2D Coulomb Gases and the Renormalized Energy

Etienne Sandier and Sylvia Serfaty

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Abstract

We study the statistical mechanics of classical two-dimensional "Coulomb gases" with general potential and arbitrary β, the inverse of the temperature. Such ensembles also correspond to random matrix models in some particular cases. The formal limit case $\beta = \infty$ corresponds to "weighted Fekete sets" and also falls within our analysis.

It is known that in such a system points should be asymptotically distributed according to a macroscopic "equilibrium measure," and that a large deviations principle holds for this, as proven by Ben Arous and Zeitouni [BZ].

By a suitable splitting of the Hamiltonian, we connect the problem to the "renormalized energy" W, a Coulombian interaction for points in the plane introduced in [SS1], which is expected to be a good way of measuring the disorder of an infinite configuration of points in the plane. By so doing, we are able to examine the situation at the microscopic scale, and obtain several new results: a next order asymptotic expansion of the partition function, estimates on the probability of fluctuation from the equilibrium measure at microscale, and a large deviations type result, which states that configurations above a certain threshhold of W have exponentially small probability. When $\beta \to \infty$, the estimate becomes sharp, showing that the system has to "crystallize" to a minimizer of W. In the case of weighted Fekete sets, this corresponds to saying that these sets should microscopically look almost everywhere like minimizers of W, which are conjectured to be "Abrikosov" triangular lattices.

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1 Introduction

We are interested in studying the probability law

$$d\mathbb{P}_n^\beta(x_1,\dots,x_n) = \frac{1}{Z_n^\beta} e^{-\frac{\beta}{2} w_n(x_1,\dots,x_n)} dx_1 \dots dx_n \tag{1.1}$$

where Z_n^β is the associated partition function, i.e. a normalizing factor such that $\mathbb{P}_n^\beta$ is a probability measure, and where

$$w_n(x_1,\dots,x_n) = -\sum_{i\neq j} \log|x_i - x_j| + n\sum_{i=1}^n V(x_i), \tag{1.2}$$

most often called the Hamiltonian. Here the x_i's belong to $\mathbb{R}^2$ (identified with the complex plane $\mathbb{C}$), $\beta > 0$ is a parameter corresponding to (the inverse of) the temperature and V is a potential satisfying some growth and regularity assumptions, which we will detail below.

The probability law $\mathbb{P}_n^\beta$ is the Gibbs measure of what is called either a classical "two-dimensional Coulomb system" or "Coulomb gas" or "two-dimensional log gas", or "two-dimensional one-component plasma", or also "Gaussian β-ensemble". It was first pointed out by Dyson [Dy] that Coulomb gases are naturally related to random matrices. This is somehow due to the fact that $e^{\sum_{i\neq j}\log|x_i-x_j|}$ is the square of the Vandermonde determinant $\prod_{i<j}|x_i - x_j|$ and thus the law $\mathbb{P}_n^\beta$, in the particular case when $V(x) = |x|^2$ and $\beta = 2$ corresponds, as shown in [G] (see also [Me], Chap. 15), to the law of eigenvalues for the Ginibre ensemble, which is the set of matrices with independent standard (complex) Gaussian entries. For the general background and references to the literature, we refer to the book by Forrester [Fo] and references therein.

The Gibbs measure $\mathbb{P}_n^\beta$ can also be studied for x_i's belonging to the real line (the one-dimensional case). In the context of statistical mechanics (general β), this corresponds to "log gases", and in the context of random matrices ($\beta = 1, 2, 4$), to Hermitian or symmetric random matrices (whose eigenvalues are always real). We examine that case in our companion paper [SS2], showing the present study can be extended to handle it. We also point out that studying $\mathbb{P}_n^\beta$ with the x_i's restricted to the unit circle and with $\beta = 1, 2, 4$ also has a random matrix interpretation: it corresponds to the so-called circular ensembles, e.g. in the $\beta = 2$ case, eigenvalues of the unitary matrices distributed according to the Haar measure. We also plan on examining this case in the future.

All the above models have been studied quite extensively in the literature, particularly from the random matrix point of view (although there one can say the planar or complex Ginibre case has attracted less attention than the real or Hermitian case), but also from the statistical mechanics point of view, more particularly in the physics literature (particularly relevant are [AJ, SM, JLM]).

The current research on the random matrix aspect in the complex case focuses on studying the more general case of random matrices with entries that are not necessarily Gaussian and showing the average behavior is the same as for the Ginibre ensemble, see e.g. [Ba, TV]. This is referred to as *universality*. We are instead limited to exact Vandermonde factors but we emphasize that our results are valid for all β, hence they are not limited to random matrices and thus for the proof we cannot rely on any explicit random matrix model. Our results also have some universality feature in the sense that they are valid for a large class of potentials V.

The function w_n can also be studied for its own sake: it can seen as the interaction energy between similarly charged particles confined by the potential V. The case where $V(x)$ is quadratic arises for instance as the interaction energy for superconducting vortices in the Ginzburg-Landau theory, in the regime where their number is fixed, bounded (see [SS2], Chapter 11). In the case where V is equal to zero on a compact set K and to $+\infty$ elsewhere, which is not treated here, the minimizers of w_n are known as Fekete points or Fekete sets, cf. the book of Saff and Totik [ST] for general reference. These are interesting in their own right – they arise mainly in polynomial interpolation – and the literature on the question of their distribution in various situations is vast. When instead V is a general smooth enough function (the situation we treat here), the minimizers of w_n are called "weighted Fekete points" or weighted Fekete sets, and are also of interest, cf. again [ST].

We will pursue the analysis of these weighted Fekete sets, which can be seen as the formal limit $\beta \to +\infty$ of (1.1), in parallel with the analysis of (1.1) for general β, and obtain new results in both cases.

In the case of the Ginibre ensemble, i.e. when $V(x) = |x|^2$ and $\beta = 1$, it is known that the "spectral measure" $\nu_n := \frac{1}{n}\sum_{i=1}^n \delta_{x_i}$ converges to the uniform measure on the unit disc. More precisely $\mathbb{P}_n^\beta$, seen as a probability on the space of probability measures on $\mathbb{C}$ (the spectral measures) converges to a Dirac mass at $\mu_0 = \frac{1}{\pi}\mathbf{1}_{B_1}dx$. This is the celebrated "circular law", attributed in this case to Ginibre, Mehta, an unpublished paper of Silverstein in 1984, and then Girko [Gi]. The large deviations from this law was established by Ben Arous and Zeitouni [BZ] (see Theorem 4 below): they showed that a large deviations principle holds with speed n^{-2} and rate function

$$I(\mu) = \int_{\mathbb{R}^2\times\mathbb{R}^2} -\log|x-y|\,d\mu(x)\,d\mu(y) + \int_{\mathbb{R}^2} |x|^2\,d\mu(x) \tag{1.3}$$

whose unique minimizer among probabilities is of course the "circle law" distribution $\frac{1}{\pi}\mathbf{1}_{B_1}\,dx$.

For the case of a general V and a general β, the same large deviations principle holds with the rate function, analogue of (1.3), being

$$I(\mu) = \int_{\mathbb{R}^2\times\mathbb{R}^2} -\log|x-y|\,d\mu(x)\,d\mu(y) + \int_{\mathbb{R}^2} V(x)\,d\mu(x). \tag{1.4}$$

This can be easily readapted from the proof of [BZ], otherwise it is proven in a much more general setting in [Be]. Again the spectral measure $\nu_n = \frac{1}{n}\sum_{i=1}^n \delta_{x_i}$ converges to the minimizer among probability measures of I, called the *equilibrium measure*, which we will denote μ_0. In the case of weighted Fekete sets, the analogue to the circular law has been known to be true for a much longer time: it was proved by Fekete, Polya and Szëgo that $\frac{1}{n}\sum_{i=1}^n \delta_{x_i}$ converges to the same equilibrium measure minimizing I (then also referred to as the electrostatic interaction energy), whose description goes back to Gauss, and was carried out with modern mathematical rigor by Frostman [FR].

We are interested in examining the "next order" behavior, or that of fluctuations around the limiting distribution μ_0. Let us mention that such questions have already been addressed, often with the point of view of deriving explicit scaling limits (e.g. [BSi, G]) or laws for certain statistics of fluctuations. One can see for example [AHM] where the authors essentially prove that the law of the linear statistics of the fluctuations is a Gaussian with specific variance and mean, or also [Rid] for related results. Our approach and results are quite different.

Recalling I and μ_0 are found through the large deviations at speed n^{-2}, we look into the speed n^{-1} and, while we do not prove a complete large deviations principle at this speed, we show there is still a sort of rate function for which a "threshhold phenomenon" holds. This analysis is based on an expansion, through a crucial but simple "splitting formula" (which we present in Section 1.1 below) of $w_n(x_1,\dots,x_n)$, as equal to $n^2 I(\mu_0) - \frac{n}{2}\log n$ plus a term of order n, which tends as $n\to+\infty$ to the "renormalized energy" W, a Coulombian interaction of points in the plane with a uniform neutralizing background, that we introduced in [SS1] and whose definition we will recall below in Section 1.2. To be more precise the limit term is the average value of W on the set of blow-up limits of the configuration of points $x_1,\dots,x_n$ at the scale $1/\sqrt{n}$. It is this average that partially plays the role of a rate function at speed n^{-1}. For a precise statement, see Theorem 5.

Another way of saying this is in the language of Γ-convergence (for a definition we refer to [Br, DM], suffice it to know that this is the right notion of convergence to ensure that minimizers of w_n converge (via their empirical measures) to minimizers of I, i.e. to μ_0): it is not very difficult to show (for a short proof, see [SS2] Prop. 11.1) that $\frac{w_n}{n^2}$ Γ-converges as $n \to \infty$ to I, defined in (1.4). Here we examine the next order in the "Γ-expansion" of w_n, i.e. we study the Γ-convergence of $\frac{1}{n}(w_n - n^2 I(\mu_0))$, and show that the Γ-limit is (the average of) W. Consequently, after blow-ups at scale $\sqrt{n}$, minimizers of w_n (i.e. weighted Fekete sets) should minimize the (average of the) renormalized energy W. For a precise statement, see Theorem 2.

Before yet giving a precise definition, let us mention that we introduced the renormalized energy W in [SS1] for the study of interaction of vortices in the context of the Ginzburg-Landau energy of superconductivity (for general reference on the topic, cf. [SS2]). Configurations that minimize the Ginzburg-Landau energy with applied magnetic field, exhibit in certain regimes "point vortices" that are densely packed (there are $n \gg 1$ of them) and are expected to arrange themselves in perfect triangular lattices (i.e. with 60° angles), named *Abrikosov lattices* after the physicist who predicted them [A]. The Abrikosov lattices are indeed observed in experiments on superconductors[1]. In [SS1] we made this partly rigorous by showing that minimizers of the Ginzburg-Landau energy have vortices that minimize the renormalized energy W after blow-up at the scale $\sqrt{n}$. The conjecture made in [SS1], also supported by some mathematical evidence (see Section 1.2), is then that the minimal value of W is achieved by the triangular lattice ; if proven true this would completely justify why vortices form these patterns. Combining this conjecture with the above conclusion that weighted Fekete sets should (after blow-up) minimize W, we thus obtain the conjecture that they also should locally form Abrikosov (triangular) lattices.

We can phrase an analogous question for Coulomb gases or random matrices. Here $\mathbb{P}_n^\beta$ induces a probability measure on the family of blow-ups of $(x_1, \dots, x_n)$ around a given origin point in E — the parameter of the family — at the scale $\sqrt{n}$, a blow-up scale after which the resulting points are typically separated by order 1 distances. In the limit $n \to \infty$ this yields a probability measure on the set of configurations of points in the plane and we may ask if, almost surely, the blow-up configurations minimize W. Our results indicate that this is not the case, however we are able to prove that there is a treshhold phenomenon, in the sense that except with exponentially small probability, the average of W is below a certain constant, itself converging to the minimum of W as $\beta \to \infty$, which indicates *crystallisation*, i.e. if the above conjecture is true, we should see Abrikosov lattices as $\beta \to \infty$.

To our knowledge, this is the first time Coulomb gases or Fekete sets are rigorously connected to triangular lattices, in agreement with predictions in the physics literature (see [AJ] and references therein).

A corollary of our way of expanding w_n is that we obtain a next order estimate of the partition function Z_n^β, a result we can already state:

Theorem 1. *Let V satisfy assumptions* (1.11) – (1.13) *below. There exist functions f_1, f_2 depending only on V, such that for any $\beta_0 > 0$ and any $\beta \geq \beta_0$, and for n larger than some n_0 depending on β_0, we have*

$$n\beta f_1(\beta) \leq \log Z_n^\beta - \left(-\frac{\beta}{2} n^2 I(\mu_0) + \frac{\beta n}{4} \log n\right) \leq n\beta f_2(\beta), \tag{1.5}$$

[1]For photos one can see `http://www.fys.uio.no/super/vortex/`

with f_1, f_2 *bounded in* $[\beta_0, +\infty)$ *and such that*

$$\lim_{\beta\to\infty} f_1(\beta) = \lim_{\beta\to\infty} f_2(\beta) = -\frac{\alpha}{2} \tag{1.6}$$

where α *is some constant related to* W*, and explicited in* (1.36) *below.*

This improves on the known results, which only gave the expansion $\log Z_n^\beta \sim \frac{\beta}{2} n^2 I(\mu_0)$. Let us recall that an exact value for Z_n^β is only known for the Ginibre ensemble case of $\beta = 2$ and $V(x) = |x|^2$: it is $Z_n^2 = n^{-\frac{1}{2}n(n+1)}\pi^n \prod_{k=1}^n k!$ (see [Me], Chap. 15). Known asymptotics allow to deduce (cf. [Fo] eq. (4.184))

$$\log Z_n^2 = -\frac{3n^2}{4} + \frac{n}{2}\log n + n(-1 + \frac{1}{2}\log 2 + \frac{3}{2}\log \pi) + O(\log n) \quad \text{as} \quad n \to \infty, \tag{1.7}$$

where we note the value of $I(\mu_0)$ is indeed $\frac{3}{4}$ for this potential. On the other hand, no exact formula exists for general potentials [2], nor for quadratic potentials if $\beta \neq 2$. This is in contrast with the one-dimensional situation for which, at least in the case of quadratic V, Z_n^β has an explicit expression for every β, given by the famous Selberg integral formulas (see e.g. [AGZ]).

In statistical mechanics language, the existence of an exact asymptotic expansion up to order n for $\log Z_n^\beta$ is essentially the existence of a thermodynamic limit. This is established in [LN] for a three-dimensional Coulomb system, and in a nonrigorous way in [SM] in two dimensions.

As suggested by the strong analogy between Coulomb gases and interacting vortices in the Ginzburg-Landau model, we will draw heavily on methods we introduced in [SS1], such as the splitting, blow-up, the use of the ergodic theorem, and particular Ginzburg-Landau tools.

The rest of the introduction is organized as follows: first we give some more notation, give the assumptions we need to make on V and state the splitting formula, then we present the definition of the renormalized energy and the main results from [SS1] that we will use, and finally we state our main results and comment on them.

1.1 The equilibrium measure and the splitting formula

We need to introduce some notation, and for this we need to describe the equilibrium measure μ_0 minimizing (1.4) among probability measures.

This description, which is now classical in potential theory (see again [ST]) says that, provided $\lim_{|x|\to+\infty} V(x) - \log|x| = +\infty$ and $\log V$ is lower semicontinuous, this equilibrium measure exists, is unique, and is characterized by the fact that there exists a constant c such that, quasi-everywhere,

$$U^{\mu_0} + \frac{V}{2} \ge c \quad \text{and} \quad U^{\mu_0} + \frac{V}{2} = c \quad \text{on } \mathrm{Supp}(\mu_0), \tag{1.8}$$

where for any measure μ, U^μ denotes the potential generated by μ, defined by

$$U^\mu(x) = -\int_{\mathbb{R}^2} \log|x-y|\, d\mu(y) = -2\pi\Delta^{-1}\mu. \tag{1.9}$$

[2] an exception is the result of [DGIL] for a quadrupole potential

Here and in all the paper, we denote by Δ^{-1} the operator of convolution by $\frac{1}{2\pi}\log|\cdot|$. It is such that $\Delta\circ\Delta^{-1}=I$, where Δ is the usual Laplacian. We denote the support of μ_0 by E.

Another way to characterize U^{μ_0} is as the solution of the following *obstacle problem*[3] : It is a superharmonic function bounded below by $c-V/2$ and harmonic outside the so-called *coincidence set*

$$E=\{U^{\mu_0}=c-V/2\}. \tag{1.10}$$

This implies in particular that U^{μ_0} is $C^{1,1}$ if V is (see [C]).

It is now a good time to state the assumptions on V that we assume are satisfied in the sequel.

$$V \text{ is } C^{1,1} \text{ and } \lim_{|x|\to+\infty} V(x)-\log|x|=+\infty, \tag{1.11}$$

$$\text{In a neighborhood of } E,\ V \text{ is } C^3 \text{ and there exists } \underline{m},\overline{m}>0 \text{ s.t. } \underline{m}\le\Delta V\le\overline{m}, \tag{1.12}$$

$$V \text{ is such that } \partial E \text{ is } C^1. \tag{1.13}$$

The assumption on the growth of V is what is needed to apply the results from [ST] and to guarantee that (1.4) has a minimizer. The other conditions are technical, they are meant to ensure that μ_0 and its support are regular enough, which we will need for example when making explicit constructions, and that μ_0 never degenerates. Indeed, assumptions (1.12)–(1.13), together with (1.8), (1.9) and the regularity of V, ensure that

$$\mu_0=m_0(x)\,dx,\quad \text{where}\quad m_0(x)=\frac{\Delta V(x)}{4\pi}\mathbf{1}_E(x). \tag{1.14}$$

hence for some $\underline{m},\overline{m}>0$ we have

$$\underline{m}\le m_0\le\overline{m}. \tag{1.15}$$

Next, we set $\zeta=U^{\mu_0}+\frac{V}{2}-c$ where c is the constant in (1.8). This function satisfies

$$\begin{cases} \Delta\zeta = \frac{1}{2}\Delta V\ \mathbf{1}_{\mathbb{R}^2\setminus E} & \\ \zeta = 0 & \text{in } E\\ \zeta > 0 & \text{in } \mathbb{R}^2\setminus E\end{cases} \tag{1.16}$$

From our assumptions on V, in particular assumption (1.12), it follows that for some $c>0$ and x in a neighbourhood of E,

$$\zeta(x)\ge c\ \mathrm{dist}(x,E)^2. \tag{1.17}$$

This follows for instance by using (1.16), (1.12) to deduce that $\Delta\zeta>\eta_1$ outside E, and then writing $\Delta\zeta$ in coordinates (r,σ) where r is the distance to E, to obtain a lower bound for ζ_{rr} outside but close enough to E (see [KS], [Fre], [Fri], [C]). In fact [C] shows that this rate is optimal.

[3]The obstacle problem is a free-boundary problem and a much-studied classical problem in the calculus of variations, for general reference see [Fri, KS].

The function ζ arises in the splitting formula for w_n which we now present. As mentioned above, expanding the probability density to the next order goes along with blowing-up the points by a factor $\sqrt{n}$. We then denote the blown-up quantities by primes. For example $x'_i = \sqrt{n}x_i$, $m_0{}'(x') = m_0(x)$, etc ....

The splitting formula, proven in Section 2 is the observation that, for any $x_1, \dots, x_n$,

$$w_n(x_1,\dots,x_n) = n^2 I(\mu_0) - \frac{n}{2}\log n + \frac{1}{\pi} W(-\nabla^\perp H', \mathbf{1}_{\mathbb{R}^2}) + 2n \sum_{i=1}^n \zeta(x_i), \tag{1.18}$$

where $\nabla^\perp := (-\partial_{x_2}, \partial_{x_1})$ and

$$H' = -2\pi \Delta^{-1} \left(\sum_{i=1}^n \delta_{x'_i} - m_0{}' \right), \tag{1.19}$$

and where, in agreement with formula (1.27) below,

$$W(-\nabla^\perp H', \mathbf{1}_{\mathbb{R}^2}) := \lim_{\eta\to 0} \left(\frac{1}{2} \int_{\mathbb{R}^2 \setminus \cup_{i=1}^n B(x'_i,\eta)} |\nabla H'|^2 + \pi n \log \eta \right). \tag{1.20}$$

Letting, for a measure ν

$$F_n(\nu) := \begin{cases} \dfrac{1}{n}\left(\dfrac{1}{\pi} W(-\nabla^\perp H', \mathbf{1}_{\mathbb{R}^2}) + 2n \displaystyle\int \zeta \, d\nu \right) & \text{if } \nu \text{ is of the form } \sum_{i=1}^n \delta_{x_i} \\ +\infty & \text{otherwise,} \end{cases} \tag{1.21}$$

the relation (1.18) can be rewritten

$$w_n(x_1,\dots,x_n) = n^2 I(\mu_0) - \frac{n}{2}\log n + n F_n\left(\sum_{i=1}^n \delta_{x_i} \right). \tag{1.22}$$

This allows to separate orders as announced since we will see that $F_n(\sum_{i=1}^n \delta_{x_i})$ is typically of order 1.

We may next cancel out leading order terms and rewrite the probability law (1.1) as

$$d\mathbb{P}_n^\beta(x_1,\dots,x_n) = \frac{1}{K_n^\beta} e^{-n\frac{\beta}{2} F_n(\sum_i \delta_{x_i})} \, dx_1 \dots dx_n \tag{1.23}$$

where

$$K_n^\beta := Z_n^\beta e^{\frac{\beta}{2}(n^2 I(\mu_0) - \frac{n}{2}\log n)}. \tag{1.24}$$

As we will see below $\log K_n^\beta$ is of order $n\beta$, which leads to Theorem 1.

1.2 The renormalized energy

We now define precisely the "renormalized energy" W, which is a way of computing the Coulomb interaction between an infinite number of point charges in the plane with a uniform neutralizing background of density m. We point out that, to our knowledge, each of the analogous Coulomb systems studied in the physics literature (e.g. [SM, AJ]) comprise a finite number of point charges, and hence implicitly extend only to a bounded domain on which there is charge neutrality. Here we do not assume any local charge neutrality.

We denote by $B(x,R)$ the ball centered at x with radius R and let $B_R = B(0,R)$.

Definition 1.1. *Let m be a nonnegative number. For any continuous function χ and any vector-field j in $\mathbb{R}^2$ such that*

$$\operatorname{curl} j = 2\pi(\nu - m), \qquad \operatorname{div} j = 0 \tag{1.25}$$

where ν has the form

$$\nu = \sum_{p\in\Lambda} \delta_p \quad \text{for some discrete set } \Lambda \subset \mathbb{R}^2, \tag{1.26}$$

we let

$$W(j,\chi) = \lim_{\eta\to 0} \left(\frac{1}{2} \int_{\mathbb{R}^2 \setminus \cup_{p\in\Lambda} B(p,\eta)} \chi |j|^2 + \pi \log \eta \sum_{p\in\Lambda} \chi(p) \right). \tag{1.27}$$

Definition 1.2. *Let m be a nonnegative number. Let j be a vector field in $\mathbb{R}^2$. We say j belongs to the admissible class $\mathcal{A}_m$* [4] *if* (1.25), (1.26) *hold and*

$$\frac{\nu(B_R)}{|B_R|} \quad \text{is bounded by a constant independent of } R > 1. \tag{1.28}$$

For any family of sets $\{\mathbf{U}_R\}_{R>0}$ in $\mathbb{R}^2$ we use the notation $\chi_{\mathbf{U}_R}$ for positive cutoff functions satisfying, for some constant C independent of R,

$$|\nabla \chi_{\mathbf{U}_R}| \le C, \quad \operatorname{Supp}(\chi_{\mathbf{U}_R}) \subset \mathbf{U}_R, \quad \chi_{\mathbf{U}_R}(x) = 1 \text{ if } d(x, {\mathbf{U}_R}^c) \ge 1. \tag{1.29}$$

Definition 1.3. *The renormalized energy W is defined, for $j \in \mathcal{A}_m$, by*

$$W(j) = \limsup_{R\to\infty} \frac{W(j, \chi_{B_R})}{|B_R|}, \tag{1.30}$$

with $\{\chi_{B_R}\}_R$ satisfying (1.29) *for the family $\{B_R\}_{R>0}$.*

In theory, many different j's could correspond to a given ν (one can always add the gradient of a harmonic function). But as it turns out, they only differ by a constant:

Lemma 1.4. *Let $m \ge 0$ and $\nu = \sum_{p\in\Lambda} \delta_p$, where $\Lambda \subset \mathbb{R}^2$ is discrete, and assume there exists j such that*

$$\operatorname{curl} j = 2\pi(\nu - m), \quad \operatorname{div} j = 0, \quad \text{and} \quad W(j) < +\infty. \tag{1.31}$$

Then any other j' satisfying (1.31) *is such that $j - j'$ is constant.*

If there exists j_ν such that (1.31) *holds and such that*

$$\lim_{R\to\infty} \fint_{B_R} j_\nu = 0, \tag{1.32}$$

then any other j satisfying (1.31) *is such that $W(j) > W(j_\nu)$.*

[4]Note that this definition slightly differs from [SS1]: $\mathcal{A}_m$ here corresponds to $\mathcal{A}_{2\pi m}$ there.

Proof. Let j, j' be as above, then $j - j'$ can be seen as as complex function of a complex variable and from (1.31) it is holomorphic. From the finiteness of $W(j)$ and $W(j')$ we deduce easily that $\int_{B_R} |j - j'|^2 \le CR^2$ and using Cauchy's estimate for the coefficients of a power series together with a mean value argument it follows that $j - j'$ is constant.

For the second statement, we deduce from the first statement that $j = j_\nu + c$ for some constant vector $c \neq 0$, and then

$$W(j, \chi_{B_R}) = W(j_\nu, \chi_{B_R}) + c \cdot \int j_\nu \chi_{B_R} + \frac{|c|^2}{2} \int \chi_{B_R},$$

so that dividing by $|B_R|$, passing to the limit as $R \to +\infty$ and in view of (1.32), we find $W(j) = W(j_\nu) + \frac{1}{2}|c|^2$. □

Note that given ν, the above lemma shows that either for all j's satisfying (1.31) the limit $\lim_{R\to\infty} \fint_{B_R} j_\nu$ exists, or it exists for none of them. Both cases may occur.

The following additional facts and remarks about W are from [SS1]:

- In the definition (1.30), the balls $\{B_R\}_R$ can be replaced by other families of (reasonable) shapes $\{\mathbf{U}_R\}_R$, this yields a definition of a renormalized energy W_U, where the letter U stands for the family $\{\mathbf{U}_R\}$.

- The value of W does not depend on $\{\chi_{B_R}\}_R$ as long as it satisfies (1.29). The corresponding statement holds for W_U under certain assumptions on the family $\{\mathbf{U}_R\}_{R>0}$.

- W is bounded below and admits a minimizer over $\mathcal{A}_1$.

- It is easy to check that if j belongs to $\mathcal{A}_m$, $m > 0$, then $j' = \frac{1}{\sqrt{m}} j(\cdot/\sqrt{m})$ belongs to $\mathcal{A}_1$ and

  $$W(j) = m\left(W(j') - \frac{\pi}{2}\log m\right). \tag{1.33}$$

  Consequently if j is a minimizer of W over $\mathcal{A}_m$, then j' minimizes W over $\mathcal{A}_1$. In particular

  $$\min_{\mathcal{A}_m} W = m\left(\min_{\mathcal{A}_1} W - \frac{\pi}{2}\log m\right). \tag{1.34}$$

- The minimizers and the value of the minimum of W_U are independent of U. However there are examples of admissible j's for which $W_U(j)$ depends on the family of shapes $\{\mathbf{U}_R\}_{R>0}$ which is used.

- If $j \in \mathcal{A}_m$ then in the neighborhood of $p \in \Lambda$ we have $\operatorname{curl} j = 2\pi(\delta_p - m)$, $\operatorname{div} j = 0$, thus we have near p the decomposition $j(x) = \nabla^\perp \log|x - p| + f(x)$ where f is C^1, and it easily follows that the limit (1.27) exists if χ is compactly supported. It also follows that j belongs to L^p_{loc} for any $p < 2$.

- Because the number of points is infinite, the interaction over large balls needs to be normalized by the volume, as in a thermodynamic limit. Thus W does not feel compact perturbations of the configuration of points. Even though the interactions are long-range, this is not difficult to justify rigorously.

- In the case $m = 1$ and when the set of points Λ is periodic with respect to some lattice $\mathbb{Z}\vec{u} + \mathbb{Z}\vec{v}$ then it can be viewed as a set of n points $a_1, \dots, a_n$ over the torus $\mathbb{T}_{(\vec{u},\vec{v})} := \mathbb{R}^2/(\mathbb{Z}\vec{u} + \mathbb{Z}\vec{v})$ with $|\mathbb{T}_{(\vec{u},\vec{v})}| = n$. In this case, the infimum of $W(j)$ among j's which satisfy (1.31) is achieved by $j_{\{a_i\}} = -\nabla^\perp h$, where h is the periodic solution to $-\Delta h = 2\pi(\sum_i \delta_{a_i} - 1)$, and

$$W(j_{\{a_i\}}) = \frac{\pi}{|\mathbb{T}_{(\vec{u},\vec{v})}|} \sum_{i \neq j} G(a_i - a_j) + \pi \lim_{x \to 0} (G(x) + \log |x|) \tag{1.35}$$

where G is the Green function of the torus with respect to its volume form, i.e. the solution to

$$-\Delta G(x) = 2\pi \left(\delta_0 - \frac{1}{|\mathbb{T}_{(\vec{u},\vec{v})}|} \right) \quad \text{in} \quad \mathbb{T}_{(\vec{u},\vec{v})}.$$

An explicit expression for G can be found via Fourier series and this leads to an explicit expression for W of the form $\sum_{i \neq j} E(a_i - a_j)$ where E is an Eisenstein series (for more details see [SS1] and also [BSe]). In this periodic setting, the expression of W is thus much simpler than (1.30) and reduces to the computation of a sum of explicit pairwise interaction.

- When the set of points Λ is itself exactly a lattice $\mathbb{Z}\vec{u} + \mathbb{Z}\vec{v}$ then W can be expressed explicitly through the Epstein Zeta function of the lattice. Moreover, using results from number theory, it is proved in [SS1], Theorem 2, that the unique minimizer of W over lattice configurations of fixed volume is the triangular lattice. This supports the conjecture that the Abrikosov triangular lattice is a global minimizer of W, with a slight abuse of language since W is not a function of the points, but of their associated current $j_{\{a_i\}}$.

This last fact allows us to think of W as a way of measuring the disorder and lack of homogeneity of a configuration of points in the plane (this point of view is pursued in [BSe] with explicit computations for random point processes). Another way to see it is to view W as measuring the distance between $\sum_{p \in \Lambda} \delta_p$ and the constant m in H^{-1}. However this distance is infinite, both because the domain is infinite and because Dirac masses do not belong to H^{-1}, which is why the actual definition is more involved and requires this "renormalized" computation using $\eta \to 0$ (hence the name, borrowed from [BBH]).

We may now define the constant α which appears in Theorem 1 and in Theorem 2 below:

$$\alpha := \frac{1}{\pi} \int_E \min_{\mathcal{A}_{m_0(x)}} W \, dx = \frac{1}{\pi} \min_{\mathcal{A}_1} W - \frac{1}{2} \int_E m_0(x) \log m_0(x) \, dx, \tag{1.36}$$

where we have used (1.34) and the fact that, from (1.14), $\int_E m_0 = 1$. Note that α only depends on V, via the integral term, and on the (so far) unknown constant $\min_{\mathcal{A}_1} W$.

1.3 Statement of main results

Our first result identifies the Γ-limit of $\{F_n\}_n$, defined in (1.21) or (1.22). This in particular allows a description of the weighted Fekete sets minimizing w_n at the microscopic level. Below we abuse notation by writing $\nu_n = \sum_{i=1}^n \delta_{x_i}$ when it should be $\nu_n = \sum_{i=1}^n \delta_{x_{i,n}}$. For such a ν, we let $\nu' = \sum_{i=1}^n \delta_{x'_i}$ be the measure in blown-up coordinates and $j_\nu = -\nabla^\perp H'$, where H' is defined by (1.19) — equivalently j_ν is the solution of div $j_\nu = 0$, $\operatorname{curl} j_\nu = 2\pi(\nu' - m_0')$ in

$\mathbb{R}^2$ which tends to 0 at infinity. (To avoid confusion, we emphasize here that ν lives at the original scale while j_ν lives at the blown-up scale.) We also let

$$P_{\nu_n} = \fint_E \delta_{(x, j_{\nu_n}(\sqrt{n}(x+\cdot))}\, dx, \tag{1.37}$$

i.e. the push-forward of the normalized Lebesgue measure on E by $x \mapsto (x, j_{\nu_n}(\sqrt{n}x + \cdot))$. It is a probability measure on $E \times L^p_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ (couples of (blow-up centers, blown-up current around this center)).

The limiting object as $n \to +\infty$ in the Γ-limit of w_n was $\nu = \lim_n \vec{u}_n$. In taking the Γ-limit of F_n, the limiting object is more complex, it is the limit P of P_{ν_n}, i.e. a Young measure akin to the Young measures on micropatterns introduced in [AM]. Note that in the rest of the paper, the probability P has nothing to do with $\mathbb{P}_n^\beta$, and depends on the realizations of the configurations of points.

We will here and below use the notation

$$D(x', R) = \nu_n \left(B\left(x, \frac{R}{\sqrt{n}}\right)\right) - n\mu_0 \left(B\left(x, \frac{R}{\sqrt{n}}\right)\right), \tag{1.38}$$

where $x' = \sqrt{n}x$ as usual.

Theorem 2 (Microscopic behavior of weighted Fekete sets)**.** *Let the potential V satisfy assumptions* (1.11)–(1.13)*. Fix $1 < p < 2$ and let $X = E \times L^p_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2)$.*

A. Lower bound. *Let $\nu_n = \sum_{i=1}^n \delta_{x_i}$ be a sequence such that $F_n(\nu_n) \le C$. Then P_{ν_n} defined by* (1.37) *is a probability measure on X and*

1. *Any subsequence of $\{P_{\nu_n}\}_n$ has a convergent subsequence converging to a probability measure on X as $n \to \infty$. We denote by P such a limit.*
2. *The first marginal of P is the normalized Lebesgue measure on E. P is invariant by $(x, j) \mapsto (x, j(\lambda(x) + \cdot))$, for any $\lambda(x)$ of class C^1 from E to $\mathbb{R}^2$ (we will say $T_{\lambda(x)}$-*invariant*).*
3. *For P almost every (x, j) we have $j \in \mathcal{A}_{m_0(x)}$.*
4. *Defining α as in* (1.36)*, it holds that*

$$\liminf_{n\to\infty} \left(F_n(\nu_n) - 2\int \zeta\, d\nu_n \right) \ge \frac{|E|}{\pi} \int W(j)\, dP(x, j) \ge \alpha. \tag{1.39}$$

B. Upper bound construction. *Conversely, assume P is a $T_{\lambda(x)}$-invariant probability measure on X whose first marginal is $\frac{1}{|E|}dx_{|E}$ and such that for P-almost every (x, j) we have $j \in \mathcal{A}_{m_0(x)}$. Then there exists a sequence $\{\nu_n = \sum_{i=1}^n \delta_{x_i}\}_n$ of empirical measures on E and a sequence $\{j_n\}_n$ in $L^p_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ such that* $\mathrm{curl}\, j_n = 2\pi(\nu_n' - m_0')$ *and such that defining P_n as in* (1.37)*, with j_n replacing j_{ν_n}, we have $P_n \to P$ as $n \to \infty$ and*

$$\limsup_{n\to\infty} F_n(\nu_n) \le \frac{|E|}{\pi} \int W_K(j)\, dP(x, j), \tag{1.40}$$

where W_K is the renormalized energy relative to the family of squares $\{K_R = [-R, R]^2\}_R$.

C. Consequences for minimizers. *If $(x_1, \dots, x_n)$ minimizes w_n for every n and $\nu_n = \sum_{i=1}^n \delta_{x_i}$, then the limit P of P_{ν_n} as defined in* (1.37) *satisfies the following.*

1. *For P-almost every (x, j), the current j minimizes W over $\mathcal{A}_{m_0(x)}$.*

2. *We have*

$$\lim_{n\to\infty} F_n(\nu_n) = \frac{|E|}{\pi} \int W(j)\, dP(x, j) = \alpha, \qquad \lim_{n\to\infty} \sum_{i=1}^{n} \mathrm{dist}^2(x_i, E) = 0.$$

3. *There exists $C > 0$ such that for every $x' \in \mathbb{R}^2$, $R > 1$ and using the notation (1.38) we have*

$$D(x', R)^2 \min\left(1, \frac{|D(x', R)|}{R^2}\right) \le Cn. \tag{1.41}$$

Note that part B of the theorem is only a partial converse to part A because the constructed j_n need not be divergence free, hence in general $j_n \neq j_{\nu_n}$. Also the lower bound (1.39) uses W and the upper bound (1.40) uses W_K. In fact (1.39) is true with W_U for any family $\{\mathbf{U}_R\}_R$ satisfying (6.2) below, and in particular we could replace W by W_K there. We recall though that both have the same minimizers and minimum value, as proved in [SS1].

This theorem is the analogue of the main result of [SS1] but for w_n rather than the Ginzburg-Landau energy. It is technically simpler to prove, except for the possibility of a nonconstant weight $m_0(x)$ which was absent from [SS1]. It can be stated as the fact that $\frac{|E|}{\pi} \int W dP$, which can be understood as the average of W with respect to all possible blow-up centers in E (chosen uniformly at random), is the Γ-limit of w_n at next order. Its minimum over all admissible probabilities is α.

The estimate (1.41) gives a control on the "discrepancy" D (between the effective number of points and the expected one) at the scale $R/\sqrt{n}$. Note that in a recent paper [AOC], the authors also study the fine behavior of weighted Fekete sets. Using completely different methods, based on Beurling-Landau densities and techniques going back to [La], they are able to show the very strong result that

$$\limsup_{R\to\infty} \limsup_{n\to\infty} \frac{D(x_n, R)}{R^2} = 0,$$

as long as $\mathrm{dist}(x_n, \partial E') \ge \log^2 n$. This shows that the density of points follows μ_0 at the microscopic scale $1/\sqrt{n}$ and thus the configurations are very rigid. This still leaves however some uncertainty about the patterns they should follow. On the contrary, our result is less precise about $D(x, R)$ since we only recover the optimal estimate when R grows faster than $n^{1/4}$, but it connects the pattern formed by the points to the Abrikosov triangular lattice via the minimization of W.

We now turn to Coulomb gases. It is straightforward from the form (1.23) and the estimate (provided by Theorem 1) $\log K_n^\beta = O(n\beta)$ where K_n^β is defined in (1.24), to deduce that $F_n \le C$ except on a set of small probability. This fact allows to derive various consequences, the first being estimates on the probability of certain rare events.

Theorem 3. *Let V satisfy assumptions* (1.11)–(1.13).

There exists a universal constant $R_0 > 0$ and $c, C > 0$ depending only on V such that: For any $\beta_0 > 0$, any n large enough depending on β_0, and any $\beta > \beta_0$, for any $x_1, \ldots, x_n \in \mathbb{R}^2$, *any* $R > R_0$*, any* $x_0' = \sqrt{n}x_0 \in \mathbb{R}^2$ *and any* $\eta > 0$*, letting* $\nu_n = \sum_{i=1}^n \delta_{x_i}$*, we have the following:*

$$\log \mathbb{P}_n^\beta \left(\left|D(x_0', R)\right| \geq \eta R^2\right) \leq -c\beta \min(\eta^2, \eta^3) R^4 + C\beta(R^2 + n) + Cn. \tag{1.42}$$

$$\log \mathbb{P}_n^\beta \left(\int \zeta \, d\nu_n \geq \eta\right) \leq -\frac{1}{2} n\beta\eta + Cn(\beta + 1). \tag{1.43}$$

Moreover, for any smooth bounded $U' = \sqrt{n}U \subset \mathbb{R}^2$*,*

$$\log \mathbb{P}_n^\beta \left(\int_{U'} \frac{D(x', R)^2}{R^2} \min\left(1, \frac{|D(x', R)|}{R^2}\right) dx' \geq \eta\right) \leq n\beta(-c\eta + C|U| + C) + Cn. \tag{1.44}$$

Finally, if $q \in [1, 2)$ *there exists* $c, C > 0$ *depending on* V *and* q *such that* $\forall \eta \geq 1$*,* $R > 0$*,*

$$\log \mathbb{P}_n^\beta \left(\left(1 + \frac{R^2}{n}\right)^{\frac{1}{2} - \frac{1}{q}} \|\nu - n\mu_0\|_{W^{-1,q}(B_{R/\sqrt{n}})} \geq \eta\sqrt{n}\right) \leq -cn\beta\eta^2 + Cn(\beta + 1), \tag{1.45}$$

where $W^{-1,q}(\Omega)$ *is the dual of the Sobolev space* $W_0^{1,p}(\Omega)$ *with* $\frac{1}{p} + \frac{1}{q} = 1$*; in particular* $W^{-1,1}$ *is the dual of Lipschitz functions.*

These estimates can roughly be read in the following way: as soon as η is large enough, the events in parentheses have probability decaying like e^{-cn}. More precisely, we bound the probability that a ball contains too many or too few points compared to the expected number $n\mu_0(B)$, but whereas the circular law does it for a *macroscopic ball*, i.e. for R comparable to $\sqrt{n}$, the estimate (1.42) is effective at *microscopic scales*, of the order of $n^{1/4}$. This is sometimes called in this context "undercrowding" or "overcrowding" of points, see [JLM, NSV, K]. In view of similar results in [JLM] and the result of [AOC], we could expect this to hold as soon as $R \gg 1$, but this seems out of reach by our method. This can also be compared with analogous estimates without error terms proven in the case of Hermitian matrices, cf. [ESY]. These results, in the Hermitian case, are proven in the general setting of Wigner matrices, i.e. Hermitian matrices with random i.i.d. entries, which do not need to be Gaussian. They only concern some fixed β however.

The estimate (1.44) gives a global version of this result: it expresses a control on the average microscopic "discrepancy" D. This control is in L^2 for large values of the discrepancy, and in L^3 for small values. The estimate (1.43) allows, in view of (1.17), to control (again with some threshhold to be beaten) the probability that some points may be far from the set E. Note that since ν_n is a non-normalized empirical measure, (1.43) ensures for example that the probability that a single point lies at a distance η from E is exponentially small as soon as η is larger than some constant. All these estimates rely on controlling D and F_n by W.

Finally, (1.45) tells us that fluctuations around the law $n\mu_0$ can be globally controlled (take for example $R = \sqrt{n}$) by $O(\sqrt{n})$ (except with exponentially small probability). We believe this estimate to be optimal. Its proof uses in a crucial manner the result of [ST], which controls, via Lorentz spaces, the difference $\nu_n - n\mu_0$ in terms of the renormalized energy W.

Our last result mostly expresses Theorem 2 in a "moderate" deviations language. Before stating it, let us recall for comparison the result of [BZ]:

Theorem 4 (Ben Arous - Zeitouni)**.** *Let* $\beta = 2$ *and* $V(x) = |x|^2$*. We denote by* $\widetilde{\mathbb{P}}_n^\beta$ *the image of the law* (1.1) *by the map* $(x_1, \dots, x_n) \mapsto \nu_n$*, where* $\nu_n = \frac{1}{n}\sum_{i=1}^n \delta_{x_i}$*. Then for any subset* A *of the set of probability measures on* $\mathbb{R}^2$ *(endowed with the topology of weak convergence), we have*

$$- \inf_{\mu \in \mathring{A}} \widetilde{I}(\mu) \le \liminf_{n\to\infty} \frac{1}{n^2} \log \widetilde{\mathbb{P}}_n^\beta(A) \le \limsup_{n\to\infty} \frac{1}{n^2} \log \widetilde{\mathbb{P}}_n^\beta(A) \le - \inf_{\mu \in \bar{A}} \widetilde{I}(\mu),$$

where $\widetilde{I} = I - \min I$*.*

Before stating our theorem, we introduce the following notation, which allows to embed $\mathbb{C}^n$ into the set of probabilities on $X = E \times L^p_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$: For any n and $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{C}^n$ we let $i(\mathbf{x}) = P_{\nu_n}$, where $\nu_n = \sum_{i=1}^n \delta_{x_i}$ and P_{ν_n} is as in (1.37), so that $i(\mathbf{x})$ is an element of $\mathcal{P}(X)$, the set of probability measures on $X = E \times L^p_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$.

Theorem 5. *For any* $n > 0$ *let* $A_n \subset \mathbb{C}^n$*. Denote*

$$A_\infty = \bigcap_{n>0} \overline{\bigcup_{m>n} i(A_m)}. \tag{1.46}$$

Then for any $\eta > 0$ *there is* $C_\eta > 0$ *depending on* V *and* η *only such that* α *being as in* (1.36)*,*

$$\limsup_{n\to\infty} \frac{\log \mathbb{P}_n^\beta(A_n)}{n} \le -\frac{\beta}{2}\Big(\frac{|E|}{\pi} \inf_{P \in A_\infty} \int W(j)\,dP(x,j) - \alpha - \eta - \frac{C_\eta}{\beta}\Big). \tag{1.47}$$

Conversely, let $A \subset \mathcal{P}(X)$ *be a set of* $T_{\lambda(x)}$*-invariant probability measures on* X *and let* $\mathring{A}$ *be the interior of* A *for the topology of weak convergence. Then for any* $\eta > 0$*, there exists a sequence of subsets* $A_n \subset E^n$ *such that*

$$-\frac{\beta}{2}\left(\frac{|E|}{\pi} \inf_{P \in \mathring{A}} \int W_K(j)\,dP(x,j) - \alpha + \eta + \frac{C_\eta}{\beta}\right) \le \liminf_{n\to\infty} \frac{\log \mathbb{P}_n^\beta(A_n)}{n}, \tag{1.48}$$

and such that for any sequence $\{\nu_n = \sum_{i=1}^n \delta_{x_i}\}_n$ *such that* $(x_1, \dots, x_n) \in A_n$ *for every* n *there exists a sequence of currents* $j_n \in L^p_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ *such that* $\operatorname{curl} j_n = 2\pi(\nu_n' - m_0')$ *and such that — defining* P_n *as in* (1.37) *with* j_n *replacing* j_{ν_n} *— we have*

$$\lim_n P_n \in \mathring{A}. \tag{1.49}$$

Note that if P_n was P_{ν_n}, then (1.49) would be equivalent to saying that $\cap_n \overline{\cup_{m>n} i(A_m)} \subset \mathring{A}$. The difference betwen P_{ν_n} and P_n is that the latter is generated by a current j_n which is not necessarily divergence free.

Compared to Theorem 4 this result can be seen as a next order (n instead of n^2) deviations result, where the average of W over blow-up centers plays the role of a rate function, with a margin which becomes small as $\beta \to \infty$. While Theorem 4 said that empirical measures at macroscopic scale converge to μ_0, except for a set of exponentially decaying probability, Theorem 5 says that within the empirical measures which do converge to μ_0, the ones with large average of W (computed after blow-up) also have exponentially decaying probability, but at the slower rate e^{-n} instead of e^{-n^2}. More precisely, there is a threshhold C/β for some $C > 0$, such that configurations satisfying

$$\frac{|E|}{\pi} \int W\,dP \ge \alpha + \frac{C}{\beta}$$

have exponentially small probability, where we recall α is also the minimum possible value of $\frac{|E|}{\pi}\int W\,dP$. Since we believe that W measures the disorder of a (limit) configuration of (blown up) points in the plane, this means that most configurations have a certain order. The threshhold, or gap, C/β tends to 0 as β tends to ∞, hence in this limit, configurations have to be closer and closer to the minimum of the average of W, or have more and more order.

Modulo the conjecture that the minimum of W is achieved by the perfect "Abrikosov" triangular lattice, this constitutes a crystallisation result. Note that to solve this conjecture, it would suffice to evaluate α, which in view of Theorem 1 is equivalent to being able to compute the asymptotics of Z_n^β as $\beta \to \infty$. More generally, the following open questions naturally arise in view of our results, and are closely related to one another:

- Prove that $\min_{\mathcal{A}_1} W$ is achieved by the triangular lattice.
- Find whether a large deviations statement is true at speed n, and if it is, find the rate function.
- Describe the limit of the probability measures $\widetilde{\mathbb{P}}_n^\beta$ on $\mathcal{P}(X)$ defined as the images of $\mathbb{P}_n^\beta$ by the embedding i.

In [SS2] we will see that all the results we have obtained here are also true in the case of points on the real line, i.e. for 1D log gases or Hermitian random matrices. There the minimization of W is solved (the minimum is the perfect lattice $\mathbb{Z}$) and the crystallisation result is complete.

The rest of the paper is organized as follows: Section 2 contains the proof of the "splitting formula". In Section 3, we present the "spreading result" from [SS1] and some first corollaries. In Section 4, we present an explicit construction which yields the lower bound on Z_n^β, whose proof is postponed to Section 7. In Section 5, we show how W controls the overcrowding/undercrowding of points, and prove Theorem 3. In Section 6 we present the ergodic averaging approach (the abstract result) and apply it to conclude the proofs of Theorem 2, 5 and 1.

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2 Proof of the splitting formula

The connection between w_n and W originates in the following computation

Lemma 2.1. *For any $x_1, \dots, x_n$ and letting $\nu = \sum_i \delta_{x_i}$ the following holds*

$$F_n(\nu) = \frac{1}{n\pi} W(-\nabla^\perp H', \mathbf{1}_{\mathbb{R}^2}) + 2\sum_{i=1}^n \zeta(x_i) = \frac{1}{n}\left(w_n(x_1,\dots,x_n) - n^2 I(\mu_0) + \frac{n}{2}\log n\right), \tag{2.1}$$

where F_n is defined in (1.21)*, W is defined in* (1.27)*, and H' is defined in* (1.19)*.*

Proof. Let $\nu = \sum_{i=1}^n \delta_{x_i}$. First we note that since ν and $n\mu_0$ have same mass and compact support we have $H(x) = O(1/|x|)$ and $\nabla H(x) = O(1/|x|^2)$ as $|x| \to +\infty$.

We prove that, denoting $\triangle$ the diagonal in $\mathbb{R}^2 \times \mathbb{R}^2$,

$$\int_{(\mathbb{R}^2\times\mathbb{R}^2)\setminus\triangle} -\log|x-y|\,d(\nu - n\mu_0)(x)\,d(\nu - n\mu_0)(y) = \frac{1}{\pi} W(-\nabla^\perp H, \mathbf{1}_{\mathbb{R}^2}). \tag{2.2}$$

First, using Green's formula,

$$\int_{B_R\setminus\cup_i B(x_i,\eta)} |\nabla H|^2 = \int_{\partial B_R} H\frac{\partial H}{\partial \nu} + \sum_i \int_{\partial B(x_i,\eta)} H\frac{\partial H}{\partial\nu} + 2\pi\int_{B_R\setminus\cup_i B(x_i,\eta)} H\,d(\nu - n\mu_0).$$

Let $H_i(x) := H(x) + \log|x - x_i|$. We have $H_i = -\log *(\nu_i - n\mu_0)$, with $\nu_i = \nu - \delta_{x_i}$, and near x_i, H_i is C^1. Therefore, using (1.19) and the boundedness of μ_0 in L^∞, we have that, as $\eta \to 0$

$$\int_{\partial B(x_i,\eta)} H\frac{\partial H}{\partial \nu} = -2\pi\log\eta + 2\pi H_i(x_i) + o(1),$$

while the integral on ∂B_R tends to 0 as $R \to +\infty$ from the decay properties of H. We thus obtain, as $\eta \to 0$ and $R \to +\infty$,

$$\int_{B_R\setminus\cup_i B(x_i,\eta)} |\nabla H|^2 = -2\pi n\log\eta + 2\pi\sum_i H_i(x_i) - 2\pi n\int H\,d\mu_0 + o(1),$$

and therefore, by definition of W,

$$W(-\nabla^\perp H, \mathbf{1}_{\mathbb{R}^2}) = \pi\sum_i H_i(x_i) - \pi n\int H\,d\mu_0. \tag{2.3}$$

Second we note that given $x \in \mathbb{R}^2$, either $x \notin \{x_i\}$ and

$$\int_{\mathbb{R}^2\setminus\{x\}} -\log|x-y|\,d(\nu - n\mu_0)(y) = H(x),$$

or $x = x_i$ and

$$\int_{\mathbb{R}^2\setminus\{x\}} -\log|x-y|\,d(\nu - n\mu_0)(y) = H_i(x_i).$$

It follows that

$$\int_{\triangle^c} -\log|x-y|\,d(\nu - n\mu_0)(x)\,d(\nu - n\mu_0)(y) = \sum_i H_i(x_i) - n\int H(x)\,d\mu_0(x),$$

which together with (2.3) proves (2.2).

On the other hand, we may rewrite w_n as

$$w_n(x_1,\dots,x_n) = \int_{\triangle^c} -\log|x-y|\,d\nu(x)\,d\nu(y) + n\int V(x)\,d\nu(x)$$

and, splitting ν as $n\mu_0 + \nu - n\mu_0$ and using the fact that $\mu_0(\triangle) = 0$, we obtain

$$w(x_1,\dots,x_n) = n^2 I(\mu_0) + 2n\int U^{\mu_0}(x)\,d(\nu - n\mu_0)(x) + n\int V(x)\,d(\nu - n\mu_0)(x)$$
$$+ \int_{\triangle^c} -\log|x-y|\,d(\nu - n\mu_0)(x)\,d(\nu - n\mu_0)(y).$$

Since $U^{\mu_0} + \frac{V}{2} = c + \zeta$ and since ν and $n\mu_0$ have same mass n, we have

$$2n \int U^{\mu_0}(x)\, d(\nu - n\mu_0)(x) + n \int V(x)\, d(\nu - n\mu_0)(x) = 2n \int \zeta\, d(\nu - \mu_0) = 2n \int \zeta\, d\nu,$$

using the fact that $\zeta = 0$ on the support of μ_0. Therefore, in view of (2.2) we have found

$$w(x_1, \ldots, x_n) = n^2 I(\mu_0) + 2n \int \zeta\, d\nu + \frac{1}{\pi} W(-\nabla^\perp H, \mathbf{1}_{\mathbb{R}^2}). \tag{2.4}$$

But, changing variables,

$$\frac{1}{2} \int_{\mathbb{R}^2 \setminus \cup_{i=1}^n B(x_i, \eta)} |\nabla H|^2 = \frac{1}{2} \int_{\mathbb{R}^2 \setminus \cup_{i=1}^n B(x_i', \sqrt{n}\eta)} |\nabla H'|^2,$$

and by adding $\pi n \log \eta$ on both sides and letting $\eta \to 0$ we deduce that $W(-\nabla^\perp H, \mathbf{1}_{\mathbb{R}^2}) = W(-\nabla^\perp H', \mathbf{1}_{\mathbb{R}^2}) - \frac{\pi}{2} n \log n$. Together with (2.4) this proves (2.1). □

3 A first lower bound on F_n and upper bound on Z_n^β

The crucial fact that we now wish to exploit is that, even though $W(j, \chi)$ does not have a sign, there are good lower bounds for F_n. This follows from the analysis of [SS1], more specifically from the following "mass spreading result", adapted from [SS1], Proposition 4.9 and Remark 4.10 (with slightly different notation), which itself is based on the so-called "ball construction method", a crucial tool in the analysis of Ginzburg-Landau equations. This result, that we will use here as a black box, says that even though the energy density associated to $W(j, \chi)$ is not positive (or even bounded below), it can be replaced by an energy-density g which is uniformly bounded below, at the expense of a negligible error.

For any set Ω, $\widehat{\Omega}$ denotes its 1-tubular neighborhood, i.e. $\{x \in \mathbb{R}^2, \mathrm{dist}(x, \Omega) < 1\}$.

Proposition 3.1. *Assume $\Omega \subset \mathbb{R}^2$ is open and (ν, j) are such that $\nu = 2\pi \sum_{p \in \Lambda} \delta_p$ for some finite subset Λ of $\widehat{\Omega}$ and $\mathrm{curl}\, j = 2\pi(\nu - m)$, $\mathrm{div}\, j = 0$ in $\widehat{\Omega}$, where $m \in L^\infty(\widehat{\Omega})$. Then there exists a measure g supported on $\widehat{\Omega}$ and such that*

- *Given any constant $0 < \rho$, there exists a family of disjoint closed balls $\mathcal{B}_\rho$ covering Supp(ν), with the sum of the radii of the balls in $\mathcal{B}_\rho$ intersecting with any ball of radius 1 bounded by ρ, and such that*

$$g \ge -C(\|m\|_{L^\infty} + 1) + \frac{1}{4} |j|^2 \mathbf{1}_{\Omega \setminus \mathcal{B}_\rho} \quad \text{in } \widehat{\Omega}, \tag{3.1}$$

  where C is a universal constant.

-

$$g = \frac{1}{2} |j|^2 \text{ outside } \cup_{p \in \Lambda} B(p, C) \tag{3.2}$$

  where C is universal.

- *For any function χ compactly supported in Ω we have*

$$\left|W(j,\chi) - \int \chi\, dg\right| \le CN(\log N + \|m\|_{L^\infty})\|\nabla\chi\|_\infty, \tag{3.3}$$

where $N = \#\{p \in \Lambda : B(p,C) \cap Supp(\nabla\chi) \neq \varnothing\}$ for some universal C.

- *For any $U \subset \Omega$,*

$$\#(\Lambda \cap U) \le C\left(1 + \|m\|_{L^\infty}^2 |\widehat{U}| + g(\widehat{U})\right). \tag{3.4}$$

Note that the result in [SS1] is not stated for any ρ but a careful inspection of the proof there allows to show that it can be readapted to make ρ arbitrarily small.

Definition 3.2. *Assume $\nu = \sum_{i=1}^n \delta_{x_i}$. Letting $\nu' = \sum_{i=1}^n \delta_{x'_i}$ be the measure in blown-up coordinates and $j_\nu = -\nabla^\perp H'$, where H' is defined by* (1.19)*, we denote by g_ν the result of applying the previous proposition to (ν', j_ν) in $\mathbb{R}^2$.*

Even though we will not use the following result in the sequel, we state it to show how we can quickly derive a first upper bound on Z_n^β from what precedes.

Proposition 3.3. *We have*

$$\log K_n^\beta \le Cn\beta + n(\log|E| + o(1)) \tag{3.5}$$

and

$$\log Z_n^\beta \le -\frac{\beta}{2} n^2 I(\mu_0) + \frac{\beta n}{4}\log n + Cn\beta + n(\log|E| + o(1)) \tag{3.6}$$

where $o(1) \to 0$ as $n \to \infty$ uniformly with respect to $\beta > \beta_0$, for any $\beta_0 > 0$, and C depends only on V.

The proof uses two lemmas.

Lemma 3.4. *For any $\nu = \sum_{i=1}^n \delta_{x_i}$, we have*

$$F_n(\nu) = \frac{1}{n\pi}\int_{\mathbb{R}^2} dg_\nu + 2\int \zeta\, d\nu, \tag{3.7}$$

where F_n is as in (1.21)*.*

Proof. This follows from (3.3) applied to χ_{B_R}, where χ_{B_R} is as in (1.29). If R is large enough then $\#\{p \in \mathrm{Supp}(\nu) : B(p,C) \cap \mathrm{Supp}(\nabla\chi) \neq \varnothing\} = 0$ and therefore (3.3) reads

$$W(j_\nu, \chi_{B_R}) = \int \chi_{B_R}\, dg_\nu.$$

Letting $R \to +\infty$ yields $W(j_\nu, \mathbf{1}_{\mathbb{R}^2}) = \int dg_\nu$ and the result, in view of (1.21). □

Lemma 3.5. *Letting ν_n stand for $\sum_{i=1}^n \delta_{x_i}$ we have, for any constant $\lambda > 0$ and uniformly w.r.t. β greater than any arbitrary positive constant β_0,*

$$\lim_{n\to\infty}\left(\int_{\mathbb{C}^n} e^{-\lambda\beta n\int \zeta\, d\nu_n}\, dx_1 \dots dx_n\right)^{\frac{1}{n}} = |E|. \tag{3.8}$$

Proof. By separation of variables, we have

$$\int_{\mathbb{C}^n} e^{-\lambda\beta n \sum_{i=1}^n \zeta(x_i)}\, dx_1 \dots dx_n = \left(\int_{\mathbb{C}} e^{-\lambda\beta n\zeta(x)}\, dx \right)^n$$

On the other hand, we have $\zeta \geq 0$ and $\{\zeta = 0\} = E$ by (1.16), hence we have $e^{-\lambda\beta n\zeta(x)} \to \mathbf{1}_E$ pointwise, as $\beta n \to \infty$. By Lebesgue's dominated convergence theorem, it follows that (3.8) holds uniformly w.r.t. $\beta > \beta_0$, for any $\beta_0 > 0$. □

Proof of Proposition 3.3. Let again ν_n stand for $\sum_{i=1}^n \delta_{x_i}$. From (3.2) we have $g_{\nu_n} \geq 0$ outside $\cup_i B(x_i, C)$ and from (3.1) we have $g_{\nu_n} \geq -C$ (depending only on $\|\mu_0\|_{L^\infty}$ hence on V) in $\cup_i B(x_i, C)$. Inserting into (3.7) we deduce that

$$F_n(\nu_n) \geq -C + 2\int \zeta\, d\nu_n,$$

where C depends only on V. Inserting into (1.23) and integrating over $\mathbb{C}^n$, we find

$$1 \leq \frac{1}{K_n^\beta} e^{Cn\beta} \int_{\mathbb{C}^n} e^{-n\beta \int \zeta\, d\nu_n}\, dx_1 \dots dx_n.$$

Inserting (3.8) and taking logarithms, it follows that

$$\log K_n^\beta \leq Cn\beta + n(\log|E| + o(1)).$$

The relation (3.6) follows using (1.24). □

4 A construction and a lower bound for Z_n^β

In this section, we construct a set of explicit configurations whose W is not too large, and show that their probability is not too small, which will lead to a lower bound on Z_n^β. This is the longest part of our proof. The method is borrowed from [SS1] but requires various adjustments that we shall detail in Section 7. We will need (1.13) in order to simplify the construction and estimates near the boundary.

We recall the following result ([SS1], Proposition 4.2). In this proposition, the notation W_K is used for the renormalized energy relative to the family of squares $K_R = [-R, R]^2$, i.e. defined by $W_K(j) = \limsup_{R\to\infty} \frac{W(j,\chi_{K_R})}{|K_R|}$ where χ_{K_R} satisfies (1.29) relative to the family of centered squares.

The following proves Theorem 2, part B and contains a bit more information useful for proving Theorem 5.

Proposition 4.1. *Let P be a $T_{\lambda(x)}$-invariant probability measure on $X = E \times L^p_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ with first marginal $dx_{|E}/|E|$ and such that for P almost every (x, j) we have $j \in \mathcal{A}_{m_0(x)}$. Then, for any $\eta > 0$, there exists $\delta > 0$ and for any n a subset $A_n \subset \mathbb{C}^n$ such that $|A_n| \geq n!(\pi\delta^2/n)^n$ and for every sequence $\{\nu_n = \delta_{y_1} + \dots + \delta_{y_n}\}_n$ with $(y_1, \dots, y_n) \in A_n$ the following holds.*

i) We have the upper bound

$$\limsup_{n\to\infty} \frac{1}{n} \left(w_n(y_1, \dots, y_n) - n^2 I(\mu_0) + \frac{n}{2}\log n \right) \leq \frac{|E|}{\pi} \int W_K(j)\, dP(x, j) + \eta. \tag{4.1}$$

ii) There exists $\{j_n\}_n$ in $L^p_{\rm loc}(\mathbb{R}^2,\mathbb{R}^2)$ such that $\operatorname{curl} j_n = 2\pi(\nu_n' - m_0')$ *and such that the image P_n of $dx_{|E}/|E|$ by the map $x \mapsto (x, j_n(\sqrt{n}x + \cdot))$ is such that*

$$\limsup_{n\to\infty} \operatorname{dist}(P_n, P) \le \eta, \tag{4.2}$$

where dist *is a distance which metrizes the topology of weak convergence on $\mathcal{P}(X)$.*

Applying the above proposition with $\eta = 1/k$ we get a subset $A_{n,k}$ in which we choose any n-tuple $(y_{i,k})_{1\le i\le n}$. This yields in turn a family $\{P_{n,k}\}$ of probability measures on X. A standard diagonal extraction argument then yields

Corollary 4.2 (Theorem 2, Part B)**.** *Under the same assumptions as Proposition 4.1, there exists a sequence $\{\nu_n = \delta_{x_1} + \cdots + \delta_{x_n}\}_n$ and a sequence $\{j_n\}_n$ in $L^p_{\rm loc}(\mathbb{R}^2,\mathbb{R}^2)$ such that* $\operatorname{curl} j_n = 2\pi(\nu_n' - m_0')$ *and*

$$\limsup_{n\to\infty} \frac{1}{n}\left(w_n(x_1,\dots,x_n) - n^2 I(\mu_0) + \frac{n}{2}\log n\right) \le \frac{|E|}{\pi}\int W_K(j)\,dP(x,j). \tag{4.3}$$

Moreover, if we assume in addition that P is $T_{\lambda(x)}$-invariant then, denoting P_n the image of $dx_{|E}/|E|$ by the map $x \mapsto (x, j_n(\sqrt{n}x + \cdot))$, we have $P_n \to P$ as $n \to +\infty$.

Another consequence of Proposition 4.1 is, recalling (1.36) and (1.24):

Corollary 4.3 (Lower bound part of Theorem 1)**.** *For any $\eta > 0$ there exists $C_\eta > 0$ such that for any $\beta > 0$ we have*

$$\liminf_{n\to+\infty} \frac{\log K_n^\beta}{n} \ge -\frac{\beta}{2}(\alpha + \eta) - C_\eta. \tag{4.4}$$

Proof of the corollary. We use Proposition 4.1. Choose $j_0 \in \mathcal{A}_1$ to be a minimizer for $\min_{\mathcal{A}_1} W$, and let P be the image of the normalized Lebesgue measure on E by the map $x \mapsto (x, \sigma_{m_0(x)} j_0)$, where

$$\sigma_m j(y) := \sqrt{m}\, j(\sqrt{m} y). \tag{4.5}$$

Then by construction P-almost every (x,j) satisfies $j \in \mathcal{A}_{m_0(x)}$ and the first marginal of P is $dx_{|E}/|E|$.

Given $\eta > 0$, applying Proposition 4.1 and using the notation there we have $|A_n| \ge n!(\delta^2/n)^n$ and from (1.23) we have

$$1 \ge \int_{A_n} \frac{1}{K_n^\beta} e^{-n\frac{\beta}{2}F_n(\nu_n)}\,dy_1 \dots dy_n, \tag{4.6}$$

where $\nu_n = \sum_{i=1}^n \delta_{y_i}$. From (1.22) and (4.1), when $(y_1,\dots,y_n) \in A_n$ we have

$$F_n(\nu_n) \le \eta + \frac{|E|}{\pi}\int W_K(j)\,dP(x,j) = \eta + \frac{1}{\pi}\int_E W_K(\sigma_{m_0(x)} j_0)\,dx.$$

From (1.33), which is true for W_K as well as W, and since $\int_E m_0 = 1$, we obtain

$$\frac{1}{\pi}\int_E W_K(\sigma_{m_0(x)} j_0)\,dx = \frac{1}{\pi} W_K(j_0) - \frac{1}{2}\int_E m_0(x)\log m_0(x)\,dx = \alpha,$$

by definition (1.36) and using the fact that W and W_K have the same minimum and minimizers (see the remark after (1.34)). We deduce

$$F_n(\nu_n) \le \eta + \alpha.$$

Together with (4.6) we find $1 \ge \frac{|A_n|}{K_n^\beta} e^{-n\frac{\beta}{2}(\eta+\alpha)}$. Taking logarithms, we find

$$\log K_n^\beta \ge \log n! + n \log \delta^2 - n \log n - \frac{1}{2} n\beta(\eta + \alpha).$$

From Stirling's formula, $\log n! \ge n \log n - Cn$ and we deduce (4.4), with $C_\eta = -\log \delta^2 + C$. Note that the dependence on η comes from δ. □

5 Consequences for fluctuations: proof of Theorem 3

In this section, we prove Theorem 3. The first step is to find, via the method first introduced in [SS3] and tools from [SS1, ST] how F_n and W control the discrepancy between ν_n and the measure $n\mu_0$.

Proposition 5.1. *Let $\nu = \sum_{i=1}^n \delta_{x_i}$, and g_ν be as in Definition 3.2. There exists a universal constant $R_0 > 0$ such that for any $R > R_0$, and any $x_0' = \sqrt{n}x_0 \in \mathbb{R}^2$, we have*

$$\int_{B_{2R}(x_0')} dg_\nu \ge cD(x_0', R)^2 \min\left(1, \frac{|D(x_0', R)|}{R^2}\right) - CR^2, \tag{5.1}$$

where $c > 0$ and C depend only on V, and where D was defined in (1.38). [5]

Proof. From Proposition 3.1 we have, defining j_ν as in Definition 3.2,

$$g_\nu \ge -C + \frac{1}{4}|j_\nu|^2 \mathbf{1}_{\mathbb{R}^2 \setminus \mathcal{B}_\rho}, \tag{5.2}$$

where $\mathcal{B}_\rho$ is a set of disjoint closed balls covering $\mathrm{Supp}(\nu')$, and the sum of the radii of the balls in $\mathcal{B}_\rho$ intersecting any given ball of radius 1 is bounded by ρ and we may take $\rho < \frac{1}{8}$.

We distinguish two cases. Either $D(x_0', R) \ge 0$ or $D(x_0', R) < 0$. Let us start with the first case. We must introduce a modified distance function to x_0', as follows. For any x, $f(x)$ is the infimum over the set of curves γ joining a point in $B_R(x_0')$ to x of the length of $\gamma \setminus \mathcal{B}_\rho$. This is also the distance to x_0' for the degenerate metric which is Euclidean outside $B_R(x_0') \cup \mathcal{B}_\rho$ and vanishes on $B_R(x_0') \cup \mathcal{B}_\rho$. We claim the following:

Lemma 5.2. *If $|x - x_0'| \ge R + 2$ then*

$$\frac{|x - x_0'| - R}{4} \le f(x) \le |x - x_0'| - R.$$

Proof. The upper bound is obvious so we turn to the lower bound. Let $\gamma(t)$ be a continuous curve joining $x = \gamma(0)$ to $B_R(x_0')$. Let us build by induction a sequence $x^0 = x, \dots, x^K$ with x^{k+1} defined as follows: let t_{k+1} be the smallest $t > t_k$ such that $\gamma(t) \notin B(x^k, 1) \cap \gamma$ or

[5] In fact R_0 could be any positive constant, and then c, C would depend on R_0 as well, but this requires to adjust ρ accordingly and we omit for simplicity to prove this fact.

$\gamma(t) \in B_R(x_0')$. This procedure terminates after a finite number of steps at $x^K \in \partial B_R(x_0')$. By triangle inequality we have

$$|x - x_0'| \le \sum_{k=0}^{K-1} |x^{k+1} - x^k| + |x^K - x_0'| \le K + R. \tag{5.3}$$

On the other hand, by property of $\mathcal{B}_\rho$, for any $0 \le k \le K-1$ we have

$$\ell(\gamma[t_k, t_{k+1}] \setminus \mathcal{B}_\rho) \ge |x^{k+1} - x^k| - 2\rho.$$

Summing this over k and using (5.3), we find

$$\ell(\gamma \setminus \mathcal{B}_\rho) \ge K - 1 - 2K\rho \ge |x - x_0'| - R - 1 - 2\rho \left(|x - x_0'| - R\right).$$

Taking the infimum over all curves γ we deduce that

$$f(x) \ge \left(|x - x_0'| - R\right)(1 - 2\rho) - 1.$$

Since by assumption $|x - x_0'| - R \ge 2$, we obtain $f(x) \ge (|x - x_0'| - R)(1 - 2\rho - \frac{1}{2})$ and the result follows since $\rho < 1/8$. □

We proceed with the proof of Proposition 5.1. Since f is Lipschitz with constant 1, almost every t is a regular value of f. For such a t the curve $\gamma_t := \{f = t\}$ is Lipschitz and does not intersect $\mathcal{B}_\rho$, since $\nabla f = 0$ there. Moreover, restating Lemma 5.2 we have

$$f(x) < t \implies x \in B_{R+4t}(x_0') \cup B_{R+2}(x_0'), \tag{5.4}$$

thus $\gamma_t \subset B_{2R}(x_0')$ if $R + 4t < 2R$, i.e. if $t < R/4$, and $R + 2 < 2R$, i.e. if $R > 2$. It follows from (5.2) and the coarea formula that if $R > 2$ then

$$\int_{B_{2R}(x_0')} dg_\nu \ge -CR^2 + \frac{1}{4} \int_{t=0}^{R/4} \left(\int_{\gamma_t} |j_\nu|^2 \right) dt. \tag{5.5}$$

We proceed by estimating the innermost integral on the right-hand side. If $t > 1/2$ then $B_{R+2} \subset B_{R+4t}$ and using (5.4) we find, using Definition 3.2 and writing D instead of $D(x_0', R)$ in the course of this proof,

$$\int_{\gamma_t} j_\nu \cdot \tau = \int_{\{f<t\}} \operatorname{curl} j_\nu \ge 2\pi\nu \left(B_{\frac{R}{\sqrt{n}}}(x_0) \right) - 2\pi n \mu_0 \left(B_{\frac{R+4t}{\sqrt{n}}}(x_0) \right)$$
$$\ge 2\pi D - C\left((R+4t)^2 - R^2 \right),$$

where we have used (1.15). The right-hand side is bounded below by πD if $R+4t < \sqrt{R^2 + cD}$ and c is small enough. Thus, if

$$1/2 < T := \frac{R}{4} \left(\sqrt{1 + c\frac{D}{R^2}} - 1 \right), \tag{5.6}$$

it follows using Cauchy-Schwarz's inequality that for every $t \in (1/2, T)$ we have

$$\int_{\gamma_t} |j_\nu|^2 \ge \pi^2 \frac{D^2}{|\gamma_t|}. \tag{5.7}$$

Inserting into (5.5) while reducing integration to $1/2 < t < \min(T, R/4)$ we obtain

$$(5.8) \qquad \int_{B_{2R}(x_0')} dg_\nu \geq -CR^2 + \pi^2 D^2 \int_{t=1/2}^{\min(T,R/4)} \frac{1}{|\gamma_t|}\, dt.$$

The evaluation of the last integral is complicated by the fact that γ_t is the level set for f rather than the usual circle, however the result will be comparable to the one we would get if we had $|\gamma_t| = 2\pi(R+t)$, this is proven as follows: From Lemma 5.2 we have for every $t \in [1/2, \min(T, R/4)]$ that $\gamma_t \subset \{x : 0 < |x - x_0'| - R < \min(4T, R)\}$. From the coarea formula and the fact that $|\nabla f| \leq 1$ it follows that

$$\int_{\frac{1}{2}}^{\min(T,R/4)} |\gamma_t|\, dt \leq |\{x : R < |x - x_0'| < R + \min(4T, R)\}| \leq CR\min(4T, R).$$

Then, using the convexity of $x \mapsto 1/x$ and Jensen's inequality in (5.8) we obtain for some $c > 0$ that

$$\int_{\frac{1}{2}}^{\min(T,R/4)} \frac{1}{|\gamma_t|}\, dt \geq c\frac{\left(\min(T, R/4) - \frac{1}{2}\right)^2}{R\min(4T, R)}.$$

Inserting into (5.8) we obtain assuming (5.6) and

$$(5.9) \qquad 1 < \min(T, R/4)$$

that

$$(5.10) \qquad \int_{B_{2R}(x_0')} dg_\nu \geq -CR^2 + c\frac{D^2}{R}\left(\min(T, R/4) - \frac{1}{2}\right).$$

One may check that (5.6), (5.9) are satisfied if $R > 4$ and $D > C_0 R$ for a large enough $C_0 > 0$. Then it is not difficult to deduce (5.1) from (5.10) by distinguishing the cases $T < R/4$ and $T \geq R/4$, i.e. $D < C_1 R^2$ and $D \geq C_1 R^2$ for a well chosen C_1. Finally, if $D < C_0 R$ then (5.1) is trivially satisfied, if C is chosen large enough.

Let us turn to the case $D(x_0', R) \leq 0$, which implies $|D(x_0', R)| \leq n\mu_0(B_{R/\sqrt{n}}(x_0)) \leq \pi R^2 \overline{m}$.

In this case we define $f(x)$ to be the distance of x to the complement of $B_R(x_0')$ with respect to the metric which is Euclidean on $B_R(x_0') \setminus \mathcal{B}_\rho$. As in Lemma 5.2 and assuming $|x - x_0'| < R - 2$ we have

$$\frac{R - |x - x_0'|}{4} \leq f(x) \leq R - |x - x_0'|.$$

and as above, if $R > 2$ and for almost every $1/2 < t < R/4$ the curve $\gamma_t = \{f = t\}$ is a Lipschitz curve which does not intersect $\mathcal{B}_\rho$ and $\{f < t\} \subset B_R(x_0') \setminus B_{R-4t}(x_0')$. It follows that, writing as before D for $D(x_0, R)$

$$(5.11) \qquad \int_{\gamma_t} j_\nu \cdot \tau = \int_{\{f<t\}} \operatorname{curl} j_\nu = \int_{B_R(x_0')} \operatorname{curl} j_\nu - \int_{B(x_0',R)\setminus\{f>t\}} \operatorname{curl} j_\nu \leq$$
$$2\pi D + 2\pi n\mu_0\left(B_{\frac{R}{\sqrt{n}}} \setminus B_{\frac{R-4t}{\sqrt{n}}}\right) \leq 2\pi D + C\left(R^2 - (R-4t)^2\right).$$

The proof then proceeds as in the first case by using Cauchy-Schwarz's inequality and integrating with respect to $t \in [1/2, \min(T, R/4)]$, where

$$T = \frac{R}{4}\left(1 - \sqrt{1 + c\frac{D}{R^2}}\right),$$

which ensures that the right-hand side in (5.11) is bounded above by πD. Note that D is nonpositive, but bounded below by $-CR^2$ hence if $c > 0$ is small enough the quantity inside the square root above is positive. □

We now proceed to the proof of Theorem 3, starting with (1.42). If $R > R_0$ and $|D(x_0', R)| \geq \eta R^2$ then from Proposition 5.1 and using the fact — from Proposition 3.1 — that g_ν is positive outside $\cup_{i=1}^n B(x_i', C)$ and that $g_\nu \geq -C$ everywhere, we deduce from (3.7) and (5.1) that

$$F_n(\nu) \geq \frac{1}{n}\left(-CR^2 + c\min(\eta^2, \eta^3)R^4\right) + 2\int \zeta\, d\nu. \tag{5.12}$$

Inserting into (1.23) we find

$$\mathbb{P}_n^\beta\left(\left|D(x_0', R)\right| \geq \eta R^2\right) \leq \frac{1}{K_n^\beta}\exp\left(C\beta R^2 - c\beta\min(\eta^2, \eta^3)R^4\right)\int e^{-n\beta\int \zeta\, d\nu}\, dx_1 \dots dx_n.$$

Then, using the lower bound (4.4) and Lemma 3.5 we deduce that if $\beta > \beta_0 > 0$ and n is large enough depending on β_0 then

$$\log \mathbb{P}_n^\beta\left(\left|D(x_0', R)\right| \geq \eta R^2\right) \leq -c\beta\min(\eta^2, \eta^3)R^4 + C\beta R^2 + Cn\beta + Cn,$$

where $c, C > 0$ depend only on V. Thus (1.42) is established.

We next prove (1.44). By Fubini's theorem, and using again the facts that g_ν is positive outside $\cup_{i=1}^n B(x_i', C)$ and $\geq -C$ everywhere we have

$$\int_{\mathbb{R}^2} dg_\nu \geq \int_{U'}\left(\fint_{B(x', 2R)} dg_\nu\right) dx' - C|U'| - Cn.$$

Combining with Proposition 5.1 it follows that

$$\int_{\mathbb{R}^2} dg_\nu \geq -C(|U'| + n) + \frac{1}{R^2}\int_{U'} -CR^2 + cD(x', R)^2 \min\left(1, \frac{|D(x', R)|}{R^2}\right) dx'.$$

i.e., changing the constants if necessary,

$$\int_{\mathbb{R}^2} dg_\nu \geq -C(|U'| + n) + \frac{c}{R^2}\int_{U'} D(x', R)^2 \min\left(1, \frac{|D(x', R)|}{R^2}\right) dx. \tag{5.13}$$

It follows, using as above (1.23), (3.7), (4.4) and Lemma 3.5, and since $|U'| = n|U|$, that

$$\log \mathbb{P}_n^\beta\left(\int_{U'} \frac{D(x', R)^2}{R^2}\min\left(1, \frac{|D(x', R)|}{R^2}\right) dx \geq \eta\right) \leq -cn\beta\eta + Cn\beta\left(|U| + 1\right) + Cn,$$

where $c, C > 0$ depend only on V, where $\beta > \beta_0 > 0$ and where $n > n_0(\beta_0)$.

We next turn to (1.43). Arguing as above, from (3.7) we have $F_n(\nu) \geq -C + 2\int \zeta\, d\nu$. Splitting $2\int \zeta\, d\nu$ as $\int \zeta\, d\nu + \int \zeta\, d\nu$, inserting into (1.23) and using (4.4) we are led to

$$\mathbb{P}_n^\beta \left(\int \zeta\, d\nu \geq \eta \right) \leq e^{-\frac{1}{2}n\beta\eta + Cn(\beta+1)} \int e^{-n\beta \int \zeta\, d\nu}\, dx_1 \ldots, dx_n,$$

where C depends only on V. Then, using Lemma 3.5 we deduce (1.43).

There remains to prove (1.45). Let $\nu = \sum_{i=1}^n \delta_{x_i}$ and χ be a nonnegative function such that $\chi = 1$ on $U := E' \cup \left(\cup_{i=1}^n B(x_i', \frac{1}{2})\right) \cup B_R$ and $\|\chi\|_\infty, \|\nabla\chi\|_\infty \leq 1$, compactly supported on $\widehat{U} = \{x : d(x, U) \leq 1\}$. We have $|\widehat{U}| \leq C(n + R^2)$, where C depends only on E, i.e. on V. Then Corollary 1.2 in [ST] asserts that, for any $q < 2$,

$$\|\sqrt{\chi} j_\nu\|_{L^q} \leq C_q (n + R^2)^{\frac{1}{q} - \frac{1}{2}} \left(W(j_\nu, \chi) + n\right)^{\frac{1}{2}}. \tag{5.14}$$

where we use that by construction, $\#\{p \in \Lambda | B(p, \frac{1}{2}) \cap \{0 \leq \chi < 1\} \neq \varnothing\} = 0$. This same fact implies that $\nu' = 0$ in the support of $1 - \chi$, hence

$$W(j_\nu, 1 - \chi) = \frac{1}{2} \int (1 - \chi) |j_\nu|^2 \geq 0.$$

In particular $W(j_\nu, \chi) \leq W(j_\nu, \chi) + W(j_\nu, 1 - \chi) = W(j_\nu, \mathbf{1}_{\mathbb{R}^2})$. It then follows from (5.14) and the fact that $F_n(\nu) = \frac{1}{\pi n} W(j_\nu, \mathbf{1}_{\mathbb{R}^2}) + 2\int \zeta\, d\nu$ (cf. (1.21)), that

$$\int_{B_R} |j_\nu|^q \leq C_q (n + R^2)^{1 - \frac{q}{2}} n^{\frac{q}{2}} \left(F_n(\nu) - 2\int \zeta\, d\nu + 1 \right)^{\frac{q}{2}}. \tag{5.15}$$

But, reasoning as above after (5.12), the probability that $F_n(\nu) - 2\int \zeta\, d\nu \geq \eta$ is bounded above for any $\beta > \beta_0 > 0$ an n large enough depending on β_0 by $\exp(-\frac{1}{2}n\beta\eta + Cn(\beta + 1))$, where C depends on V only. In view of (5.15),

$$\mathbb{P}_n^\beta \left(\int_{B_R} |j_\nu|^q \geq C_q n (\eta + 1)^{\frac{q}{2}} \left(1 + \frac{R^2}{n}\right)^{1 - \frac{q}{2}} \right) \leq \exp(-\frac{1}{2} n\beta\eta + Cn(\beta + 1)).$$

Rescaling we have $\int_\Omega |\nabla H|^q = n^{\frac{q}{2} - 1} \int_{\Omega'} |j_\nu|^q$, where $H = -2\pi\Delta^{-1}(\nu - n\mu_0)$, while $\|\nu - n\mu_0\|_{W^{-1,q}(\Omega)} \leq C\|\nabla H\|_{L^q(\Omega)}$. Therefore

$$\mathbb{P}_n^\beta \left(\left(1 + \frac{R^2}{n}\right)^{\frac{1}{2} - \frac{1}{q}} \|\nu - n\mu_0\|_{W^{-1,q}(B_{R/\sqrt{n}})} \geq C_q n^{\frac{1}{2}} (1 + \eta)^{\frac{1}{2}} \right) \leq \exp(-\frac{1}{2} n\beta\eta + Cn(\beta + 1)).$$

After a slight rewriting, this concludes the proof of Theorem 3.

6 Lower bounds via the ergodic theorem and conclusions

6.1 Abstract result via the ergodic theorem

In this section, we present the ergodic framework introduced in [SS1] for obtaining "lower bounds for 2-scale energies". We cannot directly use the result there because it is written for a uniform "macroscopic environment", which would correspond to the case where $m_0(x)$

is constant on its support (as in the circular law). To account for the possibility of varying environment or weight at the macroscopic, we can however adapt Theorem 3 of [SS1] and easily prove the following variant:

Let X denote a Polish metric space, when we speak of measurable functions on X we will always mean Borel-measurable. We assume there is a d-parameter group of transformations θ_λ acting continuously on X. More precisely we require that

- For all $u \in X$ and $\lambda, \mu \in \mathbb{R}^d$, $\theta_\lambda(\theta_\mu u) = \theta_{\lambda+\mu} u$, $\theta_0 u = u$.
- The map $(\lambda, u) \mapsto \theta_\lambda u$ is continuous with respect to each variable (hence measurable with respect to both).

Typically we think of X as a space of functions defined on $\mathbb{R}^d$ and θ as the action of translations, i.e. $\theta_\lambda u(x) = u(x + \lambda)$. Then we consider the following d-parameter group of transformations T^ε_λ acting continuously on $\mathbb{R}^d \times X$ by $T^\varepsilon_\lambda(x, u) = (x + \varepsilon\lambda, \theta_\lambda u)$. We also define $T_\lambda(x, u) = (x, \theta_\lambda u)$.

For a probability measure P on $\mathbb{R}^d \times X$ we say that P is translation-invariant if it is invariant under the action T, and we say it is $T_{\lambda(x)}$-invariant if for every function $\lambda(x)$ of class C^1, it is invariant under the mapping $(x, u) \mapsto (x, \theta_{\lambda(x)} u)$. Note that $T_{\lambda(x)}$-invariant implies translation-invariant.

Let G denote a bounded compact set in $\mathbb{R}^d$ such that

$$(6.1) \qquad |G| > 0, \quad \lim_{\varepsilon \to 0} \frac{|(G + \varepsilon x) \triangle G|}{|G|} = 0,$$

for every $x \in \mathbb{R}^2$. We let $\{f_\varepsilon\}_\varepsilon$ and f be measurable nonnegative functions on $G \times X$, and assume that for any family $\{(x_\varepsilon, u_\varepsilon)\}_\varepsilon$ such that

$$\forall R > 0, \quad \limsup_{\varepsilon \to 0} \int_{B_R} f_\varepsilon(T^\varepsilon_\lambda(x_\varepsilon, u_\varepsilon))\, d\lambda < +\infty$$

the following holds.

1. (Coercivity) $\{(x_\varepsilon, u_\varepsilon)\}_\varepsilon$ admits a convergent subsequence (note that $\{x_\varepsilon\}_\varepsilon$ subsequentially converges since G is compact).
2. (Γ-liminf) If $\{(x_\varepsilon, u_\varepsilon)\}_\varepsilon$ converges to (x, u) then
$$\liminf_{\varepsilon \to 0} f_\varepsilon(x_\varepsilon, u_\varepsilon) \geq f(x, u).$$

Then we consider an increasing family of bounded open sets $\{\mathbf{U}_R\}_{R>0}$ such that

$$(6.2) \qquad \text{(i) } \{\mathbf{U}_R\}_{R>0} \text{ is a Vitali family}, \quad (ii) \lim_{R \to +\infty} \frac{|(\lambda + \mathbf{U}_R) \triangle \mathbf{U}_R|}{|\mathbf{U}_R|} = 0$$

for any $\lambda \in \mathbb{R}^d$, where Vitali means (see [Ri]) that the intersection of the closures is $\{0\}$, that $R \mapsto |\mathbf{U}_R|$ is left continuous, and that $|\mathbf{U}_R - \mathbf{U}_R| \leq C|\mathbf{U}_R|$.

We have

Theorem 6. *Let G, X, $\{\theta_\lambda\}_\lambda$, and $\{f_\varepsilon\}_\varepsilon$, f be as above. Let*

$$F_\varepsilon(u) = \fint_G f_\varepsilon(x, \theta_{\frac{x}{\varepsilon}} u)\, dx.$$

Assume that $\{F_\varepsilon(u_\varepsilon)\}_\varepsilon$ is bounded. Let P_ε be the probability on $G \times X$ which is the image of the normalized Lebesgue measure on G under the map $x \mapsto (x, \theta_{\frac{x}{\varepsilon}} u_\varepsilon)$. Then P_ε converges to a Borel probability measure P on $G \times X$ whose first marginal is the normalized Lebesgue measure on G, which is $T_{\lambda(x)}$-invariant, such that P-a.e. (x, u) is of the form $\lim_{\varepsilon\to 0}(x_\varepsilon, \theta_{\frac{x_\varepsilon}{\varepsilon}} u_\varepsilon)$ and such that

$$\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) \geq \int f(x,u)\, dP(x,u). \tag{6.3}$$

Moreover,

$$\int f(x,u)\, dP(x,u) = \mathbf{E}^P \left(\lim_{R\to+\infty} \fint_{\mathbf{U}_R} f(x, \theta_\lambda u)\, d\lambda \right), \tag{6.4}$$

where $\mathbf{E}^P$ denotes the expectation under the probability P.

Proof. We only sketch the proof as it is very similar to [SS1]. The following points have to be checked:

1. P_ε is tight hence has a limit P. This follows from the coercivity property of f_ε as in [SS1].

2. P is $T_{\lambda(x)}$-invariant. Let Φ be bounded and continuous, and let P_λ be the push-forward of P by $(x,u) \mapsto (x, \theta_{\lambda(x)} u)$. Then from the definition of P_λ, P, P_ε, we have

$$\int \Phi(x,u)\, dP_\lambda(x,u) = \int \Phi(x, \theta_{\lambda(x)} u)\, dP(x,u) = \lim_{\varepsilon\to 0} \int \Phi(x, \theta_{\lambda(x)} u)\, dP_\varepsilon(x,u) =$$
$$\lim_{\varepsilon\to 0} \fint_G \Phi(x, \theta_{\frac{x}{\varepsilon}+\lambda(x)} u_\varepsilon)\, dx = \lim_{\varepsilon\to 0} \fint_{(I+\varepsilon\lambda)(G)} \frac{\Phi((I+\varepsilon\lambda)^{-1}(y), \theta_{\frac{y}{\varepsilon}} u_\varepsilon)}{|\det(I + \varepsilon D\lambda((I+\varepsilon\lambda)^{-1}(y))|}\, dy,$$

where the last equality follows by the change of variables $y = (I+\varepsilon\lambda)(x)$. Using the boundedness of Φ, the C^1 character of λ, the compactness of G and (6.1), we may replace $(I+\varepsilon\lambda)(G)$ by G and the denominator by 1 in the last integral and we find, using the definition of P_ε

$$\int \Phi(x,u)\, dP_\lambda(x,u) = \lim_{\varepsilon\to 0} \int \Phi((I+\varepsilon\lambda)^{-1}(x), u)\, dP_\varepsilon(x,u). \tag{6.5}$$

Since $\{P_\varepsilon\}_\varepsilon$ is tight, for any $\delta > 0$ there exists K_δ such that $P_\varepsilon({K_\delta}^c) < \delta$ for every ε. Then by uniform continuity of Φ on K_δ the map $(x,u) \mapsto \Phi((I+\varepsilon\lambda)^{-1}(x), u)$ converges uniformly on K_δ to $(x,u) \mapsto \Phi(x,u)$ and thus

$$\lim_{\varepsilon\to 0} \int_{K_\delta} \Phi((I+\varepsilon\lambda)^{-1}(x), u)\, dP_\varepsilon(x,u) = \lim_{\varepsilon\to 0} \int_{K_\delta} \Phi(x,u)\, dP_\varepsilon(x,u).$$

Since this is true for any $\delta > 0$, and using the boundedness of Φ we get

$$\lim_{\varepsilon\to 0} \int \Phi((I+\varepsilon\lambda)^{-1}(x), u)\, dP_\varepsilon(x,u) = \lim_{\varepsilon\to 0} \int \Phi(x,u)\, dP_\varepsilon(x,u) = \int \Phi(x,u)\, dP(x,u),$$

by definition of P. Thus in view of (6.5) we have $P_\lambda = P$ and P is thus $T_{\lambda(x)}$-invariant.

3. $\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) \ge \int f\, dP$. This follows from Lemma 2.2 of [SS1], since $F_\varepsilon(u_\varepsilon) = \int f_\varepsilon\, dP_\varepsilon$.

To conclude, as in [SS1], the fact that P is $T_{\lambda(x)}$-invariant (which implies T_λ-invariant) and Wiener's multiparametric ergodic theorem (see e.g. [Be]) imply that

$$\int f(x,u)\, dP(x,u) = \mathbf{E}^P\left(\lim_{R\to+\infty} \fint_{\mathbf{U}_R} f(T_\lambda(x,u))\, d\lambda\right) = \mathbf{E}^P\left(\lim_{R\to+\infty} \fint_{\mathbf{U}_R} f(x,\theta_\lambda u)\, d\lambda\right).$$

□

6.2 Proof of Theorem 2, part A

The proof follows essentially [SS1], Proposition 4.1 and below. Let $\{\nu_n\}_n$ and P_{ν_n} be as in the statement of Theorem 2. We need to prove that any subsequence of $\{P_{\nu_n}\}_n$ has a convergent subsequence and that the limit P is a $T_{\lambda(x)}$-invariant probability measure such that P-almost every (x,j) is such that $j \in \mathcal{A}_{m_0(x)}$ and (1.39) holds. Note that the fact that the first marginal of P is $dx_{|E}/|E|$ follows from the fact that, by definition, this is true of P_{ν_n}.

We thus take a subsequence of $\{P_{\nu_n}\}$ (which we don't relabel). We may assume that it has a subsequence, denoted $\bar\nu_n$, which satisfies $F_n(\bar\nu_n) \le C$, otherwise there is nothing to prove. This implies that $\bar\nu_n$ is of the form $\sum_{i=1}^n \delta_{x_{i,n}}$. We let $\bar\jmath_n$ denote the current and $\bar g_n$ the measures associated to $\bar\nu_n$ as in Definition 3.2. As usual, $\bar\nu_n' = \sum_{i=1}^n \delta_{\sqrt{n}x_{i,n}}$. Another useful consequence of $F_n(\bar\nu_n) \le C$ is that, using (2.1), we have $w_n(x_{i,n}) = n^2 I(\mu_0) + O(n\log n)$, which in turn implies (see [ST]) that

$$\frac{1}{n}\bar\nu_n \to \mu_0. \tag{6.6}$$

Step 1: We set up the framework of Section 6.1

We will use integers n instead of ε to label sequences, and the correspondence will be $\varepsilon = 1/\sqrt{n}$. We let $G = E$ and $X = \mathcal{M}_+ \times L^p_{\mathrm{loc}}(\mathbb{R}^2,\mathbb{R}^2) \times \mathcal{M}$, where $p \in (1,2)$, where $\mathcal{M}_+$ denotes the set of positive Radon measures on $\mathbb{R}^2$ and $\mathcal{M}$ the set of those which are bounded below by the constant $-C_V := -C(\|m_0\|_\infty + 1)$ of Proposition 3.1, both equipped with the topology of weak convergence.

For $\lambda \in \mathbb{R}^2$ and abusing notation we let θ_λ denote both the translation $x \to x+\lambda$ and the action

$$\theta_\lambda(\nu,j,g) = (\theta_\lambda\#\nu, j\circ\theta_\lambda, \theta_\lambda\#g).$$

Accordingly the action T^n is defined for $\lambda \in \mathbb{R}^2$ by

$$T^n_\lambda(x,\nu,j,g) = \left(x + \frac{\lambda}{\sqrt{n}}, \theta_\lambda\#\nu, j\circ\theta_\lambda, \theta_\lambda\#g\right).$$

Then we let χ be a smooth cut-off function with integral 1 and support in $B(0,1)$ and define

$$\mathbf{f}_n(x,\nu,j,g) = \begin{cases} \dfrac{1}{\pi}\displaystyle\int_{\mathbb{R}^2} \chi(y)\, dg(y) & \text{if } (\nu,j,g) = \theta_{\sqrt{n}x}(\bar\nu_n', \bar\jmath_n, \bar g_n), \\ +\infty & \text{otherwise.} \end{cases} \tag{6.7}$$

Finally we let, in agreement with Section 6.1,

$$\mathbf{F}_n(\nu, j, g) = \fint_E \mathbf{f}_n\left(x, \theta_{x\sqrt{n}}(\nu, j, g)\right) dx. \tag{6.8}$$

We have the following relation between $\mathbf{F}_n$ and F_n, as $n \to +\infty$:

$$\mathbf{F}_n(\nu, j, g) \text{ is } \begin{cases} \leq \frac{1}{|E|}\left(F_n(\bar\nu_n) - 2\int \zeta\, d\bar\nu_n\right) + o(1) & \text{if } (\nu, j, g) = (\bar\nu'_n, \bar\jmath_n, \bar g_n) \\ = +\infty & \text{otherwise.} \end{cases} \tag{6.9}$$

Indeed it is obvious from (6.7) that if $(\nu, j, g) \neq (\bar\nu'_n, \bar\jmath_n, \bar g_n)$ then $\mathbf{F}_n(\nu, j, g) = +\infty$. On the other hand, if $(\nu, j, g) = (\bar\nu'_n, \bar\jmath_n, \bar g_n)$, then from the definition of the image measure $\theta_\lambda \# \bar g_n$,

$$\mathbf{F}_n(\nu, j, g) = \frac{1}{\pi}\fint_E \int \chi(y - x\sqrt{n})\, d\bar g_n(y)\, dx = \frac{1}{\pi|E'|}\int \chi * \mathbf{1}_{E'}\, d\bar g_n.$$

Since $\chi * \mathbf{1}_{E'}$ is bounded above by 1 and is equal to 1 on $U := \{x' : \mathrm{dist}(x', \mathbb{R}^2 \setminus E') \geq 1\}$ we deduce that

$$\pi\mathbf{F}_n(\nu, j, g) \leq \frac{\bar g_n^+(\mathbb{R}^2) - \bar g_n^-(U)}{|E'|} = \frac{\bar g_n(\mathbb{R}^2) + \bar g_n^-(U^c)}{n|E|} \tag{6.10}$$
$$= \frac{\pi}{|E|}\left(F_n(\bar\nu_n) - 2\int \zeta\, d\bar\nu_n\right) + \frac{\bar g_n^-(U^c)}{n|E|}.$$

Then we note that from (3.2) in Proposition 3.1 the measure $\bar g_n^-$ is supported in the union of balls $B(x', C)$ for $x' \in \mathrm{Supp}(\bar\nu'_n)$, and bounded above by a constant. Thus $\bar g_n^-(U^c)$ is bounded by a constant times the number of balls intersecting U^c, hence by $C\bar\nu'_n\{x' : \mathrm{dist}(x', U^c) \leq C\}$. From (6.6) this is equal to $Cn\mu_0\{x : \mathrm{dist}(x, \partial E) \leq C/\sqrt{n}\} + o(n)$, which from the assumption (1.13) on E is $o(n)$. Plugging this into (6.10) proves (6.9).

Step 2: We check the hypotheses in Section 6.1

We must now check the Gamma-liminf and coercivity properties of $\{\mathbf{f}_n\}_n$.

Lemma 6.1. *Assume that $\{(x_n, \nu_n, j_n, g_n)\}_n$ converges to (x, ν, j, g). Then*

$$\liminf_n \mathbf{f}_n(x_n, \nu_n, j_n, g_n) \geq \mathbf{f}(x, \nu, j, g) := \frac{1}{\pi}\int \chi\, dg.$$

Proof. We may assume that the left-hand side is finite, in which case $\mathbf{f}_n(x_n, \nu_n, j_n, g_n) = \frac{1}{\pi}\int \chi\, dg_n$ for every large enough n, from which the result follows by passing to the limit. □

Lemma 6.2. *Assume that for any $R > 0$ we have*

$$\limsup_{n\to+\infty} \int_{B_R} \mathbf{f}_n\left(x_n + \frac{\lambda}{\sqrt{n}}, \theta_\lambda(\nu_n, j_n, g_n)\right) d\lambda < +\infty. \tag{6.11}$$

Then a subsequence of $\{(x_n, \nu_n, j_n, g_n)\}_n$ converges to some $(x, \nu, j, g) \in E \times X$.

Proof. Assume (6.11). Then the integrand there is bounded for a.e. λ and from the definition (6.7) we deduce that

$$\theta_\lambda(\nu_n, j_n, g_n) = \theta_{\sqrt{n}x_n+\lambda}(\bar\nu_n', \bar\jmath_n, \bar g_n)$$

and then that $(\nu_n, j_n, g_n) = \theta_{\sqrt{n}x_n}(\bar\nu_n', \bar\jmath_n, \bar g_n)$. Thus (6.11) gives, in view of (6.7), that for every $R > 0$ there exists $C_R > 0$ such that for any n

$$\int_{B_R}\int \chi(y - \sqrt{n}x_n - \lambda)\, d\bar g_n(y)\, d\lambda = \int \chi * \mathbf{1}_{B_R(\sqrt{n}x_n)}\, d\bar g_n < C_R.$$

This and the fact that $\bar g_n$ is bounded below implies that $\bar g_n(B_R(\sqrt{n}x_n))$ is bounded independently of n and then, using (3.4), that the same is true of $\bar\nu_n'(B_R(\sqrt{n}x_n))$. In other words $\{\nu_n = \theta_{\sqrt{n}x_n}\bar\nu_n'\}_n$ is a locally bounded sequence of (positive) measures hence converges weakly after taking a subsequence, and the same is true of $\{g_n = \theta_{\sqrt{n}x_n}\bar g_n\}_n$. On the other hand $\{x_n\}_n$ is a sequence in the compact set E hence converges modulo a subsequence.

It remains to study the convergence of $\{j_n = \bar\jmath_n\circ\theta_{\sqrt{n}x_n+\lambda}\}_n$. From (3.3) in Proposition 3.1 and the local boundedness of $\{\nu_n\}_n$ we get that $W(\bar\jmath_n, \chi * \mathbf{1}_{B_R(\sqrt{n}x_n)}) = W(j_n, \chi * \mathbf{1}_{B_R})$ is bounded independently of n for any $R > 0$ and then, using Corollary 1.2 in [ST], that $\{j_n\}_n$ is locally bounded in $L^p_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$, for any $1 \le p < 2$ hence a subsequence locally weakly converges in $L^p_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$. Strong local convergence follows as in [SS1] using elliptic regularity with $\operatorname{curl}\bar\jmath_n = \bar\nu_n' - {\mu_0}'$ and $\operatorname{div}\bar\jmath_n = 0$. □

Step 3: Conclusion

From the previous steps, we may apply Theorem 6 in this setting and we deduce in view of (6.9) that, letting Q_n denote the push-forward of the normalized Lebesgue measure on E by the map $x \mapsto (x, \theta_{\sqrt{n}x}(\bar\nu_n', \bar\jmath_n, \bar g_n))$, and $Q = \lim_n Q_n$,

$$\text{(6.12)}\quad \liminf_n \frac{1}{|E|}\left(F_n(\bar\nu_n) - 2\int \zeta\, d\bar\nu_n\right) \ge \liminf_n \mathbf{F}_n(\bar\nu_n', \bar\jmath_n, \bar g_n) \ge$$
$$\int\left(\int \chi\, dg\right) dQ(x,\nu,j,g) = \int \lim_{R\to+\infty} \fint_{B_R}\int \chi(y-\lambda)\, dg(y)\, d\lambda\, dQ(x,\nu,j,g) =$$
$$\int \lim_{R\to+\infty}\left(\frac{1}{\pi R^2}\int \chi * \mathbf{1}_{B_R}\, dg\right) dQ(x,\nu,j,g).$$

Now we use the fact that for Q-almost every (x,ν,j,g):

i) There exists a sequence $\{x_n\}_n$ in E such that $(x,\nu,j,g) = \lim_n(x_n, \theta_{\sqrt{n}x_n}(\bar\nu_n', \bar\jmath_n, \bar g_n))$.

ii) As a consequence of the above $\frac{1}{\pi R^2}\int \chi * \mathbf{1}_{B_R}\, dg$ converges to a finite limit as $R \to +\infty$.

The first point implies, since $\operatorname{curl}\bar\jmath_n = \bar\nu_n' - {m_0}'$ and $\operatorname{div}\bar\jmath_n = 0$, that $\operatorname{curl} j = \nu - m_0(x)$ and $\operatorname{div} j = 0$. The second point implies in particular using (3.4) that $\nu(B_R) \le CR^2$, proving that $(j,\nu) \in \mathcal{A}_{m_0(x)}$.

Moreover the second point implies that for any $C > 0$ we have $g(B_{R+C}\setminus B_{R-C}) = o(R^2)$ as $R \to +\infty$, and thus from point i) above

$$\lim_{R\to+\infty}\lim_{n\to+\infty}\frac{1}{R^2}\bar g_n(B_{R+C}(\sqrt{n}x_n))\setminus B_{r-C}(\sqrt{n}x_n)) = 0.$$

Using (3.4) we deduce that

$$\lim_{R\to+\infty}\lim_{n\to+\infty}\frac{1}{R^2}\bar{\nu}'_n(B_{R+C}(\sqrt{n}x_n))\setminus B_{r-C}(\sqrt{n}x_n))=0$$

and then from (3.3),

$$\lim_{R\to+\infty}\lim_{n\to+\infty}\frac{1}{R^2}\left|W(\bar{\jmath}_n,\chi*\mathbf{1}_{B_R(\sqrt{n}x_n)})-\int\chi*\mathbf{1}_{B_R(\sqrt{n}x_n)}\,d\bar{g}_n\right|=0.$$

Thus, using Lemma 4.8 in [SS1] in the limit with respect to n we deduce

$$\lim_{R\to+\infty}\frac{1}{R^2}\left|W(j,\chi*\mathbf{1}_{B_R})-\int\chi*\mathbf{1}_{B_R}\,dg\right|=0.$$

Together with (6.12) this yields, by definition of W,

$$\liminf_n\frac{1}{|E|}\left(F_n(\bar{\nu}_n)-2\int\zeta\,d\bar{\nu}_n\right)\geq\int W(j)\,dQ(x,\nu,j,g)\tag{6.13}$$

and, we recall, Q-a.e. $(j,\nu)\in\mathcal{A}_{m_0(x)}$.

Now we let P_n (resp. P) be the marginal of Q_n (resp. Q) with respect to the variables (x,j). Then the first marginal of P is the normalized Lebesgue measure on E and P-a.e. we have $j\in\mathcal{A}_{m_0(x)}$, in particular

$$W(j)\geq\min_{\mathcal{A}_{m_0(x)}}W=m_0(x)\left(\min_{\mathcal{A}_1}W-\frac{\pi}{2}\log m_0(x)\right).$$

Integrating with respect to P and noting that since only x appears on the right-hand side we may replace P by its first marginal there we find, in view of (1.36) that the lower bound (1.39) holds.

6.3 Proof of Theorem 2, completed

As mentioned above, Part B of the theorem is a direct consequence of Proposition 4.1, see Corollary 4.2.

Part C follows from the comparison of Parts A and B and the fact that W and W_K have the same minimizers and minimal value. This implies that, for minimizers, the chains of inequalities (1.39) and (1.40) are in fact equalities and that $\int W\,dP$ must be minimized hence equal to α. Since (1.39) follows from (6.13), the inequality (6.13) must also be an equality and moreover we must have $\lim_{n\to\infty}\int\zeta\,d\nu_n=0$, which in view of (1.17), implies that $\lim\sum_i \mathrm{dist}(x_i,E)^2=0$.

From the fact that $F_n(\nu_n)=O(1)$, we deduce from Proposition 5.1, 3.1 and (3.7), (arguing exactly as in the proof of Theorem 3) that there exists $C>0$ such that for every $x,R>1$, we have

$$D(x,R)^2\min\left(1,\frac{|D(x,R)|}{R^2}\right)\leq Cn.$$

This completes the proof of Theorem 2.

6.4 Deviations: proof of Theorems 5 and Theorem 1

We start with the upper bound on $\log \mathbb{P}_n^\beta$. Let A_n be a subset of $\mathbb{C}^n$. We identify points in $\mathbb{C}^n$ with measures ν_n of the form $\sum_{i=1}^n \delta_{x_i}$.

From (1.23), we have

$$\mathbb{P}_n^\beta(A_n) = \frac{1}{K_n^\beta} \int_{A_n} e^{-\frac{1}{2}\beta n F_n(\sum_{i=1}^n \delta_{x_i})}\, dx_1 \dots dx_n$$

hence

$$\frac{\log \mathbb{P}_n^\beta(A_n)}{n} = -\frac{\log K_n^\beta}{n} + \frac{1}{n} \log \int_{A_n} e^{-\frac{1}{2}\beta n F_n(\sum_{i=1}^n \delta_{x_i})}\, dx_1 \dots dx_n. \tag{6.14}$$

We deduce, writing $\widehat{F}_n(\nu) = F_n(\nu) - 2\int \zeta\, d\nu$, that

$$\frac{\log \mathbb{P}_n^\beta(A_n)}{n} \le -\frac{\log K_n^\beta}{n} + \frac{1}{n} \log \left(e^{-\frac{1}{2}\beta n \inf_{A_n} \widehat{F}_n} \int_{A_n} e^{-\beta n \int \zeta\, d\nu_n}\, dx_1 \dots dx_n \right). \tag{6.15}$$

Let ν_n such that $\widehat{F}_n(\nu_n) \le \inf_{A_n} \widehat{F}_n + 1/n$. Then from (1.39) in Theorem 2 we have, using the notations there, $\liminf_{n\to\infty} \widehat{F}_n(\nu_n) \ge \frac{|E|}{\pi} \int W(j)\, dP(x,j)$ where $P = \lim_n P_{\nu_n}$. Since $P_{\nu_n} \in i(A_n)$ by definition we have $P \in A_\infty$ since $A_\infty = \cap_{n>0} \overline{\cup_{m>n} i(A_m)}$. We may thus write

$$\liminf_{n\to+\infty} \widehat{F}_n(\nu_n) \ge \frac{|E|}{\pi} \inf_{P\in A_\infty} \int W(j)\, dP(x,j). \tag{6.16}$$

Inserting into (6.15) we are led to

$$\begin{aligned} \frac{\log \mathbb{P}_n^\beta(A_n)}{n} \le -\frac{\beta|E|}{2\pi} \inf_{P\in A_\infty} \int W(j)\, dP(x,j) - \frac{\log K_n^\beta}{n} \\ + \frac{1}{n} \log \left(\int_{\mathbb{C}^n} e^{-\beta n \int \zeta\, d\nu_n}\, dx_1 \dots dx_n \right) + o(1) \end{aligned} \tag{6.17}$$

thus in view of Lemma 3.5 and (4.4), we have established (1.47). An immediate corollary of (6.17), choosing A_n to be the full space and using $\inf \frac{1}{\pi} \int W(j)\, dP(j) = \alpha$ and Lemma 3.5, is that

$$\limsup_{n\to\infty} \frac{\log K_n^\beta}{n} \le -\frac{\beta\alpha}{2} + \log |E|. \tag{6.18}$$

We next turn to the lower bound. Fix $\eta > 0$. Given A, let $P \in \mathring{A}$ be such that

$$\int W(j)\, dP(x,j) \le \inf_{P\in\mathring{A}} \int W(j)\, dP(j) + \frac{\eta}{2}. \tag{6.19}$$

Since $P \in \mathring{A}$, if η is chosen small enough (which we assume) then $B(P, 2\eta) \subset A$, where the ball is for a distance metrizing weak convergence as in Proposition 4.1.

We then apply Proposition 4.1 to P and η. We find $\delta > 0$ and for any n large enough a set A_n such that $|A_n| \ge n!(\pi\delta^2/n)^n$ and, rewriting (4.1) with (2.1),

$$\limsup_{n\to\infty} \sup_{A_n} F_n \le \frac{|E|}{\pi} \int W_K(j)\, dP(j) + \eta. \tag{6.20}$$

Moreover, for every $(y_1,\dots,y_n)\in A_n$ and letting $\{\nu_n=\delta_{y_1}+\dots+\delta_{y_n}\}_n$, there exists $\{j_n\}_n$ in $L^p_{\rm loc}(\mathbb{R}^2,\mathbb{R}^2)$ such that $\operatorname{curl} j_n = 2\pi(\nu_n' - m_0{}')$ and such that the image P_n of $dx_{|E}/|E|$ by the map $x\mapsto (x, j_n(\sqrt{n}x+\cdot))$ is such that

$$\limsup_{n\to\infty} \operatorname{dist}(P_n,P)\le \eta. \tag{6.21}$$

In particular (1.49) holds. Moreover, inserting (6.20) and (6.19) into (1.23), we find that

$$\frac{\log \mathbb{P}_n^\beta(A_n)}{n} \ge -\frac{\log K_n^\beta}{n} - \frac{\beta|E|}{2\pi}\inf_{P\in \mathring{A}}\int W(j)\,dP(j) - \frac{1}{2}\beta\eta + \frac{1}{n}\log\left|\frac{A_n}{\sqrt{n}}\right| + o(1).$$

On the other hand, using $|A_n|\ge n!(\pi\delta^2/n)^n$ and Stirling's formula, we have $\log|A_n|\ge 2n\log\delta - Cn$. Combining with (6.18), (1.48) follows, with $C_\eta = -2\log\delta + C + \log|E|$.

Theorem 1 immediately follows by combining (6.18), (4.4) and (1.24).

7 Proof of Proposition 4.1

The construction consists of the following. We are given $\varepsilon>0$, which is the error we can afford. First we select a finite set of currents $J_1,\dots,J_N$ (N will depend on ε) which will represent the probability $P(x,j)$ with respect to its j dependence, and whose renormalized energies are well-controlled. Since P is $T_{\lambda(x)}$-invariant, we need it to be well-approximated by measures supported on the orbits of the J_i's under translations. Secondly, we work in blown-up coordinates and split the region E' (whose diameter is order $\sqrt{n}$) into many rectangles K with centers x_K and sidelengths of order $\bar{R}$ large enough. Even though we choose $\bar{R}$ to be large, it will still be very small compared to the size of E', as $n\to\infty$, so that the Diracs at $x_K/\sqrt{n}$ approximate $P(x,j)$ with respect to its x dependence. On each rectangle K, the weight $m_0{}'$ is temporarily replaced by its average m_K. Then we split each rectangle K into q^2 identical rectangles, with sidelengths of order $2R=\bar{R}/q$, where both R and q will be sufficiently large. We then select the proportion of the rectangles that corresponds to the weight that the orbit of each J_i carries in the approximation of P. In these rectangles we paste a (translated) copy of J_i at the scale m_K and suitably modified near the boundary according to a construction of [SS1] (Proposition 7.4 below) so that its tangential component on the boundary is 0 (this can be done while inducing only an error ε on W). In the few rectangles that may remain unfilled, we paste a copy of an arbitrary J_0 whose renormalized energy is finite. We perform the construction above provided we are far enough from $\partial E'$. The layer near the boundary must be treated separately, and there again an arbitrary (translated and rescaled) current can be pasted. Finally, we add a vector field to correct the discrepancy between m_K and $m_0{}'$ in each of the rectangles.

To conclude the proof of Proposition 4.1, we collect all of the estimates on the constructed vector field to show that its energy w_n is bounded above in terms of $\int W\,dP$ and that the probability measures associated to the construction have remained close to P.

7.1 Estimates on distances between probabilities

First we choose distances which metrize the topologies of $L^p_{\rm loc}(\mathbb{R}^2,\mathbb{R}^2)$ and $\mathcal{B}(X)$, the set of finite Borel measures on $X=E\times L^p_{\rm loc}(\mathbb{R}^2,\mathbb{R}^2)$. For $j_1,j_2\in L^p_{\rm loc}(\mathbb{R}^2,\mathbb{R}^2)$ we let

$$d_p(j_1,j_2)=\sum_{k=1}^\infty 2^{-k}\frac{\|j_1-j_2\|_{L^p(B(0,k))}}{1+\|j_1-j_2\|_{L^p(B(0,k))}},$$

and on X we use the product of the Euclidean distance on E and d_p, which we denote d_X. On $\mathcal{B}(X)$ we define a distance by choosing a sequence of bounded continuous functions $\{\varphi_k\}_k$ which is dense in $C_b(X)$ and we let, for any $\mu_1, \mu_2 \in \mathcal{B}(X)$,

$$d_{\mathcal{B}}(\mu_1, \mu_2) = \sum_{k=1}^{\infty} 2^{-k} \frac{|\langle \varphi_k, \mu_1 - \mu_2 \rangle|}{1 + |\langle \varphi_k, \mu_1 - \mu_2 \rangle|},$$

where we have used the notation $\langle \varphi, \mu \rangle = \int \varphi \, d\mu$.

We have the following general facts.

Lemma 7.1. *For any $\varepsilon > 0$ there exists $\eta_0 > 0$ such that if $P, Q \in \mathcal{B}(X)$ and $\|P - Q\| < \eta_0$, then $d(P, Q) < \varepsilon$. Here $\|P - Q\|$ denotes the total variation of the signed measure $P - Q$, i.e. the supremum of $\langle \varphi, P - Q \rangle$ over measurable functions φ such that $|\varphi| \le 1$.*

In particular, if $P = \sum_{i=1}^{\infty} \alpha_i \delta_{x_i}$ and $Q = \sum_{i=1}^{\infty} \beta_i \delta_{x_i}$ with $\sum_i |\alpha_i - \beta_i| < \eta_0$, then $d_{\mathcal{B}}(P, Q) < \varepsilon$.

Lemma 7.2. *Let $K \subset X$ be compact. For any $\varepsilon > 0$ there exists $\eta_1 > 0$ such that if $x \in K, y \in X$ and $d_X(x, y) < \eta_1$ then $d_{\mathcal{B}}(\delta_x, \delta_y) < \varepsilon$.*

Lemma 7.3. *Let $0 < \varepsilon < 1$. If μ is a probability measure on a set A and $f, g : A \to X$ are measurable and such that $d_{\mathcal{B}}(\delta_{f(x)}, \delta_{g(x)}) < \varepsilon$ for every $x \in A$, then*

$$d_{\mathcal{B}}(f^{\#}\mu, g^{\#}\mu) < C\varepsilon(|\log \varepsilon| + 1).$$

Proof. Take any bounded continuous function φ_k defining the distance on $\mathcal{B}(X)$. Then if $d_{\mathcal{B}}(\delta_{f(x)}, \delta_{g(x)}) < \varepsilon$ for any $x \in X$ we have in particular

$$\frac{|\varphi_k(f(x)) - \varphi_k(g(x))|}{1 + |\varphi_k(f(x)) - \varphi_k(g(x))|} \le 2^k \varepsilon.$$

It follows that

$$d_{\mathcal{B}}(f^{\#}\mu, g^{\#}\mu) \le \sum_k 2^{-k} \min(\varepsilon 2^k, 1) \le \varepsilon\left([\log_2 \varepsilon] + 1\right) + \sum_{k=[\log_2 \varepsilon]+1}^{\infty} 2^{-k} \le C\varepsilon(|\log \varepsilon| + 1).$$

□

7.2 Preliminary results

In what follows $E' = \sqrt{n}E$, ${m_0}'(x) = m_0(x/\sqrt{n})$: we work in blown-up coordinates. We consider a probability measure P on $E \times L^p_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ which is as in the proposition. We let $\widetilde{P}$ be the probability measure on $E \times \mathcal{A}_1$ which is the image of P under $(x, j) \mapsto (x, \sigma_{1/m_0(x)} j)$, so that

$$\widetilde{P} = \int \delta_x \otimes \delta_{\sigma_{1/m_0(x)} j} \, dP(x, j), \quad P = \int \delta_x \otimes \delta_{\sigma_{m_0(x)} j} \, d\widetilde{P}(x, j) \tag{7.1}$$

It is easy to check that since P is $T_{\lambda(x)}$-invariant, $\widetilde{P}$ is as well, and in particular it is translation-invariant.

The construction is based on the following statement which is a rewriting of Proposition 4.2 in [SS1] and the remark following it:

Proposition 7.4. *Let $K_R = [-R, R]^2$, let $\{\chi_R\}_R$ satisfy* (1.29) *and let W_K be the renormalized energy relative to the family $\{K_R\}_R$.*

Let $G \subset \mathcal{A}_1$ be such that there exists $C > 0$ such that for any $j \in G$ we have

$$\forall R > 1, \ \frac{\nu(K_R)}{|K_R|} < C, \tag{7.2}$$

for the associated ν's, and

$$\lim_{R \to +\infty} \frac{W(j, \chi_R)}{|K_R|} = W_K(j), \tag{7.3}$$

where the convergence is uniform w.r.t. $j \in G$.

Then for any $\varepsilon > 0$ there exists $R_0 > 0$, $\eta_2 > 0$ such that if $R > R_0$ and L is a rectangle centered at 0 whose sidelengths belong to $[2R, 2R(1+\eta_2)]$ and such that $|L| \in \mathbb{N}$, then for every $j \in G$ there exists a $j_L \in L^p_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ such that the following hold

i) $j_L = 0$ *in* L^c,

ii) There is a discrete subset $\Lambda \subset L$ such that

$$\operatorname{curl} j_L = 2\pi \left(\sum_{p \in \Lambda} \delta_p - \mathbf{1}_L \right).$$

iii) If $d(x, L^c) > R^{\frac{3}{4}}$ *then* $j_L(x) = j(x)$

iv)

$$\frac{W(j_L, \mathbf{1}_L)}{|L|} \leq W_K(j) + \varepsilon. \tag{7.4}$$

The next lemma explains how to partition E into rectangles.

Lemma 7.5. *There exists a constant $C_0 > 0$ such that, given any $R > 1$ and $q \in \mathbb{N}^*$, there exists for any $n \in \mathbb{N}^*$ a collection $\mathcal{K}_n$ of closed rectangles in E' with disjoint interiors, whose sidelengths are between $\overline{R} = 2qR$ and $\overline{R} + C_0\overline{R}/R^2$, and which are such that*

$$\{x \in E' : d(x, \partial E') \leq \overline{R}\} \subset E' \setminus \bigcup_{K \in \mathcal{K}_n} K \subset \{x \in E' : d(x, \partial E') \leq C_0 \overline{R}\}$$

and, for all $K \in \mathcal{K}_n$,

$$\int_K {m_0}' \in q^2 \mathbb{N}. \tag{7.5}$$

Proof. For each $j \in \mathbb{Z}$ we let

$$m_j(t) = \int_{x=-\infty}^{t} \int_{y=j\overline{R}}^{(j+1)\overline{R}} {m_0}'(x, y) \, dy \, dx.$$

Then each strip $\{j\overline{R} \le y < (j+1)\overline{R}\}$ is cut into rectangles $[t_{ij}, t_{(i+1),j}] \times [j\overline{R}, (j+1)\overline{R}]$ where $t_{0j} = -\infty$ and

$$t_{i+1,j} = \min\{t \ge t_{ij} + \overline{R} : m_j(t_{ij}) \in q^2\mathbb{N}\}.$$

Since by assumption (1.12) we have $m_0{}'(x) \in [\underline{m}, \overline{m}]$ for any $x \in E'$, it is not difficult to check that if such a rectangle is included in E' then

$$t_{ij} + \overline{R} \le t_{i+1,j} \le t_{ij} + \overline{R} + \frac{q^2}{\underline{m}\overline{R}},$$

and thus its sidelengths are between $\overline{R}$ and $\overline{R}+C\overline{R}/R^2$ since $\overline{R}/R^2 = 4q^2/\overline{R}$. We let $\mathcal{K}_n$ be the set of rectangles of the form $[t_{ij}, t_{(i+1),j}] \times [j\overline{R}, (j+1)\overline{R}]$ which are included in $\{x : d(x, \partial E') > \overline{R}\}$. From the above, it follows that these rectangles in fact cover the set $\{x : d(x, \partial E') > C\overline{R}\}$ for some $C > 0$ independent of $R > 1$, q. By construction each $K \in \mathcal{K}_n$ is such that

$$\int_K m_0{}' = m_j(t_{(i+1),j}) - m_j(t_{ij}) \in q^2\mathbb{N}.$$

□

The next lemma explains how to select a good subset of $L^p_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2)$.

Lemma 7.6. *Let $\tilde{P}$ be a translation invariant measure on X such that P-a.e. $j \in \mathcal{A}_1$. Then for any $\varepsilon > 0$, there exist subsets $H_\varepsilon \subset G_\varepsilon$ in $L^p_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ which are compact and such that, for any $R_\varepsilon > 0$,*

i) η_0 being given by Lemma 7.1 we have

$$\widetilde{P}(E \times G_\varepsilon{}^c) < \min(\eta_0{}^2, \eta_0\varepsilon), \quad \widetilde{P}(E \times H_\varepsilon^c) < \min(\eta_0, \varepsilon). \tag{7.6}$$

ii) For every $j \in H_\varepsilon$, there exists $\Gamma(j) \subset K_{\overline{m}R_\varepsilon}$ such that

$$|\Gamma(j)| < CR_\varepsilon{}^2\eta_0 \text{ and } \lambda \notin \Gamma(j) \implies \theta_\lambda j \in G_\varepsilon. \tag{7.7}$$

iii) The convergence in the definition of $W_K(j)$ is uniform w.r.t. $(x, j) \in G_\varepsilon$ and, writing $\mathrm{curl}\, j = 2\pi(\nu - 1)$,

$$W_K(j) \text{ and } \frac{\nu(K_R)}{R^2} \text{ are bounded uniformly w.r.t. } (x, j) \in G_\varepsilon \text{ and } R > 1. \tag{7.8}$$

iv) We have

$$d_{\mathcal{B}}(P, P'') < C\varepsilon(|\log \varepsilon| + 1), \quad \text{where} \tag{7.9}$$

$$P'' = \int_{E \times H_\varepsilon} \frac{1}{m_0(x)|K_{R_\varepsilon}|} \int_{\sqrt{m_0(x)}K_{R_\varepsilon} \setminus \Gamma(j)} \delta_x \otimes \delta_{\sigma_{m_0(x)}\theta_\mu j}\, d\mu\, d\widetilde{P}(x, j).$$

Moreover, there exists a partition of H_ε into $\cup_{i=1}^{N_\varepsilon} H_\varepsilon^i$ satisfying diam $(H_\varepsilon^i) < \eta_3$, *where η_3 is such that*

$$j \in H_\varepsilon,\ d_p(j, j') < \eta_3,\ m \in [\underline{m}, \overline{m}],\ \mu \in \sqrt{\overline{m}}K_{R_\varepsilon} \setminus \Gamma(j) \implies d_{\mathcal{B}}(\delta_{\sigma_m\theta_\mu j}, \delta_{\sigma_m\theta_\mu j'}) < \varepsilon; \tag{7.10}$$

and there exists for all i, $J_i \in H_\varepsilon^i$ such that

$$W_K(J_i) < \inf_{H_\varepsilon^i} W_K + \varepsilon. \tag{7.11}$$

Proof.

Step 1: Choice of G_ε. Since $L^p_{\rm loc}(\mathbb{R}^2,\mathbb{R}^2)$ is Polish we can always find a compact set G_ε satisfying (7.6) and $P({G_\varepsilon}^c) < \eta_0$. Then from Lemma 7.1, $P\llcorner G_\varepsilon$ (the restriction of P to G_ε) satisfies $d_{\mathcal{B}}(P, P\llcorner G_\varepsilon) < \varepsilon$.

From the translation invariance of $\widetilde{P}$ and for any λ, we have $\widetilde{P}(E \times \theta_\lambda G_\varepsilon) > 1 - \eta_0$ and therefore $d_{\mathcal{B}}(\widetilde{P}, \widetilde{P}\llcorner\theta_\lambda G_\varepsilon) < \varepsilon$. In view of (7.1), it follows that for any $\lambda \in \mathbb{R}^2$ we have $\|P - P_\lambda\| < \eta_0$ and then $d_{\mathcal{B}}(P, P_\lambda) < \varepsilon$, where

$$P_\lambda = \int_{E\times\theta_\lambda G_\varepsilon} \delta_x \otimes \delta_{\sigma_{m_0(x)} j}\, d\widetilde{P}(x,j) = \int_{E\times G_\varepsilon} \delta_x \otimes \delta_{\theta_\lambda\sigma_{m_0(x)} j}\, d\widetilde{P}(x,j).$$

Then using Lemma 7.3 we deduce that if $A \subset \mathbb{R}^2$ is any measurable set of positive measure, then

$$d_{\mathcal{B}}(P, P') < C\varepsilon(|\log \varepsilon| + 1), \quad \text{where} \quad P' = \int_{E\times G_\varepsilon} ⨍_A \delta_x \otimes \delta_{\theta_\lambda\sigma_{m_0(x)} j}\, d\lambda\, d\widetilde{P}(x,j). \tag{7.12}$$

Moreover, since P is $T_{\lambda(x)}$-invariant, choosing χ to be a smooth positive function with integral 1 supported in $B(0,1)$, the ergodic theorem (as in Section 6.1 or see again [Be]) ensures that for P-almost every (x,j) the limit

$$\lim_{R\to+\infty} \frac{1}{|K_R|} \int_{K_R} W(j(\lambda+\cdot), \chi(\lambda+\cdot))\, d\lambda$$

exists. Then $\mathbf{1}_{K_R} * \chi$ is a family of functions which satisfies (1.29) with respect to the family of squares $\{K_R\}_R$, and from the definition of the renormalized energy relative to $\{K_R\}_R$ we may rewrite the limit above as

$$W_K(j) = \lim_{R\to+\infty} \frac{1}{|K_R|} W(j, \mathbf{1}_{K_R} * \chi). \tag{7.13}$$

By Egoroff's theorem we may choose the compact set G_ε above to be such that, in addition to (7.12), the convergence in (7.13) is uniform on G_ε. In fact, since $W_K(j) < +\infty$ and $\limsup_R \nu(K_R)/R^2 < +\infty$ for P-a.e. (x,j), where $\operatorname{curl} j = 2\pi(\nu - 1)$, we may choose G_ε such that (7.8) holds.

The arguments above show that the properties (7.12), (7.8) can be satisfied for a compact set G_ε of measure arbitrarily close to 1. We choose G_ε such that (7.6) holds.

The next difficulty we have to face is that $\theta_\lambda j$ need not belong to G_ε if j does.

Step 2: Choice of H_ε. For $j \in G_\varepsilon$, let $\Gamma(j)$ be the set of λ's in $\sqrt{\overline{m}} K_{R_\varepsilon}$ such that $\theta_\lambda j \notin G_\varepsilon$. Since, from (7.6) and the translation-invariance of $\widetilde{P}$, for any $\lambda \in \mathbb{R}^2$ we have $\widetilde{P}(E\times\theta_\lambda(G_\varepsilon)^c) < {\eta_0}^2$, it follows from Fubini's theorem that

$$\int_{G_\varepsilon} |\Gamma(j)|\, d\widetilde{P}(x,j) = \int_{\sqrt{\overline{m}}K_{R_\varepsilon}} \widetilde{P}(E \times (\theta_\lambda G_\varepsilon)^c)\, d\lambda < 4\overline{m}{R_\varepsilon}^2 \min({\eta_0}^2, \eta_0\varepsilon).$$

Therefore, letting

$$H_\varepsilon = \{j \in G_\varepsilon : |\Gamma(j)| < 4\overline{m}{R_\varepsilon}^2\eta_0\}, \tag{7.14}$$

we have that (7.6) holds.

Combining (7.6) and (7.14) with Lemma 7.1, we deduce from (7.12) that (7.9) holds, where we have used the fact that $\theta_\lambda \sigma_m j = \sigma_m \theta_{\sqrt{m}\lambda} j$ to change the integration variable to $\mu = \sqrt{m_0(x)}\lambda$ in (7.12).

Step 3: Choice of $J_1, \ldots, J_{N_\varepsilon}$. We use the fact that G_ε is relatively compact in $L^p_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2)$, Lemma 7.2, and the fact that $(m, j) \mapsto \sigma_m j$ is continuous to find that there exists $\eta_4 > 0$ such that for any $m \in [\underline{m}, \overline{m}]$ and any $j \in G_\varepsilon$ it holds that

$$d_p(j, j') < \eta_4 \quad \Longrightarrow \quad d_{\mathcal{B}}\left(\delta_{\sigma_m j}, \delta_{\sigma_m j'}\right) < \varepsilon. \tag{7.15}$$

Moreover, from the continuity of $(\mu, j) \mapsto \theta_\mu j$, there exists $\eta_3 > 0$ such that

$$j \in G_\varepsilon, d_p(j, j') < \eta_3,\ \mu \in \sqrt{\overline{m}} K_{R_\varepsilon} \quad \Longrightarrow \quad d_p\left(\theta_\mu j, \theta_\mu j'\right) < \eta_4. \tag{7.16}$$

If $j \in H_\varepsilon$ and $\mu \in K\backslash...$ then $\theta_\mu j \in G_\varepsilon$ hence applying (7.15), we get (7.10).

Now we cover the relatively compact H_ε by a finite number of balls $B_1, \ldots, B_{N_\varepsilon}$ of radius $\eta_3/2$, derive from it a partition of H_ε by sets with diameter less than η_3, by letting $H_\varepsilon^1 = B_1 \cap H_\varepsilon$ and

$$H_\varepsilon^{i+1} = B_{i+1} \cap H_\varepsilon \setminus (B_1 \cup \cdots \cup B_i).$$

we then have

$$H_\varepsilon = \bigcup_{i=1}^{N_\varepsilon} H_\varepsilon^i, \quad \mathrm{diam}\,(H_\varepsilon^i) < \eta_3, \tag{7.17}$$

where the union is disjoint. Then we may choose $J_i \in H_\varepsilon^i$ such that (7.11) holds. □

7.3 Completing the construction

Step 1: Choice of R_ε. We apply Proposition 7.4 with $G = G_\varepsilon$ and $\sqrt{m}R$, where $m \in [\underline{m}, \overline{m}]$. The proposition yields $\eta_2 > 0$, $R_\varepsilon > 1$ such that for any $j \in G_\varepsilon$ and any $m \in [\underline{m}, \overline{m}]$ and any rectangle L centered at 0 with sidelengths in $[2\sqrt{m}R_\varepsilon, 2\sqrt{m}R_\varepsilon(1+\eta_2)]$, (7.4) is satisfied for some j_L, with R replaced by $\sqrt{m}R_\varepsilon$. The reason for including $\sqrt{m}$ is that we will need to scale the construction to account for the varying weight $m_0(x)$.

Since our rectangles will be obtained from Lemma 7.5 and we wish to use the approximation by j_L in them, we choose R_ε large enough so that if $m \in [\underline{m}, \overline{m}]$ and L is a rectangle centered at zero with sidelengths in $[2\sqrt{m}R_\varepsilon, 2\sqrt{m}R_\varepsilon(1 + C_0/{R_\varepsilon}^2)]$ then

$$\frac{C_0}{R_\varepsilon^2} < \eta_2, \quad \frac{C_1}{{R_\varepsilon}^2} < \eta_0, \quad K_{\sqrt{m}R_\varepsilon(1-\eta_0)} \subset \{x : d(x, L^c) > \sqrt{m}{R_\varepsilon}^{\frac{3}{4}}\} \subset K_{\sqrt{m}R_\varepsilon(1+\eta_0)}, \tag{7.18}$$

where C_0 is the constant in Lemma 7.5, $C_1 \geq 1$ is to be determined later, and η_0 is the constant in Lemma 7.1.

If $\lambda \in K_{\sqrt{m}R_\varepsilon(1-\eta_0)}$ and since $j = j_L$ if $d(x, L^c) > \sqrt{m}{R_\varepsilon}^{\frac{3}{4}}$, we deduce from (7.18) that $\theta_\lambda j_L = \theta_\lambda j$ in $B(0, \sqrt{m}{R_\varepsilon}^{\frac{3}{4}})$, so that from the definition of d_p, taking R_ε larger if necessary,

$$\forall j \in G_\varepsilon, m \in [\underline{m}, \overline{m}], \lambda \in K_{\sqrt{m}R_\varepsilon(1-\eta_0)}, \quad d_p(\theta_\lambda \sigma_m j, \theta_\lambda \sigma_m j_L) < \frac{\eta_1}{2}, \tag{7.19}$$

where η_1 comes from Lemma 7.2 applied on $\{\sigma_m j : m \in [\underline{m}, \overline{m}], j \in G_\varepsilon\}$, i.e. is such that

$$m \in [\underline{m}, \overline{m}],\ j \in G_\varepsilon, j' \in L^p_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2) \text{ and } d_p(j, j') < \eta_1 \Longrightarrow d_{\mathcal{B}}(\delta_{\sigma_m j}, \delta_{\sigma_m j'}) < \varepsilon. \tag{7.20}$$

Step 2: Choice of q_ε and the rectangles. We choose an integer q_ε large enough so that

$$\frac{N_\varepsilon}{C_1 {q_\varepsilon}^2} < \eta_0, \qquad \frac{N_\varepsilon}{{q_\varepsilon}^2} \times \max_{\substack{0 \le i \le N_\varepsilon \\ \underline{m} \le m \le \overline{m}}} W_K(\sigma_m J_i) < \varepsilon \tag{7.21}$$

where $C_1 > 1$ is to be determined later. We apply Lemma 7.5 with R_ε, q_ε and N_ε to obtain for any n a collection $\mathcal{K}_n$ of rectangles (we omit to mention the ε dependence) which cover most of E', and we also apply Lemma 7.6. We rewrite P'' given by (7.9) as

$$P'' = \sum_{K \in \mathcal{K}_n} \int_{\frac{K}{\sqrt{n}} \times H_\varepsilon} \frac{1}{m_0(x)|K_{R_\varepsilon}|} \int_{\sqrt{m_0(x)} K_{R_\varepsilon} \setminus \Gamma(j)} \delta_x \otimes \delta_{\sigma_{m_0(x)} \theta_\mu j} \, d\mu \, d\widetilde{P}(x,j). \tag{7.22}$$

Now we claim that if n is large enough and $x \in K/\sqrt{n}$, $j \in H_\varepsilon^i$, $\mu \in \sqrt{m_0(x)} K_{R_\varepsilon} \setminus \Gamma(j)$, then

$$d_{\mathcal{B}} \left(\delta_x \otimes \delta_{\sigma_{m_0(x)} \theta_\mu j}, \delta_{x_K} \otimes \delta_{\sigma_{m_K} \theta_\mu J_i} \right) < 2\varepsilon, \tag{7.23}$$

where x_K is the center of $K/\sqrt{n}$ and m_K is the average of m_0 over $K/\sqrt{m}$. Indeed, since m_0 is C^1 we have $|x - x_K| < C/\sqrt{n}$, $|m_0(x) - m_K| < C/\sqrt{n}$ thus if n is large enough, since $\theta_\mu j \in G_\varepsilon$ we find

$$d_{\mathcal{B}} \left(\delta_x \otimes \delta_{\sigma_{m_0(x)} \theta_\mu j}, \delta_{x_K} \otimes \delta_{\sigma_{m_K} \theta_\mu j} \right) < \varepsilon.$$

Moreover, since $d_p(j, J_i) < \eta_3$, we deduce from (7.10) that

$$d_{\mathcal{B}} \left(\delta_{x_K} \otimes \delta_{\sigma_{m_K} \theta_\mu j}, \delta_{x_K} \otimes \delta_{\sigma_{m_K} \theta_\mu J_i} \right) < \varepsilon,$$

which together with the previous estimate proves (7.23).

Using (7.23) together with Lemmas 7.2, 7.1, and (7.7), we deduce from (7.9) and (7.22) that $d_{\mathcal{B}}(P, P''') < C\varepsilon(|\log \varepsilon| + 1)$, where

$$\begin{aligned} P''' &= \sum_{\substack{K \in \mathcal{K}_n \\ 1 \le i \le N_\varepsilon}} \int_{\frac{K}{\sqrt{n}} \times H_\varepsilon^i} \fint_{\sqrt{m_K} K_{R_\varepsilon}} \delta_{x_K} \otimes \delta_{\sigma_{m_K} \theta_\mu J_i} \, d\mu \, d\widetilde{P}(x,j) \\ &= \sum_{\substack{K \in \mathcal{K}_n \\ 1 \le i \le N_\varepsilon}} p_{i,K} \fint_{\sqrt{m_K} K_{R_\varepsilon}} \delta_{x_K} \otimes \delta_{\sigma_{m_K} \theta_\mu J_i} \, d\mu, \end{aligned} \tag{7.24}$$

where

$$p_{i,K} = \widetilde{P} \left(\frac{K}{\sqrt{n}} \times H_\varepsilon^i \right). \tag{7.25}$$

Step 3: Choice of subrectangles and current j_n. We now replace $p_{i,K}$ in the definition (7.24) by

$$\frac{|K|}{{q_\varepsilon}^2 |E'|} n_{i,K}, \quad \text{where} \quad n_{i,K} = \left[\frac{{q_\varepsilon}^2 |E'|}{|K|} p_{i,K} \right]. \tag{7.26}$$

We have, since $\widetilde{P}(\frac{K}{\sqrt{n}} \times L^p_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)) = |K|/|E'|$,

$$\sum_{k=1}^{N_\varepsilon} n_{i,K} \le \frac{{q_\varepsilon}^2 |E'|}{|K|} \widetilde{P}\left(\frac{K}{\sqrt{n}} \times L^p_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)\right) = {q_\varepsilon}^2 \tag{7.27}$$

and

$$\left|\frac{|K_{R_\varepsilon}|}{|E'|} n_{i,K} - p_{i,K}\right| < C\left(\frac{|K|}{{q_\varepsilon}^2 |E'|} + \frac{n_{i,K}}{R_\varepsilon^2 |E'|}\right).$$

Summing with respect to i and K, using the facts that $\sum_{K \in \mathcal{K}_n} |K| < |E'|$, (7.27), and the fact that the cardinal of $\mathcal{K}_n$ is $\frac{|E'|}{4q_\varepsilon^2 R_\varepsilon^2}$, we find

$$\sum_{1 \le i \le N_\varepsilon, K \in \mathcal{K}_n} \left|\frac{|K_{R_\varepsilon}|}{|E'|} n_{i,K} - p_{i,K}\right| < C\left(\frac{N_\varepsilon}{{q_\varepsilon}^2} + \frac{1}{R_\varepsilon^4}\right).$$

We may always choose C_1 large enough in (7.18) and (7.21) so that the right-hand side is $< \eta_0$. Then Lemma 7.1 implies that $d_{\mathcal{B}}(P, P^{(4)}) < C\varepsilon(|\log \varepsilon| + 1)$ is still true after replacing $p_{i,K}$ by $\frac{|K_{R_\varepsilon}|}{|E'|} n_{i,K}$ in (7.24), i.e. where

$$P^{(4)} = \frac{1}{|E'|} \sum_{\substack{K \in \mathcal{K}_n \\ 1 \le i \le N_\varepsilon}} \frac{n_{i,K}}{m_K} \int_{\sqrt{m_K} K_{R_\varepsilon}} \delta_{x_K} \otimes \delta_{\sigma_{m_K} \theta_\mu J_i} \, d\mu. \tag{7.28}$$

Next, we divide each $K \in \mathcal{K}_n$ into a collection $\mathcal{L}_K$ of ${q_\varepsilon}^2$ identical subrectangles in the obvious way and we partition $\mathcal{L}_K$ into collections $\mathcal{L}_{K,i}$, $0 \le i \le N_\varepsilon$ such that if $k \ge 1$ then $\mathcal{L}_{K,i}$ contains $n_{i,K}$ subrectangles. This is clearly possible from (7.27). If the inequality is strict we put the extra subrectangles in $\mathcal{L}_{K,0}$, there will be $n_{0,K}$ of them and then

$$\sum_{k=0}^{N_\varepsilon} n_{k,K} = {q_\varepsilon}^2. \tag{7.29}$$

We rewrite (7.28) as

$$P^{(4)} = \frac{1}{|E'|} \sum_{\substack{K \in \mathcal{K}_n \\ 1 \le i \le N_\varepsilon \\ \tilde{L} \in \mathcal{L}_{K,i}}} \frac{1}{m_K} \int_{\sqrt{m_K} K_{R_\varepsilon}} \delta_{x_K} \otimes \delta_{\sigma_{m_K} \theta_\mu J_i} \, d\mu. \tag{7.30}$$

Now, for $\tilde{L} \in \mathcal{L}_{K,i}$, let $L = \sqrt{m_K}(\tilde{L} - x_{\tilde{L}})$, where $x_{\tilde{L}}$ denotes the center of $\tilde{L}$. From Lemma 7.5, a rectangle $K \in \mathcal{K}_n$ has sidelengths between $2q_\varepsilon R_\varepsilon$ and $2q_\varepsilon R_\varepsilon(1 + C_0/{R_\varepsilon}^2)$. Therefore L is a rectangle centered at zero with sidelengths between $2\sqrt{m_K} R_\varepsilon$ and $2\sqrt{m_K} R_\varepsilon(1 + C_0/{R_\varepsilon}^2)$, and (7.19) holds.

This, and the results of Lemma 7.6, allow us to apply Proposition 7.4 on L to any J_i, $1 \le i \le N_\varepsilon$. Note that $|L| \in \mathbb{N}$ follows from the fact that

$$|L| = m_K |\tilde{L}| = \fint_K {m_0}' \frac{|K|}{{q_\varepsilon}^2} = \frac{1}{{q_\varepsilon}^2} \int_K {m_0}'$$

and (7.5). In this way, we define currents $J_{i,L}$ which satisfy (7.4) and (7.19). We claim that, as a consequence of the latter, we have

(7.31)
$$j' = J_{i,L} \text{ on } L \implies d_{\mathcal{B}}\left(\fint_{\sqrt{m_K}K_{R_\varepsilon}} \delta_{x_K}\otimes\delta_{\sigma_{m_K}\theta_\mu J_i}\,d\mu, \frac{1}{m_K|K_{R_\varepsilon}|}\int_L \delta_{x_K}\otimes\delta_{\sigma_{m_K}\theta_\mu j'}\,d\mu\right) < C\varepsilon.$$

This goes as follows: (i) Using Lemma 7.1 and (7.7),(7.18), we find that integrating on $\sqrt{m_K}K_{(1-\eta_0)R_\varepsilon}\setminus\Gamma(J_i)$ instead of $\sqrt{m_K}K_{R_\varepsilon}$ and L induces an error of $C\varepsilon$. (ii) From (7.19), and (7.20) applied to $\theta_\mu J_i$ and $\theta_\mu j'$ we have $d_{\mathcal{B}}(\delta_{\theta_\mu J_i},\delta_{\theta_\mu j'}) < \varepsilon$ and thus in view of Lemma 7.3 we may replace $\theta_\mu J_i$ by $\theta_\mu j'$ in the integral with an error of $C\varepsilon|\log\varepsilon|$ at most. (iii) Using (7.18), (7.7) and Lemma 7.1 again, we may integrate back on $\sqrt{m_K}K_{R_\varepsilon}$ and L rather than on $K_{(1-\eta_0)R_\varepsilon}\setminus\Gamma(J_i)$, with an additional error of $C\varepsilon$. this proves (7.31).

Combining (7.31) with (7.30) and $d_{\mathcal{B}}(P,P^{(4)}) < C\varepsilon(|\log\varepsilon|+1)$, using Lemma 7.3 we find $d_{\mathcal{B}}(P,P^{(5)}) < C\varepsilon(|\log\varepsilon|+1)$, where

(7.32)
$$P^{(5)} = \frac{1}{|E'|}\sum_{\substack{K\in\mathcal{K}_n\\ 1\le i\le N_\varepsilon\\ \tilde L\in\mathcal{L}_{K,i}}}\frac{1}{m_K}\int_L \delta_{x_K}\otimes\delta_{\sigma_{m_K}\theta_\mu \tilde J_{i,L}}\,d\mu = \frac{1}{|E'|}\sum_{\substack{K\in\mathcal{K}_n\\ 1\le i\le N_\varepsilon\\ \tilde L\in\mathcal{L}_{K,i}}}\int_{L/\sqrt{m_K}} \delta_{x_K}\otimes\delta_{\theta_\lambda\sigma_{m_K}\tilde J_{i,L}}\,d\lambda,$$

where the last equality follows by changing variables to $\lambda = \mu/\sqrt{m}_K$, and where $\tilde J_{i,L}$ denotes an arbitrarily chosen element of $L^p_{\text{loc}}(\mathbb{R}^2,\mathbb{R}^2)$ such that $\tilde J_{i,L} = J_{i,L}$ on L, the constant C being independent of this choice.

If we choose an arbitrary J_0 in $\mathcal{A}_1$ and let the sum in (7.32) range over $0\le i\le N_\varepsilon$ instead of $1\le i\le N_\varepsilon$ we obtain a measure $P^{(6)}$ such that, by (7.21),

$$\|P^{(5)} - P^{(6)}\| \le \frac{1}{|E'|}\sum_{K\in\mathcal{K}_n}\frac{N_\varepsilon|K|}{{q_\varepsilon}^2} \le \eta_0,$$

hence using Lemma 7.1 we have $d_{\mathcal{B}}(P^{(5)},P^{(6)}) < \varepsilon$ and then $d_{\mathcal{B}}(P,P^{(6)}) < C\varepsilon(|\log\varepsilon|+1)$.

We now define the vector field $j_n^{\text{int}}:\mathbb{R}^2\to\mathbb{R}^2$ by letting $j_n^{\text{int}}(x) = \sigma_{m_K}J_{i,L}(x - x_{\tilde L})$ on $\tilde L = x_{\tilde L} + L/\sqrt{m_K}$, for every $K\in\mathcal{K}_n$, $0\le i\le N_\varepsilon$ and $\tilde L\in\mathcal{L}_{K,i}$. Then, for every $L\in\mathcal{L}_{K,i}$ we have $j_n^{\text{int}}(x_{\tilde L}+\cdot) = \sigma_{m_K}J_{i,L}$ on $\tilde L$, therefore we may choose $\tilde J_{i,L} = \sigma_{1/m_K}j_n^{\text{int}}(x_{\tilde L}+\cdot)$ in (7.32) and then then we may summarize the above by writing

$$\text{(7.33)}\qquad d_{\mathcal{B}}(P,P^{(6)}) < C\varepsilon(|\log\varepsilon|+1),\quad P^{(6)} = \frac{1}{|E'|}\sum_{K\in\mathcal{K}_n}\int_K \delta_{x_K}\otimes\delta_{\theta_\lambda j_n^{\text{int}}}\,d\lambda.$$

Note that since $J_{i,L} = 0$ outside L, we also have

$$\text{(7.34)}\qquad j_n^{\text{int}} = \sum_{\substack{K\in\mathcal{K}_n\\ 1\le i\le N_\varepsilon\\ \tilde L\in\mathcal{L}_{K,i}}}\sigma_{m_K}J_{i,L}(\cdot - x_{\tilde L}),\quad \operatorname{curl} j_n^{\text{int}} = 2\pi\sum_{\substack{K\in\mathcal{K}_n\\ p\in\Lambda_K}}(\delta_p - m_K),$$

where Λ_K is a finite subset of the interior of K. The second equation is satisfied in the sense of distributions on $\mathbb{R}^2$.

Step 4: Treating the boundary. Let $\hat E' := E'\setminus\cup_{K\in\mathcal{K}_n}K$. We let $t\in[0,\ell\sqrt{n}]$ denote arclength on $\partial E'$ — where ℓ is the length of ∂E — and s denote the distance to $\partial E'$, so that (t,s)

is a C^1 coordinate system on $\{x \in E' : d(x,(E')^c) < c\sqrt{n}\}$, if $c > 0$ is small enough, since the boundary of E is C^1 by (1.13). We let C_t denote the curvilinear rectangle of points with coordinates in $[0,t] \times [0, C\overline{R}_\varepsilon]$, where $\overline{R}_\varepsilon = q_\varepsilon R_\varepsilon$ and C is large enough so that $\hat{E}' \subset \{x \in E' : d(x, \partial E') < C\overline{R}_\varepsilon\}$, and define $m(t) = \int_{C_t \cap \hat{E}'} m_0{}'$. Since the distance of $\cup_{K \in \mathcal{K}_n} K$ to a given $x \in \partial E'$ is between $\overline{R}_\varepsilon$ and $C_0 \overline{R}_\varepsilon$ from Lemma 7.5 and since $m_0{}'$ is bounded above and below by (1.15), the derivative of $t \mapsto m(t)$ is between $\overline{R}_\varepsilon / C$ and $C\overline{R}_\varepsilon$ for some $C > 0$ large enough.

We let

$$k_\varepsilon = \left[\frac{\ell\sqrt{n}}{\overline{R}_\varepsilon}\right] \tag{7.35}$$

and choose $0 = t_0, \dots, t_{k_\varepsilon} = \ell\sqrt{n}$ to be such that

$$m(t_l) = \left[\frac{l}{k_\varepsilon} m(\ell\sqrt{n})\right].$$

We note that indeed $t_{k_\varepsilon} = \ell\sqrt{n}$: Since the integral of $m_0{}'$ on each square $K \in \mathcal{K}_n$ is an integer as well as the integral on E', we have $\int_{\hat{E}'} m_0{}' \in \mathbb{N}$ and therefore $m(\ell\sqrt{n}) \in \mathbb{N}$.

From the above remark about the derivative of $t \to m(t)$, we deduce that $\frac{m(\ell\sqrt{n})}{\ell\sqrt{n}}$ belongs to the interval $[\overline{R}_\varepsilon / C, C\overline{R}_\varepsilon]$ for some $C > 0$ and then it is easy to deduce that if $\sqrt{n}$ is large enough compared to $\overline{R}_\varepsilon$ then

$$n_l := m(t_{l+1}) - m(t_l) \in \left[\overline{R}_\varepsilon{}^2/C, C\overline{R}_\varepsilon{}^2\right], \quad t_{l+1} - t_l \in [\overline{R}_\varepsilon/C, C\overline{R}_\varepsilon].$$

This means that the sidelengths of the curvilinear rectangle $C_{t_{l+1}} \setminus C_{t_l}$ are comparable to $\overline{R}_\varepsilon$, and that the number of points n_l to put there in is of order $\overline{R}_\varepsilon{}^2$.

We may then include each of the sets $K_l := \hat{E}' \cap (C_{t_{l+1}} \setminus C_{t_l})$ in a ball B_l with radius in $[\overline{R}_\varepsilon/C, C\overline{R}_\varepsilon]$ and we may also choose a set of n_l points Λ_l which are at distance at least $1/C$ from each other and the complement of K_l. Let $j_l = -\nabla^\perp H$, where H solves $-\Delta H = 2\pi(\sum_{p \in \Lambda_l} \delta_p - m_l)$ in B_l and $\partial_\nu H \cdot \tau = 0$ on ∂B_l, where

$$m_l = \frac{n_l}{|K_l|} \mathbf{1}_{K_l}.$$

Then we have $\operatorname{curl} j_l = 2\pi(\sum_{p \in \Lambda_l} \delta_p - m_l)$ in B_l and $j_l \cdot \tau = 0$ on ∂B_l and we claim that for any $q \geq 1$,

$$W(j_l, \mathbf{1}_{B_l}) \leq C_\varepsilon, \quad \|j_l\|_{L^q(B_l \setminus K_l)} \leq C_{\varepsilon,q}, \tag{7.36}$$

where the constants do not depend on n, but do depend on ε through $\overline{R}_\varepsilon$. This is proved by noting that these quantities are finite, and that a compactness argument shows that the bound is uniform for any choice of points which are at distance at least $1/C$ from each other and the complement of some $K_l \subset B_l$, using for instance the explicit formulas for W in [LR]. Note that because the sets $\{K_l\}$ and the rectangles $\{K\}$ are disjoint, have measure between $\overline{R}_\varepsilon{}^2/C$ and $C\overline{R}_\varepsilon{}^2$ and diameter between $\overline{R}_\varepsilon/C$ and $C\overline{R}_\varepsilon$, we know that their overlap is bounded by a constant C independent of ε, n.

Step 5: Rectification of the weight. We rectify the weights m_K, m_l: For $K \in \mathcal{K}_n$ we let H_K solve $-\Delta H_K = 2\pi(m_0{}' - m_K)$ on K and $\partial_\nu H_K = 0$ on ∂K. Similarly we let H_l solve the same equation with B_l replacing K and m_l replacing m_K. By elliptic regularity, we deduce for any $q > 1$ that $\|\nabla H_K\|_{L^q(K)}$ (resp. $\|\nabla H_l\|_{L^q(B_l)}$) is bounded by $C_{q,\varepsilon}\|m_0{}' - m_K\|_{L^\infty(K)}$ (resp. $C_{q,\varepsilon}\|m_0{}' - m_l\|_{L^\infty(B_l)}$.) Since m_0 is C^1 we have $|\nabla m_0{}'| \le C/\sqrt{n}$, therefore $\|m_0{}' - m_K\|_{L^\infty(K)} \le C\overline{R}_\varepsilon/\sqrt{n}$, while $\|m_0{}' - m_l\|_{L^\infty(B_l)} \le C$. We deduce that

$$\|\nabla H_K\|_{L^q} \le \frac{C_{q,\varepsilon}}{\sqrt{n}}, \quad \|\nabla H_l\|_{L^q} \le C_{q,\varepsilon}. \tag{7.37}$$

We let

$$j_K = j_n^{\text{int}}{}_{|K}, \tag{7.38}$$

and

$$j_n = j_n^{\text{int}} + \sum_{K\in\mathcal{K}_n} -\nabla^\perp H_K + \sum_{i=1}^{k_\varepsilon} -\nabla^\perp H_l = \sum_{K\in\mathcal{K}_n} j_K - \nabla^\perp H_K + \sum_{l=1}^{k_\varepsilon} j_l - \nabla^\perp H_l, \tag{7.39}$$

$$\Lambda_n = \cup_{K\in\mathcal{K}}\Lambda_K \cup_{l=1}^{k_\varepsilon} \Lambda_l,$$

where j_K and H_K are set to 0 outside K and similarly for j_l, H_l outside B_l. Then $\operatorname{curl} j = 2\pi(\sum_{p\in\Lambda_n}\delta_p - m_0{}')$ in $\mathbb{R}^2$. This completes the construction of j_n.

7.4 Estimating the energy

Step 1: Energy estimate. We have

$$W(j_K, \mathbf{1}_K) = \sum_{\substack{0\le i\le N_\varepsilon \\ \tilde{L}\in\mathcal{L}_{K,i}}} W(\sigma_{m_K} J_{i,L}(\cdot - x_{\tilde{L}}), \tilde{L}).$$

From (7.4) we find, letting $L = \sqrt{m_K}(\tilde{L} - x_{\tilde{L}})$, using (7.29) and $|L| = |K|/{q_\varepsilon}^2$, that

$$W(j_K, \mathbf{1}_K) = \sum_{\substack{0\le i\le N_\varepsilon \\ \tilde{L}\in\mathcal{L}_{K,i}}} W(\sigma_{m_K} J_{i,L}, \mathbf{1}_{\tilde{L}-x_{\tilde{L}}}) \le |K| \left(\sum_{i=0}^{N_\varepsilon} \frac{n_{i,K}}{{q_\varepsilon}^2} W(\sigma_{m_K} J_i) + C\varepsilon \right). \tag{7.40}$$

We estimate the integral of $|j_n|^2$ on $\mathbb{R}^2 \setminus \cup_{p\in\Lambda_n} B(p,\eta)$. From (7.39), this integral involves on the one hand the square terms

$$\sum_{l=1}^{k_\varepsilon} \int_{(B_i)_\eta} |j_l - \nabla^\perp H_l|^2 + \sum_{K\in\mathcal{K}_n} \int_{K_\eta} |j_K - \nabla^\perp H_K|^2, \tag{7.41}$$

where $K_\eta = K \setminus \cup_{p\in\Lambda_n} B(p,\eta)$ and similarly for $(B_l)_\eta$, and on the other hand the rectangle terms

$$\sum_{\substack{K,K'\in\mathcal{K}_n \\ K\neq K'}} \int_{K_\eta\cap K'_\eta} (j_K - \nabla^\perp H_K)\cdot(j_{K'} - \nabla^\perp H_{K'}) + \sum_{1\le l\neq i\le k_\varepsilon} \cdots + \sum_{\substack{K\in\mathcal{K}_n \\ 1\le l\le k_\varepsilon}} \cdots$$

We estimate the latter as follows: Since the rectangles in $\mathcal{K}_n$ do not overlap, the first sum is equal to zero. A nonzero rectangle term must involve some B_l, and moreover a given B_l can only be present in a number of terms bounded independently of n, ε because the overlap of the balls B_l and the rectangles K is bounded. Thus from (7.35) we have at most $C\sqrt{n}/\overline{R}_\varepsilon$ nonzero rectangle terms. Moreover, since the K_l's are disjoint, and disjoint from the K's, in a rectangle term involving $B_l \cap K$ the integral can be taken over $K \setminus K_l$, and in a term involving $B_l \cap B_i$ it can be taken over $(B_l \cap B_i \setminus K_i) \cup (B_i \cap B_l \setminus K_l)$.

In any case we use Hölder's inequality and the bound $\|j_l - \nabla^\perp H_l\|_{L^q(B_l \setminus K_l)} \le C_{\varepsilon,q}$ for some $q > 2$, which follows from (7.36), (7.37), together with the bound

$$\|j_l - \nabla^\perp H_l\|_{L^{q'}(B_l)}, \ \|j_K - \nabla^\perp H_K\|_{L^{q'}(K)} \le C_{\varepsilon,q},$$

which follows from (7.40), (7.36) using Lemma 4.7 in [SS1], to conclude that each rectangle term is bounded by C_ε and then that their sum is $O(\sqrt{n})$, meaning a quantity bounded by a constant depending on ε times $\sqrt{n}$.

The limit as $\eta \to 0$ of the terms in (7.41) is estimated as above by expanding the squares and using Hölder's inequality with (7.37), (7.36), (7.40), together with the bound (7.35) to show that

$$\lim_{\eta\to 0} \frac{1}{2}\left(\sum_{l=1}^{k_\varepsilon} \int_{(B_l)_\eta} |j_l - \nabla^\perp H_l|^2 + \sum_{K\in\mathcal{K}_n} \int_{K_\eta} |j_K - \nabla^\perp H_K|^2 + \pi \# \Lambda_n \log \eta\right) \le \sum_{K\in\mathcal{K}_n} W(j_K, \mathbf{1}_K) + O(\sqrt{n}).$$

In view of the bound $O(\sqrt{n})$ for the rectangle terms and (7.40) we find using (7.29) that

$$W(j_n, \mathbf{1}_{\mathbb{R}^2}) \le \sum_{\substack{K\in\mathcal{K}_n \\ 0\le i\le N_\varepsilon}} |K| \frac{n_{i,K}}{{q_\varepsilon}^2} W(\sigma_{m_K} J_i) + Cn\varepsilon + O(\sqrt{n}). \tag{7.42}$$

Step 2: We proceed to estimating $W(j_n, \mathbf{1}_{\mathbb{R}^2})$. We have, using (7.26), (7.25), (7.21), then the fact that ${m_0}' - m_K \le C\overline{R}_\varepsilon/\sqrt{n}$ on K, then (7.11) with (1.33), then (7.11) and finally (7.1), that

$$\begin{aligned}
\sum_{i=1}^{N_\varepsilon} \frac{|K| n_{i,K}}{{q_\varepsilon}^2} W_K(\sigma_{m_K} J_i) &\le |E'| \sum_{i=1}^{N_\varepsilon} \widetilde{P}\left(\frac{K}{\sqrt{n}} \times H_\varepsilon^i\right) W(\sigma_{m_K} J_i) + |K|\varepsilon \\
&\le |E'| \sum_{i=1}^{N_\varepsilon} \int_{\frac{K}{\sqrt{n}} \times H_\varepsilon^i} W_K(\sigma_{{m_0}'(x)} J_i)\, d\widetilde{P}(x,j) + |K| \left(\frac{C}{\sqrt{n}} + \varepsilon\right) \\
&\le |E'| \sum_{i=1}^{N_\varepsilon} \int_{\frac{K}{\sqrt{n}} \times H_\varepsilon^i} W_K(\sigma_{{m_0}'(x)} j)\, d\widetilde{P}(x,j) + |K| \left(\frac{C}{\sqrt{n}} + C\varepsilon\right) \\
&= |E'| \int_{\frac{K}{\sqrt{n}} \times H_\varepsilon} W_K(j)\, dP(x,j) + |K| \left(\frac{C}{\sqrt{n}} + C\varepsilon\right).
\end{aligned} \tag{7.43}$$

Here we have used the fact that W_K is bounded below by some (negative) constant, a fact proved in [SS1] that we use below several times.

We proceed by estimating $n_{0,K}$. From (7.26) we deduce that

$$\sum_{i=1}^{N_\varepsilon} (n_{i,K} + 1) \ge \frac{{q_\varepsilon}^2 |E'|}{|K|} \widetilde{P}\left(\frac{K}{\sqrt{n}} \times H_\varepsilon\right) \ge \frac{{q_\varepsilon}^2 |E'|}{|K|} \left(\frac{|K|}{|E'|} - \widetilde{P}\left(\frac{K}{\sqrt{n}} \times {H_\varepsilon}^c\right)\right),$$

and then it follows from (7.29) that

$$n_{0,K} = {q_\varepsilon}^2 - \sum_{i=1}^{N_\varepsilon} n_{i,K} \le N_\varepsilon + \frac{{q_\varepsilon}^2 |E'|}{|K|} \widetilde{P}\left(\frac{K}{\sqrt{n}} \times {H_\varepsilon}^c\right).$$

Summing over $K \in \mathcal{K}_n$, using the fact that

$$|E' \setminus \cup_{\mathcal{K}_n} K| < C_\varepsilon \sqrt{n} \tag{7.44}$$

and then (7.21), (7.6), we find that

$$\sum_{K\in\mathcal{K}} \frac{|K|}{{q_\varepsilon}^2} n_{0,K} W_K(\sigma_{m_K} J_0) \le C|E'| \left(\tilde{P}(E \times {H_\varepsilon}^c) + \frac{1}{\sqrt{n}} + \varepsilon\right) \le Cn\left(\frac{C_\varepsilon}{\sqrt{n}} + \varepsilon\right).$$

Summing (7.43) with respect to $K \in \mathcal{K}_n$ and adding the above estimate we find, in view of (7.44), (7.42) and (7.6), that

$$W(j_n, \mathbf{1}_{\mathbb{R}^2}) \le n|E| \int_{E\times L^p_{\mathrm{loc}}} W_K(j)\, dP(x,j) + Cn\left(\varepsilon + \frac{C_\varepsilon}{\sqrt{n}}\right). \tag{7.45}$$

Step 3: Energy bound for $(x_1, \dots, x_n)$. From (7.45), the constructed currents $\{j_n\}$ and points $\{\Lambda_n\}_n$ satisfy $\mathrm{curl}\, j_n = 2\pi\left(\sum_{p\in\Lambda_n} \delta_p - m_0'\right)$ in $\mathbb{R}^2$ with $\#\Lambda_n = n$ and

$$\limsup_n \frac{W(j_n, \mathbf{1}_{\mathbb{R}^2})}{n} \le |E| \int W_K(j)\, dP(x,j) + C\varepsilon. \tag{7.46}$$

Now let $\{x_i\}_i = \{p/\sqrt{n}\}_{p\in\Lambda_n}$ be the points in Λ_n in the initial scale, and let $\nu_n = \sum_i \delta_{x_i}$. Since $\Lambda_n \subset E'$ by construction, we have $\mathrm{Supp}(\nu_n) \subset E$ and thus $\int \zeta\, d\nu = 0$. Moreover, defining H_n' by (1.19), we have that $-\Delta H_n' = \mathrm{curl}\, j_n$ and we may thus write $j_n = -\nabla^\perp H_n' - \nabla f_n$ for some function f_n. But $j_n = 0$ outside of E, by construction, while H_n' decays fast at infinity by its definition (1.19) and the fact that the right-hand side of (1.19) has integral 0. Letting $U_\eta = \cup_{p\in\Lambda_n} B(p,\eta)$, we first have

$$\int_{B_R\setminus U_\eta} |j_n|^2 - \int_{B_R\setminus U_\eta} |j_n + \nabla f_n|^2 = -2\int_{B_R\setminus U_\eta} (j_n + \nabla f_n)\cdot \nabla f_n + \int_{B_R\setminus U_\eta} |\nabla f_n|^2.$$

Since $j_n \in L^q_{loc}$ for any $q < 2$ and since $f_n \in W^{1,q}_{loc}(\mathbb{R}^2)$ for all q, the last two terms on the right-hand side converge as $\eta \to 0$ to the integrals over B_R. Also integrating by parts, using the Jacobian structure and the decay of f_n and j_n, we have $\int_{B_R} (j_n + \nabla f_n)\cdot\nabla f_n = \int_{B_R} -\nabla^\perp H_n' \cdot \nabla f_n \to 0$ as $R \to +\infty$. Therefore, letting $\eta \to 0$ then $R \to +\infty$ in the above yields

$$W(j_n, \mathbf{1}_{\mathbb{R}^2}) - W(-\nabla^\perp H_n', \mathbf{1}_{\mathbb{R}^2}) \ge \int |\nabla f_n|^2 \ge 0.$$

Together with (7.46), we deduce in view of (2.1) that

$$\limsup_{n\to\infty} \frac{1}{n}\left(w_n(x_1,\dots,x_n) - n^2 I(\mu_0) + \frac{n}{2}\log n\right) \le \frac{|E|}{\pi} \int W_K(j)\, dP(x,j) + C\varepsilon. \tag{7.47}$$

Step 4: Existence of A_n. We claim that if n is large enough and if $j \in L^p_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ is such that

$$d_p(j(\sqrt{n}x + \cdot), j_n^{\text{int}}(\sqrt{n}x + \cdot)) < \eta_1/2 \tag{7.48}$$

for any $x \in E \setminus F$ for some set F satisfying $|F| < \eta_0 |E|$, then

$$d_{\mathcal{B}}\left(\fint_E \delta_x \otimes \delta_{\theta_{\sqrt{n}x} j}\, dx, \fint_E \delta_x \otimes \delta_{\theta_{\sqrt{n}x} j_n^{\text{int}}}\, dx \right) < C\varepsilon(|\log \varepsilon| + 1). \tag{7.49}$$

This would follow immediately from Lemmas 7.1, 7.2 and 7.3 if $\theta_{\sqrt{n}x} j_n^{\text{int}}$ belonged to some compact set independent of $x \notin F$ and n. In our case we note that if x belongs to some $\tilde{L} \in \mathcal{L}_{K,i}$, where $K \in \mathcal{K}_n$ and $0 \le i \le N_\varepsilon$, then

$$j_n^{\text{int}}(x + \cdot) = \sigma_{m_K} J_{i,L}(\cdot + x - x_{\tilde{L}}).$$

Moreover, since $J_i \in H_\varepsilon$, from (7.14) it follows that if $x - x_{\tilde{L}} \notin \Gamma(J_i)/\sqrt{m_K}$ then $j' := \sigma_{m_K} J_i(\cdot + x - x_{\tilde{L}}) \in G_\varepsilon$. If in addition, $\text{dist}(x, \partial \tilde{L}) > \eta_0 R_\varepsilon$, then we deduce from (7.19) that $d_p\left(j_n^{\text{int}}(x + \cdot), j'\right) < \eta_1/2$ and $d_p\left(j(x + \cdot), j'\right) < \eta_1$. Lemma 7.2 then yields $d_{\mathcal{B}}(\delta_{j_n^{\text{int}}(x+\cdot)}, \delta_{j'}) < \varepsilon$ and $d_{\mathcal{B}}(\delta_{j(x+\cdot)}, \delta_{j'}) < \varepsilon$ thus

$$d_{\mathcal{B}}(\delta_{j_n^{\text{int}}(x+\cdot)}, \delta_{j(x+\cdot)}) < 2\varepsilon.$$

In view of Lemma 7.3 we find

$$d_{\mathcal{B}}\left(\frac{1}{|E|} \int_{E\setminus\tilde{F}} \delta_x \otimes \delta_{\theta_{\sqrt{n}x} j}\, dx, \frac{1}{|E|} \int_{E\setminus\tilde{F}} \delta_x \otimes \delta_{\theta_{\sqrt{n}x} j_n^{\text{int}}}\, dx \right) < C\varepsilon(|\log \varepsilon| + 1),$$

where $\tilde{F}$ is the union of F and of the union with respect to $0 \le i \le N_\varepsilon$, $K \in \mathcal{K}_n$ and $\tilde{L} \in \mathcal{L}_{K,i}$ of $\frac{1}{\sqrt{n}}\left(x_{\tilde{L}} + \Gamma(J_i)/\sqrt{m_K}\right)$, of $\frac{1}{\sqrt{n}}\{x \in \tilde{L} : \text{dist}(x, \partial\tilde{L}) \le \eta_0 R_\varepsilon\}$, and of $E \setminus \cup_{\mathcal{K}_n} \frac{K}{\sqrt{n}}$. It turns out that $|\tilde{F}| < C\eta_0$ if n is large enough, C being of course independent of ε, and thus using Lemma 7.1 we deduce (7.49). The claim is proved.

To prove the existence of the set A_n, we note that the currents J_i used in constructing j_n^{int} depend on ε but are independent on n. Then they are truncated to obtain $J_{i,K}$ where the sidelengths of L are in $[R_\varepsilon/C, CR_\varepsilon]$, i.e. in an interval independent of n. It follows at once that there exists $\delta > 0$ such that the points in L may be perturbed by an amount δ so that for every i, K and $\tilde{L} \in \mathcal{L}_{K,i}$ the perturbed $J^p_{i,L}$ is at a distance at most $\eta_1/4$ of $J_{i,K}$, for every n. Then in view of (7.44) and (7.37) it follows that for n large enough the resulting j_n^p will satisfy (7.48) for x far enough from $\partial E'$, i.e. outside a set of proportion relative to $|E'|$ tending to 0 as $n \to \infty$. We deduce that j_n^p satisfies (7.49), hence if n is large enough

$$d_{\mathcal{B}}(P_{j_n^p}, P) < C\varepsilon(|\log \varepsilon| + 1).$$

The same reasoning implies that if we let $\{x_1, \dots, x_n\}$ be the points in Λ_n in original coordinates, then perturbing the points in Λ_n by an amount $\delta > 0$ small enough, i.e. perturbing the x_i's by an amount $\delta/\sqrt{n}$ at most we obtain points y_i such that $w_n(y_i) \le w_n(x_i) + \varepsilon$. Since the ordering of the points is irrelevant, we let S_n denote the set of permutations of $1 \dots n$ and define

$$A_n = \{(y_1, \dots, y_n) : \exists\, \sigma \in S_n |x_i - y_{\sigma(j)}| < \delta.$$

Then, given $\eta > 0$, from the previous discussion and choosing $\varepsilon > 0$ small enough we have for any n and any $(y_1, \dots, y_n) \in A_n$ that (4.1) is satisfied and the existence of j_n such that $\text{curl}\, j_n = 2\pi\left(\sum_i \delta_{y_i'} - m_0'\right)$ and such that $\{P_{j_n}\}_n$ satisfies (4.2).

This concludes the proof of Proposition 4.1.

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Etienne Sandier
Université Paris-Est,
LAMA – CNRS UMR 8050,
61, Avenue du Général de Gaulle, 94010 Créteil. France
& Institut Universitaire de France
`sandier@u-pec.fr`

Sylvia Serfaty
UPMC Univ Paris 06, UMR 7598 Laboratoire Jacques-Louis Lions,
Paris, F-75005 France ;
CNRS, UMR 7598 LJLL, Paris, F-75005 France
& Courant Institute, New York University
251 Mercer st, NY NY 10012, USA
`serfaty@ann.jussieu.fr`