The Godunov scheme for scalar conservation laws with discontinuous bell-shaped flux functions

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November 28, 2011

Abstract

We consider hyperbolic scalar conservation laws with discontinuous flux function of the type

$$\partial_t u + \partial_x f(x,u) = 0 \text{ with } f(x,u) = f_L(u)\mathbb{1}_{\mathbb{R}^-}(x) + f_R(u)\mathbb{1}_{\mathbb{R}^+}(x).$$

Here $f_{L,R}$ are compatible bell-shaped flux functions as appear in numerous applications. In [1] and [2], it was shown that several notions of solution make sense, according to a choice of the so-called (A,B)-connection. In this note, we remark that every choice of connection (A,B) corresponds to a limitation of the flux under the form $f(u)|_{x=0} \le \bar{F}$, first introduced in [3]. Hence we derive a very simple and cheap to compute explicit formula for the the Godunov numerical flux across the interface $\{x=0\}$, for each choice of the connection. This gives a simple-to-use numerical scheme governed only by the parameter $\bar{F}$. A numerical illustration is provided.

Keywords Scalar conservation laws, discontinuous flux functions, flux limitation, Godunov scheme, L^1 dissipative germs.

1 Introduction

Since it arises in several real life applications like traffic flow modeling [4], multiphase flows in porous media [5, 6, 7, 8] or water treatment [9], the Cauchy problem of the type

$$\partial_t u + \partial_x f(x,u) = 0, \qquad u(\cdot,0) = u_0, \tag{1}$$

where the flux function f is discontinuous w.r.t. the space variable have been widely studied during the last 20 years. A particular attention has been paid to the most simple case, i.e.

$$f(x,u) = f_L(u)\mathbb{1}_{\mathbb{R}^-}(x) + f_R(u)\mathbb{1}_{\mathbb{R}^+}(x). \tag{2}$$

In the sequel, we assume that the flux functions $f_{L,R}$ are *compatible* and *bell-shaped*, i.e.,

(A1) the functions $f_{L,R}$ are Lipschitz continuous and such that $f_L(0) = f_R(0)$, $f_L(1) = f_R(1)$;

(A2) there exists $b_{L,R} \in [0,1]$ such that $f'_{L,R}(u)(u - b_{L,R}) < 0$ for a.e. $u \in [0,1]$.

We also require the following condition on the initial data u_0:

(A3) u_0 is a measurable function satisfying $0 \le u_0(x) \le 1$ for a.e. $x \in \mathbb{R}$.

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For such a problem, it is natural to consider entropy solutions in the sense of Kružkov [10] away from the flux discontinuity at $x = 0$, i.e., functions $u \in L^\infty(\mathbb{R} \times \mathbb{R}_+; [0,1])$ such that (3) holds with $q_{L,R}(u,\kappa) := \operatorname{sign}(u-\kappa)(f_{L,R}(u) - f_{L,R}(\kappa))$. It has been pointed out in [1] that prescribing the balance of the fluxes at the interface is not sufficient to ensure uniqueness of a solution of the problem (1). Namely, some entropy criterion has to be fulfilled by the solution at the interface, and different physical contexts lead to different interface coupling criteria and thus to different notions of solution. In Section 2, we give a short introduction to the problem by following the theory introduced in [2] and extensively developed in [11]. We re-interpret the "(A,B)-connections" of [1, 2] in terms of interface flux constraints "$f(u)|_{x=0} \le \bar{F}_{(A,B)}$" introduced in [3]. Due to this idea of flux limitation at the interface, in Section 3 we establish an explicit formula for the flux at the interface corresponding to any Riemann problem. This yields the flux for the Godunov scheme for approximation of solutions to problem (1) for any choice of interface coupling (i.e., for any choice of a connection (A,B) or of an interface flux constraint $\bar{F} = \bar{F}_{(A,B)}$).

2 Connections, flux limitation and L^1 dissipative germs

Definition 2.1 (Connections and L^1 dissipative germs; see [1, 2] and [11]) *For $f_{L,R}$ satisfying* (**A1**), (**A2**)*, a couple $(A,B) \in [0,1]^2$ is said to be a* connection *if $A \in [b_L, 1]$, $B \in [0, b_R]$ and $f_L(A) = f_R(B)$. We define the corresponding L^1* dissipative germ $\mathcal{G}_{(A,B)}$ *(cf. [11]) to be the singleton $\{(A,B)\}$, and we set*

$$\mathcal{G}^*_{(A,B)} = \Big\{ (c_L, c_R) \in [0,1]^2 \ \ s.t. \ \ f_L(c_L) = f_R(c_R) \ \ and \ \ q_R(c_R, B) - q_L(c_L, A) \le 0 \Big\}.$$

We denote by $\mathcal{U} \subset [0,1]^2$ the set of all the connections corresponding to the flux functions f_L, f_R. Finally, we define the optimal connection $(A^{\rm opt}, B^{\rm opt})$ *by*

$$(A^{\rm opt}, B^{\rm opt}) \in \mathcal{U}, \ \textit{with either} \ \ A^{\rm opt} = b_L \ \textit{or} \ B^{\rm opt} = b_R.$$

As it was shown in [12], under Assumption **(A2)**, a function $u \in L^\infty(\mathbb{R}^* \times \mathbb{R}_+; [0,1])$ satisfying (3) admits one-sided traces $\gamma_{L,R}(u) \in L^\infty(\mathbb{R}_+)$ achieved in a strong sense. This permits to give the next definition.

Definition 2.2 ($\mathcal{G}_{(A,B)}$-entropy solution) *A function $u \in L^\infty(\mathbb{R}^* \times \mathbb{R}_+; [0,1])$ is said to be a $\mathcal{G}_{(A,B)}$-*entropy solution *of (1),(2) if it satisfies*

$$\forall \kappa \in [0,1] \quad \partial_t |u - \kappa| + \partial_x q_{L,R}(u,k) \le 0 \qquad in \ \ \mathcal{D}'(\Omega_{L,R}), \tag{3}$$

and for a.e. $t > 0$, one has $(\gamma_L(u)(t), \gamma_R(u)(t)) \in \mathcal{G}^*_{(A,B)}$.

The theory developed in [11] shows that for all $(A,B) \in \mathcal{U}$, there exists a unique $\mathcal{G}_{(A,B)}$-entropy solution to problem (1) in the sense of Definition 2.2. Equivalent characterizations of the $\mathcal{G}_{(A,B)}$-entropy solutions in terms of up-to-the-interface entropy inequalities were used in [11, 2]. In this paper, we will rather benefit from the point of view developed in [3] and then in [13, 14]; to this end, we establish the link between connections and flux limitation at the interface. We need more notations (see Fig. 1). For $(A,B) \in \mathcal{U}$,

$$\text{set} \quad \bar{F}_{(A,B)} := f_L(A) = f_R(B); \quad \text{notice that } \bar{F}_{(A^{\rm opt}, B^{\rm opt})} = \bar{F}^{\rm opt} := \max\nolimits_{(A,B)\in\mathcal{U}} \bar{F}_{(A,B)}.$$

The set $\mathcal{U}$ of connections can be parametrized by $\bar{F}$ (we write $\bar{F}_{(A,B)}$ or $(A_{\bar{F}}, B_{\bar{F}})$ to stress this link) which takes values in $[\bar{F}^{\rm barr}, \bar{F}^{\rm opt}] := \big[\max(f_{L,R}(0), f_{L,R}(1)), \ \min(f_L(b_L), f_R(b_R)) \big]$.
Set $\mathcal{O} := \mathcal{G}^*_{(A^{\rm opt}, B^{\rm opt})}$. Then $\mathcal{O} \setminus \{(A^{\rm opt}, B^{\rm opt})\}$ is the set of all couples $(a,b) \in [0,1]^2 \setminus \mathcal{U}$ such that $f_L(a) = f_R(b)$. In contrast to *under-compressive states* $(A,B) \in \mathcal{U}$, every couple $(a,b) \in \mathcal{O}$ will be called an *over-compressive state* (note that $(A^{\rm opt}, B^{\rm opt}) \in \mathcal{U} \cap \mathcal{O}$ is both under- and over-compressive). We have

$$\mathcal{G}^*_{(A,B)} = \{(A,B)\} \cup \mathcal{O}_{\bar{F}_{(A,B)}}, \ \ \text{where} \ \ \mathcal{O}_{\bar{F}_{(A,B)}} := \big\{ (c_L, c_R) \in \mathcal{O} \ \ \text{s.t.} \ \ f_L(c_L) = f_R(c_R) \le \bar{F}_{(A,B)} \big\} \tag{4}$$

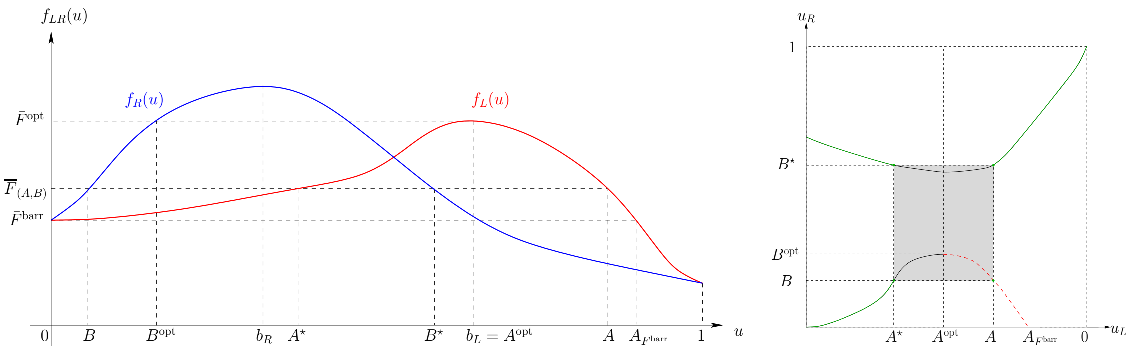


Figure 1: On the left hand side, the two flux functions $f_{L,R}$ have been plotted together. A choice $(A,B) \in \mathcal{U}$ of connection is drawn, as well as the particular values $A^\star$ and $B^\star$ such that $f_L(A^\star) = f_R(B^\star) = \bar{F}_{(A,B)}$. As it plays a particularly important role, the connection $(A^{\mathrm{opt}}, B^{\mathrm{opt}})$ is also represented. On the right hand side, we have drawn the sets $\mathcal{U}$ (red dashed line), $\mathcal{O} \equiv \mathcal{O}_{(A^{\mathrm{opt}},B^{\mathrm{opt}})}$ (solid line) and its subset $\mathcal{O}_{(A,B)}$ (green solid line, outside of the grey rectangle). The grey rectangle represents the open set of $(u_L, u_R) \in [0,1]^2$ that fail to satisfy $[\, f_L(u_L) \le \bar{F}_{(A,B)} \,] \,\&\, [\, f_R(u_R) \le \bar{F}_{(A,B)} \,]$.

is a restriction of $\mathcal{O}$. The connection (A,B) is the only under-compressive state belonging to $\mathcal{G}^*_{(A,B)}$. From (4), we readily see that $\mathcal{O}_{\bar{F}}$ depends in a monotone way on $\bar{F} \in [\bar{F}^{\mathrm{barr}}, \bar{F}^{\mathrm{opt}}]$.

In [3] (see also [13, 14]), L^1-contractive semigroups of solutions were constructed even for the classical case $f_L = f_R$, by imposing an interface flux constraint of the form $f_{L,R}(\gamma_{L,R}(u)) \le \bar{F}$ at $\{x = 0\}$. In the case $f_L \ne f_R$, the situation is exactly similar. Namely, each connection (A,B) makes appear a set of trace couples $\mathcal{G}^*_{(A,B)}$ satisfying (4), so that the different $\mathcal{G}_{(A,B)}$-entropy solutions for (1),(2) for different $(A,B) \in \mathcal{U}$ correspond to different levels $\bar{F}_{(A,B)}$ of interface flux constraint. Kružkov solutions (in the case $f_L = f_R$) and optimal entropy solutions (in the general case) shall be seen as the unconstrained ones.

3 The Godunov scheme

Consider the Riemann problem (1),(2) with initial datum $u_0 = u_L 1\!\!1_{\mathbb{R}^-} + u_R 1\!\!1_{\mathbb{R}^+}$. Let us compute the flux across the interface $\{x = 0\}$ of the $\mathcal{G}_{(A,B)}$-entropy solution u of the Riemann problem in order to be able to build the Godunov scheme (see [15]). Note that such scheme is proved to be convergent in [11].

In the case $f_L = f_R$, the numerical scheme proposed in [13] used the flux $\min\{\bar{F}, \mathcal{F}(u_L, u_R)\}$, i.e., a given interface numerical flux $\mathcal{F}(\cdot,\cdot)$ for the unconstrained problem was limited to a given maximal value $\bar{F}$. Moreover, in the particular case where $\mathcal{F}(\cdot,\cdot)$ is the Godunov flux for the unconstrained problem, it is shown in [14] that the resulting scheme for the constrained problem is also the Godunov one. Here, we show that the same property holds for general $f_{L,R}$, namely, the Godunov flux through the interface $\{x = 0\}$ corresponding to the $\mathcal{G}_{(A,B)}$-entropy solution is the Godunov flux corresponding to the optimal entropy solution on which we apply the constraint afterwards. Notice that, in addition, an explicit formula for the Godunov flux for the optimal entropy solution is well known since [16].

Theorem 3.1 (Main result) *The Godunov flux for $\mathcal{G}_{(A,B)}$-entropy solutions at the interface $x = 0$ is given by*

$$\mathcal{F}(u_L, u_R) = \min\left(\bar{F}_{(A,B)}, f_L\left(\min(u_L, b_L)\right), f_R\left(\max(u_R, b_R)\right)\right). \tag{5}$$

Moreover, whenever $\mathcal{F}^{\mathrm{opt}}(u_L, u_R) > \bar{F}_{(A,B)}$, i.e., the constraint is active, one has

$$\gamma_L(u) = A, \qquad \gamma_R(u) = B.$$

Proof: As it has been explicitly stated in [16], it follows from the *bell-shaped* behavior of the flux functions (see Assumption **(A2)**) that the flux of the $(A^{\rm opt}, B^{\rm opt})$-entropy solution of the above Riemann problem across the discontinuity $\{x = 0\}$ is given by

$$\mathcal{F}^{\rm opt}(u_L, u_R) = \min\left(f_L\left(\min(u_L, b_L)\right), f_R\left(\max(u_R, b_R)\right)\right). \tag{6}$$

We have two possibilities. First, assume that $\mathcal{F}^{\rm opt}(u_L, u_R) \leq \bar{F}_{(A,B)}$; we see from Fig. 1 that the traces couple $(\gamma_L(u^{\rm opt}), \gamma_R(u^{\rm opt}))$ belongs to $\mathcal{O} \cap \mathcal{G}^*_{(A,B)} = \mathcal{O}_{\bar{F}_{(A,B)}}$. Therefore, in this case the $\mathcal{G}_{(A^{\rm opt},B^{\rm opt})}$-entropy solution of the Riemann problem coincides with the $\mathcal{G}_{(A,B)}$-entropy solution. Therefore, in the case under consideration the flux across the interface, which is given by formula (6), is also given by formula (5).

Second, assume that $\mathcal{F}^{\rm opt}(u_L, u_R) > \bar{F}_{(A,B)}$, so that $(A, B) \neq (A^{\rm opt}, B^{\rm opt})$. In this case, one has

$$f_L(A) < f_L(\gamma_L(u^{\rm opt})) = \min\big(f_L(\min(u_L, b_L))\,,\, f_L(\max(b_L, \gamma_L(u^{\rm opt})))\big). \tag{7}$$

Denoting by $A^\star \in [0, b_L]$ and $B^\star \in [b_R, 0]$ the values with $f_L(A^\star) = f_L(A) = \bar{F}_{(A,B)} = f_R(B^\star) = f_R(B)$ (see Fig. 1), one deduces from (7) that $u_L > A^\star$ and $u_R < B^\star$. Therefore, using **(A2)**, one obtains that

$$f_L(A) = \bar{F}_{(A,B)} = \min\left(f_L(\min(u_L, b_L)), f_L(\max(b_L, A))\right).$$

Similarly, one obtains that $f_R(B) = \bar{F}_{(A,B)} = \min\left(f_R(\min(B, b_R)), f_R(\max(b_R, u_R))\right)$. These two relations imply that the boundary $\{x = 0\}$ is characteristic for each of the Cauchy-Dirichlet problems

$$\begin{cases} \partial_t u + \partial_x f_L(u) = 0 \text{ in } \mathcal{D}'(\mathbb{R}^-_* \times \mathbb{R}^+_*), \\ u(x,0) = u_L \text{ for } x < 0, \\ u(0,t) = A \text{ for } t > 0; \end{cases} \qquad \begin{cases} \partial_t u + \partial_x f_R(u) = 0 \text{ in } \mathcal{D}'(\mathbb{R}^+_* \times \mathbb{R}^+_*), \\ u(x,0) = u_R \text{ for } x > 0, \\ u(0,t) = B \text{ for } t > 0. \end{cases} \tag{8}$$

This ensures that the boundary conditions prescribed in (8) are fulfilled in a strong sense by the function u (see [17]). Defining u as the juxtaposition of the entropy solutions of problems (8) in the sense of [17], we see that u satisfies (3) and it takes the initial datum $u_L 1\!\!1_{\mathbb{R}^-} + u_R 1\!\!1_{\mathbb{R}^+}$. Moreover, we have $(\gamma_L(u)(t), \gamma_R(u)(t)) = (A, B) \in \mathcal{G}^*_{(A,B)}$ for all $t > 0$, ensuring that u is the unique $\mathcal{G}_{(A,B)}$-entropy solution (see [2, 11]) to the Riemann problem under study. Thus the Godunov flux for this Riemann problem is the flux of u across the interface. The latter is given by $\mathcal{F}(u_L, u_R) = \bar{F}_{(A,B)}$, so that formula (5) is true also in this case. □

4 Numerical example

In order to illustrate our purpose we compute the approximate solution corresponding to a case where the constraint on the flux at the interface is active for some initial laps of time, then it becomes inactive. The flux functions $f_{L,R}$, the flux constraint $\bar{F}$ and the initial data u_0 are defined by

$$f_L(u) = 2u(1-u), \qquad f_R(u) = u(1-u), \qquad \bar{F} = 0.125, \qquad u_0(x) = 0.5 + 0.5\sin(4\pi x).$$

Define the Godunov numerical fluxes on each side from the interface $\{x = 0\}$:

$$G_{L,R}(u, v) = \min\left(f_{L,R}(\min(u, b_{L,R})), f_{L,R}(\max(v, b_{L,R}))\right),$$

and take Δt,$\Delta x > 0$ such that

$$\frac{L_f \Delta t}{\Delta x} \leq 1 - \xi \tag{9}$$

for some Lispchitz constant L_f of both $f_{L,R}$ and some $\xi \in (0, 1)$. Then the Godunov scheme is given by

$$u^{n+1}_{j+1/2} = u^n_{j+1/2} + \frac{\Delta t}{\Delta x}\left(F^n_j - F^n_{j+1}\right), \qquad \forall j \in \mathbb{Z}, \forall n \in \mathbb{N}.$$

with

$$F^n_j = \begin{cases} G_L(u^n_{j-1/2}, u^n_{j+1/2}) & \text{if } j < 0, \\ G_R(u^n_{j-1/2}, u^n_{j+1/2}) & \text{if } j > 0, \\ \mathcal{F}(u^n_{-1/2}, u^n_{1/2}) & \text{if } j = 0, \end{cases}$$

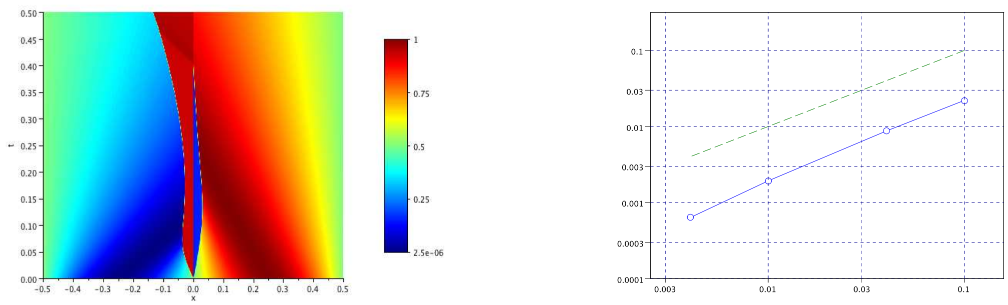
where the interface Godunov flux function $\mathcal{F}$ is given by (5). The discrete solution u_h is then given by

$$u_h(x,t) = u^n_{j+1/2} \quad \text{ if } (x,t) \in (j\Delta x, (j+1)\Delta x) \times (n\Delta t, (n+1)\Delta t).$$

By Theorem 3.1, this is the Godunov scheme for (1),(2); its convergence is therefore justified in [11, Sect. 4.8 and Sect.$\dot{6}$.3]. More precisely, under the CFL condition (9) the approximate solution u_h takes values in $[0,1]$ and

$$u_h \to u \text{ in } L^1_{loc}\,(\mathbb{R}\times\mathbb{R}^+) \text{ as } \Delta x, \Delta t \to 0.$$

The reference solution $u_{h_{\rm ref}}$, presented on Fig. 2a, is computed with the Godunov scheme with the values $\Delta x = 10^{-3}$ and $\Delta t = 5 * 10^{-4}$. It appears that the flux limitation constraint is active for $t \leq 0.4$, and then it becomes inactive.



(a) The reference solution $u_{h_{\rm ref}}$

(b) $\log \|u_h - u_{h_{\rm ref}}\|_{L^1}$ as a function of $\log(\Delta x)$

On Fig. 2b, we plot $\log \|u_h - u_{h_{\rm ref}}\|_{L^1(K)}$ for $K = [-0.5, 0.5] \times [0, 0.5]$ as a function of $\log(\Delta x)$ (solid blue line). We observe the slope +1. Since the flux functions $f_{L,R}$ are genuinely nonlinear, the expected convergence order of the Godunov scheme in each of the subdomains $\Omega_{L,R}$ is 1, i.e.

$$\|u_h - u\|_{L^1(K)} \leq C(K)\Delta x,$$

for all compact subset K of $\mathbb{R}\times\mathbb{R}_+$ lying far enough from the interface. This estimate seems to be preserved in a neighbourhood of the interface, which means that our numerical treatment of the flux discontinuity does not damage the convergence rate of the scheme.

5 Conclusion

As a conclusion, remark that the numerical fluxes of the Godunov scheme given by formula (5) are cheap to compute. In particular, no integration is needed to compute the solution of the Godunov scheme, in contrast, e.g., to the Engquist-Osher type scheme proposed in [2].

Moreover, the scheme based on (5) readily adapts to any level $\bar{F}$ of interface flux constraint. We refer to [5] (see also [6, 7]) for an example of determination of the level of constraint in the setting of Buckley-Leverett equations for two-phase flow in a two-rocks' medium. Indeed, in this model all the values $\bar{F} \in [\bar{F}^{\rm barr}, \bar{F}^{\rm opt}]$ can appear as physically motivated ones, depending on the behaviour of the capillary pressure profiles on each side from the interface.

Acknowledgements

The work of the first author was supported by the Young Researchers project CoToCoLa of the French ANR agency. The work of the second author was supported by GNR MoMaS, CNRS-2439 (PACEN/CNRS, ANDRA, BRGM, CEA, EDF, IRSN).

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