Local null controllability of a fluid-solid interaction problem in dimension 3

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Abstract

We are interested by the three-dimensional coupling between an incompressible fluid and a rigid body. The fluid is modeled by the Navier-Stokes equations, while the solid satisfies the Newton's laws. In the main result of the paper we prove that, with the help of a distributed control, we can drive the fluid and structure velocities to zero and the solid to a reference position provided that the initial velocities are small enough and the initial position of the structure is close to the reference position. This is done without any condition on the geometry of the rigid body.

1 Introduction

1.1 Statement of problem

We consider a rigid structure immersed in a viscous incompressible fluid. At time t, we denote by $\Omega_S(t)$ the domain occupied by the structure. The structure and the fluid are contained in a fixed bounded domain $\Omega \subset \mathbb{R}^3$. Let $\mathcal{O} \subset\subset \Omega$ be the control domain. We suppose that the boundaries of $\Omega_S(0)$ and Ω are smooth (C^4 for instance) and that

$$\Omega_S(0) \subset \Omega \setminus \mathcal{O},\ d(\partial(\Omega \setminus \mathcal{O}), \overline{\Omega_S(0)}) \geqslant \delta_0 > 0 \tag{1}$$

For any $t > 0$, we note $\Omega_F(t) := \Omega \setminus \overline{\Omega_S(t)}$ the region occupied by the fluid and $\widetilde{\mathcal{O}} \subset\subset \mathcal{O}$ an open set. The time evolution of the eulerian velocity u and the pressure p of the fluid is governed by the incompressible Navier-Stokes equations: $\forall\, t > 0$, $\forall\, x \in \Omega_F(t)$

$$\begin{cases} (u_t + (u \cdot \nabla)u)(t,x) - \nabla \cdot \sigma(u,p)(t,x) = v(t,x)\zeta(x), \\ \nabla \cdot u(t,x) = 0. \end{cases} \tag{2}$$

The stress tensor is given by

$$\sigma(u,p) := 2\mu\epsilon(u) - pId,$$

where $\epsilon(u) := \frac{1}{2}(\nabla u + \nabla u^t)$ and the viscosity coefficient μ is supposed to be positive. The function $\zeta \in C_c^2(\mathcal{O})$ satisfies $\zeta = 1$ in $\widetilde{\mathcal{O}}$ and v is a control force which acts over the system through $\mathcal{O}$.

At time t, the motion of the rigid structure is given by the position $b(t) \in \mathbb{R}^3$ of the center of mass and by a rotation (orthogonal) matrix $Q(t) \in \mathbb{M}_{3\times 3}(\mathbb{R})$. The domain $\Omega_S(t)$ is given by $\chi_S(t, \Omega_S(0))$, where χ_S denotes the flow associated to the motion of the structure:

$$\chi_S(t,y) = b(t) + Q(t)Q_0^{-1}(y - b_0) \quad \forall y \in \Omega_S(0).$$

Here, Q_0 and b_0 are respectively the initial rotation matrix and the initial position of the solid.

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Let $r:(0,T)\to\mathbb{R}^3$ be the angular velocity. Then, the rotation matrix is the solution of the following system:

$$\begin{cases} \dfrac{dQ}{dt}(t) = (r\times Q)(t) \quad t\in(0,T), \\ Q(0)=Q_0. \end{cases} \tag{3}$$

For the equations of the structure, we denote by $m>0$ the mass of the rigid structure and $J(t)\in\mathbb{M}_{3\times 3}(\mathbb{R})$ its tensor of inertia at time t. This tensor is given by

$$J(t)d\cdot\tilde{d} = \int_{\Omega_S(0)} \big(d\times Q(t)(y-b_0)\big)\cdot\big(\tilde{d}\times Q(t)(y-b_0)\big)\,dy \quad \forall d,\,\tilde{d}\in\mathbb{R}^3. \tag{4}$$

One can prove that

$$J(t)d\cdot d \geqslant C_J|d|^2 \text{ for all } d\in\mathbb{R}^3,$$

where C_J is a positive constant independent of $t>0$. The equations of the structure motion are given by the balance of linear and angular momentum. We have, for all $t\in(0,T)$

$$\begin{cases} m\ddot{b} = \displaystyle\int_{\partial\Omega_S(t)} \sigma(u,p)n\,d\gamma, \\ J\dot{r} = \big(Jr\big)\times r + \displaystyle\int_{\partial\Omega_S(t)} (x-b)\times(\sigma(u,p)n)\,d\gamma. \end{cases} \tag{5}$$

In these equations, n is the outward unit normal to $\partial\Omega_S(t)$. On the boundary of the fluid, the eulerian velocity has to satisfy a no-slip boundary condition. Therefore, we have, for all $t>0$

$$\begin{cases} u(t,x)=0, \forall\, x\in\partial\Omega, \\ u(t,x)=\dot{b}(t)+r(t)\times(x-b(t)), \forall\, x\in\partial\Omega_S(t). \end{cases} \tag{6}$$

The system is completed by the following initial conditions:

$$u(0,\cdot)=u_0 \text{ in } \Omega_F(0),\ b(0)=b_0,\ \dot{b}(0)=b_1,\ r(0)=r_0, \tag{7}$$

which satisfy

$$u_0\in H^1(\Omega_F(0)),\ \nabla\cdot u_0=0 \text{ in } \Omega_F(0),\ u_0=0 \text{ on } \partial\Omega,\ u_0(x)=b_1+r_0\times(x-b_0),\ x\in\partial\Omega_S(0). \tag{8}$$

Let us now recall some of the most relevant results in interaction problems between a rigid structure and an incompressible fluid.

A local result was proved in [12], while the existence of global weak solutions is proved in [5] and [6] (with variable density) and [18] ($2D$, with variable density); in this last paper, the existence of a solution is proved even beyond collisions. Later, the existence and uniqueness of strong global solutions in $2D$ was proved in [19] as well as the local in time existence and uniqueness of strong solutions in $3D$.

In this paper, we prove the local null controllability of system (2)-(7). The same result was proved in [4] and in [16] in dimension 2 provided that $\Omega_S(0)$ satisfies some geometric properties. For the Burgers equation with a moving particle in dimension 1, the local null controllability was proved in [7]. In the absence of a solid, the local exact controllability to the trajectories of the Navier-Stokes equations was proved in [15]. This result was later improved in [9].

We state now the main result of this paper:

Theorem 1 *There exists $\delta>0$ such that for any (u_0,b_0,b_1,r_0,Q_0) satisfying (8), $u_0\in H^2(\Omega_F(0))$ and*

$$\|u_0\|_{H^2(\Omega_F(0))}+|b_0|+|b_1|+|r_0|+|Q_0-Id|<\delta, \tag{9}$$

there exists a control $v\in L^2(0,T;H^1(\Omega))$ such that the solution of (2)-(7) satisfies

$$u(T,\cdot)=0 \text{ in } \Omega_F(T),\ b(T)=0,\ \dot{b}(T)=0,\ r(T)=0,\ Q(T)=Id.$$

The proof of this result is based on a fixed-point argument. For this matter, we first consider a linearized system for which we prove the existence of controls in $L^2(0,T;H^1(\Omega))$ which drive the velocities to zero and the position of the structure to the desired reference position $(b(T),Q(T))=(0,Id)$.

This null controllability result is established with the help of a Carleman inequality for the associated adjoint system. To prove this Carleman inequality, we use a different and more concise method than the one presented in [15] and [9] and used in [4] and [16]: we first consider the parabolic equation satisfied by the curl of the solution (where the pressure does not appear) and establish a Carleman inequality for this parabolic problem in terms of two boundary integrals concerning some traces of the velocity. These boundary terms are then estimated thanks to regularity results which are stated and proved in the Appendix at the end of the paper.

1.2 A problem linearized with respect to the fluid velocity

Let us introduce

$$\left\{\begin{array}{l}(\hat{b},\hat{r})\in H^2(0,T)\times H^1(0,T)\\ (\hat{b},\dot{\hat{b}},\hat{r})_{|t=0}=(b_0,b_1,r_0).\end{array}\right. \tag{10}$$

This allows us to define the following domains:

$$\widehat{\Omega}_S(t):=\hat{b}(t)+\widehat{Q}(t)Q_0^{-1}(\Omega_S(0)-b_0)$$

and $\widehat{\Omega}_F(t):=\Omega\setminus\overline{\widehat{\Omega}_S(t)}$, where $\widehat{Q}$ is the solution of (3) with r replaced by $\hat{r}$. We suppose that the solid domain stays far away from $\partial(\Omega\setminus\mathcal{O})$:

$$\exists\delta_1>0: d(\overline{\widehat{\Omega}_S(t)},\partial(\Omega\setminus\mathcal{O}))\geqslant\delta_1\ \ \forall t\in[0,T]. \tag{11}$$

Let us now define several notations which we will use all along the paper. We introduce the following spaces of functions defined on moving domains: for r, $p\in\mathbb{N}$,

$$L^2(L^2):=\left\{u \text{ measurable }: \int_0^T\int_{\widehat{\Omega}_F(s)}|u|^2\,dx\,ds<+\infty\right\},$$

$$L^2(H^p):=\left\{u\in L^2(L^2): \int_0^T\|u\|^2_{H^p(\widehat{\Omega}_F(s))}\,ds<+\infty\right\},$$

$$H^r(H^p):=\left\{u\in L^2(L^2): \int_0^T\sum_{\beta=0}^{r}\|\partial_t^\beta u\|^2_{H^p(\widehat{\Omega}_F(s))}\,ds<+\infty\right\},$$

with the natural associated norms coming from the definition. On the other hand, we define

$$C^0(L^2):=\{u \text{ such that } \widetilde{u}(s,x):=u(s,x)1_{\widehat{\Omega}_F(s)}\in C^0([0,T];L^2(\Omega))\}$$

and

$$C^r(H^p):=\{u:\partial_t^\beta\partial_x^\alpha u\in C^0(L^2),\ \forall\, 0\leqslant\beta\leqslant r,\ \forall\, 0\leqslant|\alpha|\leqslant p\}$$

with the associated norms given by

$$\|u\|_{C^0(L^2)}:=\max_{t\in(0,T)}\|u(t)\|_{L^2(\widehat{\Omega}_F(t))}=\max_{t\in(0,T)}\|\widetilde{u}(t)\|_{L^2(\Omega)}$$

and

$$\|u\|_{C^r(H^p)}:=\sum_{\beta=0}^{r}\max_{t\in(0,T)}\|\partial_t^\beta u(t)\|_{H^p(\widehat{\Omega}_F(t))}.$$

Let us now consider a velocity $\hat{u}$ satisfying

$$\begin{cases} \hat{u} \in \hat{Z} := H^1(L^6) \cap L^2(W^{2,6}),\ \nabla \cdot \hat{u} = 0 \quad x \in \widehat{\Omega}_F(t), \\ \hat{u}(t,x) = (\dot{\hat{b}}(t) + \hat{r}(t) \times (x - \hat{b}(t)))1_{\partial\widehat{\Omega}_S(t)}(x) \quad x \in \partial\widehat{\Omega}_F(t). \end{cases} \tag{12}$$

Let us also introduce the spaces

$$\hat{Y}_k := L^2(H^{2+k}) \cap H^{1+k/2}(L^2)$$

for $k \in [-2,2]$. Observe that $\hat{Y}_k$ is continuously imbedded in $H^1(H^k)$.

Now, we consider the following linear system around $(\hat{u}, \hat{b}, \hat{r})$: for all $t \in (0,T)$

$$\begin{cases} u_t(t,x) + (\hat{u}\cdot\nabla)u(t,x) - \nabla\cdot\sigma(u,p)(t,x) = v(t,x)\zeta(x) & x \in \widehat{\Omega}_F(t), \\ \nabla \cdot u(t,x) = 0 & x \in \widehat{\Omega}_F(t), \\ u(t,x) = 0 & x \in \partial\Omega, \\ u(t,x) = \dot{b}(t) + r(t) \times (x - \hat{b}(t)) & x \in \partial\widehat{\Omega}_S(t), \\ m\ddot{b}(t) = \displaystyle\int_{\partial\widehat{\Omega}_S(t)} (\sigma(u,p)n)(t,x)\,d\gamma, & \\ (\hat{J}\dot{r})(t) = ((\hat{J}\hat{r}) \times r)(t) + \displaystyle\int_{\partial\widehat{\Omega}_S(t)} (x - \hat{b}(t)) \times (\sigma(u,p)n)(t,x)\,d\gamma, & \\ u_{|t=0} = u_0 \text{ in } \Omega_F(0),\ b(0) = b_0,\ \dot{b}(0) = b_1,\ r(0) = r_0., & \end{cases} \tag{13}$$

where $\hat{J}$ is defined by (4) with Q replaced by $\widehat{Q}$. The rotation matrix Q is then defined by (3).

As we will see in Section 3, we will be interested in driving the solution of (13) to zero by means of $L^2(0,T;H^1(\Omega))$ controls. In order to do this, we will first obtain $L^2((0,T)\times\Omega)$ controls supported in a smaller open set $\mathcal{O}_2 \subset\subset \widetilde{\mathcal{O}}$ for the following linear system:

$$\begin{cases} u^*_t(t,x) + (\hat{u}\cdot\nabla)u^*(t,x) - \nabla\cdot\sigma(u^*,p^*)(t,x) = v^*(t,x)1_{\mathcal{O}_2}(x) & x \in \widehat{\Omega}_F(t), \\ \nabla \cdot u^*(t,x) = 0 & x \in \widehat{\Omega}_F(t), \\ u^*(t,x) = 0 & x \in \partial\Omega, \\ u^*(t,x) = \dot{b}^*(t) + r^*(t) \times (x - \hat{b}(t)) & x \in \partial\widehat{\Omega}_S(t), \\ m\ddot{b}^*(t) = \displaystyle\int_{\partial\widehat{\Omega}_S(t)} (\sigma(u^*,p^*)n)(t,x)\,d\gamma, & \\ (\hat{J}\dot{r}^*)(t) = ((\hat{J}\hat{r}) \times r^*)(t) + \displaystyle\int_{\partial\widehat{\Omega}_S(t)} (x - \hat{b}(t)) \times (\sigma(u^*,p^*)n)(t,x)\,d\gamma, & \\ u^*_{|t=0} = u_0 \text{ in } \Omega_F(0),\ b^*(0) = b_0,\ \dot{b}^*(0) = b_1,\ r^*(0) = r_0. & \end{cases} \tag{14}$$

Notice that the control force is slightly different from the one in (13).

In order to prove the null controllability of this system, we will prove a Carleman inequality for its adjoint system. Let us introduce this system:

$$
\begin{cases}
-\varphi_t(t,x) - (\hat{u}\cdot\nabla)\varphi(t,x) - \nabla\cdot\sigma(\varphi,\pi)(t,x) = 0 & x \in \widehat{\Omega}_F(t),\\
\nabla\cdot\varphi(t,x) = 0 & x \in \widehat{\Omega}_F(t),\\
\varphi(t,x) = 0 & x \in \partial\Omega,\\
\varphi(t,x) = \dot{a}(t) + \omega(t)\times(x-\hat{b}(t)) & x \in \partial\widehat{\Omega}_S(t),\\
m\ddot{a}(t) = -\displaystyle\int_{\partial\widehat{\Omega}_S(t)} (\sigma(\varphi,\pi)n)(t,x)\,d\gamma,\\
\dfrac{d}{dt}(\hat{J}\omega)(t) = ((\hat{J}\hat{r})\times\omega)(t) - \displaystyle\int_{\partial\widehat{\Omega}_S(t)} (x-\hat{b}(t))\times(\sigma(\varphi,\pi)n)(t,x)\,d\gamma,\\
\varphi_{|t=T} = \varphi_T \text{ in } \widehat{\Omega}_F(T),\ a(T) = a_0^T,\ \dot{a}(T) = a_1^T,\ \omega(T) = \omega_T.
\end{cases}
\tag{15}
$$

In the sequel, we will suppose that $\varphi_T \in L^2(\widehat{\Omega}_F(T))$ and a_0^T, a_1^T, $\omega_T \in \mathbb{R}^3$.

The paper is organized as follows: in Section 2, we state and prove the Carleman inequality satisfied by the adjoint system. In Section 3, we deduce from this inequality an observability inequality and a controllability result for the linearized system. At last, in Section 4, we prove the null controllability of the non-linear system using a fixed point theorem.

2 Carleman inequality for the adjoint system

Let us first introduce the weight functions which we will use in the proof. Let $\beta \in C^0(W^{2,\infty}) \cap C^1(W^{1,\infty})$ satisfy

$$
\beta = 0 \text{ on } \partial\widehat{\Omega}_F(t),\ \beta > 0 \text{ in } \widehat{\Omega}_F(t),\ |\nabla\beta| \geqslant c_0 > 0 \text{ in } \widehat{\Omega}_F(t)\setminus\overline{\mathcal{O}}_0,
$$
$$
\frac{\partial\beta}{\partial n} \leqslant -c_1 < 0 \text{ on } \partial\Omega,\ \frac{\partial\beta}{\partial n} \geqslant c_2 > 0 \text{ on } \partial\widehat{\Omega}_S(t),
$$

where $\mathcal{O}_0 \subset\subset \mathcal{O}_2$ is an open set. The existence of a function β satisfying the previous properties is proved in [4]. Let now λ be a positive parameter, $M := \|\beta\|_{C^0(L^\infty)}$ and

$$
\begin{aligned}
\alpha(t,x) &:= \frac{e^{(2k+2)\lambda M} - e^{\lambda(2kM+\beta(t,x))}}{t^k(T-t)^k},\ \xi(t,x) = \frac{e^{\lambda(2kM+\beta(t,x))}}{t^k(T-t)^k},\\
\alpha^*(t) &:= \frac{e^{(2k+2)\lambda M} - e^{2k\lambda M}}{t^k(T-t)^k},\ \xi^*(t) = \frac{e^{2k\lambda M}}{t^k(T-t)^k}.
\end{aligned}
\tag{16}
$$

Here, $k \geqslant 24$ is a constant.

Then, we can prove the following Carleman inequality:

Proposition 2 *Let $(\hat{u}, \hat{b}, \hat{r})$ be such that (10), (11) and (12) are satisfied. Then, there exist two constants C_1 (depending on Ω, $\mathcal{O}$, δ_0 and $\|\hat{u}\|_{\hat{Z}}, \|\hat{b}\|_{W^{1,\infty}(0,T)}, \|\hat{r}\|_{L^\infty(0,T)}$) and $C_2 > 0$ (just depending on Ω, $\mathcal{O}$ and δ_0) such that for all $\varphi_T \in L^2(\widehat{\Omega}_F(T))$ and all a_0^T, a_1^T, $\omega_T \in \mathbb{R}^3$ we have*

$$
\begin{aligned}
&s^4\lambda^6 \int_0^T \int_{\widehat{\Omega}_F(t)} e^{-2s\alpha}\xi^5|\varphi|^2 dx\,dt + s^4\lambda^5 \int_0^T e^{-2s\alpha^*}(\xi^*)^4(|\dot{a}|^2 + |\omega|^2)\,dt\\
&\leqslant C_2 s^5\lambda^6 \int_0^T \int_{\mathcal{O}_2} e^{-2s\alpha}\xi^5|\varphi|^2 dx\,dt,
\end{aligned}
\tag{17}
$$

for all $\lambda \geqslant C_1$ and all $s \geqslant C_1(T^k + T^{2k})$, where $(\varphi, \pi, a, \omega)$ is the solution to (15).

Proof: All along the proof, C (resp. $\widehat{C}$) will stand for a positive constant just depending on Ω, $\mathcal{O}$ and δ_0 (resp. on Ω, $\mathcal{O}$, δ_0 and $\|\hat{u}\|_{\hat{Z}}, \|\hat{b}\|_{W^{1,\infty}(0,T)}, \|\hat{r}\|_{L^\infty(0,T)}$).

A) Carleman estimate for the heat equation

Let us apply the curl operator to the equation satisfied by (φ, π):

$$-(\nabla \times \varphi)_t - (\hat{u}\cdot\nabla)(\nabla\times\varphi) - \mu\Delta(\nabla\times\varphi) = L(\hat{u},\varphi) \ \text{ in } \widehat{\Omega}_F(t), \tag{18}$$

where the right-hand side satisfies

$$|L(\hat{u},\varphi)| \leqslant C|\nabla\hat{u}||\nabla\varphi| \ \text{ in } \widehat{\Omega}_F(t).$$

Therefore, $\nabla\times\varphi$ fulfills a system of three heat equations. For this kind of systems, Carleman inequalities are well-understood since [10]. Here, we are going to use an inequality which has been proved in [4] (see section 2.1 in that reference). More precisely, we use the first inequality in page 21 of [4] by observing that, for the second term of the third line of that inequality, we have $e^{-2s\mathcal{V}^*}\gamma^2 = |\nabla w\tau|^2$ (τ is the tangential vector field) with the notations of [4]. Using this for $\psi := e^{-2s\alpha}\nabla\times\varphi$, we can deduce

$$\begin{aligned}
&s^3\lambda^4\int_0^T\int_{\widehat{\Omega}_F(t)}\xi^3|\psi|^2dx\,dt + s\lambda^2\int_0^T\int_{\widehat{\Omega}_F(t)}\xi|\nabla\psi|^2dx\,dt + s^3\lambda^3\int_0^T\int_{\partial\widehat{\Omega}_S(t)}(\xi^*)^3|\psi|^2d\gamma\,dt\\
&+s\lambda\int_0^T\int_{\partial\widehat{\Omega}_S(t)}\xi^*|\nabla\psi n|^2d\gamma\,dt \leqslant C\Bigg(s^3\lambda^4\int_0^T\int_{\mathcal{O}_0}\xi^3|\psi|^2dx\,dt + s\lambda^2\int_0^T\int_{\mathcal{O}_0}\xi|\nabla\psi|^2dx\,dt\\
&+s\lambda\int_0^T\int_{\partial\widehat{\Omega}_S(t)}\xi^*|\nabla\psi\tau|^2d\gamma\,dt + s\lambda^2\int_0^T\int_{\partial\widehat{\Omega}_S(t)}\xi^*|\nabla\psi n||\psi|d\gamma\,dt\\
&+\int_0^T\int_{\partial\widehat{\Omega}_S(t)}|\nabla\psi n||\psi_t + (\hat{u}\cdot\nabla)\psi|d\gamma\,dt + \int_0^T\int_{\widehat{\Omega}_F(t)}e^{-2s\alpha}|\nabla\hat{u}|^2|\nabla\varphi|^2dx\,dt\Bigg),
\end{aligned}\tag{19}$$

for all $\lambda \geqslant \widehat{C}$ and all $s \geqslant C(T^k + T^{2k})$. Let us use Young's inequality for the fourth and fifth terms in the right-hand side of this inequality. This yields:

$$\begin{aligned}
&s^3\lambda^4\int_0^T\int_{\widehat{\Omega}_F(t)}\xi^3|\psi|^2dx\,dt + s\lambda^2\int_0^T\int_{\widehat{\Omega}_F(t)}\xi|\nabla\psi|^2dx\,dt + s^3\lambda^3\int_0^T\int_{\partial\widehat{\Omega}_S(t)}(\xi^*)^3|\psi|^2d\gamma\,dt\\
&+s\lambda\int_0^T\int_{\partial\widehat{\Omega}_S(t)}\xi^*|\nabla\psi n|^2d\gamma\,dt \leqslant C\Bigg(s^3\lambda^4\int_0^T\int_{\mathcal{O}_0}\xi^3|\psi|^2dx\,dt + s\lambda^2\int_0^T\int_{\mathcal{O}_0}\xi|\nabla\psi|^2dx\,dt\\
&+s\lambda\int_0^T\int_{\partial\widehat{\Omega}_S(t)}\xi^*|\nabla\psi\tau|^2d\gamma\,dt + s^{-1}\lambda^{-1}\int_0^T\int_{\partial\widehat{\Omega}_S(t)}(\xi^*)^{-1}|\psi_t + (\hat{u}\cdot\nabla)\psi|^2d\gamma\,dt\\
&+\int_0^T\int_{\widehat{\Omega}_F(t)}e^{-2s\alpha}|\nabla\hat{u}|^2|\nabla\varphi|^2dx\,dt\Bigg),
\end{aligned}\tag{20}$$

for all $\lambda \geqslant \widehat{C}$ and all $s \geqslant C(T^k + T^{2k})$. We can estimate the term $(\hat{u}\cdot\nabla)\psi$ in the fourth integral of the right-hand side of the previous inequality thanks to (12) (since $\hat{Z} \subset C^0(L^\infty)$):

$$s^{-1}\lambda^{-1}\int_0^T\int_{\partial\widehat{\Omega}_S(t)}(\xi^*)^{-1}|(\hat{u}\cdot\nabla)\psi|^2d\gamma\,dt \leqslant \varepsilon s\lambda\int_0^T\int_{\partial\widehat{\Omega}_S(t)}\xi^*|\nabla\psi|^2d\gamma\,dt$$

for $\varepsilon > 0$ small enough provided that $\lambda \geqslant \widehat{C}_\varepsilon$ and $s \geqslant C_\varepsilon T^{2k}$.

We come back to φ now. Observe that the boundary term concerning ψ_t in (20) can be bounded as follows:

$$s^{-1}\lambda^{-1}\int_0^T\int_{\partial\widehat{\Omega}_S(t)}(\xi^*)^{-1}|\psi_t|^2 d\gamma\,dt \leqslant 2\left(s^{-1}\lambda^{-1}\int_0^T\int_{\partial\widehat{\Omega}_S(t)} e^{-2s\alpha^*}(\xi^*)^{-1}|\nabla\times\varphi_t|^2 d\gamma\,dt\right.$$
$$\left.+s^{-1}\lambda^{-1}\int_0^T\int_{\partial\widehat{\Omega}_S(t)}|\partial_t(e^{-s\alpha^*})|^2(\xi^*)^{-1}|\nabla\times\varphi|^2 d\gamma\,dt\right).$$

Since $|\alpha_t^*|\leqslant \widehat{C}(T+T^2)(\xi^*)^{1+1/k}$, the last term can be absorbed by the third integral in the left-hand side of (20) by taking $\lambda\geqslant\widehat{C}$ and $s\geqslant C(T^{2k-1}+T^{2k})$. Thus, using also that $\nabla\beta\cdot\tau=0$ on $\partial\widehat{\Omega}_S(t)$ for the third term in the right-hand side of (20), we can rewrite estimate (20) in the following way:

$$\begin{aligned}
&s^3\lambda^4\int_0^T\int_{\widehat{\Omega}_F(t)} e^{-2s\alpha}\xi^3|\nabla\times\varphi|^2dx\,dt + s\lambda^2\int_0^T\int_{\widehat{\Omega}_F(t)} e^{-2s\alpha}\xi|\nabla(\nabla\times\varphi)|^2dx\,dt\\
&\leqslant C\left(s^3\lambda^4\int_0^T\int_{\mathcal{O}_0} e^{-2s\alpha}\xi^3|\nabla\times\varphi|^2dx\,dt + s\lambda^2\int_0^T\int_{\mathcal{O}_0} e^{-2s\alpha}\xi|\nabla(\nabla\times\varphi)|^2dx\,dt\right.\\
&+s\lambda\int_0^T\int_{\partial\widehat{\Omega}_S(t)} e^{-2s\alpha^*}\xi^*|\nabla(\nabla\times\varphi)\tau|^2d\gamma\,dt + s^{-1}\lambda^{-1}\int_0^T\int_{\partial\widehat{\Omega}_S(t)} e^{-2s\alpha^*}(\xi^*)^{-1}|\nabla\times\varphi_t|^2d\gamma\,dt\\
&\left.+\int_0^T\int_{\widehat{\Omega}_F(t)} e^{-2s\alpha}|\nabla\hat{u}|^2|\nabla\varphi|^2dx\,dt\right),
\end{aligned} \tag{21}$$

for $\lambda\geqslant\widehat{C}$ and $s\geqslant C(T^k+T^{2k})$.

Let us obtain estimates on the second term in the right-hand side of (21). Let $\mathcal{O}_1$ be an open set with $\mathcal{O}_0\subset\subset\mathcal{O}_1\subset\subset\mathcal{O}_2$ and $\theta_0\in C_c^2(\mathcal{O}_1)$ be a positive function satisfying $\theta_0(x)=1$ for all $x\in\mathcal{O}_0$. We apply the curl operator to the first equation in (15):

$$-(\nabla\times\varphi)_t-\nabla\times[(\hat{u}\cdot\nabla)\varphi]-\mu\Delta(\nabla\times\varphi)=0 \text{ in } \widehat{\Omega}_F(t).$$

Then, if we set $\rho(t,x):=s\lambda^2\theta_0(x)e^{-2s\alpha(t,x)}\xi(t,x)$, we multiply this equation by $\rho\nabla\times\varphi$ and we integrate by parts in $\mathcal{O}_0$, we obtain:

$$\begin{aligned}
&-\frac{1}{2}\frac{d}{dt}\int_{\mathcal{O}_0}\rho|\nabla\times\varphi|^2dx+\frac{1}{2}\int_{\mathcal{O}_0}\rho_t|\nabla\times\varphi|^2dx+\int_{\mathcal{O}_0}\rho[(\hat{u}\cdot\nabla)(\nabla\times\varphi)]\cdot(\nabla\times\varphi)dx\\
&+\int_{\mathcal{O}_0}[(\nabla\rho)\times(\nabla\times\varphi)]\cdot[(\hat{u}\cdot\nabla)\varphi]dx+\int_{\mathcal{O}_0}\rho\Delta\varphi\cdot[(\hat{u}\cdot\nabla)\varphi]dx+\mu\int_{\mathcal{O}_0}\rho|\nabla(\nabla\times\varphi)|^2dx\\
&-\frac{\mu}{2}\int_{\mathcal{O}_0}\Delta\rho|\nabla\times\varphi|^2dx=0,
\end{aligned} \tag{22}$$

for $t\in(0,T)$. Next, we integrate between $t=0$ and $t=T$, we use $\hat{u}\in\hat{Z}\subset C^0(L^\infty)$ and

$$|\rho_t|+|\Delta\rho|\leqslant Cs^3\lambda^4\xi^3e^{-2s\alpha}\quad\text{for } s\geqslant C(T^k+T^{2k}) \text{ and } \lambda\geqslant C.$$

This leads to:

$$\begin{aligned}
&s\lambda^2\int_0^T\int_{\mathcal{O}_0} e^{-2s\alpha}\xi|\nabla(\nabla\times\varphi)|^2dx\,dt\\
&\leqslant\widehat{C}\left(s^3\lambda^4\int_0^T\int_{\mathcal{O}_1} e^{-2s\alpha}\xi^3|\nabla\times\varphi|^2dx\,dt+s\lambda^2\int_0^T\int_{\mathcal{O}_1} e^{-2s\alpha}\xi^{-1}|\nabla\varphi|^2dx\,dt\right),
\end{aligned} \tag{23}$$

for $\lambda\geqslant\widehat{C}$ and $s\geqslant C(T^k+T^{2k})$. Observe that the second term in the right-hand side of this estimate can be bounded by the last one in (21) by taking $s\geqslant CT^{2k}$. Next, we estimate the local term on $\nabla\times\varphi$. In order to do this, let $\theta_1 \in C_c^2(\mathcal{O}_2)$ be a positive function satisfying $\theta_1(x) = 1$ for all $x \in \mathcal{O}_1$. Then, integrating by parts in

$$s^3\lambda^4 \int_0^T \int_{\mathcal{O}_2} \theta_1 e^{-2s\alpha}\xi^3 (\nabla \times \varphi)\cdot(\nabla \times \varphi)\,dx\,dt + s\lambda^2 \int_0^T \int_{\mathcal{O}_2} \theta_1 e^{-2s\alpha}\xi(\nabla\varphi)\cdot(\nabla\varphi)\,dx\,dt$$

we can prove that

$$\begin{aligned}
&s^3\lambda^4 \int_0^T \int_{\mathcal{O}_1} e^{-2s\alpha}\xi^3|\nabla\times\varphi|^2 dx\,dt + s\lambda^2 \int_0^T\int_{\mathcal{O}_1} e^{-2s\alpha}\xi|\nabla\varphi|^2\,dx\,dt \leqslant C_\varepsilon s^5\lambda^6 \int_0^T\int_{\mathcal{O}_2} e^{-2s\alpha}\xi^5|\varphi|^2\,dx\,dt\\
&+\varepsilon\left(s\lambda^2\int_0^T\int_{\mathcal{O}_2} e^{-2s\alpha}\xi|\nabla(\nabla\times\varphi)|^2dx\,dt + s^3\lambda^4\int_0^T\int_{\mathcal{O}_2} e^{-2s\alpha}\xi^3|\nabla\times\varphi|^2dx\,dt\right)
\end{aligned}$$

for $\lambda \geqslant C$ and $s \geqslant CT^{2k}$.

Thus, combining this with (21) and (23), we obtain

$$\begin{aligned}
&s^3\lambda^4\int_0^T\int_{\widehat{\Omega}_F(t)} e^{-2s\alpha}\xi^3|\nabla\times\varphi|^2dx\,dt + s\lambda^2\int_0^T\int_{\widehat{\Omega}_F(t)} e^{-2s\alpha}\xi|\nabla(\nabla\times\varphi)|^2dx\,dt\\
&\leqslant C\left(s^5\lambda^6\int_0^T\int_{\mathcal{O}_2} e^{-2s\alpha}\xi^5|\varphi|^2dx\,dt + s\lambda\int_0^T\int_{\partial\widehat{\Omega}_S(t)} e^{-2s\alpha^*}\xi^*|\nabla(\nabla\times\varphi)\tau|^2d\gamma\,dt\right.\\
&\left.+s^{-1}\lambda^{-1}\int_0^T\int_{\partial\widehat{\Omega}_S(t)} e^{-2s\alpha^*}(\xi^*)^{-1}|\nabla\times\varphi_t|^2d\gamma\,dt + \int_0^T\int_{\widehat{\Omega}_F(t)} e^{-2s\alpha}|\nabla\hat{u}|^2|\nabla\varphi|^2dx\,dt\right),
\end{aligned} \tag{24}$$

for $\lambda \geqslant \widehat{C}$ and $s \geqslant C(T^k + T^{2k})$.

B) Elliptic estimates

Since $\nabla\cdot\varphi = 0$, observe that φ satisfies the following boundary-value problem:

$$\begin{cases} \Delta\varphi = -\nabla\times(\nabla\times\varphi) := f_0 & \text{in } \widehat{\Omega}_F(t),\\ \varphi = (\dot{a} + \omega\times(x-\hat{b}))1_{\partial\widehat{\Omega}_S(t)} := g_0 & \text{on } \partial\widehat{\Omega}_F(t). \end{cases}$$

- Applying classical elliptic estimates, we have

$$\|\varphi\|_{H^1(\widehat{\Omega}_F(t))} \leqslant \widehat{C}(\|f_0\|_{H^{-1}(\widehat{\Omega}_F(t))} + \|g_0\|_{H^{1/2}(\partial\widehat{\Omega}_F(t))}) \leqslant \widehat{C}(\|\nabla\times\varphi\|_{L^2(\widehat{\Omega}_F(t))} + |\dot{a}| + |\omega|).$$

which directly leads to

$$\begin{aligned}
&s^3\lambda^4\int_0^T\int_{\widehat{\Omega}_F(t)} e^{-2s\alpha^*}(\xi^*)^3|\nabla\varphi|^2dx\,dt\\
&\leqslant \widehat{C}\left(s^3\lambda^4\int_0^T\int_{\widehat{\Omega}_F(t)} e^{-2s\alpha}\xi^3|\nabla\times\varphi|^2dx\,dt + s^3\lambda^4\int_0^T e^{-2s\alpha^*}(\xi^*)^3(|\dot{a}|^2+|\omega|^2)dt\right)
\end{aligned} \tag{25}$$

- We apply now the classical elliptic Carleman estimate which can be proved as in [10]:

$$\begin{aligned}
&\kappa^4\lambda^6\int_{\widehat{\Omega}_F(t)} \exp\{2\kappa e^{\lambda\beta}\}e^{4\lambda\beta}|\varphi|^2dx + \kappa^2\lambda^4\int_{\widehat{\Omega}_F(t)} \exp\{2\kappa e^{\lambda\beta}\}e^{2\lambda\beta}|\nabla\varphi|^2dx + \kappa^4\lambda^5\int_{\partial\widehat{\Omega}_F(t)} e^{2\kappa}|\varphi|^2d\gamma\\
&\leqslant C\left(\kappa^4\lambda^6\int_{\mathcal{O}_2}\exp\{2\kappa e^{\lambda\beta}\}e^{4\lambda\beta}|\varphi|^2dx + \kappa\lambda^2\int_{\widehat{\Omega}_F(t)}\exp\{2\kappa e^{\lambda\beta}\}e^{\lambda\beta}|f_0|^2dx + \kappa^2\lambda^3e^{2\kappa}\int_{\partial\widehat{\Omega}_F(t)}|\partial_\kappa g_0|^2d\gamma\right),
\end{aligned}$$

for any $\kappa \geqslant \widehat{C}$ and any $\lambda \geqslant \widehat{C}$. Combining this with H^2 elliptic estimates, we deduce that

$$\kappa^4\lambda^6 \int_{\widehat{\Omega}_F(t)} \exp\{2\kappa e^{\lambda\beta}\} e^{4\lambda\beta}|\varphi|^2 dx + \kappa^2\lambda^4 \int_{\widehat{\Omega}_F(t)} \exp\{2\kappa e^{\lambda\beta}\} e^{2\lambda\beta}|\nabla\varphi|^2 dx + \kappa^4\lambda^5 \int_{\partial\widehat{\Omega}_F(t)} e^{2\kappa}|\varphi|^2 d\gamma$$
$$+\lambda^2 \int_{\widehat{\Omega}_F(t)} \exp\{2\kappa e^{\lambda\beta}\}|D^2\varphi|^2 dx \leqslant C\left(\kappa^4\lambda^6 \int_{\mathcal{O}_2} \exp\{2\kappa e^{\lambda\beta}\} e^{4\lambda\beta}|\varphi|^2 dx\right.$$
$$\left.+\kappa\lambda^2 \int_{\widehat{\Omega}_F(t)} \exp\{2\kappa e^{\lambda\beta}\} e^{\lambda\beta}|\Delta\varphi|^2 dx + \kappa^2\lambda^3 e^{2\kappa}(|\dot{a}|^2 + |\omega|^2)\right),$$

for any $\kappa \geqslant \widehat{C}$ and any $\lambda \geqslant \widehat{C}$, where we have used that

$$\|\varphi\|_{H^{3/2}(\partial\widehat{\Omega}_F(t))} \leqslant \widehat{C}(|\dot{a}| + |\omega|).$$

We set $\kappa := \dfrac{se^{2k\lambda M}}{t^k(T-t)^k}$ and we multiply the previous inequality by

$$\exp\left\{-2s\frac{e^{(2k+2)\lambda M}}{t^k(T-t)^k}\right\}.$$

This yields

$$\lambda^2 \int_0^T \int_{\widehat{\Omega}_F(t)} e^{-2s\alpha}(s^4\lambda^4\xi^4|\varphi|^2 + s^2\lambda^2\xi^2|\nabla\varphi|^2 + |D^2\varphi|^2) dx\, dt + s^4\lambda^5 \int_0^T \int_{\partial\widehat{\Omega}_F(t)} e^{-2s\alpha^*}(\xi^*)^4|\varphi|^2 d\gamma\, dt$$
$$\leqslant C\left(s^4\lambda^6 \int_0^T \int_{\mathcal{O}_2} e^{-2s\alpha}\xi^4|\varphi|^2 dx\, dt + s\lambda^2 \int_0^T \int_{\widehat{\Omega}_F(t)} e^{-2s\alpha}\xi|\Delta\varphi|^2 dx\, dt\right. \tag{26}$$
$$\left.+s^2\lambda^3 \int_0^T e^{-2s\alpha^*}(\xi^*)^2(|\dot{a}|^2 + |\omega|^2) d\gamma\, dt\right),$$

for $\lambda \geqslant \widehat{C}$ and $s \geqslant \widehat{C}(T^k + T^{2k})$. Observe that the terms $|\nabla\varphi|^2$ and $|D^2\varphi|^2$ in the left-hand side of (26) allow to absorb the last term in (24) taking $\lambda \geqslant \widehat{C}$ and $s \geqslant CT^{2k}$ and using $\hat{u} \in C^0(W^{1,3})$. Combining this with (25) and (24), we obtain

$$s^4\lambda^6 \int_0^T \int_{\widehat{\Omega}_F(t)} e^{-2s\alpha}\xi^4|\varphi|^2 dx\, dt + s^3\lambda^4 \int_0^T \int_{\widehat{\Omega}_F(t)} e^{-2s\alpha^*}(\xi^*)^3|\nabla\varphi|^2 dx\, dt$$
$$+s^4\lambda^5 \int_0^T \int_{\partial\widehat{\Omega}_F(t)} e^{-2s\alpha^*}(\xi^*)^4|\varphi|^2 d\gamma\, dt \leqslant C\left(s^5\lambda^6 \int_0^T \int_{\mathcal{O}_2} e^{-2s\alpha}\xi^5|\varphi|^2 dx\, dt\right.$$
$$+s\lambda \int_0^T \int_{\partial\widehat{\Omega}_S(t)} e^{-2s\alpha^*}\xi^*|\nabla(\nabla\times\varphi)\tau|^2 d\gamma\, dt + s^{-1}\lambda^{-1} \int_0^T \int_{\partial\widehat{\Omega}_S(t)} e^{-2s\alpha^*}(\xi^*)^{-1}|\nabla\times\varphi_t|^2 d\gamma\, dt \tag{27}$$
$$\left.+s^3\lambda^4 \int_0^T e^{-2s\alpha^*}(\xi^*)^3(|\dot{a}|^2 + |\omega|^2) dx\, dt\right),$$

for $\lambda \geqslant \widehat{C}$ and $s \geqslant \widehat{C}(T^k + T^{2k})$.

We notice that

$$\int_{\partial\widehat{\Omega}_S(t)} |\varphi|^2 d\gamma \geqslant \widehat{C}(|\dot{a}|^2 + |\omega|^2).$$

The proof of this inequality is given in [3] (lemma 1, section 4.1). This allows to absorb the last term in the right-hand side of (27) thanks to $s \geqslant \widehat{C}T^{2k}$. For the moment, we have:

$$
\begin{aligned}
&s^4\lambda^6 \int_0^T \int_{\widehat{\Omega}_F(t)} e^{-2s\alpha}\xi^4|\varphi|^2 dx\, dt + s^3\lambda^4 \int_0^T \int_{\widehat{\Omega}_F(t)} e^{-2s\alpha^*}(\xi^*)^3|\nabla\varphi|^2 dx\, dt \\
&+s^4\lambda^5 \int_0^T e^{-2s\alpha^*}(\xi^*)^4(|\dot{a}|^2+|\omega|^2)d\gamma\, dt \leqslant C\left(s^5\lambda^6 \int_0^T \int_{\mathcal{O}_2} e^{-2s\alpha}\xi^5|\varphi|^2 dx\, dt\right. \\
&\left.+s\lambda \int_0^T \int_{\partial\widehat{\Omega}_S(t)} e^{-2s\alpha^*}\xi^*|\nabla(\nabla\times\varphi)\tau|^2 d\gamma\, dt + s^{-1}\lambda^{-1} \int_0^T \int_{\partial\widehat{\Omega}_S(t)} e^{-2s\alpha^*}(\xi^*)^{-1}|\nabla\times\varphi_t|^2 d\gamma\, dt\right),
\end{aligned}
\tag{28}
$$

for $\lambda \geqslant \widehat{C}$ and $s \geqslant \widehat{C}(T^k + T^{2k})$.

The rest of the proof is dedicated to the estimate of the two boundary terms

$$B_1 := s\lambda \int_0^T \int_{\partial\widehat{\Omega}_S(t)} e^{-2s\alpha^*}\xi^*|\nabla(\nabla\times\varphi)\tau|^2 d\gamma\, dt$$

and

$$B_2 := s^{-1}\lambda^{-1} \int_0^T \int_{\partial\widehat{\Omega}_S(t)} e^{-2s\alpha^*}(\xi^*)^{-1}|\nabla\times\varphi_t|^2 d\gamma\, dt.$$

C) Estimate of B_1

Let us define on $(0,T)$

$$\theta_1 := s^{1/2}\lambda^{1/2}e^{-s\alpha^*}(\xi^*)^{1/2}$$

and set $(\varphi^*, \pi^*, \dot{a^*}, \omega^*) := \theta_1(\varphi, \pi, \dot{a}, \omega)$ together with $a^*(T) = 0$. These functions satisfy

$$
\begin{cases}
-\varphi^*_t(t,x) - (\hat{u}\cdot\nabla)\varphi^*)(t,x) - \nabla\cdot\sigma(\varphi^*,\pi^*)(t,x) = -\dot{\theta}_1\varphi & x \in \widehat{\Omega}_F(t), \\
\nabla\cdot\varphi^*(t,x) = 0 & x \in \widehat{\Omega}_F(t), \\
\varphi^*(t,x) = 0 & x \in \partial\Omega, \\
\varphi^*(t,x) = \dot{a^*}(t) + \omega^*(t)\times(x-\hat{b}(t)) & x \in \partial\widehat{\Omega}_S(t), \\
m\ddot{a^*}(t) = -\displaystyle\int_{\partial\widehat{\Omega}_S(t)} (\sigma(\varphi^*,\pi^*)n)(t,x)\, d\gamma + m\dot{\theta}_1\dot{a}, \\
\dfrac{d}{dt}(\hat{J}\omega^*)(t) = ((\hat{J}\hat{r})\times\omega^*)(t) - \displaystyle\int_{\partial\widehat{\Omega}_S(t)} (x-\hat{b}(t))\times(\sigma(\varphi^*,\pi^*)n)(t,x)\, d\gamma + \hat{J}\dot{\theta}_1\omega, \\
\varphi^*_{|t=T} = 0 \text{ in } \widehat{\Omega}_F(T),\ a^*(T) = \dot{a^*}(T) = 0,\ \omega^*(T) = 0.
\end{cases}
\tag{29}
$$

Here, we apply Corollary 9 (stated in the Appendix) with $k_0 = 13/9$ and we deduce the existence of a constant $\widehat{C}$ such that

$$
\begin{aligned}
&\|\theta_1\varphi\|_{L^2(H^{23/9})} + \|\theta_1\varphi\|_{H^1(H^{5/9})} + \|\theta_1\dot{a}\|_{H^{23/18}(0,T)} + \|\theta_1\omega\|_{H^{23/18}(0,T)} \\
&\qquad \leqslant \widehat{C}(\|\dot{\theta}_1\varphi\|_{L^2(H^{5/9})} + \|\dot{\theta}_1\varphi\|_{H^{5/18}(L^2)} + \|\dot{\theta}_1\dot{a}\|_{H^{5/18}} + \|\dot{\theta}_1\omega\|_{H^{5/18}}).
\end{aligned}
\tag{30}
$$

Since $23/9 > 5/2$, $B_1 \leqslant \widehat{C}\|\theta_1\varphi\|^2_{L^2(H^{23/9})}$, it suffices to estimate all four terms in the right-hand side of (30).

C.1) Estimate of $\|\dot{\theta}_1\varphi\|_{L^2(H^{5/9})}$

After an interpolation argument, we have

$$\|\varphi\|_{H^{5/9}(\widehat{\Omega}_F(t))} \leqslant \widehat{C}\|\varphi\|^{4/9}_{L^2(\widehat{\Omega}_F(t))}\|\varphi\|^{5/9}_{H^1(\widehat{\Omega}_F(t))}.$$

Multiplying this inequality by $\dot{\theta}_1$, we obtain

$$\dot{\theta}_1\|\varphi\|_{H^{5/9}(\widehat{\Omega}_F(t))} \leqslant \widehat{C}s^{2/9}(\xi^*)^{2/9-4/(9k)}(\dot{\theta}_1)^{4/9}\|\varphi\|^{4/9}_{L^2(\widehat{\Omega}_F(t))}s^{-2/9}(\xi^*)^{-2/9+4/(9k)}(\dot{\theta}_1)^{5/9}\|\varphi\|^{5/9}_{H^1(\widehat{\Omega}_F(t))}.$$

Applying now Young's inequality, we get

$$\dot{\theta}_1\|\varphi\|_{H^{5/9}(\widehat{\Omega}_F(t))} \leqslant (\varepsilon s^{1/2}(\xi^*)^{1/2-1/k}\dot{\theta}_1\|\varphi\|_{L^2(\widehat{\Omega}_F(t))} + \widehat{C}_\varepsilon s^{-2/5}(\xi^*)^{-2/5+4/(5k)}\dot{\theta}_1\|\varphi\|_{H^1(\widehat{\Omega}_F(t))}). \tag{31}$$

Observe that

$$|\dot{\theta}_1| \leqslant s^{3/2}\lambda^{1/2}e^{-s\alpha^*}(\xi^*)^{3/2+1/k},$$

with $s \geqslant \widehat{C}(T^k + T^{2k})$.

Integrating in time inequality (31), we obtain

$$\|\dot{\theta}_1\varphi\|^2_{L^2(H^{5/9})} \leqslant \varepsilon s^4\lambda\int_0^T\int_{\widehat{\Omega}_F(t)} e^{-2s\alpha^*}(\xi^*)^4|\varphi|^2dx\,dt + C_\varepsilon s^{11/5}\lambda\int_0^T\int_{\widehat{\Omega}_F(t)} e^{-2s\alpha^*}(\xi^*)^{11/5+18/(5k)}|\nabla\varphi|^2dx\,dt.$$

These two terms can be absorbed by the left hand-side of the Carleman inequality (28) provided that $k \geqslant 9$, $s \geqslant C(T^k + T^{2k})$ and $\lambda \geqslant 1$.

C.2) Estimate of $\|\dot{\theta}_1\varphi\|_{H^{5/8}(L^2)}$

Observe that

$$\|\dot{\theta}_1\varphi\|^2_{L^2(L^2)} \leqslant \widehat{C}s^3\lambda\int_0^T\int_{\widehat{\Omega}_F(t)} e^{-2s\alpha^*}(\xi^*)^{3+2/k}|\varphi|^2dx\,dt$$

and

$$\|\dot{\theta}_1\varphi\|^2_{H^1(L^2)} \leqslant \widehat{C}\left(s^3\lambda\int_0^T\int_{\widehat{\Omega}_F(t)} e^{-2s\alpha^*}(\xi^*)^{3+2/k}|\varphi_t|^2dx\,dt + s^5\lambda\int_0^T\int_{\widehat{\Omega}_F(t)} e^{-2s\alpha^*}(\xi^*)^{5+4/k}|\varphi|^2dx\,dt\right).$$

By an interpolation argument due to [20], we get

$$\begin{aligned}
\|\dot{\theta}_1\varphi\|^2_{H^{5/18}(L^2)} \leqslant \widehat{C}\lambda\Bigg(&\int_0^T\int_{\widehat{\Omega}_F(t)} e^{-2s\alpha^*}(s^3(\xi^*)^{3+2/k}|\varphi|^2)^{13/18}(s^3(\xi^*)^{3+2/k}|\varphi_t|^2)^{5/18}dx\,dt\\
&+\int_0^T\int_{\widehat{\Omega}_F(t)} e^{-2s\alpha^*}(s^3(\xi^*)^{3+2/k}|\varphi|^2)^{13/18}(s^5(\xi^*)^{5+4/k}|\varphi|^2)^{5/18}dx\,dt\Bigg)\\
= \widehat{C}\lambda\Bigg(&\int_0^T\int_{\widehat{\Omega}_F(t)} e^{-2s\alpha^*}(s^4(\xi^*)^4|\varphi|^2)^{13/18}(s^{2/5}(\xi^*)^{2/5+36/(5k)}|\varphi_t|^2)^{5/18}dx\,dt\\
&+\int_0^T\int_{\widehat{\Omega}_F(t)} e^{-2s\alpha^*}(s^4(\xi^*)^4|\varphi|^2)^{13/18}(s^{12/5}(\xi^*)^{12/5+46/(5k)}|\varphi|^2)^{5/18}dx\,dt\Bigg).
\end{aligned}$$

In these two integrals, we apply Young inequality with parameters 18/13 and 18/5 and we find

$$\begin{aligned}
\|\dot{\theta}_1\varphi\|^2_{H^{5/18}(L^2)} \leqslant{}& \widehat{C}\lambda s^4\int_0^T\int_{\widehat{\Omega}_F(t)} e^{-2s\alpha^*}(\xi^*)^4|\varphi|^2dx\,dt + \widehat{C}\lambda s^{2/5}\int_0^T\int_{\widehat{\Omega}_F(t)} e^{-2s\alpha^*}(\xi^*)^{2/5+36/(5k)}|\varphi_t|^2dx\,dt\\
&+\widehat{C}\lambda s^{12/5}\int_0^T\int_{\widehat{\Omega}_F(t)} e^{-2s\alpha^*}(\xi^*)^{12/5+46/(5k)}|\varphi|^2dx\,dt.
\end{aligned}$$

The first and third integrals can be absorbed by the left hand side of (28) taking $\lambda \geqslant \widehat{C}$, $s \geqslant C(T^k + T^{2k})$ and $k \geqslant 23/2$ while the second integral is absorbed by the second term in the left-hand side of (30) (squared) taking $\lambda \geqslant \widehat{C}$, $s \geqslant C(T^k + T^{2k})$ and $k \geqslant 24$.

C.3) Estimate of $\|\dot\theta_1 \dot a\|_{H^{5/18}}$

We have

$$\|\dot\theta_1 \dot a\|^2_{L^2(0,T)} \leqslant \widehat{C} s^3 \lambda \int_0^T e^{-2s\alpha^*}(\xi^*)^{3+2/k}|\dot a|^2 dt$$

and

$$\|\dot\theta_1 \dot a\|^2_{H^1(0,T)} \leqslant \widehat{C}\left(s^3\lambda \int_0^T e^{-2s\alpha^*}(\xi^*)^{3+2/k}|\ddot a|^2 dt + s^5\lambda \int_0^T e^{-2s\alpha^*}(\xi^*)^{5+4/k}|\dot a|^2 dt\right).$$

Using again an interpolation argument due to [20], we get

$$\|\dot\theta_1 \dot a\|^2_{H^{5/18}(0,T)} \leqslant \widehat{C}\lambda \left(\int_0^T e^{-2s\alpha^*}(s^3(\xi^*)^{3+2/k}|\dot a|^2)^{13/18}(s^3(\xi^*)^{3+2/k}|\ddot a|^2)^{5/18} dt \right.$$
$$\left. + \int_0^T e^{-2s\alpha^*}(s^3(\xi^*)^{3+2/k}|\dot a|^2)^{13/18}(s^5(\xi^*)^{5+4/k}|\dot a|^2)^{5/18} dt \right).$$

We apply Young inequality with parameters 18/13 and 18/5 and we find

$$\begin{aligned}\|\dot\theta_1 \dot a\|^2_{H^{5/18}(0,T)} \leqslant{}& \widehat{C}\lambda s^{32/9} \int_0^T e^{-2s\alpha^*}(\xi^*)^{32/9+23/(9k)}|\dot a|^2 dt + \widehat{C}\lambda s^{2/5}\int_0^T e^{-2s\alpha^*}(\xi^*)^{2/5+36/(5k)}|\ddot a|^2 dt \\ &+\widehat{C}\lambda s^4 \int_0^T e^{-2s\alpha^*}(\xi^*)^4|\dot a|^2 dt.\end{aligned} \tag{32}$$

The first and third integrals can be absorbed by the left-hand side of (28) taking $\lambda \geqslant \widehat{C}$, $s \geqslant C(T^k + T^{2k})$ and $k \geqslant 23/4$ while the second integral can be absorbed with the third term in the left-hand side of (30) provided that $\lambda \geqslant \widehat{C}$, $s \geqslant C(T^k + T^{2k})$ and $k \geqslant 12$.

C.4) Estimate of $\|\dot\theta_1 \omega\|_{H^{5/18}}$

In order to estimate this term, we proceed exactly as in step **C.3)**.

To conclude paragraph **C)**, we put together steps **C.1)-C.4)** and this gives the following estimate on B_1:

$$\begin{aligned}B_1 \leqslant \varepsilon \Bigg(& s^4\lambda^6 \int_0^T\int_{\widehat\Omega_F(t)} e^{-2s\alpha}\xi^4|\varphi|^2 dx\,dt + s^3\lambda^4 \int_0^T\int_{\widehat\Omega_F(t)} e^{-2s\alpha^*}(\xi^*)^3|\nabla\varphi|^2 dx\,dt \\ &+ s^4\lambda^5\int_0^T e^{-2s\alpha^*}(\xi^*)^4(|\dot a|^2+|\omega|^2)d\gamma\,dt\Bigg)\end{aligned} \tag{33}$$

for $\lambda \geqslant \widehat{C}_\varepsilon$ and $s \geqslant \widehat{C}_\varepsilon(T^k + T^{2k})$ for $k \geqslant 24$.

D) Estimate of B_2

Let us define on $[0,T]$

$$\theta_2 := s^{-1/2}\lambda^{-1/2}e^{-s\alpha^*}(\xi^*)^{-1/2}.$$

Then, $\theta_2(\varphi, \pi, \dot a, \omega)$ satisfy system (29) with θ_1 replaced by θ_2. We notice that

$$|B_2| \leqslant C\|\theta_2\varphi_t\|^2_{L^2(H^{14/9})},$$

since $14/9 > 3/2$.

Let us apply Corollary 9 for $k_0 = 4/9$. For our system, the compatibility condition (71) is satisfied since, thanks to the weight function θ_2, all the initial conditions are equal to zero. This yields, in particular:

$$\|\theta_2\varphi_t\|_{L^2(H^{14/9})} \leqslant \widehat{C}(\|\dot\theta_2\varphi\|_{L^2(H^{14/9})\cap H^{7/9}(L^2)} + \|\dot\theta_2 \dot a\|_{H^{7/9}(0,T)} + \|\dot\theta_2\omega\|_{H^{7/9}(0,T)}).$$

Observe that

$$\|\dot\theta_2\varphi\|_{L^2(H^{14/9})\cap H^{7/9}(L^2)} + \|\dot\theta_2\dot a\|_{H^{7/9}(0,T)} + \|\dot\theta_2\omega\|_{H^{7/9}(0,T)} \leqslant C(\|\dot\theta_2\varphi\|_{\hat Y_0} + \|\dot\theta_2\dot a\|_{H^1(0,T)} + \|\dot\theta_2\omega\|_{H^1(0,T)}).$$

Applying now Proposition 7 to $\dot\theta_2(\varphi,\pi,\dot a,\omega)$, we deduce

$$\|\theta_2\varphi_t\|_{L^2(H^{14/9})} \leqslant \widehat{C}(\|\ddot\theta_2\varphi\|_{L^2(L^2)} + \|\ddot\theta_2\dot a\|_{L^2(0,T)} + \|\ddot\theta_2\omega\|_{L^2(0,T)}). \tag{34}$$

Using the definition of the weight functions (see (16)), we obtain

$$|\ddot\theta_2| \leqslant \widehat{C}(s\xi^*)^{3/2+2/k}\lambda^{-1/2}e^{-s\alpha^*} \quad \text{for } s \geqslant \widehat{C}(T^k+T^{2k}).$$

This readily implies that the first (resp. second and third) norm in the right-hand side of (34) is absorbed by the first (resp. third) integral in the left-hand side of (28) provided that $\lambda \geqslant \widehat{C}$, $s \geqslant \widehat{C}(T^k+T^{2k})$ and $k \geqslant 4$.

Consequently, we have proved for B_2 that

$$B_2 \leqslant \varepsilon\left(s^4\lambda^6\int_0^T\int_{\widehat\Omega_F(t)} e^{-2s\alpha}\xi^4|\varphi|^2dx\,dt + s^4\lambda^5\int_0^T e^{-2s\alpha^*}(\xi^*)^4(|\dot a|^2+|\omega|^2)d\gamma\,dt\right) \tag{35}$$

for $\lambda \geqslant \widehat{C}_\varepsilon$ and $s \geqslant \widehat{C}_\varepsilon(T^k+T^{2k})$ for $k \geqslant 4$.

Thus combining (33) and (35) with (28), we obtain the desired inequality (17). This concludes the proof of Proposition 2.

3 Controllability problems

3.1 Observability inequalities for the adjoint system

Proposition 3 *There exists a constant $C_1 > 0$ depending on $\|\hat u\|_{\hat Z}$, $\|\hat b\|_{W^{1,\infty}(0,T)}$, $\|\hat r\|_{L^\infty(0,T)}$ such that for any $(\varphi_T, a_0^T, a_1^T, \omega_T)$ with $\varphi_T \in L^2(\widehat\Omega_F(T))$ and any $(\hat u, \hat b, \hat r)$ satisfying (10)-(12), the solution (φ,π,a,ω) of (15) satisfies*

$$\|\varphi(0,\cdot)\|^2_{L^2(\Omega_F(0))} + |\dot a(0)|^2 + |\omega(0)|^2 \leqslant C_1\iint_{(0,T)\times\mathcal{O}_2}|\varphi|^2dx\,dt. \tag{36}$$

Proof: The proof relies on an energy inequality for system (15). Indeed, let us multiply the equation of φ by φ and integrate in space. Using the equations of a and ω, this yields

$$-\frac{1}{2}\frac{d}{dt}\int_{\widehat\Omega_F(t)}|\varphi|^2dx + \int_{\widehat\Omega_F(t)}|\nabla\varphi|^2dx - \frac{m}{2}\frac{d}{dt}|\dot a|^2 - \frac{1}{2}\frac{d}{dt}(\hat J\omega\cdot\omega) = \frac{1}{2}\dot{\hat J}\omega\cdot\omega.$$

Thus, for any $0 \leqslant t_1 < t_2 \leqslant T$, we have

$$\int_{\widehat\Omega_F(t_1)}|\varphi(t_1)|^2dx + |\dot a(t_1)|^2 + |\omega(t_1)|^2 \leqslant \widehat{C}\left(\int_{\widehat\Omega_F(t_2)}|\varphi(t_2)|^2dx + |\dot a(t_2)|^2 + |\omega(t_2)|^2\right).$$

Combining this with the Carleman inequality (17) and using the properties of the weight function α (see (16)), we obtain (36) in a classical way.

The observability inequality (36) will not allow to lead the center of mass a to zero at time $t = T$ and the rotation matrix Q to the identity at time $t = T$. For this matter, we will improve this observability inequality (see (39) below), following the ideas of [17]. We first introduce some auxiliary problems. Let us denote by e_k the k-th element of the canonic basis in $\mathbb{R}^3$ for $k=1,2,3$. Let us consider $(\varphi^{(j)},\pi^{(j)},a^{(j)},\omega^{(j)})$ the solution of

$$\begin{cases} -\varphi^{(j)}_t(t,x)-(\hat u\cdot\nabla)\varphi^{(j)})(t,x)-\nabla\cdot\sigma(\varphi^{(j)},\pi^{(j)})(t,x)=0 & x\in\widehat\Omega_F(t),\\ \nabla\cdot\varphi^{(j)}(t,x)=0 & x\in\widehat\Omega_F(t),\\ \varphi^{(j)}(t,x)=0 & x\in\partial\Omega,\\ \varphi^{(j)}(t,x)=\dot a^{(j)}(t)+\omega^{(j)}(t)\times(x-\hat b(t)) & x\in\partial\widehat\Omega_S(t),\\ m(\ddot a^{(j)}(t)+e_j)=-\displaystyle\int_{\partial\widehat\Omega_S(t)}(\sigma(\varphi^{(j)},\pi^{(j)})n)(t,x)\,d\gamma, \\ \dfrac{d}{dt}(\hat J\omega^{(j)})(t)=((\hat J\hat r)\times\omega^{(j)})(t)-\displaystyle\int_{\partial\widehat\Omega_S(t)}(x-\hat b(t))\times(\sigma(\varphi^{(j)},\pi^{(j)})n)(t,x)\,d\gamma, \\ \varphi^{(j)}_{|t=T}=0 \text{ in } \widehat\Omega_F(T),\ a^{(j)}(T)=\dot a^{(j)}(T)=\omega^{(j)}(T)=0, \end{cases} \tag{37}$$

for $j=1,2,3$ and the solution of

$$\begin{cases} -\varphi^{(j)}_t(t,x)-(\hat u\cdot\nabla)\varphi^{(j)})(t,x)-\nabla\cdot\sigma(\varphi^{(j)},\pi^{(j)})(t,x)=0 & x\in\widehat\Omega_F(t),\\ \nabla\cdot\varphi^{(j)}(t,x)=0 & x\in\widehat\Omega_F(t),\\ \varphi^{(j)}(t,x)=0 & x\in\partial\Omega,\\ \varphi^{(j)}(t,x)=\dot a^{(j)}(t)+\omega^{(j)}(t)\times(x-\hat b(t)) & x\in\partial\widehat\Omega_S(t),\\ m\ddot a^{(j)}(t)=-\displaystyle\int_{\partial\widehat\Omega_S(t)}(\sigma(\varphi^{(j)},\pi^{(j)})n)(t,x)\,d\gamma, \\ \dfrac{d}{dt}(\hat J\omega^{(j)})(t)+e_{j-3}=((\hat J\hat r)\times\omega^{(j)})(t)-\displaystyle\int_{\partial\widehat\Omega_S(t)}(x-\hat b(t))\times(\sigma(\varphi^{(j)},\pi^{(j)})n)(t,x)\,d\gamma, \\ \varphi^{(j)}_{|t=T}=0 \text{ in } \widehat\Omega_F(T),\ a^{(j)}(T)=\dot a^{(j)}(T)=\omega^{(j)}(T)=0, \end{cases} \tag{38}$$

for $j=4,5,6$.

Using the duality between systems (37)-(38) and (14), we obtain

$$\int_0^T\int_{\mathcal{O}_2} v^*\cdot\varphi^{(j)}dx\,dt=-\int_{\Omega_F(0)}u_0\cdot\varphi^{(j)}_{|t=0}\,dx-m\dot a^{(j)}(0)\cdot b_1-\hat J(0)r_0\cdot\omega^{(j)}(0)+m(b^*_j(T)-b_{0,j})$$

for $j=1,2,3$ and

$$\int_0^T\int_{\mathcal{O}_2} v^*\cdot\varphi^{(j)}dx\,dt=-\int_{\Omega_F(0)}u_0\cdot\varphi^{(j)}_{|t=0}\,dx-m\dot a^{(j)}(0)\cdot b_1-\hat J(0)r_0\cdot\omega^{(j)}(0)+\int_0^T r^*_{j-3}(t)\,dt$$

for $j=4,5,6$.

Observe that $b^*_j(T)=0$ for $j=1,2,3$ is equivalent to the fact that v^* satisfies three conditions depending on u_0,b_0,b_1 and r_0. On the other hand, if we define θ_0 and (x_0,x_1,x_2) respectively the angle and the axis of the rotation matrix Q_0, we have that

$$Q_0=\exp\left[\begin{pmatrix}0 & -x_2\theta_0 & x_1\theta_0\\ x_2\theta_0 & 0 & -x_0\theta_0\\ -x_1\theta_0 & x_0\theta_0 & 0\end{pmatrix}\right].$$

Remark also that from (3) we get:

$$Q^*(t)=\exp\left[\int_0^t\begin{pmatrix}0 & -r^*_3 & r^*_2\\ r^*_3 & 0 & -r^*_1\\ -r^*_2 & r^*_1 & 0\end{pmatrix}(\tau)\,d\tau\right]Q_0.$$

Thus, $Q^*(T) = Id$ will hold if

$$\int_0^T r_3^*(t)\,dt = -x_2\theta_0,\ \int_0^T r_2^*(t)\,dt = -x_1\theta_0,\ \int_0^T r_1^*(t)\,dt = -x_0\theta_0,$$

which is equivalent to three conditions on the control v^* depending on u_0, b_1, r_0 and Q_0.

As a conclusion, enforcing that $b^*(T) = 0$ and $Q^*(T) = Id$ is equivalent to

$$\int_0^T \int_{\mathcal{O}_2} v^*(t,x) \cdot \varphi^{(j)}(t,x)\,dx\,dt = C^{(j)} \quad \forall 1 \leqslant j \leqslant 6,$$

for some $C^{(j)} \in \mathbb{R}$ depending on the initial conditions. Observe that the set of functions v^* satisfying this system of equations is nonempty. Indeed, assume that a linear combination of $\{\varphi^{(j)}\}_{1\leqslant j\leqslant 6}$ cancels on $\mathcal{O}_2$, then according to the unique continuation property of the fluid problem proved in [8], it cancels on the whole fluid domain. Then due to the solid equations, we can show that the coefficients of the linear combination are null (we refer to [4] for more details).

We define the orthogonal projection P from $L^2((0,T)\times\Omega)$ into $span(1_{\mathcal{O}_2}(\varphi^{(j)})_{1\leqslant j\leqslant 6})$:

$$\int_0^T \int_{\mathcal{O}_2} (v^* - P(v^*)) \cdot \varphi^{(j)}\,dx\,dt = 0 \quad 1 \leqslant j \leqslant 6.$$

We also consider the operators $P^{(j)}$ satisfying

$$P(v^*) = \sum_{j=1}^{6} P^{(j)}(v^*)\varphi^{(j)}.$$

Proposition 4 *There exists a constant $C_1 > 0$ depending on $\|\hat{u}\|_{\hat{Z}}$, $\|\hat{b}\|_{W^{1,\infty}(0,T)}$, $\|\hat{r}\|_{L^\infty(0,T)}$ such that for any $(\varphi_T,\, a_0^T,\, a_1^T,\, \omega_T)$ with $\varphi_T \in L^2(\widehat{\Omega}_F(T))$ and any $(\hat{u}, \hat{b}, \hat{r})$ satisfying (10)-(12), the solution $(\varphi, \pi, a, \omega)$ of (15) satisfies*

$$\|\varphi(0,\cdot)\|^2_{L^2(\Omega_F(0))} + |\dot{a}(0)|^2 + |\omega(0)|^2 + \sum_{j=1}^{6} |P^{(j)}(\varphi)|^2 \leqslant C_1 \iint_{(0,T)\times\mathcal{O}_2} |\varphi - P(\varphi)|^2 dx\,dt \tag{39}$$

The idea of the proof is to argue by contradiction and use the Carleman inequality (17). This is done in the same way as in [7] (see Proposition 3.2 therein) and [4] (see Proposition 5 therein), so we omit the proof.

3.2 Controllability of system (13)

In this paragraph, we prove the null controllability of system (13):

Proposition 5 *Let (u_0, b_0, b_1, r_0) satisfy (8), $u_0 \in H^2(\Omega_F(0))$ and $(\hat{u}, \hat{b}, \hat{r})$ satisfy (10)-(12). Then, there exists a control $v \in L^2(0,T; H^1(\Omega))$ such that the solution (u, p, b, r) to the problem (13) satisfies*

$$u(T,\cdot) = 0 \ \text{ in } \widehat{\Omega}_F(T),\ b(T) = 0,\ \dot{b}(T) = 0,\ \omega(T) = 0,\ Q(T) = Id, \tag{40}$$

where Q is given by (3). Moreover, there exists a constant $K_0 > 0$ such that

$$\|v\|_{L^2(0,T;H^1(\Omega))} \leqslant K_0(\|u_0\|_{H^2(\Omega_F(0))} + |b_0| + |b_1| + |r_0|). \tag{41}$$

Proof: From the observability inequality (39), it is classical to prove the existence of a control $v^* \in L^2((0,T)\times\Omega)$ such that the solution

$$(u^*, p^*, b^*, r^*) \in (L^2(H^2) \cap C^0(H^1)) \times L^2(H^1) \times H^2(0,T) \times H^1(0,T)$$

of (14) satisfies (40) and (41) for the $L^2(L^2)$ norm (see Proposition 4.1 in [7] or Proposition 6 in [4]). In the sequel of this proof, $\widehat{C}$ denotes a generic positive constant which may depend on $\|\hat{u}\|_{\hat{Z}}, \|\hat{b}\|_{W^{1,\infty}(0,T)}$ and $\|\hat{r}\|_{L^\infty(0,T)}$.

Let us now modify the control v^* into a $L^2(H^1)$ control and such that (40) and (41) are still satisfied. For this purpose, let $(\bar{u}, \bar{p}, \bar{b}, \bar{r})$ be the solution of (13) with null control. From Corollary 9 for $k_0 = 1$, we have that

$$(\bar{u}, \bar{p}, \bar{b}, \bar{r}) \in (L^2(H^3) \cap C^0(H^2)) \times L^2(H^2) \times H^{5/2}(0,T) \times H^{3/2}(0,T)$$

and there exists $K > 0$ such that

$$\|\bar{u}\|_{L^2(H^3)} + \|\bar{u}\|_{C0(H^2)} + \|\bar{p}\|_{L^2(H^2)} + \|\bar{b}\|_{H^{5/2}(0,T)} + \|\bar{r}\|_{H^{3/2}(0,T)} \leqslant \widehat{C}(\|u_0\|_{H^2(\Omega_F(0))} + |b_0| + |b_1| + |r_0|). \quad (42)$$

We consider now a function $\eta_0 \in C^1([0,T])$ such that $\eta_0(t) = 1$, $t \in [0, T/2]$, $\eta_0(t) = 0$, $t \in [3T/4, T]$ and $\eta_0(t) \geqslant 0$, $t \in [0,T]$. Then, the function

$$(w, q, c, s) := (u^* - \eta_0\bar{u}, p^* - \eta_0\bar{p}, b^* - \eta_0\bar{b}, r^* - \eta_0\bar{r})$$

satisfies the four first identities of (40) and

$$\begin{cases} w_t(t,x) + (\hat{u}\cdot\nabla)w)(t,x) - \nabla\cdot\sigma(w,q)(t,x) = F_0(t,x) + v^* 1_{\mathcal{O}_2} & x \in \widehat{\Omega}_F(t), \\ \nabla\cdot w(t,x) = 0 & x \in \widehat{\Omega}_F(t), \\ w(t,x) = 0 & x \in \partial\Omega, \\ w(t,x) = \dot{c}(t) + s(t) \times (x - \hat{b}(t)) + F_1(t) & x \in \partial\widehat{\Omega}_S(t), \\ m\ddot{c}(t) = \displaystyle\int_{\partial\widehat{\Omega}_S(t)} (\sigma(w,q)n)(t,x)\, d\gamma + F_2(t), \\ (\hat{J}\dot{s})(t) = ((\hat{J}\hat{r}) \times s)(t) + \displaystyle\int_{\partial\widehat{\Omega}_S(t)} (x - \hat{b}(t)) \times (\sigma(w,q)n)(t,x)\, d\gamma + F_3(t), \\ w_{|t=0} = 0 \text{ in } \Omega_F(0),\ c(0) = 0,\ \dot{c}(0) = 0,\ s(0) = 0 \end{cases} \quad (43)$$

where $F_0 := -\eta_{0,t}\bar{u} \in L^2(H^2)$, $F_1 = \eta_{0,t}\bar{b}$, $F_2 := -m(\eta_{0,tt}\bar{b} + 2\eta_{0,t}\dot{\bar{b}}) \in H^1(0,T)$ and $F_3 := -\eta_{0,t}\hat{J}\bar{r} \in H^1(0,T)$. Thanks to (42), we have that

$$\|F_0\|_{L^2(H^2)} + \|F_1\|_{H^1(0,T)} + \|F_2\|_{H^1(0,T)} + \|F_3\|_{H^1(0,T)} \leqslant \widehat{C}(\|u_0\|_{H^2(\Omega_F(0))} + |b_0| + |b_1| + |r_0|). \quad (44)$$

Using this estimate and Proposition 7, we obtain

$$\|w\|_{L^2(H^2)} + \|w\|_{H^1(L^2)} + \|q\|_{L^2(H^1)} + \|c\|_{H^2(0,T)} + \|s\|_{H^1(0,T)} \leqslant \widehat{C}(\|u_0\|_{H^2(\Omega_F(0))} + |b_0| + |b_1| + |r_0|). \quad (45)$$

We consider $\mathcal{O}_3$ and $\mathcal{O}_4$ two open sets such that

$$\mathcal{O}_2 \subset\subset \mathcal{O}_3 \subset\subset \mathcal{O}_4 \subset\subset \widetilde{\mathcal{O}}.$$

Let $\theta \in C_c^2(\mathcal{O}_4)$ be a function satisfying $\theta(x) = 1$ for every $x \in \mathcal{O}_3$. We introduce the variables

$$(\tilde{w}, \tilde{q}, \tilde{c}, \tilde{s}) := ((1-\theta)w, (1-\theta)q, c, s),$$

which satisfy the four first identities of (40) and fulfill the following system:

$$\begin{cases} \tilde{w}_t(t,x) + (\hat{u}\cdot\nabla)\tilde{w})(t,x) - \mu\Delta\tilde{w}(t,x) + \nabla\tilde{q}(t,x) = F_0(t,x) + G_0(t,x), & x \in \widehat{\Omega}_F(t), \\ \nabla\cdot\tilde{w}(t,x) = -\nabla\theta\cdot w & x \in \widehat{\Omega}_F(t), \\ \tilde{w}(t,x) = 0 & x \in \partial\Omega, \\ \tilde{w}(t,x) = \dot{\tilde{c}}(t) + \tilde{s}(t)\times(x-\hat{b}(t)) & x \in \partial\widehat{\Omega}_S(t), \\ m\ddot{\tilde{c}}(t) = \displaystyle\int_{\partial\widehat{\Omega}_S(t)} (\sigma(\tilde{w},\tilde{q})n)(t,x)\,d\gamma + F_2(t), \\ (\hat{J}\dot{\tilde{s}})(t) = ((\hat{J}\hat{r})\times\tilde{s})(t) + \displaystyle\int_{\partial\widehat{\Omega}_S(t)} (x-\hat{b}(t))\times(\sigma(\tilde{w},\tilde{q})n)(t,x)\,d\gamma + F_3(t), \\ \tilde{w}_{|t=0} = 0 \text{ in } \Omega_F(0),\ \tilde{c}(0)=0,\ \dot{\tilde{c}}(0)=0,\ \tilde{s}(0)=0, \end{cases} \tag{46}$$

with

$$G_0 := -\theta F_0 - (\hat{u}\cdot\nabla\theta)w + \mu(2(\nabla\theta\cdot\nabla)w + \Delta\theta w) - q\nabla\theta.$$

Here, we have used that $(1-\theta)v^* 1_{\mathcal{O}_2} \equiv 0$. Using (12), the properties of θ, (44) and (45), we have that

$$\mathrm{Supp}(G_0) \subset \mathcal{O}_4,\ \|G_0\|_{L^2(H^1)} \leqslant \widehat{C}(\|u_0\|_{H^2(\Omega_F(0))} + |b_0| + |b_1| + |r_0|). \tag{47}$$

Let us now lift the divergence condition. This divergence condition satisfies

$$\mathrm{Supp}(\nabla\theta\cdot w) \subset\subset \mathcal{O}_4,\ \int_{\mathcal{O}_4} \nabla\theta\cdot w\,dx = 0,\ \nabla\theta\cdot w \in L^2(H^2)\cap H^1(L^2).$$

Using [2] (Theorem 2.4, page 72 with $m = r = 2$), there exists a lifting $U \in H^1(H_0^1(\mathcal{O}_4)) \cap L^2(H_0^3(\mathcal{O}_4))$ satisfying

$$\nabla\cdot U = \nabla\theta\cdot w \text{ in } \mathcal{O}_4,\ \|U\|_{H^1(H^1)} + \|U\|_{L^2(H^3)} \leqslant \widehat{C}(\|w\|_{H^1(L^2)} + \|w\|_{L^2(H^2)}). \tag{48}$$

Moreover, since $w_{|t=0} = w_{t=T} = 0$ in $\mathcal{O}_4$, we have that $U_{|t=0} = U_{|t=T} = 0$ in $\mathcal{O}_4$. Let us still call U its extension by zero to Ω. We consider now the system satisfied by $(W := \tilde{w} - U, \tilde{q}, \tilde{c}, \tilde{s})$:

$$\begin{cases} W_t(t,x) + (\hat{u}\cdot\nabla)W)(t,x) - \nabla\cdot\sigma(W,\tilde{q})(t,x) = F_0(t,x) + G_1(t,x), & x \in \widehat{\Omega}_F(t), \\ \nabla\cdot W(t,x) = 0 & x \in \widehat{\Omega}_F(t), \\ W(t,x) = 0 & x \in \partial\Omega, \\ W(t,x) = \dot{\tilde{c}}(t) + \tilde{s}(t)\times(x-\hat{b}(t)) & x \in \partial\widehat{\Omega}_S(t), \\ m\ddot{\tilde{c}}(t) = \displaystyle\int_{\partial\widehat{\Omega}_S(t)} (\sigma(W,\tilde{q})n)(t,x)\,d\gamma + F_2(t), \\ (\hat{J}\dot{\tilde{s}})(t) = ((\hat{J}\hat{r})\times\tilde{s})(t) + \displaystyle\int_{\partial\widehat{\Omega}_S(t)} (x-\hat{b}(t))\times(\sigma(W,\tilde{q})n)(t,x)\,d\gamma + F_3(t), \\ W_{|t=0} = 0 \text{ in } \Omega_F(0),\ \tilde{c}(0)=0,\ \dot{\tilde{c}}(0)=0,\ \tilde{s}(0)=0, \end{cases} \tag{49}$$

with

$$G_1 := G_0 - U_t - (\hat{u}\cdot\nabla)U + \mu\Delta U.$$

From the definition of θ, (47), (48) and the fact that $\hat{u} \in \widehat{Z}$, it is clear that

$$\mathrm{Supp}(G_1) \subset \mathcal{O}_4,\ \|G_1\|_{L^2(H^1)} \leqslant \widehat{C}(\|u_0\|_{H^2(\Omega_F(0))} + |b_0| + |b_1| + |r_0|).$$

Consequently, $G_1 = \zeta G_1$ and $v := G_1$ satisfies (41). Finally, $(u,p,b,r) := (W + \eta_0\bar{u}, \tilde{q} + \eta_0\bar{p}, \tilde{c} + \eta_0\bar{b}, \tilde{s} + \eta_0\bar{r})$ with the control force v solves system (13) and, since $r = r^*$, $Q(T) = Q^*(T) = Id$, (40) holds.

4 Local null controllability

To prove Theorem 1, we perform a fixed-point argument for a multivalued map (see [22], Theorem 9.B, page 452):

Theorem 6 *Assume that the multivalued map $\Lambda : K \to 2^K$ satisfies:*

- *Λ is upper semi-continuous.*
- *K is a nonempty, compact, convex set in a locally convex space X.*
- *The set $\Lambda(x)$ is nonempty, closed and convex for all $x \in K$.*

Then, Λ has a fixed-point.

We are going to apply this theorem in the fixed domain $\Omega_F(0)$. More precisely, let

$$\begin{aligned} K := \{(z,b,r) \in (L^2(0,T;W^{2,6}(\Omega_F(0))) \cap H^1(0,T;L^6(\Omega_F(0)))) \times H^2(0,T) \times H^1(0,T) \\ \text{such that } \nabla\cdot z = 0 \text{ in } \Omega_F(0),\ z = 0 \text{ on } \partial\Omega_F(0) \quad \text{and} \\ \|z\|_{L^2(0,T;W^{2,6}(\Omega_F(0)))} + \|z\|_{H^1(0,T;L^6(\Omega_F(0)))} + \|b\|_{H^2(0,T)} + \|r\|_{H^1(0,T)} \leqslant R\} \end{aligned} \tag{50}$$

for some small $R > 0$ and

$$X := L^2(0,T;H^1(\Omega_F(0))) \times C^1([0,T]) \times C^0([0,T]).$$

In order to define Λ, we consider $(\hat{z},\hat{b},\hat{r}) \in K$. We define the associated flow in the solid domain:

$$\hat{\chi}(t,y) = \hat{b}(t) + \hat{Q}(t)Q_0^{-1}(y-b_0)\ \forall y \in \Omega_S(0). \tag{51}$$

Then, the solid domain is given by $\widehat{\Omega}_S(t) := \hat{\chi}(t,\Omega_S(0))$ for each $t > 0$. Observe that condition (11) is satisfied for R small enough. Next, we define the eulerian velocity $\hat{u}_S \in H^1(H^3)$ as the solution, together with $\hat{q}_S$, of

$$\begin{cases} -\mu\Delta\hat{u}_S + \nabla\hat{q}_S = 0 & \text{in } \widehat{\Omega}_F(t), \\ \nabla\cdot\hat{u}_S = 0 & \text{in } \widehat{\Omega}_F(t), \\ \hat{u}_S(t,x) = \dot{\hat{b}}(t) + \hat{r}(t)\times(x-\hat{b}(t)) & \text{on } \partial\widehat{\Omega}_S(t), \\ \hat{u}_S = 0 & \text{on } \partial\Omega. \end{cases} \tag{52}$$

It satisfies

$$\|\hat{u}_S\|_{H^1(H^3)} + \|\hat{q}_S\|_{H^1(H^2)} \leqslant C(\|\hat{b}\|_{H^2(0,T)} + \|\hat{r}\|_{H^1(0,T)}) \tag{53}$$

for some $C > 0$.

Now, we extend the flow $\hat{\chi}$ to the fluid domain:

$$\begin{cases} \dfrac{\partial\hat{\chi}(t,y)}{\partial t} = (\hat{u}_S \circ \hat{\chi})(t,y) \quad \forall y \in \Omega_F(0), \\ \hat{\chi}(0,y) = y \quad \forall y \in \Omega_F(0). \end{cases} \tag{54}$$

This flow satisfies

$$\|\hat{\chi} - id\|_{H^2(H^3)} \leqslant C(\|\hat{b}\|_{H^2(0,T)} + \|\hat{r}\|_{H^1(0,T)}) \leqslant CR, \tag{55}$$

for some $C > 0$.

Next, we consider $\hat{u} \in \hat{Z}$ defined by

$$\hat{u}(t,x) := \hat{u}_S(t,x) + (\nabla\hat{\chi}(t,\hat{\chi}^{-1}(t,x)))\,\hat{z}(t,\hat{\chi}^{-1}(t,x))\ \forall x \in \widehat{\Omega}_F(t).$$

This vector field satisfies $\nabla\cdot\hat{u} = 0$ in $\widehat{\Omega}_F(t)$, $\hat{u} = 0$ on $\partial\Omega$. Moreover, there exists $C > 0$ such that

$$\|\hat{u}\|_{\hat{Z}} \leqslant C(\|\hat{z}\|_{L^2(0,T;W^{2,6}(\Omega_F(0)))\cap H^1(0,T;L^6(\Omega_F(0)))} + \|\hat{b}\|_{H^2(0,T)} + \|\hat{r}\|_{H^1(0,T)}).$$

This velocity vector field being given, according to Proposition 5, we can construct a control $v \in L^2(0,T;H^1(\Omega))$ and a solution (u,p,b,r) of system (13) which satisfy (40) and (41). From Proposition 7 (with $g_0 = g_2 = g_3 = 0$ and $g_1 = v\zeta(x)$), we have that $(u,p,b,r) \in \hat{Y}_0 \times L^2(H^1) \times H^2(0,T) \times H^1(0,T)$ and

$$\|(u,p,b,r)\|_{\hat{Y}_0 \times L^2(H^1)\times H^2(0,T)\times H^1(0,T)} \leqslant \widehat{C}(\|v\|_{L^2((0,T)\times\Omega)} + \|u_0\|_{H^1(\Omega_F(0))} + |b_0| + |b_1| + |r_0|). \tag{56}$$

Let (u_S, q_S) be defined by (52) with the boundary condition on $\partial\widehat{\Omega}_S(t)$ replaced by $\dot{b}(t) + r(t) \times (x - \hat{b}(t))$. Then (u_S, q_S) satisfies (53) with $(\hat{b}, \hat{r})$ replaced by (b, r) and $(u - u_S, p - q_S)$ is the solution of the following system:

$$\begin{cases} (u-u_S)_t(t,x) - \nabla\cdot\sigma(u-u_S, p-q_S)(t,x) = v\zeta(x) - (\hat{u}\cdot\nabla)u(t,x) - u_{S,t}(t,x) & x \in \widehat{\Omega}_F(t), \\ \nabla\cdot(u-u_S)(t,x) = 0 & x \in \widehat{\Omega}_F(t), \\ (u-u_S)(t,x) = 0 & x \in \partial\widehat{\Omega}_F(t), \\ (u-u_S)(0,x) = u_0(x) - u_S(0,x) & x \in \Omega_F(0). \end{cases} \tag{57}$$

Since $v \in L^2(0,T;H^1(\Omega))$, $\hat{u} \in \hat{Z}$, $u \in \hat{Y}_0$ and u_S satisfies (53), we have that the right-hand side of this system belongs to $L^2(L^6)$.

Finally, we define

$$z(t,y) := (\nabla\hat{\chi})^{-1}(t,y)(u-u_S)(t,\hat{\chi}(t,y)) \quad \forall y \in \Omega_F(0)$$

and $h(t,y) := (p-q_S)(t,\hat{\chi}(t,y))$, $\forall y \in \Omega_F(0)$. We notice that (z,h) satisfies

$$\begin{cases} z_t - \nabla\cdot\sigma(z,h) = F & \text{in } (0,T)\times\Omega_F(0), \\ \nabla\cdot z = 0 & \text{in } (0,T)\times\Omega_F(0), \\ z = 0 & \text{on } (0,T)\times\partial\Omega_F(0), \\ z(0,x) = u_0(x) - u_S(0,x) & \text{in } \Omega_F(0), \end{cases} \tag{58}$$

where

$$\|F\|_{L^2(0,T;L^6(\Omega_F(0)))} \leqslant C(\|v\|_{L^2(0,T;H^1(\Omega))} + \|\nabla\hat{\chi} - Id\|_{C^0([0,T]\times\overline{\Omega_F(0)})}(\|z\|_{K_1} + \|\nabla h\|_{L^2(0,T;L^6(\Omega_F(0)))}))$$
$$+\widehat{C}(\|u\|_{\hat{Y}_0} + \|p\|_{L^2(H^1)} + \|b\|_{H^2(0,T)} + \|r\|_{H^1(0,T)}).$$

Here, K_1 stands for the first component of the space K, which was defined in (50). Now, we decompose $F = F_1 + \nabla F_2$ with $F_1 \in L^2(0,T;L^6(\Omega_F(0)))$ satisfying $\nabla\cdot F_1 = 0$ in $(0,T)\times\Omega_F(0)$, $F_1\cdot n = 0$ on $(0,T)\times\partial\Omega_F(0)$, $F_2 \in L^2(0,T;W^{1,6}(\Omega_F(0)))$ and

$$\|F_1\|_{L^2(0,T;L^6(\Omega_F(0)))} + \|\nabla F_2\|_{L^2(0,T;L^6(\Omega_F(0)))} \leqslant C\|F\|_{L^2(0,T;L^6(\Omega_F(0)))}.$$

Then, we apply Theorem 2.8 in [11] to $(z, h - F_2)$ with right-hand side F_1 and we obtain that

$$\|(z,\nabla h)\|_{K_1\times L^2(0,T;L^6(\Omega_F(0)))} \leqslant C(\|F\|_{L^2(0,T;L^6(\Omega_F(0)))} + \|u_0 - u_S(0,\cdot)\|_{H^2(\Omega_F(0))}).$$

Using now that $(\hat{u}, \hat{b}, \hat{r})$ belongs to K, (55) and (56), we deduce that

$$\|(z,\nabla h)\|_{K_1\times L^2(0,T;L^6(\Omega_F(0)))} \leqslant \widehat{C}R\|(z,\nabla h)\|_{K_1\times L^2(0,T;L^6(\Omega_F(0)))}$$
$$+ \widehat{C}(\|v\|_{L^2(0,T;H^1(\Omega))} + \|u_0\|_{H^2(\Omega_F(0))} + |b_0| + |b_1| + |r_0|).$$

Thanks to (41) and (56), we obtain

$$\|(z,b,r)\|_K \leqslant C(\|u_0\|_{H^2(\Omega_F(0))} + |b_0| + |b_1| + |r_0|). \tag{59}$$

With all these ingredients, we define

$$\Lambda(\hat{z}, \hat{b}, \hat{r}) = \{(z,b,r) \in K : (u,p,b,r) \text{ satisfies (13) for some } p \text{ and } v, \text{ (40) and (41)}\}.$$

• We directly have that $\Lambda : K \to 2^K$ from (59) and taking δ in (9) sufficiently small.

• Let us now prove that Λ is upper semi-continuous. For this, let $A \subset K$ be a closed subset. We have to prove that $\Lambda^{-1}(A)$ is also closed.

Let $(\hat{z}_n, \hat{b}_n, \hat{r}_n) \subset \Lambda^{-1}(A)$ such that $(\hat{z}_n, \hat{b}_n, \hat{r}_n) \to (\hat{z}, \hat{b}, \hat{r})$ in X. We intend to prove that $(\hat{z}, \hat{b}, \hat{r}) \in \Lambda^{-1}(A)$, that is to say, that there exists $(z, b, r) \in A$ such that $(z, b, r) \in \Lambda(\hat{z}, \hat{b}, \hat{r})$. Let $(z_n, b_n, r_n) \in \Lambda(\hat{z}_n, \hat{b}_n, \hat{r}_n) \subset A$. From the definition of K, we have that there exists a subsequence $(z_{\psi(n)}, b_{\psi(n)}, r_{\psi(n)})$ such that

$$(z_{\psi(n)}, b_{\psi(n)}, r_{\psi(n)}) \rightharpoonup (z, b, r) \text{ in } K \quad \text{and} \quad (z_{\psi(n)}, b_{\psi(n)}, r_{\psi(n)}) \to (z, b, r) \text{ in } X. \tag{60}$$

Since A is closed, we have that $(z, b, r) \in A$. It remains to prove that $(z, b, r) \in \Lambda(\hat{z}, \hat{b}, \hat{r})$.

First, we observe that $\hat{\chi}_{\psi(n)} \to \hat{b} + \hat{Q}Q_0^{-1}(y - b_0)$ in $C^1([0,T]; H^3(\Omega_S(0)))$ (see (51)). Let us prove that

$$\hat{\chi}_{\psi(n)} \to \hat{\chi} \text{ in } C^1([0,T]; H^3(\Omega_F(0))). \tag{61}$$

For this, we consider the Stokes system fulfilled by

$$(\hat{u}_{S,\psi(n)} \circ \hat{\chi}_{\psi(n)}, \hat{q}_{S,\psi(n)} \circ \hat{\chi}_{\psi(n)}) - (\hat{u}_S \circ \hat{\chi}, \hat{q}_S \circ \hat{\chi}). \tag{62}$$

Since $(z_{\psi(n)}, b_{\psi(n)}, r_{\psi(n)})$ belongs to K, $\hat{u}_{S,\psi(n)}$ satisfies (53) and $\hat{\chi}_{\psi(n)}$ satisfies (55), one can see that the $H^1(\Omega_F(0))$-norm of the right-hand side and the $H^2(\Omega_F(0))$-norm of the divergence condition of this system can be estimated by

$$CR(\|\hat{u}_{S,\psi(n)} \circ \hat{\chi}_{\psi(n)} - \hat{u}_S \circ \hat{\chi}\|_{H^3(\Omega_F(0))} + \|\hat{q}_{S,\psi(n)} \circ \hat{\chi}_{\psi(n)} - \hat{q}_S \circ \hat{\chi}\|_{H^2(\Omega_F(0))} + \|\hat{\chi}_{\psi(n)} - \hat{\chi}\|_{H^3(\Omega_F(0))}).$$

As long as the boundary term is concerned, we have that

$$\hat{u}_{S,\psi(n)} \circ \hat{\chi}_{\psi(n)} - \hat{u}_S \circ \hat{\chi} = \dot{\hat{b}}_{\psi(n)} - \dot{\hat{b}} + (\hat{r}_{\psi(n)} - \hat{r}) \times (\hat{Q}(y - b_0)) + \hat{r}_{\psi(n)} \times (\hat{Q}_{\psi(n)} - \hat{Q})(y - b_0), \tag{63}$$

which tends to zero strongly in $C^0([0,T]; H^{5/2}(\partial\Omega_S(0)))$. Consequently, thanks to (54), we obtain

$$\hat{u}_{S,\psi(n)} \circ \hat{\chi}_{\psi(n)} - \hat{u}_S \circ \hat{\chi} \to 0 \quad \text{in } C^0([0,T]; H^3(\Omega_F(0)))$$

and (61). Taking a look again at the Stokes system satisfied by (62), we see that the $H^1(0,T; H^1(\Omega_F(0)))$-norm of the right-hand side and the $H^1(0,T; H^2(\Omega_F(0)))$-norm of the divergence are estimated by

$$CR(\|\hat{u}_{S,\psi(n)} \circ \hat{\chi}_{\psi(n)} - \hat{u}_S \circ \hat{\chi}\|_{H^1(0,T;H^3(\Omega_F(0)))} + \|\hat{q}_{S,\psi(n)} \circ \hat{\chi}_{\psi(n)} - \hat{q}_S \circ \hat{\chi}\|_{H^1(0,T;H^2(\Omega_F(0)))}$$
$$+ \|\hat{\chi}_{\psi(n)} - \hat{\chi}\|_{H^1(0,T;H^3(\Omega_F(0)))}).$$

For the boundary term (63), we deduce that its $H^1(0,T; H^{5/2}(\partial\Omega_S(0)))$-norm is bounded independently of n. As a consequence, up to a subsequence, we obtain

$$\hat{u}_{S,\psi(n)} \circ \hat{\chi}_{\psi(n)} - \hat{u}_S \circ \hat{\chi} \rightharpoonup 0 \quad \text{in } H^1(0,T; H^3(\Omega_F(0))).$$

In the same way, one can prove that

$$u_{S,\psi(n)} \circ \hat{\chi}_{\psi(n)} - u_S \circ \hat{\chi} \to 0 \quad \text{in } C^0([0,T]; H^3(\Omega_F(0))), \; u_{S,\psi(n)} \circ \hat{\chi}_{\psi(n)} - u_S \circ \hat{\chi} \rightharpoonup 0 \quad \text{in } H^1(0,T; H^3(\Omega_F(0))). \tag{64}$$

We recall the definition of $u_{\psi(n)}$:

$$u_{\psi(n)} \circ \hat{\chi}_{\psi(n)} = u_{S,\psi(n)} \circ \hat{\chi}_{\psi(n)} + (\nabla\hat{\chi}_{\psi(n)}) z_{\psi(n)} \text{ in } \Omega_F(0).$$

Thanks to (60), (61) and (64), one can pass to the limit in the system satisfied by

$$(u_{\psi(n)} \circ \hat{\chi}_{\psi(n)}, p_{\psi(n)} \circ \hat{\chi}_{\psi(n)}, b_{\psi(n)}, r_{\psi(n)})$$

and we deduce that (u, p, b, r) satisfies system (13).

• For each $(\hat{z}, \hat{b}, \hat{r}) \in K$, $\Lambda(\hat{z}, \hat{b}, \hat{r})$ is closed in X. Indeed, let $(z_n, b_n, r_n) \in \Lambda(\hat{z}, \hat{b}, \hat{r})$ be such that

$$(z_n, b_n, r_n) \to (z, b, r) \text{ in } X.$$

Then, arguing as in the previous paragraph, one can show that $(z, b, r) \in \Lambda(\hat{z}, \hat{b}, \hat{r})$. In fact, the same convergences can be proved in a simpler way since the domains do not depend on n.

Thus, we can apply Theorem 6 and obtain the existence of a fixed point to Λ. This concludes the proof of Theorem 1.

Appendix

In this Appendix, we will establish some regularity results for a fluid-structure system similar to (13):

$$
\begin{cases}
w_t(t,x) + (\hat{u}\cdot\nabla)w(t,x) - \nabla\cdot\sigma(w,q)(t,x) = g_1(t,x) & x\in\widehat{\Omega}_F(t),\\
\nabla\cdot w(t,x) = 0 & x\in\widehat{\Omega}_F(t),\\
w(t,x) = 0 & x\in\partial\Omega,\\
w(t,x) = \dot{c}(t) + s(t)\times(x-\hat{b}(t)) + g_0(t,x) & x\in\partial\widehat{\Omega}_S(t),\\
m\ddot{c}(t) = \displaystyle\int_{\partial\widehat{\Omega}_S(t)}(\sigma(w,q)n)(t,x)\,d\gamma + g_2(t),\\
(\hat{J}\dot{s})(t) = ((\hat{J}\hat{r})\times s)(t) + \displaystyle\int_{\partial\widehat{\Omega}_S(t)}(x-\hat{b}(t))\times(\sigma(w,q)n)(t,x)\,d\gamma + g_3(t),\\
w_{|t=0} = w_0 \text{ in } \Omega_F(0),\ c(0)=c_0,\ \dot{c}(0)=c_1,\ s(0)=s_0.
\end{cases}
\tag{65}
$$

Proposition 7 *Assume that (w_0,c_0,c_1,s_0) satisfies (8) and let $(\hat{u},\hat{b},\hat{r})$ satisfy (10) (with (b_0,b_1,r_0) replaced by (c_0,c_1,s_0)) and (11)-(12). Moreover, let us suppose that $g_0\in L^2(H^2)$, the trace of g_0 belongs to $H^1(0,T;L^2(\partial\widehat{\Omega}_S(t)))$, $g_1\in L^2(L^2)$, $g_2\in L^2(0,T)$ and $g_3\in L^2(0,T)$. Then, there exists $\widehat{C}$ (depending on Ω,δ_0 and $\|\hat{u}\|_{\hat{Z}},\|\hat{b}\|_{H^2(0,T)},\|\hat{r}\|_{H^1(0,T)}$) such that the solution of (65) satisfies*

$$(w,q,c,s)\in\hat{Y}_0\times L^2(H^1)\times H^2(0,T)\times H^1(0,T)$$

and

$$
\begin{aligned}
&\|(w,q,c,s)\|_{\hat{Y}_0\times L^2(H^1)\times H^2(0,T)\times H^1(0,T)} \leqslant \widehat{C}(\|g_0\|_{L^2(H^2)} + \|g_0\|_{H^1(0,T;L^2(\partial\widehat{\Omega}_S(t)))} + \|g_1\|_{L^2(L^2)}\\
&+\|g_2\|_{L^2(0,T)} + \|g_3\|_{L^2(0,T)} + \|w_0\|_{H^1(\Omega_F(0))} + |c_0| + |c_1| + |s_0|).
\end{aligned}
\tag{66}
$$

Proof: First, we prove that $w\in L^2(H^1)\cap C^0(L^2)$ together with $c\in W^{1,\infty}(0,T)$, $s\in L^\infty(0,T)$. Then, we will prove $(w,q,c,s)\in\hat{Y}_0\times L^2(H^1)\times H^2(0,T)\times H^1(0,T)$.

First Step: We multiply the equation of w in (65) by w and we integrate in $\widehat{\Omega}_F(t)$. After an integration by parts and using the equations of the solid, this yields:

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\widehat{\Omega}_F(t)}|w|^2dx + \mu\int_{\widehat{\Omega}_F(t)}|\nabla w|^2dx + \frac{m}{2}\frac{d}{dt}|\dot{c}|^2 + \frac{1}{2}\frac{d}{dt}(\hat{J}s\cdot s)\\
&= \int_{\widehat{\Omega}_F(t)} w\,g_1\,dx + \dot{c}\cdot g_2 + s\cdot g_3 + \frac{1}{2}\dot{\hat{J}}s\cdot s - \int_{\partial\widehat{\Omega}_S(t)}(\sigma(w,q)n)\cdot g_0\,d\gamma.
\end{aligned}
$$

We integrate in t, we use that $\hat{J}s\cdot s\geqslant C|s|^2$ for some $C>0$ and we obtain

$$
\begin{aligned}
&\|w\|_{L^2(H^1)} + \|w\|_{L^\infty(L^2)} + \|c\|_{W^{1,\infty}(0,T)} + \|s\|_{L^\infty(0,T)} \leqslant \widehat{C}_\varepsilon\big(\|g_0\|_{L^2(H^2)} + \|g_1\|_{L^1(L^2)} + \|g_2\|_{L^1(0,T)}\\
&+\|g_3\|_{L^1(0,T)} + \|w_0\|_{L^2(\Omega_F(0))} + |c_0| + |c_1| + |s_0|\big) + \varepsilon\|(w,q)\|_{\hat{Y}_0\times L^2(H^1)},
\end{aligned}
\tag{67}
$$

for any $\varepsilon>0$.

Second Step: We multiply the equation of w by w_t and we integrate in $\widehat{\Omega}_F(t)$. After some computations, we obtain that:

$$
\begin{aligned}
&\int_{\widehat{\Omega}_F(t)}|w_t|^2dx + \frac{1}{2}\frac{d}{dt}\int_{\widehat{\Omega}_F(t)}|\nabla w|^2dx + m|\ddot{c}|^2 + \hat{J}\dot{s}\cdot\dot{s} = -\int_{\widehat{\Omega}_F(t)} w_t\cdot(\hat{u}\cdot\nabla)w\,dx + \frac{1}{2}\int_{\widehat{\Omega}_F(t)}\hat{u}\cdot\nabla|\nabla w|^2dx\\
&+\int_{\widehat{\Omega}_F(t)} w_t\cdot g_1\,dx - \int_{\partial\widehat{\Omega}_S(t)}[s\times(\hat{r}\times(x-\hat{b})) - \nabla w(\dot{\hat{b}} + \hat{r}\times(x-\hat{b}))]\sigma(w,q)n\,d\gamma + \dot{s}\cdot(\hat{J}\hat{r}\times s + g_3) + \ddot{c}g_2.\\
&-\int_{\partial\widehat{\Omega}_S(t)}(\sigma(w,q)n)\cdot(g_{0,t} + (\hat{u}\cdot\nabla)g_0)\,d\gamma.
\end{aligned}
$$

Using the continuity of the trace operator, we obtain

$$\begin{aligned}&\int_{\widehat{\Omega}_F(t)} |w_t|^2 dx + \frac{d}{dt}\int_{\widehat{\Omega}_F(t)} |\nabla w|^2 dx + |\ddot{c}|^2 + |\dot{s}|^2 \leqslant \varepsilon(\|w\|^2_{H^2(\widehat{\Omega}_F(t))} + \|q\|^2_{H^1(\widehat{\Omega}_F(t))})\\ &+\widehat{C}_\varepsilon\left(\int_{\widehat{\Omega}_F(t)} |\nabla w|^2\, dx + \int_{\partial\widehat{\Omega}_S(t)} |g_{0,t}|^2\, d\gamma + \|g_0\|^2_{H^2(\widehat{\Omega}_F(t))} + \int_{\widehat{\Omega}_F(t)} |g_1|^2\, dx + |g_2|^2 + |g_3|^2 + |s|^2\right),\end{aligned} \tag{68}$$

for $\varepsilon > 0$ small enough. Now, we regard the equation of w as a stationary system:

$$\begin{cases} -\nabla\cdot\sigma(w,q)(t,x) = g_1(t,x) - w_t(t,x) - (\hat{u}\cdot\nabla)w(t,x) & x \in \widehat{\Omega}_F(t),\\ \nabla\cdot w(t,x) = 0 & x \in \widehat{\Omega}_F(t),\\ w(t,x) = 0 & x\in\partial\Omega,\\ w(t,x) = \dot{c}(t) + s(t)\times(x-\hat{b}(t)) + g_0(t,x) & x\in\partial\widehat{\Omega}_S(t).\end{cases}$$

We can show that for a. e. $t\in(0,T)$, we have

$$\|w\|_{H^2(\widehat{\Omega}_F(t))}+\|q\|_{H^1(\widehat{\Omega}_F(t))} \leqslant \widehat{C}(\|g_0\|_{H^2(\widehat{\Omega}_F(t))}+\|g_1\|_{L^2(\widehat{\Omega}_F(t))}+\|w_t\|_{L^2(\widehat{\Omega}_F(t))}+\|\nabla w\|_{L^2(\widehat{\Omega}_F(t))}+|\dot{c}|+|s|). \tag{69}$$

Indeed, let $\hat{\chi}_e \in C^1([0,T];C^2(\overline{\Omega}))$ such that

$$\begin{cases} \hat{\chi}_e(t,y) = \hat{b}(t) + \hat{Q}(t)Q_0^{-1}(y-b_0) \quad \forall y\in\Omega_S(0),\\ \hat{\chi}_e(t,y) = y \quad \forall y \in\partial\Omega,\\ \exists\hat{\chi}_e^{-1}\in C^1([0,T];C^2(\overline{\Omega}))/\hat{\chi}_e(t,\hat{\chi}_e^{-1}(t,x)) = x,\ \forall t\in(0,T),\ \forall x\in\Omega,\\ \|\hat{\chi}_e - id\|_{C^1([0,T];C^2(\overline{\Omega}))} \leqslant C(\|\hat{b}\|_{W^{1,\infty}(0,T)} + \|\hat{r}\|_{L^\infty(0,T)}).\end{cases}$$

Then, the variable $(w\circ\hat{\chi}_e, q\circ\hat{\chi}_e)$ satisfies a stationary Stokes system in $\Omega_F(0)$. Here, we can apply classical estimates for the Stokes operator (see, for instance, [21]). For the right-hand side of the Stokes problem, we take into account that the terms of the form

$$(\nabla\hat{\chi}_e - Id)(D^2 w\circ\hat{\chi}_e + \nabla q\circ\hat{\chi}_e),$$

can be estimated in L^2 by $\varepsilon(\|w\circ\hat{\chi}_e\|_{H^2(\Omega_F(0))} + \|q\circ\hat{\chi}_e\|_{H^1(\Omega_F(0))})$ in $(0,T_0)\times\Omega_F(0)$ provided that T_0 is chosen small enough in terms of $\|\hat{b}\|_{W^{1,\infty}(0,T)} + \|\hat{r}\|_{L^\infty(0,T)}$.

On the other hand, the divergence condition equals $(\nabla w\circ\hat{\chi}_e(\nabla\hat{\chi}_e - Id))$, which is estimated in H^1 by $\varepsilon\|w\circ\hat{\chi}_e\|_{H^2(\Omega_F(0))}$. Repeating this process $[T/T_0]+1$ times allows to establish (69).

Finally, combining (69) with (67)-(68) and applying Gronwall's Lemma, we obtain the desired estimate (66).

Let us now establish the existence of more regular solutions when $g_0\equiv 0$. In order to do this, we suppose that $w_0\in H^\varsigma(\Omega_F(0))$ for $\varsigma > 5/2$ and we define some new functions. Let us note $J_0 = J_{|t=0}$ and

$$q_1 := -(\hat{u}_{|t=0}\cdot\nabla)(c_1 + s_0\times(x-b_0))1_{\partial\Omega_S(0)} + \Delta w_0 + g_{1|t=0} \quad \text{on } \partial\Omega_F(0).$$

Then, we first define the triplet $(\tilde{c}_1, \tilde{s}_0, q_0)$ by

$$\tilde{c}_1 := \frac{1}{m}\int_{\partial\Omega_S(0)} \sigma(w_0,q_0)n\, d\gamma + \frac{1}{m}g_2(0),$$

$$\tilde{s}_0 := J_0^{-1}[(J_0 r_0)\times s_0 + \int_{\partial\Omega_S(0)} (x-b_0)\times\sigma(w_0,q_0)n\, d\gamma + g_3(0)].$$

and

$$\begin{cases} \Delta q_0 = -\nabla \cdot [(\hat{u}_{|t=0} \cdot \nabla) w_0] + \nabla \cdot g_{1|t=0} & \text{in } \Omega_F(0), \\ \dfrac{\partial q_0}{\partial n} = -(\tilde{c}_1 + \tilde{s}_0 \times (x - b_0)) \cdot n 1_{\partial \Omega_S(0)} + q_1 \cdot n & \text{on } \partial \Omega_F(0). \end{cases}$$

Using the fact that J_0 is positive definite, one can easily check that this system has a unique solution $(\tilde{c}_1, \tilde{s}_0, q_0)$ satisfying

$$|\tilde{c}_1| + |\tilde{s}_0| + \|q_0\|_{H^2(\Omega_F(0))} \leqslant \widehat{C}(|s_0| + |c_1| + \|w_0\|_{H^3(\Omega_F(0))} + \|g_1\|_{\hat{Y}_0} + \|g_2\|_{H^1(0,T)} + \|g_3\|_{H^1(0,T)}). \quad (70)$$

Finally,

$$\tilde{w}_0 := g_{1|t=0} + \nabla \cdot \sigma(w_0, q_0) - (\hat{u}_{|t=0} \cdot \nabla) w_0.$$

Let us introduce the following compatibility condition:

$$\tilde{w}_0(x) = (\tilde{c}_1 + \tilde{s}_0 \times (x - b_0) + [(c_1 + s_0 \times (x - b_0)) \cdot \nabla](c_1 + s_0 \times (x - b_0) - w_0(x)) 1_{\partial \Omega_S(0)}(x), \ x \in \partial \Omega_F(0). \quad (71)$$

Proposition 8 *Let $g_0 \equiv 0$, $g_1 \in \hat{Y}_0$ and $g_2, g_3 \in H^1(0,T)$. Assume that $w_0 \in H^3(\Omega_F(0))$, $(w_0, c_0, c_1, s_0, g_1, g_2, g_3)$ satisfy (8) and (71) and let $(\hat{u}, \hat{b}, \hat{r})$ satisfy (10) (with (b_0, b_1, r_0) replaced by (c_0, c_1, s_0)) and (11)-(12). Then, there exists $\widehat{C}$ (depending on Ω, δ_0 and $\|\hat{u}\|_{\hat{Z}}, \|\hat{b}\|_{H^2(0,T)}, \|\hat{r}\|_{H^1(0,T)}$) such that the solution of (65) satisfies*

$$(w, q, c, s) \in \hat{Y}_2 \times (L^2(H^3) \cap H^1(H^1)) \times H^3(0,T) \times H^2(0,T)$$

and

$$\begin{aligned} &\|(w, q, c, s)\|_{\hat{Y}_2 \times (L^2(H^3) \cap H^1(H^1)) \times H^3(0,T) \times H^2(0,T)} \\ &\leqslant \widehat{C}(\|g_1\|_{\hat{Y}_0} + \|g_2\|_{H^1(0,T)} + \|g_3\|_{H^1(0,T)} + \|w_0\|_{H^3(\Omega_F(0))} + |c_0| + |c_1| + |s_0|). \end{aligned} \quad (72)$$

Proof: Let us differentiate system (65) with respect to the time variable. This yields

$$\begin{cases} w_{tt}(t,x) + (\hat{u} \cdot \nabla) w_t(t,x) - \nabla \cdot \sigma(w_t, q_t)(t,x) = \tilde{g}_1(t,x) & x \in \widehat{\Omega}_F(t), \\ \nabla \cdot w_t(t,x) = 0 & x \in \widehat{\Omega}_F(t), \\ w_t(t,x) = 0 & x \in \partial \Omega, \\ w_t(t,x) = \ddot{c}(t) + \dot{s}(t) \times (x - \hat{b}(t)) + \tilde{g}_0(t,x) & x \in \partial \widehat{\Omega}_S(t), \\ m \dddot{c}(t) = \displaystyle\int_{\partial \widehat{\Omega}_S(t)} (\sigma(w_t, q_t) n)(t,x) \, d\gamma + \tilde{g}_2(t), \\ (\hat{J} \ddot{s})(t) = ((\hat{J} \hat{r}) \times \dot{s})(t) + \displaystyle\int_{\partial \widehat{\Omega}_S(t)} (x - \hat{b}(t)) \times (\sigma(w_t, q_t) n)(t,x) \, d\gamma + \tilde{g}_3(t), \\ w_{t|t=0} = \tilde{w}_0 \ \text{ in } \Omega_F(0), \, \dot{c}(0) = c_1, \, \ddot{c}(0) = \tilde{c}_1, \, \dot{s}(0) = \tilde{s}_0, \end{cases} \quad (73)$$

where

$$\tilde{g}_1 := g_{1,t} - (\hat{u}_t \cdot \nabla) w, \ \ \tilde{g}_0 := (\hat{u} \cdot \nabla)(\dot{c} + s \times (x - \hat{b}) - w),$$

$$\tilde{g}_2 := g_{2,t} + \int_{\partial \widehat{\Omega}_S(t)} (\hat{u} \cdot \nabla) \sigma(w,q) n \, d\gamma + \int_{\partial \widehat{\Omega}_S(t)} \sigma(w,q)(\hat{r} \times n) \, d\gamma,$$

$$\tilde{g}_3 := g_{3,t} - \dot{\hat{J}} \dot{s} - s \times \frac{d}{dt}(\hat{J} \hat{r}) + \int_{\partial \widehat{\Omega}_S(t)} (\hat{r} \times (x - \hat{b})) \times \sigma(w,q) n \, d\gamma$$

$$+ \int_{\partial \widehat{\Omega}_S(t)} (x - \hat{b}) \times (\hat{u} \cdot \nabla) \sigma(w,q) n \, d\gamma + \int_{\partial \widehat{\Omega}_S(t)} (x - \hat{b}) \times \sigma(w,q)(\hat{r} \times n) \, d\gamma.$$

Observe now that, thanks to (12) and (71), we have that

$$w_{t|t=0} = (\ddot{c}(0) + \dot{s}(0) \times (x - b_0) + \tilde{g}_{0|t=0}) 1_{\partial \Omega_S(0)} \quad \text{on } \partial \Omega_F(0).$$

This allows to apply estimate (66) to (73):

$$\|(w_t, q_t, \dot{c}, \dot{s})\|_{\hat{Y}_0\times L^2(H^1)\times H^2(0,T)\times H^1(0,T)} \leqslant \widehat{C}(\|\tilde{g}_0\|_{L^2(H^2)} + \|\tilde{g}_0\|_{H^1(0,T;L^2(\partial\widehat{\Omega}_S(t)))} + \|\tilde{g}_1\|_{L^2(L^2)} \\ +\|\tilde{g}_2\|_{L^2(0,T)} + \|\tilde{g}_3\|_{L^2(0,T)} + \|\tilde{w}_0\|_{H^1(\Omega_F(0))} + |c_1| + |\tilde{c}_1| + |\tilde{s}_0|). \tag{74}$$

Then, from classical estimates for the stationary Stokes system, we find

$$\|(w, q, c, s)\|_{\hat{Y}_2\times(L^2(H^3)\cap H^1(H^1))\times H^3(0,T)\times H^2(0,T)} \leqslant \widehat{C}(\|\tilde{g}_0\|_{L^2(H^2)} + \|\tilde{g}_0\|_{H^1(0,T;L^2(\partial\widehat{\Omega}_S(t)))} \\ +\|\tilde{g}_1\|_{L^2(L^2)} + \|\tilde{g}_2\|_{L^2(0,T)} + \|\tilde{g}_3\|_{L^2(0,T)} + \|\tilde{w}_0\|_{H^1(\Omega_F(0))} + |c_1| + |\tilde{c}_1| + |\tilde{s}_0|). \tag{75}$$

Let us now estimate $\tilde{g}_i$ $(0 \leqslant i \leqslant 3)$.

- Estimate of $\tilde{g}_0$.

First,

$$\|\tilde{g}_0\|_{L^2(H^2)} \leqslant C\|\hat{u}\|_{L^2(H^2)}(\|\dot{c}\|_{L^\infty(0,T)} + \|s\|_{L^\infty(0,T)}) + \|(\hat{u}\cdot\nabla)w\|_{L^2(H^2)}. \tag{76}$$

For the last term in this inequality, we have for $0 < \delta < 1/2$

$$\begin{aligned}\|(\hat{u}\cdot\nabla)w\|_{L^2(H^2)} &\leqslant C(\|\hat{u}\|_{L^2(H^2)}\|\nabla w\|_{C^0(C^0)} + \|\hat{u}\|_{C^0(H^1)}\|\nabla w\|_{L^2(W^{1,\infty})} + \|\hat{u}\|_{C^0(H^1)}\|\nabla w\|_{L^2(W^{2,3})}) \\ &\leqslant C\|\hat{u}\|_{\hat{Y}_0}(\|w\|_{C^0(H^{5/2+\delta})} + \|w\|_{L^2(H^{7/2+\delta})} + \|w\|_{L^2(H^{7/2})} \\ &\leqslant \varepsilon(\|w\|_{C^0(H^3)} + \|w\|_{L^2(H^4)}) + \widehat{C}_\varepsilon(\|w\|_{C^0(H^2)} + \|w\|_{L^2(H^2)}),\end{aligned} \tag{77}$$

for any $\varepsilon > 0$.

Then, we use that

$$\tilde{g}_{0,t}(t,x) = [(\hat{u}\cdot\nabla)(\dot{c} + s\times(x-\hat{b}) - w)]_t(t,x) \quad x \in \widehat{\Omega}_F(t).$$

Taking traces in this identity and using (12), we deduce for $0 < \delta < 1/2$

$$\|\tilde{g}_0\|_{H^1(0,T;L^2(\partial\widehat{\Omega}_S(t)))} \leqslant C\|\hat{u}\|_{H^1(0,T;L^2(\partial\widehat{\Omega}_S(t)))}(\|\dot{c}\|_{L^\infty(0,T)} + \|s\|_{L^\infty(0,T)} + \|w\|_{C^0([0,T];L^\infty(\partial\widehat{\Omega}_S(t)))}) \\ +C\|\hat{u}\|_{C^0([0,T];L^\infty(\partial\widehat{\Omega}_S(t)))}(\|\dot{c}\|_{H^1(0,T)} + \|s\|_{H^1(0,T)} + \|w\|_{H^1(H^{3/2+\delta})}) \\ \leqslant \widehat{C}_\varepsilon(\|c\|_{H^2(0,T)} + \|s\|_{H^1(0,T)} + \|w\|_{L^\infty(H^1)} + \|w\|_{H^1(L^2)}) + \varepsilon(\|w\|_{C^0(H^3)} + \|w\|_{H^1(H^2)}) \tag{78}$$

for any $\varepsilon > 0$.

- Estimate of $\tilde{g}_1$.

We have for $0 < \delta < 1/2$

$$\|(\hat{u}_t\cdot\nabla)w\|_{L^2(L^2)} \leqslant \|\hat{u}_t\|_{L^2(L^2)}\|\nabla w\|_{C^0(C^0)} \leqslant \widehat{C}\|w\|_{C^0(H^{5/2+\delta})} \leqslant \widehat{C}_\varepsilon\|w\|_{C^0(H^1)} + \varepsilon\|w\|_{C^0(H^3)}, \tag{79}$$

for any $\varepsilon > 0$.

- Estimate of $\tilde{g}_2$.

Using (12), we obtain

$$\|\tilde{g}_2\|_{L^2(0,T)} \leqslant (\|\hat{u}\|_{C^0([0,T];L^\infty(\partial\widehat{\Omega}_S(t)))} + \|\hat{r}\|_{L^\infty})(\|w\|_{L^2(H^{5/2+\delta})} + \|q\|_{L^2(H^{3/2+\delta})}) + \|g_2\|_{H^1(0,T)},$$

for any $0 < \delta < 1/2$. Thus,

$$\|\tilde{g}_2\|_{L^2(0,T)} \leqslant \widehat{C}_\varepsilon(\|w\|_{L^2(H^2)} + \|q\|_{L^2(H^1)} + \|g_2\|_{H^1(0,T)}) + \varepsilon(\|w\|_{L^2(H^4)} + \|q\|_{L^2(H^3)}), \tag{80}$$

for any $\varepsilon > 0$

- Estimate of $\tilde{g}_3$.

Analogously as for $\tilde{g}_2$, we easily obtain

$$\|\tilde{g}_3\|_{L^2(0,T)} \leqslant \widehat{C}_\varepsilon(\|w\|_{L^2(H^2)} + \|q\|_{L^2(H^1)} + \|g_3\|_{H^1(0,T)} + \|s\|_{H^1(0,T)}) + \varepsilon(\|w\|_{L^2(H^4)} + \|q\|_{L^2(H^3)}), \tag{81}$$

for any $\varepsilon > 0$.

Reassembling estimates (76)-(81), and combining with (75), we find

$$\|(w,q,c,s)\|_{\hat{Y}_2\times(L^2(H^3)\cap H^1(H^1))\times H^3(0,T)\times H^2(0,T)} \leqslant \widehat{C}_\varepsilon(\|w\|_{\hat{Y}_0} + \|q\|_{L^2(H^1)} + \|c\|_{H^2(0,T)} + \|s\|_{H^1(0,T)}$$
$$+\|g_1\|_{\hat{Y}_0} + \|g_2\|_{H^1(0,T)} + \|g_3\|_{H^1(0,T)} + \|w_0\|_{H^3(\Omega_F(0))} + |c_1| + |\tilde{c}_1| + |\tilde{s}_0|) + \varepsilon\|(w,q)\|_{\hat{Y}_2\times(L^2(H^3)\cap H^1(H^1))},$$

for any $\varepsilon > 0$. Applying Proposition 7 in order to estimate the four first terms and taking ε small enough, we obtain the desired inequality (72).

Corollary 9 *Let $k_0 \in [0,2]\setminus\{\frac{1}{2}\}$. Let $g_0 \equiv 0$, $g_1 \in L^2(H^{2-k_0}) \cap H^{1-k_0/2}(L^2)$ and g_2, $g_3 \in H^{1-k_0/2}(0,T)$. Assume that $w_0 \in H^{3-k_0}(\Omega_F(0))$, $(w_0, c_0, c_1, s_0, g_1, g_2, g_3)$ satisfy (8) and condition (71) if $k_0 < 1/2$. Furthermore, let $(\hat{u}, \hat{b}, \hat{r})$ satisfy (10) (with (b_0, b_1, r_0) replaced by (c_0, c_1, s_0)) and (11)-(12). Then, there exists $\widehat{C}$ (depending on Ω, δ_0 and $\|\hat{u}\|_{\hat{Z}}, \|\hat{b}\|_{H^2(0,T)}, \|\hat{r}\|_{H^1(0,T)}$) such that the solution of (65) satisfies*

$$(w,q,c,s) \in \hat{Y}_{2-k_0} \times (L^2(H^{3-k_0}) \cap H^{1-k_0/2}(H^1)) \times H^{3-k_0/2}(0,T) \times H^{2-k_0/2}(0,T)$$

and

$$\|(w,q,c,s)\|_{\hat{Y}_{2-k_0}\times(L^2(H^{3-k_0})\cap H^{1-k_0/2}(H^1))\times H^{3-k_0/2}(0,T)\times H^{2-k_0/2}(0,T)}$$
$$\leqslant \widehat{C}(\|g_1\|_{\hat{Y}_{-k_0}} + \|g_2\|_{H^{1-k_0/2}(0,T)} + \|g_3\|_{H^{1-k_0/2}(0,T)} + \|w_0\|_{H^{3-k_0}(\Omega_F(0))} + |c_0| + |c_1| + |s_0|).$$

The proof of this corollary is classical and it stands on interpolation arguments between Proposition 7 (with parameter $k_0/2$) and Proposition 8 (with parameter $1 - k_0/2$) (we refer to [20] and [1]).

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