The very fast solution of a special second order ODE with exponentially decaying forcing and applications.

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Abstract

Let b, c, p be arbitrary positive constants and let $f \in C(\mathbb{R}^+)$ be such that for some $\lambda > c, F > 0$ we have $|f(t)| \le F \exp(-\lambda t)$. Then all solutions x of

$$x'' + cx' + b|x|^p x = f(t) \tag{E}$$

tend to 0 as well as x' as t tends to infinity. Moreover there exists a unique maximal solution y of (E) defined for t large enough such that for some constant $C > 0$ we have $|y(t)| + |y'(t)| \le C \exp(-\lambda t)$. Finally all other solutions of (E) decay to 0 either like e^{-ct} or like $(1+t)^{-\frac{1}{p}}$ as t tends to infinity.

Résumé

Soient b, c, p trois constantes positives quelconques et $f \in C(\mathbb{R}^+)$ tel que pour certains $\lambda > c, F > 0$ on ait $|f(t)| \le F \exp(-\lambda t)$. Alors toute solution x def

$$x'' + cx' + b|x|^p x = f(t) \tag{E}$$

tend vers 0 ainsi que x' lorsque t tend vers l'infini. De plus, il existe une solution maximale unique y de (E) definie pour t assez grand telle que pour un certain $C > 0$ on ait $|y(t)| + |y'(t)| \le C \exp(-\lambda t)$. Finalement toute autre solution de (E) tend vers 0 soit comme e^{-ct}, soit comme $(1+t)^{-\frac{1}{p}}$ lorsque t tend vers l'infini.

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1 Introduction.

In this paper we consider the solutions on $\mathbb{R}^+$ of the second order scalar ODE

$$x'' + cx' + b|x|^p x = f(t) \tag{1}$$

where b, c, p are arbitrary positive constants. By a suitable scaling in t, more precisely by setting

$$x(t) = (\frac{c^2}{b})^{\frac{1}{p}} u(ct)$$

this equation reduces to

$$u'' + u' + |u|^p u = h(t) \tag{2}$$

with

$$h(t) = \frac{1}{c^2}(\frac{b}{c^2})^{\frac{1}{p}} f(\frac{t}{c})$$

The case $f = 0$ has been studied for instance in [5]: in this case all solutions are non-oscillatory, tend to 0 at least as $(1+t)^{-\frac{1}{p}}$ and there are, up to time translation, exactly 2 solutions which decay as e^{-ct}. In addition if f is bounded, all solutions of (1) are global and bounded on $\mathbb{R}^+$ and if f tends to 0 at infinity, it is classical to verify that

$$\lim_{t\to\infty} (|x(t)| + |x'(t)|) = 0$$

The object of this paper is to investigate what happens for (1) when f decays at infinity faster than e^{-ct}. In this case we expect the trivial solution to be replaced by a unique fast decaying solution and it is reasonable to conjecture the existence of 3 rates of decay: the fast rate, e^{-ct} and $(1+t)^{-\frac{1}{p}}$. For results of this type in a more nonlinear situation but without forcing term, cf [7]. The case of a linear restoring force and nonlinear damping

$$x'' + c|x'|^q x' + \omega^2 x = f(t)$$

has been studied in [6] as an illustration of the role of rapidly decaying solution to investigate the behavior of general solutions. However this last equation is less difficult since apart from the special solution, the total energy of all other solutions decay at the same rate $(1+t)^{-\frac{2}{q}}$. Finally this work has been partly motivated by some earlier general studies where a fast decaying forcing term appears, cf. e.g. [1, 3, 4, 5, 6, 8, 9, 10].

2 Main results.

Theorem 2.1. *Assume that f has the following decay property*

$$\exists \lambda > c, \quad \exists F > 0, \quad \forall t \geq 0 \quad |f(t)| \leq F \exp(-\lambda t) \tag{3}$$

Then there exists a unique solution y of (1) *such that*

i) y satisfies (1) *on some interval $(T_0, +\infty)$ where $T_0 \in [-\infty, +\infty)$.*
ii) For some constant $C > 0$ we have

$$\forall t > T_0, \quad |y(t)| + |y'(t)| \leq C \exp(-\lambda t). \tag{4}$$

iii) y cannot be extended to a solution of (1) *on $(T_1, +\infty)$ when $T_1 < T_0$.*

Theorem 2.2. *Under the hypotheses of theorem 2.1, as t tends to infinity any solution x of* (1) *other than y satisfies either*

$$|x(t)| \sim e^{-ct} \text{ and } |x'(t)| \sim e^{-ct} \tag{5}$$

or

$$|x(t)| \sim t^{-\frac{1}{p}} \text{ and } |x'(t)| \sim t^{-\frac{p-1}{p}} \tag{6}$$

where the sign $\sim$ written between two positive functions f, g defined on a half line $J = [a, +\infty)$ means that $\frac{f}{g}$ and $\frac{g}{f}$ are both bounded on $J' = [A, \infty)$ for some $A \geq a$.

3 Existence of the very fast solution.

This section is devoted to the proof of the first part of Theorem 2.1. We consider the case of equation (2) and we assume

$$\exists H > 0, \quad \forall t \geq 0 \quad |h(t)| \leq H \exp(-\gamma t) \tag{7}$$

where $\gamma = \frac{\lambda}{c} > 1$. First we show that for $H < H_0$ small enough, there is a solution z of (2) on $\mathbb{R}^+$ such that

$$\sup_{t \geq 0} (|z(t)| + |z'(t)|) \exp(\gamma t) < \infty$$

To this end we rewrite equation (2) in the form

$$(e^t u')' = e^t (h(t) - |u|^p u)$$

which is equivalent to

$$u'(t) = -e^{-t} \int_t^\infty e^s (h(s) - |u(s)|^p u(s)) ds$$

which means that we look for

$$z(t) = -\int_t^\infty w(s) ds$$

where $w = z'$ solves the integral equation

$$w(t) = -e^{-t} \int_t^\infty e^s \Big[h(s) + \Big| \int_s^\infty w(r) dr \Big|^p \int_s^\infty w(r) dr \Big] ds \tag{8}$$

We introduce

$$X = \{f \in C(\mathbb{R}^+ \quad \sup_{t\geq 0} |f(t)| \exp(\gamma t) < \infty\}$$

endowed with the norm defined by

$$\forall f \in X, \quad \|f\| = \sup_{t\geq 0} |f(t)| \exp(\gamma t)$$

setting

$$(\mathcal{T}w)(t) = -e^{-t} \int_t^\infty e^s \Big[h(s) + \Big| \int_s^\infty w(r)dr)\Big|^p \int_s^\infty w(r)dr\Big] ds \tag{9}$$

it is not difficult to check that $\mathcal{T}(X) \subset X$ with

$$\forall w \in X, \quad \|\mathcal{T}w\| \leq \frac{H}{\gamma - 1} + \frac{\|w\|^{p+1}}{\gamma^{p+1}[(p+1)\gamma - 1]}$$

In addition one easily checks

$$\forall w_1 \in X, \forall w_2 \in X \quad \|\mathcal{T}w_1 - \mathcal{T}w_2\| \leq (p+1)\frac{\max\{\|w_1\|^p, \|w_2\|^p\}\|w_1 - w_2\|}{\gamma^{p+1}[(p+1)\gamma - 1]}$$

In particular $\mathcal{T}$ is a contraction on the ball $B_r = \{w \in X, \|w\| \leq r\}$ for $r \leq r_0$ small enough. Choosing now r_1 small enough so that $j(r) := r - \dfrac{r^{p+1}}{\gamma^{p+1}[(p+1)\gamma - 1]} > 0$ for $r \leq r_1$ and setting $r_2 = \min\{r_0, r_1\}$, it follows easily that for $H \leq (\gamma - 1)j(r_2) =: \eta$, $\mathcal{T}$ is a contraction from the ball B_{r_2} into itself. The fixed point of $\mathcal{T}$ is our solution. Finally, for any H given, we can choose T so large that

$$\forall t \geq 0 \quad |h(t+T)| \leq \eta \exp(-\gamma t)$$

and applying the result to the translated equation we obtain a solution starting for $t \geq T$. The corresponding maximal solution fulfills all the conditions.

4 A non oscillation result.

Theorem 5.2 implies in particular that the solution $x(t)$ has a constant sign for t large. In this section we prove some preliminary results in this direction. These results rely on the following simple property

Lemma 4.1. *Let $J = [t_0, \infty)$ and $a \in C(J)$ be such that*

$$\sup_{t\in J} a(t) < \frac{1}{4} \tag{10}$$

Then the solution ρ of

$$\rho' + \rho + \rho^2 = -a(t) \tag{11}$$

with $\rho(t_0) = -\frac{1}{2}$ exists globally on J.

Proof. Let $\rho \in C^1[t_0, T^*)$ be the maximal solution of (11) starting from $-\frac{1}{2}$. We set

$$\forall t \in [0, T^* - t_0), \quad q(t) := \rho(t + t_0) + \frac{1}{2}$$

Then $q(0) = 0$ and

$$q' + q^2 = \frac{1}{4} - a(t) \geq \eta > 0$$

we establish that

$$\forall t \in [0, T^* - t_0), \quad q(t) \geq 0$$

Indeed since $q'(0) = \eta > 0$, q is positive near 0. Assuming first that q does not remain nonnegative throughout $[0, T^* - t_0)$, let

$$\tau = \inf\{t \in [0, T^* - t_0), \quad q(t) < 0\}$$

Then clearly $\tau \in (0, T^* - t_0)$ and we have $q(\tau) = 0, q'(\tau) \geq \eta > 0$. But then $q < 0$ on $(\tau - \varepsilon, \tau)$ for some $\varepsilon > 0$, contradicting the definition of τ. Therefore

$$\forall t \in [0, T^* - t_0), \quad q(t) \geq 0$$

On the other hand by the equation

$$\forall t \in [0, T^* - t_0), \quad q(t) \leq \frac{t}{4} - \int_0^t a(s)ds$$

Finally

$$\forall t \in [t_0, T^*), \quad -\frac{1}{2} \leq \rho(t) \leq -\frac{1}{2} + \frac{t}{4} - \int_0^t a(s)ds$$

Hence $T^* = \infty$ □

Lemma 4.2. *Let $a \in C(\mathbb{R}^+)$ be such that*

$$\limsup_{t \to \infty} a(t) < \frac{1}{4} \tag{12}$$

Then any solution $v \in C^2(\mathbb{R}^+)$ of

$$v'' + v' + a(t)v = 0 \tag{13}$$

is such that either $v \equiv 0$ or v has a constant sign for t large.

Proof. First we select t_0 and $\eta > 0$ such that

$$\forall t \in J = [t_0, \infty), \quad a(t) \leq \frac{1}{4} - \eta$$

We set $v(t) = e^{\mu(t)}w(t)$, so that (13) becomes

$$w'' + (2\mu' + 1)w' + (\mu'' + \mu' + \mu'^2 + a(t))w = 0$$

We select $\mu(t) = \int_{t_0}^t \rho(s)ds$ defined for $t \in J$ and given by lemma 4.1, therefore we find

$$\forall t \in J = [t_0, \infty), \quad w'' + (2\mu' + 1)w' = 0 \tag{14}$$

Then either $w' \equiv 0$ or w' has a constant sign and never vanishes. In the second case w has at most one zero on J and so is v. In the first case w is constant: if the constant is not 0, it follows that v has a constant sign, otherwise $v \equiv 0$. This concludes the proof □

5 Proofs of the main results.

The existence part of theorem 2.1 was established in Section 3. The uniqueness part of theorem 2.1 and the result of theorem 2.2 will both follow from Lemma 4.2 and the following simple preliminary result

Lemma 5.1. *Let $\varphi \in C^1(J)$ be such that*

$$\forall t \in J, \quad \varphi'(t) + \varphi(t) > 0$$

Then $\varphi \in C^1(J)$ has at most 1 zero in J.

Proof. The function $w(t) = e^t\varphi(t)$ is increasing and therefore has at most 1 zero in J. □

Proposition 5.2. *Under the hypotheses of theorem 2.1, any solution x of* (1) *other than y is such that for some $\delta > 0$ and $T > 0$*

$$\forall t \geq T, \quad \inf\{|x'(t)|, |x(t)|\} \geq \delta e^{-ct} \tag{15}$$

Proof. Let z be the fast solution of (2) and $u = z + v$ any other solution. Then v satisfies (13) with

$$a(t) := \frac{|v + z|^p(v + z) - |z|^p z}{v(t)}$$

whenever $v(t) \neq 0$ since

$$|a(t)| \leq (p + 1)(|z|^p + |u|^p)$$

it is clear that a tends to 0 at infinity. Applying Lemma 4.2, we find that v has a constant sign for t large and in addition $av = |v + z|^p(v + z) - |z|^p z$ does not vanish for the same values of t. Now by using Lemma 5.1 with $\varphi = v'$ or $\varphi = -v'$, we can see that v' has a constant sign for t large. Since v and v' tend to 0 at infinity we obtain

$$v(t) = -\int_t^\infty v'(s)ds.$$

Assuming for instance $v' < 0$ on $J_1 = [t_1, \infty)$ we have $v > 0$ on J and since $a > 0$ we find

$$\forall t \in J_1, \quad (e^t v'(t))' = -e^t a(t) v(t) < 0$$

then

$$\forall t \in J_1, \quad e^t v'(t) < e^{t_1} v'(t_1) =: -2\sigma < 0$$
$$\forall t \in J_1, \quad v'(t) < -\sigma e^{-t}$$

and then

$$\forall t \in J_1, \quad u'(t) = v'(t) + z'(t) < -2\sigma e^{-t} + Ce^{-\gamma t}$$

Since $\gamma > 1$ we have for t_2 large enough

$$\forall t \geq t_2, \quad u'(t) < -\sigma e^{-t}$$

and finally

$$\forall t \geq t_2, \quad u(t) = -\int_t^\infty u'(s)ds > \sigma e^{-t}$$

This concludes the proof of Proposition 5.2 and this implies the uniqueness part of theorem 2.1 □

Proof of Theorem 2.2. Taking account of the conclusion of Proposition 5.2, the proof is a simple adaptation from [5], Lemma 5, p. 318-320 in a slightly different case. For completeness we recall here the main steps. We start with the equation (13) satisfied by $v = u - z$ and since v is non-oscillatory we introduce $w = \dfrac{v'}{v}$ which is a negative solution of

$$w' + w^2 + w + a(t) = 0 \tag{16}$$

Just as in [5], Proposition 2, p. 300-302 we can show easily that either

$$-\infty \leq \limsup_{t\to\infty} w(t) \leq -1$$

or

$$\lim_{t\to\infty} w(t) = 0.$$

The first case implies immediately

$$|v(t)| \leq C(\nu)\exp(-\nu t)$$

for any $\nu \in (0, 1)$, then by a bootstrap argument using the value of $a(t)$ it is easy to conclude that

$$|v(t)| + |v'(t)| \leq C\exp(-t)$$

and the proof is finished in this case.

In the second case wez note that as t goes to infinity

$$a(t) \sim |v(t)|^p \sim |u(t)|^p$$

By integrating (13) we deduce easily, since $|v'|$ is negligeable with respect to $|v|$

$$|v(t)| \sim \int_t^\infty |v(s)|^{p+1} ds \tag{17}$$

At this point we note that the upper estimate

$$|v(t)| + t|v'(t)| \leq Kt^{-\frac{1}{p}}$$

follows by a simple differential inequality as in [2]. Therefore the RHS in (17) is finite and this inequality yields, by letting

$$y(t) := \int_t^\infty |v(s)|^{p+1} ds$$

first the inequality

$$y' \geq -c_1 y^{p+1}$$

and then

$$|v(t)| \sim y(t) \geq c_2 t^{-\frac{1}{p}}$$

By comparing the already known estimates we obtain

$$-w(t) = |\frac{v'(t)}{v(t)}| \leq K_1 t^{-1}; \quad 0 \leq a(t) \sim t^{-1}$$

By plugging this information in the equation (16) we find since $w^2(t) = O(t^2)$

$$-(w' + w) \sim t^{-1}$$

hence

$$(e^t w(t))' \leq -\delta e^t t^{-1}$$

for some $\delta > 0$. By integrating on $[A, t)$ for some $A > 0$ we deduce

$$-e^t w(t) + e^A w(A) \geq \delta \int_A^t \frac{e^s}{s} ds \geq \frac{\delta}{t} \int_A^t e^s ds = \frac{\delta}{t}(e^t - e^A)$$

Hence in particular

$$\forall t \geq A, \quad -w(t) \geq \frac{\delta}{t} - e^{-t} e^A (\frac{\delta}{t} + w(A))$$

therefore for t large enough we have

$$|v'(t)| \geq \frac{\delta}{2t} |v(t)|$$

and the last inequality follows.

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