# Oscillatory behavior near blow-up of the solutions to some second order nonlinear ODE

Faouzia Aloui(*)

**Résumé:** On étudie les propriétés oscillatoires des solutions d'une équation différentielle ordinaire non linéaire scalaire du second ordre:

$$u''(t) + |u(t)|^{\beta} u(t) + g(u'(t)) = 0, \quad t \leq 0 \tag{0.1}$$

oũ $\beta$ est une constante positive et $g : \mathbf{R} \to \mathbf{R}$ une fonction croissante, et localement lipschitzienne comparable à $|v|^{\alpha} v$, $\alpha > 0$.

**Abstract:** We study the oscillation properties of solutions to the nonlinear scalar second order ODE:

$$u''(t) + |u(t)|^{\beta} u(t) + g(u'(t)) = 0, \quad t \leq 0 \tag{0.2}$$

where $\beta$ is a positive constant and $g : \mathbf{R} \to \mathbf{R}$ is an increasing and locally lipschitz function behaving globally like $|v|^{\alpha} v$, $\alpha > 0$.

**(*)UPMC Univ Paris 6, UMR 7598, Laboratoire Jacques Louis Lions, 4 place jussieu, 75252 Paris cedex 05, France**

## 1 Introduction and main results

We consider first the following equation:

$$u''(t) + |u|^{\beta} u(t) + g(u'(t)) = 0 \qquad (1.1)$$

where $t \in \mathbb{R}$, $\beta > 0$ and $g$ is a locally lipschitz continuous function which satisfies the following hypotheses:

$$\exists c > 0, \quad \forall v, \quad |g(v)| \leq c|v|^{\alpha+1} \qquad (1.2)$$

$$\exists \eta > 0, \quad \forall v, \quad g(v)v \geq \eta |v|^{\alpha+2} \qquad (1.3)$$

For some $\alpha > 0$.
The behavior and the oscillation properties of solutions of the equation (1.1) are determined by some relationship between $\alpha$ and $\beta$.

Let us recall the known results in the model case where $g(u') = b|u'|^{\alpha} u'$, then (1.1) becomes:

$$u'' + |u|^{\beta} u + b|u'|^{\alpha} u' = 0 \qquad (1.4)$$

A detailed study of the large time behavior of solutions to (1.4), when $t \geq 0$, was carried out by **Haraux** in **[1]** for all positive values $\alpha$ and $\beta$.
The type of equation (1.4) represents the motion of an oscillator subject to a nonlinear damping and a nonlinear restoring force and all solutions are global and decay to zero as $t \to \infty$.
To investigate the behavior of solutions to (1.4), **Haraux** in **[1]** used a method of polar coordinates which showed that the main condition deciding the oscillatory character of non-trivial solutions is the relative position of $\alpha$ and $\frac{\beta}{\beta+2}$.
The results of **[1]** can be summarized as follows:
**a**) Assume either

$$\alpha > \frac{\beta}{\beta+2}$$

or

$$\alpha = \frac{\beta}{\beta+2}; \quad b < (\beta+2)\left(\frac{\beta+2}{2\beta+2}\right)^{\frac{\beta+1}{\beta+2}}$$

Then any solution $u(t)$ to (1.4) which is not identically 0 changes sign on each interval $(T, \infty)$ and so does $u'(t)$.
**b**) if $\alpha < \frac{\beta}{\beta+2}$
Then any solution $u(t)$ of (1.4) which is not identically 0 has a finite number of zeroes on $(0, \infty)$.
Moreover for $t$ large enough, $u'(t)$ has the sign opposite to that of $u(t)$ and $u''(t)$ has the same sign as $u(t)$.

**c**) if $\alpha = \frac{\beta}{\beta+2}$; $\quad b \geq (\beta+2)(\frac{\beta+2}{2\beta+2})^{\frac{\beta+1}{\beta+2}}$
Then any solution $u(t)$ of (1.4) which is not identically 0 has at most one zero on $(0, \infty)$.

By changing $u(t)$ to $u(-t)$, the question of qualitative behavior for $t \leq 0$ of maximal solutions to (1.1) leads to study the antidissipative equation :

$$u'' + |u|^{\beta} u = g^{\sim}(u') \tag{1.5}$$

for $t \geq 0$, with $g^{\sim}(v) = -g(-v)$, therefore $g^{\sim}$ satisfies the same hypotheses as $g$ cited in the introduction, with the same values of $c$ and $\eta$.

First, we can state the known results of the model case with $g^{\sim}(u') = b|u'|^{\alpha} u'$, so that we consider the equation:

$$u'' + |u|^{\beta} u = b|u'|^{\alpha} u' \tag{1.6}$$

when $t \geq 0$.
For any solution of (1.6), the maximal existence interval of $u$ is of the form $I = (0, T)$ where $T = T(u) \in \mathbf{R} \cup +\infty$.
The behavior (oscillatory or non-oscillatory) of solutions of (1.6), was studied by **Balabane, Jazar** and **Souplet** in **[3]** and in **[4]** using a method completely different from that of **[1]**.
The results of **[3]** and **[4]** are the following:

**i**) if $\alpha \geq \beta > 1$ or $\alpha > \beta = 1$ the equation (1.6) has one nontrivial global solution, unique up to a sign and a time translation.
Any other solution blows-up in finite time.
**ii**) Assume either

$$1 < \alpha < \frac{\beta}{\beta+2}$$

or

$$\alpha = \frac{\beta}{\beta+2}; \quad b < (\beta+2)(\frac{\beta+2}{2\beta+2})^{\frac{\beta+1}{\beta+2}}$$

then all nontrivial solutions are nonglobal, they have oscillatory finite- time blow-up , meaning that $T < \infty$ and

$$\liminf_{t \to T} u(t) = \liminf_{t \to T} u'(t) = -\infty, \qquad \limsup_{t \to T} u(t) = \limsup_{t \to T} u'(t) = +\infty.$$

Moreover, the energy defined by $E(t) = \frac{u'^2}{2} + \frac{|u|^{\beta+2}}{\beta+2}$ blows-up at the rate:

$$C_1(T-t)^{\frac{-2}{\alpha}} \leq E(t) \leq C_2(T-t)^{\frac{-2}{\alpha}}$$

as $t \to T$, for some $C_1, C_2 > 0$.
**iii**) if $\frac{\beta}{\beta+2} < \alpha < \inf(1, \beta)$, then all nontrivial solutions are nonglobal and

have a constant sign for $t$ close to the blowing-up time. The positive solutions blow-up as follows:

$$u'(t) \sim (T-t)^{-\frac{1}{\alpha}}, \quad u(t) \sim (T-t)^{-\frac{1-\alpha}{\alpha}} \quad as \quad t \to T. \tag{1.7}$$

Except for an exceptional solution, unique up to a sign and a time translation, which blows-up at a faster rate than (1.7), more precisely if $u \neq 0$, we have:

$$u'(t) \sim (T-t)^{-\frac{\beta}{\beta-\alpha}}, \quad u(t) \sim (T-t)^{-\frac{\alpha}{\beta-\alpha}} \quad as \quad t \to T.$$

**iv**) if $\alpha = \frac{\beta}{\beta+2}$; $\quad b \geq (\beta+2)(\frac{\beta+2}{2\beta+2})^{\frac{\beta+1}{\beta+2}}$, then all nontrivial solutions blow-up in finite time $T$, they have a constant sign and:

$$C_1(T-t)^{-\frac{2}{\beta}} \leq |u(t)| \leq C_2(T-t)^{-\frac{2}{\beta}} \quad as \quad t \to T$$

$$C'_1(T-t)^{-\frac{2+\beta}{\beta}} \leq |u'(t)| \leq C'_2(T-t)^{-\frac{2+\beta}{\beta}} \quad as \quad t \to T$$

for some positive constants $C_1$, $C_2$, $C'_1$, $C'_2$.

The objective of this paper is to recover the oscillatory (or non-oscillatory) properties of solutions of (1.6) when $t \in [0, T]$ by the same method as [1] when $1 < \alpha < \beta$. Moreover, we generalize the results to (1.5) with a general function $g$ satisfying (1.2) – (1.3).

The plan is the following: section 2 contains basic energy estimates of solutions of the equation (1.5) when $t \in [0, T]$. Section 3 and 4 is devoted to study the oscillatory (or non-oscillatory) behavior of these solutions.

## 2 Basic energy estimates for equation (1.5)

We define the energy of (1.5) by:

$$E(t) = \frac{u'^2}{2} + \frac{|u|^{\beta+2}}{\beta+2}.$$

From (1.3), we deduce that the energy is non-decreasing with:

$$E'(t) = u'g^{\sim}(u') > 0$$

whenever $u' \neq 0$.

In the following section, we prove that the maximal existence time of $u \neq 0$ is $T < \infty$.

**Theorem 2.1.** *Let $1 < \alpha < \beta$ and $u \neq 0$ be a solution of* (1.5)*, then $u$ blows-up in finite time $T < \infty$. Moreover*

***i)*** *if $\alpha \leq \frac{\beta}{\beta+2}$, then $\exists\ C_0, C_1 > 0$ such that:*

$$\forall t \in [0,T], \quad C_0(T-t)^{-\frac{2}{\alpha}} \leq E(t) \leq C_1(T-t)^{-\frac{2}{\alpha}} \tag{2.1}$$

***ii)*** *if $\frac{\beta}{\beta+2} < \alpha < \beta$, then $\exists\ C' > 0$ such that:*

$$\forall t \in [0,T], \quad E(t) \leq C'(T-t)^{-\frac{(\beta+2)(\alpha+1)}{\beta-\alpha}} \tag{2.2}$$

**Remark 2.2.**

1) For $\alpha \geq \beta$, it is shown by **Souplet** in **[**3**]** that the equation (1.5) has one nontrivial global unbounded solution, up to a sign and a time-translation.

2) By using (1.3), it is easy to prove that the only solution of (1.5) to be global and bounded is the zero solution, so if $u \neq 0$, $E$ is unbounded. Then, if $u \neq 0$

$$\lim_{t \to T} u^2(t) + u'^2(t) = +\infty.$$

3) Moreover, under the additional assumption $g \in C^1$ and $g' > 0$, it is easy to prove as in **[**3**]** that if:

$$\lim u(t) = +\infty \quad (resp - \infty)$$

then

$$\lim u'(t) = +\infty \quad (resp - \infty)$$

and

$$u'' > 0 \quad (resp < 0) \quad as \quad t \to T.$$

***Proof of theorem 2.1:*** we consider the energy functional:

$$F(t) = E(t) - \epsilon|u|^\lambda uu' \tag{2.3}$$

where $\lambda > 0$, $\epsilon > 0$ .
By Young's inequality, we find

$$||u|^\lambda uu'| \le |u|^{2(\lambda+1)} + |u'|^2$$

we assume that

$$2(\lambda+1) \le \beta + 2 \Leftrightarrow \lambda \le \frac{\beta}{2}.$$

In order that,

$$\forall u \in \mathbf{R}, \quad |u|^{2(\lambda+1)} \le \max(|u|^{\beta+2}, 1) \le |u|^{\beta+2} + 1$$

Then, we obtain the existence of $K > 0$ such that

$$-C + (1 - K\epsilon)E(t) \le F(t) \le (1 + K\epsilon)E(t) + C'$$

for $t \in [0, T_{max})$.
Then, assuming $\epsilon$ small enough, we have

$$\frac{1}{2}E(t) - C_1 \le F(t) \le 2E(t) + C_2, \quad \forall t \in [0, T]. \tag{2.4}$$

Let us differentiate $F$, we have:

$$F'(t) = \frac{d}{dt}E(t) - \epsilon(\lambda+1)|u|^\lambda u'^2 - \epsilon|u|^\lambda uu''$$

$$F'(t) = u'g^{\sim}(u') + \epsilon|u|^{\beta+\lambda+2} - \epsilon(\lambda+1)|u|^\lambda u'^2 - \epsilon|u|^\lambda ug^{\sim}(u') \tag{2.5}$$

By using Young's inequality with exponents $\frac{\alpha+2}{\alpha}$ and $\frac{\alpha+2}{2}$ in the third term, we have:

$$|u|^\lambda u'^2 \le \rho|u|^{\lambda(\frac{\alpha+2}{\alpha})} + c(\rho)|u'|^{\alpha+2}$$

We assume that

$$\lambda(\frac{\alpha+2}{\alpha}) \le \beta + \lambda + 2 \Leftrightarrow \lambda \le \frac{\alpha}{2}(\beta+2)$$

In order that,

$$\forall u \in \mathbf{R}, \quad |u|^{\lambda(\frac{\alpha+2}{\alpha})} \le |u|^{\beta+\lambda+2} + 1$$

We choose $\rho$ small enough, then

$$-\epsilon(\lambda+1)|u|^\lambda u'^2 \ge -\frac{\epsilon}{4}|u|^{\beta+\lambda+2} - P\epsilon|u'|^{\alpha+2} - \delta_1 \tag{2.6}$$

By using Young's inequality with exponents $\alpha+2$ and $\frac{\alpha+2}{\alpha+1}$ in the last term, we have

$$|u|^{\lambda}ug^{\sim}(u') \leq \rho|u|^{(\lambda+1)(\alpha+2)} + c'(\rho)|g^{\sim}(u')|^{\frac{\alpha+2}{\alpha+1}}$$

By using (1.2), we find

$$|u|^{\lambda}ug^{\sim}(u') \leq \rho|u|^{(\lambda+1)(\alpha+2)} + c'(\rho)c|u'|^{\alpha+2}$$

We assume that

$$(\lambda+1)(\alpha+2) \leq \beta+\lambda+2 \Leftrightarrow \lambda \leq \frac{\beta-\alpha}{\alpha+1}$$

In order that,

$$\forall u \in \mathbf{R}, \quad |u|^{(\lambda+1)(\alpha+2)} \leq |u|^{\beta+\lambda+2} + 1$$

For $\rho$ small enough, then

$$-\epsilon|u|^{\lambda}ug^{\sim}(u') \geq -\frac{\epsilon}{4}|u|^{\beta+\lambda+2} - \epsilon P'|u'|^{\alpha+2} - \delta_2 \tag{2.7}$$

From (1.3) and by substituting the inequalities (2.6) and (2.7) in (2.5), we obtain

$$\begin{aligned} F'(t) &\geq (\eta - P\epsilon - P'\epsilon)|u'|^{\alpha+2} + \frac{\epsilon}{2}|u|^{\beta+\lambda+2} - M \\ &\geq (\eta - Q\epsilon)|u'|^{\alpha+2} + \frac{\epsilon}{2}|u|^{\beta+\lambda+2} - M \end{aligned}$$

With $Q = P + P'$
For $\epsilon$ small enough, we have

$$F'(t) \geq \frac{\epsilon}{2}(|u'|^{\alpha+2} + |u|^{\beta+\lambda+2}) - M$$

where

$$\lambda = \min(\frac{\beta}{2}, \frac{\beta-\alpha}{\alpha+1}, \frac{\alpha}{2}(\beta+2))$$

Set

$$\sigma = \min(\frac{\alpha+2}{2}, 1 + \frac{\beta-\alpha}{(\alpha+1)(\beta+2)})$$

Then, by using the following inequality

$$(x+y)^{\sigma} \leq c(\sigma)(x^{\sigma} + y^{\sigma})$$

We have by (2.4)

$$F'(t) \geq \frac{\epsilon}{2}c^{-1}(\sigma)c_1 E(t)^{\sigma} - M' \geq \frac{\epsilon}{4}c_2 F(t)^{\sigma} - M' \tag{2.8}$$

where $c_2 = c^{-1}(\sigma)c_1$ and $M' > 0$.
Since $E$ is unbounded so is $F$, Then $\exists T^* < T$ for which $\frac{\epsilon}{4}c_2 F(t)^{\sigma} > 2M'$ for

$t \in (T^*, T)$.
Therefore,

$$F'(t) \geq \frac{\epsilon}{4} c_3 F(t)^{\sigma} \tag{2.9}$$

and (2.9) implies that $T < \infty$.

We distinguish two cases:
**i**) $\alpha \leq \frac{\beta}{\beta+2}$, then $\frac{\alpha}{2}(\beta+2) \leq \frac{\beta}{2}$ and $\frac{\beta-\alpha}{\alpha+1} \geq \frac{\beta - \frac{\beta}{\beta+2}}{\frac{\beta}{\beta+2}+1} = \frac{\beta}{2}$
Then we choose $\lambda = \frac{\alpha}{2}(\beta+2)$, we have:

$$\beta + \lambda + 2 = (\beta+2)(\frac{\alpha+2}{2})$$

and

$$\sigma = \frac{\alpha+2}{2}$$

Then, by (2.9), we have

$$F'(t) \geq \frac{\epsilon}{4} c_3 F(t)^{\frac{\alpha+2}{2}}$$

We have

$$\frac{d}{dt} F(t)^{-\frac{\alpha}{2}} = -\frac{\alpha}{2} \frac{d}{dt} F(t) F(t)^{-\frac{\alpha}{2}-1} \leq -\frac{\alpha}{8} \epsilon c_3$$

By integrating from $t$ to $\tau$, we find

$$\int_t^{\tau} \frac{d}{ds} F(s)^{-\frac{\alpha}{2}} ds \leq \int_t^{\tau} -\epsilon c_4 ds$$

with $c_4 = \frac{\alpha}{8} c_3$ Then,

$$F(\tau)^{-\frac{\alpha}{2}} - F(t)^{-\frac{\alpha}{2}} \leq -\epsilon c_4 (\tau - t)$$

Since

$$F(\tau) \to +\infty \quad if \quad \tau \to T$$

Then

$$F(\tau)^{-\frac{\alpha}{2}} \to 0 \quad if \quad \tau \to T$$

Therefore, by letting $\tau \to T$

$$F(t) \leq \epsilon^{-\frac{2}{\alpha}} c_4' (T-t)^{-\frac{2}{\alpha}}$$

We assume $c_5 = \epsilon^{-\frac{2}{\alpha}} c_4'$, then

$$F(t) \leq c_5 (T-t)^{-\frac{2}{\alpha}}$$

By using (2.4), we deduce

$$E(t) \leq C_1(T-t)^{-\frac{2}{\alpha}} \tag{2.10}$$

with $C_1 > 2c_5$, for $t$ close to $T$

For the converse inequality, we have

$$E'(t) = u'g(u') \leq c|u'|^{\alpha+2} \leq cKE(t)^{\frac{\alpha+2}{2}}$$

Then

$$\frac{d}{dt}E(t)^{-\frac{\alpha}{2}} = -\frac{\alpha}{2}\frac{d}{dt}E(t)E(t)^{-\frac{\alpha+2}{2}} \geq -\frac{\alpha}{2}cK$$

By integrating from $t$ to $\tau$, we find

$$E(\tau)^{-\frac{\alpha}{2}} - E(t)^{-\frac{\alpha}{2}} \geq -K(\tau - t)$$

Since $E(\tau) \to +\infty$ if $\tau \to T$, then

$$E(t) \geq C_0(T-t)^{-\frac{2}{\alpha}} \tag{2.11}$$

Therefore, from (2.10) and (2.11), we find

$$C_0(T-t)^{-\frac{2}{\alpha}} \leq E(t) \leq C_1(T-t)^{-\frac{2}{\alpha}}$$

**ii**) if $\frac{\beta}{\beta+2} < \alpha < \beta$, we have $\frac{\alpha}{2}(\beta+2) > \frac{\beta}{2}$ and $\frac{\beta-\alpha}{\alpha+1} < \frac{\beta}{2}$
Then $\lambda = \frac{\beta-\alpha}{\alpha+1}$,we have

$$\beta + \lambda + 2 = (\beta+2)(1 + \frac{\beta-\alpha}{(\alpha+1)(\beta+2)})$$

Moreover, we have

$$1 + \frac{\beta-\alpha}{(\alpha+1)(\beta+2)} - \frac{\alpha}{2} - 1 < 0$$

and

$$\sigma = 1 + \frac{\beta-\alpha}{(\alpha+1)(\beta+2)}$$

Then

$$F'(t) \geq \frac{\epsilon}{2}c^{-1}(\alpha,\beta)c_2F(t)^{1+\frac{\beta-\alpha}{(\alpha+1)(\beta+2)}}$$

We have

$$\frac{d}{dt}F(t)^{-\frac{\beta-\alpha}{(\alpha+1)(\beta+2)}} = -\frac{\beta-\alpha}{(\alpha+1)(\beta+2)}\frac{d}{dt}F(t)F(t)^{-1-\frac{\beta-\alpha}{(\alpha+1)(\beta+2)}}$$

Therefore,

$$\frac{d}{dt}F(t)^{-\frac{\beta-\alpha}{(\alpha+1)(\beta+2)}} \leq -\frac{\beta-\alpha}{(\alpha+1)(\beta+2)}\epsilon c_3$$

By integrating from $t$ to $\tau$, we have

$$\int_t^\tau \frac{d}{ds} F(s)^{-\frac{\beta-\alpha}{(\alpha+1)(\beta+2)}} ds \leq \int_t^\tau -\epsilon c_3 ds$$

Then, as $i)$ , we find if $\tau \to T$

$$F(t) \leq \epsilon^{-\frac{(\alpha+1)(\beta+2)}{\beta-\alpha}} c_4 (T-t)^{-\frac{(\alpha+1)(\beta+2)}{\beta-\alpha}}$$

Then we assume $C_1' = \epsilon^{-\frac{(\alpha+1)(\beta+2)}{\beta-\alpha}} c_4$ and we use (2.4), we find

$$E(t) \leq C'(T-t)^{-\frac{(\alpha+1)(\beta+2)}{\beta-\alpha}}$$

□

## 3 Oscillation of solutions of (1.5) near blow-up for $\alpha$ small:

To establish the oscillatory character of solutions of (1.5), we can use the method from **[1]**. We obtain the following result:

**Theorem 3.1.** *Assume either*

$$1 < \alpha < \frac{\beta}{\beta+2}$$

*Or*

$$\alpha = \frac{\beta}{\beta+2}; \quad c < (\beta+2)(\frac{\beta+2}{2\beta+2})^{\frac{\beta+1}{\beta+2}}$$

*Then, all nontrivial solutions of* (1.5) *have oscillatory blow-up at time $T$ and:*

$$\limsup_{t\to T} u(t) = \limsup_{t\to T} u'(t) = +\infty; \quad \liminf_{t\to T} u(t) = \liminf_{t\to T} u'(t) = -\infty$$

***Proof of theorem 3.1****:* Since the energy of (1.5) is positive $\forall\, t \in [0,T]$, we introduce as in **[1]** the polar coordinates:

$$(\frac{2}{\beta+2})^{\frac{1}{2}} |u|^{\frac{\beta}{2}} u = r(t)\cos\theta(t); \quad u'(t) = r(t)\sin\theta(t)$$

where $r$ and $\theta$ are $C^1$ functions and $r(t) = E(t)^{\frac{1}{2}} > 0$.
We have

$$u''(t) = -|u|^\beta u + g^\sim(u') = r'(t)\sin\theta(t) + r(t)\theta'(t)\cos\theta(t) \tag{3.1}$$

$$(\frac{\beta+2}{2})^{\frac{1}{2}}|u|^{\frac{\beta}{2}}u' = r'(t)\cos\theta(t) - r(t)\theta'(t)\sin\theta(t) \tag{3.2}$$

Then
(3.1)* $\cos\theta$ - (3.2)*$\sin\theta$, give

$$\theta' = -(\frac{\beta+2}{2})^{\frac{\beta+1}{\beta+2}} r(t)^{\frac{\beta}{\beta+2}} |\cos\theta|^{\frac{\beta}{\beta+2}} + g^{\sim}(r\sin\theta)\frac{\cos\theta}{r}$$

Use (1.2), we have

$$\theta' \leq -(\frac{\beta+2}{2})^{\frac{\beta+1}{\beta+2}} r(t)^{\frac{\beta}{\beta+2}} |\cos\theta|^{\frac{\beta}{\beta+2}} + cr^{\alpha}|\sin\theta|^{\alpha+1}\cos\theta$$

Since $\alpha < \frac{\beta}{\beta+2}$ and if $t \to T$, $r(t) \sim C(T-t)^{-\frac{1}{\alpha}}$, then $r^{\alpha}|\cos\theta| \leq \varrho r^{\frac{\beta}{\beta+2}}|\cos\theta|^{\frac{\beta}{\beta+2}}$, as $t \to T$.
Therefore

$$\theta' \leq -\xi(T-t)^{-\frac{\beta}{\alpha(\beta+2)}}|\cos\theta|^{\frac{\beta}{\beta+2}}, \quad as \quad t \to T.$$

Where $\frac{\beta}{\alpha(\beta+2)} > 1$.
In the case $\alpha = \frac{\beta}{\beta+2}$; $\quad c < (\beta+2)(\frac{\beta+2}{2\beta+2})^{\frac{\beta+1}{\beta+2}}$, we have

$$\begin{aligned}\theta' &\leq -r^{\alpha}((\frac{\beta+2}{2})^{\frac{\beta+1}{\beta+2}}|\cos\theta|^{\alpha} - c|\sin\theta|^{\alpha+1}\cos\theta) \\ &\leq -r^{\alpha}|\cos\theta|^{\alpha}((\frac{\beta+2}{2})^{\frac{\beta+1}{\beta+2}} - c|\sin\theta|^{\alpha+1}|\cos\theta|^{1-\alpha})\end{aligned}$$

We assume

$$f(\theta) = |\sin\theta|^{\alpha+1}|\cos\theta|^{1-\alpha}, \quad \theta \in \mathbf{R}$$

Then, we have

$$\max_{\theta\in\mathbf{R}} f(\theta) = (\frac{1}{\beta+2})^{\frac{1}{\beta+2}}(\frac{\beta+1}{\beta+2})^{\frac{\beta+1}{\beta+2}}$$

Hence

$$(\frac{\beta+2}{2})^{\frac{\beta+1}{\beta+2}} - c|\sin\theta|^{\alpha+1}|\cos\theta|^{1-\alpha} \geq (\frac{\beta+2}{2})^{\frac{\beta+1}{\beta+2}} - c(\frac{1}{\beta+2})^{\frac{1}{\beta+2}}(\frac{\beta+1}{\beta+2})^{\frac{\beta+1}{\beta+2}}$$

Therefore

$$(\frac{\beta+2}{2})^{\frac{\beta+1}{\beta+2}} - c(\frac{1}{\beta+2})^{\frac{1}{\beta+2}}(\frac{\beta+1}{\beta+2})^{\frac{\beta+1}{\beta+2}} > 0$$

$$\Leftrightarrow c < (\beta+2)(\frac{\beta+2}{2\beta+2})^{\frac{\beta+1}{\beta+2}}$$

Then, we find in both cases for $t \to T$ ,

$$\theta' \leq -\xi(T-t)^{-1}|\cos\theta|^{\frac{\beta}{\beta+2}}$$

We introduce the following function:

$$H(s) = \int_a^s \frac{du}{|\cos u|^{\frac{\beta}{\beta+2}}}$$

We assume that $u$ does not vanish if $t \to T$, and for $t \in [t_0, T]$, $\theta(t) \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $H(\theta(t) = K(t)$ is differentiable and we have

$$\forall t_0 \leq t \leq T, \quad K'(t) \leq -\xi(T-t)^{-1}$$

We integrate from $t_0$ to $t$

$$H(\theta(t)) \leq H(\theta(t_0)) - \xi \log(T - t_0) + \xi \log(T-t)$$

If $t \to T$, we find $H(\theta(t)) \to -\infty$ which is impossible since $H$ is nonnegative. Then this contradiction proves that $u$ vanishes on each half-line. Since $u'$ cannot vanish at the same time, $u$ must change sign.If $u'$ and $u''$ vanish at the same time, the equation shows that $u$ vanishes also, this contradiction implies that $u'$ changes sign also.
Finally, by (2.2) we have, $\lim_{t\to T} u^2(t) + u'^2(t) = +\infty$. Then, since $u(t)$ and $u'(t)$ have oscillatory blow-up, we deduce:

$$\limsup_{t\to T} u(t) = \limsup_{t\to T} u'(t) = +\infty$$

and

$$\liminf_{t\to T} u(t) = \liminf_{t\to T} u'(t) = -\infty.$$

□

# 4 Non oscillation of solutions of (1.5) for $\alpha$ large

In this section, we generalize to (1.5) the non-oscillation result of **[3]** by the method of **[1]**. We obtain:

**Theorem 4.1.** *Assuming*

$$\frac{\beta}{\beta+2} < \alpha < \beta$$

*Then any solution $(u(t), u'(t))$ has a finite number of zeroes in $(T-\epsilon, T)$, $\epsilon > 0$, and blow-up as $t \to T$. Also,assuming $g \in C^1$ and $g' > 0$, if $t \to T$, $u(t)$, $u'(t)$ and $u''(t)$ have the same sign.*

***Proof of theorem 4.1:*** Let

$$G(s) = \int_0^s |\sin u|^{\alpha} \sin u \quad \cos u \quad du$$

We differentiate $G$, we obtain

$$[G(\theta(t))]' = \theta' |\sin\theta|^{\alpha} \sin\theta\cos\theta$$

$$[G(\theta(t))]' = -(\frac{\beta+2}{2})^{\frac{\beta+1}{\beta+2}} r(t)^{\frac{\beta}{\beta+2}} |\cos\theta|^{\frac{\beta}{\beta+2}} |\sin\theta|^{\alpha} \sin\theta\cos\theta \quad + \quad \frac{g^{\sim}(r\sin\theta)}{r} |\sin\theta|^{\alpha} \sin\theta\cos^2\theta$$

Then, from (1.3), we obtain

$$[G(\theta(t))]' = -(\frac{\beta+2}{2})^{\frac{\beta+1}{\beta+2}} r(t)^{\frac{\beta}{\beta+2}} |\cos\theta|^{\frac{\beta}{\beta+2}} |\sin\theta|^{\alpha} \sin\theta\cos\theta \quad + \quad \eta r^{\alpha} \sin^{2(\alpha+1)}\theta\cos^2\theta$$

By using the Young's inequality with exponents 2 in the first term , we have

$$[G(\theta(t))]' \geq -C' r^{\frac{2\beta}{\beta+2}-\alpha}$$

Then, using (2.2)

$$[G(\theta(t))]' \geq -C'' r^{\frac{2\beta}{\beta+2}-\alpha} \geq -C''(T-t)^{-\mu}.$$

With

$$\begin{aligned} 0 \leq \mu &= (\frac{2\beta}{\beta+2} - \alpha)(\frac{(\alpha+1)(\beta+2)}{2(\beta-\alpha)}) \\ &= \alpha + 1 - \frac{\alpha\beta(\alpha+1)}{2(\beta-\alpha)} \\ &= 1 + \alpha[1 - \frac{\beta(\alpha+1)}{2(\beta-\alpha)}] \\ &= 1 + \alpha[\frac{\beta - \alpha(\beta+2)}{2(\beta-\alpha)}] < 1 \end{aligned}$$

To finish the proof we shall use the following lemma ($cf.$**[**1**]** for proof).

**Lemma 4.2.** *Assuming $\theta \in C^1(a,T)$ and $G$ be a non constant $\tau$-periodic function.*
*We assume $h \in L^1(a,T)$ and*

$$(G(\theta(t)))' \geq h(t), \quad \forall t \in [a,T].$$

*Then for $t_1 \leq t < T$, $\theta(t)$ remains in some interval of length $\leq \tau$.*
*Moreover, if $G$ has a finite number of zeroes in $[0,\tau]$, then $\theta(t)$ has a limit if $t \to T$.*

***End of the proof of theorem 4.1***: From the **lemma**(4.2), $\theta(t)$ has a limit as $t \to T$.
We distinguish two cases:
**case1**: if this limit differs from $\frac{\pi}{2}$ (mod $\pi$), $u \sim Cr^{\frac{2}{\beta+2}} > 0$ if $t \to T$, then $u$ has a constant sign.

**case2**: in the opposite case, we have, if $t \to T$, $|u'(t)| \sim r(t) > 0$, then $u'(t)$ does not vanish if $t \to T$ and therefore $u(t)$ has a constant sign if $t \to T$.
If $u'(t)$ has several zeroes in $(T-\epsilon, T)$, then $u''$ has different signs at two successive zeroes of $u'$, from equation (1.5) $u$ must have different signs also, which is impossible.
Then, $u'(t)$ has a constant sign as $t \to T$.
Since $u(t)$ and $u'(t)$ deos not change sign if $t \to T$, we conclude by (2.2) that:

$$\lim_{t\to T} u(t) = \lim_{t\to T} u'(t) = \pm\infty.$$

And by (2.2), we deduce that $u$, $u'$ and $u''$ have the same sign if $t \to T$. $\square$

**Theorem 4.3.** *Assuming*

$$\alpha = \frac{\beta}{\beta+2}; \quad b \geq b_0 = (\beta+2)(\frac{\beta+2}{2\beta+2})^{\frac{\beta+1}{\beta+2}}$$

*Then any solution* $u(t)$ *of* (1.6) *blows-up in finite time* $T$ *and has a finite number of zeroes in* $[0, T]$.

***Proof of theorem 4.3:***

$$\begin{aligned}\theta' &= -r^{\alpha}((\frac{\beta+2}{2})^{\frac{\beta+1}{\beta+2}}|\cos\theta|^{\alpha} - b|\sin\theta|^{\alpha}\sin\theta\cos\theta) \\ &= -r^{\alpha}|\cos\theta|^{\alpha}((\frac{\beta+2}{2})^{\frac{\beta+1}{\beta+2}} - b|\sin\theta|^{\alpha+1}|\cos\theta|^{1-\alpha})\end{aligned}$$

**i)**If $b = b_0$, we have

$$(\frac{\beta+2}{2})^{\frac{\beta+1}{\beta+2}} - b|\sin\theta|^{\alpha+1}|\cos\theta|^{1-\alpha} \geq (\frac{\beta+2}{2})^{\frac{\beta+1}{\beta+2}} - b_0(\frac{1}{\beta+2})^{\frac{1}{\beta+2}}(\frac{\beta+1}{\beta+2})^{\frac{\beta+1}{\beta+2}} = 0$$

Therefore, the coefficient of $-r^{\alpha}$ is positive, then $\theta$ is non-increasing.
The distance of two zereos of $h(\theta) = (\frac{\beta+2}{2})^{\frac{\beta+1}{\beta+2}}|\cos\theta|^{\alpha} - b|\sin\theta|^{\alpha}\sin\theta\cos\theta$ other than $\frac{\pi}{2}(mod\pi)$ is not more than $\pi$, therefore we have two cases:

**case1:** if $\theta(t)$ remains in an interval of length less than $\pi$, so $\theta$ is bounded and from above it is non-increasing then it converges to a limit as $t \to T$.

**case2:** if $\theta(t)$ coincides with one of these zeroes if $t \to T$, due to existence and uniqueness for the $ODE$ satisfied by $\theta(t)$near the non-trivial equilibria, it must remains constant and $u$ never vanishes.

**ii)** If $b > b_0$, the coefficient of $-r^{\alpha}$ still has nontrivial zeroes.We have two cases:

**case1:** if $\theta$ does not take any of the corresponding values, then, it is bounded and since the coefficient of $-r^\alpha$ is negative near the trivial zeroes, $\theta$ is non-decreasing and therefore it is convergent.

**case2:** if $\theta$ takes one of the corresponding values, hence it remains constant and $u$ never vanishes. □

**Remark 4.4.** For the equation (1.1) when $t \geq 0$, by the same calculation as **[**1**]**, it is easy to show that the solutions to (1.1) have the same oscillatory (or non-oscillatory) behavior as in (1.4) in the noncritcal case $\alpha \neq \frac{\beta}{\beta+2}$. In the case $\alpha = \frac{\beta}{\beta+2}$, the result will depend on $c$ and $\eta$.

□

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