Oscillatory behavior near blow-up of the solutions to some second order nonlinear ODE

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Résumé: On étudie les propriétés oscillatoires des solutions d’une équation différentielle ordinaire non linéaire scalaire du second ordre:

\[ u''(t) + |u(t)|^\beta u(t) + g(u'(t)) = 0, \quad t \leq 0 \] (0.1)

ou \( \beta \) est une constante positive et \( g: \mathbb{R} \to \mathbb{R} \) une fonction croissante, et localement lipschitzienne comparable à \( |v|^\alpha v \), \( \alpha > 0 \).

Abstract: We study the oscillation properties of solutions to the nonlinear scalar second order ODE:

\[ u''(t) + |u(t)|^\beta u(t) + g(u'(t)) = 0, \quad t \leq 0 \] (0.2)

where \( \beta \) is a positive constant and \( g: \mathbb{R} \to \mathbb{R} \) is an increasing and locally lipschitz function behaving globally like \( |v|^\alpha v \), \( \alpha > 0 \).

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1 Introduction and main results

We consider first the following equation:

\[ u''(t) + |u|^{\beta} u(t) + g(u'(t)) = 0 \]  

(1.1)

where \( t \in \mathbb{R} \), \( \beta > 0 \) and \( g \) is a locally lipschitz continuous function which satisfies the following hypotheses:

\[ \exists c > 0, \quad \forall v, \quad |g(v)| \leq c|v|^\alpha + 1 \]  

(1.2)

\[ \exists \eta > 0, \quad \forall v, \quad g(v)v \geq \eta|v|^\alpha + 2 \]  

(1.3)

For some \( \alpha > 0 \).

The behavior and the oscillation properties of solutions of the equation (1.1) are determined by some relationship between \( \alpha \) and \( \beta \).

Let us recall the known results in the model case where \( g(u') = b|u'|^{\alpha}u' \), then (1.1) becomes:

\[ u'' + |u|^{\beta} u + b|u'|^{\alpha} u' = 0 \]  

(1.4)

A detailed study of the large time behavior of solutions to (1.4), when \( t \geq 0 \), was carried out by Haraux in [1] for all positive values \( \alpha \) and \( \beta \).

The type of equation (1.4) represents the motion of an oscillator subject to a nonlinear damping and a nonlinear restoring force and all solutions are global and decay to zero as \( t \to \infty \).

To investigate the behavior of solutions to (1.4), Haraux in [1] used a method of polar coordinates which showed that the main condition deciding the oscillatory character of non-trivial solutions is the relative position of \( \alpha \) and \( \frac{\beta}{\beta + 2} \).

The results of [1] can be summarized as follows:

a) Assume either

\[ \alpha > \frac{\beta}{\beta + 2} \]

or

\[ \alpha = \frac{\beta}{\beta + 2}; \quad b < (\beta + 2)(\frac{\beta + 2}{2\beta + 2})^\frac{\beta + 1}{\beta + 2} \]

Then any solution \( u(t) \) to (1.4) which is not identically 0 changes sign on each interval \((T, \infty)\) and so does \( u'(t) \).

b) if \( \alpha < \frac{\beta}{\beta + 2} \)

Then any solution \( u(t) \) of (1.4) which is not identically 0 has a finite number of zeroes on \((0, \infty)\).

Moreover for \( t \) large enough, \( u'(t) \) has the sign opposite to that of \( u(t) \) and \( u''(t) \) has the same sign as \( u(t) \).
(1) if $\alpha = \frac{\beta}{\beta + 2}$; \quad b \geq (\beta + 2)(\frac{\beta + 2}{\beta + 2})^{\frac{\beta + 1}{2}}$

Then any solution $u(t)$ of (1.4) which is not identically 0 has at most one zero on $(0, \infty)$.

By changing $u(t)$ to $u(-t)$, the question of qualitative behavior for $t \leq 0$ of maximal solutions to (1.1) leads to study the antidual equation:

$$u'' + |u|^\beta u = g^\sim(u')$$

(1.5)

for $t \geq 0$, with $g^\sim(v) = -g(-v)$, therefore $g^\sim$ satisfies the same hypotheses as $g$ cited in the introduction, with the same values of $c$ and $\eta$.

First, we can state the known results of the model case with $g^\sim(u') = b|u'|^\alpha u'$, so that we consider the equation:

$$u'' + |u|^\beta u = b|u'|^\alpha u'$$

(1.6)

when $t \geq 0$.

For any solution of (1.6), the maximal existence interval of $u$ is of the form $I = (0, T)$ where $T = T(u) \in \mathbb{R} \cup +\infty$.

The behavior (oscillatory or non-oscillatory) of solutions of (1.6), was studied by Balabane, Jazar and Souplet in [3] and in [4] using a method completely different from that of [1].

The results of [3] and [4] are the following:

i) if $\alpha \geq \beta > 1$ or $\alpha > \beta = 1$ the equation (1.6) has one nontrivial global solution, unique up to a sign and a time translation.

Any other solution blows-up in finite time.

ii) Assume either

$$1 < \alpha < \frac{\beta}{\beta + 2}$$

or

$$\alpha = \frac{\beta}{\beta + 2}; \quad b < (\beta + 2)(\frac{\beta + 2}{\beta + 2})^{\frac{\beta + 1}{2}}$$

then all nontrivial solutions are nonglobal, they have oscillatory finite-time blow-up , meaning that $T < \infty$ and

$$\liminf_{t \to T} u(t) = \liminf_{t \to T} u'(t) = -\infty, \quad \limsup_{t \to T} u(t) = \limsup_{t \to T} u'(t) = +\infty.$$

Moreover, the energy defined by $E(t) = \frac{u'^2}{2} + \frac{|u|^{\beta+2}}{\beta+2}$ blows-up at the rate:

$$C_1(T - t)^{-\frac{\alpha}{2}} \leq E(t) \leq C_2(T - t)^{-\frac{\alpha}{2}}$$

as $t \to T$, for some $C_1, C_2 > 0$.

iii) if $\frac{\beta}{\beta + 2} < \alpha < \inf(1, \beta)$, then all nontrivial solutions are nonglobal and
have a constant sign for $t$ close to the blowing-up time. The positive solutions blow-up as follows:

$$u'(t) \sim (T - t)^{-\frac{1}{\alpha}}, \quad u(t) \sim (T - t)^{-\frac{1-\alpha}{\alpha}} \quad \text{as} \quad t \to T. \quad (1.7)$$

Except for an exceptional solution, unique up to a sign and a time translation, which blows-up at a faster rate than $(1.7)$, more precisely if $u \neq 0$, we have:

$$u'(t) \sim (T - t)^{-\frac{\beta}{\beta - \alpha}}, \quad u(t) \sim (T - t)^{-\frac{\alpha}{\beta - \alpha}} \quad \text{as} \quad t \to T.$$

iv) if $\alpha = \frac{\beta}{\beta + 2}$; $b \geq (\beta + 2)(\frac{\beta + 2}{2\beta + 2})^{\frac{\beta + 1}{\beta + 2}}$, then all nontrivial solutions blow-up in finite time $T$, they have a constant sign and:

$$C_1(T - t)^{-\frac{2}{\beta}} \leq |u(t)| \leq C_2(T - t)^{-\frac{2}{\beta}} \quad \text{as} \quad t \to T$$

$$C_1'(T - t)^{-\frac{2+\beta}{\beta}} \leq |u'(t)| \leq C_2'(T - t)^{-\frac{2+\beta}{\beta}} \quad \text{as} \quad t \to T$$

for some positive constants $C_1, C_2, C_1', C_2'$.

The objective of this paper is to recover the oscillatory (or non-oscillatory) properties of solutions of (1.6) when $t \in [0, T]$ by the same method as [1] when $1 < \alpha < \beta$. Moreover, we generalize the results to (1.5) with a general function $g$ satisfying (1.2) – (1.3).

The plan is the following: section 2 contains basic energy estimates of solutions of the equation (1.5) when $t \in [0, T]$. Section 3 and 4 is devoted to study the oscillatory (or non-oscillatory) behavior of these solutions.

2 Basic energy estimates for equation (1.5)

We define the energy of (1.5) by:

$$E(t) = \frac{u'^2}{2} + \frac{|u|^\beta + 2}{\beta + 2}.$$

From (1.3), we deduce that the energy is non-decreasing with:

$$E'(t) = u'g(u') > 0$$

whenever $u' \neq 0$.

In the following section, we prove that the maximal existence time of $u \neq 0$ is $T < \infty$. 

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Theorem 2.1. Let \( 1 < \alpha < \beta \) and \( u \neq 0 \) be a solution of (1.5), then \( u \) blows-up in finite time \( T < \infty \). Moreover

i) if \( \alpha \leq \frac{\beta}{\beta + 2} \), then \( \exists C_0, C_1 > 0 \) such that:
\[
\forall t \in [0, T], \quad C_0 (T - t)^{-\frac{2}{\alpha}} \leq E(t) \leq C_1 (T - t)^{-\frac{2}{\alpha}} \tag{2.1}
\]

ii) if \( \frac{\beta}{\beta + 2} < \alpha < \beta \), then \( \exists C' > 0 \) such that:
\[
\forall t \in [0, T], \quad E(t) \leq C'(T - t)^{-\frac{(\beta + 2)(\alpha + 1)}{\beta - \alpha}} \tag{2.2}
\]

Remark 2.2.

1) For \( \alpha \geq \beta \), it is shown by Souplet in [3] that the equation (1.5) has one nontrivial global unbounded solution, up to a sign and a time-translation.

2) By using (1.3), it is easy to prove that the only solution of (1.5) to be global and bounded is the zero solution, so if \( u \neq 0 \), \( E \) is unbounded. Then, if \( u \neq 0 \)
\[
\lim_{t \to T} u^2(t) + u'^2(t) = +\infty.
\]

3) Moreover, under the additional assumption \( g \in C^1 \) and \( g' > 0 \), it is easy to prove as in [3] that if:
\[
\lim_{t \to T} u(t) = +\infty \quad (\text{resp} - \infty)
\]
then
\[
\lim_{t \to T} u'(t) = +\infty \quad (\text{resp} - \infty)
\]
and
\[
u'' > 0 \quad (\text{resp} < 0) \quad \text{as} \quad t \to T.
\]
Proof of theorem 2.1: we consider the energy functional:

\[ F(t) = E(t) - \epsilon |u|^\lambda u'^2 \]  

(2.3)

where \( \lambda > 0, \epsilon > 0 \).

By Young's inequality, we find

\[ ||u|^\lambda u'|| \leq ||u|^{2(\lambda + 1)} + |u'|^2 \]

we assume that

\[ 2(\lambda + 1) \leq \beta + 2 \iff \lambda \leq \frac{\beta}{2}. \]

In order that,

\[ \forall u \in \mathbb{R}, \quad |u|^{2(\lambda + 1)} \leq \max(|u|^\beta, 1) \leq |u|^\beta + 1 \]

Then, we obtain the existence of \( K > 0 \) such that

\[ -C + (1 - K\epsilon)E(t) \leq F(t) \leq (1 + K\epsilon)E(t) + C' \]

for \( t \in [0, T_{\max}) \).

Then, assuming \( \epsilon \) small enough, we have

\[ \frac{1}{2}E(t) - C_1 \leq F(t) \leq 2E(t) + C_2, \quad \forall t \in [0, T]. \]  

(2.4)

Let us differentiate \( F \), we have:

\[ F'(t) = \frac{d}{dt}E(t) - \epsilon(\lambda + 1)|u|^\lambda u'^2 - \epsilon|u|^\lambda uu'' \]

\[ F'(t) = u'g^\sim(u') + \epsilon|u|^\beta u'^2 + (\lambda + 1)|u|^\lambda u'^2 - \epsilon|u|^\lambda uu'' \]  

(2.5)

By using Young's inequality with exponents \( \frac{a + 2}{\alpha} \) and \( \frac{a + 2}{2} \) in the third term, we have:

\[ |u|^\lambda u'^2 \leq \rho|u|^\lambda + c(\rho)|u'|^{\alpha + 2} \]

We assume that

\[ \lambda\left(\frac{a + 2}{\alpha}\right) \leq \beta + \lambda + 2 \iff \lambda \leq \frac{\alpha}{2}(\beta + 2) \]

In order that,

\[ \forall u \in \mathbb{R}, \quad |u|^\lambda \leq |u|^\beta + \lambda + 2 + 1 \]

We choose \( \rho \) small enough, then

\[ -\epsilon(\lambda + 1)|u|^\lambda u'^2 \geq -\frac{\epsilon}{4}|u|^\beta + \lambda + 2 - P\epsilon|u'|^{\alpha + 2} - \delta_1 \]  

(2.6)
By using Young’s inequality with exponents $\alpha + 2$ and $\frac{\alpha + 2}{\alpha + 1}$ in the last term, we have

$$|u|^{\lambda} u g \sim (u') \leq \rho |u|^{(\lambda+1)(\alpha+2)} + c'(\rho) |g^\sim (u')|^{\frac{\alpha + 2}{\alpha + 1}}$$

By using (1.2), we find

$$|u|^{\lambda} u g \sim (u') \leq \rho |u|^{(\lambda+1)(\alpha+2)} + c'(\rho)|u'|^{\alpha+2}$$

We assume that

$$(\lambda + 1)(\alpha + 2) \leq \beta + \lambda + 2 \iff \lambda \leq \frac{\beta - \alpha}{\alpha + 1}$$

In order that,

$$\forall u \in \mathbb{R}, \quad |u|^{(\lambda+1)(\alpha+2)} \leq |u|^\beta + \lambda + 2 + 1$$

For $\rho$ small enough, then

$$-\epsilon |u|^{\lambda} u g \sim (u') \geq -\epsilon |u|^\beta + \lambda + 2 - \epsilon P' |u'|^{\alpha+2} - \delta_2 \quad (2.7)$$

From (1.3) and by substituting the inequalities (2.6) and (2.7) in (2.5), we obtain

$$F'(t) \geq (\eta - Pe - P'\epsilon)|u'|^{\alpha+2} + \frac{\epsilon}{2} |u|^\beta + \lambda + 2 - M$$

$$\geq (\eta - Q\epsilon)|u'|^{\alpha+2} + \frac{\epsilon}{2} |u|^\beta + \lambda + 2 - M$$

With $Q = P + P'$

For $\epsilon$ small enough, we have

$$F'(t) \geq \frac{\epsilon}{2}(|u'|^{\alpha+2} + |u|^\beta + \lambda + 2) - M$$

where

$$\lambda = \min\left(\frac{\beta}{2}, \frac{\beta - \alpha}{\alpha + 1}, \frac{\alpha}{2}(\beta + 2)\right)$$

Set

$$\sigma = \min\left(\frac{\alpha + 2}{2}, 1 + \frac{\beta - \alpha}{(\alpha + 1)(\beta + 2)}\right)$$

Then, by using the following inequality

$$(x + y)^\sigma \leq c(\sigma)(x^\sigma + y^\sigma)$$

We have by (2.4)

$$F'(t) \geq \frac{\epsilon}{2} c^{-1}(\sigma)c_1 E(t)^\sigma - M' \geq \frac{\epsilon}{4} c_2 F(t)^\sigma - M'$$

(2.8)

where $c_2 = c^{-1}(\sigma)c_1$ and $M' > 0$.

Since $E$ is unbounded so is $F$, then $\exists T^* < T$ for which $\frac{\epsilon}{4} c_2 F(t)^\sigma > 2M'$ for
\[ t \in (T^*, T). \]

Therefore,

\[ F'(t) \geq \frac{\epsilon}{4}c_3 F(t)^\sigma \]  \hspace{2cm} (2.9)

and (2.9) implies that \( T < \infty \).

We distinguish two cases:

i) \( \alpha \leq \frac{\beta}{\beta + 2} \), then \( \frac{\alpha}{2}(\beta + 2) \leq \frac{\beta}{2} \) and \( \frac{\beta - \alpha}{\alpha + 1} \geq \frac{\beta - \beta}{\frac{\beta}{2} + 1} = \frac{\beta}{2} \)

Then we choose \( \lambda = \frac{\alpha}{2}(\beta + 2) \), we have:

\[ \beta + \lambda + 2 = (\beta + 2)(\frac{\alpha + 2}{2}) \]

and

\[ \sigma = \frac{\alpha + 2}{2} \]

Then, by (2.9), we have

\[ F'(t) \geq \frac{\epsilon}{4}c_3 F(t)^{\frac{\alpha + 2}{2}} \]

We have

\[ \frac{d}{dt} F(t)^{-\frac{\alpha}{2}} = -\frac{\alpha}{2} F(t) F(t)^{-\frac{\alpha}{2} - 1} \leq -\frac{\alpha}{8} c_3 \]

By integrating from \( t \) to \( \tau \), we find

\[ \int_{t}^{\tau} \frac{d}{ds} F(s)^{-\frac{\alpha}{2}} ds \leq \int_{t}^{\tau} -\epsilon c_4 ds \]

with \( c_4 = \frac{\alpha}{8} c_3 \) Then,

\[ F(\tau)^{-\frac{\alpha}{2}} - F(t)^{-\frac{\alpha}{2}} \leq -\epsilon c_4 (\tau - t) \]

Since

\[ F(\tau) \to +\infty \quad \text{if} \quad \tau \to T \]

Then

\[ F(\tau)^{-\frac{\alpha}{2}} \to 0 \quad \text{if} \quad \tau \to T \]

Therefore, by letting \( \tau \to T \)

\[ F(t) \leq \epsilon^{-\frac{\alpha}{2}} c_4' (T - t)^{-\frac{\alpha}{2}} \]

We assume \( c_5 = \epsilon^{-\frac{\alpha}{2}} c_4' \), then

\[ F(t) \leq c_5 (T - t)^{-\frac{\alpha}{2}} \]
By using (2.4), we deduce
\[ E(t) \leq C_1(T-t)^{-\frac{2}{\alpha}} \]  
(2.10)
with \( C_1 > 2c_3 \), for \( t \) close to \( T \).

For the converse inequality, we have
\[ E'(t) = u'g(u') \leq c|u'|^{\alpha+2} \leq cKE(t)^{\frac{\alpha+2}{2}} \]
Then
\[ \frac{d}{dt} E(t)^{-\frac{\alpha}{2}} = -\frac{\alpha}{2} \frac{d}{dt} E(t)^{-\frac{\alpha+2}{2}} \geq -\frac{\alpha}{2} cK \]
By integrating from \( t \) to \( \tau \), we find
\[ E(\tau)^{-\frac{\alpha}{2}} - E(t)^{-\frac{\alpha}{2}} \geq -K(\tau-t) \]
Since \( E(\tau) \to +\infty \) if \( \tau \to T \), then
\[ E(t) \geq C_0(T-t)^{-\frac{2}{\alpha}} \]  
(2.11)
Therefore, from (2.10) and (2.11), we find
\[ C_0(T-t)^{-\frac{2}{\alpha}} \leq E(t) \leq C_1(T-t)^{-\frac{2}{\alpha}} \]

ii) if \( \frac{\beta}{\beta+2} < \alpha < \beta \), we have \( \frac{\alpha}{2}(\beta+2) > \frac{\beta}{2} \) and \( \frac{\beta-\alpha}{\alpha+1} < \frac{\beta}{2} \)
Then \( \lambda = \frac{\beta-\alpha}{\alpha+1} \), we have
\[ \beta + \lambda + 2 = (\beta+2)(1 + \frac{\beta-\alpha}{(\alpha+1)(\beta+2)}) \]
Moreover, we have
\[ 1 + \frac{\beta-\alpha}{(\alpha+1)(\beta+2)} - \frac{\alpha}{2} - 1 < 0 \]
and
\[ \sigma = 1 + \frac{\beta-\alpha}{(\alpha+1)(\beta+2)} \]
Then
\[ F'(t) \geq \frac{\epsilon}{2} c^{-1}(\alpha, \beta)c_2 F(t)^{1+\frac{\beta-\alpha}{(\alpha+1)(\beta+2)}} \]
We have
\[ \frac{d}{dt} F(t)^{-\frac{\beta-\alpha}{(\alpha+1)(\beta+2)}} = -\frac{\beta-\alpha}{(\alpha+1)(\beta+2)} \frac{d}{dt} F(t)F(t)^{-1-\frac{\beta-\alpha}{(\alpha+1)(\beta+2)}} \]
Therefore,
\[ \frac{d}{dt} F(t)^{-\frac{\beta-\alpha}{(\alpha+1)(\beta+2)}} \leq -\frac{\beta-\alpha}{(\alpha+1)(\beta+2)} c_3 \epsilon \]
By integrating from $t$ to $\tau$, we have

$$\int_t^\tau \frac{d}{ds} F(s)^{-\frac{\beta-\alpha}{(\alpha+1)(\beta+2)}} ds \leq \int_t^\tau -\epsilon c_3 ds$$

Then, as $\epsilon$, we find if $\tau \to T$

$$F(t) \leq \epsilon^{-\frac{(\alpha+1)(\beta+2)}{\beta-\alpha}} c_4 (T-t)^{-\frac{(\alpha+1)(\beta+2)}{\beta-\alpha}}$$

Then we assume $C_1' = \epsilon^{-\frac{(\alpha+1)(\beta+2)}{\beta-\alpha}} c_4$ and we use (2.4), we find

$$E(t) \leq C'(T-t)^{-\frac{(\alpha+1)(\beta+2)}{\beta-\alpha}}$$

\[\square\]

3 Oscillation of solutions of (1.5) near blow-up for $\alpha$ small:

To establish the oscillatory character of solutions of (1.5), we can use the method from [1]. We obtain the following result:

**Theorem 3.1.** Assume either

$$1 < \alpha < \frac{\beta}{\beta+2}$$

Or

$$\alpha = \frac{\beta}{\beta+2}; \quad c < (\beta+2)(\beta+2)(\beta+1)\frac{1}{(\beta+2)^2}$$

Then, all nontrivial solutions of (1.5) have oscillatory blow-up at time $T$ and:

$$\limsup_{t \to T} u(t) = \limsup_{t \to T} u'(t) = +\infty; \quad \liminf_{t \to T} u(t) = \liminf_{t \to T} u'(t) = -\infty$$

**Proof of theorem 3.1:** Since the energy of (1.5) is positive $\forall \ t \in [0, T]$, we introduce as in [1] the polar coordinates:

$$\left(\frac{2}{\beta+2}\right)^\frac{1}{2}|u|^{\frac{\beta}{2}} u = r(t) \cos \theta(t); \quad u'(t) = r(t) \sin \theta(t)$$

where $r$ and $\theta$ are $C^1$ functions and $r(t) = E(t)^{\frac{1}{2}} > 0$.

We have

$$u''(t) = -|u|^\beta u + g^\sim(u') = r'(t) \sin \theta(t) + r(t) \theta'(t) \cos \theta(t) \quad (3.1)$$
\[
\left(\frac{\beta + 2}{2}\right)^{\frac{1}{2}} |u|^\frac{\beta}{2} u' = r'(t) \cos \theta(t) - r(t) \theta'(t) \sin \theta(t) \tag{3.2}
\]

Then

\[
(3.1)^* \cos \theta - (3.2)^* \sin \theta, \text{ give}
\]

\[
\theta' = -\left(\frac{\beta + 2}{2}\right)^{\frac{\beta + 1}{\beta + 2}} r(t)^{\frac{\beta}{\beta + 2}} |\cos \theta|^{\frac{\beta}{\beta + 2}} + g(t) \sin \theta \frac{\cos \theta}{r}
\]

Use (1.2), we have

\[
\theta' \leq -\left(\frac{\beta + 2}{2}\right)^{\frac{\beta + 1}{\beta + 2}} r(t)^{\frac{\beta}{\beta + 2}} |\cos \theta|^{\frac{\beta}{\beta + 2}} + \alpha |\sin \theta|^{\alpha + 1} \cos \theta
\]

Since \(\alpha < \frac{\beta}{\beta + 2}\) and if \(t \to T\), \(r(t) \sim C(T-t)^{-\frac{\beta}{\beta + 2}}\), then \(r^\alpha |\cos \theta| \leq g r^\frac{\beta}{\beta + 2} |\cos \theta|^{\frac{\beta}{\beta + 2}}\), as \(t \to T\).

Therefore

\[
\theta' \leq -\xi (T-t)^{-\frac{\beta}{\alpha(\beta + 2)}} |\cos \theta|^{\frac{\beta}{\beta + 2}}, \quad \text{as } t \to T.
\]

Where \(\frac{\beta}{\alpha(\beta + 2)} > 1\).

In the case \(\alpha = \frac{\beta}{\beta + 2}\): \(c < (\beta + 2)(\frac{\beta + 2}{\beta^2 + 2})^{\beta + 1}\), we have

\[
\theta' \leq -r^\alpha ((\frac{\beta + 2}{2})^{\frac{\beta + 1}{\beta + 2}} |\cos \theta|^{\alpha} - c |\sin \theta|^{\alpha + 1} \cos \theta)
\]

\[
\leq -r^\alpha |\cos \theta|^{\alpha} ((\frac{\beta + 2}{2})^{\frac{\beta + 1}{\beta + 2}} - c |\sin \theta|^{\alpha + 1} |\cos \theta|^{1 - \alpha})
\]

We assume

\[
f(\theta) = |\sin \theta|^{\alpha + 1} |\cos \theta|^{1 - \alpha}, \quad \theta \in \mathbb{R}
\]

Then, we have

\[
\max_{\theta \in \mathbb{R}} f(\theta) = (\frac{1}{\beta + 2})^{\frac{\beta + 1}{\beta + 2}} (\frac{\beta + 1}{\beta + 2})^{\frac{\beta + 1}{\beta + 2}}
\]

Hence

\[
(\frac{\beta + 2}{2})^{\frac{\beta + 1}{\beta + 2}} - c|\sin \theta|^{\alpha + 1} |\cos \theta|^{1 - \alpha} \geq (\frac{\beta + 2}{2})^{\frac{\beta + 1}{\beta + 2}} - c(\frac{1}{\beta + 2})^{\frac{\beta + 1}{\beta + 2}} (\frac{\beta + 1}{\beta + 2})^{\frac{\beta + 1}{\beta + 2}}
\]

Therefore

\[
(\frac{\beta + 2}{2})^{\frac{\beta + 1}{\beta + 2}} - c(\frac{1}{\beta + 2})^{\frac{1}{\beta + 2}} (\frac{\beta + 1}{\beta + 2})^{\frac{\beta + 1}{\beta + 2}} > 0
\]

\[
\Leftrightarrow c < (\beta + 2)(\frac{\beta + 2}{2\beta + 2})^{\beta + 1}
\]

Then, we find in both cases for \(t \to T\),

\[
\theta' \leq -\xi(T-t)^{-1} |\cos \theta|^{\frac{\beta}{\beta + 2}}
\]
We introduce the following function:

\[ H(s) = \int_a^s \frac{du}{\cos u\sqrt{\beta + 2}} \]

We assume that \( u \) does not vanish if \( t \to T \), and for \( t \in [t_0, T] \), \( \theta(t) \in (-\frac{\pi}{2}, \frac{\pi}{2}) \), \( H(\theta(t)) = K(t) \) is differentiable and we have

\[ \forall t_0 \leq t \leq T, \quad K'(t) \leq -\xi(T - t)^{-1} \]

We integrate from \( t_0 \) to \( t \)

\[ H(\theta(t)) \leq H(\theta(t_0)) - \xi \log(T - t_0) + \xi \log(T - t) \]

If \( t \to T \), we find \( H(\theta(t)) \to -\infty \) which is impossible since \( H \) is nonnegative. Then this contradiction proves that \( u \) vanishes on each half-line. Since \( u' \) cannot vanish at the same time, \( u \) must change sign. If \( u' \) and \( u'' \) vanish at the same time, the equation shows that \( u \) vanishes also, this contradiction implies that \( u' \) changes sign also.

Finally, by (2.2) we have, \( \lim_{t \to T} u^2(t) + u'^2(t) = +\infty \). Then, since \( u(t) \) and \( u'(t) \) have oscillatory blow-up, we deduce:

\[ \limsup_{t \to T} u(t) = \limsup_{t \to T} u'(t) = +\infty \]

and

\[ \liminf_{t \to T} u(t) = \liminf_{t \to T} u'(t) = -\infty. \]

\[ \square \]

4 Non oscillation of solutions of (1.5) for \( \alpha \) large

In this section, we generalize to (1.5) the non-oscillation result of [3] by the method of [1]. We obtain:

**Theorem 4.1.** Assuming

\[ \frac{\beta}{\beta + 2} < \alpha < \beta \]

Then any solution \((u(t), u'(t))\) has a finite number of zeroes in \((T - \epsilon, T)\), \( \epsilon > 0 \), and blow-up as \( t \to T \). Also, assuming \( g \in C^1 \) and \( g' > 0 \), if \( t \to T \), \( u(t), u'(t) \) and \( u''(t) \) have the same sign.

**Proof of theorem 4.1:** Let

\[ G(s) = \int_0^s |\sin u|^\alpha \sin u \cos u \ du \]
We differentiate $G$, we obtain

$$\frac{d}{dt} G(\theta(t)) = \theta'(t) \sin \theta(\theta(t)) \sin \theta \cos \theta$$

Then, from (1.3), we obtain

$$\frac{d}{dt} G(\theta(t)) = -\frac{\beta + 2}{2} \frac{\beta + 1}{|\cos \theta|} \sin \theta \sin \theta \cos \theta + \frac{g^\sim (r \sin \theta)}{r} \sin \theta \sin \theta \cos \theta + \eta r^\alpha \sin^{2(\alpha + 1)} \theta \cos^2 \theta$$

By using the Young’s inequality with exponents 2 in the first term, we have

$$\frac{d}{dt} G(\theta(t)) \geq -C r^{\frac{2\beta}{\beta + 2} - \alpha}$$

Then, using (2.2)

$$\frac{d}{dt} G(\theta(t)) \geq -C'' r^{\frac{2\beta}{\beta + 2} - \alpha} \geq -C'' (T - t)^{-\mu}.$$

With

$$0 \leq \mu = \left(\frac{2\beta}{\beta + 2} - \alpha\right) \left(\frac{(\alpha + 1)(\beta + 2)}{2(\beta - \alpha)}\right)$$

$$= \alpha + 1 - \frac{\alpha \beta (\alpha + 1)}{2(\beta - \alpha)}$$

$$= 1 + \frac{\alpha [1 - \frac{\beta (\alpha + 1)}{2(\beta - \alpha)}]}{2(\beta - \alpha)}$$

$$= 1 + \frac{\alpha [\frac{\beta - \alpha (\beta + 2)}{2(\beta - \alpha)}]}{2(\beta - \alpha)} < 1$$

To finish the proof we shall use the following lemma (cf. [1] for proof).

**Lemma 4.2.** Assuming $\theta \in C^1(a, T)$ and $G$ be a non constant $\tau$-periodic function.

We assume $h \in L^1(a, T)$ and

$$(G(\theta(t)))' \geq h(t), \quad \forall t \in [a, T].$$

Then for $t_1 \leq t < T$, $\theta(t)$ remains in some interval of length $\leq \tau$. Moreover, if $G$ has a finite number of zeroes in $[0, \tau]$, then $\theta(t)$ has a limit if $t \to T$.

**End of the proof of theorem 4.1:** From the lemma (4.2), $\theta(t)$ has a limit as $t \to T$.

We distinguish two cases:

**case1:** if this limit differs from $\frac{\pi}{2}$ (mod $\pi$), $u \sim Cr^{\frac{2\beta}{\beta + 2}} > 0$ if $t \to T$, then $u$ has a constant sign.
**case 2:** in the opposite case, we have, if \( t \to T \), \(|u'(t)| \sim r(t) > 0 \), then \( u'(t) \) does not vanish if \( t \to T \) and therefore \( u(t) \) has a constant sign if \( t \to T \).

If \( u'(t) \) has several zeroes in \((T - \epsilon, T)\), then \( u'' \) has different signs at two successive zeroes of \( u' \), from equation (1.5) \( u \) must have different signs also, which is impossible.

Then, \( u'(t) \) has a constant sign as \( t \to T \).

Since \( u(t) \) and \( u'(t) \) do not change sign if \( t \to T \), we conclude by (2.2) that:

\[
\lim_{t \to T} u(t) = \lim_{t \to T} u'(t) = \pm \infty.
\]

And by (2.2), we deduce that \( u, u' \) and \( u'' \) have the same sign if \( t \to T \).

**Theorem 4.3.** Assuming

\[
\alpha = \frac{\beta}{\beta + 2}; \quad b \geq b_0 = (\beta + 2)\left(\frac{\beta + 2}{2\beta + 2}\right)^{\frac{\beta + 1}{\beta + 2}}
\]

Then any solution \( u(t) \) of (1.6) blows-up in finite time \( T \) and has a finite number of zeroes in \([0, T] \).

**Proof of theorem 4.3:**

\[
\theta' = -r^\alpha((\beta + 2)\left(\frac{\beta + 2}{2\beta + 2}\right)^{\frac{\beta + 1}{\beta + 2}}|\cos \theta|^\alpha - b|\sin \theta|^\alpha \sin \theta \cos \theta)
\]

\[
= -r^\alpha|\cos \theta|^\alpha((\beta + 2)\left(\frac{\beta + 2}{2\beta + 2}\right)^{\frac{\beta + 1}{\beta + 2}} - b|\sin \theta|^\alpha + 1|\cos \theta|^{1-\alpha})
\]

i) If \( b = b_0 \), we have

\[
(\beta + 2)\left(\frac{\beta + 2}{2\beta + 2}\right)^{\frac{\beta + 1}{\beta + 2}} - b|\sin \theta|^\alpha + 1|\cos \theta|^{1-\alpha} \geq (\beta + 2)\left(\frac{\beta + 2}{2\beta + 2}\right)^{\frac{\beta + 1}{\beta + 2}} - b_0\left(\frac{1}{\beta + 2}\right)\left(\frac{1}{\beta + 2}\right)^{\frac{\beta + 1}{\beta + 2}} = 0
\]

Therefore, the coefficient of \(-r^\alpha\) is positive, then \( \theta \) is non-increasing.

The distance of two zeroes of \( h(\theta) = (\beta + 2)\left(\frac{\beta + 2}{2\beta + 2}\right)^{\frac{\beta + 1}{\beta + 2}}|\cos \theta|^\alpha - b|\sin \theta|^\alpha \sin \theta \cos \theta \)
other than \( \frac{\pi}{2} \) (mod \( \pi \)) is not more than \( \pi \), therefore we have two cases:

**case 1:** if \( \theta(t) \) remains in an interval of length less than \( \pi \), so \( \theta \) is bounded and from above it is non-increasing then it converges to a limit as \( t \to T \).

**case 2:** if \( \theta(t) \) coincides with one of these zeroes if \( t \to T \), due to existence and uniqueness for the ODE satisfied by \( \theta(t) \) near the non-trivial equilibria, it must remains constant and \( u \) never vanishes.

ii) If \( b > b_0 \), the coefficient of \(-r^\alpha\) still has nontrivial zeroes. We have two cases:
**case 1**: if $\theta$ does not take any of the corresponding values, then, it is bounded and since the coefficient of $-r^\alpha$ is negative near the trivial zeroes, $\theta$ is non-decreasing and therefore it is convergent.

**case 2**: if $\theta$ takes one of the corresponding values, hence it remains constant and $u$ never vanishes.

**Remark 4.4.** For the equation (1.1) when $t \geq 0$, by the same calculation as [1], it is easy to show that the solutions to (1.1) have the same oscillatory (or non-oscillatory) behavior as in (1.4) in the noncritical case $\alpha \neq \frac{\beta}{\beta+2}$. In the case $\alpha = \frac{\beta}{\beta+2}$, the result will depend on $c$ and $\eta$.

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**Références**