Spreading properties and complex dynamics for monostable reaction-diffusion equations

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Abstract

This paper is concerned with the study of the large-time behavior of the solutions u of a class of one-dimensional reaction-diffusion equations with monostable reaction terms f, including in particular the classical Fisher-KPP nonlinearities. The nonnegative initial data $u_0(x)$ are chiefly assumed to be exponentially bounded as x tends to $+\infty$ and separated away from the unstable steady state 0 as x tends to $-\infty$. On the one hand, we give some conditions on u_0 which guarantee that, for some $\lambda > 0$, the quantity $c_\lambda = \lambda + f'(0)/\lambda$ is the asymptotic spreading speed, in the sense that $\lim_{t\to+\infty} u(t, ct) = 1$ (the stable steady state) if $c < c_\lambda$ and $\lim_{t\to+\infty} u(t, ct) = 0$ if $c > c_\lambda$. These conditions are fulfilled in particular when $u_0(x)\, e^{\lambda x}$ is asymptotically periodic as $x \to +\infty$. On the other hand, we also construct examples where the spreading speed is not uniquely determined. Namely, we show the existence of classes of initial conditions u_0 for which the $\omega-$limit set of $u(t, ct + x)$ as t tends to $+\infty$ is equal to the whole interval $[0, 1]$ for all $x \in \mathbb{R}$ and for all speeds c belonging to a given interval (γ_1, γ_2) with large enough $\gamma_1 < \gamma_2$.

Keywords: reaction-diffusion equations; spreading speeds; propagation phenomena.
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1 Introduction

We study in this paper the large-time behavior of the solutions of monostable reaction-diffusion equations of the type

$$\begin{cases} \partial_t u - \partial_{xx} u = f(u), & t > 0,\ x \in \mathbb{R}, \\ u(0, x) = u_0(x) & \text{for a.e. } x \in \mathbb{R}, \end{cases} \tag{1.1}$$

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where the reaction term $f : [0,1] \to \mathbb{R}$ is a $\mathcal{C}^1$ function such that

$$f(0) = f(1) = 0,\ \ f(s) > 0 \text{ if } s \in (0,1),\ \ f'(0) > 0, \tag{1.2}$$

and u_0 is a measurable initial datum such that $u_0 \not\equiv 0$, $u_0 \not\equiv 1$ and $0 \le u_0(x) \le 1$ for almost every $x \in \mathbb{R}$ (the quantity u stands for a normalized density in the applications in population dynamics models, see e.g. [4, 32, 39]). Under these hypotheses, the Cauchy problem (1.1) is well-posed, the solution u is classical for $t > 0$, and $u(t,x) \in (0,1)$ for all $t > 0$, $x \in \mathbb{R}$.

This type of equation has first been investigated by Fisher [12] and Kolmogorov, Petrovski and Piskunov [24] in the 30's. Among other results, these authors proved that, in dimension 1, when $f(s) = s(1-s)$ and u_0 is the Heaviside function, that is $u_0(x) = 1$ if $x < 0$ and 0 if $x > 0$, then

$$\begin{cases} \displaystyle\min_{x \le ct} u(t,x) \to \ 1 \ \text{ as } t \to +\infty & \text{if } c < c^*, \\ \displaystyle\max_{x \ge ct} u(t,x) \to \ 0 \ \text{ as } t \to +\infty & \text{if } c > c^*, \end{cases} \tag{1.3}$$

with $c^* = 2$ in this case. Such properties are called *spreading properties* and the quantity c^* is called the *spreading speed* associated with the initial datum u_0. This result has been extended by Aronson and Weinberger [1] in the 70's to multidimensional media and positive nonlinearities satisfying (1.2). In particular, it is proved in [1] that, in dimension 1, formula (1.3) still holds when u_0 is the Heaviside function, for some positive real number c^* which only depends on f.

Travelling fronts

For general functions f satisfying (1.2), this threshold c^* also turns out to be the minimal speed of existence of travelling fronts solutions of equation (1.1). Namely, we say that a solution u of (1.1) is a travelling front if it can be written as

$$u(t,x) = U_c(x - ct),$$

with $U_c(-\infty) = 1$, $U_c(+\infty) = 0$ and $0 < U_c < 1$ in $\mathbb{R}$. In this case, we say that c is the speed of the travelling front solution u. It is well known [1, 15] that if f satisfies (1.2), then there exists a speed c^* such that there exists a travelling front solution of (1.1) with speed c if and only if $c \ge c^*$. Furthermore, if f satisfies the now-called Fisher-KPP assumption, that is if

$$0 < f(s) \le f'(0)s \text{ for all } s \in (0,1), \tag{1.4}$$

then $c^* = 2\sqrt{f'(0)}$. For general functions f satisfying (1.2), one has

$$c^* \ge 2\sqrt{f'(0)},$$

see [1, 15]. Lastly, for each $c \ge c^*$, the profile U_c associated with the travelling front of speed c is decreasing on $\mathbb{R}$ and unique up to translation and, if $c > c^*$, there exists $M > 0$ such that

$$U_c(z) \sim M\, e^{-\lambda z} \text{ as } z \to +\infty, \tag{1.5}$$

where $\lambda = (c - \sqrt{c^2 - 4f'(0)})/2$ is the smallest root of the equation $\lambda^2 - \lambda c + f'(0) = 0$. When $c = c^* > 2\sqrt{f'(0)}$, then $U_{c^*}(z) \sim M\, e^{-\overline{\lambda} z}$ as $z \to +\infty$ for some $M > 0$, where $\overline{\lambda}$ is the largest root of $\overline{\lambda}^2 - \overline{\lambda} c^* + f'(0) = 0$. When $c = c^* = 2\sqrt{f'(0)}$, then $U_{c^*}(z) \sim (M\, z + M')\, e^{-\lambda^* z}$ as $z \to +\infty$, where $\lambda^* = \sqrt{f'(0)} = c^*/2$ and either $M > 0$, or $M = 0$ and $M' > 0$. Notice here that the nondegeneracy of f at 0, that is the condition $f'(0) > 0$, guarantees the exponential behavior of the travelling fronts as they approach 0. Thus, the estimates of the spreading speeds, as defined below, are expected to be given in terms of the exponential decay rate of the initial condition. If $f'(0) = 0$, then the non-critical travelling fronts have in general an algebraic decay and the convergence to the travelling fronts depends on the algebraic decay rate of the initial condition and exponentially decaying initial conditions will then travel with the minimal speed (see [22, 38] for some results in that direction).

Definition of minimal and maximal spreading speeds for front-like initial data

Before stating our main results in the next section, we define in this section the notions of minimal and maximal spreading speeds for the solutions u of (1.1) with initial conditions $u_0 : \mathbb{R} \to [0,1]$ which are much more general than the Heaviside function. We state here some elementary comparisons between the spreading speeds and we recall the known standard examples for which the spreading speeds are well determined. It would have been natural to also consider heterogeneous reaction-diffusion equations as well as equations in higher dimensions. We chose to present our results in the homogeneous one-dimensional setting for problem (1.1) for the sake of simplicity of the presentation, and also because this one-dimensional homogeneous framework already captures new and interesting complex propagation phenomena at large time. However, in the appendix, we briefly mention some extensions of our main results to more general heterogeneous and higher-dimensional situations.

Coming back to problem (1.1), the initial data u_0 we consider are front-like, in the sense of the following definition.

Definition 1.1 *We say that a function $u_0 \in L^\infty(\mathbb{R})$ is* front-like *if $0 \le u_0(x) \le 1$ for a.e. $x \in \mathbb{R}$ and there exist $x_- \in \mathbb{R}$ and $\delta > 0$ such that*

$$u_0(x) \ge \delta \text{ for a.e. } x < x_- \text{ and } \lim_{x \to +\infty} \|u_0\|_{L^\infty(x,+\infty)} = 0.$$

The term front-like means that the values of $u_0(x)$ as $x \to \pm\infty$ (up to a negligible set) are strictly ordered, although the front-like initial u_0 may not be nonincreasing on $\mathbb{R}$ even up to a set of zero measure. However, these very mild conditions still guarantee that $u(t,x) \to 0$ as $x \to +\infty$ for every $t > 0$, from standard parabolic estimates and the maximum principle.

For such initial data, we still expect the solutions of the Cauchy problem (1.1) to spread, that is the stable state 1 to invade the unstable steady state 0. At first glance, we could think that a property like (1.3) still holds, where c^* would in general be replaced with a quantity $w > 0$ which would depend on u_0. A natural question, which is fundamental for the applications in biology or ecology, would then be to compute the speed w of this invasion. In fact, it turns out that some complex dynamics may occur in general. The mild conditions in Definition 1.1 give rise to a large variety of propagation phenomena at large time, some of them being of a completely new type. Thus, in order to quantify the spreading, we are led to introduce two natural quantities: the minimal and the maximal spreading speeds.

Definition 1.2 *For a given front-like function* u_0*, we define the* minimal *and* maximal spreading speeds $w_*(u_0)$ *and* $w^*(u_0)$ *of the solution* u *of* (1.1) *as*

$$\begin{aligned} w_*(u_0) &= \sup\Big\{c\in\mathbb{R},\ \inf_{x\le ct} u(t,x)\to 1 \ \textit{as}\ t\to+\infty\Big\},\\ w^*(u_0) &= \inf\ \Big\{c\in\mathbb{R},\ \sup_{x\ge ct} u(t,x)\to 0 \ \textit{as}\ t\to+\infty\Big\}. \end{aligned}$$

It immediately follows from Definition 1.2 that, for any given front-like function u_0,

$$\begin{cases} \displaystyle\inf_{x\le ct} u(t,x)\to 1 \text{ as } t\to+\infty \text{ for all } c<w_*(u_0),\\ \displaystyle\sup_{x\ge ct} u(t,x)\to 0 \text{ as } t\to+\infty \text{ for all } c>w^*(u_0) \end{cases}$$

if $w^*(u_0)$ is finite. Actually, we will see below that $w_*(u_0)$ can never be $-\infty$, but that $w^*(u_0)$, and $w_*(u_0)$, are sometimes equal to $+\infty$.

Computation of the spreading speeds in the standard cases

Let us now give some general comparisons and a list of standard examples for which these quantities can be explicitly computed. First, when there is a real number A such that

$$u_0(x)=\sigma\in(0,1] \text{ for a.e. } x<A \text{ and } u_0(x)=0 \text{ for a.e. } x>A,$$

it is then well known [1, 24] that $w_*(u_0)=w^*(u_0)=c^*$, where

$$c^* \text{ is the minimal speed of existence of travelling fronts solutions.}$$

Using this fact and the parabolic maximum principle, as any front-like function is bounded from below by a space shift of the Heaviside function multiplied by some $\sigma\in(0,1]$, we get that

$$c^*\le w_*(u_0)\le w^*(u_0)\le+\infty \tag{1.6}$$

for any front-like initial datum u_0.

In general, the spreading speeds are strictly larger than c^*. For example, for any speed $c\ge c^*$, if $u(t,x)=U_c(x-ct)$ is a travelling front solution of speed c, then

$$w_*(u(0,\cdot))=w_*(U_c)=w^*(u(0,\cdot))=w^*(U_c)=c.$$

Set now

$$\lambda^*=\min\big\{\lambda>0,\ \lambda^2-\lambda c^*+f'(0)=0\big\}=\frac{c^*-\sqrt{c^{*2}-4f'(0)}}{2}, \tag{1.7}$$

which is a well defined real number since $c^* \geq 2\sqrt{f'(0)}$, consider

$$u_0(x) = \min\left(\sigma, \theta\, e^{-\lambda x}\right) \ \text{ for all } x \in \mathbb{R} \tag{1.8}$$

with $\sigma \in (0,1]$, $\theta > 0$ and $\lambda \in (0, \lambda^*)$, and define

$$c_\lambda = \lambda + \frac{f'(0)}{\lambda}.$$

When f satisfies (1.2) and $f'(s) \leq f'(0)$ for all $s \in [0,1]$, it has been proved through probabilistic methods by McKean [30] and through PDE's methods by Kametaka [20] that the solution u of (1.1) satisfies

$$\sup_{x\in\mathbb{R}} \left|u(t,x) - U_{c_\lambda}(x - c_\lambda t + x_0)\right| \to 0 \text{ as } t \to +\infty, \tag{1.9}$$

where U_{c_λ} is the travelling front profile with speed c_λ, satisfying (1.5), and $x_0 = -\lambda^{-1}\ln(\theta/M)$. This property implies that

$$w_*(u_0) = w^*(u_0) = c_\lambda = \lambda + \frac{f'(0)}{\lambda}. \tag{1.10}$$

When $\lambda \geq \lambda^*$ in (1.8), McKean [30] and Kametaka [20] proved a similar convergence, namely that

$$\sup_{x\in\mathbb{R}} \left|u(t,x) - U_{c^*}(x - c^* t + m(t))\right| \to 0 \text{ as } t \to +\infty, \tag{1.11}$$

where $m(t)/t \to 0$ as $t \to +\infty$. This implies (1.3) and leads to $w_*(u_0) = w^*(u_0) = c^*$. These limits (1.9) and (1.11) have been extended by Uchiyama [40] to general monostable functions f fulfilling (1.2) and to front-like initial data satisfying $\lim_{x\to+\infty} u_0(x+x_0)/u_0(x) = e^{-\lambda x_0}$ for all $x_0 \in \mathbb{R}$ (see also [10, 25, 31, 35, 36] for further results and more precise convergence estimates).

On the other hand, Bramson [7] and Lau [26] investigated spreading properties for more general front-like initial data, using respectively probabilistic and PDE tools, when f satisfies (1.2) and $f'(s) \leq f'(0)$ for all $s \in [0,1]$. They proved that if u_0 is a front-like initial datum such that there exist $h > 0$ and $0 < \lambda < \lambda^* = \sqrt{f'(0)}$ such that

$$\lim_{x\to+\infty} \frac{1}{x} \ln\left(\int_x^{(1+h)x} u_0(y)dy\right) = -\lambda, \tag{1.12}$$

then $w_*(u_0) = w^*(u_0) = c_\lambda = \lambda + f'(0)/\lambda$. This result is more general than the one of Uchiyama [40], but it requires the nonlinearity f to satisfy $f'(s) \leq f'(0)$ for all $s \in [0,1]$. This property simplifies the analysis since it is known that the linearization near $u = 0$ does govern the global dynamics of the equation in this case. However, we believe that Bramson's and Lau's results could be extended from the KPP framework to that of (1.2), using comparison with KPP nonlinearities. As will be seen in the Section 3, we will use in this paper other assumptions and tools, which still guarantee the uniqueness of the spreading speeds in the general monostable case (1.2). Furthermore, we also show that complex dynamics may occur in general.

Lastly, if u_0 is front-like and

$$u_0(x)\, e^{\varepsilon x} \to +\infty \text{ as } x \to +\infty \text{ for all } \varepsilon > 0, \tag{1.13}$$

then it follows from the maximum principle and (1.10) that $w_*(u_0) \geq \varepsilon + f'(0)/\varepsilon$ for all $\varepsilon > 0$, whence $w_*(u_0) = w^*(u_0) = +\infty$. In this case, together with the Fisher-KPP assumption (1.4), Hamel and Roques [17] also computed the position of the level sets of the function $u(t, \cdot)$ as $t \to +\infty$, according to the precise asymptotic behavior of $u_0(x)$ as $x \to +\infty$.

To sum up, the spreading speeds $w_*(u_0)$ and $w^*(u_0)$ are explicitly known when the front-like initial data u_0 are exponentially decaying near $+\infty$, or when they fulfill (1.12) under the additional condition that f satisfies $f'(s) \leq f'(0)$ for all $s \in [0, 1]$. It is important to notice that, in all aforementioned examples, one has $w_*(u_0) = w^*(u_0)$. This leads to the following natural questions, that we investigate in the present paper:

- is it possible to compute $w_*(u_0)$ and $w^*(u_0)$ for more general initial conditions, given a nonlinearity f satisfying (1.2) only ?
- is it always true that $w_*(u_0) = w^*(u_0)$?

Remark 1.3 Throughout the paper, the initial conditions u_0 are assumed to be front-like in the sense of Definition 1.1. Obviously, when $0 \leq u_0 \leq 1$, $u_0 \not\equiv 0$ and $u_0(x) \to 0$ as $x \to \pm\infty$, then left and right minimal and maximal spreading speeds could be defined and similar results as the ones stated in the next section could be obtained. One of the reasons lies on the fact that $u(t, x) \to 1$ as $t \to +\infty$ locally uniformly in $x \in \mathbb{R}$ (as a matter of fact, $\min_{|x| \leq ct} u(t, x) \to 1$ as $t \to +\infty$ for all $c \in [0, c^*)$, see [1]). Thus, the spreading properties to the left and to the right only depend on the behavior of the tails of u_0 at $\pm\infty$.

2 Main results

We first consider the class of front-like functions u_0 such that

$$u_0(x) = O\big(e^{-\Lambda(x)x}\big) \text{ as } x \to +\infty \text{ with } \lim_{x \to +\infty} \Lambda(x) = \lambda \in [0, +\infty]. \tag{2.14}$$

We first look for some conditions on u_0 which guarantee that $w_*(u_0) = w^*(u_0) = c_\lambda$. In other words, we want to know whether u satisfies the same spreading property as the solution associated with the initial datum $x \mapsto \min(\sigma, \theta\, e^{-\lambda x})$, for some $\sigma \in (0, 1]$ and $\theta > 0$. If $\lambda \in [\lambda^*, +\infty]$, where $\lambda^* > 0$ was defined in (1.7), then, as already emphasized, it follows from the maximum principle and [40] that $w_*(u_0) = w^*(u_0) = c^*$ (it is actually sufficient to suppose that $\liminf_{x \to +\infty} \Lambda(x) \geq \lambda^*$). We thus restrict ourselves to the case

$$0 \leq \lambda < \lambda^*.$$

The condition we will exhibit on u_0 depends on the function $x \mapsto \rho(x) := u_0(x)\, e^{\lambda x}$ (for x sufficiently large). Basically, this condition requires the solution of the heat equation associated with the initial datum ρ (extended by 1 in a neighborhood of $-\infty$) to be uniformly away from 0 for each fixed $t > 0$.

To make the arguments work, we shall use an additional assumption on the nonlinearity f near 0. Namely, we assume that there exist $C > 0, \gamma > 0$ and $s_0 \in (0,1)$ so that

$$\forall s \in [0, s_0],\ \ f(s) \geq f'(0)s - Cs^{1+\gamma}. \tag{2.15}$$

Note that this hypothesis is fulfilled in particular if f is of class $\mathcal{C}^{1+\gamma}$ in a neighborhood of 0.

We first deal with the case $0 < \lambda < \lambda^*$ in (2.14).

Theorem 2.1 *Let f satisfy* (1.2) *and* (2.15)*, let $\lambda \in (0, \lambda^*)$ and let u_0 be a front-like function such that there exist $x_0 \in \mathbb{R}$, a nonnegative bounded function $\rho : (x_0, +\infty) \to [0, +\infty)$ and a function $\Lambda : (x_0, +\infty) \to \mathbb{R}$ so that*

$$u_0(x) = \rho(x)\, e^{-\Lambda(x)x} \ \textit{for a.e. } x > x_0 \ \ \textit{and} \ \ \Lambda(x) \to \lambda \ \textit{as } x \to +\infty.$$

Let $\overline{\rho} : \mathbb{R} \to [0, +\infty)$ be defined by $\overline{\rho}(x) = 1$ for $x < x_0$ and $\overline{\rho}(x) = \rho(x)$ for $x > x_0$. Lastly, let ζ be the solution of the heat equation

$$\begin{cases} \partial_t \zeta - \partial_{xx}\zeta = 0, & t > 0,\ x \in \mathbb{R}, \\ \zeta(0, x) = \overline{\rho}(x) & \textit{for a.e. } x \in \mathbb{R}. \end{cases} \tag{2.16}$$

If there exists a time $T > 0$ such that

$$\inf_{x \in \mathbb{R}} \zeta(T, x) > 0, \tag{2.17}$$

then

$$w_*(u_0) = w^*(u_0) = \lambda + \frac{f'(0)}{\lambda}. \tag{2.18}$$

Remark 2.2 Note that the existence of a time $T > 0$ such that $\inf_{x\in\mathbb{R}} \zeta(T, x) > 0$ is equivalent to

$$\inf_{x \in \mathbb{R}} \zeta(t, x) > 0 \ \text{ for all } t > 0$$

and even

$$\inf_{t' \geq t} \left(\inf_{x \in \mathbb{R}} \zeta(t', x) \right) > 0 \ \text{ for all } t > 0. \tag{2.19}$$

Indeed, if there exist $t_0 > 0$ and a sequence $(x_n)_{n\in\mathbb{N}}$ of real numbers such that $\zeta(t_0, x_n) \to 0$ as $n \to +\infty$, then the Schauder parabolic estimates and the strong parabolic maximum principle imply that, up to extraction of a subsequence, $\zeta(t, x + x_n) \to 0$ as $n \to +\infty$ locally uniformly in $(t, x) \in (0, +\infty) \times \mathbb{R}$. Thus, if $\inf_{x\in\mathbb{R}} \zeta(T, x) > 0$ for some $T > 0$, then $\inf_{x\in\mathbb{R}} \zeta(t, x) > 0$ for all $t > 0$ and the parabolic maximum principle yields (2.19). Observe also that, by linearity of the heat equation and by the maximum principle, condition (2.17) remains unchanged if the function $\overline{\rho}$ is set to be equal to any given positive real number $\eta > 0$, instead of 1, on $(-\infty, x_0)$. Thus, what really matters in condition (2.17) is the behavior of $\rho(x)$ as $x \to +\infty$.

Under the only monostability and behavior-at-0 conditions (1.2) and (2.15), Theorem 2.1 then gives a sufficient condition on u_0 for the solution u of (1.1) to spread at speed $c_\lambda = \lambda + f'(0)/\lambda$. It is immediate to see that the boundedness assumption of ρ is necessary for Theorem 2.1 to hold in general. For instance, if $\rho(x) = e^{\varepsilon x}$ for $x > x_0$ with $\varepsilon \in (0, \lambda)$, then, by writing $u_0(x) = \widetilde{\rho}(x)\, e^{-\widetilde{\Lambda}(x)\, x}$ for a.e. $x > x_0$ with $\widetilde{\Lambda}(x) = \Lambda(x) - \varepsilon$ and $\widetilde{\rho}(x) = 1$ for $x > x_0$, the conclusion of Theorem 2.1 yields $w_*(u_0) = w^*(u_0) = c_{\lambda-\varepsilon} > c_\lambda$. Similarly, the condition (2.17) is obviously necessary for the conclusion to hold in general: indeed, if $\rho(x) = e^{-\varepsilon x}$ for $x > x_0$, with $\varepsilon \in (0, \lambda^* - \lambda)$, then (2.17) is not fulfilled and $w_*(u_0) = w^*(u_0) = c_{\lambda+\varepsilon} < c_\lambda$. However, this condition (2.17) is also not necessary in general for the conclusion (2.18) to hold. That is, there are examples for which (2.17) is violated and (2.18) still holds. For instance, choose any positive measurable function ρ on $(x_0, +\infty)$ such that

$$\rho(x) \to 0 \text{ and } |\ln \rho(x)| = o(x) \text{ as } x \to +\infty;$$

then $\zeta(t, x) \to 0$ as $x \to +\infty$ for all $t > 0$ and conclusion (2.18) still holds since Theorem 2.1 can be applied by writting u_0 as $u_0(x) = e^{-\widetilde{\Lambda}(x)x}$ for a.e. $x \geq x_0$ with

$$\widetilde{\Lambda}(x) = \Lambda(x) - \frac{\ln \rho(x)}{x} \to \lambda \text{ as } x \to +\infty.$$

On the other hand, what is much less obvious is to see that there are examples for which (2.17) is violated while the maximal spreading speed $w^*(u_0)$ is still equal to c_λ and the minimal spreading speed $w_*(u_0)$ is strictly less. That will be the purpose of Theorem 2.6 which is stated at the end of this section.

Before doing so, we first state an immediate corollary of Theorem 2.1, concerning the particular case of a function ρ which is the restriction on $(x_0, +\infty)$ of a function having an average: we say that a function $g \in L^\infty(\mathbb{R})$ admits an average $g_m \in \mathbb{R}$ if

$$\frac{1}{h}\int_x^{x+h} g(z)\, dz \to g_m \text{ as } h \to +\infty \text{ uniformly in } x \in \mathbb{R}.$$

Corollary 2.3 *Let f, λ, u_0, x_0, ρ and Λ be as in Theorem* 2.1 *and assume furthermore that ρ can be extended on $\mathbb{R}$ to a bounded nonnegative function having a positive average. Then, the conclusion* (2.18) *holds automatically.*

In particular, Corollary 2.3 covers the case of nonnegative functions $\rho \in L^\infty(\mathbb{R})$ which are periodic, almost-periodic or uniquely ergodic, assuming that their average, which exists, is positive. Under the assumptions of Corollary 2.3, it is easy to check that u_0 satisfies the condition (1.12) of Bramson [7] and Lau [26]. Hence, if, in addition to (1.2) and (2.15), the function f is such that $f'(s) \leq f'(0)$ for all $s \in [0, 1]$, the proof of Corollary 2.3 gives then an alternate approach of that of Bramson and Lau.

We point out that, in Theorem 2.1 or in Corollary 2.3, the function ρ may vanish on sequences of sets with positive measure on $[A, +\infty)$ for all large A, in which case the function u_0 cannot be bounded from below by a positive constant times any function $e^{-\lambda x}$ for large x. A typical example is when ρ is periodic and vanishes periodically, as in the joint

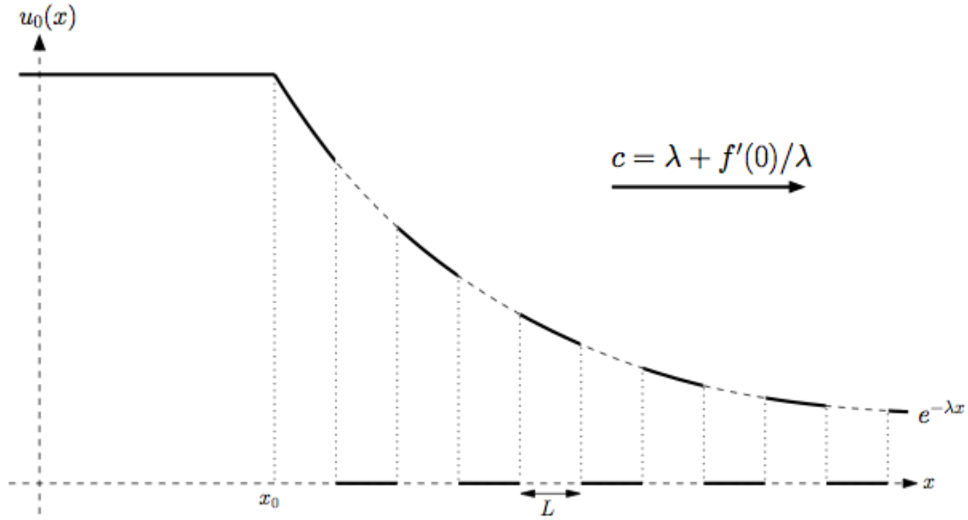


figure. However, for the conclusion (2.18) to hold, the function ρ cannot be too close to 0 on a too large set, this is roughly speaking the meaning of condition (2.17). The simplest example is when ρ is periodic: the function ρ may vanish periodically but, unless it vanishes almost everywhere, the spreading speeds $w_*(u_0)$ and $w^*(u_0)$ are equal to c_λ.

Another enlightening application of Theorem 2.1 is the following one. Let $0<\lambda_1<\lambda_2<\lambda^*$ be fixed, let ρ_1 and ρ_2 be two given bounded nonnegative periodic functions with positive averages, let $u_{1,0}$ and $u_{2,0}$ be two given front-like functions such that $u_{1,0}(x) = \rho_1(x)\,e^{-\lambda_1 x}$ and $u_{2,0}(x) = \rho_2(x)\,e^{-\lambda_2 x}$ for large x, and let u_1 and u_2 be the solutions of (1.1) with initial conditions $u_{1,0}$ and $u_{2,0}$, respectively. It follows from Theorem 2.1 (actually, Corollary 2.3) that u_1 and u_2 spread at the speeds $c_{\lambda_1} = \lambda_1 + f'(0)/\lambda_1$ and $c_{\lambda_2} = \lambda_2 + f'(0)/\lambda_2$, respectively. Let now u_0 be a front-like function such that

$$u_0(x) = \rho_1(x)\,e^{-\lambda_1 x} + \rho_2(x)\,e^{-\lambda_2 x} \ \text{ for large } x$$

and let u be the solution of (1.1) with initial condition u_0. Since u_0 is equal to a linear combination of the functions $u_{1,0}$ and $u_{2,0}$ near $+\infty$, one could have thought that u would have spread at a speed which would have been a sort of average of c_{λ_1} and c_{λ_2}. This is actually not the case, since Theorem 2.1 implies that

$$w_*(u_0) = w^*(u_0) = c_{\lambda_1}.$$

In other words, u spreads at the largest speed, that is the one given only from the slowest exponential decay. Indeed, for large x, $u_0(x) = \rho(x)\,e^{-\lambda_1 x}$, where $\rho(x) = \rho_1(x) + \rho_2(x)\,e^{-(\lambda_2-\lambda_1)x}$ is bounded near $+\infty$; since $\rho \geq \rho_1 \geq 0$ and ρ_1 has a positive average, the condition (2.17) is fulfilled and the conclusion (2.18) holds with $\lambda = \lambda_1$.

When, in Theorem 2.1, the function Λ is equal to the constant λ, the method we use to prove Theorem 2.1 gives actually more than (2.18) under assumption (2.17). Namely, it implies that the solution u of (1.1) is asymptotically almost trapped between two travelling fronts solutions with speed $c_\lambda = \lambda + f'(0)/\lambda$, in the following sense.

Proposition 2.4 *Let f, λ, u_0, x_0, ρ and Λ be as in Theorem* 2.1 *and assume furthermore that $\Lambda = \lambda$ on $[x_0, +\infty)$ and that* (2.17) *holds. Then there exist two real numbers x_1 and x_2 such that*

$$U_{c_\lambda}(x + x_1) \le \liminf_{t\to+\infty} u(t, x + c_\lambda t) \le \limsup_{t\to+\infty} u(t, x + c_\lambda t) \le U_{c_\lambda}(x + x_2) \qquad (2.20)$$

uniformly in $x \in \mathbb{R}$.

This result means that the solution u is, at large time, as close as wanted from two shifts of the travelling front U_{c_λ} in the moving frame with speed c_λ (in the case when Λ depends on x, the conclusion is not true in general, see the comment below on the position of the level sets of u at large time). However, even when Λ is constant, formula (2.20) does not mean that $u(t, \cdot + c_\lambda t)$ is truly trapped between two shifts of U_{c_λ}, even for large t. Indeed, for instance, if $0 < \mathrm{esssup}_{\mathbb{R}} u_0 = M_0 < 1$, then $\sup_{\mathbb{R}} u(t, \cdot) \le M(t)$ for all $t \ge 0$, where $\dot{M}(t) = f(M(t))$ for all $t \ge 0$ and $M(0) = M_0$. Since $M(t) < 1$ for all $t \ge 0$ and since $U_{c_\lambda}(-\infty) = 1$, the function $u(t, \cdot + c_\lambda t)$ can never be larger than any shift of U_{c_λ}. Proposition 2.4 does not mean either that the solution in the moving frame, that is $u(t, \cdot + c_\lambda t)$, converges to a shift of the front U_{c_λ}. The solution may well oscillate without converging between two shifts of the front U_{c_λ}, as proved by Bages, Martinez and Roquejoffre [2, 29] under the additional assumption that f is concave. Actually, more precise estimates of the time-dependent shift, which also hold for more geneal periodic equations in cylindrical domains, are given in [2, 29] when u_0 is assumed to be trapped between two finite shifts of the same front U_{c_λ}.

Lastly, when u_0 is not exponentially bounded as $x \to +\infty$, in the sense of (1.13), then $w_*(u_0) = w^*(u_0) = +\infty$, as already noticed. More generally speaking, Theorem 2.1 still holds when $\lambda = 0$, if the condition (2.17) is fulfilled, implying that the solution u spreads with infinite speed.

Corollary 2.5 *Under the same notations as in Theorem* 2.1 *but with $\lambda = 0$, and under the assumption* (2.17)*, one has*

$$w_*(u_0) = w^*(u_0) = +\infty. \qquad (2.21)$$

It is possible to reformulate the above results in terms of the level sets of the solution u of the Cauchy problem (1.1). Namely, given a front-like initial condition u_0, define the level set of u for a value $m \in (0, 1)$ at a time $t > 0$, as follows:

$$E_m(t) = \big\{x \in \mathbb{R},\ u(t, x) = m\big\}.$$

For a given $m \in (0, 1)$, this set can be empty, but it is easy to see that it is non-empty and compact when t is sufficiently large. Now, from Definition 1.2 and under the hypotheses of Theorem 2.1 with $0 < \lambda < \lambda^*$ (resp. Corollary 2.5 with $\lambda = 0$), we can reformulate the conclusions (2.18) and (2.21) into:

$$\forall\, m \in (0, 1),\ \lim_{t\to+\infty} \frac{1}{t} \min E_m(t) = \lim_{t\to+\infty} \frac{1}{t} \max E_m(t) = c_\lambda = \lambda + \frac{f'(0)}{\lambda} \qquad (2.22)$$

with the convention that $c_0 = +\infty$. In other words, for any $m \in (0,1)$ and any family of real numbers $(x_m(t))_{t>0}$ such that $u(t, x_m(t)) = m$ for large t, then $x_m(t)/t \to c_\lambda$ as $t \to +\infty$. Thus, the quantity c_λ is the asymptotic time-averaged speed of *all* level sets of u. We mention here that another notion of speed, that of bulk burning rate defined, under additional assumptions on u_0, as the integral of $\partial_t u$ over $\mathbb{R}$, was also introduced in [9] (see also [23] for further estimates). The bulk burning rate can then be viewed as a space-averaged speed and, of course, the bulk burning rate and the spreading speeds defined in Definition 1.2 coincide at large time when the solution u converges globally to a travelling front.

As far as Proposition 2.4 is concerned, its conclusion (2.20) implies in particular that, for all $m \in (0,1)$,

$$\limsup_{t\to+\infty} \big|\max E_m(t) - c_\lambda t\big| < +\infty \ \text{ and } \ \limsup_{t\to+\infty} \big|\min E_m(t) - c_\lambda t\big| < +\infty, \tag{2.23}$$

Property (2.23) is clearly stronger than (2.22). Both (2.22) and (2.23) also yield formula (2.18), since, as it can be easily seen, $\liminf_{x\to-\infty} u(t,x) \to 1$ as $t \to +\infty$ and $\lim_{x\to+\infty} u(t,x) = 0$ for all $t \geq 0$. However, it is worth noticing here that, in general, the only assumptions of Theorem 2.1 do not guarantee that the level sets $E_m(t)$ stay at finite distance as $t \to +\infty$ from the position $c_\lambda t$ for each fixed $m \in (0,1)$. For instance, if u_0 is front-like and $u_0(x) = e^{-\Lambda(x)x}$ for large x with $\lim_{x\to+\infty} \Lambda(x) = \lambda \in (0, \lambda^*)$ and $\lim_{x\to+\infty}(\Lambda(x) - \lambda)x = +\infty$ (resp. $-\infty$), it then follows from the comparison principle and the general convergence results (1.9), that

$$\max E_m(t) - c_\lambda t \to -\infty \ \big(\text{resp. } \min E_m(t) - c_\lambda t \to +\infty\big) \text{ as } t \to +\infty$$

for all value $m \in (0,1)$, while $w_*(u_0) = w^*(u_0) = c_\lambda$ from Theorem 2.1. On the other hand, if $|\Lambda(x) - \lambda| = O(x^{-1})$ as $x \to +\infty$ and if (2.17) is fulfilled, then Proposition 2.4 and the maximum principle imply that (2.20) holds, whence (2.23).

In all above results, the solutions u of (1.1) have a well defined spreading speed, that is $w_*(u_0) = w^*(u_0)$, and this quantity is explicitely expressed in terms of the asymptotic behavior of the front-like initial condition at $+\infty$. We now exhibit a class of front-like initial data u_0 for which $w_*(u_0) < w^*(u_0)$. We not only prove that for some range of speeds c, the functions $t \mapsto u(t, ct + x)$ do not converge as $t \to +\infty$, but also that their $\omega-$limit sets are the whole interval $[0,1]$. We recall that the $\omega-$limit set as $t \to +\infty$ of a function $t \mapsto g(t) \in [0,1]$ defined in a neighborhood of $+\infty$ is the set of all $s \in [0,1]$ for which there exists a sequence $t_n \to +\infty$ such that $g(t_n) \to s$ as $n \to +\infty$. Given a function f satisfying (1.2), we denote

$$M_f = \max_{s\in[0,1]} f'(s) > 0.$$

From comparisons with KPP-type nonlinearities, it follows that $c^* \leq 2\sqrt{M_f}$, where c^* is the minimal speed of travelling fronts with nonlinearity f (see also [15]).

Theorem 2.6 *Let f satisfy* (1.2) *and* (2.15) *and let $\gamma_1 < \gamma_2$ be given in the interval $[2\sqrt{M_f}, +\infty]$. Then there exists a front-like function u_0 such that*

$$\gamma_1 = w_*(u_0) < w^*(u_0) = \gamma_2.$$

Furthermore, for any $c \in (\gamma_1, \gamma_2)$*, any* $x \in \mathbb{R}$ *and any* $m \in (0,1)$*, the* $\omega-$*limit set of the function* $t \mapsto u(t, ct+x)$ *as* $t \to +\infty$ *is equal to the whole interval* $[0,1]$ *and the* $\omega-$*limit sets of the functions* $t \mapsto t^{-1} \min E_m(t)$ *and* $t \mapsto t^{-1} \max E_m(t)$ *are equal to the whole interval* $[\gamma_1, \gamma_2]$*.*

The initial data u_0 are constructed in such a way that they oscillate as $x \to +\infty$ between the two exponential functions $e^{-\lambda_1 x}$ and $e^{-\lambda_2 x}$ on larger and larger space-intervals, with $\gamma_1 = c_{\lambda_1}$ (or $\lambda_1 = \lambda^*$ if $\gamma_1 = c^*$) and $\gamma_2 = c_{\lambda_2}$. The proof then shows that the solution u of (1.1) oscillates on larger and larger time-intervals between two approximate solutions moving with speeds close to γ_1 and γ_2, so that the averaged speeds of the level sets, namely $\min E_m(t)/t$ and $\max E_m(t)/t$, oscillate infinitely many times between γ_1 and γ_2. Therefore, the level sets do not converge in speed to any real number as $t \to +\infty$. We refer to Section 4 for the details. It is worth noticing that, for such monostable problems, this completely new and highly non-trivial oscillating dynamics is present even in the simplest case of the one-dimensional homogeneous equation (1.1). It also holds for general monostable functions f satisfying (1.2) and (2.15), provided that the chosen speeds γ_1 and γ_2 are large enough. Notice that, in the case when $f'(s) \le f'(0)$ for all $s \in [0,1]$, then $M_f = f'(0)$ and $c^* = 2\sqrt{M_f}$. Hence, in this case, the speeds γ_1 and γ_2 can take any values between c^* and $+\infty$.

The proofs of the above results rely firstly on the maximum principle and on the construction of suitable sub- and supersolutions for the Cauchy problem (1.1). The gaussian decay of the heat kernel plays a crucial role in the proof of Theorem 2.6. We have to estimate sharply the time-depending behavior of $u(t,x)$ as $x \to +\infty$ and we prove that these tails force the solution to spread at the desired approximated speeds on large time-intervals. It is important to point out that, even if the spreading properties are determined through the asymptotic behavior of $u(t,x)$ as $x \to +\infty$, that is as $u \to 0$, the function f may not need to be concave or even of the KPP type (1.4).

Remark 2.7 Similar propagation phenomena have been shown by Hamel and Sire [19] for ignition-type nonlinearities arising in combustion theory (see e.g.[21]), that is functions f for which there exists $\theta \in (0,1)$ such that

$$f(s) = 0 \text{ if } s \in [0,\theta] \cup \{1\}, \;\; f(s) > 0 \text{ if } s \in (\theta, 1). \tag{2.24}$$

It is known that for such nonlinearities, for all $\alpha \in [0,\theta)$, there exists a unique speed $\widetilde{c}_\alpha$ so that there exists a travelling front that connects α to 1 with speed $\widetilde{c}_\alpha$, see [21]. The front-like initial data $u_0 : \mathbb{R} \to [0,1]$ are then defined as follows: $\liminf_{x\to-\infty} u_0(x) > \theta$ and $\limsup_{x\to+\infty} u_0(x) < \theta$ (up to a negligible set). For such nonlinearities f, the same definition for the minimal spreading speed $w_*(u_0)$ as in Definition 1.2 is taken, but the maximal spreading speed $w^*(u_0)$ is now defined by

$$w^*(u_0) = \inf\Big\{c \in \mathbb{R}, \; \limsup_{t\to+\infty} \Big(\sup_{x \ge ct} u(t,x)\Big) \le \theta\Big\},$$

since the whole interval $[0,\theta]$ corresponds to the set of weakly unstable zeroes of the function f. More general heterogeneous problems in higher dimensions have been considered in [19], but, as far as the homogeneous one-dimensional equation (1.1) is concerned, the results of Hamel and Sire are the following ones: if u_0 is front-like and if $u_0(x) - p(x) \to 0$ as $x \to +\infty$, where p is a periodic function with periodic average $\overline{p}$, then $w_*(u_0) = w^*(u_0) = \widetilde{c}_{\overline{p}}$. But in the general case, the authors constructed a class of initial data u_0 such that $w_*(u_0) < w^*(u_0)$. Moreover, their construction gives that, for such u_0, for any $c \in (w_*(u_0), w^*(u_0))$ and any $x \in \mathbb{R}$, the ω-limit set of $t \mapsto u(t, ct + x)$ is $[\alpha, 1]$, where $\alpha \in [0, \theta)$ is defined by $\widetilde{c}_\alpha = w_*(u_0)$. It is interesting to notice that, despite their similarities, the results and proofs of [19] and the present paper are different in nature. For instance, a front-like initial condition u_0 which oscillates periodically at $+\infty$ between two constants α and β in $[0, \theta)$ for equation (1.1) with (2.24) leads to a solution u spreading at an average speed belonging to the open interval $(\widetilde{c}_\alpha, \widetilde{c}_\beta)$. On the other hand, under the condition (1.2), a front-like initial condition which oscillates periodically at $+\infty$ between two exponential tails $e^{-\lambda_1 x}$ and $e^{-\lambda_2 x}$ with $0 < \lambda_1 < \lambda_2 < \lambda^*$ leads to a solution u spreading at the speed c_{λ_1}. Furthermore, although the equation (1.1) reduces to the heat equation when $u < \theta$ under assumption (2.24), the propagation phenomena and the proofs in this case are rather nonlinear in nature, whereas the spreading properties stated in Theorems 2.1 and 2.6 of the present paper are chiefly determined by the asymptotic behavior of u_0 when it approaches 0 and then by the linearization of (1.1) around $u = 0$, even if the function f does not satisfy the KPP assumption (1.4). Lastly, we mention that when the nonlinearity f is of the bistable type on $[0, 1]$, that is when there exists $\theta \in (0, 1)$ such that

$$f(0) = f(\theta) = f(1) = 0,\ f < 0 \text{ on } (0, \theta),\ f > 0 \text{ on } (\theta, 1),\ f'(0) < 0,\ f'(1) < 0,$$

then the situation is much simpler: there is a unique (up to shifts) travelling front $U_c(x - ct)$ connecting 0 to 1, with a unique speed c, and, for any "front-like" initial datum u_0, namely $0 \le u_0(x) \le 1$ for a.e. $x \in \mathbb{R}$ and $\liminf_{x \to -\infty} u_0(x) > \theta > \limsup_{x \to +\infty} u_0(x)$, then $u(t, x)$ converges to $U_c(x - ct + x_0)$ uniformly in $x \in \mathbb{R}$ as $t \to +\infty$, for some $x_0 \in \mathbb{R}$, see [11]. Thus, in the bistable case, the solutions u spread at the unique speed c for all front-like initial conditions u_0. We refer to [42] for a much more complete picture in heterogeneous media.

3 The case when the spreading speed is unique

This section is devoted to the proof of Theorem 2.1 and its corollaries. It is based on the construction of sub- and supersolutions moving asymptotically at the speed c_λ, and on the basic interpretation of the solutions of the linearized problem (3.25) below in terms of the solutions of the heat equation (2.16). As a matter of fact, we first prove Theorem 2.1 when the function Λ is a constant. Namely, we prove Proposition 2.4, which implies the conclusion (2.18) of Theorem 2.1 when $\Lambda(x) = \lambda$ in a neighborhood of $+\infty$.

Proof of Proposition 2.4. As $0 < \lambda < \lambda^*$, one has $\lambda^2 - \lambda c^* + f'(0) > 0$. In other words, $c_\lambda > c^*$. As recalled in the introduction, we know that there exists a travelling front solution $U_{c_\lambda}(x - c_\lambda t)$ of (1.1) with speed c_λ, and such that $U_{c_\lambda}(x) \sim M\, e^{-\lambda x}$ as $x \to +\infty$, for some $M > 0$. Since ρ is bounded, there exists then $\widetilde{M} > 0$ such that $u_0(x) \le \overline{u}_0(x)$ for a.e. $x \in \mathbb{R}$, where

$$\overline{u}_0(x) = \min\big(1, \widetilde{M}\, e^{-\lambda x}\big).$$

Let now $\overline{u}$ be the solution of (1.1) with initial condition $\overline{u}_0$. Because of (1.9), there exists a real number x_2 such that

$$\sup_{x\in\mathbb{R}} \Big|\overline{u}(t,x) - U_{c_\lambda}(x - c_\lambda t + x_2)\Big| \to 0 \ \text{ as } t \to +\infty.$$

But the maximum principle yields

$$u(t,x) \le \overline{u}(t,x) \text{ for all } t > 0 \text{ and } x \in \mathbb{R},$$

which provides the right inequality in (2.20). Furthermore, for any speed c such that $c > c_\lambda$, one has

$$\begin{aligned} 0 \le \limsup_{t\to+\infty}\Big(\max_{x\ge ct} u(t,x)\Big) \le \limsup_{t\to+\infty}\Big(\max_{x\ge ct} \overline{u}(t,x)\Big) &= \limsup_{t\to+\infty}\Big(\max_{x\ge ct} U_{c_\lambda}(x - c_\lambda t + x_2)\Big) \\ &= \limsup_{t\to+\infty} U_{c_\lambda}(ct - c_\lambda t + x_2) = 0, \end{aligned}$$

which implies that $w^*(u_0) \le c_\lambda$.

In order to prove the left inequality in (2.20), and consequently $w_*(u_0) \ge c_\lambda$, consider the solution ξ of the linear problem

$$\begin{cases} \partial_t \xi - \partial_{xx}\xi = f'(0)\xi, & t > 0,\ x \in \mathbb{R}, \\ \xi(0,x) = \overline{\rho}(x)\, e^{-\lambda x} & \text{for a.e. } x \in \mathbb{R}. \end{cases} \tag{3.25}$$

From the definition of c_λ, the maximum principle yields

$$\xi(t,x) \le \|\overline{\rho}\|_{L^\infty(\mathbb{R})}\, e^{-\lambda(x - c_\lambda t)} \text{ for all } t > 0 \text{ and } x \in \mathbb{R}. \tag{3.26}$$

The key-point here is to observe that the function $(t,x) \mapsto e^{-\lambda(x-c_\lambda t)}\zeta(t, x-2\lambda t)$ solves (3.25), since ζ solves (2.16). Thus, by uniqueness, one has

$$\xi(t,x) = e^{-\lambda(x-c_\lambda t)}\zeta(t, x - 2\lambda t) \tag{3.27}$$

for all $t > 0$ and $x \in \mathbb{R}$.

Remember now that $s_0 \in (0,1)$, $\gamma > 0$ and $C > 0$ are given in (2.15). Define $P(\beta) = \beta^2 - \beta c_\lambda + f'(0)$ for all $\beta \in \mathbb{R}$. This function P is decreasing on the interval $[0,\lambda]$ since $\lambda > 0$ is its smallest simple zero, and one has $2\lambda < c_\lambda$. Choose $\varepsilon > 0$ and $\kappa \in (0,1]$ small enough so that

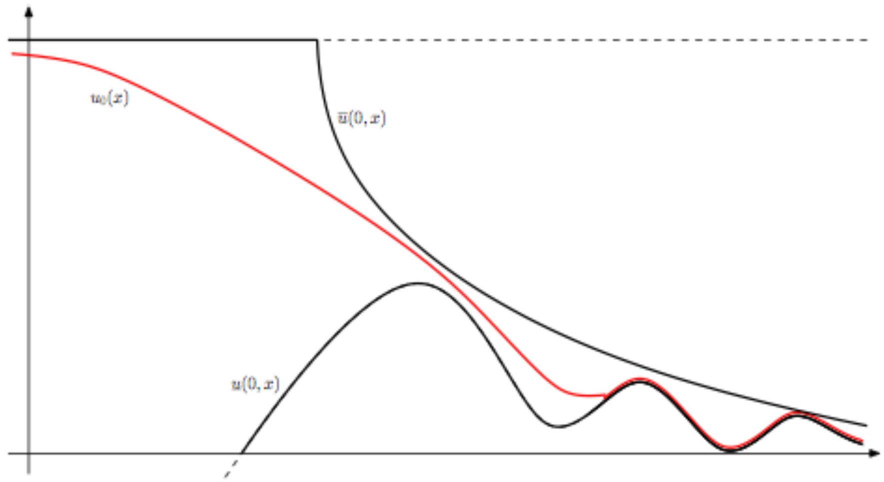
$$(1+\gamma)\,\lambda \ge \lambda + \varepsilon \text{ and } (c_\lambda - 2\lambda - \varepsilon)\,\varepsilon \ge \kappa^\gamma.$$

Next, owing to (3.26), choose $A > 0$ large enough so that $A \ge C\,\|\overline{\rho}\|^{1+\gamma}_{L^\infty(\mathbb{R})}$ and

$$\forall\,(t,x) \in (0,+\infty)\times\mathbb{R},\ \Big(\xi(t,x) > A\,e^{-(\lambda+\varepsilon)(x-c_\lambda t)}\Big) \Longrightarrow \Big(x \ge \max(c_\lambda t, x_0)\Big)$$

and

$$\kappa\left(\xi(t,x) - A\,e^{-(\lambda+\varepsilon)(x-c_\lambda t)}\right) \le s_0 \text{ for all } t > 0 \text{ and } x \in \mathbb{R}.$$



Lastly, set

$$\underline{u}(t,x) = \max\Big(0, \kappa\left(\xi(t,x) - A\,e^{-(\lambda+\varepsilon)(x-c_\lambda t)}\right)\Big) \tag{3.28}$$

in $[0,+\infty)\times\mathbb{R}$ (see the joint figure for a schematic shape of the functions $\underline{u}$, u and $\overline{u}$ at time $t=0$). It follows that

$$\Omega = \big\{(t,x)\in(0,+\infty)\times\mathbb{R},\ \underline{u}(t,x)>0\big\} \subset \big\{(t,x)\in(0,+\infty)\times\mathbb{R},\ x\geq \max(c_\lambda t, x_0)\big\}$$

and

$$\sup_{(t,x)\in(0,+\infty)\times\mathbb{R}} \underline{u}(t,x) \leq s_0.$$

Let us then check that $\underline{u}$ is a subsolution for problem (1.1). When $(t,x)\in\Omega$, one has

$$\begin{array}{rcl} \partial_t\underline{u}(t,x) - \partial_{xx}\underline{u}(t,x) - f'(0)\,\underline{u}(t,x) & = & \big((\lambda+\varepsilon)^2 - (\lambda+\varepsilon)c_\lambda + f'(0)\big)\,\kappa\,A\,e^{-(\lambda+\varepsilon)(x-c_\lambda t)} \\ & = & -(c_\lambda - 2\lambda - \varepsilon)\,\varepsilon\,\kappa\,A\,e^{-(\lambda+\varepsilon)(x-c_\lambda t)} \\ & \leq & -(c_\lambda - 2\lambda - \varepsilon)\,\varepsilon\,\kappa\,A\,e^{-(1+\gamma)\lambda(x-c_\lambda t)} \\ & \leq & -\kappa^{1+\gamma} A\,e^{-(1+\gamma)\lambda(x-c_\lambda t)} \\ & \leq & -C\,\underline{u}(t,x)^{1+\gamma} \end{array}$$

from (3.26) and the choice of ε, κ and A. Therefore, $\partial_t\underline{u} - \partial_{xx}\underline{u} \leq f(\underline{u})$ in Ω because of (2.15). It also follows from the definition of $\xi(0,\cdot)$ and from the choice of A and the inequality $0<\kappa\leq 1$, that $\underline{u}(0,x)\leq u(0,x)$ for a.e. $x\in\mathbb{R}$. Summing up, as $\underline{u}=0$ in $(0,+\infty)\times\mathbb{R}\setminus\Omega$, the function $\underline{u}$ is a subsolution of (1.1). Thus

$$u(t,x)\geq \underline{u}(t,x)\ \text{ for all } (t,x)\in(0,+\infty)\times\mathbb{R} \tag{3.29}$$

from the maximum principle.

Lastly, let $\tau>0$ be any positive real number. On the one hand, the maximum principle implies that $\liminf_{x\to-\infty} u(t,x)\geq\theta(t)$ for all $t>0$, where $\dot\theta(t) = f(\theta(t))$ in $[0,+\infty)$ and $\theta(0) = \liminf_{x\to-\infty}\big(\mathrm{essinf}_{(-\infty,x)} u_0\big)>0$. Hence

$$\liminf_{x\to-\infty} u(t,x) \geq \theta(t) > 0 \text{ for all } t>0. \tag{3.30}$$

Since u is a continuous positive function on $(0,+\infty)\times\mathbb{R}$, there exists then $\eta\in(0,1)$ such that $u(\tau,x)\ge\eta$ for all $x\le 0$. On the other hand, remember from Remark 2.2 and assumption (2.17) that $\eta':=\inf_{x\in\mathbb{R}}\zeta(\tau,x)>0$. Therefore, it follows from (3.27), (3.28) and (3.29) that

$$1\ge u(\tau,x)\ge v_0(x):=\eta\,\mathbf{1}_{(-\infty,0]}(x) + \max\big(0,\kappa\,\eta'\,e^{-\lambda(x-c_\lambda\tau)}-\kappa\,A\,e^{-(\lambda+\varepsilon)(x-c_\lambda\tau)}\big)\,\mathbf{1}_{(0,+\infty)}(x)\ge 0 \tag{3.31}$$

for all $x\in\mathbb{R}$. Let v denote the solution of (1.1) with initial condition v_0. The maximum principle implies that

$$u(t,x)\ge v(t-\tau,x)\text{ for all }t\ge\tau\text{ and }x\in\mathbb{R}. \tag{3.32}$$

But v_0 is front-like and $v_0(x)\sim\eta''\,e^{-\lambda x}$ as $x\to+\infty$, with $\eta''=\kappa\,\eta'\,e^{\lambda\,c_\lambda\tau}>0$. It follows then from [40] that there exists $\widetilde{x}_1\in\mathbb{R}$ such that

$$v(t,x+c_\lambda t)\to U_{c_\lambda}(x+\widetilde{x}_1)\text{ uniformly in }x\text{ as }t\to+\infty. \tag{3.33}$$

The inequality (3.32) then gives the left inequality in (2.20) with $x_1=\widetilde{x}_1+c_\lambda\tau$. The proof of Proposition 2.4 is thereby complete. Observe finally that, as done above for $w^*(u_0)$, the left inequality in (2.20) also yields $w_*(u_0)\ge c_\lambda$, since $U_{c_\lambda}(-\infty)=1$. Eventually, $w_*(u_0)=w^*(u_0)=c_\lambda$. $\Box$

Remark 3.1 The arguments used in the proof of Proposition 2.4, namely the construction of the subsolution $\underline{u}$ and the inequality (3.31), imply that, for all $t>0$, $u(t,\cdot)$ is front-like and $\liminf_{x\to+\infty}u(t,x)\,e^{\lambda x}>0$. As a matter of fact, the quantity $u(t,x)\,e^{\lambda x}$ is also bounded as $x\to+\infty$ for all $t>0$. Indeed, denote $L=\sup_{s\in(0,1]}f(s)/s$. It follows from the maximum principle that

$$u(t,x)\le\widetilde{M}\,e^{(\lambda^2+L)t-\lambda x}\quad\text{for all }t>0\text{ and }x\in\mathbb{R}, \tag{3.34}$$

where $0<\widetilde{M}=\|u_0(x)\,e^{\lambda x}\|_{L^\infty(\mathbb{R})}<+\infty$. Thus, $\limsup_{x\to+\infty}u(t,x)\,e^{\lambda x}<+\infty$ for all $t>0$. Furthermore, if f satisfies (1.4), then $L=f'(0)$ and (3.34) implies that $u(t,x)\le\widetilde{M}\,e^{-\lambda(x-c_\lambda t)}$ for all $t>0$ and $x\in\mathbb{R}$, which directly gives $w^*(u_0)\le c_\lambda$.

Remark 3.2 In the proof of Proposition 2.4, the left inequality in (2.20) implies immediately that $w_*(u_0)\ge c_\lambda$. The proof uses (3.32) and the convergence result (3.33), given that v_0 is front-like and decays with the right exponential decay $e^{-\lambda x}$ as $x\to+\infty$. However, one could also derive the weaker inequality $w_*(u_0)\ge c_\lambda$ without referring to the stronger properties (3.32) and (3.33), using only (3.29) and (3.30). Indeed, with the same notations as in the proof of Proposition 2.4, remember that $\inf_{[T,+\infty)\times\mathbb{R}}\zeta\ge\inf_{\mathbb{R}}\zeta(T,\cdot)>0$ and choose $\delta_0>0$ large enough such that $\inf_{[T,+\infty)\times\mathbb{R}}\zeta\ge 2\,A\,\kappa^{-1}\,e^{-\varepsilon\delta_0}$. It follows from (3.27), (3.28) and (3.29) that, for all $t\ge T$,

$$u(t,c_\lambda t+\delta_0)\ge\kappa\,e^{-\lambda\delta_0}\zeta(t,(c_\lambda-2\lambda)t+\delta_0)-A\,e^{-(\lambda+\varepsilon)\delta_0}\ge A\,e^{-(\lambda+\varepsilon)\delta_0}>0. \tag{3.35}$$

Let $\tau>0$ be arbitrary and define

$$Q=\big\{(t,x)\in[\tau,+\infty)\times\mathbb{R},\ x\le c_\lambda t+\delta_0\big\}.$$

Since the function u is continuous and positive in $(0,+\infty)\times\mathbb{R}$, it follows from (3.30) and (3.35) that $\alpha := \inf_{(t,x)\in\partial Q} u(t,x) \in (0,1)$. Since $f(\alpha)>0$, the weak maximum principle yields $u\geq\alpha$ in Q. Consider now any real number c such that $c<c_\lambda$ and assume by contradiction that there exist $\varepsilon_0>0$ and a sequence $(t_n,x_n)_{n\in\mathbb{N}}$ in $(0,+\infty)\times\mathbb{R}$ such that

$$t_n\to+\infty \text{ as } n\to+\infty, \text{ and } x_n\leq ct_n \text{ and } u(t_n,x_n)\leq 1-\varepsilon_0 \text{ for all } n\in\mathbb{N}.$$

Set $v_n(t,x)=u(t+t_n,x+x_n)$. From the Schauder parabolic estimates, the functions v_n converge in $\mathcal{C}^{1,2}_{loc}(\mathbb{R}\times\mathbb{R})$, up to extraction of a subsequence, to a solution v_∞ of

$$\partial_t v_\infty - \partial_{xx} v_\infty = f(v_\infty) \text{ in } \mathbb{R}\times\mathbb{R}$$

such that $0\leq v_\infty\leq 1$. Moreover, for all $(t,x)\in\mathbb{R}\times\mathbb{R}$, there exist $n_0\in\mathbb{N}$ large enough so that $(t+t_n,x+x_n)\in Q$ for all $n\geq n_0$, since $x_n\leq ct_n$ and $c<c_\lambda$. Thus $v_\infty\geq\alpha$ in $\mathbb{R}\times\mathbb{R}$. In particular, it follows from the maximum principle that $v_\infty(t,x)\geq\omega(t-t_0)$ for all $t_0\in\mathbb{R}$ and for all $(t,x)\in[t_0,+\infty)\times\mathbb{R}$, where $\dot\omega(t)=f(\omega(t))$ in $[0,+\infty)$ and $\omega(0)=\alpha$. Since $\omega(+\infty)=1$, one concludes, by passing to the limit as $t_0\to-\infty$, that $v_\infty(t,x)\geq 1$ for all $(t,x)\in\mathbb{R}\times\mathbb{R}$, which contradicts $v_\infty(0,0)\leq 1-\varepsilon_0$. Thus, $\inf_{x\leq ct} u(t,x)\to 1$ as $t\to+\infty$ for all $c<c_\lambda$, whence $w_*(u_0)\geq c_\lambda$.

Proof of Theorem 2.1. Take $\varepsilon>0$ such that $\lambda+\varepsilon<\lambda^*$, and then $x_1\geq\max(x_0,0)$ large enough so that $\Lambda(x)\leq\lambda+\varepsilon$ for all $x>x_1$. Set

$$v_{\varepsilon,0}(x)=\begin{cases} u_0(x) & \text{if } x\leq x_1,\\ \rho(x)\,e^{-(\lambda+\varepsilon)x} & \text{if } x>x_1.\end{cases}$$

Then $v_{\varepsilon,0}\leq u_0$ a.e. in $\mathbb{R}$ and thus $w_*(u_0)\geq w_*(v_{\varepsilon,0})$, using the maximum principle and Definition 1.2 of the minimal spreading speed w_*. Moreover, we know from Proposition 2.4 that

$$w_*(v_{\varepsilon,0})=\lambda+\varepsilon+\frac{f'(0)}{\lambda+\varepsilon}.$$

Hence, $w_*(u_0)\geq\lambda+\varepsilon+f'(0)/(\lambda+\varepsilon)$ for all $\varepsilon>0$, which gives $w_*(u_0)\geq\lambda+f'(0)/\lambda$. A similar argument leads to the inequality $w^*(u_0)\leq\lambda+f'(0)/\lambda$. The inequality $w_*(u_0)\leq w^*(u_0)$ completes the proof. $\Box$

Proof of Corollary 2.3. Consider the solution ζ of (2.16), that is

$$\begin{cases} \partial_t\zeta-\partial_{xx}\zeta=0, & t>0,\ x\in\mathbb{R},\\ \zeta(0,x)=\overline{\rho}(x) & \text{for a.e. } x\in\mathbb{R},\end{cases}$$

where the function ρ is now assumed to be a bounded, nonnegative function on $\mathbb{R}$ with positive average ρ_m, and $\overline{\rho}(x)=\rho(x)$ if $x>x_0$ and $\overline{\rho}(x)=1$ if $x<x_0$. Choose any real number $\eta>0$ such that $\eta\leq 1$ and $\eta\|\rho\|_{L^\infty(\mathbb{R})}\leq 1$. Thus, $\overline{\rho}(x)\geq\eta\,\rho(x)$ for a.e. $x\in\mathbb{R}$ and

$$\zeta(t,x)\geq\widetilde{\zeta}(t,x) \text{ for all } (t,x)\in(0,+\infty)\times\mathbb{R},$$

where $\widetilde{\zeta}$ denotes the solution of the heat equation $\partial_t\widetilde{\zeta} = \partial_{xx}\widetilde{\zeta}$ with initial condition $\eta\,\rho$. But it is elementary to see that $\widetilde{\zeta}(t,x) \to \eta\,\rho_m$ as $t \to +\infty$ uniformly in $x \in \mathbb{R}$ (we give a quick proof in the next paragraph for the sake of completeness). Since $\eta > 0$ and ρ_m is positive by assumption, condition (2.17) is fulfilled and we can thus apply Theorem 2.1, which provides (2.18).

Let us now check that $\widetilde{\zeta}(t,x) \to \eta\,\rho_m$ as $t \to +\infty$ uniformly in $x \in \mathbb{R}$. By linearity, it is sufficient to consider the case $\eta = 1$. For all $(x,y) \in \mathbb{R}^2$, set $R_x(y) = \int_x^y \rho(z)dx$. For all $(t,x) \in (0,+\infty) \times \mathbb{R}$, one has

$$\widetilde{\zeta}(t,x) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{\frac{-y^2}{4t}} \rho(x-y)dy = \frac{-1}{\sqrt{4\pi t}} \int_{\mathbb{R}} \frac{y}{2t} e^{\frac{-y^2}{4t}} R_x(x-y)dy$$

after integrating by parts (notice that $|R_x(x-y)| = O(|y|)$ as $|y| \to +\infty$). Let $\varepsilon > 0$ be arbitrary. Since ρ is assumed to have the average ρ_m, there exists $A > 0$ such that $|-y^{-1}R_x(x-y) - \rho_m| \leq \varepsilon$ for all $|y| \geq A$ and for all $x \in \mathbb{R}$. Thus, for all $(t,x) \in (0,+\infty) \times \mathbb{R}$,

$$\begin{aligned} |\widetilde{\zeta}(t,x) - \rho_m| \;\leq\; & \frac{1}{\sqrt{4\pi t}} \int_{-A}^{A} \frac{|y|}{2t} e^{-\frac{y^2}{4t}} |R_x(x-y)|dy + \frac{\varepsilon}{\sqrt{4\pi t}} \int_{|y|\geq A} \frac{y^2}{2t} e^{-\frac{y^2}{4t}} dy \\ & + \rho_m \times \Big| \frac{1}{\sqrt{4\pi t}} \int_{|y|\geq A} \frac{y^2}{2t} e^{-\frac{y^2}{4t}} dy - 1 \Big|. \end{aligned}$$

Since $|R_x(x-y)| \leq \|\rho\|_{L^\infty(\mathbb{R})} \times |y|$ for all $(x,y) \in \mathbb{R}^2$, the first term of the right-hand side converges to 0 as $t \to +\infty$, uniformly in $x \in \mathbb{R}$. The other two terms are independent of x and converge to ε and 0, respectively, as $t \to +\infty$. Thus, $|\widetilde{\zeta}(t,x) - \rho_m| \leq 2\varepsilon$ for t large enough, uniformly in $x \in \mathbb{R}$. This provides the desired result. □

Proof of Corollary 2.5. The same kind of argument as in the proof of Theorem 2.1 implies that for all $\varepsilon \in (0, \lambda^*)$, one has

$$w_*(u_0) \geq \varepsilon + \frac{f'(0)}{\varepsilon}.$$

We get the conclusion (2.21) by letting $\varepsilon \to 0^+$. □

4 Complex dynamics and intervals of spreading speeds

This section is devoted to the proof of Theorem 2.6. That is, we construct explicit examples of front-like initial conditions u_0 for which the minimal and maximal spreading speeds $w_*(u_0)$ and $w^*(u_0)$ are any two given strictly ordered numbers between $2\sqrt{M_f}$ and $+\infty$, where $M_f = \max_{s\in[0,1]} f'(s)$. The constructed functions u_0 oscillate at $+\infty$ between two exponentially decaying functions, with different exponential rates. The intervals of oscillation are larger and larger. They are chosen in such a way that, during some suitable time-intervals and on some space-intervals, the Gaussian estimates of the difference between the solution u and two approximated fronts is negligible.

Proof of Theorem 2.6. Let $\gamma_1 < \gamma_2$ be given in the closed interval $[2\sqrt{M_f}, +\infty] \subset [c^*, +\infty]$. If $\gamma_1 > c^*$, let $\lambda_1 \in (0, \lambda^*)$ be such that $c_{\lambda_1} = \gamma_1$, that is $\lambda_1 = (\gamma_1 - \sqrt{\gamma_1^2 - 4f'(0)})/2$. If $\gamma_1 = c^*$, set $\lambda_1 = \lambda^*$. Let also λ_2 be the unique real number in $[0, \lambda^*)$ such that $c_{\lambda_2} = \gamma_2$ (with the convention that $c_0 = +\infty$). In all cases, there holds

$$0 \leq \lambda_2 < \lambda_1 \leq \lambda^*.$$

Let $(\lambda_{2,n})_{n\in\mathbb{N}}$ be the sequence defined by

$$\forall\, n \in \mathbb{N}, \quad \begin{cases} \lambda_{2,n} = \lambda_2 & \text{if } \lambda_2 > 0, \\ \lambda_{2,n} = \dfrac{\lambda_1}{n+2} & \text{if } \lambda_2 = 0. \end{cases}$$

Let now $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be any two increasing sequences of positive real numbers such that

$$0 < x_n < y_n < \frac{\lambda_1}{\lambda_{2,n}}\, y_n < x_{n+1} - 1 < x_{n+1} \ \text{ for all } n \in \mathbb{N}$$

and

$$\lim_{n\to+\infty} \frac{y_n}{x_n} = \lim_{n\to+\infty} \frac{x_{n+1}}{(\lambda_1/\lambda_{2,n})y_n} = +\infty. \tag{4.36}$$

Typical examples are $x_n = (2n+n_0)!$ and $y_n = (2n+1+n_0)!$ if $\lambda_2 > 0$ (resp. $x_n = ((2n+n_0)!)^2$ and $y_n = ((2n+1+n_0)!)^2$ if $\lambda_2 = 0$), for some large enough integer n_0.

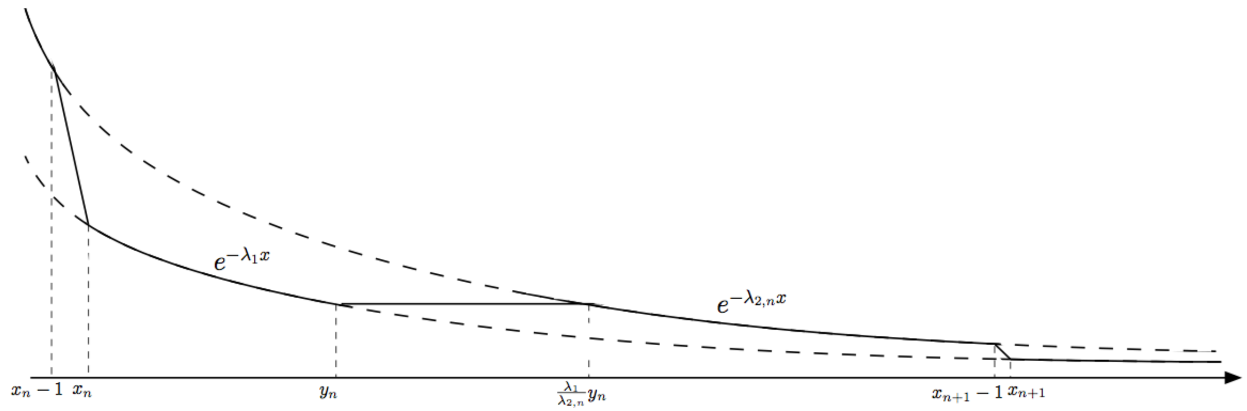
Given any such sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$, we define the function $u_0 : x \mapsto u_0(x)$ as follows:

$$u_0(x) = \begin{cases} \min(1, e^{-\lambda_1 x}) & \text{if } x < x_0, \\ e^{-\lambda_1 x} & \text{if } x_n \leq x < y_n, \\ e^{-\lambda_1 y_n} & \text{if } y_n \leq x < \dfrac{\lambda_1}{\lambda_{2,n}}\, y_n, \\ e^{-\lambda_{2,n} x} & \text{if } \dfrac{\lambda_1}{\lambda_{2,n}}\, y_n \leq x < x_{n+1} - 1, \\ e^{-\lambda_1 x_{n+1}} + \big(e^{-\lambda_{2,n}(x_{n+1}-1)} - e^{-\lambda_1 x_{n+1}}\big)\,(x_{n+1} - x) & \text{if } x_{n+1} - 1 \leq x < x_{n+1}, \end{cases}$$

see the joint figure below. The function u_0 is thus continuous, front-like in the sense of Definition 1.1, non-increasing in $\mathbb{R}$, and $u_0(-\infty) = 1$. Let u be the solution of (1.1) with the initial condition u_0 and let us check that the conclusion of Theorem 2.6 holds with this choice of u_0.

The function u_0 oscillates between $e^{-\lambda_1 x}$ and $e^{-\lambda_2 x}$ (or $e^{-\lambda_{2,n} x}$ if $\lambda_2 = 0$) as $x \to +\infty$. It is also glued between these two exponentially decaying functions between y_n and $(\lambda_1/\lambda_{2,n})y_n$ and between $x_{n+1} - 1$ and x_{n+1} in such a way that it is nonincreasing. This monotonicity property will be inherited at all positive times, which reduces the level sets $E_m(t)$ to singletons (and will then help in the calculations of their positions). Namely, the strong maximum principle implies that, for every $t > 0$, the function $u(t, \cdot)$ is decreasing on $\mathbb{R}$, and $u(t, -\infty) = 1$, $u(t, +\infty) = 0$. Therefore, for every $t > 0$ and $m \in (0, 1)$, the level set $E_m(t)$ reduces to a singleton

$$E_m(t) = \big\{x_m(t)\big\}.$$



Furthermore, the functions $t \mapsto x_m(t)$ are all (at least) of class C^1 on $(0, +\infty)$ from the implicit function theorem.

Since u_0 is front-like and

$$e^{-\lambda_1 x} \leq u_0(x) \leq e^{-\lambda_2 x} \ \text{ for all } x \geq 0,$$

it follows from the maximum principle, together with [40] (or Theorem 2.1) and the general comparisons (1.6), that

$$\gamma_1 \leq w_*(u_0) \leq w^*(u_0) \leq \gamma_2.$$

It also follows from the definitions of the spreading speeds that, for every $m \in (0, 1)$,

$$\gamma_1 \leq w_*(u_0) \leq \liminf_{t \to +\infty} \frac{x_m(t)}{t} \leq \limsup_{t \to +\infty} \frac{x_m(t)}{t} \leq w^*(u_0) \leq \gamma_2. \tag{4.37}$$

Next, let $\underline{u}_0$ and $\overline{u}_0$ be the two functions defined on $\mathbb{R}$ by

$$\underline{u}_0(x) = \begin{cases} 1 & \text{if } x < 0, \\ e^{-\lambda_1 x} & \text{if } x \geq 0 \end{cases} \quad \text{and } \overline{u}_0(x) = \begin{cases} 1 & \text{if } x < 0, \\ e^{-\lambda_{2,n} x} & \text{if } x_n \leq x < x_{n+1}. \end{cases} \tag{4.38}$$

Observe that, if $\lambda_2 > 0$, then $\overline{u}_0(x) = e^{-\lambda_2 x}$ for all $x \geq 0$. The function $\underline{u}_0$ is obviously front-like, as is the function $\overline{u}_0$ if $\lambda_2 > 0$. If $\lambda_2 = 0$, then $\lambda_{2,n} = \lambda_1/(n+2)$, whence $\lambda_{2,n} x_n \to +\infty$ as $n \to +\infty$ (since $x_{n+1}/x_n \to +\infty$ as $n \to +\infty$) and $\overline{u}_0(x) \to 0$ as $x \to +\infty$. In other words, the function $\overline{u}_0$ is front-like whenever λ_2 is positive or 0. Let $\underline{u}$ and $\overline{u}$ be the solutions of (1.1) with initial conditions $\underline{u}_0$ and $\overline{u}_0$. Since $0 \leq \underline{u}_0 \leq u_0 \leq \overline{u}_0 \leq 1$ on $\mathbb{R}$, the maximum principle yields

$$0 \leq \underline{u}(t, x) \leq u(t, x) \leq \overline{u}(t, x) \leq 1 \ \text{ for all } t \geq 0 \text{ and } x \in \mathbb{R}.$$

Furthermore, as already recalled in Section 1, it follows from Uchiyama [40] that

$$\sup_{x \in \mathbb{R}} \left| \underline{u}(t, x) - U_{\gamma_1}(x - \gamma_1 t + m_1(t)) \right| \to 0 \text{ as } t \to +\infty, \tag{4.39}$$

where $m_1(t)/t \to 0$ as $t \to +\infty$ (moreover, if $\gamma_1 > c^*$, then $m_1(t)$ can be chosen to be a constant real number x_1 in the above formula). Similarly, if $\gamma_2 < +\infty$ (that is, $\lambda_2 > 0$), then there exists $x_2 \in \mathbb{R}$ such that

$$\sup_{x\in\mathbb{R}} \Big|\overline{u}(t,x) - U_{\gamma_2}(x-\gamma_2 t + x_2)\Big| \to 0 \text{ as } t \to +\infty. \tag{4.40}$$

Let us now prove that these two approximated travelling fronts $U_{\gamma_1}(x-\gamma_1 t + m_1(t))$ and $U_{\gamma_2}(x-\gamma_2 t + x_2)$ (if $\gamma_2 < +\infty$) are closer and closer to u on some larger and larger space-intervals during some larger and larger intervals of time. That will be sufficient to derive the conclusion of Theorem 2.6 (at least if $\gamma_2 < +\infty$, the case $\gamma_2 = +\infty$ requiring a special treatment).

To do so, denote

$$v = u - \underline{u} \ge 0 \text{ and } w = \overline{u} - u \ge 0 \text{ on } [0,+\infty)\times\mathbb{R}.$$

Choose any sequences $(t_n)_{n\in\mathbb{N}}$ and $(t'_n)_{n\in\mathbb{N}}$ of positive real numbers such that

$$x_n < t_n \le t'_n < y_n \text{ for all } n\in\mathbb{N} \ \text{ and } \ \lim_{n\to+\infty}\frac{t_n}{x_n} = \lim_{n\to+\infty}\frac{y_n}{t'_n} = +\infty.$$

Such sequences exist since $y_n/x_n \to +\infty$ as $n \to +\infty$. For instance, a particular choice is: $t_n = x_n^{1-\theta} y_n^{\theta}$ and $t'_n = x_n^{1-\theta'} y_n^{\theta'}$ with $0 < \theta \le \theta' < 1$. We now claim that

$$\max_{t\in[t_n,t'_n]} \Big(\max_{x\in\left[(2\sqrt{M_f}+\varepsilon)t,\gamma t\right]} v(t,x)\Big) \to 0 \ \text{ as } n\to+\infty \tag{4.41}$$

for any two positive real numbers ε and γ such that

$$2\sqrt{M_f} + \varepsilon \le \gamma.$$

This property will imply that the solution u is close to $\underline{u}$ and then to the approximated front $U_{\gamma_1}(x-\gamma_1 t + m_1(t))$ on sequences of time-intervals $[t_n, t'_n]$ and on some space-intervals, provided that the ratio between the position and the time belongs to $[2\sqrt{M_f}+\varepsilon, \gamma]$. Since $\varepsilon > 0$ can be arbitrarily small, the equality $w_*(u_0) = \gamma_1$ will follow.

In order to prove (4.41), let $\varepsilon > 0$ and $\gamma > 0$ be as above and denote, for all $n \in \mathbb{N}$,

$$E_n^{\varepsilon,\gamma} = \Big\{(t,x)\in(0,+\infty)\times\mathbb{R},\ t_n \le t \le t'_n,\ (2\sqrt{M_f}+\varepsilon)t \le x \le \gamma t\Big\}.$$

Observe that

$$0 \le v(0,x) = u_0(x) - \underline{u}_0(x) \le \sum_{n\in\mathbb{N}} \mathbf{1}_{[y_n,x_{n+1}]}(x) \ \text{ for all } x\in\mathbb{R}$$

and that

$$\partial_t v(t,x) - \partial_{xx} v(t,x) = f(u(t,x)) - f(\underline{u}(t,x)) \le M_f\, v(t,x) \ \text{ for all } (t,x)\in(0,+\infty)\times\mathbb{R},$$

owing to the definition of $M_f = \max_{s\in[0,1]} f'(s)$ and the nonnegativity of v. The maximum principle implies then that, for all $(t,x)\in(0,+\infty)\times\mathbb{R}$,

$$0 \le v(t,x) \le \frac{e^{M_f t}}{\sqrt{4\pi t}} \sum_{n\in\mathbb{N}} \int_{y_n}^{x_{n+1}} e^{-\frac{(x-y)^2}{4t}}\,dy. \tag{4.42}$$

Then, choose $n_1\in\mathbb{N}$ such that

$$x_n \le (2\sqrt{M_f}+\varepsilon)\,t_n \le \gamma\,t'_n \le y_n \ \text{ for all } n\ge n_1.$$

For any $n\ge n_1$ and $(t,x)\in E_n^{\varepsilon,\gamma}$, one then has

$$x_n \le (2\sqrt{M_f}+\varepsilon)\,t_n \le (2\sqrt{M_f}+\varepsilon)\,t \le x \le \gamma\,t \le \gamma\,t'_n \le y_n,$$

whence

$$\begin{aligned} 0 \le v(t,x) \ &\le\ \frac{e^{M_f t}}{\sqrt{4\pi t}} \times \Big(\int_{-\infty}^{x_n} e^{-\frac{(x-y)^2}{4t}}\,dy + \int_{y_n}^{+\infty} e^{-\frac{(x-y)^2}{4t}}\,dy\Big) \\ &= \frac{e^{M_f t}}{\sqrt{\pi}} \int_{-\infty}^{\frac{x_n-x}{\sqrt{4t}}} e^{-z^2}dz + \frac{e^{M_f t}}{\sqrt{\pi}} \int_{\frac{y_n-x}{\sqrt{4t}}}^{+\infty} e^{-z^2}dz, \end{aligned} \tag{4.43}$$

from (4.42). But

$$\frac{x_n-x}{\sqrt{4t}} \le \frac{x_n-(2\sqrt{M_f}+\varepsilon)t}{\sqrt{4t}} = -\sqrt{t}\times\Big(\sqrt{M_f}+\frac{\varepsilon}{2}-\frac{x_n}{2t}\Big) \le -\sqrt{t}\times\Big(\sqrt{M_f}+\frac{\varepsilon}{2}-\frac{x_n}{2t_n}\Big)$$

and $x_n/t_n\to 0$ as $n\to+\infty$. Therefore, there exists $n_2\ge n_1$ such that, for all $n\ge n_2$ and $(t,x)\in E_n^{\varepsilon,\gamma}$,

$$\frac{x_n-x}{\sqrt{4t}} \le -\sqrt{M_f t} \le -\sqrt{M_f t_n} < 0.$$

On the other hand,

$$\int_A^{+\infty} e^{-z^2}dz \le \frac{e^{-A^2}}{2A}$$

for all $A>0$. Therefore,

$$\frac{e^{M_f t}}{\sqrt{\pi}} \int_{-\infty}^{\frac{x_n-x}{\sqrt{4t}}} e^{-z^2}dz \le \frac{e^{M_f t}}{\sqrt{\pi}} \times \int_{-\infty}^{-\sqrt{M_f t}} e^{-z^2}dz \le \frac{e^{M_f t}}{\sqrt{\pi}} \times \frac{e^{-M_f t}}{\sqrt{4M_f t}} \le \frac{1}{\sqrt{4\pi M_f t_n}} \tag{4.44}$$

for all $n\ge n_2$ and $(t,x)\in E_n^{\varepsilon,\gamma}$. As far as the second term in the right-hand side of (4.43) is concerned, one knows that

$$\frac{y_n-x}{\sqrt{4t}} \ge \frac{y_n-\gamma t'_n}{2\sqrt{t'_n}} \ge \frac{y_n}{4\sqrt{t'_n}} \text{ for all } n \text{ large enough,}$$

since $y_n/t'_n\to+\infty$ as $n\to+\infty$. Thus, there exists $n_3\ge n_2$ such that

$$\frac{e^{M_f t}}{\sqrt{\pi}} \int_{\frac{y_n-x}{\sqrt{4t}}}^{+\infty} e^{-z^2}dz \le \frac{e^{M_f t'_n}}{\sqrt{\pi}} \int_{\frac{y_n}{4\sqrt{t'_n}}}^{+\infty} e^{-z^2}dz \le \frac{e^{M_f t'_n}}{\sqrt{\pi}} \times e^{-\frac{y_n^2}{16\,t'_n}} \times \frac{2\sqrt{t'_n}}{y_n} \tag{4.45}$$

for all $n \geq n_3$ and $(t,x) \in E_n^{\varepsilon,\gamma}$. Combining (4.43), (4.44) and (4.45), one infers that, for all $n \geq n_3$,

$$\max_{(t,x)\in E_n^{\varepsilon,\gamma}} v(t,x) \leq \frac{1}{\sqrt{4\pi M_f t_n}} + \frac{e^{M_f t'_n}}{\sqrt{\pi}} \times e^{-\frac{y_n^2}{16\,t'_n}} \times \frac{2\sqrt{t'_n}}{y_n}.$$

But the right-hand side converges to 0 as $n \to +\infty$, since t_n, y_n/t'_n and $y_n/\sqrt{t'_n}$ all converge to $+\infty$ as $n \to +\infty$. This provides (4.41).

Putting together (4.39), (4.41) and the fact that $U_{\gamma_1}(+\infty) = 0$, it follows that, for all $A \in \mathbb{R}$ and $(2\sqrt{M_f} \leq)\, \gamma_1 < c < \gamma$,

$$\max_{t\in[t_n,t'_n]} \Big(\max_{x\in[ct+A,\gamma t]} u(t,x) \Big) \to 0 \ \text{ as } n \to +\infty.$$

In particular,

$$\max_{t\in[t_n,t'_n]} u(t,ct+x) \to 0 \text{ as } n \to +\infty \text{ for all } c > \gamma_1 \text{ and } x \in \mathbb{R}. \tag{4.46}$$

Since $u(t,\cdot)$ is decreasing for all $t > 0$, one actually gets that

$$\max_{t\in[t_n,t'_n]} \Big(\max_{x\in[ct+A,+\infty)} u(t,x) \Big) \to 0 \ \text{ as } n \to +\infty$$

for all $A \in \mathbb{R}$ and $c > \gamma_1$. Therefore, for all $m \in (0,1)$, $\liminf_{t\to+\infty} x_m(t)/t \leq \gamma_1$ and eventually

$$\liminf_{t\to+\infty} \frac{x_m(t)}{t} = \gamma_1 \tag{4.47}$$

because of (4.37). Furthermore, $w_*(u_0) \leq \gamma_1$, and (4.37) also yields the equality

$$w_*(u_0) = \gamma_1.$$

Let us now prove that $w^*(u_0) = \gamma_2$ and $\limsup_{t\to+\infty} x_m(t)/t = \gamma_2$ for all $m \in (0,1)$. Remember the definition of $\overline{u}_0$ in (4.38), and that $w = \overline{u} - u \geq 0$ in $[0,+\infty) \times \mathbb{R}$. Choose any sequences $(\tau_n)_{n\in\mathbb{N}}$ and $(\tau'_n)_{n\in\mathbb{N}}$ of positive real numbers such that

$$\frac{\lambda_1}{\lambda_{2,n}}\, y_n < \tau_n \leq \tau'_n < x_{n+1} - 1 \text{ for all } n \in \mathbb{N} \ \text{ and } \ \lim_{n\to+\infty} \frac{\tau_n}{(\lambda_1/\lambda_{2,n})y_n} = \lim_{n\to+\infty} \frac{x_{n+1}}{\tau'_n} = +\infty.$$

Such sequences exist because of (4.36). Since $w(0,\cdot) = \overline{u}_0 - u_0 = 0$ on all the intervals $[(\lambda_1/\lambda_{2,n})y_n, x_{n+1} - 1]$ for all $n \in \mathbb{N}$, the same arguments as for the function v imply that

$$\max_{t\in[\tau_n,\tau'_n]} \Big(\max_{x\in\left[(2\sqrt{M_f}+\varepsilon)t,\gamma t\right]} w(t,x) \Big) \to 0 \ \text{ as } n \to +\infty \tag{4.48}$$

for any two positive real numbers ε and γ such that $2\sqrt{M_f} + \varepsilon \leq \gamma$.

Consider first the case $\gamma_2 < +\infty$ (that is, $\lambda_2 > 0$). It follows then from (4.40), (4.48) and $U_{\gamma_2}(-\infty) = 1$ that, for all $A \in \mathbb{R}$ and $2\sqrt{M_f} < c' < c < \gamma_2$,

$$\min_{t\in[\tau_n,\tau'_n]} \Big(\min_{x\in[c't,ct+A]} u(t,x) \Big) \to 1 \ \text{ as } n \to +\infty.$$

Since $u(t,\cdot)$ is decreasing for all $t>0$, one actually gets that

$$\min_{t\in[\tau_n,\tau'_n]}\Big(\min_{x\in(-\infty,ct+A)} u(t,x)\Big)\to 1\ \text{ as } n\to+\infty$$

for all $A\in\mathbb{R}$ and $c<\gamma_2$. In particular,

$$\min_{t\in[\tau_n,\tau'_n]} u(t,ct+x)\to 1 \text{ as } n\to+\infty \text{ for all } c<\gamma_2 \text{ and } x\in\mathbb{R}. \tag{4.49}$$

Furthermore, for all $m\in(0,1)$, $\limsup_{t\to+\infty} x_m(t)/t\geq\gamma_2$ and eventually

$$\limsup_{t\to+\infty}\frac{x_m(t)}{t}=\gamma_2$$

because of (4.37). Lastly, $w^*(u_0)\geq\gamma_2$, and (4.37) yields

$$w^*(u_0)=\gamma_2.$$

Lastly, consider the case $\gamma_2=+\infty$ (that is, $\lambda_2=0$). Let η be any real number in the interval $(0,\lambda^*)$. Let $n_\eta\in\mathbb{N}$ be such that $0<\lambda_{2,n}<\eta$ for all $n\geq n_\eta$. Define the function $u_0^\eta:\mathbb{R}\to[0,1]$ by

$$u_0^\eta(x)=\begin{cases}1 & \text{if } x<0,\\ 0 & \text{if } 0\leq x<\dfrac{\lambda_1}{\lambda_{2,n_\eta}}\,y_{n_\eta},\\ e^{-\eta x} & \text{if } \dfrac{\lambda_1}{\lambda_{2,n}}\,y_n\leq x<x_{n+1}-1 \text{ with } n\geq n_\eta,\\ 0 & \text{if } x_{n+1}-1\leq x<\dfrac{\lambda_1}{\lambda_{2,n+1}}\,y_{n+1} \text{ with } n\geq n_\eta.\end{cases}$$

From the choice of n_η and u_0, one has $u_0\geq u_0^\eta$ on $\mathbb{R}$, whence

$$u(t,x)\geq u^\eta(t,x)\ \text{ for all } t>0 \text{ and } x\in\mathbb{R} \tag{4.50}$$

from the maximum principle, where u^η denotes the solution of the equation (1.1) with initial condition u_0^η. Define now

$$\overline{u}_0^\eta(x)=\begin{cases}1 & \text{if } x<\dfrac{\lambda_1}{\lambda_{2,n_\eta}}\,y_{n_\eta},\\ e^{-\eta x} & \text{if } x\geq\dfrac{\lambda_1}{\lambda_{2,n_\eta}}\,y_{n_\eta}\end{cases}$$

and let $\overline{u}^\eta$ be the solution of problem (1.1) with initial condition $\overline{u}_0^\eta$. Since $\overline{u}_0^\eta\geq u_0^\eta$ on $\mathbb{R}$, the maximum principle yields

$$w^\eta(t,x)=\overline{u}^\eta(t,x)-u^\eta(t,x)\geq 0\ \text{ for all } t>0 \text{ and } x\in\mathbb{R}.$$

Furthermore, since $\overline{u}_0^\eta = u_0^\eta$ on the intervals $\big[(\lambda_1/\lambda_{2,n})y_n, x_{n+1}-1\big)$ for all $n \geq n_\eta$, the same arguments as above imply that

$$\max_{t\in[\tau_n,\tau'_n]}\Big(\max_{x\in\big[(2\sqrt{M_f}+\varepsilon)t,\gamma t\big]} w^\eta(t,x)\Big)\to 0 \ \text{ as } n\to+\infty$$

for all $\varepsilon>0$ and $\gamma<+\infty$ such that $2\sqrt{M_f}+\varepsilon\leq\gamma$. On the other hand, because of Uchiyama [40], there exists $x_\eta\in\mathbb{R}$ such that

$$\sup_{x\in\mathbb{R}}\Big|\overline{u}^\eta(t,x)-U_{c_\eta}(x-c_\eta t+x_\eta)\Big|\to 0 \text{ as } t\to+\infty,$$

where $c_\eta=\eta+f'(0)/\eta$. Since $U_{c_\eta}(-\infty)=1$, one then infers that

$$\min_{t\in[\tau_n,\tau'_n]}\Big(\min_{x\in[c't,ct+A]} u^\eta(t,x)\Big)\to 1 \ \text{ as } n\to+\infty$$

for all $A\in\mathbb{R}$ and $2\sqrt{M_f}<c'<c<c_\eta$. Remember now that $u\geq u^\eta$ from (4.50) and that $u(t,\cdot)$ is decreasing for all $t>0$. Therefore,

$$\min_{t\in[\tau_n,\tau'_n]}\Big(\min_{x\in(-\infty,ct+A]} u(t,x)\Big)\to 1 \ \text{ as } n\to+\infty \tag{4.51}$$

for all $A\in\mathbb{R}$ and $c<c_\eta$. Since $\eta\in(0,\lambda^*)$ can be chosen arbitrarily small and $c_\eta\to+\infty$ as $\eta\to0^+$, it follows that (4.51) holds for all $c\in\mathbb{R}$ and $A\in\mathbb{R}$. In particular,

$$\min_{t\in[\tau_n,\tau'_n]} u(t,ct+x)\to 1 \text{ as } n\to+\infty \text{ for all } c<+\infty \text{ and } x\in\mathbb{R}. \tag{4.52}$$

Moreover, $\limsup_{t\to+\infty} x_m(t)/t=+\infty$ for all $m\in(0,1)$ and $w^*(u_0)=+\infty$.

As a conclusion, whenever γ_2 is finite or $+\infty$, there always holds $w^*(u_0)=\gamma_2$, and $\limsup_{t\to+\infty} x_m(t)/t=\gamma_2$ for all $m\in(0,1)$. Because of (4.47), one concludes that, for all $m\in(0,1)$, the ω-limit set of the (continuous on $(0,+\infty)$) function $t\mapsto x_m(t)/t$ is equal to the whole interval $[\gamma_1,\gamma_2]$. Lastly, the limits (4.46), (4.49) and (4.52) imply that, for all $c\in(\gamma_1,\gamma_2)$ and $x\in\mathbb{R}$, the ω-limit set of the (continuous on $(0,+\infty)$) function $t\mapsto u(t,ct+x)$ is equal to the whole interval $[0,1]$. The proof of Theorem 2.6 is thereby complete. □

Remark 4.1 If $\gamma_1>c^*$, then the quantity $m_1(t)$ appearing in (4.39) can be chosen to be a constant real number x_1. Together with the inequality $u\geq\underline{u}$ and formula (4.49) applied with $c=\gamma_1<\gamma_2$, it follows that, for each $x\in\mathbb{R}$, the ω-limit set of the function $t\mapsto u(t,\gamma_1t+x)$ is equal to the interval $[U_{\gamma_1}(x+x_1),1]$. Similarly, if $\gamma_2<+\infty$, then (4.40) and formula (4.46) applied with $c=\gamma_2>\gamma_1$ imply that, for each $x\in\mathbb{R}$, the ω-limit set of the function $t\mapsto u(t,\gamma_2t+x)$ is equal to the interval $[0,U_{\gamma_2}(x+x_2)]$.

Remark 4.2 The complex dynamics shown in Theorem 2.6 for the nonlinear equation (1.1) resembles that already known for the pure heat equation $\partial_t\zeta=\partial_{xx}\zeta$. Namely, there are initial conditions $\zeta_0\in L^\infty(\mathbb{R})$, which oscillate between $\mathrm{essinf}_{\mathbb{R}}\zeta_0$ and $\mathrm{esssup}_{\mathbb{R}}\zeta_0$ on larger and larger intervals, and for which the ω-limit set of the function $t \mapsto \zeta(t,x)$ is equal to the whole interval $[\mathrm{essinf}_{\mathbb{R}}\zeta_0, \mathrm{esssup}_{\mathbb{R}}\zeta_0]$ for each $x \in \mathbb{R}$. This phenomenon was first pointed out by Collet and Eckmann [8]. Somehow, for the nonlinear equation (1.1), the complex dynamics appears when the initial condition u_0 oscillates on larger and larger intervals between two exponentially decaying functions with different decay rates. For such u_0, the proof of Theorem 2.6 shows that the solution u oscillates between the two *nonlinear* travelling fronts whose speeds are associated to the two decay rates of u_0.

5 Appendix. Extensions to heterogeneous higher-dimensional problems

In the appendix, we just mention without proof some possible extensions of the results of the previous sections to more general equations. Similar theorems can indeed be established with the same type of methods, concerning more general heterogeneous equations in higher dimensions for which (pulsating) travelling fronts still exist.

To be more precise, consider the Cauchy problem

$$\begin{cases} \partial_t u - \mathrm{div}(A(z)\nabla u) + q(z)\cdot\nabla u &= f(z,u), \quad t>0,\ z\in\overline{\Omega}, \\ \nu(z)A(z)\nabla u &= 0, \qquad\quad\ t>0,\ z\in\partial\Omega, \\ u(0,z) &= u_0(z) \qquad \text{for a.e. } z\in\Omega, \end{cases} \tag{5.53}$$

where $\Omega \subset \mathbb{R}^N$ is an unbounded domain of class $C^{2,\alpha}$ (with $\alpha > 0$), periodic in d directions and bounded in the remaining variables. That is, there are an integer $d \in \{1, \cdots, N\}$ and d positive real numbers $L_1, \ldots, L_d$ such that

$$\begin{cases} \exists\ R \geq 0, \quad \forall\ z=(x,y)\in\overline{\Omega}, \quad |y|\leq R, \\ \forall\ k \in L_1\mathbb{Z}\times\cdots\times L_d\mathbb{Z}\times\{0\}^{N-d}, \quad \overline{\Omega}=\overline{\Omega}+k, \end{cases}$$

where $x = (x_1, \cdots, x_d)$, $y = (x_{d+1}, \cdots, x_N)$ and $|\cdot|$ denotes the euclidean norm. Typical examples of such domains are the whole space $\mathbb{R}^N$ with or without periodic perforations, or infinite cylinders with constant or periodically undulating sections. We denote by ν the outward unit normal on $\partial\Omega$, and $\xi B\xi' = \sum_{1\leq i,j\leq N} \xi_i B_{ij} \xi'_j$ for any two vectors $\xi = (\xi_i)_{1\leq i\leq N}$ and $\xi' = (\xi'_i)_{1\leq i\leq N}$ in $\mathbb{R}^N$ and any $N \times N$ matrix $B = (B_{ij})_{1\leq i,j\leq N}$ with real entries. The symmetric matrix field $A = (A_{ij})_{1\leq i,j,\leq N}$ is assumed to be of class $C^{1,\alpha}(\overline{\Omega})$ and uniformly positive definite. The vector field $q = (q_i)_{1\leq i\leq N}$ is assumed to be of class $C^{0,\alpha}(\overline{\Omega})$ and divergence-free. The reaction term $f : \overline{\Omega} \times [0,1] \to \mathbb{R}$, $(z,s) \mapsto f(z,s)$ is continuous, of class $C^{0,\alpha}$ with respect to z uniformly in $s \in [0,1]$, and of class C^1 with respect to s uniformly in $z \in \overline{\Omega}$. All functions A_{ij}, q_i and $f(\cdot,s)$ (for all $s \in [0,1]$) are assumed to be periodic in $\overline{\Omega}$, in the sense that they all satisfy

$$w(x+k,y) = w(x,y) \text{ for all } z=(x,y)\in\overline{\Omega} \text{ and } k\in L_1\mathbb{Z}\times\cdots\times L_d\mathbb{Z}.$$

We further assume that q has zero average, that

$$f(z,0)=f(z,1)=0,\ \partial_s f(z,0)>0 \text{ for all } z\in\overline{\Omega},\ f>0 \text{ on } \overline{\Omega}\times(0,1)$$

and that there exist $0 < s_0 < s_1 < 1$, $\gamma > 0$, $C > 0$ such that $f(z,s) \geq \partial_s f(z,0)\, s - C\, s^{1+\gamma}$ on $\overline{\Omega} \times [0,s_0]$ and $f(z,\cdot)$ is nonincreasing on $[s_1,1]$ for all $z \in \overline{\Omega}$.

For this problem, the usual notion of travelling fronts does not hold anymore in general, and it is replaced with that of pulsating travelling fronts. Namely, given a unit vector $e \in \mathbb{R}^d \times \{0\}^{N-d}$, a pulsating travelling front connecting 0 to 1, travelling in the direction e with (mean) speed $c \in \mathbb{R}^*$, is a time-global classical solution $U_c : \mathbb{R} \times \overline{\Omega} \to (0,1)$ of (5.53) such that

$$\begin{cases} u(t,z) = U_c(z \cdot e - ct, z) \text{ for all } (t,z) \in \mathbb{R} \times \overline{\Omega}, \\ U_c(s,\cdot) \text{ is periodic in } \overline{\Omega} \text{ for all } s \in \mathbb{R}, \\ U_c(s,z) \underset{s \to +\infty}{\longrightarrow} 0,\ U_c(s,z) \underset{s \to -\infty}{\longrightarrow} 1, \text{ uniformly in } z \in \overline{\Omega}. \end{cases}$$

It is known that, for each direction e, there is a minimal speed $c^*(e) > 0$ such that pulsating travelling fronts U_c in the direction e exist if and only if $c \geq c^*(e)$, see [3, 41]. Furthermore, if f also satisfies the generalized KPP assumption $f(z,s) \leq \partial_s f(z,0)s$ on $\overline{\Omega} \times [0,1]$, then the fronts U_c with speed c are unique up to shifts in time, see [18]. Under the KPP assumption, the speed $c^*(e)$ is given by $c^*(e) = \min_{\lambda>0} k_e(\lambda)/\lambda$, where $k_e(\lambda)$ is the principal eigenvalue of the operator

$$\psi \mapsto \mathrm{div}(A\nabla\psi) - 2\lambda eA\nabla\psi - q \cdot \nabla\psi + [-\lambda \mathrm{div}(Ae) + \lambda q \cdot e + \lambda^2 eAe + \partial_s f(z,0)]\psi \quad (5.54)$$

acting on the set of $C^2(\overline{\Omega})$ periodic functions ψ such that $\nu A\nabla\psi = \lambda(\nu Ae)\psi$ on $\partial\Omega$ (the principal eigenfunction $\psi = \psi_{e,\lambda}$ is positive in $\overline{\Omega}$ unique up to multiplication by positive constants), see [6]. More generally speaking, with or without the KPP assumption, the inequality

$$c^*(e) \geq \min_{\lambda>0} \frac{k_e(\lambda)}{\lambda}$$

always holds, see [3].

The Cauchy problem (5.53), where $u_0 : \Omega \to \mathbb{R}$ is measurable and satisfies $0 \leq u_0 \leq 1$ a.e. in Ω and $u_0 \not\equiv 0$, $u_0 \not\equiv 1$ a.e. in Ω,[1] was first considered when the initial condition u_0 is compactly supported. In this case, the solution u spreads in any given unit direction $e \in \mathbb{R}^d \times \{0\}^{N-d}$ with the speed

$$C^*(e) = \min_{e' \in \mathbb{R}^d \times \{0\}^{N-d},\ e' \cdot e > 0} \frac{c^*(e')}{e' \cdot e} > 0,$$

in the sense that, as $t \to +\infty$, $u(t, c\,t\,e + z) \to 1$ for any $0 \leq c < C^*(e)$ and $u(t, c\,t\,e + z) \to 0$ for any $c > C^*(e)$ locally uniformly in z such that $c\,t\,e + z \in \overline{\Omega}$ (see [5, 13, 14, 28, 41]).

In this appendix, given a direction e in $\mathbb{R}^d \times \{0\}^{N-d}$, we consider the case when the initial condition u_0 is front-like in the direction e uniformly with respect to the orthogonal directions, that is

$$\liminf_{M \to -\infty} \Big(\operatorname*{essinf}_{\Omega \cap \{z \cdot e < M\}} u_0 \Big) > 0 \text{ and } \lim_{M \to +\infty} \|u_0\|_{L^\infty(\Omega \cap \{z \cdot e > M\})} = 0.$$

[1]The strong maximum principle then yields $0 < u(t,z) < 1$ for all $t > 0$ and $z \in \overline{\Omega}$.

The natural extension of the minimal and maximal spreading speeds in the given direction e, uniformly with respect to the orthogonal directions, is the following one:

$$\begin{aligned} w_*(u_0) &= \sup\Big\{c\in\mathbb{R},\ \inf_{z\in\overline{\Omega},\, z\cdot e\le ct} u(t,z)\to 1 \text{ as } t\to+\infty\Big\},\\ w^*(u_0) &= \inf\ \Big\{c\in\mathbb{R},\ \sup_{z\in\overline{\Omega},\, z\cdot e\ge ct} u(t,z)\to 0 \text{ as } t\to+\infty\Big\}. \end{aligned}$$

When u_0 is front-like in the direction e and is such that $u_0=0$ a.e. in $\Omega\cap\{z\cdot e>M\}$ for some $M\in\mathbb{R}$, then $w_*(u_0)=w^*(u_0)=c^*(e)$, as proved by Weinberger [41] (see also [33, 34] for further results in space-time periodic media, [37] for space periodic and time-limit periodic media and [27] for abstract monotone evolution systems). When u_0 is front-like in the direction e and exponentially decreasing as $z\cdot e\to+\infty$, the exact estimates of the spreading speeds have been established only in the KPP case (see [34, 37]). In the general monostable case, the spreading speeds $w_*(u_0)$ and $w^*(u_0)$ are still expected to be finite and to strongly depend on the exponential decay of u_0 and on that of the fronts U_c. In order to quantity these statements, one needs to introduce a few additional notations. Let $\lambda^*(e)>0$ be the smallest root of the equation $k_e(\lambda)=c^*(e)\lambda$. It was proved in [16] that, if $c>c^*(e)$, then any pulsating travelling front U_c with speed c in the direction e is such that

$$\ln U_c(s,z)\sim-\lambda s \text{ as } s\to+\infty \text{ uniformly in } z\in\overline{\Omega},$$

where $\lambda\in(0,\lambda^*(e))$ is the smallest root of the equation $k_e(\lambda)=c\lambda$. The map $c\mapsto\lambda$ is decreasing, one-to-one and onto from $(c^*(e),+\infty)$ onto $(0,\lambda^*(e))$. Furthermore, if u_0 decays exactly as a given front $U_c(z\cdot e,z)$ as $z\cdot e\to+\infty$ and is not far from 1 as $z\cdot e\to-\infty$, then $u(t,z)$ converges to this front $U_c(z\cdot e-ct,z)$ as $t\to+\infty$ uniformly in $z\in\overline{\Omega}$. Even if the exact exponential decay of the fronts U_c as they approach 0 is not know in general (it is however in the generalized KPP case even for the minimal speed $c^*(e)$, leading to more precise stability results, see [16, 18]), the aforementioned logarithmic equivalent is enough to show that similar results as in Section 2 are still valid for the problem (5.53).

Namely, the following statements generalize the results of Section 2. In the sequel, u_0 denotes a front-like initial condition in a given unit direction $e\in\mathbb{R}^d\times\{0\}^{N-d}$.

- If $u_0(z)=O(e^{-\Lambda(z)\,z\cdot e})$ as $z\cdot e\to+\infty$ with $\liminf_{z\in\Omega,\,z\cdot e\to+\infty}\Lambda(z)\ge\lambda^*(e)$, then

$$w_*(u_0)=w^*(u_0)=c^*(e).$$

- If there exist $\lambda\in(0,\lambda^*(e))$, $M\in\mathbb{R}$, a nonnegative bounded function ρ defined on $\Omega\cap\{z\cdot e>M\}$ and a function $\Lambda:\Omega\cap\{z\cdot e>M\}\to\mathbb{R}$ such that

$$u_0(z)=\rho(z)\,e^{-\Lambda(z)\,z\cdot e} \text{ a.e. in } \Omega\cap\{z\cdot e>M\}$$

  and $\Lambda(z)\to\lambda$ as $z\cdot e\to+\infty$, and if there exists $T>0$ such that

$$\inf_{\overline{\Omega}}\widetilde{\zeta}(T,\cdot)>0, \tag{5.55}$$

where $\widetilde{\zeta}$ is the solution of the linear equation

$$\begin{cases} \partial_t\widetilde{\zeta} - \operatorname{div}(A\nabla\widetilde{\zeta}) - 2\nabla(\ln\psi_{e,\lambda})A\nabla\widetilde{\zeta} + 2\lambda eA\nabla\widetilde{\zeta} + q\cdot\nabla\widetilde{\zeta} = 0 & t>0,\ z\in\overline{\Omega}, \\ \nu A\nabla\widetilde{\zeta} = 0 & t>0,\ z\in\partial\Omega \end{cases}$$

with initial condition $\widetilde{\zeta}(0,z)=\rho(z)$ if $z\cdot e>M$ and $\widetilde{\zeta}(0,z)=1$ if $z\cdot e<M$ in Ω, then

$$w_*(u_0)=w^*(u_0)=\frac{k_e(\lambda)}{\lambda}.$$

Notice that the condition (5.55) is equivalent to the condition (2.17) given in Theorem 2.1 for the solution ζ of (2.16) in the case $N=1$, $A=1$ and $q=0$, since, in this particular case, $\psi_{e,\lambda}$ is constant and $\widetilde{\zeta}(t,x)=\zeta(t,x-2\lambda t)$. Notice also that (5.55) is equivalent to $\inf_{\overline{\Omega}}\widetilde{\zeta}(t,\cdot)>0$ for all $t>0$ and even $\inf_{[t,+\infty)\times\overline{\Omega}}\widetilde{\zeta}>0$ for all $t>0$. If the function ρ can be extended to a bounded nonnegative function having a positive average, then (5.55) is fulfilled automatically, whence $w_*(u_0)=w^*(u_0)=c$. Furthermore, if $\Lambda(z)\to 0$ as $z\cdot e\to+\infty$ and if (5.55) is satisfied, then $w_*(u_0)=w^*(u_0)=+\infty$. Lastly, if f satisfies the generalized KPP condition $f(z,s)\le\partial_sf(z,0)s$ on $\overline{\Omega}\times[0,1]$, if $\Lambda=\lambda$ in $\Omega\cap\{z\cdot e>M\}$ and if (5.55) is fulfilled, then

$$\liminf_{t\to+\infty}\big(u(t,z)-U_c(z\cdot e-ct+\tau_1,z)\big)\ge 0$$

and

$$\limsup_{t\to+\infty}\big(u(t,z)-U_c(z\cdot e-ct+\tau_2,z)\big)\le 0$$

uniformly in $z\in\overline{\Omega}$, for some $\tau_1,\ \tau_2\in\mathbb{R}$, where U_c denotes the profile of the (unique up to time-shifts) pulsating travelling front with speed $c=k_e(\lambda)/\lambda$ in the direction e.

- For any large enough speeds $\gamma_1<\gamma_2\le+\infty$, there exist front-like initial conditions u_0 such that

$$\gamma_1=w_*(u_0)<w^*(u_0)=\gamma_2.$$

Furthermore, for any $c\in(\gamma_1,\gamma_2)$, any $M\in\mathbb{R}$ and any $m\in(0,1)$, the $\omega-$limit sets of the functions $t\mapsto\inf_{\Omega\cap\{z\cdot e=ct+M\}}u(t,\cdot)$ and $t\mapsto\sup_{\Omega\cap\{z\cdot e=ct+M\}}u(t,\cdot)$ as $t\to+\infty$ are equal to the whole interval $[0,1]$ and the $\omega-$limit sets of the functions $t\mapsto t^{-1}\inf\{z\cdot e,\ u(t,z)=m\}$ and $t\mapsto t^{-1}\sup\{z\cdot e,\ u(t,z)=m\}$ are equal to the whole interval $[\gamma_1,\gamma_2]$.

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