Mean curvature flow with obstacles

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Abstract

We consider the evolution of fronts by mean curvature in the presence of obstacles. We construct a weak solution to the flow by means of a variational method, corresponding to an implicit time-discretization scheme. Assuming the regularity of the obstacles, in the two-dimensional case we show existence and uniqueness of a regular solution before the onset of singularities. Finally, we discuss an application of this result to the positive mean curvature flow.

1   Introduction

Motivated by several models in physics, biology and material science, there has been a growing interest in recent years towards the rigorous analysis of front propagation in heterogeneous media, see [26, 8, 18, 21, 13] and references therein. In this paper, we analyze the evolution by mean curvature of an interface in presence of hard obstacles which can stop the motion. Even if this is a prototypical model of energy driven front propagation in a medium with obstacles, to our knowledge there are no rigorous results concerning existence, uniqueness and regularity of the flow. On the other hand, we mention that the corresponding stationary problem, the so-called *obstacle problem*, has been studied in great detail, see [25, 12] and references therein.

To be more precise, given an open set $\Omega \subset \mathbb{R}^n$, we consider the evolution of a hypersurface $\partial E(t)$, with the constraint $E(t) \subset \Omega$ for all $t \geq 0$, where Ω is an open subset of $\mathbb{R}^n$ and $\mathbb{R}^n \setminus \Omega$ represents the obstacles. The corresponding geometric equation formally reads (we refer to Section 4 for a precise definition):

$$v(x) = \begin{cases} \kappa(x) & \text{if } x \in \Omega \\ \max(\kappa(x), 0) & \text{if } x \in \partial \Omega \end{cases} \tag{1}$$

where v and κ denote respectively the normal inward velocity and the mean curvature of $\partial E(t)$. Notice that the right-hand side of (1) is discontinuous on $\partial \Omega$, so that the classical

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viscosity theory [19] does not apply to this case (see however [20, 9] for a possible approach in this direction).

We are particularly interested in existence and uniqueness of smooth (that is $C^{1,1}$) solutions to (1). We tackle this problem by means of a variational method first introduced in [5, 24] (see also [6] for a simpler description of the same approach), which is based on an implicit time-discretization scheme for (1).

After showing the consistency of the scheme with regular solutions (Theorem 4.7), we obtain a comparison principle and uniqueness of smooth solutions in any dimensions (Corollary 4.8). Moreover, in the two-dimensional case we are also able to prove local in time existence of solutions (Theorem 5.3). Notice that in general one cannot expect existence of regular solutions for all time, due to the presence of singularities of the flow (even in dimension 2). On the other hand, due to the presence of the obstacles, regular solutions do not necessarily vanish in finite time and may exist for all times. Eventually, we apply our result to the positive curvature flow in two dimensions, obtaining a short time existence and uniqueness result (Corollary 6.5) for $C^{1,1}$-regular flows. Indeed, such evolution can be seen as a curvature flow where the obstacle is given by the complementary of the initial set.

We point out that the study of the positive curvature flow in Section 6 is related to some biological models which originally motivated our work: in several recent studies of actomyosin cable contraction in morphogenesis and tissue repair there is increasing evidence that the contractile structure forms only in the positive curvature part of the boundary curve (see [4, 3] and references therein). Since the contraction of such actomyosin structures can be associated with curvature terms (see [22, 1, 2]), this leads very naturally to consider the positive curvature flow problem.

Notice that a set evolving according to this law is always nonincreasing with respect to inclusion, which is a feature not satisfied by the usual curvature flow. This shows why assembling the contractile structure only in the positive curvature portion of the boundary (instead of all around) and thus doing positive curvature flow (instead of usual curvature flow) is an interesting way to evolve from the biological point of view: it corresponds to making our wound (or hole) close in a manner where we never abandon any portion of the surface we have already managed to cover since we started closing.

We also remark that the positive curvature flow is useful in the context of image analysis [27, p. 204], and appears naturally in some differential games [23].

2 Notation

Given an open set $A \subseteq \mathbb{R}^n$, a function $u \in L^1(A)$ whose distributional gradient Du is a Radon measure with finite total variation in A is called a function of bounded variation, and the space of such functions will be denoted by $BV(A)$. The total variation of Du on A turns out to be

$$\sup \left\{ \int_A u \operatorname{div} z \, dx : z \in C_0^\infty(A; \mathbb{R}^n), \ |z(x)| \leq 1 \ \forall x \in A \right\}, \tag{2}$$

and will be denoted by $|Du|(A)$ or by $\int_A |Du|$. The map $u \to |Du|(A)$ is $L^1(A)$-lower semi-continuous, and $BV(A)$ is a Banach space when endowed with the norm $\|u\| := \int_A |u| \, dx + |Du|(A)$. We refer to [7] for a comprehensive treatment of the subject.

We say that a set E satisfies the exterior (resp. interior) R-ball condition, for some $R > 0$, if for any $x \in \partial E$ there exists a ball $B_R(x')$, with $x \in \partial B_R(x')$ and $B_R(x') \cap E = \emptyset$ (resp. $B_R(x') \subseteq E$). Notice that a set E with compact boundary satisfies both the interior and the exterior R-ball condition, for some $R > 0$, if and only if ∂E is of class $C^{1,1}$.

3 The implicit scheme

Following the celebrated papers [5, 24], we shall define an implicit time discrete scheme for (1). As a preliminary step, we consider solutions of the Total Variation minimization problem with obstacles; the scheme is then defined in Definition 4.2 below.

Let $B \subset \mathbb{R}^n$ be an open set and let $v : B \to [-\infty, \infty)$ be a measurable function, with $v^+ \in L^2(B)$. Following [5, 15, 24], given $h > 0$ and $f \in L^2(B)$, we let $S_{h,v}(f, B) \in L^2(B) \cap BV(B)$ be the unique minimizer of the problem

$$\min_{u \geq v} \int_B |Du| + \frac{1}{2h} \int_B (u - f)^2 \, dx. \tag{3}$$

We have the following comparison result (see [15, Lemma 2.1]).

Proposition 3.1. *The operator $S_{h,\cdot}(\cdot, B)$ is monotone, in the sense that $u_1 = S_{h,v_1}(f_1, B) \geq u_2 = S_{h,v_2}(f_2, B)$ whenever $f_1 \geq f_2$ and $v_1 \geq v_2$ a.e.*

Proof. The idea is simply to compare the sum of the energies of u_1 and u_2, with the sum of the energy of $u_1 \wedge u_2$ (which is admissible in the problem defining u_2) and of $u_1 \vee u_2$ (which is admissible in the problem defining u_1). The conclusion follows from the uniqueness of the solution to (3). $\square$

Proposition 3.2. *Assume $f, v^+ \in L^\infty(B)$: then $u = S_{h,v}(f, B) \in L^\infty(B)$ and*

$$\|S_{h,v}(f, B)\|_{L^\infty(B)} \leq \max\left(\|f\|_{L^\infty(B)}, \|v^+\|_{L^\infty(B)}\right).$$

Proof. Again, the proof is trivial. It is enough check that the energy of $u_M = (u \vee -M) \wedge M$ is less than the energy of u, while u_M is admissible as soon as $M \geq \max\left(\|f\|_{L^\infty(B)}, \|v^+\|_{L^\infty(B)}\right)$. $\square$

Theorem 3.3. *Let $v : \mathbb{R}^n \to [-\infty, +\infty)$ be a measurable function with $v^+ \in L^\infty_{\mathrm{loc}}(\mathbb{R}^n)$, $f \in L^\infty_{\mathrm{loc}}(\mathbb{R}^n)$, and $h > 0$. There exists a unique function $u \in L^\infty_{\mathrm{loc}}(\mathbb{R}^n) \cap BV_{\mathrm{loc}}(\mathbb{R}^n)$, which we shall denote by $S_{h,v}f$, such that for all $R > 0$ and $p \in (n, +\infty)$ there holds*

$$\lim_{M \to \infty} \|u - S_{h,v}(f, B_M)\|_{L^p(B_R)} = 0.$$

This function is characterized by the fact that $u \geq v$ a.e., and for any R and any $\varphi \in BV(\mathbb{R}^n)$ with support in B_R and $u + \varphi \geq v$ a.e.,

$$\int_{B_R} |Du| + \frac{1}{2h} \int |u - f|^2 \, dx \;\leq\; \int_{B_R} |D(u + \varphi)| + \frac{1}{2h} \int |u + \varphi - f|^2 \, dx.$$

Proof. We shall show a bit more: for any $M > 0$, let us denote by u_M an arbitrary local minimizer of (3), in the sense that

$$\int_{B_M} |Du_M| + \frac{1}{2h} \int |u_M - f|^2 \, dx \le \int_{B_M} |D(u_M + \varphi)| + \frac{1}{2h} \int |u_M + \varphi - f|^2 \, dx \quad (4)$$

for any $\varphi \in BV(B_M)$ with compact support. We will show that $(u_M)_{M \ge 2R}$ is a Cauchy sequence in $L^p(B_R)$, provided $p > n$.

To start, let us consider $\psi : \mathbb{R} \to \mathbb{R}_+$ a smooth, nondecreasing and bounded function with $0 \le \psi(s) \le Cs^+$ for any s. Let $M' > M > 0$, and let $\varphi \in C_c^\infty(B_M; \mathbb{R}_+)$, which we extend by zero to $B_{M'}$. We denote $u = u_M$, $u' = u_{M'}$. Let $t > 0$: observe that

$$u'(x) + t\psi(u(x) - u'(x))\varphi(x) \ge u'(x) \ge v(x)$$

$$\begin{aligned}
u(x) - t\psi(u(x) - u'(x))\varphi(x) &\ge u(x) - tC \sup \varphi \, (u(x) - u'(x))^+ \\
&\ge u(x) - (u(x) - u'(x))^+ \\
&= \min\{u(x), u'(x)\} \\
&\ge v(x)
\end{aligned}$$

for almost every $x \in \mathbb{R}^n$, as soon as $t \le (C \sup \varphi)^{-1}$.

Hence, we deduce from (4) that for t small enough,

$$\int_{B_M} |D(u - t\psi(u - u')\varphi| + \frac{1}{2h} \int_{B_M} |u - t\psi(u - u')\varphi - f|^2 \, dx$$
$$\ge \int_{B_M} |Du| + \frac{1}{2h} \int_{B_M} |u - f|^2 \, dx$$

and

$$\int_{B_M} |D(u' + t\psi(u - u')\varphi| + \frac{1}{2h} \int_{B_M} |u' + t\psi(u - u')\varphi - f|^2 \, dx$$
$$\ge \int_{B_M} |Du'| + \frac{1}{2h} \int_{B_M} |u' - f|^2 \, dx,$$

which we sum to obtain

$$\frac{t}{h} \int_{B_M} (u - u')\psi(u - u')\varphi \, dx \le \frac{t^2}{2h} \int_{B_M} g^2 \, dx$$
$$+ \int_{B_M} |Du - t\psi'(u - u')(Du - u')\varphi - t\psi(u - u')\nabla\varphi|$$
$$+ |D(u' + t\psi'(u - u')(Du - Du')\varphi + t\psi(u - u')\nabla\varphi| - |Du| - |Du'| \,.$$

For $\rho \le t\|\varphi\|_\infty \|\psi'\|_\infty \le 1$ and t small enough, the integrand in the right-hand side has the form

$$|p - \rho(p - p') - tq| + |p' + \rho(p - p') + tq| - |p| - |q|$$
$$\le 2t|q| + (1 - \rho)|p| + \rho|p'| + (1 - \rho)|p'| + \rho|p| - |p| - |q| = 2t|q|$$

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and we obtain

$$\frac{t}{h}\int_{B_M}(u-u')\psi(u-u')\varphi\,dx \;\le\; \frac{t^2}{2h}\int_{B_M}g^2\,dx \;+\; 2t\int_{B_M}\psi(u-u')|\nabla\varphi|\,dx\,.$$

Dividing by t and letting $t \to 0$, we deduce

$$\int_{B_M}(u-u')\psi(u-u')\varphi\,dx \;\le\; 2h\int_{B_M}\psi(u-u')|\nabla\varphi|\,dx\,. \tag{5}$$

Now, by approximation this remains valid for $\psi(s) = (s^+)^{p-1}$, while we can take $\varphi(x) = \varphi_0(|x|/M)^p$, for some $\varphi_0 \in C_c^\infty([0,1);\mathbb{R}_+)$ which is 1 on $[0,1/2]$. It follows from (5) and Hölder's inequality that

$$\int_{B_M}\left[(u-u')^+\varphi_0(|x|/M)\right]^p\,dx$$

$$\le 2h\int_{B_M}\left[(u-u')^+\varphi_0(|x|/M)\right]^{p-1}\frac{p}{R}\left|\varphi_0'(|x|/M)\right|\,dx$$

$$\le 2h\left[\int_{B_M}\left[(u-u')^+\varphi_0(|x|/M)\right]^p\right]^{1-\frac{1}{p}}\left[\left(\frac{p}{R}\right)^p\left|\varphi_0'(|x|/M)\right|^p\right]^{\frac{1}{p}}\,.$$

Hence:

$$\left\|(u-u')^+\varphi_0\left(\frac{|\cdot|}{M}\right)\right\|_{L^p(B_M)} \;\le\; \frac{2hp\,\omega_n^{1/p}}{M^{1-n/p}}$$

with ω_n the volume of the unit ball. Exchanging the roles of u and u' in the previous proof, we find that

$$\|u_M - u_{M'}\|_{L^p(B_{M/2})} \;\le\; \frac{2hp\,\omega_n^{1/p}}{M^{1-n/p}}\,. \tag{6}$$

As in particular u_M (or $u_{M'}$) could, in this calculation, have been chosen to be the minimizer $S_{h,v}(f,B_M)$, which is bounded by Proposition 3.2, we obtain that $u_{M'} \in L^p(B_{M/2})$ (as well as u_M). Hence, choosing $R > 0$, we see that $(u_M)_{M \ge 2R}$ defines a Cauchy sequence in $L^p(B_R)$, provided $p > n$. It follows that it converges to some limit $u \in L^p(B_R)$. As R is arbitrary, we build in this way a function u which clearly satisfies the thesis of the theorem. $\square$

Corollary 3.4. *Assume $f \ge f'$, $v \ge v'$, $h > 0$, then $S_{h,v}(f) \ge S_{h,v'}(f')$.*

Proof. It follows from Proposition 3.1 and the definition of $S_{h,v}$. $\square$

Corollary 3.5. *If f, v are uniformly continuous on $\mathbb{R}^n$, with a modulus of continuity $\omega(\cdot)$, then $S_{h,v}f$ is also uniformly continuous with the same modulus of continuity.*

Proof. It follows from the previous corollary. For $z \in \mathbb{R}^n$, let $v'(x) := v(x-z)-\omega(|z|) \le v(x)$ and $f'(x) := f(x-z)-\omega(|z|) \le f(x)$. Then, $S_{h,v'}f' = S_{h,v}f(\cdot - z) - \omega(|z|) \le S_{h,v}f$, which shows the corollary. $\square$

Observe that, if f, v are uniformly continuous, then $S_{h,v}(f, B)$ satisfies the elliptic equation

$$-\operatorname{div} z + \frac{u - f}{h} = 0 \qquad \text{on } \{x \in B : \ u(x) > v(x)\}. \tag{7}$$

where the vector field z satisfies $|z| = 1$ and $z = Du/|Du|$ whenever $|Du| \neq 0$.

Proposition 3.6. *Assume that $f(x) \to \infty$ as $|x| \to \infty$, and let $s \in \mathbb{R}$. Then the set $\{S_{h,v}(f) < s\}$ is the minimal solution of the problem*

$$\min_{E \subset \{v < s\}} P(E) + \int_E \frac{f - s}{h} \, dx. \tag{8}$$

Similarly, the set $\{S_{h,v}(f) \leq s\}$ is the maximal solution of

$$\min_{E \subset \{v \leq s\}} P(E) + \int_E \frac{f - s}{h} \, dx. \tag{9}$$

Proof. Let $M > 0$ and consider the set $E_M^s = \{S_{h,v}(f, B_M) < s\}$. Reasoning as in [11] (see also [16, Sec. 2.2.2]) one can show that E_M^s is the minimal solution of

$$\min_{E \subset B_M \cap \{v < s\}} P(E, B_M) + \int_E \frac{f - s}{h} \, dx.$$

Since f is coercive, the sets E_M^s do not depend on M for M big enough, and coincide with the set $\{S_{h,v}(f) < s\}$, so that the result follows letting $M \to +\infty$.
The second assertion regarding the set $\{S_{h,v}(f) \leq s\}$ can be proved analogously. $\qquad \square$

4 Mean curvature flow with obstacles

Let us give a precise definition of the flow (1). Given a set $E \subset \mathbb{R}^n$ we denote by

$$d_E(x) := \operatorname{dist}(x, E) - \operatorname{dist}(x, \mathbb{R}^n \setminus E) \qquad x \in \mathbb{R}^n$$

the signed distance function from E, which is negative inside E and positive outside.

Definition 4.1. *Given a family of sets $E(t)$, $t \in [0, T]$, we set*

$$d(x, t) := d_{E(t)}(x).$$

We say that $E(t)$ is a $C^{1,1}$ supersolution of (1) if there exists a bounded open set $U \subset \mathbb{R}^n$ such that $E(t) \subset \Omega$ and $\partial E(t) \subset U$ for all $t \in [0, T]$,

$$d \in \operatorname{Lip}(U \times [0, T])$$
$$|\nabla^2 d| \in L^\infty(U \times [0, T]) \tag{10}$$

and

$$\frac{\partial d}{\partial t} \geq \Delta d + O(d) \qquad \text{a.e. in } U \times [0, T]. \tag{11}$$

We say that $E(t)$ is a $C^{1,1}$ subsolution of (1) if (11) is replaced by

$$\frac{\partial d}{\partial t} \leq \Delta d + O(d) \qquad \text{a.e. in } (U \times [0, T]) \cap \{d > d_\Omega\}, \tag{12}$$

and we say that $E(t)$ is a $C^{1,1}$ solution of (1) if it is both a supersolution and a subsolution.

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We now fix an open set $\Omega \subset \mathbb{R}^n$ (representing the complement of the obstacle) and a compact set $E \subseteq \Omega$. The case when E^c is compact can be treated with minor modifications.

Since E is compact, without loss of generality we can assume that Ω is bounded. Indeed, as it will be clear from the sequel, replacing Ω with $\Omega \cap B_M$ will not affect our construction, provided $B_M \supset E$.

Definition 4.2. *Let $h > 0$ and set*

$$T_h E := \{S_{h,d_\Omega} d_E < 0\}. \tag{13}$$

Given $t > 0$, we let

$$E_h(t) := T_h^{[t/h]} E$$

be the discretized evolution of E defined by the scheme T_h.

Notice that $T_h E$ is an open subset of Ω and, by Proposition 3.6, $T_h E$ is the minimal solution of the geometric problem

$$\min_{F \subseteq \Omega} P(F) + \frac{1}{h} \int_F d_E \, dx \tag{14}$$

or equivalently

$$\min_{F \subseteq \Omega} P(F) + \frac{1}{h} \int_{F \triangle E} |d_E| \, dx.$$

When $\Omega = \mathbb{R}^n$ this corresponds to the implicit scheme introduced in [5, 24] for the mean curvature flow. Here, from (7) it also follows that $T_h E$ satisfies

$$\kappa + \frac{d_E}{h} = 0 \qquad \text{on} \quad \partial T_h E \setminus \partial \Omega. \tag{15}$$

Remark 4.3. Observe that from Proposition 3.1 it follows

$$E_1 \subset E_2 \Rightarrow T_h E_1 \subset T_h E_2.$$

Moreover, by Corollary 3.4 we have $S_{h,d_\Omega} d_E \geq S_{h,-\infty} d_E$ which implies $T_h E \subseteq \widetilde{T}_h E := \{S_{h,-\infty} d_E < 0\}$. Notice that $\widetilde{T}_h E$ is the scheme introduced in [5, 24] for the (unconstrained) mean curvature flow.

From the general regularity theory for minimizers of the perimeter with a smooth obstacle [25, 12] we have the following result.

Proposition 4.4. *Let $\partial \Omega$ be of class $C^{1,1}$, $E \subseteq \Omega$ and $h > 0$. Then there exists a closed set $\Sigma \subset \partial T_h E \cap \Omega$ such that $\mathcal{H}^s(\Sigma) = 0$ for all $s > n - 8$, $\partial T_h E \setminus \Sigma$ is of class $C^{1,1}$, and $(\partial T_h E \cap \Omega) \setminus \Sigma$ is analytic.*

Proposition 4.5. *Let $\partial \Omega$ be of class $C^{1,1}$. Then there exists $C(\Omega) > 0$ such that*

$$T_h E = \{S_{h,-\infty}(d_E + Ch\chi_{\Omega^c}) < 0\}$$

for all $C \geq C(\Omega)$. In particular $T_h E$ is a minimizer of the prescribed curvature problem

$$\min_F P(F) + C|F \setminus \Omega| + \frac{1}{h} \int_F d_E \, dx. \tag{16}$$

Proof. We recall that $S_{h,-\infty}(d_E + Ch\chi_{\Omega^c})$ is the limit, as $M \to \infty$, of the minimizer u_M of the variational problem

$$\min_{u \in BV(B_M)} \int_{B_M} |Du| + \frac{1}{2h} \int_{B_M} (u - d_E - Ch\chi_{\Omega^c})^2 dx \tag{17}$$

From Proposition 3.6 it follows that $T_h E$ is the minimal solution to (14), while

$$\bar{F} = \{S_{h,-\infty}(d_E + Ch\chi_{\Omega^c}) < 0\}$$

is the minimal solution to (16). If $\bar{F} \subset \Omega$, then $|\bar{F} \setminus \Omega| = 0$ and both $\bar{F}$ and $T_h E$ solve the same problem, and they must therefore coincide.

In order to show that $\bar{F} \subset \Omega$, it is enough to find a positive constant $\tilde{C}$ such that for all $x \notin \Omega$, $u_M \geq \tilde{C} > 0$ for M large enough.

By assumption, Ω satisfies an exterior R-ball condition, for some $R > 0$, that is, for any $x \notin \Omega$, there is a ball $B_R(x')$ with $x \in B_R(x')$ and $B_R(x') \cap \Omega = \emptyset$. If M is large enough, we also have $B_R(x') \in B_{M/2}$. Since $E \subset \Omega$, $d_E + hC\chi_{\Omega^c} \geq hC\chi_{B_R(x')}$, so that u_M is larger than the minimizer u' of

$$\min_{u \in BV(B_M)} \int_{B_M} |Du| + \frac{1}{2h} \int_{B_M} (u - hC\chi_{B_R(x')})^2 dx$$

If $C > n/R$, then it is well known that for M large enough, $u' \geq (C - n/R)h$ a.e. in $\chi_{B_R(x')}$ [11]. The thesis then follows. $\qquad\square$

4.1 Existence of weak solutions

As a consequence of Proposition 4.5, when $\partial\Omega$ is of class $C^{1,1}$ the scheme enters the framework considered in [17]. In that case, we can also deduce the existence of weak solutions in the sense of [5, 24]. We observe that the results in [6, p. 226] and, as a consequence, [6, Th. 3.3] still apply and we can deduce the (approximate) $1/(n+1)$–Hölder-continuity in time of the discrete flow starting from an initial set E_0. Then, up to subsequences, we can pass to the limit and deduce the existence of a flow $E(t)$, which is Hölder-continuous in time in $L^1(\Omega)$, and in a weak sense a solution of (1).

4.2 Consistency of the scheme

The main result of this section (Th. 4.7) is showing that the implicit scheme is consistent with regular evolutions, according to the following definition.

Definition 4.6. *The scheme T_h is consistent if and only if*

1. *If $E(\cdot)$ is a supersolution (see Def. 4.1) in an interval $[t_1, t_2]$, then for any $t \in [t_1, t_2]$, any Hausdorff limit of $T_h^n E(t_1)$, $n \to \infty$, $h \to 0$, $nh \to t - t_1$, contains $E(t)$.*

2. *If $E(\cdot)$ is a subsolution, this inclusion is reversed.*

Theorem 4.7. *The scheme T_h is consistent.*

Proof. The proof consists in building, arbitrarily close to $\partial E(t)$, strict super and subsolutions of class C^2, of the curvature flow with forcing term $C\chi_{\Omega^c}$, for C large enough. Then, the consistency result in [17, Th. 3.3] applies.

Step 1. Let E be a subsolution on $[t_1, t_2]$ in the sense of Definition 4.1, let U be the neighborhood associated to $\partial E(t)$. Without loss of generality we can assume $t_1 = 0$.

Observe that there exists $\rho > 0$ such that $\{|d(\cdot, t)| \leq \rho\} \subset U$ for all $t \in [0, t_2]$, and the sets $\partial\Omega$, $\partial\{d(\cdot, t) \leq s\}$, $|s| \leq \rho$, satisfy the interior and exterior ρ-ball condition for all times (in particular $\partial E(t)$ satisfies the condition with radius 2ρ).

Let $c_\rho \geq (d-1)/\rho^2$, and for $\varepsilon > 0$ small, let

$$d_\varepsilon(x, t) = d(x, t) - \varepsilon - 4c_\rho\varepsilon t \qquad t \in [0, t_2].$$

Observe that for ε small enough, $\{|d_\varepsilon(\cdot, t)| \leq \rho/2\} \subset \{|d(\cdot, t)| \leq \rho\}$ for all t. The constant c_ρ is precisely chosen so that in this set, the curvature of two level surfaces $\{d(\cdot, t) = s\}$ and $\{d(\cdot, t) = s'\}$ at points along the same normal vector $\nabla d(\cdot, t)$ differ by at most $c_\rho|s - s'|$. We have, for a.e. $t \in (0, t_2)$ and $x \in \{|d(\cdot, t)| \leq \rho\} \subset U$,

$$\frac{\partial d_\varepsilon}{\partial t}(x, t) = \frac{\partial d}{\partial t}(\Pi_{\partial E(x,t)}(x), t) - 4c_\rho\varepsilon,$$

thus:

- If $\Pi_{\partial E(x,t)}(x) \in \Omega$, then (by Definition 4.1)

$$\frac{\partial d_\varepsilon}{\partial t}(x, t) \leq \Delta d_\varepsilon(x, t) - 4c_\rho\varepsilon + c_\rho|d| \leq \Delta d_\varepsilon(x, t) + c_\rho|d_\varepsilon| + c_\rho(-4\varepsilon + \varepsilon(1 + 4c_\rho t))$$

so that if $t \leq \bar{t} = \min(t_2, 1/(2c_\rho))$ and $|d_\varepsilon| \leq \varepsilon/2$,

$$\frac{\partial d_\varepsilon}{\partial t}(x, t) \leq \Delta d_\varepsilon(x, t) - c_\rho\frac{\varepsilon}{2}. \tag{18}$$

- While if $\Pi_{\partial E(x,t)}(x) \in \partial\Omega$, then $d = d_\Omega$ and almost surely $\partial d/\partial t = 0$, so that $\partial d_\varepsilon/\partial t = -4c_\rho\varepsilon$. On the other hand, there is a constant $\bar{C}$ large enough (of order $1/\rho$, and admissible for Proposition 4.5) such that $|\Delta d_\varepsilon| \leq \bar{C}$ a.e. in $\{|d(\cdot, t)| < \rho\}$, and we deduce

$$0 = \frac{\partial d_\varepsilon}{\partial t}(x, t) \leq \Delta d_\varepsilon(x, t) + \bar{C} - 4c_\rho\varepsilon. \tag{19}$$

Moreover, if $d_\varepsilon \geq -\varepsilon/2$, we have that $d_\Omega = d \geq 4c_\rho\varepsilon t + \varepsilon/2$.

Consider a function g_ε which is $\bar{C}$ in $\{d_\Omega \geq \varepsilon/2\}$, 0 in Ω, and smoothly decreasing from $\bar{C}$ to 0 as d_Ω decreases from $\varepsilon/2$ to 0: we deduce from (18) and (19) that

$$\frac{\partial d_\varepsilon}{\partial t} \leq \Delta d_\varepsilon + g_\varepsilon - c_\rho\frac{\varepsilon}{2}$$

a.e. in $\{(x, t) : |d_\varepsilon(x, t)| \leq \varepsilon/2, t \in (0, \bar{t})\}$. We have built a strict subflow, as close as we want from $\partial E(t)$, for $t \in [0, \bar{t}]$. The fact that $\bar{t}$ could be less than t_2 is not an issue, as we will see in the end of the next step. On the other hand, the consistency result in [17] requires that

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d is at least C^2 in space, which is not the case here (and the proof does not extend to $C^{1,1}$ regularity). For this, we need an additional smoothing of the surface, which we perform in a second step.

Step 2. Now consider a spatial mollifier $\varphi_\eta(x) = \eta^{-n}\varphi(x/\eta)$, with $\eta << \varepsilon$. For all time let $d_\varepsilon^\eta = \varphi_\eta * d_\varepsilon$, which is still Lipschitz in t and now, smooth in x. If η is small enough, and since g_ε is continuous, we have

$$\frac{\partial d_\varepsilon^\eta}{\partial t} \leq \Delta d_\varepsilon^\eta + g_\varepsilon - c_\rho \frac{\varepsilon}{4}$$

for a.e. x, t with $|d_\varepsilon(x,t)| \leq \varepsilon/2 - \eta$. We can rewrite this equation as a curvature motion equation with some error term, as follows:

$$\frac{\partial d_\varepsilon^\eta}{\partial t} \leq |\nabla d_\varepsilon^\eta|\left(\operatorname{div}\frac{\nabla d_\varepsilon^\eta}{|\nabla d_\varepsilon^\eta|} + g_\varepsilon\right) - c_\rho\frac{\varepsilon}{4} + g_\varepsilon(1 - |\nabla d_\varepsilon^\eta|) + \frac{(D^2 d_\varepsilon^\eta \nabla d_\varepsilon^\eta) \cdot \nabla d_\varepsilon^\eta}{|\nabla d_\varepsilon^\eta|^2}. \tag{20}$$

Now, we have that

$$1 \geq |\nabla d_\varepsilon^\eta| \geq 1 - c\eta \tag{21}$$

almost everywhere, for some constant $c > 0$, of order $1/\rho$. Hence, if η is small enough, we have

$$g_\varepsilon(1 - |\nabla d_\varepsilon^\eta|) \leq c_\rho \varepsilon/16. \tag{22}$$

We claim that the following estimates holds: there exists a constant $c > 0$ (of order $1/\rho^2$) such that

$$|D^2 d_\varepsilon^\eta \nabla d_\varepsilon^\eta| \leq c\eta. \tag{23}$$

This will be shown later on (see *Step 3*). Using (21) and (23), we find that

$$\frac{(D^2 d_\varepsilon^\eta \nabla d_\varepsilon^\eta) \cdot \nabla d_\varepsilon^\eta}{|\nabla d_\varepsilon^\eta|^2} \leq c_\rho \varepsilon/16$$

if η is small enough. Thus (20) becomes, using (22),

$$\frac{\partial d_\varepsilon^\eta}{\partial t} \leq |\nabla d_\varepsilon^\eta|\left(\operatorname{div}\frac{\nabla d_\varepsilon^\eta}{|\nabla d_\varepsilon^\eta|} + g_\varepsilon\right) - c_\rho\frac{\varepsilon}{8}. \tag{24}$$

Since $|D^2 d_\varepsilon| \leq 1/\rho$ for a.e. t and x with $|d_\varepsilon(x,t)| \leq \varepsilon/2$, this is also true for $|D^2 d_\varepsilon^\eta|$ (for $|d_\varepsilon(x,t)| \leq \varepsilon/2 - \eta$), and using (21) we can easily deduce that the boundaries of the level sets $E_\varepsilon(t) = \{d_\varepsilon^\eta(\cdot, t) \leq 0\}$ have an interior and an exterior ball condition with radius $\rho/2$. Together with (24), and using $g_\varepsilon \leq \bar{C}\chi_{\Omega^c}$, we find that $E_\varepsilon(t)$, $0 \leq t \leq \bar{t}$, is a strict subflow for the motion with normal speed $V = -\kappa - \bar{C}\chi_{\Omega^c}$, and [17, Th. 3.3] holds. We deduce that there exists $h_0 > 0$ such that if $h < h_0$, $\overline{T}_h(E_\varepsilon(t)) \subseteq E_\varepsilon(t + h)$ for any $t \in [0, \bar{t} - h]$, where $\overline{T}_h$ is the evolution scheme defined by

$$\overline{T}_h E = \left\{S_{h,-\infty}(d_E + \bar{C}h\chi_{\Omega^c}) < 0\right\}$$

for any bounded set E. (It corresponds to the time-discretization of the mean curvature flow with discontinuous forcing term $-C\chi_{\Omega^c}$.) Recall that if $E \subset \Omega$, Proposition 4.5 shows that

$\overline{T}_h E = T_h E \subset \Omega$. In particular, for the subflow $E(\cdot)$ considered here, he have $T_h^n(E(0)) = \overline{T}_h^n E(0)$, for all n and $h > 0$. By induction, it follows that as long as $nh \leq \bar{t}$,

$$T_h^n E(0) = \overline{T}_h^n E(0) \subseteq E_\varepsilon(nh),$$

hence $T_h^{\lfloor t/h \rfloor} E(0)$ is in a 3ε-neighborhood of $E(t)$. Since $\bar{t}$ only depends on $\rho > 0$ (the regularity of the subflow $E(\cdot)$), we can split $[0, t_2]$ into a finite number of intervals of size at most $\bar{t}$ and reproduce this construction on each interval, making sure that the ε parameter of each interval is less than one third of the ε of the next interval.

We deduce that for any $\delta > 0$, if $h > 0$ is small enough, then $T_h^n E(0) \subset \{d_{E(nh)} \leq \delta\}$, for $0 \leq nh \leq t_2$. This shows the consistency of T_h with subflows, assuming (23) holds.

Step 3: Proof of estimate (23). Recall that since d_ε is a distance function, $|\nabla d_\varepsilon| = 1$ almost everywhere. Now, let us compute, for $\eta > 0$ small and $x, y \in \{d(\cdot, t) \leq \varepsilon/2 - \eta\}$:

$$|\nabla d_\varepsilon^\eta(x,t)|^2 - |\nabla d_\varepsilon^\eta(y,t)|^2 = (\nabla d_\varepsilon^\eta(x,t) - \nabla d_\varepsilon^\eta(y,t)) \cdot (\nabla d_\varepsilon^\eta(x,t) + \nabla d_\varepsilon^\eta(y,t))$$

$$= \int_{B_\eta} \int_{B_\eta} (\nabla d_\varepsilon(x - z, t) - \nabla d_\varepsilon(y - z, t)) \cdot (\nabla d_\varepsilon(x - z', t) + \nabla d_\varepsilon(y - z', t)) \, \varphi_\eta(z) \varphi_\eta(z') \, dz \, dz'.$$
$$(25)$$

As $|D^2 d_\varepsilon| \leq 1/\rho$, $\nabla d_\varepsilon(\cdot, t)$ is $1/\rho$-Lipschitz, using $|\nabla d_\varepsilon(x - z, t)|^2 - |\nabla d_\varepsilon(y - z, t)|^2 = 0$ it follows

$$(\nabla d_\varepsilon(x - z, t) - \nabla d_\varepsilon(y - z, t)) \cdot (\nabla d_\varepsilon(x - z', t) + \nabla d_\varepsilon(y - z', t))$$

$$\leq |\nabla d_\varepsilon(x - z, t) - \nabla d_\varepsilon(y - z, t)| \frac{2}{\rho} |z - z'| \leq \frac{2}{\rho^2} |x - y||z - z'|$$

and it follows from (25) that

$$|\nabla d_\varepsilon^\eta(x,t)|^2 - |\nabla d_\varepsilon^\eta(y,t)|^2 \leq \frac{4}{\rho^2} |x - y| \eta.$$

We deduce (letting $y \to x$) that

$$2|D^2 d_\varepsilon^\eta(x,t) \nabla d_\varepsilon^\eta(x,t)| \leq \frac{4}{\rho^2} \eta,$$

which is estimate (23).

Step 4. Consistency with superflows: the proof is almost identical (reversing the signs and inequalities), but simpler for superflows. Indeed, all the sets we now consider stay in Ω and we do not need to take into account the constraint or the forcing term $\bar{C} \chi_{\Omega^c}$.

$\square$

We can define a generalized flow as limit of the scheme T_h as $h \to 0$. Given an initial set $E \subseteq \Omega$, for all $t \geq 0$ we let

$$E_h(t) = T_h^{[t/h]} E \qquad \text{and} \qquad E_h = \bigcup_{t \geq 0} E_h(t) \times \{t\} \subset \mathbb{R}^n \times [0, +\infty). \qquad (26)$$

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Then there exists a sequence $(h_k)_{k\geq 1}$ such that both E_{h_k} and $\mathbb{R}^n \times [0,+\infty) \setminus E_{h_k} = {}^c E_{h_k}$ converge in the Hausdorff distance (locally in time) to E^* and ${}^c E_*$ respectively.

From Corollary 3.4 and Theorem 4.7 we obtain a comparison and uniqueness result for solutions of (1).

Corollary 4.8. *Let $E_1(t)$ and $E_2(t)$ be respectively a sub- and a supersolution of (1) for $t \in [0,T]$, in the sense of Definition 4.1. Then, if $E_1(0) \subseteq E_2(0)$, it follows that $E_1(t) \subseteq E_2(t)$ for all $t \in [0,T]$. In particular, if ∂E is compact and of class $C^{1,1}$, there exists at most one solution $E(t)$ starting from E. Moreover, by Remark 4.3, $E(t)$ is contained in the solution to the (unconstrained) mean curvature flow starting from E.*

5 Short time existence and uniqueness in dimension two

In this section we assume $n = 2$ and $\partial\Omega$ of class $C^{1,1}$.

The following result follows as in [11, Lemma 7].

Lemma 5.1. *Let $h > 0$ and let $E \subseteq \Omega$ with ∂E of class $C^{1,1}$. Let δ_E be the maximum $\delta > 0$ such that both ∂E and $\partial\Omega$ satisfy the δ-ball condition, and let $u = S_{h,d_\Omega} d_E$. Then, for all $\delta' \in (0, \delta_E)$ we have*

$$|u - d_E| \leq \frac{h}{\delta_E - \delta'} \qquad \text{in } \{|d_E| \leq \delta'\} \tag{27}$$

for all $h < (\delta_E - \delta')^2/3$.

Lemma 5.2. *Let $E \subseteq \Omega$ with ∂E of class $C^{1,1}$. Then, there exists $\delta > 0$ and $T > 0$ such that*

$$\partial E_h(t) \text{ satisfies the } \delta\text{-ball condition for all } t \in [0,T]. \tag{28}$$

Proof. Let δ_E be as in Lemma 5.1, and let $K = 2/\delta_E$. By Lemma 5.1, applied with $\delta' = Kh$, we get

$$\mathrm{d}_{\mathcal{H}}(\partial T_h E, \partial E) \leq \frac{h}{\delta_E - Kh} \leq \frac{h}{\delta_E}\left(1 + \frac{K}{\delta_E} h + \widehat{C}\frac{K^2}{\delta_E^2} h^2\right)$$

for all $h \leq h_0 := \delta_E^2/12$, where the constant $\widehat{C} > 0$ is independent of E. Recalling (15) and Proposition 4.4, we get

$$\|\kappa\|_{L^\infty(\partial T_h E)} \leq \frac{1}{\delta_E}\left(1 + \frac{K}{\delta_E} h + \widehat{C}\frac{K^2}{\delta_E^2} h^2\right)$$

which implies

$$\begin{aligned}
\delta_{T_h E} &\geq \min\left(\frac{1}{\|\kappa\|_{L^\infty(\partial T_h E)}}, \delta_E - \mathrm{d}_{\mathcal{H}}(\partial T_h E, \partial E)\right) \\
&\geq \delta_E \cdot \min\left(1 - \frac{h}{\delta_E^2}\left(1 + \frac{K}{\delta_E} h + \widehat{C}\frac{K^2}{\delta_E^2} h^2\right), \left(1 + \frac{K}{\delta_E} h + \widehat{C}\frac{K^2}{\delta_E^2} h^2\right)^{-1}\right)
\end{aligned} \tag{29}$$

for all $h \leq h_0$. By iterating (29) we obtain (28). $\qquad\square$

We now prove a short time existence and uniqueness result for solutions to (1).

Theorem 5.3. *Let $\partial\Omega$ be of class $C^{1,1}$ and let $E \subseteq \Omega$ with ∂E of class $C^{1,1}$. Then there exists $T > 0$ such that (1) admits a unique $C^{1,1}$ solution $E(t)$ on $[0,T]$ with $E(0) = E$.*

Proof. Let E_h be as in (26) and let

$$d_h(t) = \left(1 + \left[\frac{t}{h}\right] - \frac{t}{h}\right) d_{E_h(t)} + \left(\frac{t}{h} - \left[\frac{t}{h}\right]\right) d_{E_h(t+h)}.$$

By Lemmas 5.1 and 5.2 there exist an open set $U \subset \mathbb{R}^n$ and $T > 0$ such that $\partial E_h(t) \subset U$ for all $t \in [0,T]$ and $|\nabla^2 d_h| \in L^\infty(U \times [0,T])$; moreover, recalling (27) we also have $d_h \in \mathrm{Lip}(U \times [0,T])$. By the Arzelà-Ascoli Theorem the functions d_h converge uniformly in $U \times [0,T]$, up to a subsequence as $h \to 0$, to a limit function $d \in \mathrm{Lip}(U \times [0,T])$ such that $|\nabla^2 d| \in L^\infty(U \times [0,T])$ and $|\nabla d| = 1$ in $U \times [0,T]$. Letting $E(t) = \{x : d(x,t) < 0\}$, for all $t \in [0,T]$ we then have $E(0) = E$, $E(t) \subset \Omega$ and $\partial E(t)$ is of class $C^{1,1}$.

It remains to show that (11) and (12) hold in $U \times [0,T]$. From Theorem 4.7 it follows that, given a supersolution $\widetilde{E}(t)$ on $[t_1, t_2] \subset [0,T]$ with $\widetilde{E}(t_1) \subseteq E(t_1)$, we have $\widetilde{E}(t) \subseteq E(t)$ for all $t \in [t_1, t_2]$, and the same holds with reversed inclusions if $\widetilde{E}(t)$ is a subsolution. This implies that

$$\frac{\partial d}{\partial t} = \Delta d \qquad \text{a.e. in } (U \times [0,T]) \cap \{d > d_\Omega\} \cap \{d = 0\},$$

which proves (12). Observe that, by parabolic regularity, $\partial E(t) \cap \Omega$ is an analytic curve and the equality holds everywhere.

As we have

$$\frac{\partial d}{\partial t} = 0 \qquad \text{a.e. in } (U \times [0,T]) \cap \{d = d_\Omega\},$$

the proof of (11) amounts to show

$$\Delta d \leq 0 \qquad \text{a.e. in } (U \times [0,T]) \cap \{d = d_\Omega\}. \tag{30}$$

Assume by contradiction that there exist $(\bar{x}, \bar{t}) \in (U \times (0,T)) \cap \{d = d_\Omega\}$ such that

$$\frac{\partial d}{\partial t}(\bar{x}, \bar{t}) = 0 < \Delta d(\bar{x}, \bar{t}) = \Delta d_\Omega(\bar{x}). \tag{31}$$

Without loss of generality we can assume $d(\bar{x}, \bar{t}) = d_\Omega(\bar{x}) = 0$, and d_Ω is twice differentiable (in the classical sense) at $\bar{x}$.

Let us take an open set $\widetilde{\Omega} \supset \Omega$ with (compact) boundary of class C^∞ and such that

$$\bar{x} \in \partial\widetilde{\Omega} \qquad \text{and} \qquad \Delta d_{\widetilde{\Omega}}(\bar{x}) \geq \Delta d_\Omega(\bar{x}) > 0.$$

We let $\widetilde{\Omega}(t)$, for $t \in [0, \tau]$ and $\tau > 0$, be the evolution by curvature of $\widetilde{\Omega}$ [5], and observe that $\widehat{E}(t) = \widetilde{\Omega}(t - \bar{t})$, $t \in [\bar{t}, \bar{t} + \tau]$, is a subsolution in the sense of Definition 4.1. In particular, by Theorem 4.7

$$E(t) \subseteq \widehat{E}(t) \qquad \text{for all } t \in [\bar{t}, \bar{t} + \tau],$$

but this implies, letting $\hat{d}(x, t) = d_{\widehat{E}(t)}(x)$ and recalling (31),

$$0 = \frac{\partial d}{\partial t}(\bar{x}, \bar{t}) \geq \frac{\partial \hat{d}}{\partial t}(\bar{x}, \bar{t}) = \Delta d_{\widetilde{\Omega}}(\bar{x}) \geq \Delta d_\Omega(\bar{x}) > 0,$$

leading to a contradiction. This proves (30) and thus (11).

Finally, the uniqueness of $E(t)$ follows from Corollary 4.8. $\qquad\square$

Remark 5.4. Notice that in Theorem 5.3 it is enough to assume that Ω satisfies the exterior R-ball condition for some $R > 0$, which is a weaker assumption than requiring $\partial\Omega$ to be of class $C^{1,1}$. Indeed, we can approximate Ω with the sets

$$\Omega_\rho := \bigcup_{B_\rho(x) \subseteq \Omega} B_\rho(x) \qquad \rho > 0.$$

Notice that $\Omega_\rho \subseteq \Omega$ and $\partial\Omega_\rho$ is of class $C^{1,1}$, for all $\rho > 0$. If we take ρ small enough so that $E \subseteq \Omega_\rho$ then, by Theorem 5.3 applied with Ω replaced by Ω_ρ, we obtain a solution $E_\rho(t)$ on $[0, T_\rho]$. However, $E_\rho(t)$ is also a solution of the original problem, with constraint Ω instead of Ω_ρ, since Ω_ρ is a subsolution to (1) in the sense of Definition 4.1.

6 Positive mean curvature flow

In this section we consider the geometric equation

$$v = \max(\kappa, 0). \tag{32}$$

Notice that, by passing to the complementary set, (32) includes the evolution by negative mean curvature $v = \min(\kappa, 0)$.

Definition 6.1. *Given a family of sets $E(t)$, $t \in [0, T]$, we set*

$$d(x, t) := d_{E(t)}(x).$$

We say that $E(t)$ is a $C^{1,1}$ solution of (32) if there exists a bounded open set $U \subset \mathbb{R}^n$ such that $\partial E(t) \subset U$ for all $t \in [0, T]$,

$$d \in \mathrm{Lip}(U \times [0, T]) \qquad |\nabla^2 d| \in L^\infty(U \times [0, T])$$

and

$$\frac{\partial d}{\partial t} = \max\left(\Delta d, 0\right) + O(d) \qquad \text{a.e. in } U \times [0, T]. \tag{33}$$

Lemma 6.2. *Let $E_1(t)$ and $E_2(t)$, with $t \in [t_1, t_2]$, be two $C^{1,1}$ solutions of (32) in the sense of Definition 6.1. Then, if $E_1(t_1) \subseteq E_2(t_1)$, it follows that $E_1(t) \subseteq E_2(t)$ for all $t \in [t_1, t_2]$. In particular, if ∂E is compact and of class $C^{1,1}$, there exists at most one solution $E(t)$ starting from E.*

Proof. Notice that it is enough to prove the thesis with $t_2 = t_1 + \tau$, for some $\tau > 0$, since the general claim then follows by iteration. Fix $\varepsilon > 0$, let $d_\varepsilon(x, t) := d_{E_2(t)}(x) + \varepsilon + C\varepsilon t$ and let

$$E_\varepsilon(t) := \{x : d_\varepsilon(x, t) \leq 0\} \qquad t \in [t_1, t_1 + \tau],$$

where the positive constants C, τ will be determined later. Notice that ∂E_ε is compact and of class $C^{1,1}$ for all ε small enough, and $E_\varepsilon(t) \to E_2(t)$ as $\varepsilon \to 0$. A direct computation gives

$$\frac{\partial d_\varepsilon}{\partial t} \geq \max\left(\Delta d_\varepsilon + \varepsilon\left(C - CK^2\tau - K^2\right), 0\right) + O(d_\varepsilon) \qquad \text{a.e. in } U \times [t_1, t_1 + \tau], \tag{34}$$

where

$$K = \sup_{x \in [t_1, t_2]} \|\Delta d_{E_2(t)}\|_{L^\infty(\partial E_2(t))}.$$

If we choose $C = 2K^2$ and $\tau < 1/C$, (34) implies that $E_\varepsilon(t)$ is a supersolution of (32). Letting $D_\varepsilon(t) := \mathrm{dist}(\partial E_1(t), \partial E_\varepsilon(t))$, we have that D_ε is Lipschitz continuous, $D_\varepsilon(0) \geq \varepsilon$ and $D'_\varepsilon(t) \geq 0$ for a.e. $t \in [t_1, t_1 + \tau]$. As a consequence, $E_\varepsilon(t) \subseteq E_1(t)$ for all $t \in [t_1, t_1 + \tau]$, and the thesis follows by letting $\varepsilon \to 0$. $\qquad\square$

Remark 6.3. Notice that the viscosity theory [19] applies to equation (32), since the function $\kappa \to \max(\kappa, 0)$ is continuous. Then, Lemma 6.2 implies that, if the initial set has compact boundary of class $C^{1,1}$, the corresponding viscosity solution does not create fattening, i.e. is unique, before the onset of singularities. Corollary 6.5 below will establish the existence of such $C^{1,1}$ solutions.

Given $E \subset \mathbb{R}^n$ and $h > 0$, we set $E_h^0 = \widetilde{E}_h^0 = E$ and, by iteration,

$$
\begin{aligned}
\widetilde{E}_h^n &:= \left\{ S_{h, d_{\widetilde{E}_h^{n-1}}} \big(d_{\widetilde{E}_h^{n-1}} \big) < 0 \right\} \\
E_h^n &:= \left\{ S_{h, d_E} \big(d_{E_h^{n-1}} \big) < 0 \right\}
\end{aligned}
\tag{35}
$$

for all $n \in \mathbb{N}$. We also let $\widetilde{E}_h(t) := \widetilde{E}_h^{[t/h]}$ and $E_h(t) := E_h^{[t/h]}$. Notice that $E_h(t)$ is the discretized evolution corresponding to the mean curvature flow with obstacle $\Omega = E$ (see Definition 4.2), while $\widetilde{E}_h(t)$ is an implicit scheme for (32).

Proposition 6.4. *Let $h > 0$ and let $E \subset \mathbb{R}^n$ be a set with compact boundary. Then*

$$\widetilde{E}_h(t) = E_h(t) \qquad for\ all \quad t \geq 0 \,.$$

In particular

$$E_h(t_2) \subseteq E_h(t_1) \qquad for\ all \quad t_1 \leq t_2. \tag{36}$$

Proof. We have to show that $\widetilde{E}_h^n = E_h^n$ for all $n \in \mathbb{N}$. By the definition we have $\widetilde{E}_h^1 = E_h^1 =: F$. If we also show that $\widetilde{E}_h^2 = E_h^2$, then the thesis follows by iteration. As $d_F \geq d_E$, by Proposition 3.1 we have that $S_{h, d_E}(d_F) \geq S_{h, d_E}(d_E)$, so that

$$E_h^2 = \{ S_{h, d_E}(d_F) < 0 \} \subset \{ S_{h, d_E}(d_E) < 0 \} = F. \tag{37}$$

By Proposition 3.6 we know that E_h^2 is the minimal solution of

$$\min_{X \subset E} P(X) + \frac{1}{h} \int_X d_F \, dx \,.$$

Recalling (37) it then follows that E_h^2 is also the minimal solution of

$$\min_{X \subset F} P(X) + \frac{1}{h} \int_X d_F \, dx$$

and hence coincides with $\widetilde{E}_h^2$, again by Proposition 3.6. $\qquad\square$

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Proposition 6.4 implies that the evolution (32), with initial set E, can be seen as a particular case of (1) with $\Omega = E$. As a consequence, from Theorem 5.3 we get a short time existence result for regular solutions to (32).

Corollary 6.5. *Let $E \subset \mathbb{R}^2$ with compact boundary of class $C^{1,1}$. Then there exists $T > 0$ such that (32) admits a unique solution $E(t)$ on $[0,T]$ with $E(0) = E$ and $\partial E(t)$ a compact set of class $C^{1,1}$ for all $t \in [0,T]$. Moreover*

$$E(t_2) \subseteq E(t_1) \qquad for \ all \quad t_1 \leq t_2. \tag{38}$$

Proof. Thanks to Theorem 5.3 there exist $T > 0$ and a unique solution $E(t)$ of (1) on $[0,T]$, with $E(0) = E = \Omega$ and $\partial E(t)$ of class $C^{1,1}$. By Proposition 6.4, for all $\bar{t} \in [0,T)$, $E(t)$ is the solution of (1) on $[\bar{t},T]$ with obstacle $\Omega = E(\bar{t})$. In particular, letting as above $d(x,t) = d_{E(t)}(x)$ and recalling (11), this implies

$$\frac{\partial d}{\partial t} = \max\left(\Delta d + O(d), 0\right) \qquad \text{a.e. in } U \times [0,T],$$

that is, $E(t)$ is the solution of (32) in the sense of Definition 6.1.
The uniqueness of $E(t)$ follows from Lemma 6.2, and (38) follows from (36). $\qquad\square$

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