (g_\varepsilon)_+(E) \le Ce_\varepsilon(\widehat{E}). \tag{6.4}$$

4. *Letting*

$$f_\varepsilon = e_\varepsilon - \frac{1}{2}|\log\,\varepsilon|\nu_\varepsilon,$$

for every Lipschitz function ξ vanishing on $\partial\Omega_\varepsilon$ and every $0 < \eta \le 1$ we have

$$\int_{\Omega_\varepsilon} \xi\, d(f_\varepsilon - g_\varepsilon) \le C\int_{\Omega_\varepsilon} \widehat{\nabla\xi}\, \left[d|\nu_\varepsilon| + (\beta+\eta)\, d(g_\varepsilon)_+ + \frac{|\log\eta|^2}{\eta}\, dx\right] + C\int_{\widehat{\partial\Omega_\varepsilon}} \widehat{\xi} e_\varepsilon. \tag{6.5}$$

5. *For any measurable set $E \subset \Omega_\varepsilon$ and every $0 < \eta \le 1$ we have*

$$|\nu_\varepsilon|(E) \le C\left(\eta(g_\varepsilon)_+(\widehat{E}) + \frac{1}{\eta}|\widehat{E}| + \frac{e_\varepsilon(\widehat{E}\cap\widehat{\partial\Omega_\varepsilon})}{|\log\,\varepsilon|}\right), \quad |\nu_\varepsilon|(E) \le C\frac{e_\varepsilon(\widehat{E})}{|\log\,\varepsilon|}. \tag{6.6}$$

6. *Assuming $|u_\varepsilon| \le 1$ in Ω_ε, then for every ball B_R of radius R such that $B_{R+C} \subset \Omega_\varepsilon$ and every $p < 2$,*

$$\int_{B_R} |j_\varepsilon|^p \le C_p\left((g_\varepsilon)_+(B_{R+C}) + R^2\right).$$

7. *Assume $|u_\varepsilon| \le 1$, that dist $(0, \partial\Omega_\varepsilon) \to +\infty$ as $\varepsilon \to 0$ and that for any $R > 0$*

$$\limsup_{\varepsilon\to 0} g_\varepsilon(\mathbf{U}_R) < +\infty, \tag{6.7}$$

where $\mathbf{U}_R$ is any family satisfying (1.4)–(1.5).

Then, up to extraction of a subsequence and for any $p < 2$, the vorticities $\{\mu(u_\varepsilon, A_\varepsilon)\}_\varepsilon$ converge in $W^{-1,p}_{loc}(\mathbb{R}^2)$ to a measure ν of the form $2\pi\sum_{p\in\Lambda}\delta_p$, where Λ is a discrete subset of $\mathbb{R}^2$, the currents $\{j(u_\varepsilon, A_\varepsilon)\}_\varepsilon$ converge weakly in $L^p_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ to j, and the induced fields $\{\mathrm{curl}\, A_\varepsilon\}_\varepsilon$ converge weakly in $L^2_{\mathrm{loc}}(\mathbb{R}^2)$ to h which are such that

$$\mathrm{curl}\, j = \nu - h, \quad \text{in } \mathbb{R}^2.$$

8. *If we replace the assumption* (6.13) *by the stronger assumption*

$$\limsup_{\varepsilon\to 0} g_\varepsilon(\mathbf{U}_R) < CR^2,$$

where C is independent of R, then the limit j of the currents satisfies, for any $p < 2$,

$$\limsup_{R\to+\infty} \fint_{\mathbf{U}_R} |j|^p\, dx < +\infty. \tag{6.8}$$

Moreover for every family $\chi_{\mathbf{U}_R}$ satisfying (1.3) we have

$$\liminf_{\varepsilon\to 0} \int_{\mathbb{R}^2} \frac{\chi_{\mathbf{U}_R}}{|\mathbf{U}_R|}\, dg_\varepsilon \ge \left(\frac{W(j, \chi_{\mathbf{U}_R})}{|\mathbf{U}_R|} + \frac{1}{2}\fint_{\mathbf{U}_R} h^2 + \frac{\gamma}{2\pi}\fint_{\mathbf{U}_R} h\right) + o_R(1), \tag{6.9}$$

where γ is the constant in (1.52) *and $o_R(1)$ is a function tending to* 0 *as $R \to +\infty$.*

The main point is that g_ε is a modification of $f_\varepsilon = e_\varepsilon - \frac{1}{2}|\log \varepsilon|\nu_\varepsilon$ with a small error (measured by (6.5)) which, contrarily to f_ε, is bounded below according to item 2.

Now, using the notation of Section 5, we blow up at the scale $\ell_\varepsilon = \sqrt{h_{\text{ex}}}$, letting $x' = \ell_\varepsilon x$, $\varepsilon' = \varepsilon/\ell_\varepsilon$, and

$$u'_\varepsilon(x') = u_\varepsilon(x), \quad A'_\varepsilon(x') = \ell_\varepsilon(A_\varepsilon(x) - \nabla^\perp h_{\varepsilon,N}(x)). \tag{6.10}$$

Then we deduce from Theorem 6 applied in $\{\Omega'_\varepsilon\}_\varepsilon$ to $\{(u'_\varepsilon, A'_\varepsilon)\}_\varepsilon$

Proposition 6.1. *Assume that as* $\varepsilon \to 0$

$$\frac{h_{\text{ex}}}{|\log \varepsilon|} \to \lambda \in [\lambda_\Omega, +\infty], \quad \log|\log \varepsilon| \ll h_{\text{ex}} - \lambda_\Omega |\log \varepsilon| \quad \textit{and} \quad h_{\text{ex}} \leq \frac{1}{\varepsilon^\beta},$$

where β *is small enough, and that* $G_\varepsilon(u_\varepsilon, A_\varepsilon) \leq C h_{\text{ex}} |\log \varepsilon'|$. *We also assume that* (1.47) *holds. Then there exists* $N \in \{N_0^-, N_0^+\}$ *such that there exist measures* $\{g'_\varepsilon\}_\varepsilon$ *defined on* Ω'_ε *satisfying the following four properties, for any family* $\{\mathbf{U}_R\}_R$ *of sets satisfying* (1.4), (1.5) *and any family of functions* $\{\chi_{\mathbf{U}_R}\}_R$ *satisfying* (1.3).

1. g'_ε *is bounded below by a — not necessarily positive — constant* C *independent of* ε.
2. *Defining* F'_ε *as in* (5.14) *we have, writing* ω'_ε *for* $\omega'_{\varepsilon,N}$ *and letting* $\tilde\omega'_\varepsilon = \{x \mid d(x, {\omega'_\varepsilon}^c) \geq 2\}$,

$$\liminf_{\varepsilon\to 0} \frac{F'_\varepsilon(u'_\varepsilon, A'_\varepsilon) - g'_\varepsilon(\tilde\omega'_\varepsilon)}{|\omega'_\varepsilon|} \geq 0. \tag{6.11}$$

and for any $1 \leq p < 2$,

$$\int_{\Omega'_\varepsilon} |j'_\varepsilon|^p \leq C_p \left(F'_\varepsilon(u'_\varepsilon, A'_\varepsilon) + |\omega'_\varepsilon|\right), \ \frac{1}{{\ell_\varepsilon}^2}\int_{\Omega'_\varepsilon} |\text{curl}\, A'_\varepsilon - m_\varepsilon \mathbf{1}_{\omega'_\varepsilon}|^2 \leq C\left(F'_\varepsilon(u'_\varepsilon, A'_\varepsilon) + |\omega'_\varepsilon|\right), \tag{6.12}$$

where $j'_\varepsilon := (iu'_\varepsilon, \nabla_{A'_\varepsilon} u'_\varepsilon)$.

3. *If* $\{x'_\varepsilon\}_\varepsilon$ *satisfies dist* $(x'_\varepsilon, (\omega'_\varepsilon)^c) \to +\infty$ *and*

$$\forall R > 0, \ \limsup_{\varepsilon\to 0} g'_\varepsilon(x'_\varepsilon + \mathbf{U}_R) < +\infty, \tag{6.13}$$

then, up to extraction of a subsequence, the translated vorticities $\{\mu'_\varepsilon(x'_\varepsilon + \cdot)\}_\varepsilon$ *— where* $\mu'_\varepsilon := \mu(u'_\varepsilon, A'_\varepsilon)$ *— converge in* $W^{-1,p}_{loc}(\mathbb{R}^2)$ *to a measure* ν *of the form* $2\pi \sum_{p\in\Lambda} \delta_p$, *where* Λ *is a discrete subset of* $\mathbb{R}^2$, *and the translated currents* $\{j'_\varepsilon(x'_\varepsilon + \cdot)\}_\varepsilon$ *converge in* $L^p_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ *for any* $p < 2$ *to* j *such that* div $j = 0$ *and* curl $j = \nu - m_\lambda$.

4. *If, in addition, we assume that there exists* $C > 0$ *such that for any* $R > 0$

$$\limsup_{\varepsilon\to 0} \frac{g'_\varepsilon(x'_\varepsilon + \mathbf{U}_R)}{|\mathbf{U}_R|} < C, \tag{6.14}$$

then $j \in \mathcal{A}_{m_\lambda}$ *(*$\mathcal{A}_{m_\lambda}$ *is defined in Definition 1.1) and*

$$\limsup_{R\to+\infty} \fint_{\mathbf{U}_R} |j|^p \, dx < +\infty, \tag{6.15}$$

and

$$\liminf_{R\to\infty} \liminf_{\varepsilon\to 0} \frac{1}{|\mathbf{U}_R|} \int \chi_{\mathbf{U}_R}(x - x'_\varepsilon)\, dg'_\varepsilon(x) \geq \limsup_{R\to\infty} \frac{W(j, \chi_{\mathbf{U}_R})}{|\mathbf{U}_R|} + \frac{\gamma}{2\pi} m_\lambda. \tag{6.16}$$

Proof of Proposition 6.1. We apply Theorem 6 to $\{(u'_\varepsilon, A'_\varepsilon)\}$ on Ω'_ε. First, we check that (6.2) holds. Since $A'_\varepsilon(x') = \ell_\varepsilon(A_\varepsilon - \nabla^\perp h_{\varepsilon,N})(x)$,

$$G^0_{\varepsilon'}(u'_\varepsilon, A'_\varepsilon) = \frac{1}{2}\int_\Omega |\nabla_{A_\varepsilon - \nabla^\perp h_{\varepsilon,N}} u_\varepsilon|^2 + \ell_\varepsilon^2(\operatorname{curl} A_\varepsilon - \Delta h_{\varepsilon,N})^2 + \frac{(1-|u_\varepsilon|^2)^2}{2\varepsilon^2}$$
$$\leq 2G_\varepsilon(u_\varepsilon, A_\varepsilon) + \int_\Omega |\nabla h_{\varepsilon,N}|^2 + \frac{1}{h_{\rm ex}}|h_{\rm ex} + \mu_{\varepsilon,N} - h_{\varepsilon,N}|^2,$$

using the fact that $-\Delta h_{\varepsilon,N} = \mu_{\varepsilon,N} - h_{\varepsilon,N}$. If $N \in \{N_0^-, N_0^+\}$, then (5.25) implies that $\|h_{\rm ex} - h_{\varepsilon,N}\|^2_{H^1} \leq C h_{\rm ex}|\log\varepsilon'|$, while

$$\int_\Omega \frac{\mu^2_{\varepsilon,N}}{h_{\rm ex}} = \int_\Omega h_{\rm ex} {m_\varepsilon}^2 \leq C h_{\rm ex}.$$

Therefore, if $h_{\rm ex} \leq \varepsilon^{-\beta}$ and $G_\varepsilon(u_\varepsilon, A_\varepsilon) \leq C h_{\rm ex}|\log\varepsilon'|$, then $G^0_{\varepsilon'}(u'_\varepsilon, A'_\varepsilon) \leq C\varepsilon^{-2\beta}$ if ε is small enough. We conclude by noting that if $h_{\rm ex} \leq \varepsilon^{-\frac{1}{2}}$ then $\varepsilon' \leq \varepsilon^{3/4}$ hence if β is small enough, Theorem 6 applies.

It gives us a spread out density $(g'_\varepsilon)_0$ of e'_ε and a measure $\nu'_\varepsilon = 2\pi\sum_i d_i\delta_{a_i}$ depending only on u'_ε, hence on u_ε, but *not* on our choice of N. Noting that $\frac{1}{2\pi}\nu'_\varepsilon(\Omega'_\varepsilon)$ is an integer, we let

$$\begin{cases} N = N_0^+ & \text{if } \frac{1}{2\pi}\nu'_\varepsilon(\Omega'_\varepsilon) \geq N_0^+ \\ N = N_0^- & \text{if } \frac{1}{2\pi}\nu'_\varepsilon(\Omega'_\varepsilon) \leq N_0^- \end{cases} \tag{6.17}$$

and this is the choice of N we make from now on.

Testing $\mu(u'_\varepsilon, A'_\varepsilon) - \nu'_\varepsilon$ against ${\zeta_\varepsilon}' - c_{\varepsilon,N}$ which is in $C^{0,1}_0(\Omega'_\varepsilon)$, we find, in view of (5.14)

$$F'_\varepsilon(u'_\varepsilon, A'_\varepsilon) = \frac{1}{2}\int_{\Omega'_\varepsilon} |\nabla_{A'_\varepsilon} u'_\varepsilon|^2 + \frac{1}{{\ell_\varepsilon}^2}|\operatorname{curl} A'_\varepsilon - m_\varepsilon \mathbf{1}_{\omega'_\varepsilon}|^2 + \frac{(1-|u'_\varepsilon|^2)^2}{2(\varepsilon')^2}$$
$$- \int_{\Omega'_\varepsilon} {\zeta_\varepsilon}'\, d\nu'_\varepsilon + c_{\varepsilon,N}\int_{\Omega'_\varepsilon}(d\nu'_\varepsilon - m_\varepsilon \mathbf{1}_{\omega'_\varepsilon}) + o(1). \tag{6.18}$$

Note that $m_\varepsilon|\omega'_\varepsilon| = \mu_{\varepsilon,N}(\omega_{\varepsilon,N}) = 2\pi N$. But, from the choice (6.17), if $\frac{1}{2\pi}\nu'_\varepsilon(\Omega'_\varepsilon) \geq N_0^+$ then we have $N = N_0^+ \geq N_0$ and thus $c_{\varepsilon,N} \geq 0$ by (5.4), hence

$$c_{\varepsilon,N}\int_{\Omega'_\varepsilon}(d\nu'_\varepsilon - m_\varepsilon \mathbf{1}_{\omega'_\varepsilon}) \geq 2\pi c_{\varepsilon,N}(N_0^+ - N) = 0.$$

If on the other hand $\frac{1}{2\pi}\nu'_\varepsilon(\Omega'_\varepsilon) \leq N_0^-$ then by (5.4) we have $c_{\varepsilon,N} \leq 0$ hence

$$c_{\varepsilon,N}\int_{\Omega'_\varepsilon}(d\nu'_\varepsilon - m_\varepsilon \mathbf{1}_{\omega'_\varepsilon}) \geq 2\pi c_{\varepsilon,N}(N_0^- - N) = 0.$$

So in both cases, the last term in (6.18) is nonnegative and we are led to

$$F'_\varepsilon(u'_\varepsilon, A'_\varepsilon) \geq \frac{1}{2}\int_{\Omega'_\varepsilon} |\nabla_{A'_\varepsilon} u'_\varepsilon|^2 + \frac{1}{{\ell_\varepsilon}^2}|\operatorname{curl} A'_\varepsilon - m_\varepsilon \mathbf{1}_{\omega'_\varepsilon}|^2 + \frac{(1-|u'_\varepsilon|^2)^2}{2(\varepsilon')^2} - \int_{\Omega'_\varepsilon} {\zeta_\varepsilon}'\, d\nu'_\varepsilon + o(1). \tag{6.19}$$

[1]

To the spread out density $(g'_\varepsilon)_0$ we add

$$(g'_\varepsilon)_1 = \frac{1}{{\ell_\varepsilon}^2}\left(\operatorname{curl} A'_\varepsilon - m_\varepsilon \mathbf{1}_{\omega'_\varepsilon}\right)^2 - \frac{1}{2}(\operatorname{curl} A'_\varepsilon)^2$$

and call the result g'_ε. Using the notation (6.1) we may rewrite (6.19) as

$$F'_\varepsilon(u'_\varepsilon, A'_\varepsilon) \geq \int_{\Omega'_\varepsilon} e'_\varepsilon + (g'_\varepsilon)_1 - \int_{\Omega'_\varepsilon} {\zeta_\varepsilon}' \, d\nu'_\varepsilon + o(1). \tag{6.20}$$

We now check that g'_ε satisfies the required properties. Since $m_\varepsilon \in (0,1]$ we have that $(g'_\varepsilon)_1$ is bounded below by a universal constant if, for instance, $\ell_\varepsilon < 1/2$, which is true for small ε since $\lim_{\varepsilon\to 0} \ell_\varepsilon = 0$.

1. Since $(g'_\varepsilon)_0$ and $(g'_\varepsilon)_1$ are both bounded below by a constant independent of ε', so is g'_ε.
2. Item 2 will be proven in full below. We prove here the case $\lambda_\Omega < \lambda < +\infty$, which is technically simpler. Using ${f_\varepsilon}' = e'_\varepsilon - \frac{1}{2}|\log \varepsilon'|\nu'_\varepsilon$, rewrite (6.20) as

$$F'_\varepsilon(u'_\varepsilon, A'_\varepsilon) \geq \int_{\Omega'_\varepsilon} \xi \, df'_\varepsilon + \int_{\Omega'_\varepsilon} (1-\xi) e'_\varepsilon + \int_{\Omega'_\varepsilon} (g'_\varepsilon)_1 + o(1), \tag{6.21}$$

where $\xi = 2{\zeta_\varepsilon}'/|\log \varepsilon'|$. Then, from (6.5) applied to $-\xi$ and since from (5.17), we have $\xi = 1$ on ω'_ε, it follows that

$$\int_{\Omega'_\varepsilon} \xi \, d{f_\varepsilon}' \geq \int_{\Omega'_\varepsilon} \xi \, d(g'_\varepsilon)_0 - \|\nabla \xi\|_\infty \int_{\widehat{\Omega'_\varepsilon \setminus \omega'_\varepsilon}} d(|\nu'_\varepsilon| + (g'_\varepsilon)_0^+). \tag{6.22}$$

Since again $\xi = 1$ on ω'_ε, we may write

$$\int \xi \, d(g'_\varepsilon)_0 = (g'_\varepsilon)_0(\tilde\omega'_\varepsilon) + \int_{\Omega'_\varepsilon \setminus \tilde\omega'_\varepsilon} \xi \, d(g'_\varepsilon)_0^+ - \int_{\Omega'_\varepsilon \setminus \tilde\omega_\varepsilon} \xi \, d(g'_\varepsilon)_0^-. \tag{6.23}$$

Let $A = \widehat{\Omega'_\varepsilon \setminus \omega'_\varepsilon} \cap \{\xi > 1/2\}$. Since, from (5.17), $\|\nabla\xi\|_\infty \leq C\ell_\varepsilon$, we have $\widehat{A} \subset \tilde{A} := \{\xi > 1/4\} \setminus \tilde\omega'_\varepsilon$ if ε is small enough. Using (6.6) with $\eta = 1$ we deduce that

$$|\nu'_\varepsilon|\left(\widehat{\Omega'_\varepsilon \setminus \omega'_\varepsilon} \cap \{\xi > 1/2\}\right) \leq C(g'_\varepsilon)_0^+(\tilde{A}) + C|\Omega'_\varepsilon| \leq C \int_{\Omega'_\varepsilon \setminus \tilde\omega'_\varepsilon} \xi \, d(g'_\varepsilon)_0^+ + Ch_{\text{ex}}. \tag{6.24}$$

The same bound is trivially true for $(g'_\varepsilon)_0^+\left(\widehat{\Omega'_\varepsilon \setminus \omega'_\varepsilon} \cap \{\xi > 1/2\}\right)$.

On the other hand, letting $B = \widehat{\Omega'_\varepsilon \setminus \omega'_\varepsilon} \cap \{\xi \leq 1/2\}$, we have $\xi < 2/3$ on $\widehat{B}$ if ε is small enough, therefore using (6.6) we find that

$$|\nu'_\varepsilon|\left(\widehat{\Omega'_\varepsilon \setminus \omega'_\varepsilon} \cap \{\xi \leq 1/2\}\right) \leq Ce_\varepsilon(\{\xi < 2/3\}) \leq C \int (1-\xi) e'_\varepsilon, \tag{6.25}$$

[1] At this point we could also choose N to be $\frac{1}{2\pi}\nu'_\varepsilon(\Omega'_\varepsilon)$ i.e. the total degree of u_ε. Then the term in factor of $c_{\varepsilon,N}$ in (6.18) is 0, and we still have (6.19). We may then proceed with an unchanged proof of the lower bound with that N. Alternatively, we may analyse further the positive term $c_{\varepsilon,N} \int_{\Omega'_\varepsilon} (d\nu'_\varepsilon - m_\varepsilon \mathbf{1}_{\omega'_\varepsilon})$ that has been discarded.

and the same bound is true for $(g'_\varepsilon)_0^+\left(\widehat{\Omega'_\varepsilon\setminus\omega'_\varepsilon}\cap\{\xi>1/2\}\right)$, using (6.4). We have thus obtained that the negative terms in the right-hand side of (6.22) can be absorbed (since $\|\nabla\xi\|_\infty\le C\ell_\varepsilon=o(1)$) in the positive terms of (6.21) and (6.23). There remains to absorb the negative term of the right-hand side of (6.23).

Moreover for any $n>0$ and letting $A_n=\{\xi>1-1/n\}$ we have if ε is small enough that $\widehat{A_n^c}\subset A_{2n}^c$ and $C/|\log\varepsilon|<1/4n$. Thus, splitting into A_n and A_n^c, and using the fact that $(g'_\varepsilon)_0^-\le C$ and (6.4) we have

$$\int_{\Omega'_\varepsilon\setminus\tilde\omega_\varepsilon}\xi\,d(g'_\varepsilon)_0^-\le\frac{C}{|\log\varepsilon|}e'_\varepsilon(A_{2n}^c)+(g'_\varepsilon)_0^-(A_n\setminus\tilde\omega_\varepsilon)\le\frac12\int(1-\xi)e'_\varepsilon+C|A_n\setminus\tilde\omega_\varepsilon|. \tag{6.26}$$

Using the fact that $\|\nabla\xi\|_\infty=o(1)$, and the fact that $(g'_\varepsilon)_1\ge0$ outside ω'_ε and is bounded below by $-C$ in $\omega'_\varepsilon\setminus\tilde\omega'_\varepsilon$ we deduce from (6.21)–(6.26) that for any $n>0$

$$F'_\varepsilon(u'_\varepsilon,A'_\varepsilon)\ge(g'_\varepsilon)_0(\tilde\omega'_\varepsilon)+(g'_\varepsilon)_1(\tilde\omega'_\varepsilon)-C|A_n\setminus\tilde\omega_\varepsilon|+o(|\log\varepsilon|).$$

Then (6.11) follows by dividing by $|\omega'_\varepsilon|$ and taking the limit first as $\varepsilon\to0$ and then as $n\to+\infty$, noting that from (5.15), (5.3), (1.36), we have

$$A_n\setminus\tilde\omega_\varepsilon=\{x\in\omega'_\varepsilon\mid d(x,{\omega'_\varepsilon}^c)<2\}\cup\left\{x\in\omega'_\varepsilon\mid m_{\varepsilon,N}<\frac{h_{\varepsilon,N}(\ell_\varepsilon x)}{h_{\rm ex}}<m_{\varepsilon,N}+\frac{|\log\varepsilon'|}{2n}\right\}$$

and thus — using Lemma 5.6, (5.21) — that $\limsup_{\varepsilon\to0}\frac{|A_n\setminus\tilde\omega_\varepsilon|}{|\log\varepsilon|}$ tends to 0 as $n\to+\infty$.

3. For notational simplicity, we assume $x'_\varepsilon=0$. Since $(g'_\varepsilon)_1$ is bounded below, $\fint_{\mathbf{U}_R}g'_\varepsilon$ being bounded above independently of ε for any $R>0$ implies that the same is true for $\fint_{\mathbf{U}_R}(g'_\varepsilon)_0$ hence (6.7) is satisfied and from item 7 in Theorem 6 we deduce the convergence of the currents (locally weak L^p), vorticities ($W^{-1,p}_{\rm loc}$) and fields (weak $L^2_{\rm loc}$) to j, ν and h satisfying $\operatorname{curl} j=\nu-h$. Since we assume (1.47) we have $\operatorname{div} j(u_\varepsilon,A_\varepsilon)=0$. But a direct computation (see Lemma 5.9) gives $\operatorname{div} j_{1,\varepsilon}=\operatorname{div} j(u_\varepsilon,A_\varepsilon)+\operatorname{div}((1-|u_\varepsilon|^2)\nabla^\perp h_{\varepsilon,N})=0$ in $\omega_{\varepsilon,N}$ since $\nabla h_{\varepsilon,N}=0$ in $\omega_{\varepsilon,N}$. At the blown-up scale this means that when $d(0,(\omega'_\varepsilon)^c)\to+\infty$, we have for any $R>0$ and ε small enough $\operatorname{div} j'_\varepsilon=0$ in B_R. We deduce that $\operatorname{div} j'_\varepsilon\to0$ strongly in $W^{-1,p}_{\rm loc}(\mathbb{R}^2)$ and thus $\operatorname{div} j=0$. Since $\operatorname{curl} j'_\varepsilon=\mu'_\varepsilon+\operatorname{curl} A'_\varepsilon$ we also have that $\operatorname{curl} j'_\varepsilon$ is compact in $W^{-1,p}_{\rm loc}(\mathbb{R}^2)$ for $p<2$. It follows that j'_ε is compact in $L^p_{\rm loc}$ and the convergence of j'_ε is strong.

   Moreover we have

$$\limsup_{\varepsilon\to0}\fint_{\mathbf{U}_R}\left(\frac{1}{{\ell_\varepsilon}^2}\left(\operatorname{curl} A'_\varepsilon-m_\varepsilon\mathbf{1}_{\omega'_\varepsilon}\right)^2-\frac12(\operatorname{curl} A'_\varepsilon)^2\right)<+\infty \tag{6.27}$$

   for every $R>0$, from which we easily deduce that the limit h of $\{\operatorname{curl} A'_\varepsilon\}_\varepsilon$ is equal to m_λ, where $\lambda\in[\lambda_\Omega,+\infty]$ is the limit of $h_{\rm ex}/|\log\varepsilon|$. Indeed under the hypothesis $d(0,(\omega'_\varepsilon)^c)\to+\infty$, we have $m_\varepsilon\mathbf{1}_{\omega'_\varepsilon}(x'_\varepsilon+\cdot)\to m_\lambda$ locally from (5.12). We thus have $\operatorname{curl} j=\nu-m_\lambda$.

4. Again we assume $x'_\varepsilon=0$. The hypothesis that $\limsup_{\varepsilon\to0}\fint_{\mathbf{U}_R}g'_\varepsilon$ is bounded above independently of R implies as above that the same is true for $\limsup_{\varepsilon\to0}\fint_{\mathbf{U}_R}(g'_\varepsilon)_0$.

Then item 8 of Theorem 6 applies and we obtain (6.8), hence (6.15), and from (6.9) we get

$$\liminf_{\varepsilon\to 0} \int \frac{\chi_{\mathbf{U}_R}}{|\mathbf{U}_R|}\, d(g'_\varepsilon)_0 \geq \frac{W(j,\chi_{\mathbf{U}_R})}{|\mathbf{U}_R|} + \frac{1}{2}\fint_{\mathbf{U}_R} {m_\lambda}^2 + \frac{\gamma}{2\pi}\fint_{\mathbf{U}_R} m_\lambda + o_R(1), \tag{6.28}$$

where $\lim_{R\to+\infty} o_R(1) = 0$. On the other hand (6.27) implies that $\operatorname{curl} A'_\varepsilon \to m_\lambda$ locally strongly in L^2, hence

$$\liminf_{\varepsilon\to 0} \int \frac{\chi_{\mathbf{U}_R}}{|\mathbf{U}_R|}\, d(g'_\varepsilon)_1 \geq -\frac{1}{2}{m_\lambda}^2.$$

Together with (6.28), this proves (6.16).

To prove that $j \in \mathcal{A}_{m_\lambda}$ we integrate $\operatorname{curl} j = \nu - m_\lambda$ over B_{R+t} and B_{R-t} to obtain

$$\pi m_\lambda (R-t)^2 + \int_{\partial B_{R-t}} j\cdot\tau = \nu(B_{R-t}) \leq \nu(B_R) \leq \nu(B_{R+t}) = \pi m_\lambda (R+t)^2 + \int_{\partial B_{R+t}} j\cdot\tau.$$

Then, a mean-value argument and (6.15), with $p = 1$, allow to deduce the existence of $t \in [0, \sqrt{R}]$ such that

$$\int_{\partial B_{R-t}\cup\partial B_{R-t}} |j| \leq C R^{3/2},$$

and we deduce that $\nu(B_R) \sim \pi m_\lambda R^2$ as $R \to +\infty$, and so $j \in \mathcal{A}_{m_\lambda}$.

□

It remains to prove item 2 in all generality using Theorem 6.

Proof of item 2 in Proposition 6.1 in the general case. Recall that from its definition (5.15), ζ'_ε achieves its maximum $\frac{1}{2}|\log\varepsilon'|$ on ω'_ε, and is equal to $c_{\varepsilon,N}$ on $\partial\Omega'_\varepsilon$. Thus

$$\xi = 2\frac{\zeta'_\varepsilon}{|\log\varepsilon'|}$$

achieves its maximum 1 on ω'_ε and its minimum $c_{\varepsilon,N}/|\log\varepsilon'|$, which from (5.12) is $o(1)$, on $\partial\Omega'_\varepsilon$.

We let

$$E_1 = \{x \in \Omega'_\varepsilon, \xi > 1-\delta\}, \quad E_2 = \{x \in \Omega'_\varepsilon, \xi > 1-2\delta\},$$

and recall that

$$\tilde{\omega}'_\varepsilon = \{x \in \omega'_\varepsilon, \operatorname{dist}\,(x, (\omega'_\varepsilon)^c) \geq 2\}.$$

We will need the following lemma.

Lemma 6.2. *For any $M > 0$ and ε small enough there exist $\delta > 0$ such that*

$$\delta > \frac{M}{|\log\varepsilon'|}, \quad \widehat{E}_1 \subset E_2, \quad |\tilde{\omega}'_\varepsilon| > M|\widehat{E}_2 \setminus \tilde{\omega}'_\varepsilon|. \tag{6.29}$$

Proof. First we treat the case where $h_{\rm ex}/|\log\varepsilon| \to \lambda \in (\lambda_\Omega, +\infty]$. In this case, using (5.21) in Proposition 5.6 we find

$$\lim_{\delta\to 0} \frac{|\{1-2\delta h_{\rm ex}/|\log\varepsilon'| < \xi < 1\}|}{|\omega'_\varepsilon|} = 0,$$

uniformly with respect to $\varepsilon \le \varepsilon_0$, if ε_0 is small enough. Since $h_{\rm ex}/|\log\varepsilon'|$ has the same limit as $h_{\rm ex}/|\log\varepsilon|$, it is bounded away from 0 as $\varepsilon \to 0$. We deduce easily that δ may be chosen small enough so that

$$\frac{|\{1-3\delta h_{\rm ex}/|\log\varepsilon'| < \xi < 1\}|}{|\omega'_\varepsilon|} \le \frac{2}{M}, \tag{6.30}$$

for any $\varepsilon \le \varepsilon_0$.

Then we note, since from (5.15) we have $|\nabla\xi| \le |\log\varepsilon'|^{-\frac12}$, that $\widehat{E}_1 \subset E_2$ holds for ε small enough, as well as $\widehat{E}_2 \subset \{1-3\delta h_{\rm ex}/|\log\varepsilon'| < \xi\}$. It follows, in view of (6.30) and since $\omega'_\varepsilon = \{\xi = 1\}$, that for ε small enough

$$\frac{|\widehat{E}_2 \setminus \omega'_\varepsilon|}{|\omega'_\varepsilon|} \le \frac{2}{M}. \tag{6.31}$$

To conclude we note that $|\omega'_\varepsilon \setminus \tilde\omega'_\varepsilon| \le |\{d(x,\partial\omega'_\varepsilon) \le 2\ell_\varepsilon\}| = o(|\omega'_\varepsilon|)$, where we have used (5.21) and scaled. Thus if ε is small enough and using (6.31) we find $|\tilde\omega'_\varepsilon| > M|\widehat{E}_2 \setminus \tilde\omega'_\varepsilon|$.

If $h_{\rm ex}/|\log\varepsilon| \to +\infty$ then we choose $\delta = 1/2$. As above, $\|\nabla\xi\|_\infty \to 0$ implies that $\widehat{E}_1 \subset E_2$ for ε small enough, and of course $\delta > M/|\log\varepsilon'|$ is satisfied for ε small enough depending on M. Moreover (5.19) implies that $d(\omega_\varepsilon, \Omega^c) \to 0$ as $\varepsilon \to 0$, and $d(\tilde\omega_\varepsilon, (\omega_\varepsilon)^c) = 2\ell_\varepsilon \to 0$ as well. Therefore $|\Omega \setminus \tilde\omega_\varepsilon| = o(|\tilde\omega_\varepsilon|)$ and after scaling $|\Omega'_\varepsilon \setminus \tilde\omega'_\varepsilon| = o(|\tilde\omega'_\varepsilon|)$, and in particular $|\widehat{E}_2 \setminus \tilde\omega'_\varepsilon| = o(|\tilde\omega'_\varepsilon|)$.

If $h_{\rm ex}/|\log\varepsilon| \to \lambda_\Omega$, we choose $\delta = cL_\varepsilon^2$ with $c > 0$ independent of ε to be chosen small enough depending on M. From (5.22) applied with $M = \frac{c}{2\lambda_\Omega}$ and $\delta = \frac{\eta}{2\lambda_\Omega}$, and rewritten in terms of

$$\xi(x) = 1 - 2\frac{h_{\varepsilon,N}(\ell_\varepsilon x) - \min_\Omega h_{\varepsilon,N}}{|\log\varepsilon'|}$$

we deduce that

$$1 - \xi(x) \le cL_\varepsilon^2 \frac{h_{\rm ex}}{\lambda_\Omega |\log\varepsilon'|} \implies U_Q\left(\frac{\ell_\varepsilon x - x_0}{L_\varepsilon}\right) \le \frac{c+\eta}{2\lambda_\Omega}.$$

Therefore, since $\delta = cL_\varepsilon^2$ and $\lambda_\Omega = \lim_\varepsilon h_{\rm ex}/|\log\varepsilon'|$, for any $\eta > 0$ and if ε is small enough then

$$1 - \xi(x) \le 2\delta \implies U_Q\left(\frac{\ell_\varepsilon x - x_0}{L_\varepsilon}\right) \le \frac{c+\eta}{2\lambda_\Omega}.$$

Since $\ell_\varepsilon/L_\varepsilon \to 0$ from Proposition 5.6 this in turn implies that if ε is small enough and $|y| < 1$ then

$$U_Q\left(\frac{\ell_\varepsilon x - x_0}{L_\varepsilon} + \frac{\ell_\varepsilon}{L_\varepsilon} y\right) < \frac{c+2\eta}{2\lambda_\Omega}.$$

Using (5.22) again, this implies that $\xi(x+y) > 1 - 2(c+3\eta)L_\varepsilon^2$. Choosing $\eta = c/6$ we have proven that $\widehat{E_2} \subset \{\xi > 1 - 3\delta\}$ if ε is small enough. Similarly, we may prove that $\widehat{E_2^c} \subset \{\xi < 1 - \delta\} = E_1^c$. We further deduce that, as $\varepsilon \to 0$,

$$|\widehat{E_2}| < \left(\frac{L_\varepsilon}{\ell_\varepsilon}\right)^2 \left|\left\{U_Q < \frac{c}{2\lambda_\Omega}\right\}\right| + o\left(\frac{L_m}{\ell_\varepsilon}\right)^2. \tag{6.32}$$

Moreover from item 4 in Proposition 5.6 we have $|\tilde{\omega}| \sim \left(\frac{L_m}{\ell_\varepsilon}\right)^2$. With (6.32), and since $|\{U_Q < \frac{2c}{\lambda_\Omega}\}| \to |E_Q| = 1$ as $c \to 0$, and $\tilde{\omega} \subset \widehat{E_2}$, we deduce that $|\widehat{E_2} \setminus \tilde{\omega}|/|\tilde{\omega}|$ can be made arbitrarily small by choosing c, and then ε, small enough. This proves (6.29) and the lemma. $\square$

Returning to our proof, we start from (6.20), and we bound from below

$$\int_{\Omega'_\varepsilon} e'_\varepsilon - \zeta_\varepsilon{}' \, d\nu'_\varepsilon.$$

Let C_0 be the constant in (6.6) i.e. such that for any set $E \subset \Omega'_\varepsilon$

$$|\nu_\varepsilon|(E) \leq \frac{C_0}{|\log \varepsilon|} e_\varepsilon(\widehat{E}) \tag{6.33}$$

and assume that $C_0 > 2$.

For notational simplicity we now write Ω, e, ν, g, Ω, $\tilde{\omega}_\varepsilon$, ζ instead of Ω'_ε, e'_ε, ν'_ε, $(g'_\varepsilon)_0$, ω'_ε, $\tilde{\omega}'_\varepsilon$, ζ'_ε and we let $f = e - \frac{1}{2}|\log \varepsilon'|\nu$. Since $\xi = o(1)$ on $\partial\Omega$ and $\|\nabla \xi\|_\infty = o(1)$ (recall (5.12), (5.17)), for any $\eta > 0$ and ε small enough there exists a smooth positive cut-off function $\chi \leq 1$ such that $|\nabla \chi| \leq \eta$, such that $\chi = 0$ on $\{\xi(x) \leq \frac{1}{8C_0}\}$ and such that $\chi = 1$ on $\{\xi(x) \geq \frac{1}{4C_0}\}$. Moreover, $\{x \mid d(x, \partial\Omega) \leq 2\} \subset \{\xi \leq \frac{1}{8C_0}\}$.

We then note that since $f = e - \frac{1}{2}|\log \varepsilon'|\nu$ and $\zeta = \frac{1}{2}|\log \varepsilon'|\xi$, we may write

$$e - \zeta\nu = (1-\chi)\left(e - \frac{1}{2}\xi|\log\varepsilon'|\nu\right) + \chi\left(\xi(f-g) + \xi g + (1-\xi)e\right). \tag{6.34}$$

Step 1: We first study

$$\begin{aligned} &\int_\Omega \chi\left(\xi \, d(f-g) + \xi \, dg + (1-\xi)e\right) \\ &\quad = \int_{\tilde{\omega}} \chi\xi \, dg + \left(\int_{(\tilde{\omega})^c} \chi\xi \, dg_+ + \int_\Omega \chi(1-\xi)e\right) + \left(\int_\Omega \chi\xi \, d(f-g) - \int_{(\tilde{\omega})^c} \chi\xi \, dg_-\right). \end{aligned} \tag{6.35}$$

The first parenthesis on the right-hand side contains positive terms, while the second one contains negative terms. We use (6.4), (6.6) to bound from below the negative terms, and use the positive terms to balance them. From (6.35) and (6.5) (applied with $\eta = 1$), and since $\chi = \xi = 1$ on $\tilde{\omega}$ by construction and $\chi = 0$ on $\{\text{dist}\,(x, \partial\Omega) \geq 2\}$, we have

$$\int_\Omega \chi(\xi \, d(f-g) + \xi \, dg + (1-\xi)e) \geq \int_{\tilde{\omega}} dg + \frac{3}{4}\int_\Omega \chi(1-\xi)e + \frac{1}{2}\int_{(\tilde{\omega})^c} \chi\xi \, dg_+ + I_1 + I_2, \tag{6.36}$$

where

$$\begin{aligned} I_1 &= -C\int_\Omega \widehat{\nabla(\chi\xi)} \, dg_+ + \frac{1}{4}\int_{(\tilde{\omega})^c} \chi\xi \, dg_+ + \frac{1}{8}\int_\Omega \chi(1-\xi)e, \\ I_2 &= -C\int_\Omega \widehat{\nabla(\chi\xi)} \, d|\nu| - \int_{(\tilde{\omega})^c} \chi\xi \, dg_- + \frac{1}{4}\int_{(\tilde{\omega})^c} \chi\xi \, dg_+ + \frac{1}{8}\int_\Omega \chi(1-\xi)e. \end{aligned}$$

We first study I_1. Since $\nabla(\chi\xi) = 0$ in ω, we have $\{\widehat{\nabla(\chi\xi)} \neq 0\} \subset (\tilde{\omega})^c$. In addition $|\nabla(\chi\xi)| \leq \eta$ if ε is small enough since $|\chi| \leq 1, |\xi| \leq 1, |\nabla\chi| \leq \eta$ and $|\nabla\xi| = o(1)$. Thus, noting that $\chi = 1$ on $\{\xi \geq \frac{1}{2}\}$,

$$2\int_\Omega \widehat{\nabla(\chi\xi)}\, dg_+ \leq \eta \int_{(\tilde{\omega})^c \cap \{\xi \geq \frac{1}{2}\}} dg_+ + \eta \int_{(\tilde{\omega})^c \cap \{\xi \leq \frac{1}{2}\}} dg_+$$
$$\leq \eta \int_{(\tilde{\omega})^c \cap \{\xi \geq \frac{1}{2}\}} \xi\chi\, dg_+ + C\eta \int_\Omega (1-\xi)e$$

where we have used (6.4). We thus obtain, choosing η small enough,

$$I_1 \geq -\frac{1}{8}\int_\Omega (1-\chi)(1-\xi)e.$$

We turn to I_2 and split the negative contributions over E_2 and E_2^c. Using Lemma 6.2, choosing M large enough and using (6.4), (6.6) and the above, we have

$$\int_{E_2^c} \left(\widehat{\nabla(\chi\xi)}\, d|\nu| + \chi\xi\, dg_-\right) \leq \frac{Ce(\widehat{E_2^c})}{|\log\varepsilon'|} \leq \frac{1}{8}\int_{\widehat{E_2^c}} (1-\xi)e,$$

since $\widehat{E_2^c} \subset E_1^c$ and $\delta > M/|\log\varepsilon'|$.

On the other hand, using (6.6) with $\eta = 1$ and the fact that $\nabla(\xi\chi) = 0$ in $\tilde{\omega}$ and $|\nabla(\chi\xi)| \leq \eta + o(1)$ we have, since $\widehat{E_2 \setminus \omega} \subset \widehat{E}_2 \setminus \tilde{\omega}$ and by choosing η small enough,

$$\int_{E_2} \widehat{\nabla(\chi\xi)}\, d|\nu| \leq C\eta \int_{\widehat{E_2}\setminus\tilde{\omega}} (dg_+ + 1) \leq \frac{1}{4}\int_{\tilde{\omega}^c} \chi\xi\, dg_+ + C\eta|\widehat{E_2} \setminus \tilde{\omega}|,$$

while using the fact that g is bounded below (hence $g_- \leq C$), we have

$$\int_{\tilde{\omega}^c \cap E_2} \chi\xi\, dg_- \leq C|E_2 \setminus \tilde{\omega}|.$$

Combining all the above, we deduce that choosing η small enough we have, if ε is small, that

$$I_2 \geq -\frac{1}{8}\int_\Omega (1-\chi)(1-\xi)e - C|\widehat{E}_2 \setminus \tilde{\omega}|.$$

From Lemma 6.2, it follows that $\forall M > 0$ and ε small enough, $I_2 \geq -\frac{C}{M}|\tilde{\omega}| - \frac{1}{8}\int_\Omega (1-\chi)(1-\xi)e$.

Inserting the bounds for I_1 and I_2 into (6.36) we find

$$\int_\Omega \chi\left(\xi\, d(f-g) + \xi\, dg + (1-\xi)e\right) \geq g(\tilde{\omega}) + \frac{3}{4}\int_\Omega \chi(1-\xi)e + \frac{1}{2}\int_{(\tilde{\omega})^c} \chi\xi\, dg_+ - \frac{1}{4}\int_\Omega (1-\chi)(1-\xi)e - \frac{C}{M}|\tilde{\omega}|. \tag{6.37}$$

Step 2: We examine $\int_\Omega (1-\chi)(e - \frac{1}{2}\xi|\log\varepsilon'|\, d\nu)$. Since $|\xi| \leq \frac{1}{4C_0}$ on the support of $1-\chi$ we have there

$$e - \frac{1}{2}\xi|\log\varepsilon'|\nu \geq e - \frac{1}{8C_0}|\log\varepsilon'||\nu|.$$

On the other hand, from item 2 of Theorem 6 we have $g_+ \le e + \frac{1}{2}|\log\varepsilon'||\nu|$ hence on the support of $1-\chi$ we have

$$e - \frac{1}{2}\xi|\log\varepsilon'|\nu \ge (1 - \frac{1}{4C_0})e + \frac{g_+}{4C_0} - \frac{2}{8C_0}|\log\varepsilon'||\nu|$$

and thus

$$\int_\Omega (1-\chi)(e - \frac{1}{2}\xi|\log\varepsilon'|\,d\nu) \ge (1 - \frac{1}{4C_0})\int_\Omega (1-\chi)e + \frac{1}{4C_0}\left(\int_\Omega (1-\chi)\left(dg_+ - |\log\varepsilon'|\,d|\nu|\right)\right).$$

Step 3: Inserting the above and (6.37) in (6.34) we obtain

$$\int_\Omega e - \zeta\,d\nu \ge g(\tilde\omega) + \frac{3}{4}\int_\Omega \chi(1-\xi)e + \frac{1}{2}\int_{(\tilde\omega)^c}\chi\xi\,dg_+ - \frac{1}{4}\int_\Omega (1-\chi)(1-\xi)e - \frac{C}{M}|\tilde\omega| + (1-\frac{1}{4C_0})\int_\Omega(1-\chi)e + \frac{1}{4C_0}\int_\Omega (1-\chi)\,dg_+ - \frac{|\log\varepsilon'|}{4C_0}\int_\Omega (1-\chi)\,d|\nu|. \tag{6.38}$$

Since $0 \le 1-\xi \le 1 + c_{\varepsilon,N}/|\log\varepsilon'| = 1 + o(1)$, and $C_0 > 2$ we have

$$-\frac{1}{4}\int_\Omega (1-\chi)(1-\xi)e + \frac{3}{4}\int_\Omega \chi(1-\xi)e + (1-\frac{1}{4C_0})\int_\Omega (1-\chi)e \ge \left((1+o(1))(1-\frac{1}{4C_0}) - \frac{1}{4}\right)\int_\Omega (1-\chi)(1-\xi)e + \frac{3}{4}\int_\Omega\chi(1-\xi)e \ge \frac{1}{2}\int_\Omega (1-\xi)e.$$

On the other hand, by (6.33) and choice of χ we have

$$\frac{|\log\varepsilon'|}{4C_0}\int_\Omega (1-\chi)\,d|\nu| \le \frac{1}{4}\int_{\widehat{\{\xi<\frac{1}{4C_0}\}}} e.$$

Since $|\nabla\xi| = o(1)$ we know that $\widehat{\{\xi < \frac{1}{4C_0}\}} \subset \{\xi < \frac{1}{4}\}$ for ε small enough and thus

$$\frac{|\log\varepsilon'|}{4C_0}\int_\Omega (1-\chi)\,d|\nu| \le \frac{1}{3}\int_\Omega (1-\xi)e.$$

We have obtained

$$-\frac{1}{4}\int_\Omega (1-\chi)(1-\xi)e + \frac{3}{4}\int_\Omega \chi(1-\xi)e + (1-\frac{1}{4C_0})\int_\Omega (1-\chi)e - \frac{|\log\varepsilon'|}{4C_0}\int_\Omega (1-\chi)\,d|\nu| \ge \frac{1}{6}\int_\Omega (1-\xi)e \ge 0. \tag{6.39}$$

For the terms in (6.38) involving g, g_+, we use the fact that $\xi > \frac{1}{8C_0}$ on the support of χ:

$$g(\tilde\omega) + \frac{1}{4C_0}\int_\Omega (1-\chi)\,dg_+ + \frac{1}{2}\int_{(\tilde\omega)^c}\chi\xi\,dg_+ \ge \frac{1}{16C_0}g_+(\Omega) - g_-(\tilde\omega).$$

We may also bound this term below by $g(\tilde\omega)$. Combining with (6.39) and (6.38) we are led either to

$$\int_\Omega e - \zeta\, d\nu \geq g(\tilde\omega) - \frac{C}{M}|\tilde\omega|, \tag{6.40}$$

where M can be arbitrarily large, or, using $g_- \leq C$ and keeping $\frac{1}{6}\int(1-\xi)e$, to

$$\int_\Omega e - \zeta\, d\nu \geq \frac{1}{16C_0} g_+(\Omega) - C|\tilde\omega| + \frac{1}{6}\int_\Omega (1-\xi)e. \tag{6.41}$$

Step 4: Conclusion. We return to (6.20) and recall that the letter g stood for $(g'_\varepsilon)_0$. We deduce from (6.40)

$$F'_\varepsilon(u'_\varepsilon, A'_\varepsilon) \geq (g'_\varepsilon)_0(\tilde\omega'_\varepsilon) + (g'_\varepsilon)_1(\Omega'_\varepsilon) - \frac{C}{M}|\tilde\omega'_\varepsilon| + o(1).$$

Since $(g'_\varepsilon)_1 \geq 0$ on $\Omega'_\varepsilon \setminus \omega'_\varepsilon$, since $(g'_\varepsilon)_1 \geq -C$ on ω'_ε and since $|\omega'_\varepsilon \setminus \tilde\omega'_\varepsilon| = o(|\omega'_\varepsilon|)$, we have $(g'_\varepsilon)_1(\Omega'_\varepsilon) \geq (g'_\varepsilon)_1(\tilde\omega'_\varepsilon) + o(\omega'_\varepsilon)$. Then we deduce from (6.40), letting $\varepsilon \to 0$ and then $M \to +\infty$, that

$$F'_\varepsilon(u'_\varepsilon, A'_\varepsilon) \geq g'_\varepsilon(\tilde\omega'_\varepsilon) + o(|\omega'_\varepsilon|),$$

proving (6.11).

To prove (6.12) we combine (6.41) with item 6 in Theorem 6 to obtain

$$F'_\varepsilon(u'_\varepsilon, A'_\varepsilon) \geq \frac{1}{C_p}\int_{\{d(x,(\Omega'_\varepsilon)^c)\geq C\}} |j'_\varepsilon|^p - C|\tilde\omega'_\varepsilon| + \frac{1}{6}\int_{\Omega'_\varepsilon}(1-\xi)e'_\varepsilon.$$

This proves the first inequality in (6.12), noting that $(1-\xi) \geq 1/2$ on $\{d(x,(\Omega'_\varepsilon)^c) \geq C\}$ if ε is small enough and that $e'_\varepsilon \geq \frac{1}{2}|j'_\varepsilon|^2$. Finally, from (6.40),

$$F'_\varepsilon(u'_\varepsilon, A'_\varepsilon) \geq (g'_\varepsilon)_0(\tilde\omega'_\varepsilon) + (g'_\varepsilon)_1(\Omega'_\varepsilon) + o(|\omega'_\varepsilon|), \tag{6.42}$$

and it is straightforward to check that

$$(g'_\varepsilon)_1 \geq \begin{cases} \frac{1}{2{\ell_\varepsilon}^2}(\operatorname{curl} A'_\varepsilon - m_\varepsilon)^2 & \text{on } \omega'_\varepsilon \\ \frac{1}{2{\ell_\varepsilon}^2}(\operatorname{curl} A'_\varepsilon)^2 & \text{on } \Omega'_\varepsilon \setminus \omega'_\varepsilon. \end{cases}$$

Therefore

$$(g'_\varepsilon)_1(\Omega'_\varepsilon) \geq \int_{\Omega'_\varepsilon} \frac{(\operatorname{curl} A'_\varepsilon - m_\varepsilon)^2}{2{\ell_\varepsilon}^2} - C|\omega'_\varepsilon|,$$

which together with (6.42) finishes the proof of (6.12). □

6.2 The case of small applied field

Here, by small applied field we mean fields h_{ex} that satisfy the assumptions of Proposition 6.1. We are ready to define the appropriate space and Γ-converging functions to apply the scheme described in Section 2. We choose $X = L^p_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2) \times \mathcal{M}_0$, where $\mathcal{M}_0$ denotes the set of measures μ such that $\mu + C$ is a positive locally bounded measure on $\mathbb{R}^2$, $-C$ being the constant bounding from below g_ε in item 1 of Proposition 6.1. We consider on $L^p_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ the strong topology and on $\mathcal{M}_0$ that of weak convergence. The space of positive measures equipped with the weak convergence is metrizable hence the space X is a Polish space. We will typically denote by $\mathbf{x}$ an element of X.

There is a natural action θ of $\mathbb{R}^2$ on X by translations:

$$\theta_\lambda\,(j(\cdot), g(\cdot)) = (j(\lambda + \cdot), g(\lambda + \cdot))$$

which is clearly continuous with respect to the couple (j, g) as well as with respect to λ.

Denoting as above ω'_ε the rescaled coincidence set, we know from Lemma 5.6 that it satisfies (1.14). Now consider $\{(u_\varepsilon, A_\varepsilon)\}_\varepsilon$ satisfying the hypothesis of Proposition 6.1, then the proposition provides us with measures g'_ε defined on Ω'_ε. We also have the rescaled current $j'_\varepsilon = \operatorname{curl}(iu'_\varepsilon, \nabla_{A'_\varepsilon} u'_\varepsilon)$ where $(u'_\varepsilon, A'_\varepsilon)$ are as in Proposition 6.1. We define functions $\{\mathbf{f}_\varepsilon\}_\varepsilon$ on X as follows.

Having chosen a smooth positive function $\chi : \mathbb{R}^2 \to \mathbb{R}$ with support in the unit ball and integral equal to 1, we let

$$\mathbf{f}_\varepsilon(\mathbf{x}) = \begin{cases} \displaystyle\int \chi(y - x)\, dg'_\varepsilon(y) & \text{if } \exists x \in \omega'_\varepsilon \text{ s.t. } \mathbf{x} = (j'_\varepsilon(x + \cdot), g'_\varepsilon(x + \cdot)) \\ +\infty & \text{otherwise.} \end{cases}$$

Note that since j'_ε and g'_ε vanish outside a compact set, there is at most one x such that $\mathbf{x} = (j'_\varepsilon(x + \cdot), g'_\varepsilon(x + \cdot))$ unless $\mathbf{x} = 0$, in which case we let $\mathbf{f}_\varepsilon(\mathbf{x}) = +\infty$.

The third statement of Proposition 6.1 implies that $\{\mathbf{f}_\varepsilon\}_\varepsilon$ satisfies the requirement of coercivity (1.15). Indeed assume that $\{\mathbf{x}_\varepsilon\}_\varepsilon$ is a sequence in X such that

$$\limsup_{\varepsilon \to 0} \int_{B_R} \mathbf{f}_\varepsilon(\theta_\lambda \mathbf{x}_\varepsilon)\, d\lambda < +\infty \tag{6.43}$$

for every $R > 0$, then in particular this integral is finite if ε is small enough, which implies that $\mathbf{f}_\varepsilon(\theta_\lambda \mathbf{x}_\varepsilon) < +\infty$ for almost every $\lambda \in B_R$. Thus there exists $\{x_\varepsilon\}_\varepsilon$ such that $\mathbf{x}_\varepsilon = (j'_\varepsilon(x_\varepsilon + \cdot), g'_\varepsilon(x_\varepsilon + \cdot))$ and $\lambda + x_\varepsilon \in \omega'_\varepsilon$ for almost every $\lambda \in B_R$, when ε is small enough. In particular the distance of x_ε to $\mathbb{R}^2 \setminus \omega'_\varepsilon$ is larger than R for ε small enough, and (6.43) reads

$$\forall R > 0, \quad \limsup_{\varepsilon \to 0} \int_{B_R} \int \chi(y - \lambda - x_\varepsilon)\, dg'_\varepsilon(y)\, d\lambda < +\infty.$$

Integrating first w.r.t. λ we find

$$\limsup_{\varepsilon \to 0} \int \chi_R(y - x_\varepsilon)\, dg'_\varepsilon(y) < +\infty,$$

where $\chi_R = \chi * \mathbf{1}_{B_R}$.

Since g'_ε is bounded from below independently of ε, and since χ_R is a positive function equal to 1 on B_{R-1}, this implies that the second part of (6.13) is satisfied. Thus, up to a subsequence, the currents $j'_\varepsilon(x_\varepsilon + \cdot)$ converge in $L^p_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$. Going to a further subsequence, $\{g'_\varepsilon(x_\varepsilon + \cdot)\}_\varepsilon$ converges in $\mathcal{M}$.

Finally, χ was chosen smooth, it is clear that the Γ-liminf requirement (1.16) is satisfied if we define the function $\mathbf{f}$ on X by

$$\mathbf{f}(j, g) = \int \chi\, dg. \tag{6.44}$$

Now Theorem 3 applies and combined with Proposition 6.1 gives:

Proposition 6.3. *Assume that* $\{(u_\varepsilon, A_\varepsilon)\}_\varepsilon$ *satisfy* (1.47) *and*

$$G_\varepsilon(u_\varepsilon, A_\varepsilon) \le \min_{N\in\{N_0^-,N_0^+\}} G_\varepsilon^N + CN_0$$

for some constant C*, and assume* $h_{\rm ex}$ *satisfy the hypothesis of Proposition 6.1 and* (5.29)*. Let* $(u'_\varepsilon, A'_\varepsilon)$ *and* N *be as in Proposition 6.1. Using the notation above, and letting* P_ε *be the probability measure on* $L^p_{\rm loc}(\mathbb{R}^2,\mathbb{R}^2)$ *which is the push-forward of the normalized uniform measure on* ω'_ε *by the map* $x \mapsto j'_\varepsilon(x+\cdot)$*, we have the following.*

1. *A subsequence of* $\{P_\varepsilon\}_\varepsilon$ *weakly converges to a translation-invariant probability measure* P *on* $L^p_{\rm loc}(\mathbb{R}^2,\mathbb{R}^2)$ *such that* P*-a.e.* $j \in \mathcal{A}_{m_\lambda}$*.*
2. *For any family* $\{\mathbf{U}_R\}_{R>0}$ *satisfying* (1.4)*,* (1.5)*, we have*

$$\liminf_{\varepsilon\to 0} \frac{F'_\varepsilon(u'_\varepsilon, A'_\varepsilon)}{|\omega'_\varepsilon|} \ge \int W_U(j)\,dP(j) + m_\lambda \frac{\gamma}{2\pi} \tag{6.45}$$

and

$$G_\varepsilon(u_\varepsilon, A_\varepsilon) \ge G_\varepsilon^N + N\left(\frac{2\pi}{m_\lambda}\int W_U(j)\,dP(j) + \gamma + o(1)\right). \tag{6.46}$$

Remark 6.4. *We claim that under the hypothesis of this proposition, the limit of* P_ε *which is the image of the normalized Lebesgue measure on* ω'_ε *by the map*

$$\varphi : x \mapsto j'_\varepsilon(\cdot - x)$$

is unchanged if, in the definition of φ*, we replace* $A'_\varepsilon(y) = \ell_\varepsilon(A_\varepsilon(x+\ell_\varepsilon y) - \nabla^\perp h_{\varepsilon,N}(x+\ell_\varepsilon y))$ *by* $\ell_\varepsilon A_\varepsilon(x+\ell_\varepsilon y)$*, i.e. define* P_ε *as in Theorem 5.*

Indeed, denote by ψ *the corresponding modification of* φ *and by* Q_ε *the ensuing modification of* P_ε*. From* (1.14)*, there exists* $\tilde\omega'_\varepsilon$ *such that* $\tilde\omega'_\varepsilon + B_R \subset \omega'_\varepsilon$ *for any* $R>0$ *if* ε *is small enough, and such that* $|\tilde\omega'_\varepsilon| \sim |\omega'_\varepsilon|$*. Since* $\tilde\omega'_\varepsilon \subset \omega'_\varepsilon$ *and* $|\tilde\omega'_\varepsilon| \sim |\omega'_\varepsilon|$*, replacing* ω'_ε *by* $\tilde\omega'_\varepsilon$ *in the definition of either* P_ε *or* Q_ε *does not change their limit. Then, since* $\nabla^\perp h_{\varepsilon,N}(x+\ell_\varepsilon y) = 0$ *for any* $x \in \tilde\omega'_\varepsilon$ *and* $y \in B_R$ *and if* ε *is small enough, we find that* P_ε *and* Q_ε*, seen as measures on* $L^p(B_R)$*, coincide if* ε *is small enough depending on* R*. It follows that their limits in* $L^p_{\rm loc}(\mathbb{R}^2,\mathbb{R}^2)$ *are equal, proving the claim.*

Proof of Proposition 6.3. It is immediate that $(u_\varepsilon, A_\varepsilon)$ satisfies the hypotheses of Proposition 6.1. Let N be given by Proposition 6.1. By Lemma 5.8, we have, for that N, $G_\varepsilon(u_\varepsilon, A_\varepsilon) = G_\varepsilon^N + F_\varepsilon(u_\varepsilon, A_{1,\varepsilon}) + o(N)$. But from the upper bound assumption, $G_\varepsilon(u_\varepsilon, A_\varepsilon) \le G_\varepsilon^N + CN_0$ so we deduce

$$F_\varepsilon(u_\varepsilon, A_{1,\varepsilon}) \le CN_0 \simeq CN = O(h_{\rm ex}|\omega_{\varepsilon,N}|).$$

In blown-up coordinates, this means $F'_\varepsilon(u'_\varepsilon, A'_\varepsilon) \le C|\tilde\omega'_\varepsilon|$.

On the other hand, from (6.11) in Proposition 6.1, we have

$$F'_\varepsilon(u'_\varepsilon, A'_\varepsilon) \ge g'_\varepsilon(\tilde\omega'_\varepsilon) - o(|\tilde\omega'_\varepsilon|),$$

where we recall $\tilde\omega'_\varepsilon = \{d(x,(\omega'_\varepsilon)^c) \ge 2\}$. Also, since $\int \chi = 1$, since χ has support in B_1 and since g'_ε is bounded below,

$$g'_\varepsilon(\omega'_\varepsilon) \ge \int_{\tilde\omega'_\varepsilon} \mathbf{f}_\varepsilon(\theta_\lambda j'_\varepsilon, \theta_\lambda g'_\varepsilon)\,d\lambda - o(|\omega'_\varepsilon|),$$

where $\bar\omega'_\varepsilon = \{d(x,(\omega'_\varepsilon)^c)\ge 3\}$. Combining these facts, we have

$$F_\varepsilon(j'_\varepsilon, g'_\varepsilon) := \fint_{\bar\omega'_\varepsilon} \mathbf{f}_\varepsilon(\theta_\lambda j'_\varepsilon, \theta_\lambda g'_\varepsilon)\, d\lambda \le C$$

and we now apply Theorem 3 to these functionals. We deduce that the measures $\{Q_\varepsilon\}_\varepsilon$, where Q_ε is the image under

$$x \mapsto (j'_\varepsilon(x+\cdot), g'_\varepsilon(x+\cdot))$$

of the uniform normalized probability measure on $\bar\omega'_\varepsilon$, converge to a translation invariant probability measure Q on X and

$$\liminf_{\varepsilon\to 0} \frac{F'_\varepsilon(u'_\varepsilon, A'_\varepsilon)}{|\bar\omega'_\varepsilon|} \ge \int_X \left(\lim_{R\to+\infty} \fint_{\mathbf{U}_R} \mathbf{f}(\theta_\lambda \mathbf{x})\right) dQ(\mathbf{x}).$$

Moreover, since from (1.14) we have $|\omega'_\varepsilon| \sim |\bar\omega'_\varepsilon|$, we may replace $\bar\omega'_\varepsilon$ with ω'_ε in the definition of Q_ε and obtain the same limit. But, writing as above $\chi_{\mathbf{U}_R} = \chi * \mathbf{1}_{\mathbf{U}_R}$, we have as above from (6.44)

$$\lim_{R\to+\infty} \fint_{\mathbf{U}_R} \mathbf{f}(\theta_\lambda \mathbf{x})\, d\lambda = \lim_{R\to+\infty} \frac{1}{|\mathbf{U}_R|} \int \chi_{\mathbf{U}_R}(x)\, dg(x),$$

where $\mathbf{x} = (j,g)$. Now, if $\mathbf{f}(\mathbf{x})$ is finite and $\mathbf{x}$ is in the support of Q, then there exists (see Remark 1.6) a sequence $\{x_\varepsilon\}_\varepsilon$ such that $(j'_\varepsilon(x_\varepsilon+\cdot), g'_\varepsilon(x_\varepsilon+\cdot))$ converges to $\mathbf{x}$ in X, with (6.14) satisfied with $\mathbf{U}_R$. It follows from Proposition 6.1 that $j \in \mathcal{A}_{m_\lambda}$ and that (6.16) is satisfied, thus

$$\begin{aligned}\lim_{R\to+\infty} \frac{1}{|\mathbf{U}_R|} \int \chi_{\mathbf{U}_R}(x)\, dg(x) &= \lim_{R\to+\infty} \lim_{\varepsilon\to 0} \frac{1}{|\mathbf{U}_R|} \int \chi_{\mathbf{U}_R}(x - x_\varepsilon)\, dg'_\varepsilon(x)\\ &\ge \limsup_{R\to\infty} \frac{W(j, \chi_{\mathbf{U}_R})}{|\mathbf{U}_R|} + m_\lambda \frac{\gamma}{2\pi} = W_U(j) + m_\lambda \frac{\gamma}{2\pi}.\end{aligned}$$

Letting $P_\varepsilon(j)$ and $P(j)$ denote the marginals of the measures Q_ε and Q with respect to the first variable, we immediately deduce that $P_\varepsilon \to P$ and that (6.45) is satisfied. Replacing (6.45) in (5.30) we find (6.46), since $|\omega'_\varepsilon| = h_{\rm ex}|\omega_{\varepsilon,N}| = \frac{2\pi N}{m_{\varepsilon,N}} = \frac{2\pi N}{m_\lambda} + o(N)$. □

Remark 6.5. *As in Remark 2.3 we can also obtain a result of equipartition of energy of F'_ε on ω'_ε.*

6.3 The case of larger applied field

We prove that the conclusions of Proposition 6.3 hold for larger fields as well, that is fields which do not satisfy the assumptions of Proposition 6.1. The technical difficulty is that the applied field is too large to have the energy upper bound that is needed to apply the result of Theorem 6, so we use the strategy of [SS4] Chapter 8: average over smaller balls where most of the time the local energy is small enough to apply Theorem 6 and the result for small applied fields. Note that in this regime, by item 6 in Proposition 5.6 we have for $N \in \{N_0^-, N_0^+\}$, $G_\varepsilon^N = G_\varepsilon^{N_0} + o(h_{\rm ex}) = \frac12 |\Omega| h_{\rm ex} |\log \varepsilon'| + o(h_{\rm ex})$.

Proposition 6.6. *Assume that*

$$|\log \varepsilon|^4 \ll h_{\rm ex} \ll 1/\varepsilon^2$$

and that $G_\varepsilon(u_\varepsilon, A_\varepsilon) \le \frac{h_{\rm ex}}{2}|\Omega||\log\varepsilon'| + Ch_{\rm ex}$. We also assume that $|u_\varepsilon| \le 1$ and that A_ε is a critical point of $G_\varepsilon(u_\varepsilon,\cdot)$. Then, using the same notation as in Propositions 6.1 and 6.3, we have, for any U

$$G_\varepsilon(u_\varepsilon, A_\varepsilon) \ge \frac{1}{2}|\Omega| h_{\rm ex}|\log\varepsilon'| + |\Omega| h_{\rm ex}\left(\int W_U(j)\,dP(j) + \frac{\gamma}{2\pi} + o(1)\right), \tag{6.47}$$

and P-a.e. $j \in \mathcal{A}_1$.

Step 1: Blow-up. The proof follows the ideas in Chapter 8 of [SS4, SS5]. We recall the rescaling formula from there. Define $\tilde x = x/\sigma$ and

$$\tilde u_\varepsilon(\tilde x) = u_\varepsilon(x), \quad \tilde A_\varepsilon(\tilde x) = \sigma A_\varepsilon(x), \quad \tilde\Omega = \frac{\Omega}{\sigma}, \quad \tilde\varepsilon = \frac{\varepsilon}{\sigma}, \quad \tilde h_{\rm ex} = \sigma^2 h_{\rm ex}.$$

Then, denoting $B_x^\sigma = B(x,\sigma)$, we have up to translation $G_\varepsilon(u_\varepsilon, A_\varepsilon, B_x^\sigma) = \tilde G_\varepsilon(\tilde u_\varepsilon, \tilde A_\varepsilon, B^1_{\tilde x})$, where $G_\varepsilon(u_\varepsilon, A_\varepsilon, B_x^\sigma)$ denotes the Ginzburg-Landau energy restricted to B_x^σ, and

$$\tilde G_\varepsilon(\tilde u_\varepsilon, \tilde A_\varepsilon, B_1) = \frac{1}{2}\int_{B_1} |\nabla_{\tilde A_\varepsilon}\tilde u_\varepsilon|^2 + \frac{1}{\sigma^2}\left(\operatorname{curl}\tilde A_\varepsilon - \tilde h_{\rm ex}\right)^2 + \frac{\left(1-|\tilde u_\varepsilon|^2\right)^2}{2\tilde\varepsilon^2}.$$

As in [SS4], Chap 8, if $|\log\varepsilon|^4 \ll h_{\rm ex} \ll 1/\varepsilon^2$ we may choose $\sigma \ll 1$ such that

$$\tilde h_{\rm ex} = (\log\tilde\varepsilon)^4 \qquad |\log\tilde\varepsilon| \sim |\log\varepsilon'|, \tag{6.48}$$

so that $\sigma^2 \log^4\frac{1}{\sigma} = \varepsilon^2 h_{\rm ex}$. □

Step 2: Fubini. We give a formulation of the energy which follows from Fubini's theorem:

$$G_\varepsilon(u_\varepsilon, A_\varepsilon, \Omega) = \int_{x\in\mathbb{R}^2} \frac{G_\varepsilon(u_\varepsilon, A_\varepsilon, B_x^\sigma \cap \Omega)}{|B_x^\sigma|}. \tag{6.49}$$

Note that if we restrict the integration to the set Ω_σ of those x's such that $B_x^\sigma \subset \Omega$, then we have an inequality.

For $x \in \Omega_\sigma$ we define P_ε^x to be the push-forward of the normalized Lebesgue measure on B_x^σ by the map $x \mapsto j'_\varepsilon(\frac{x}{\ell_\varepsilon} + \cdot)$, where again $j'_\varepsilon = (iu'_\varepsilon, \nabla_{A'_\varepsilon} u'_\varepsilon)$, and $u'_\varepsilon, A'_\varepsilon$ are as in (6.10). It is an element of $\mathcal{P}$, the set of probability measures on $X = L^p_{\rm loc}(\mathbb{R}^2, \mathbb{R}^2)$. On $\mathcal{P}$ we define the function

$$\mathbf{f}_\varepsilon(P) = \begin{cases} \dfrac{G_\varepsilon(u_\varepsilon, A_\varepsilon, B_x^\sigma)}{|B_x^\sigma|} - \dfrac{h_{\rm ex}}{2}|\log\varepsilon'| & \text{if } \exists x \in \Omega_\sigma \text{ s.t. } P = P_\varepsilon^x \\ +\infty & \text{otherwise.} \end{cases}$$

We also define Q_ε to be the push-forward under $x \mapsto P_\varepsilon^x$ of the normalized Lebesgue measure on Ω_σ. It is a probability measure on $\mathcal{P}$. Now (6.49) becomes, after subtracting $\frac{h_{\rm ex}}{2}|\Omega_\sigma||\log\varepsilon'|$,

$$G_\varepsilon(u_\varepsilon, A_\varepsilon, \Omega) - \frac{h_{\rm ex}}{2}|\Omega_\sigma||\log\varepsilon'| = |\Omega_\sigma|\int_{\mathcal{P}} \mathbf{f}_\varepsilon(P)\,dQ_\varepsilon(P).$$

Note that since $|\Omega\setminus\Omega_\sigma| = O(\sigma)$, and since $\sigma|\log\varepsilon'| = o(1)$ — this easily follows from (6.48) — we deduce from the above that

$$G_\varepsilon(u_\varepsilon, A_\varepsilon, \Omega) - \frac{h_{\rm ex}}{2}|\Omega||\log\varepsilon'| = o(h_{\rm ex}) + |\Omega_\sigma|\int_{\mathcal{P}} \mathbf{f}_\varepsilon(P)\,dQ_\varepsilon(P). \tag{6.50}$$

□

Step 3: Γ*-convergence of* $\frac{1}{h_{\rm ex}}\mathbf{f}_\varepsilon$. Assume that P_ε is a probability measure such that $\mathbf{f}_\varepsilon(P_\varepsilon)\le Ch_{\rm ex}$ and that $P_\varepsilon\in\mathcal{P}$ converges to P. Then since $\mathbf{f}_\varepsilon(P_\varepsilon)<\infty$, there exists for each ε some $x_\varepsilon\in\Omega_\sigma$ such that $P_\varepsilon=P_\varepsilon^{x_\varepsilon}$ and

$$\begin{aligned}\mathbf{f}_\varepsilon(P_\varepsilon)&=\frac{G_\varepsilon(u_\varepsilon,A_\varepsilon,B^\sigma_{x_\varepsilon})}{|B^\sigma|}-\frac{h_{\rm ex}}{2}|\log\varepsilon'|\\ &=\frac{1}{\sigma^2}\left(\frac{\tilde G_\varepsilon(\tilde u_\varepsilon,\tilde A_\varepsilon,B^1_{\tilde x_\varepsilon})}{|B_1|}-\frac{\sigma^2h_{\rm ex}}{2}|\log\varepsilon'|\right)\\ &=\frac{1}{\sigma^2}\left(\frac{\tilde G_\varepsilon(\tilde u_\varepsilon,\tilde A_\varepsilon,B^1_{\tilde x_\varepsilon})}{|B_1|}-\frac{\tilde h_{\rm ex}}{2}|\log\varepsilon'|\right).\end{aligned}\tag{6.51}$$

Since $\mathbf{f}_\varepsilon(P_\varepsilon)\le Ch_{\rm ex}$ we get $\tilde G_\varepsilon(\tilde u_\varepsilon,\tilde A_\varepsilon,B^1_{\tilde x_\varepsilon})\le C(h_{\rm ex}\sigma^2+h_{\rm ex}\sigma^2|\log\varepsilon'|)=C\tilde h_{\rm ex}(1+|\log\varepsilon'|)$ and in view of (6.48), note that $\varepsilon'=\varepsilon\sqrt{h_{\rm ex}}=\tilde\varepsilon\sqrt{\tilde h_{\rm ex}}$, we may apply Proposition 6.3 with ε replaced by $\tilde\varepsilon$, $h_{\rm ex}$ by $\tilde h_{\rm ex}$, etc. This way we find that, $\tilde N$ being given by Proposition 6.3, $\tilde P$ is concentrated on $\mathcal{A}_1$ and that

$$\begin{aligned}\tilde G_\varepsilon(\tilde u_\varepsilon,\tilde A_\varepsilon,B^1_{\tilde x_\varepsilon})\ge\frac12\|h_{\tilde\varepsilon,\tilde N}-\tilde h_{\rm ex}\|^2_{H^1(B^1_{\tilde x_\varepsilon})}+\pi\tilde N|\log\varepsilon'|\\ +\tilde h_{\rm ex}|\omega_{\tilde\varepsilon}|\left(\int W_U(j)\,d\tilde P(j)+\frac{\gamma}{2\pi}+o(1),\right)\end{aligned}\tag{6.52}$$

where $h_{\tilde\varepsilon,\tilde N}$ is the solution of the obstacle problem (1.35), replacing Ω with $B^1_{\tilde x_\varepsilon}$, $h_{\rm ex}$ with $\tilde h_{\rm ex}$ and ε with $\tilde\varepsilon$; and $\mu_{\tilde\varepsilon,\tilde N}=-\Delta h_{\tilde\varepsilon,\tilde N}+h_{\tilde\varepsilon,\tilde N}$, $\omega_{\tilde\varepsilon}=\mathrm{Supp}(\mu_{\tilde\varepsilon,\tilde N})$. We can further check, since $\tilde h_{\rm ex}=|\log\tilde\varepsilon|^4$, that from (5.19), (5.26) in Proposition 5.6, we have, as $\tilde\varepsilon\to0$,

$$|\omega_{\tilde\varepsilon}|=|B_1|+o(1)\qquad\|h_{\tilde\varepsilon,\tilde N}-\tilde h_{\rm ex}\|^2_{H^1}+\pi\tilde N|\log\varepsilon'|=\frac12\tilde h_{\rm ex}|B_1||\log\varepsilon'|+o(\tilde h_{\rm ex}).\tag{6.53}$$

In (6.52), $\tilde P$ is such that $\tilde P$-a.e. $j\in\mathcal{A}_1$, and is the limit of the probability measures $\{P_{\tilde\varepsilon}\}_\varepsilon$. Using Remark 6.4, the measure $\tilde P$ is the limit of the image of the normalized Lebesgue measure on $\omega_{\tilde\varepsilon}$ by $\varphi:x\mapsto j(u_x,A_x)$, where

$$u_x(y)=\tilde u_\varepsilon(x+\ell_\varepsilon y)=u_\varepsilon(\sigma x+\ell_\varepsilon y),\quad A_x(y)=\ell_\varepsilon\tilde A_\varepsilon(x+\ell_\varepsilon y)=\ell_\varepsilon u_\varepsilon(\sigma x+\ell_\varepsilon y),$$

while P_ε is the image of the normalized Lebesgue measure on $B^\sigma_{\tilde x_\varepsilon}$ by the usual blow-up map $x\mapsto j'_\varepsilon(x+\cdot)$ which, changing the variable to $\tilde x$, is equal to the image of the normalized Lebesgue measure on $B^1_{\tilde x_\varepsilon}$ by φ. From $|\omega_{\tilde\varepsilon}|\sim|B_1|$ and $\omega_{\tilde\varepsilon}\subset B^1_{\tilde x_\varepsilon}$, we then deduce that $\{P_\varepsilon\}_\varepsilon$ and $\{P_{\tilde\varepsilon}\}_\varepsilon$ have the same limit, i.e. that $\tilde P=P$. Then (6.52), (6.53) yield

$$\tilde G_\varepsilon(\tilde u_\varepsilon,\tilde A_\varepsilon,B_1^{\tilde x_\varepsilon})\ge\frac12\tilde h_{\rm ex}|B_1|\log\frac{1}{\tilde\varepsilon\sqrt{\tilde h_{\rm ex}}}+\tilde h_{\rm ex}|B_1|\left(\int W_U(j)\,dP(j)+\frac{\gamma}{2\pi}\right)+o(\tilde h_{\rm ex}),$$

where $P=\lim_\varepsilon P_\varepsilon$ is concentrated on $\mathcal{A}_1$. Then, from (6.51),

$$\liminf_{\varepsilon\to0}\frac{1}{h_{\rm ex}}\mathbf{f}_\varepsilon(P_\varepsilon)\ge\int W_U(j)\,dP(j)+\frac{\gamma}{2\pi}.\tag{6.54}$$

□

Step 4: $\{Q_\varepsilon\}_\varepsilon$ is tight. A consequence of (6.54) and Proposition 4.1 below is that the function $\mathbf{f}_\varepsilon$ is bounded below independently of ε, thus the results of Section 2 apply. The measures Q_ε are regular from their definition hence given $\delta > 0$ there exists compact sets K_ε in $\mathcal{P}$ such that $Q_\varepsilon(K_\varepsilon) > 1 - \delta$, and since ${h_{\rm ex}}^{-1} \int \mathbf{f}_\varepsilon(P)\, dQ_\varepsilon(P) < C$ from (6.50) and the energy upper bound, we can also require that ${h_{\rm ex}}^{-1}\mathbf{f}_\varepsilon < 1/\delta$ on K_ε.

Now assume $P_\varepsilon \in K_\varepsilon$ for each $\varepsilon > 0$. Then $\mathbf{f}_\varepsilon(P_\varepsilon)/h_{\rm ex}$ is bounded independently of ε and therefore from the previous step and after taking a subsequence, $P_\varepsilon \to P$. This shows that the hypotheses of Lemma 2.1 are satisfied and thus that any subsequence of $\{Q_\varepsilon\}_\varepsilon$ has a convergent subsequence. In addition, we deduce that Q-almost every P satisfies that P-almost every $j \in \mathcal{A}_1$. □

Step 5: Conclusion. Combining the convergence of $\{Q_\varepsilon\}_\varepsilon$ and (6.54), we deduce with the help of Lemma 2.2 that

$$\liminf_{\varepsilon\to 0} \frac{1}{h_{\rm ex}} \int \mathbf{f}_\varepsilon(P)\, dQ_\varepsilon(P) \geq \int \left(\int W_U(j)\, dP(j) \right) dQ(P) + \frac{\gamma}{2\pi}. \tag{6.55}$$

It remains to show that

$$\int \left(\int W_U(j)\, dP(j) \right) dQ(P) = \int W_U(j)\, d\overline{P}(j), \tag{6.56}$$

where $\overline{P} = \lim_\varepsilon P_\varepsilon$ and P_ε is the push-forward of the normalized uniform measure on ω'_ε by the map $x \mapsto j'_\varepsilon(x + \cdot)$ where j'_ε is the current defined from (6.10).

But, if φ is a continuous and bounded function on X, by definition of Q_ε we have

$$\int \varphi\, dP_\varepsilon = \int \left(\int \varphi\, dP \right) dQ_\varepsilon(P) + o(1),$$

since $|\omega'_\varepsilon| \sim \Omega_\sigma$. Hence, passing to the limit, we find

$$\int \varphi\, d\overline{P} = \int \left(\int \varphi\, dP \right) dQ(P).$$

It is straightforward to check that this equality extends to positive measurable functions, and in particular to W, which was proven to be measurable in Proposition 4.1. This proves (6.56). In addition since Q almost every P satisfies $j \in \mathcal{A}_1$, and since $\overline{P} = \int P\, dQ(P)$, we also have that $\overline{P}$ almost every $j \in \mathcal{A}_1$. Renaming $\overline{P}$ by P, and since $|\Omega_\sigma| \sim |\Omega|$ as $\varepsilon \to 0$, combining (6.50), (6.55) and (6.56) proves (6.47). □

6.4 Proof of Theorem 5

We may now combine the results of Propositions 6.3, 6.6, Lemma 5.8 and the upper bound of Theorem 7 below to prove Theorem 5.

Assertion 2 of the Theorem is part of Theorem 7 below.

For Assertion 1, in view of Remark 6.4, propositions 6.3 and 6.6 imply, for a suitable choice of $N \in \{N_0^-, N_0^+\}$, the convergence of P_ε to a translation invariant measure P on $L^p_{\rm loc}$ such that (1.49) holds.

We next prove (1.50). We have seen in the subsections just above that the upper bound condition implies $F_\varepsilon(u_\varepsilon, A_{1,\varepsilon}) \le Ch_{\rm ex}|\omega_{\varepsilon,N}|$ or, blowing-up, ${F_\varepsilon}'(u'_\varepsilon, A'_\varepsilon) \le C|\omega'_\varepsilon|$. We may write $\mu(u_\varepsilon, A_\varepsilon) - \mu_{\varepsilon,N} = \operatorname{curl} j_{1,\varepsilon} + \alpha + \beta$, where

$$\alpha = \operatorname{curl} A_{1,\varepsilon} - \mu_{\varepsilon,N}, \quad \beta = \mu(u_\varepsilon, A_\varepsilon) - \operatorname{curl} A_{1,\varepsilon} - \operatorname{curl} j_{1,\varepsilon}.$$

From (6.12) we have $\|\alpha\|_{L^2}^2 \le C({F_\varepsilon}' + |\omega'_\varepsilon|) \le Ch_{\rm ex}|\omega_{\varepsilon,N}|$, hence $\|\alpha\|_{W^{-1,p}} \le C\sqrt{N}$. The same bound holds for β from (5.33). Finally, from (6.12), we have $\int |j'|^p \le C(F'_\varepsilon + |\omega'_\varepsilon|) \le CN$. Rescaling this relation, we get

$$\|j\|_{L^p(\Omega)} \le {h_{\rm ex}}^{\frac12 - \frac1p}\|j'\|_{L^p(\Omega'_\varepsilon)} \le C{h_{\rm ex}}^{\frac12-\frac1p}N^{\frac1p} \le CN^{\frac12}$$

where we have used the fact that $N \le \frac{h_{\rm ex}|\Omega|}{2\pi} \le Ch_{\rm ex}$. Therefore $\|\operatorname{curl} j_{1,\varepsilon}\|_{W^{-1,p}} \le C\sqrt{N}$, which concludes the proof of (1.50).

It remains to prove the statement concerning minimizers $\{(u_\varepsilon, A_\varepsilon)\}_\varepsilon$ of G_ε. In the case of small applied fields $h_{\rm ex} \le \varepsilon^{-\beta}$, Corollary 7.1 gives $G_\varepsilon(u_\varepsilon, A_\varepsilon) \le \min_{N\in\{N_0^-, N_0^+\}} G_\varepsilon^N + CN_0$ hence Proposition 6.3 applies. Comparing the lower bound (6.46) to the upper bound (7.2) in Theorem 7, we deduce that N minimizes the right-hand side and that

$$\int W_U(j)\, dP(j) \le \min_{\mathcal{A}_{m_\lambda}} W.$$

Since P is supported on $\mathcal{A}_{m_\lambda}$, we obtain that P-a.e. j minimizes W_U over $\mathcal{A}_{m_\lambda}$. Since minimizers of W_U are independent of U, we have the result.

In the case of large applied fields $|\log\varepsilon|^4 \ll \frac{1}{\varepsilon^2}$, Corollary 7.2 yields

$$\min G_\varepsilon \le \frac{h_{\rm ex}}{2}|\Omega||\log\varepsilon'| + |\Omega|h_{\rm ex}\left(\min_{\mathcal{A}_1} W + \frac{\gamma}{2\pi} + o(1)\right),$$

thus Proposition 6.6 applies and comparing the above to (6.47), we deduce that there is equality and that

$$\int W_U(j)\, dP(j) \le \min_{\mathcal{A}_1} W.$$

We again deduce that P-almost every j minimizes W over $\mathcal{A}_1$. From (5.26) in Proposition 5.6, we get that (1.49) holds.

Note that Theorem 5 implies Theorem 4 since, from (5.25) in Proposition 5.6, if $h_{\rm ex} = \lambda|\log\varepsilon|$ with $\lambda > \lambda_\Omega$ then $G_\varepsilon^N - G_\varepsilon^{N_0} = O(1)$ as $\varepsilon \to 0$.

Remark 6.7. *If we had chosen from the beginning $N = \bar N = \frac{1}{2\pi}\nu_\varepsilon(\Omega'_\varepsilon)$ as indicated in footnote in the proof of Proposition 6.1, the we would obtain the lower bound* (1.49) *with that $\bar N$. Combining with the upper bound of Theorem 7 we may deduce that $G_\varepsilon^{\bar N} = \min_{N\in\mathbb{N}} G_\varepsilon^N + o(\bar N) + o(N_0)$. A careful examination of the variation of G_ε^N with N, based on Lemma 5.4 should then allow to obtain that $\bar N = N_0^-$ or N_0^+ up to an error which is quantified by the examination of the growth of $G_\varepsilon^N = G_\varepsilon^{N_0}$ (this is quite delicate, though). We expect this error to be* 0 *for small enough applied fields, in particular for $h_{\rm ex} < H_{c_1} + O(\sqrt{|\log\varepsilon|})$.*

7 Upper bound

In this section, we use the notation of Section 5, in particular the definitions of $h_{\varepsilon,N}, \omega_{\varepsilon,N}$... can be found there. We make here, and here only, the assumption that Ω is convex. This guarantees the smoothness of the solutions to the obstacle problem (A.1), see below. We do not believe this is a serious restriction, but the possible presence of cusps in the coincidence set would certainly add technical difficulties to our construction.

We prove the upper bound matching the lower bound of Theorem 5 (recall the definition of G_ε^N in (1.41)):

Theorem 7. *Assume that Ω is convex and that (1.48) holds. Then for any family of integers $\{N\}$ depending on ε and satisfying*

$$1 \ll N \leq \frac{|\Omega| h_{\mathrm{ex}}}{2\pi}, \quad \text{as } \varepsilon \to 0 \tag{7.1}$$

the following holds:

1. *There exists $(u_\varepsilon, A_\varepsilon)$ such that, as $\varepsilon \to 0$,*

$$G_\varepsilon(u_\varepsilon, A_\varepsilon) \leq G_\varepsilon^N + N\left(\frac{2\pi}{m_\lambda} \min_{\mathcal{A}_{m_\lambda}} W + \gamma + o(1)\right), \tag{7.2}$$

where $\lambda \in [\lambda_\Omega, +\infty]$ is the limit of $h_{\mathrm{ex}}/|\log \varepsilon|$ as $\varepsilon \to 0$, where $\mathcal{A}_{m_\lambda}$ is as in Definition 1.1, and where $\varepsilon' = \varepsilon\sqrt{h_{\mathrm{ex}}}$.

2. *Let $1 < p < 2$ be given. For any probability P on $L^p_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ which is invariant under the action of translations and concentrated on $\mathcal{A}_{m_\lambda}$, there exists $(u_\varepsilon, A_\varepsilon)$ such that, letting P_ε be the push-forward of the normalized Lebesgue measure on $\omega_{\varepsilon,N}$ by the map $x \mapsto \frac{1}{\sqrt{h_{\mathrm{ex}}}} j(u_\varepsilon, A_\varepsilon)\left(x + \frac{\cdot}{\sqrt{h_{\mathrm{ex}}}}\right)$, we have as $\varepsilon \to 0$, $P_\varepsilon \to P$ weakly and*

$$G_\varepsilon(u_\varepsilon, A_\varepsilon) \leq G_\varepsilon^N + N\left(\frac{2\pi}{m_\lambda}\int W_K(j)\, dP(j) + \gamma + o(1)\right). \tag{7.3}$$

Corollary 7.1. *Under the same assumptions, we have*

$$\min G_\varepsilon \leq \min_{N \in \{N_0^-, N_0^+\}} G_\varepsilon^N + CN_0 \leq Ch_{\mathrm{ex}}|\log \varepsilon'|.$$

Proof. To obtain an upper bound, we apply the result above with $N = N_0^-$ and $N = N_0^+$ (recall that N_0 is not necessarily an integer) and use (5.25) in Proposition 5.6. □

Corollary 7.2. *Under the same assumptions, if $|\log \varepsilon|^4 \ll h_{\mathrm{ex}} \ll \frac{1}{\varepsilon^2}$, we have*

$$\min G_\varepsilon \leq \frac{1}{2}|\Omega| h_{\mathrm{ex}}|\log \varepsilon'| + h_{\mathrm{ex}}|\Omega|\left(\min_{\mathcal{A}_1} W + \frac{\gamma}{2\pi} + o(1)\right).$$

Proof. This follows from (7.2) applied with $N = N_0^-$, (5.26) in Proposition 5.6, $|\omega_{\varepsilon,N}| \leq |\Omega|$ and $\lambda = +\infty$. □

We now prove Theorem 7.

7.1 Properties of $\omega_{\varepsilon,N}$

The convexity of Ω guarantees that the coincidence sets $\{\omega_m\}_m$ for the minimizers of (A.1) are convex (Friedman-Phillips, [FP], see also [Ka], [DM]). Then the density criterion of Caffarelli [Caf] and regularity improvement of Kinderlehrer-Nirenberg [KN] and Isakov [Is] imply that it is in fact analytic for any m. We state a density estimate which is uniform with respect to $m \in (\underline{h}_0, 1]$. This is the only place where we use the assumption that Ω is convex.

Lemma 7.3. *Assume Ω is convex (so that* (1.31) *is satisfied), and let L_m be as in Proposition A.1. Then there exists $\alpha > 0$ and $r_0 > 0$ such that for any $m \in (\underline{h}_0, 1]$, any $r < r_0$, and any $x \in \omega_m$*

$$\frac{|\omega_m \cap B(x, L_m r)|}{|B(x, L_m r)|} \geq \alpha.$$

Proof. We call $d(r)$ the density ratio above. Since ω_m is convex, $r \mapsto d(r)$ is decreasing. But from Proposition A.1, the diameter of ω_m is bounded by CL_m and $|\omega_m| \geq cL_m^2$, where $c, C > 0$ are independent of $m \in (\underline{h}_0, 1]$. Therefore $d(C) \geq c/C^2$. Letting $\alpha = c/C^2$ and $r_0 = C$ proves the lemma. □

Now assume the hypothesis of Theorem 7 are satisfied. Then Lemma 5.1 applies and $\mu_{\varepsilon,N} := -\Delta h_{\varepsilon,N} + h_{\varepsilon,N} = m_{\varepsilon,N} h_{\rm ex} \mathbf{1}_{\omega_{\varepsilon,N}}$ where $m_{\varepsilon,N}$ satisfies (5.2) and $m_{\varepsilon,N} h_{\rm ex} |\omega_{\varepsilon,N}| = 2\pi N$. Let

$$\ell'_\varepsilon = \frac{1}{\sqrt{m_{\varepsilon,N} h_{\rm ex}}}.$$

We have $(\ell'_\varepsilon)^{-2} |\omega_{\varepsilon,N}| \in 2\pi\mathbb{N}$.

Rescaling the previous lemma we find

Corollary 7.4. *There exists $\alpha > 0$ such that for any $R > 0$, any ε small enough depending on R, and $x \in \omega_{\varepsilon,N}$ we have $|\omega_{\varepsilon,N} \cap B(x, R\ell'_\varepsilon)| \geq \alpha |B(x, R\ell'_\varepsilon)|$.*

Proof. From Proposition 5.6 we have $\ell'_\varepsilon \ll L_\varepsilon$, where L_ε is the value of L_m corresponding to $m = m_{\varepsilon,N}$. Thus, if ε is small enough, we have $R\ell'_\varepsilon \leq r_0 L_\varepsilon$, and the previous lemma applies. □

7.2 Definition of the test current

The construction follows similar lines as [SS4], Chapters 7 and 10, but the estimates must be more precise as only an error of $o(1)$ per vortex is allowed. From now on we assume (1.31), (1.48) and (7.1). We write for simplicity ω_ε instead of $\omega_{\varepsilon,N}$ and m_ε instead of $m_{\varepsilon,N}$.

Let $R \in 4\pi\mathbb{N}$ be given. We tile $\mathbb{R}^2$ in the obvious way by a collection $\{\mathcal{K}_i\}_i$ of squares of sidelength $2R\ell'_\varepsilon$. We let

$$I = \{i, \mathcal{K}_i \subset \omega_\varepsilon, \text{dist } (\mathcal{K}_i, \partial\omega_\varepsilon) \geq \ell'_\varepsilon\}, \quad \tilde{\omega}_\varepsilon = \cup_{i \in I} \mathcal{K}_i \quad \hat{\omega}_\varepsilon = \omega_\varepsilon \setminus \tilde{\omega}_\varepsilon.$$

To prove the first item of the theorem, we apply Corollary 4.4 to a minimizer of W to find j_R in K_R such that $j_R \cdot \tau$ on ∂K_R where $K_R = [-R, R]^2$. To prove the second item, let P concentrated on $\mathcal{A}_{m_\lambda}$ be given, and let us define $\bar{P}$ to be the push-forward of P under the rescaling $j \mapsto \frac{1}{\sqrt{m_\lambda}} j\left(\frac{\cdot}{\sqrt{m_\lambda}}\right)$. Then $\bar{P}$ is concentrated on $\mathcal{A}_1$ and from (1.9), we have

$$\int W_K(j)\, d\bar{P}(j) = \frac{1}{m_\lambda} \int W_K(j)\, dP(j) + \frac{1}{4} \log m_\lambda. \tag{7.4}$$

We then apply Corollary 4.5 to the probability $\bar{P}$, it gives again a j_R in K_R such that $j_R \cdot \tau = 0$ on ∂K_R. We continue the construction in the same way in either of these two cases.

We next extend j_R by periodicity to $\mathbb{R}^2$ and, denoting by c_0 the center of $\mathcal{K}_{i_0}$, where $i_0 \in I$ is arbitrary, we let

$$\tilde{j}_\varepsilon(x) = \begin{cases} \frac{1}{\ell'_\varepsilon} j_R\left(\frac{x - c_0}{\ell'_\varepsilon}\right) & \text{in } \tilde{\omega}_\varepsilon \\ 0 & \text{in } \mathbb{R}^2 \setminus \tilde{\omega}_\varepsilon. \end{cases}$$

In particular, letting $\widetilde{\Lambda} = (c_0 + \ell'_\varepsilon \Lambda_R) \cap \tilde{\omega}_\varepsilon$, where Λ_R denotes the support of ν_R, and since $\tilde{j}_\varepsilon \cdot \tau = 0$ on $\partial\tilde{\omega}_\varepsilon$, we have

$$\operatorname{curl} \tilde{j}_\varepsilon = 2\pi \sum_{p \in \widetilde{\Lambda}} \delta_p - m_\varepsilon h_{\mathrm{ex}} \mathbf{1}_{\tilde{\omega}_\varepsilon}, \quad \text{in } \mathbb{R}^2.$$

We define a current $\hat{j}_\varepsilon$ as follows. First we note that since $|\mathcal{K}_i|, |\omega_\varepsilon| \in 2\pi {\ell'_\varepsilon}^2 \mathbb{N}$, we have $|\hat{\omega}_\varepsilon| \in 2\pi {\ell'_\varepsilon}^2 \mathbb{N}$. Then, using Corollary 7.4 we may — we omit the cumbersome details — find disjoint measurable sets $\mathcal{C}_1, \dots, \mathcal{C}_n$ and $y_i \in \mathcal{C}_i$ such that, for some $c, C > 0$ independent of R, ε,

$$\mathbf{1}_{\hat{\omega}_\varepsilon} = \sum_i \mathbf{1}_{\mathcal{C}_i}, \quad |\mathcal{C}_i| = 2\pi {\ell'_\varepsilon}^2, \quad B(y_i, c\ell'_\varepsilon) \subset \mathcal{C}_i \subset B(y_i, C\ell'_\varepsilon). \tag{7.5}$$

We let $j_i = -\nabla^\perp f_i$, where

$$\begin{cases} -\Delta f_i = 2\pi \delta_{y_i} - m_\varepsilon h_{\mathrm{ex}} \mathbf{1}_{\mathcal{C}_i} & \text{in } B(y_i, C\ell'_\varepsilon) \\ \partial_\nu f_i = 0 & \text{on } \partial B(y_i, C\ell'_\varepsilon), \end{cases} \tag{7.6}$$

and then, letting $j_i = 0$ on $\mathbb{R}^2 \setminus B(y_i, C\ell'_\varepsilon)$, we let $\hat{j}_\varepsilon = \sum_{i=1}^n j_i$. We have, letting $\widehat{\Lambda} = \{y_1, \dots, y_n\}$,

$$\operatorname{curl} \hat{j}_\varepsilon = 2\pi \sum_{p \in \widehat{\Lambda}} \delta_p - m_\varepsilon h_{\mathrm{ex}} \mathbf{1}_{\hat{\omega}_\varepsilon}.$$

Finally we let $j_\varepsilon = \tilde{j}_\varepsilon + \hat{j}_\varepsilon$, $\Lambda = \widetilde{\Lambda} \cup \widehat{\Lambda}$. We have

Proposition 7.5. *The current j_ε satisfies*

$$\begin{cases} \operatorname{curl} j_\varepsilon = 2\pi \sum_{p \in \Lambda} \delta_p - h_{\mathrm{ex}} m_\varepsilon \mathbf{1}_{\omega_\varepsilon} & \text{in } \mathbb{R}^2 \\ j_\varepsilon = 0 & \text{on } \mathbb{R}^2 \setminus \Omega. \end{cases} \tag{7.7}$$

Moreover

$$\limsup_{\eta \to 0} \frac{1}{m_\varepsilon h_{\mathrm{ex}} |\omega_\varepsilon|} \left(\frac{1}{2} \int_{\Omega \setminus \cup_{p \in \Lambda} B(p, \eta \ell'_\varepsilon)} |j_\varepsilon|^2 + \pi \# \Lambda \log \eta \right) \le \frac{W(j_R, \mathbf{1}_{K_R})}{R^2} + o_\varepsilon(1), \tag{7.8}$$

where $\lim_{\varepsilon \to 0} o_\varepsilon(1) = 0$.

Finally, there exists $\eta_0 > 0$ such that for any ε small enough, any $p \in \Lambda$ and any $q \in [1, +\infty)$ we have

$$\|j_\varepsilon - \nabla^\perp \log|\cdot - p|\|_{L^q(B(p, \eta_0 \ell'_\varepsilon))} \le C_q {\ell'_\varepsilon}^{\frac{2}{q} - 1}. \tag{7.9}$$

Proof. The fact that $\operatorname{curl} j_\varepsilon = 2\pi \sum_{p\in\Lambda} \delta_p - h_{\rm ex} m_\varepsilon$ is obvious, and $j_\varepsilon = 0$ on Ω^c follows from the definition of j_ε and the fact that $d(\omega_\varepsilon, \Omega^c) \geq C\ell'_\varepsilon$ if ε is small enough. This is a consequence of the fact that on ω_ε we have $h_{\varepsilon,N} = m_{\varepsilon,N} h_{\rm ex}$ while on $\partial\Omega$ we have $h_{\varepsilon,N} = h_{\rm ex}$. The difference is

$$\Delta = h_{\rm ex}(m_{0,\varepsilon} - m_{\varepsilon,N} + 1 - m_{0,\varepsilon}) = c_{\varepsilon,N} + \frac{1}{2}|\log\varepsilon'| \approx \frac{1}{2}|\log\varepsilon'|,$$

using (5.12). It follows using (5.20) that $d(\omega_\varepsilon, \Omega^c) \geq \sqrt{|\log\varepsilon'|/h_{\rm ex}} \gg \ell'_\varepsilon$.

We estimate $\hat{j}_\varepsilon$. From (7.6), $f_i(y) = -\log|y - y_i| + g_i((y - y_i)/\ell'_\varepsilon)$, where g_i solves $\Delta g_i(x) = \mathbf{1}_{\mathcal{C}_i}(y_i + \ell'_\varepsilon x)$ in $B(0, C)$ and $\partial_\nu g_i = 0$ on $\partial B(0, C)$. Since $\mathbf{1}_{\mathcal{C}_i} \in L^\infty$, elliptic regularity implies that $\|\nabla g_i\|_{L^q} \leq C_q$ for every $q \in [1, +\infty)$. We easily deduce that

$$\|j_i - \nabla^\perp \log|\cdot - y_i|\|_{L^q(B(y_i, C\ell'_\varepsilon))} \leq C_q {\ell'_\varepsilon}^{\frac{2}{q}-1}. \tag{7.10}$$

Since $j_i = 0$ outside $B(y_i, C\ell'_\varepsilon)$ we deduce

$$\|j_i\|_{L^q(\mathbb{R}^2 \setminus B(y_i, c\ell'_\varepsilon))} \leq C_q {\ell'_\varepsilon}^{\frac{2}{q}-1}\left(1 + c^{2-q}\right), \tag{7.11}$$

$$\|j_i\|_{L^2(\mathbb{R}^2 \setminus B(y_i, c\ell'_\varepsilon))} \leq C\left(1 + \log\frac{1}{c}\right). \tag{7.12}$$

Then we compute estimates for $\tilde{j}_\varepsilon$. Since j_R is defined independently of ε, there exists $\eta_0 > 0$ (depending on R) which bounds from below the distances between the points in Λ_R and between Λ_R and ∂K_R. Since $\operatorname{div} j_R = 0$ in K_R we have that $\|j_R - \nabla^\perp \log|\cdot - y|\|_{L^q(B(y,\eta_0))} \leq C_q$ for any $y \in \Lambda_R$ and moreover $j_R \in C^\infty_{\rm loc}(K_R \setminus \Lambda_R)$. It follows that $\forall y \in \widetilde{\Lambda}$ and $\forall q \in [1, +\infty)$ we have

$$\|\tilde{j}_\varepsilon - \nabla^\perp \log|\cdot - y|\|_{L^q(B(y, \eta_0 \ell'_\varepsilon))} \leq C_q {\ell'_\varepsilon}^{\frac{2}{q}-1}. \tag{7.13}$$

Moreover, since j_R is uniformly locally bounded in $L^{q'}$, for every $q' \in [1, 2)$, we have for every $x \in \mathbb{R}^2$ and every $M > 0$

$$\|\tilde{j}_\varepsilon\|_{L^{q'}(B(x, M\ell'_\varepsilon))} \leq C_{q'} M^{\frac{2}{q'}} {\ell'_\varepsilon}^{\frac{2}{q'}-1}. \tag{7.14}$$

We are ready to derive estimates for $j_\varepsilon = \tilde{j}_\varepsilon + \sum_{i=1}^n j_i$. First we note that from (7.5) we have that $|y_i - y_j|$ and $d(y_i, \tilde{\omega}_\varepsilon)$ are bounded below by $c\ell'_\varepsilon$, if $i \neq j$, therefore since $\operatorname{Supp} j_i \subset B(y_i, C\ell'_\varepsilon)$ and $\operatorname{Supp} \tilde{j}_\varepsilon \subset \tilde{\omega}_\varepsilon$ the overlap number of the supports of $\tilde{j}_\varepsilon$ and the j_i's is bounded by a constant C independent of R, ε. Moreover $\tilde{\omega}_\varepsilon$ is included in the complement of $\cup_i B(y_i, c\ell'_\varepsilon)$.

We have $\int_{\mathbb{R}^2} \hat{j}_\varepsilon \cdot \tilde{j}_\varepsilon = \sum_i \int_{\mathbb{R}^2} j_i \cdot \tilde{j}_\varepsilon$, but the number of i's for which the integrals above are nonzero is bounded by $C\frac{|\operatorname{Supp}\tilde{j}_\varepsilon \cap \operatorname{Supp}\hat{j}_\varepsilon|}{{\ell'_\varepsilon}^2}$ since $\operatorname{Supp}\tilde{j}_\varepsilon \cap \operatorname{Supp}\hat{j}_\varepsilon \subset \{d(x, \omega_\varepsilon^c) \leq C\ell'_\varepsilon\}$ and using Proposition 5.6, item 4. Applying Hölder's inequality to each of these nonzero integrals, and using (7.11), (7.14), we deduce

$$\int_{\mathbb{R}^2} \hat{j}_\varepsilon \cdot \tilde{j}_\varepsilon \leq C\frac{|\operatorname{Supp}\tilde{j}_\varepsilon \cap \operatorname{Supp}\hat{j}_\varepsilon|}{{\ell'_\varepsilon}^2} \times C_q {\ell'_\varepsilon}^{\frac{2}{q}-1} \times C_{q'} {\ell'_\varepsilon}^{\frac{2}{q'}-1} \leq C m_\varepsilon h_{\rm ex} o_\varepsilon(1), \tag{7.15}$$

Note that still from Proposition 5.6, item 4, $|\hat{\omega}_\varepsilon| = o_\varepsilon(1)|\omega_\varepsilon|$ and, since $\#\widehat{\Lambda} = {\ell'_\varepsilon}^{-2}(2\pi)^{-1}|\hat{\omega}_\varepsilon|$ and from (7.10) applied with $q = 2$ and (7.12), we deduce

$$\left|\frac{1}{2}\int_{\mathbb{R}^2\setminus\cup_{p\in\widehat{\Lambda}}B(p,\eta\ell'_\varepsilon)} |\hat{j}_\varepsilon|^2 + \pi\#\widehat{\Lambda}\log\eta\right| \le C\#\widehat{\Lambda} \le Cm_\varepsilon h_{\rm ex}|\hat{\omega}_\varepsilon| = o_\varepsilon(1)h_{\rm ex}|\omega_\varepsilon|. \tag{7.16}$$

Also, since by definition

$$W(j_R, \mathbf{1}_{K_R}) = \lim_{\eta\to 0}\left(\frac{1}{2}\int_{K_R\setminus\cup_{p\in\Lambda_R}B(p,\eta)} |j_R|^2 + \pi\#(\Lambda_R\cap K_R)\log\eta\right),$$

we have, multiplying by the number of squares in $\tilde{\omega}_\varepsilon$, which is $|\tilde{\omega}_\varepsilon|{\ell'_\varepsilon}^{-2}/R^2 = h_{\rm ex}m_\varepsilon|\tilde{\omega}_\varepsilon|/R^2$, that

$$\lim_{\eta\to 0}\frac{1}{m_\varepsilon h_{\rm ex}|\tilde{\omega}_\varepsilon|}\left(\frac{1}{2}\int_{\mathbb{R}^2\setminus\cup_{p\in\widetilde{\Lambda}}B(p,\eta\ell'_\varepsilon)} |\tilde{j}_\varepsilon|^2 + \pi\#\widetilde{\Lambda}\log\eta\right) = \frac{W(j_R, \mathbf{1}_{K_R})}{R^2}, \tag{7.17}$$

and the limit is uniform in ε since, despite the notation, the left-hand side only depends on R. From (7.15), (7.16) and (7.17) we deduce

$$\limsup_{\eta\to 0}\frac{1}{m_\varepsilon h_{\rm ex}|\omega_\varepsilon|}\left(\frac{1}{2}\int_{\mathbb{R}^2\setminus\cup_{p\in\Lambda}B(p,\eta\ell'_\varepsilon)} |j_\varepsilon|^2 + \pi\#\Lambda\log\eta\right) \le \left(\frac{W(j_R, \mathbf{1}_{K_R})}{R^2}\frac{|\tilde{\omega}_\varepsilon|}{|\omega_\varepsilon|} + o_\varepsilon(1)\right),$$

and this holds uniformly in ε. Since $|\hat{\omega}_\varepsilon| = o_\varepsilon(1)|\omega_\varepsilon|$, which implies that $|\tilde{\omega}_\varepsilon| = (1-o_\varepsilon(1))|\omega_\varepsilon|$, we obtain (7.8). Then (7.9) follows from (7.10), (7.13). □

7.3 Definition of the test-configuration

We next find $A_{1,\varepsilon}$ such that $\operatorname{curl} A_{1,\varepsilon} = m_\varepsilon h_{\rm ex}\mathbf{1}_{\omega_\varepsilon} = \mu_{\varepsilon,N}$, and set

$$A_\varepsilon = A_{1,\varepsilon} + \nabla^\perp h_{\varepsilon,N}.$$

To define u_ε, we start by defining its phase φ_ε by requiring

$$\nabla\varphi_\varepsilon = A_{1,\varepsilon} + j_\varepsilon. \tag{7.18}$$

Indeed, denoting by Θ the phase of $\prod_{p\in\Lambda}\frac{z-p}{|z-p|}$, we have by (7.7)

$$\operatorname{curl}(A_{1,\varepsilon} + j_\varepsilon - \nabla\Theta) = \mu_{\varepsilon,N} + 2\pi\sum_{p\in\Lambda}\delta_p - m_\varepsilon h_{\rm ex}\mathbf{1}_{\omega_\varepsilon} - 2\pi\sum_{p\in\Lambda}\delta_p = 0,$$

therefore $A_{1,\varepsilon} + j_\varepsilon - \nabla\Theta$ is the gradient of a function ψ, we may then let $\varphi_\varepsilon = \Theta + \psi$, this function is well-defined modulo 2π in $\Omega\setminus\Lambda$ and satisfies (7.18). Hence $e^{i\varphi_\varepsilon}$ is well-defined in $\Omega\setminus\Lambda$ and $\nabla\varphi_\varepsilon = A_{1,\varepsilon} + j_\varepsilon$.

Fixing $M > 1$, we then define

$$u_\varepsilon(x) = e^{i\varphi_\varepsilon(x)} \quad \text{in } \Omega\setminus\cup_{p\in\Lambda}B(p, M\varepsilon)$$
$$u_\varepsilon(x) = \tfrac{1}{f(M)}f\left(\tfrac{|x-p|}{\varepsilon}\right)e^{i\varphi(x)} \quad \text{in } B(p, M\varepsilon),$$

where f is the modulus of the unique radial degree-one vortex $u_0(r,\theta) = f(r)e^{i\theta}$ (see [BBH, Mi, HH]). f is increasing from 0 to M, so that in particular we deduce $|u_\varepsilon| \le 1$ everywhere.

This construction is possible since, for fixed R, the distances between the points in Λ are bounded below by $\eta_0\ell'_\varepsilon$ (for some η_0 possibly smaller than the one used before) and $\ell'_\varepsilon = \frac{1}{\sqrt{m_\varepsilon h_{\rm ex}}} \gg \varepsilon$ since we assume $h_{\rm ex} \ll \frac{1}{\varepsilon^2}$. The test-configuration $(u_\varepsilon, A_\varepsilon)$ is now defined and there remains to evaluate its energy and show that it satisfies (7.2), respectively (7.3).

7.4 Splitting of the energy of $(u_\varepsilon, A_\varepsilon)$

According to Proposition 5.3, we have the relation

$$G_\varepsilon(u_\varepsilon, A_\varepsilon) = G_\varepsilon^N + F_\varepsilon(u_\varepsilon, A_{1,\varepsilon}) - \int_\Omega (1-|u_\varepsilon|^2)|\nabla h_{\varepsilon,N}|^2,$$

where

$$F_\varepsilon(u_\varepsilon, A_{1,\varepsilon}) = \frac{1}{2}\int_\Omega |\nabla_{A_{1,\varepsilon}} u|^2 + (\operatorname{curl} A_{1,\varepsilon} - \mu_{\varepsilon,N})^2 + \frac{(1-|u_\varepsilon|^2)^2}{2\varepsilon^2}$$
$$+ \int_\Omega (h_{\varepsilon,N} - h_{\rm ex} - c_{\varepsilon,N})\mu(u_\varepsilon, A_{1,\varepsilon}) + c_{\varepsilon,N}\int_\Omega (\mu(u_\varepsilon, A_{1,\varepsilon}) - \mu_{\varepsilon,N}).$$

First we observe that $\int_\Omega (1-|u_\varepsilon|^2)|\nabla h_{\varepsilon,N}|^2 = 0$ since $\nabla h_{\varepsilon,N} = 0$ in ω_ε and $|u_\varepsilon| = 1$ outside $\cup_{p\in\Lambda} B(p, M\varepsilon)$, which is included in ω_ε if ε is small enough. Thus $G_\varepsilon(u_\varepsilon, A_\varepsilon) = G_\varepsilon^N + F_\varepsilon(u_\varepsilon, A_{1,\varepsilon})$. There remains to evaluate $F_\varepsilon(u_\varepsilon, A_{1,\varepsilon})$. By definition

$$\mu(u_\varepsilon, A_{1,\varepsilon}) = \operatorname{curl}(iu_\varepsilon, \nabla_{A_{1,\varepsilon}} u_\varepsilon) + \operatorname{curl} A_{1,\varepsilon} = \operatorname{curl}(|u_\varepsilon|^2(\nabla\varphi_\varepsilon - A_{1,\varepsilon})) + \mu_{\varepsilon,N}$$
$$= \operatorname{curl}(|u_\varepsilon|^2 j_\varepsilon) + \mu_{\varepsilon,N}.$$

Since $j_\varepsilon = 0$ on Ω^c, we have $\int_\Omega \mu(u_\varepsilon, A_{1,\varepsilon}) = \int_\Omega \mu_{\varepsilon,N} = 2\pi N$. Moreover, a direct computation shows that $\mu(u, A) = 0$ where $|u| \equiv 1$ so $\mu(u_\varepsilon, A_{1,\varepsilon})$ is supported in ω_ε, where $h_{\varepsilon,N} - h_{\rm ex} - c_{\varepsilon,N} = -\frac{1}{2}|\log\varepsilon'|$ (see (5.3)), so we deduce

$$\int_\Omega (h_{\varepsilon,N} - h_{\rm ex} - c_{\varepsilon,N})\mu(u_\varepsilon, A_{1,\varepsilon}) + c_{\varepsilon,N}\int_\Omega (\mu(u_\varepsilon, A_{1,\varepsilon}) - \mu_{\varepsilon,N}) = -\pi N|\log\varepsilon'|.$$

On the other hand, by choice of φ_ε,

$$\int_\Omega |\nabla_{A_{1,\varepsilon}} u_\varepsilon|^2 = \int_\Omega |\nabla|u_\varepsilon||^2 + \int_\Omega |u_\varepsilon|^2|\nabla\varphi_\varepsilon - A_{1,\varepsilon}|^2 = \int_\Omega |\nabla|u_\varepsilon||^2 + \int_\Omega |u_\varepsilon|^2|j_\varepsilon|^2.$$

Recalling that $\operatorname{curl} A_{1,\varepsilon} - \mu_{\varepsilon,N} = 0$, we are thus led to

$$G_\varepsilon(u_\varepsilon, A_\varepsilon) = G_\varepsilon^N - \pi N|\log\varepsilon'| + \frac{1}{2}\int_\Omega |u_\varepsilon|^2|j_\varepsilon|^2 + |\nabla|u_\varepsilon||^2 + \frac{(1-|u_\varepsilon|^2)^2}{2\varepsilon^2}. \tag{7.19}$$

It remains to estimate the terms on the right-hand side.

Lemma 7.6 (Energy in $B(p, M\varepsilon)$)**.** *For every $p \in \Lambda$, we have*

$$\frac{1}{2}\int_{B(p,M\varepsilon)} |u_\varepsilon|^2|j_\varepsilon|^2 + |\nabla|u_\varepsilon||^2 + \frac{(1-|u_\varepsilon|^2)^2}{2\varepsilon^2} = \pi\log M + \gamma + o_M(1) + o_\varepsilon(1) \tag{7.20}$$

where $o_M(1) \to 0$ as $M \to \infty$ and $o_\varepsilon(1) \to 0$ as $\varepsilon \to 0$, and γ is the constant of (1.52)*.*

Proof. From (1.52), and since $u_0(r,\theta) = f(r)e^{i\theta}$, we have

$$\gamma = \lim_{M\to+\infty}\left(\frac{1}{2}\int_0^M \left(f'^2 + \frac{f^2}{r^2} + \frac{(1-f^2)^2}{2}\right) 2\pi r\,dr - \pi\log M\right).$$

On the other hand, from (7.9) in Proposition 7.5 and since $|u_\varepsilon| \le 1$, we have for any $p \in \Lambda$, by Hölder's inequality

$$\int_{B(p,M\varepsilon)} |u_\varepsilon|^2 \left| j_\varepsilon - \nabla^\perp \log|\cdot - p| \right|^2 \le$$

$$|B(p,M\varepsilon)|^{1-\frac{2}{q}} \|j_\varepsilon - \nabla^\perp \log|\cdot-p|\|^2_{L^q(B(p,M\varepsilon))} \le \left(\frac{M\varepsilon}{\ell'_\varepsilon}\right)^{2-\frac{4}{q}} = o_\varepsilon(1),$$

choosing $q > 2$ and since $\ell'_\varepsilon \gg \varepsilon$. Moreover, choosing $q > 2$ and $\frac{1}{q} + \frac{1}{q'} = 1$, we have by Hölder again

$$\left| \int_{B(p,M\varepsilon)} |u_\varepsilon|^2 \nabla^\perp \log|\cdot-p| \cdot (j_\varepsilon - \nabla^\perp \log|\cdot - p|) \right| \le$$

$$\|j_\varepsilon - \nabla^\perp \log|\cdot-p|\|_{L^q(B(p,M\varepsilon))} \|\frac{1}{|x|}\|_{L^{q'}(B(0,\varepsilon))} \le C(\ell'_\varepsilon)^{\frac{2}{q}-1} \varepsilon^{\frac{2}{q'}-1} = o_\varepsilon(1)$$

where we used (7.9) and the fact that $\ell'_\varepsilon \gg \varepsilon$ by (1.48). Finally, since

$$\int_{B(p,M\varepsilon)} |u_\varepsilon|^2 \left|\nabla^\perp \log|\cdot-p|\right|^2 = \int_{B(p,M\varepsilon)} \frac{f^2(|x|/\varepsilon)}{|x|^2}\, dx,$$

we deduce that for each $p \in \Lambda$,

$$\int_{B(p,M\varepsilon)} |u_\varepsilon|^2 |j_\varepsilon|^2 = \int_{B(0,M\varepsilon)} \frac{f^2(|x|/\varepsilon)}{|x|^2} + o_\varepsilon(1).$$

Following [SS4] p. 210, we deduce that (7.20) holds. □

Next, we consider the energy in the annuli $B(p, \ell'_\varepsilon \eta) \setminus B(p, M\varepsilon)$, which are disjoint when $\eta < \eta_0$.

Lemma 7.7 (Energy in the annuli)**.** *For every $p \in \Lambda$, we have*

$$\frac{1}{2}\int_{B(p,\ell'_\varepsilon\eta)\setminus B(p,M\varepsilon)} |u_\varepsilon|^2|j_\varepsilon|^2 + |\nabla|u_\varepsilon||^2 + \frac{(1-|u_\varepsilon|^2)^2}{2\varepsilon^2} \le \pi \log\frac{\eta\ell'_\varepsilon}{M\varepsilon} + C\eta. \tag{7.21}$$

Proof. Since $|u_\varepsilon| = 1$ on the annulus $A = B(p, \ell'_\varepsilon\eta) \setminus B(p, M\varepsilon)$ only the first term in (7.21) needs to be bounded. Using (7.9) in Proposition 7.5 we have for $q > 2$

$$\int_A |j_\varepsilon - \nabla^\perp \log|\cdot-p||^2 \le |A|^{1-\frac{2}{q}} \|j_\varepsilon - \nabla^\perp \log|\cdot-p|\|^2_{L^q(A)} \le C_q \eta^{2-\frac{4}{q}}.$$

A similar argument yields, for any $q' \in [1, 2)$,

$$\left| \int_A (j_\varepsilon - \nabla^\perp \log|\cdot-p|) \cdot \nabla^\perp \log|\cdot-p| \right| \le C_{q'} \eta^{\frac{2}{q'}-1}.$$

We deduce, choosing for instance $q = 4$, $q' = 1$ above,

$$\frac{1}{2}\int_A |j_\varepsilon|^2 \le \frac{1}{2}\int_A |\nabla^\perp \log|\cdot-p||^2 + C\eta = \pi\log\frac{\eta\ell'_\varepsilon}{M\varepsilon} + C\eta,$$

proving (7.21). □

From (7.19), (7.8), (7.20), (7.21), and letting $M \to +\infty$, $\eta \to 0$, we deduce, since $m_\varepsilon \to m_\lambda$ as $\varepsilon \to 0$ and since $2\pi N = m_\varepsilon h_{\rm ex}|\omega_\varepsilon|$ and $\frac{\ell'_\varepsilon}{\varepsilon} = \frac{1}{\sqrt{m_\varepsilon \varepsilon'}}$ that for any $R > 0$

$$G_\varepsilon(u_\varepsilon, A_\varepsilon) \le G_\varepsilon^N + N\left(\pi \log \frac{1}{\sqrt{m_\lambda}} + \gamma\right) + 2\pi N \frac{W(j_R, \mathbf{1}_{K_R})}{|K_R|} + o_\varepsilon(N). \tag{7.22}$$

We recall our choice of j_R. To prove item 1 of the theorem j_R was the result of applying Corollary 4.4 to a minimizer of W, hence from (4.6) was such that

$$\limsup_{R\to\infty} \frac{W(j_R, \mathbf{1}_{K_R})}{|K_R|} \le \min_{\mathcal{A}_1} W,$$

so that letting $R \to \infty$ in (7.22) we find

$$G_\varepsilon(u_\varepsilon, A_\varepsilon) \le G_\varepsilon^N + N\left(2\pi(\min_{\mathcal{A}_1} W - \frac{1}{4}\log m_\lambda) + \gamma + o(1)\right).$$

In view of (1.12), this proves (7.2).

To prove item 2 of the theorem, we chose j_R given by Corollary 4.5 applied to $\bar{P}$, so that, using (7.4),

$$\limsup_{R\to\infty} \frac{W(j_R, \mathbf{1}_{K_R})}{|K_R|} \le \int W_K(j)\, d\bar{P}(j) = \frac{1}{m_\lambda}\int W_K(j)\, dP(j) + \frac{1}{4}\log m_\lambda.$$

By letting $R \to \infty$ in (7.22), the result (7.3) follows.

To conclude the proof of the theorem, it remains to show that $P_\varepsilon \to P$, where P_ε is the push-forward of the normalized Lebesgue measure on $\omega_{\varepsilon,N}$ by $x \mapsto \frac{1}{\sqrt{h_{\rm ex}}} j(u_\varepsilon, A_\varepsilon)\left(x + \frac{\cdot}{\sqrt{h_{\rm ex}}}\right)$. Equivalently it suffices to show that $\bar{P}_\varepsilon \to \bar{P}$ where $\bar{P}_\varepsilon$ is the push-forward of the normalized Lebesgue measure on $\omega_{\varepsilon,N}$ by $x \mapsto \frac{1}{\sqrt{m_\lambda h_{\rm ex}}} j(u_\varepsilon, A_\varepsilon)(x + \frac{\cdot}{\sqrt{m_\lambda h_{\rm ex}}})$. Let Φ be a continuous function on $L^p_{\rm loc}(\mathbb{R}^2, \mathbb{R}^2)$. By definition

$$\int \Phi(j)\, d\bar{P}_\varepsilon(j) = \fint_{\omega_{\varepsilon,N}} \Phi\left(\frac{1}{\sqrt{m_\lambda h_{\rm ex}}} j(u_\varepsilon, A_\varepsilon)(x + \frac{\cdot}{\sqrt{m_\lambda h_{\rm ex}}})\right) dx.$$

For any $\eta > 0$, we also have

$$\int \Phi(j)\, d\bar{P}_\varepsilon(j) = \fint_{\{\text{dist }(x, \partial\tilde{\omega_\varepsilon} \ge \eta)\}} \Phi\left(\frac{1}{\sqrt{m_\lambda h_{\rm ex}}} j(u_\varepsilon, A_\varepsilon)(x + \frac{\cdot}{\sqrt{m_\lambda h_{\rm ex}}})\right) dx + o_\eta(1)$$

where $o_\eta(1) \to 0$ as $\eta \to 0$. On the other hand by definition of $(u_\varepsilon, A_\varepsilon)$, $j(u_\varepsilon, A_\varepsilon) = j_\varepsilon + (|u_\varepsilon|^2 - 1)j_\varepsilon - |u_\varepsilon|^2 \nabla^\perp h_{\varepsilon,N}$. But $h_{\varepsilon,N}$ is constant in $\omega_{\varepsilon,N}$, and $(|u_\varepsilon|^2 - 1)j_\varepsilon \to 0$ in L^p so we deduce that in $\{\text{dist }(x, \partial\tilde{\omega_\varepsilon}) \ge \eta\}$ we have $\frac{1}{\sqrt{m_\lambda h_{\rm ex}}} j(u_\varepsilon, A_\varepsilon)(x + \frac{\cdot}{\sqrt{m_\lambda h_{\rm ex}}}) \to j_R$ in $L^p_{\rm loc}(\mathbb{R}^2, \mathbb{R}^2)$, as $\varepsilon \to 0$. It follows that

$$\int \Phi(j)\, d\bar{P}_\varepsilon(j) = \fint_{K_R} \Phi(j_R(x.))\, dx + o_\varepsilon(1) + o_\eta(1) = \int \Phi(j)\, dP_R(j) + o_\varepsilon(1) + o_\eta(1),$$

where we have used the periodicity of j_R, and where P_R is as in Corollary 4.5. But $P_R \to \bar{P}$ as $R \to \infty$ so letting $\varepsilon \to 0$, $\eta \to 0$, and $R \to +\infty$, we obtain the desired result.

A Additional results on the obstacle problem

Here we gather a few results on the obstacle problem that we need at various places in the paper. Although these results may be known to experts, we have not been able to find them in the literature, so they may be of independent interest. We focus on the particular type of obstacle problem we are concerned with, which is an obstacle problem with constant obstacle. More precisely, letting Ω be any smooth bounded domain in $\mathbb{R}^2$, for any $m \in (-\infty, 1]$ we denote by H_m the minimizer of

$$\min_{H-1\in H_0^1(\Omega)} (1-m)\int_\Omega |-\Delta H + H| + \frac{1}{2}\int_\Omega |\nabla H|^2 + |H-1|^2. \tag{A.1}$$

By convex duality and the maximum principle (cf. [Br, BS]) it is equivalent to the obstacle problem

$$\min_{\substack{H-1\in H_0^1(\Omega)\\ H\geq m}} \frac{1}{2}\int_\Omega |\nabla H|^2 + H^2. \tag{A.2}$$

We also define the coincidence set

$$\omega_m = \{x \in \Omega | H_m(x) = m\}.$$

For general references on obstacle problems we refer for example to [KS]. (A.2) is a standard obstacle problem where the obstacle is constant and equal to m, and we are interested in the properties of the coincidence set ω_m as m varies. It is known that

$$-\Delta H_m + H_m = m\mathbf{1}_{\omega_m}.$$

Note that the regularity of ω_m for fixed m is well-known (see [Caf, BK] or the survey [Mo]), however this is not sufficient for our purposes, since we need estimates which are uniform in m. More precisely let us define h_0 to be the minimizer of the unconstrained problem, i.e. the solution of

$$\begin{cases} -\Delta h_0 + h_0 = 0 & \text{in } \Omega \\ h_0 = 1 & \text{on } \partial\Omega, \end{cases}$$

and set

$$\underline{h}_0 = \min_\Omega h_0.$$

Then the situation is as follows:

1. If $m < \underline{h}_0$ then $\omega_m = \varnothing$ and $H_m = h_0$. Thus

$$\underline{h}_0 = 1 - \frac{1}{2\lambda_\Omega} \tag{A.3}$$

   where λ_Ω is as in Section 1.6.

2. If $\underline{h}_0 < m \leq 1$ then $\omega_m \neq \varnothing$. Moreover, as $m \nearrow 1$, $\omega_m \to \Omega$, and as $m \searrow \underline{h}_0$, ω_m reduces to the set of points where h_0 achieves its minimum $\underline{h}_0$. If we assume in addition that $\underline{h}_0$ is achieved at a unique point x_0 then as $m \searrow \underline{h}_0$, ω_m is expected to shrink down to x_0 in an ellipse shape.

Our task here is to establish more precisely this behaviour, in particular obtain some uniform estimates of convergence of ω_m (after blow-up at a suitable scale) to an ellipse. We recall that we make the assumption (1.31). It is standard (see [ST]) that there exists an ellipse E_Q of measure 1 and a nonnegative function U_Q defined in $\mathbb{R}^2$ such that

$$\Delta U_Q = \frac{\Delta Q}{2}\mathbf{1}_{\mathbb{R}^2\setminus E_Q}, \quad \{U_Q = 0\} = E_Q. \tag{A.4}$$

One may check that this U_Q is unique and that

$$U_Q = \mathrm{Cst} + \frac{Q}{2} - \frac{\Delta Q}{4\pi}\log * \mathbf{1}_{E_Q}.$$

This ellipse will be shown to be the limit of the coincidence sets ω_m as $m \searrow \underline{h}_0$, rescaled at a scale L_m which is not completely obvious to guess since it contains a logarithmic factor, more precisely

$$L_m \sim \sqrt{\frac{\pi(m-\underline{h}_0)}{\underline{h}_0|\log(m-\underline{h}_0)|}} \quad \text{as } m \searrow \underline{h}_0. \tag{A.5}$$

Our main result is the following.

Proposition A.1. *Let H_m be as above the minimizer of* (A.2). *The following holds.*

1. *The coincidence set ω_m is empty if $m < \underline{h}_0$ and has positive measure if $m > \underline{h}_0$. H_m is increasing with m, the coincidence set ω_m as well, and $m \mapsto m|\omega_m|$ is a continuous, strictly increasing bijection from $[\underline{h}_0, 1]$ to $[0, |\Omega|]$.*

2. *If K is any compact subset of $(\underline{h}_0, 1)$ then, uniformly with respect to $m \in K$,*

$$\lim_{\delta\to 0}\frac{|\{x \mid d(x,\partial\omega_m) < \delta\}|}{|\omega_m|} = 0, \quad \lim_{\delta\to 0}\frac{|\{m < H_m < m+\delta\}|}{|\omega_m|} = 0.$$

3. *For any $\delta > 0$, if $1-m$ is small enough then $x \in \Omega \setminus \omega_m$ implies that $d(x,\partial\Omega)^2 < (1-m)(2+\delta)$. In particular there exists a constant C depending only on Ω such that*

$$|\Omega\setminus\omega_m| \le C\sqrt{1-m} \qquad \|\nabla H_m\|_{L^\infty(\Omega)} \le C\sqrt{1-m}. \tag{A.6}$$

4. *Assuming (1.31), there is a length L_m such that*

$$L_m{}^2|\log L_m| \sim 2\pi(m-\underline{h}_0)/\underline{h}_0 \quad \text{as } m \to \underline{h}_0 \tag{A.7}$$

and such that for any $M, \delta > 0$, if m is sufficiently close to $\underline{h}_0$ then

$$\begin{gathered}\{d(x,\omega_Q{}^c) > \delta L_m\} \subset \omega_m \subset \{d(x,\omega_Q) < \delta L_m\},\\ \left\{\frac{H_m - m}{L_m{}^2} \le M\right\} \subset x_0 + L_m\{U_Q \le M+\delta\},\\ \left\{\frac{H_m - m}{L_m{}^2} \ge M\right\} \subset x_0 + L_m\{U_Q \ge M-\delta\}.\end{gathered} \tag{A.8}$$

where $\omega_Q = x_0 + L_m E_Q$. In particular $|\omega_m| \sim L_m{}^2$ as $m \to \underline{h}_0$.

The main technique is the construction of barriers:

Lemma A.2. *Let H_m be as above. Let h be a continuous function.*

Interior barrier. *If $h \geq H_m$ on $\partial\Omega$, if $h \geq m$ in Ω and if $-\Delta h + h \geq 0$ in Ω, then*

$$h \geq H_m \text{ in } \Omega, \text{ in particular } \{h = m\} \subset \omega_m.$$

Exterior barrier. *If $h \leq H_m$ on $\partial\Omega$, if $h \geq m$ in Ω and if $-\Delta h + h \leq m\mathbf{1}_{\{h=m\}}$ in Ω, then*

$$h \leq H_m \text{ in } \Omega, \text{ in particular } \omega_m \subset \{h = m\}.$$

Both assertions are easy and standard applications of the maximum principle.

We now turn to the proof of the proposition. First recall that from a well known result of J. Frehse ([Fr]), H_m is $C^{1,1}$ and, as a consequence, $-\Delta H_m + H_m = m\mathbf{1}_{\omega_m}$.

Proof of 1). If $m < \beta$, then $H_m + (\beta - m)$ is an interior barrier for h_β, thus $\omega_m \subset \omega_\beta$ (hence ω_m is increasing in m). If $m < \underline{h}_0$ then h_0 is an exterior barrier for H_m hence $\omega_m = \varnothing$. If $m > \underline{h}_0$ and ω_m is negligible, then $-\Delta H_m + H_m = m\mathbf{1}_{\omega_m} = 0$ in Ω and $H_m = 1$ on $\partial\Omega$ hence $H_m = h_0$, which is impossible. By Lemma A.2 too, for $m \leq m'$ we have $H_m \leq H_{m'} \leq H_m + (m' - m)$, so H_m is increasing in m, and also $H_{m'} \to H_m$ uniformly in Ω as m' tends to m from above, and then in the distributions sense too. Hence, using $-\Delta(H_{m'} - H_m) + (H_{m'} - H_m) \geq m\mathbf{1}_{\omega_{m'} \setminus \omega_m}$, we find that $|\omega_{m'} \setminus \omega_m|$ tends to zero as $m' \to m$. It follows that $m \mapsto m|\omega_m|$ is a continuous strictly increasing function of m. For $m = \underline{h}_0$ it is equal to 0, while for $m = 1$, it is immediate that $H_m = 1$ and $\omega_m = \Omega$, so $m \mapsto m|\omega_m|$ maps $[\underline{h}_0, 1]$ to $[0, |\Omega|]$ bijectively. □

Proof of 2). Let $\omega_{m,\delta} = \{x \mid d(x, \partial\omega_m) < \delta\}$. Then it is true that $\lim_{\delta\to 0} |\omega_{m,\delta}| = 0$, indeed $\cap_{\delta>0}\omega_{m,\delta} = \partial\omega_m$ and $|\partial\omega_m| = 0$, see [BK]. From the previous step $\lim_{\delta\to 0} |\omega_{m+\delta} \setminus \omega_{m-\delta}| = 0$. Thus for any $\varepsilon > 0$ and any $m \in K$ there exists $\beta > m$ such that $|\omega_\beta \setminus \omega_m| < \varepsilon$ and then there exists $\delta > 0$ such that $|\omega_{\beta,\delta}| < \varepsilon$ and $|\omega_{m,\delta}| < \varepsilon$. Then for any $m' \in [m, \beta]$, it holds that

$$\omega_{m',\delta} \subset \omega_{m,\delta} \cup (\omega_\beta \setminus \omega_m) \cup \omega_{\beta,\delta}$$

hence $|\omega_{m',\delta}| < 3\varepsilon$. Then by the compactness of K, $\lim_{\delta\to 0} |\omega_{m,\delta}| = 0$ uniformly in $m \in K$, which is what we want since from 1) we have that $\inf_{m\in K} |\omega_m| > 0$.

Similarly, we have $\cap_{\delta>0}\{m < H_m < m + \delta\} = \varnothing$ and, since $m' \geq m$ implies $H_m \leq H_{m'} \leq H_m + (m' - m)$, we have

$$\{m' < H_{m'} < m' + \delta\} \subset \{m < H_m < m' + \delta\}.$$

Thus for any $m \in K$ and $\varepsilon > 0$, taking δ such that $|\{m < H_m < m + 2\delta\}| < \varepsilon$ we find $|\{m' < H_{m'} < m' + \delta\}| < \varepsilon$ for any $m' \in [m, m + \delta]$ and it follows as above that $\lim_{\delta\to 0} |\{m < H_m < m + \delta\}| = 0$ uniformly in $m \in K$. □

Proof of 3). We let $d(x) = d(x, \partial\Omega)$ and

$$h(x) = \begin{cases} (1 - m)\frac{(d-\eta)^2}{\eta^2} + m & \text{if } d(x) \leq \eta \\ m & \text{if } d(x) > \eta. \end{cases}$$

We claim that if η is well-chosen and $1-m$ is small enough, the function h is an interior barrier for H_m. Indeed $h(x) = f \circ d$ hence $\Delta h = f''(d)|\nabla d|^2 + f'(d)\Delta d$ with

$$f'(d) = \frac{2(1-m)}{\eta^2}(d-\eta)\mathbf{1}_{d\le\eta}, \quad f''(d) = \frac{2(1-m)}{\eta^2}\mathbf{1}_{d\le\eta}.$$

In addition we have $|\nabla d| = 1$ and $|\Delta d| \le \kappa_\Omega$, where κ_Ω is the maximum of the curvature on $\partial\Omega$. Thus $-\Delta h + h$ is trivially positive on $\{d \ge \eta\}$ while on the set $\{d < \eta\}$ we have

$$-\Delta h + h \ge -2\frac{1-m}{\eta^2} - 2\frac{1-m}{\eta}\kappa_\Omega + m.$$

Letting $\eta^2 = (2+\delta)(1-m)$, the right-hand side is positive for m close enough to 1 (depending on δ). Then h is an interior barrier for H_m and we deduce that $\{d^2 > (2+\delta)(1-m)\} \subset \omega_m$. The first result in (A.6), i.e. $|\Omega\backslash\omega_m| \le C\sqrt{1-m}$, follows immediately, for some appropriate C depending on Ω. The second assertion is a consequence of the first one, together with the estimate

$$\|H_m\|_{C^{1,1}(\Omega)} \le C,$$

with C independent of m (see [BK]). Indeed, either $x \in \omega_m$ and then $\nabla H_m(x) = 0$, or $x \notin \omega_m$ and since there exists $y \in \omega_m$ such that $|x-y| \le C\sqrt{1-m}$, the $C^{1,1}$ estimate implies that

$$|\nabla H_m(x)| \le |\nabla H_m(y)| + C\sqrt{1-m} = C\sqrt{1-m}.$$

□

Proof of 4). Let $\omega_Q = x_0 + L_m E_Q$, for some L_m to be specified below, and define h to be the solution of $-\Delta h + h = m\mathbf{1}_{\omega_Q}$ in Ω and $h = 1$ on $\partial\Omega$. Then we may express h as

$$h(x) = m\int G_\Omega(x,y)\mathbf{1}_{\omega_Q}(y)\,dy + h_0(x),$$

where $G_\Omega(\cdot,y)$ is the solution of $-\Delta G_\Omega + G_\Omega = \delta_y$ in Ω and $G_\Omega = 0$ on $\partial\Omega$. We further split G_Ω as

$$G_\Omega(x,y) = -\frac{1}{2\pi}\log|x-y| + S_\Omega(x,y),$$

where $S_\Omega(\cdot,y)$ solves $-\Delta S_\Omega + S_\Omega = (2\pi)^{-1}\log|\cdot - y|$ in Ω, thus is C^1 locally in Ω.

Replacing accordingly in the expression of h and writing $x = x_0 + L_m x'$, we obtain (using the fact that the volume of E_Q is 1)

$$h(x) = \underline{h}_0 - \frac{m}{2\pi}{L_m}^2\log L_m + {L_m}^2 m S_\Omega(x_0,x_0) + {L_m}^2\left(\frac{1}{2}Q(x') - \frac{m}{2\pi}\log*\mathbf{1}_{E_Q}(x')\right) + R(x),$$

where

$$R(x) = \left(h_0(x) - \underline{h}_0 - \frac{1}{2}{L_m}^2 Q(x')\right) + m\int\left(S_\Omega(x,y) - S_\Omega(x_0,x_0)\right)\mathbf{1}_{\omega_Q}(y)\,dy.$$

The first term in $R(x)$ is the remainder of the Taylor expansion of order 2 of h_0 at x_0. It is therefore $O(|x-x_0|^3)$ and its derivatives are $O(|x-x_0|^2)$ since h_0 is analytic. Since $|\omega_Q| = {L_m}^2$ and since $y \in \omega_Q \implies |y-x_0| \le CL_m$, the second term in $R(x)$ is clearly $O({L_m}^2|x-x_0|)$ and, since S_Ω is C^1, differentiating under the integral sign shows its derivatives are $O({L_m}^2)$.

Differentiating twice allows to bound its second derivatives by $\|D^2 S_\Omega\|_{L^q} \|\mathbf{1}_{\omega_Q}\|_{L^{q'}}$ and using the equation satisfied by S_Ω we have that $\|D^2 S_\Omega\|_{L^q}$ for every $q < +\infty$, while $\|\mathbf{1}_{\omega_Q}\|_{L^{q'}} = {L_m}^{\frac{2}{q'}}$. Thus the second derivatives of the second term are bounded by $C_p {L_m}^p$, for every $p < 2$. To sum up, for any $p < 2$ we have

(A.9)
$$R(x) = O\left(L_m^3 + |x-x_0|^3\right), \ \nabla R(x) = O\left(L_m^2 + |x-x_0|^2\right), \ \nabla^2 R(x) = O\left(L_m^p + |x-x_0|\right).$$

Then we note that $\frac{1}{2}Q - \frac{\underline{h}_0}{2\pi} \log * \mathbf{1}_{E_Q}$ is equal to $U_Q + C_{\Omega,Q}$ where $C_{\Omega,Q}$ is a constant. Indeed, note that since $Q = D^2 h_0(x_0)$ and $-\Delta h_0 + h_0 = 0$ we have $\frac{1}{2}\Delta Q = \Delta h_0(x_0) = h_0(x_0) = \underline{h}_0$, and the claim follows from (A.4). Thus, if we choose for L_m a solution of the following equation

$$m - \underline{h}_0 = -\frac{\underline{h}_0}{2\pi} {L_m}^2 \log L_m + {L_m}^2 \left(\underline{h}_0 S_\Omega(x_0, x_0) + C_{\Omega,Q}\right), \tag{A.10}$$

we have

$$h(x) = m + {L_m}^2 U_Q(x') + R'(x), \tag{A.11}$$

where

$$R'(x) = R(x) + {L_m}^2 (\underline{h}_0 - m) \left(\frac{1}{2\pi} \log * \mathbf{1}_{E_Q}(x') + \frac{1}{2\pi} \log L_m - S_\Omega(x_0, x_0) \right).$$

Since $|\underline{h}_0 - m| \le C {L_m}^2 \log L_m$, we easily deduce that R' satisfies the same properties as R, i.e. (A.9). Returning to (A.10), since the term ${L_m}^2 \log L_m$ dominates, we deduce that ${L_m}^2 |\log L_m| \sim 2\pi(m - \underline{h}_0)/\underline{h}_0$ as $m \to \underline{h}_0$ and that (A.7) and (A.5) holds. From (A.11) and (A.9) for R', we deduce that for any $M, \delta > 0$ and if $m - \underline{h}_0$ is small enough, then

$$\{U_Q < M - \delta\} \subset \left\{ x' \mid \frac{h(x_0 + L_m x') - m}{{L_m}^2} < M \right\} \subset \{U_Q < M + \delta\}. \tag{A.12}$$

Indeed (A.9), (A.11) imply local uniform convergence of $\frac{h(x_0 + L_m x') - m}{{L_m}^2}$ to U_Q as $m - \underline{h}_0$ decreases to 0, which implies that L_m decreases to 0 by (A.5). Thus it suffices to check that $h(x_0 + L_m x') \le m + M {L_m}^2$ implies a uniform bound for x'. This is the case because the equation satisfied by h implies that $\|h - h_0\|_\infty \to 0$ as $m \searrow \underline{h}_0$. Thus $h(x) \le m + M {L_m}^2$ implies that $x - x_0$ is small as $m \searrow \underline{h}_0$. Then, $|x - x_0|^3 = o(|x - x_0|^2) = o({L_m}^2 + {L_m}^2 U_Q(x'))$ since $U_Q(x') \sim \frac{1}{2} Q(x')$ as $|x'| \to +\infty$. Therefore $R'(x) = o({L_m}^2 + {L_m}^2 U_Q(x'))$ and we deduce from (A.11) a bound for $U_Q(x')$, hence for x'. Note in particular that the minimum of $h(x_0 + L_m x')$ is achieved in a fixed compact set of $\mathbb{R}^2$, and then (A.9), (A.11) yield

$$\underline{h} := \min_\Omega h = O({L_m}^3 + m). \tag{A.13}$$

Next, we use h to construct an exterior and an interior barrier for H_m. The exterior barrier is defined by $h_{\text{out}} = \max(h - \delta {L_m}^2, m)$. Where $h_{\text{out}} = m$ it is obvious that $-\Delta h_{\text{out}} + h_{\text{out}} = m$. Then by definition $h_{\text{out}}(x) \ne m$ implies that $h(x) - \delta {L_m}^2 > m$ and then from (A.11)–(A.9) if $m - \underline{h}_0$ is small enough, we must have $x \notin \omega_Q$ (recall that $U_Q = 0$ in E_Q). Therefore where $h_{\text{out}} > m$ we have

$$-\Delta h_{\text{out}} + h_{\text{out}} = -\Delta h + h - \delta {L_m}^2 = -\delta {L_m}^2.$$

Therefore $-\Delta h_{\text{out}} + h_{\text{out}} \leq m\mathbf{1}_{\{h_{\text{out}}=m\}}$. The other properties of exterior barriers are trivially verified by h_{out} thus $h_{\text{out}} \leq H_m$ and then, using (A.12),

$$\begin{aligned} \text{(A.14)} \qquad & \omega_m \subset \{h_{\text{out}} = m\} = \{h \leq m + \delta {L_m}^2\} \subset \{x_0 + L_m y \mid U_Q(y) \leq 2\delta\} \\ & \{H_m \leq m + M{L_m}^2\} \subset \{h_{\text{out}} \leq m + M{L_m}^2\} \subset \{h \leq m + (M+\delta){L_m}^2\} \\ & \qquad \subset \{x_0 + L_m y \mid U_Q(y) \leq M + 2\delta\}. \end{aligned}$$

For the interior barrier, let $\chi : \mathbb{R}_+ \to \mathbb{R}_+$ be smooth and such that $\chi = 1$ in a neighborhood of 0 and $\chi_{|[1,+\infty)} = 0$. Then let $\underline{h} = \min(m, \min_\Omega h)$ and

$$h_{\text{int}}(x) = m + (h(x) - \underline{h})\varphi(x), \quad \varphi(x) = \chi\left(\frac{d(x, {\omega_Q}^c)}{\delta L_m}\right).$$

We have $-\Delta h_{\text{int}} + h_{\text{int}} = -\Delta h + m + h - \underline{h} = m - \underline{h} \geq 0$ on ${\omega_Q}^c$ and $h_{\text{int}} = 1 + (m - \underline{h}) \geq 1$ on $\partial\Omega$. Moreover, $h_{\text{int}} \geq m$. It remains to check that $-\Delta h_{\text{int}} + h_{\text{int}} \geq 0$ in ω_Q.

Since $U_Q = 0$ on E_Q and using (A.11),(A.9), we have on ω_Q

$$\Delta h_{\text{int}} = \varphi\Delta h + 2\nabla\varphi \cdot \nabla h + (h - \underline{h})\Delta\varphi = O\left(L_m + \frac{{L_m}^2}{\delta} + \frac{{L_m}^3}{\delta^2}\right).$$

Indeed, since $U_Q = 0$ on ω_Q, we have $h - \underline{h} = O({L_m}^3)$ and $\nabla(h - \underline{h}) = O({L_m}^2)$, $\Delta(h - \underline{h}) = O(L_m)$ in ω_Q. It follows that if $m - \underline{h}_0$ is small enough depending on δ, then $-\Delta h_{\text{int}} + h_{\text{int}} \geq m/2$ in ω_Q, finishing the proof that h_{int} is an interior barrier for H_m. Then $h_{\text{int}} \geq H_m$ in Ω and it follows easily using (A.12) that

$$\begin{aligned} \text{(A.15)} \qquad & \{d(x, {\omega_Q}^c) > \delta L_m\} \subset \{h_{\text{int}} = m\} \subset \omega_m \\ & \{H_m \geq m + M{L_m}^2\} \subset \{h_{\text{int}} \geq m + M{L_m}^2\} = \{h \geq m + M{L_m}^2\} \\ & \qquad \subset \{x_0 + L_m y \mid U_Q(y) \geq M - \delta\}. \end{aligned}$$

The relation $\{h_{\text{int}} \geq m + M{L_m}^2\} = \{h \geq m + M{L_m}^2\}$ follows from the fact that $h_{\text{int}} = h$ outside ω_Q and that if $m - \underline{h}_0$ is small, then both sets are disjoint from ω_Q: Indeed if $x \in \omega_Q$ then from (A.9), (A.11) we have $h(x) - m = O({L_m}^3)$ and $h_{\text{int}}(x) - m = O({L_m}^3)$ from (A.13). Then, (A.8) follows from (A.14), (A.15). The fact that $|\omega_m| \sim {L_m}^2$ as $m \to \underline{h}_0$ is an easy consequence.

□

References

[ACO] G. Alberti, R. Choksi, F. Otto, Uniform Energy Distribution for an Isoperimetric Problem With Long-range Interactions. *Journal Amer. Math. Soc.* **22**, no 2 (2009), 569-605.

[AM] G. Alberti, S. Müller, A new approach to variational problems with multiple scales. *Comm. Pure Appl. Math.* **54**, no. 7 (2001), 761-825.

[AS] A. Aftalion, S. Serfaty, Lowest Landau level approach in superconductivity for the Abrikosov lattice close to H_{c_2}. *Selecta Math* **13** (2007), no. 2, 183–202.

[Ay] H. Aydi, Doctoral Dissertation. Université Paris-12, 2004.

[AyS] H. Aydi, E. Sandier, Vortex analysis of the periodic Ginzburg-Landau model. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **26** (2009), no. 4, 1223–1236.

[Be] M. E. Becker, Multiparameter groups of measure-preserving transformations: a simple proof of Wiener's ergodic theorem. *Ann Probability* **9**, No 3 (1981), 504–509.

[BSS] L. Berlyand, E. Sandier, S. Serfaty, in preparation.

[BBH] F. Bethuel, H. Brezis, F. Hélein, *Ginzburg-Landau Vortices*, Birkhäuser, 1994.

[Bi] P. Billingsley, *Convergence of probability measures. Second edition.* Wiley & Sons, New York, 1999.

[BoS] A. Borodin, S. Serfaty, Renormalized Energy Concentration in Random Matrices, preprint.

[Br] H. Brezis, Problèmes unilatéraux. *J. Math. Pures Appl. (9)*, **51**, (1972), 1–168.

[BK] H. Brezis, D. Kinderlehrer The smoothness of Solutions to Nonlinear Variational Inequalities. *Indiana Univ. Math. J.* **23** (1973/74), 831–844.

[BS] H. Brezis, S. Serfaty, A variational formulation for the two-sided obstacle problem with measure data. *Commun. Contemp. Math.* **4** (2002), no. 2, 357–374.

[Caf] L. Caffarelli, The regularity of free boundaries in higher dimensions. *Acta Math.* **139** (1977), no. 3-4, 155–184.

[CF] L.A. Caffarelli, A. Friedman, Convexity of solutions of semilinear elliptic equations, *Duke Math. J.* **52**, Number 2 (1985), 431-456.

[Cas] J. W. S. Cassels, On a problem of Rankin about the Epstein zeta-function. *Proc. Glasgow Math. Assoc.* **4** (1959), 73–80.

[CO] X. Chen, Y. Oshita, An application of the modular function in nonlocal variational problems. *Arch. Rat. Mech. Anal.* **186** (2007) 109–132.

[Co] D. L. Cohn, *Measure Theory*, Birkhaüser, 1980.

[CK] H. Cohn, A. Kumar, Universally optimal distribution of points on spheres. *J. Amer. Math. Soc.* **20** (2007), no. 1, 99–148.

[DeG] P. G. DeGennes, *Superconductivity of metal and alloys.* Benjamin, New York and Amsterdam, 1966.

[Di] P. H. Diananda, Notes on two lemmas concerning the Epstein zeta-function. *Proc. Glasgow Math. Assoc.* **6**, (1964), 202–204.

[DM] J. Dolbeault, R. Monneau, Convexity Estimates for Nonlinear Elliptic Equations and Application to Free Boundary Problems. *Ann. Inst. H. Poincaré, Analyse Non Linéaire* **19**, No 6, 903-926 (2002).

[Du] R. Dudley, *Real analysis and probability*, Cambridge Studies in Advanced Mathematics, 74, Cambridge University Press, 2002.

[En1] V. Ennola, A remark about the Epstein zeta function. *Proc Glasg Math Assoc.* **6**, (1964), 198–201.

[En2] V. Ennola, On a problem about the Epstein zeta-function. *Proc. Cambridge Philos. Soc.* **60** (1964), 855–875.

[FH1] S. Fournais, B. Helffer, On the third critical field in Ginzburg-Landau theory. *Comm. Math. Phys.* **266**, 153-196 (2006).

[FH2] S. Fournais, B. Helffer, *Spectral Methods in Surface Superconductivity*, to appear in Progress in Nonlinear Differential Equations, Birkhäuser.

[Fr] J. Frehse, On the regularity of the solution of a second order variational inequality. *Boll. Un. Mat. Ital. (4)* **6** (1972), 312–315.

[FP] A. Friedman, D. Phillips, The free boundary of a semilinear elliptic equation. *Trans. Amer. Math. Soc.* **282** (1984), no. 1, 153–182.

[GMS1] D. Goldman, C. Muratov, S. Serfaty, The Γ-limit of the two dimensional Ohta-Kawasaki energy. Part I: Droplet density, `http://arxiv.org/abs/1201.0222`.

[GMS2] D. Goldman, C. Muratov, S. Serfaty, The Γ-limit of the two dimensional Ohta-Kawasaki energy. Part II: Derivation of the renormalized energy, in preparation.

[HH] R. M. Hervé, M. Hervé, Étude qualitative des solutions réelles d'une équation différentielle liée à l'équation de Ginzburg-Landau. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **11** (1994), no. 4, 427–440.

[Is] V. M. Isakov, Inverse theorems on the smoothness of potentials. *Differencial'nye Uravnenija* **11** (1975), 66–74, 202.

[Je] R. L. Jerrard, Lower bounds for generalized Ginzburg-Landau functionals. *SIAM J. Math. Anal.* **30** (1999), no. 4, 721-746.

[JS] R. L. Jerrard, H. M. Soner, The Jacobian and the Ginzburg-Landau energy. *Calc. Var. Partial Differential Equations* **14** (2002), no. 2, 151–191.

[Ka] B. Kawohl, When are solutions to nonlinear elliptic boundary value problems convex?, *Comm. PDE* **10** (1985), 1213-1225.

[KN] D. Kinderlehrer, L. Nirenberg, Regularity in free boundary problems. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **4** (1977), no. 2, 373–391.

[KS] D. Kinderlehrer, G. Stampacchia, *An introduction to variational inequalities and their applications*, Pure and Applied Mathematics, 88. Academic Press, 1980.

[KL] N. Korevaar, J.L. Lewis, Convex solutions of certain elliptic equations have constant rank Hessians. *Arch. Rational Mech. Anal.* **97** (1987), No. 1, 19–32.

[La1] S. Lang, *Elliptic functions*, Springer-Verlag, New York, 1987.

[La2] S. Lang, *Introduction to Arakelov theory*, Springer-Verlag, New York, 1988.

[Mi] P. Mironescu, Les minimiseurs locaux pour l'équation de Ginzburg-Landau sont à symétrie radiale. *C. R. Acad. Sci. Paris Sér. I Math.* **323** (1996), no. 6, 593–598.

[Mo] R. Monneau, A brief overview on the obstacle problem. *European Congress of Mathematics, Vol. II* , 303–312, Progr. Math., 202, Birkhäuser, Basel, 2001

[Mont] H. L. Montgomery, Minimal Theta functions. *Glasgow Math J.* **30**, (1988), No. 1, 75-85, (1988).

[Mu] C. Muratov, Droplet phases in non-local Ginzburg-Landau models with Coulomb repulsion in two dimensions, to appear in *Comm. Math. Phys.*

[Rad] C. Radin, The ground state for soft disks. *J. Stat. Phys.* **26** (2), (1981), 367–372.

[Ran] R. A. Rankin, A minimum problem for the Epstein zeta function,. *Proc. Glasgow Math. Assoc, 1* (1953), 149-158.

[Ri] N. M. Rivière, Singular integrals and multiplier operators. *Ark. Mat.* **9**, 243–278. (1971).

[ST] E. Saff, V. Totik, *Logarithmic potentials with external fields*, Springer-Verlag, Berlin, 1997.

[SST] D. Saint-James, G. Sarma, E. J. Thomas, *Type-II superconductivity.* Pergamon Press, Oxford, 1969.

[Sa] E. Sandier, Lower bounds for the energy of unit vector fields and applications. *J. Funct. Anal.* **152** (1998), No. 2, 379–403.

[SS1] E. Sandier, S. Serfaty, Global Minimizers for the Ginzburg-Landau Functional below the First critical Magnetic Field. *Annales Inst. H. Poincaré, Anal. non linéaire* **17**, (2000), No 1, 119-145.

[SS2] E. Sandier, S. Serfaty, The decrease of bulk-superconductivity close to the second critical field in the Ginzburg-Landau model. *SIAM J. Math. Anal.* **34** (2003), No. 4, 939–956.

[SS3] E. Sandier, S. Serfaty, Improved Lower Bounds for Ginzburg-Landau Energies via Mass Displacement, to appear in *Analysis and PDE*, `http://arxiv.org/abs/1011.4616`.

[SS4] E. Sandier, S. Serfaty, *Vortices in the Magnetic Ginzburg-Landau Model*, Birkhäuser, 2007.

[SS5] Erratum to the above, available at `www.ann.jussieu.fr/∼serfaty/publis.html`

[SS6] E. Sandier, S. Serfaty, 2D Coulomb Gases and the Renormalized Energy, preprint.

[SS7] E. Sandier, S. Serfaty, 1D Log Gases and the Renormalized Energy, in preparation.

[ST] S. Serfaty, I. Tice, Lorentz Space Estimates for the Coulombian Renormalized Energy, to appear in *Comm. Contemp. Math.*

[St] M. Struwe, On the asymptotic behavior of minimizers of the Ginzburg-Landau model in 2 dimensions. *Diff. and Int. Equations* **7** (1994). No 5-6, 1613-1624.

[Th] F. Theil, A proof of crystallization in two dimensions. *Comm. Math. Phys.* **262** (2006), No. 1, 209–236.

[Ti] M. Tinkham, *Introduction to superconductivity.* Second edition. McGraw-Hill, New York, 1996.

Etienne Sandier
Université Paris-Est,
LAMA – CNRS UMR 8050,
61, Avenue du Général de Gaulle, 94010 Créteil. France
& Institut Universitaire de France
sandier@u-pec.fr

Sylvia Serfaty
UPMC Univ Paris 06, UMR 7598 Laboratoire Jacques-Louis Lions,
Paris, F-75005 France ;
CNRS, UMR 7598 LJLL, Paris, F-75005 France
& Courant Institute, New York University
251 Mercer st, NY NY 10012, USA
serfaty@ann.jussieu.fr