An incompressible fluid in a weakly deformable shell

Part I: Analysis of the model

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Abstract: We consider the flow of a viscous incompressible fluid in a pipe when the boundary is a deformable shell of Naghdi type. We prove that the corresponding system of partial differential equations has a solution when the deformation of the shell is smooth and small enough. We propose an algorithm that uncouples the unknowns and prove its convergence.

Résumé: Nous considérons l'écoulement d'un fluide visqueux incompressible dans un tuyau dont la frontière est une coque déformable de type Naghdi. Nous prouvons que le système d'équations aux dérivées partielles correspondant admet une solution lorsque la déformation de la coque est assez régulière et suffisamment petite. Nous proposons un algorithme qui permet de découpler les inconnues et démontrons sa convergence.

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1. Introduction.

The aim of this work is to study the flow of a viscous incompressible fluid governed by the Navier-Stokes equations when part of the solid boundary of the domain is a deformable shell of Naghdi type. This clearly leads to the coupling of two equations:
(i) The fluid has an action on the shell. So it deforms this shell according to the Naghdi law.
(ii) The computation domain for the fluid depends on the shell, and the Navier-Stokes equations must be solved on this shell-dependent domain.
So, the resulting model is made of two systems of partial differential equations, one set on a two-dimensional domain and the other one set on a three-dimensional domain.

A huge amount of mathematical work has been performed on the coupling of fluids with different types of elastic structures, see for instance [6], [12], [15], [16] and the references therein. A large number of them deal with a solid immersed in a fluid, which is a rather different problem. Moreover, the work concerning fluids inside a shell (see [7] and the references therein for the presentation of different shell models) seems more limited, we refer to [8], [14], and [17] for basic papers on this subject. The reason for this is that coupling a shell model with the Stokes or Navier-Stokes equations is much easier when the deformation of the shell is expressed in Cartesian coordinates, which has only been recently established for the Naghdi model, see [3] and [4]. Relying on this result, we are in a position to write the full model in the stationary case. Next we prove that it admits a solution for a specific but realistic three-dimensional geometry. This requires some further assumptions on the deformation of the shell, first that it is smooth enough (we refer to [1] for first results on this subject) which requires some restrictions on the domain, second that it is small enough which only depends on the properties of the elastic material.

However, writing a discretization of the full system seems very hard due to the complexity of the coupling. Thus, we propose an iterative algorithm that uncouples the two equations. A large number of such algorithms have been studied, see [9], [10] and the references therein, however the algorithm we work with is derived from the well-posedness proof in a straightforward way. The convergence of this algorithm is geometric, so it seems to be an efficient tool for the discretization of the model as will be analyzed in the second part of this work.

An outline of the paper is as follows.

- Section 2 is devoted to the derivation of the full model.
- The analysis of the resulting system of partial differential equations is performed in Section 3.
- In Section 4, we propose an algorithm that uncouples the Navier-Stokes and shell equations and prove its convergence.

2. The coupled model.

We begin with some notation. Let Ω be a bounded connected domain in $\mathbb{R}^3$ with boundary $\partial\Omega$. The generic point in Ω is denoted by $\boldsymbol{X} = (X, Y, Z)$. We assume without restriction that Ω is contained in the rectangular parallelepiped

$$\Omega_0 =]-X_0, X_0[\times]-Y_0, Y_0[\times]-Z_0, Z_0[,$$

where X_0, Y_0 and Z_0 are positive real numbers, and that its boundary $\partial\Omega$ is made of three disjoint parts:
(i) the inflow part Γ_{in} is connected, has a Lipschitz-continuous boundary and is contained in the plane $X = -X_0$,
(ii) the outflow part Γ_{out} is connected, has a Lipschitz-continuous boundary and is contained in the plane $X = X_0$,
(iii) the "shell" part Γ is contained in the domain $\overline{\Omega}_0$.
On the other hand, we introduce a bounded rectangle $\omega =]-x_0, x_0[\times]-y_0, y_0[$ in $\mathbb{R}^2$ (the generic point in ω being now $\boldsymbol{x} = (x, y)$). Denoting by $W^{m,p}(\omega)$ the Sobolev spaces on ω and by $W^{m,p}_\sharp(\omega)$ their subspaces made of functions which are periodic of period $2x_0$ with respect to the variable x, we assume that there exists a one-to-one mapping $\boldsymbol{\varphi}$ in $W^{2,\infty}_\sharp(\omega)^3$ with values in $\overline{\Omega}_0$ (this periodicity is necessary to close the pipe) such that the two vectors

$$\boldsymbol{a}_\alpha(\boldsymbol{x}) = (\partial_\alpha \boldsymbol{\varphi})(\boldsymbol{x}) \tag{2.1}$$

are linearly independent at each point $\boldsymbol{x}$ of $\overline{\omega}$. The midsurface of the shell $\boldsymbol{\varphi}(\overline{\omega})$ coincides with Γ. Note that these conditions are very general and are not in contradiction with the fact

$$\partial\Omega = \Gamma_{\text{in}} \cup \Gamma_{\text{out}} \cup \Gamma, \tag{2.2}$$

as illustrated in the following figure. From now on, condition (2.2) is supposed to hold.

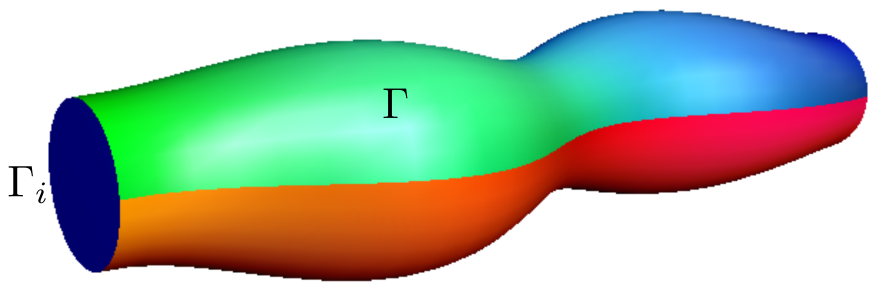

Figure 1. An example of domain Ω

Remark 2.1. In the previous definition, the part Γ of $\partial\Omega$ is the midsurface of the shell. This means that the thickness of the shell t is neglected, since in a more complex model only the upper or lower surface of the shell shoud be included in $\partial\Omega$. However neglecting t seems realistic in a large number of physical situations.

The unknowns of the model are the velocity $\boldsymbol{u}$ and the pressure p of the fluid, together with the displacement $\boldsymbol{d}$ of the shell and the rotation $\boldsymbol{r}$ of its orthogonal fibers. We denote by $\Gamma_{\boldsymbol{d}}$ the midsurface $(\boldsymbol{\varphi} + \boldsymbol{d})(\overline{\omega})$ and by $\Omega_{\boldsymbol{d}}$ the bounded domain such that its boundary satisfies

$$\partial\Omega_{\boldsymbol{d}} = \Gamma_{\text{in}} \cup \Gamma_{\text{out}} \cup \Gamma_{\boldsymbol{d}}. \tag{2.3}$$

The domain $\Omega_{\boldsymbol{d}}$ is occupied by the fluid, so that we consider the standard Navier–Stokes equations on this domain, provided with no-slip boundary conditions,

$$\begin{cases} -\nu_f\,\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \mathbf{grad}\,p = \boldsymbol{f} & \text{in } \Omega_{\boldsymbol{d}},\\ \operatorname{div}\boldsymbol{u} = 0 & \text{in } \Omega_{\boldsymbol{d}},\\ \boldsymbol{u} = \boldsymbol{g}_{\text{in}} & \text{on } \Gamma_{\text{in}},\\ \boldsymbol{u} = \boldsymbol{g}_{\text{out}} & \text{on } \Gamma_{\text{out}},\\ \boldsymbol{u} = \boldsymbol{0} & \text{on } \Gamma_{\boldsymbol{d}}. \end{cases} \tag{2.4}$$

The coefficient ν_f is a positive constant which represents the viscosity of the fluid. The data are a density of body forces $\boldsymbol{f}$ and the boundary terms $\boldsymbol{g}_{\text{in}}$ and $\boldsymbol{g}_{\text{out}}$. Standard density arguments (indeed, it nearly follows from the previous assumptions that $\Omega_{\boldsymbol{d}}$ has a Lipschitz-continuous boundary) yield that this problem admits the equivalent variational formulation

Find $(\boldsymbol{u},p)$ *in* $H^1(\Omega_{\boldsymbol{d}})^3 \times L^2_\circ(\Omega_{\boldsymbol{d}})$ *such that*

$$\boldsymbol{u} = \boldsymbol{g}_{\text{in}} \quad \text{on } \Gamma_{\text{in}}, \qquad \boldsymbol{u} = \boldsymbol{g}_{\text{out}} \quad \text{on } \Gamma_{\text{out}}, \qquad \text{and} \qquad \boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_{\boldsymbol{d}}, \tag{2.5}$$

and

$$\begin{aligned} &\forall \boldsymbol{v} \in H^1_0(\Omega_{\boldsymbol{d}})^3, \qquad a_{f,\boldsymbol{d}}(\boldsymbol{u},\boldsymbol{v}) + c_{f,\boldsymbol{d}}(\boldsymbol{u};\boldsymbol{u},\boldsymbol{v}) + b_{f,\boldsymbol{d}}(\boldsymbol{v},p) = \langle \boldsymbol{f},\boldsymbol{v}\rangle,\\ &\forall q \in L^2_\circ(\Omega_{\boldsymbol{d}}), \qquad b_{f,\boldsymbol{d}}(\boldsymbol{u},q) = 0, \end{aligned} \tag{2.6}$$

where the bilinear forms $a_{f,\boldsymbol{d}}(\cdot,\cdot)$ and $b_{f,\boldsymbol{d}}(\cdot,\cdot)$ are defined by

$$\begin{aligned} a_{f,\boldsymbol{d}}(\boldsymbol{u},\boldsymbol{v}) &= \nu_f \int_{\Omega_{\boldsymbol{d}}} \big(\mathbf{grad}\,\boldsymbol{u}\big)(\boldsymbol{X}) : \big(\mathbf{grad}\,\boldsymbol{v}\big)(\boldsymbol{X})\,d\boldsymbol{X},\\ b_{f,\boldsymbol{d}}(\boldsymbol{v},\boldsymbol{q}) &= -\int_{\Omega_{\boldsymbol{d}}} \big(\operatorname{div}\boldsymbol{v}\big)(\boldsymbol{X})q(\boldsymbol{X})\,d\boldsymbol{X}, \end{aligned} \tag{2.7}$$

while the trilinear form $c_{f,\boldsymbol{d}}(\cdot;\cdot,\cdot)$ is given by

$$c_{f,\boldsymbol{d}}(\boldsymbol{w};\boldsymbol{u},\boldsymbol{v}) = \int_{\Omega_{\boldsymbol{d}}} \big((\boldsymbol{w}\cdot\nabla)\boldsymbol{u}\big)(\boldsymbol{X}) \cdot \boldsymbol{v}(\boldsymbol{X})\,d\boldsymbol{X}. \tag{2.8}$$

Here, $\langle\cdot,\cdot\rangle$ stands for the duality pairing between $H^{-1}(\Omega_{\boldsymbol{d}})^3$ and $H^1_0(\Omega_{\boldsymbol{d}})^3$ and, as standard, $L^2_\circ(\Omega_{\boldsymbol{d}})$ denotes the space of functions in $L^2(\Omega_{\boldsymbol{d}})$ with a null integral on $\Omega_{\boldsymbol{d}}$.

Let us now consider the shell model which requires some further notation. We first assume that the mapping $\boldsymbol{\varphi}$ is bi-Lipschitzian, i.e., that there exist positive constants $c_\flat$ and $c_\sharp$ such that

$$\forall \boldsymbol{x} \in \overline{\omega}, \forall \boldsymbol{y} \in \overline{\omega}, \quad c_\flat\,|\boldsymbol{x}-\boldsymbol{y}| \le |\boldsymbol{\varphi}(\boldsymbol{x}) - \boldsymbol{\varphi}(\boldsymbol{y})| \le c_\sharp\,|\boldsymbol{x}-\boldsymbol{y}|, \tag{2.9}$$

where $|\cdot|$ stands for the Euclidean norm in $\mathbb{R}^n$, $n=2$ or 3. Owing to definition (2.1), the vector defined by cross product

$$\boldsymbol{a}_3(\boldsymbol{x}) = \frac{\boldsymbol{a}_1(\boldsymbol{x}) \wedge \boldsymbol{a}_2(\boldsymbol{x})}{|\boldsymbol{a}_1(\boldsymbol{x}) \wedge \boldsymbol{a}_2(\boldsymbol{x})|}$$

is the unit normal vector on the midsurface at point $\boldsymbol{\varphi}(\boldsymbol{x})$. The vectors $\boldsymbol{a}_i(\boldsymbol{x})$ define the local covariant basis at point $\boldsymbol{\varphi}(\boldsymbol{x})$. The contravariant basis $\boldsymbol{a}^i(\boldsymbol{x})$ is defined by the relations $\boldsymbol{a}_i \cdot \boldsymbol{a}^j = \delta_i^j$ where δ_i^j is the Kronecker symbol (as usual, Greek indices and exponents take their values in the set $\{1,2\}$ and Latin indices and exponents take their values in the set $\{1,2,3\}$). Note that all these vectors belong to $W^{1,\infty}(\omega)^3$. The first fundamental form of the surface is given in contravariant components by

$$a^{\alpha\beta} = \boldsymbol{a}^\alpha \cdot \boldsymbol{a}^\beta.$$

We set $a(\boldsymbol{x}) = |\boldsymbol{a}_1(\boldsymbol{x}) \wedge \boldsymbol{a}_2(\boldsymbol{x})|^2$ so that $\sqrt{a(\boldsymbol{x})}$ is the area element of the midsurface in the chart $\boldsymbol{\varphi}$. The thickness of the shell, denoted by t, is a positive continuous function on $\overline{\omega}$, periodic with respect to x.

In the case of a homogeneous, isotropic material with Young modulus $E > 0$ and Poisson ratio ν_s, $0 \leq \nu_s < \frac{1}{2}$, the contravariant components of the elasticity tensor $a^{\alpha\beta\rho\sigma}$ are given by

$$a^{\alpha\beta\rho\sigma} = \frac{E}{2(1+\nu_s)}(a^{\alpha\rho}a^{\beta\sigma} + a^{\alpha\sigma}a^{\beta\rho}) + \frac{E\nu_s}{1-\nu_s^2}a^{\alpha\beta}a^{\rho\sigma}. \tag{2.10}$$

This tensor satisfies the usual symmetry properties and is uniformly strictly positive. In this context and with the notation $\boldsymbol{D} = (\boldsymbol{d}, \boldsymbol{r})$, the covariant components of the change of metric tensor read

$$\gamma_{\alpha\beta}(\boldsymbol{d}) = \frac{1}{2}(\partial_\alpha \boldsymbol{d} \cdot \boldsymbol{a}_\beta + \partial_\beta \boldsymbol{d} \cdot \boldsymbol{a}_\alpha), \tag{2.11}$$

the components of the change of transverse shear tensor read

$$\delta_{\alpha 3}(\boldsymbol{D}) = \frac{1}{2}(\partial_\alpha \boldsymbol{d} \cdot \boldsymbol{a}_3 + \boldsymbol{r} \cdot \boldsymbol{a}_\alpha), \tag{2.12}$$

and the covariant components of the change of curvature tensor read

$$\chi_{\alpha\beta}(\boldsymbol{D}) = \frac{1}{2}(\partial_\alpha \boldsymbol{d} \cdot \partial_\beta \boldsymbol{a}_3 + \partial_\beta \boldsymbol{d} \cdot \partial_\alpha \boldsymbol{a}_3 + \partial_\alpha \boldsymbol{r} \cdot \boldsymbol{a}_\beta + \partial_\beta \boldsymbol{r} \cdot \boldsymbol{a}_\alpha), \tag{2.13}$$

see [4]. Note that all these quantities make sense for shells with little regularity, and are easily expressed with the Cartesian coordinates of the unknowns and geometric data.

In order to ensure that formula (2.2) is compatible with definition (2.3), we assume that the shell is clamped on all its "physical" boundary, which leads to introduce the space

$$H^1_\diamond(\omega) = \big\{ v \in H^1_\sharp(\omega);\ v(x,-y_0) = v(x,y_0) = 0,\ -x_0 \leq x \leq x_0 \big\}.$$

With this definition, both unknowns $\boldsymbol{d}$ and $\boldsymbol{r}$ belong to $H^1_\diamond(\omega)^3$. So, we consider the function space introduced in [4], which is appropriate in the context of shells,

$$\mathbb{V}(\omega) = \big\{ \boldsymbol{E} = (\boldsymbol{e}, \boldsymbol{s}) \in H^1_\diamond(\omega)^3 \times H^1_\diamond(\omega)^3;\ \boldsymbol{s} \cdot \boldsymbol{a}_3 = 0 \text{ in } \omega \big\}. \tag{2.14}$$

The variational formulation of the problem corresponding to the linear Naghdi model for shells enclosing a fluid is given by, for any data $\boldsymbol{k}$ in $L^2(\omega)^3$,

Find $\boldsymbol{D} = (\boldsymbol{d}, \boldsymbol{r})$ in $\mathbb{V}(\omega)$ such that

$$\forall \boldsymbol{E} \in \mathbb{V}(\omega), \quad a_s\big(\boldsymbol{D}, \boldsymbol{E}\big) = \mathcal{L}(\boldsymbol{E}), \tag{2.15}$$

where the bilinear form $a_s(\cdot,\cdot)$ is defined by (with the standard Einstein summation convention for repeated indices and exponents)

$$\begin{aligned} a_s(\boldsymbol{D},\boldsymbol{E}) = \int_\omega \Big\{ t a^{\alpha\beta\rho\sigma} \Big[\gamma_{\alpha\beta}(\boldsymbol{d})\gamma_{\rho\sigma}(\boldsymbol{e}) + \frac{t^2}{12}\chi_{\alpha\beta}(\boldsymbol{D})\chi_{\rho\sigma}(\boldsymbol{E}) \Big] \\ + 2t\frac{E}{1+\nu_s} a^{\alpha\beta}\delta_{\alpha 3}(\boldsymbol{D})\delta_{\beta 3}(\boldsymbol{E}) \Big\} \sqrt{a}\, d\boldsymbol{x}, \end{aligned} \tag{2.16}$$

and the linear form $\mathcal{L}(\cdot)$ is given by

$$\mathcal{L}(\boldsymbol{E}) = \int_\omega \boldsymbol{k}\cdot\boldsymbol{e}\sqrt{a}\, d\boldsymbol{x} + \int_{\Gamma_{\boldsymbol{d}}} (\nu_f\, \partial_n \boldsymbol{u} - p\boldsymbol{n})(\boldsymbol{\tau})\cdot(\boldsymbol{e}\circ(\boldsymbol{\varphi}+\boldsymbol{d})^{-1})(\boldsymbol{\tau})\, d\boldsymbol{\tau}. \tag{2.17}$$

Note that, in this last definition, the mapping $\boldsymbol{\varphi}+\boldsymbol{d}$ maps $\overline{\omega}$ onto $\Gamma_{\boldsymbol{d}}$, so that the two integrals can be written on the same domain: For instance, we have, with the notation $\breve{\boldsymbol{u}} = \boldsymbol{u}\circ(\boldsymbol{\varphi}+\boldsymbol{d})$ and $\breve{p} = p\circ(\boldsymbol{\varphi}+\boldsymbol{d})$,

$$\int_{\Gamma_{\boldsymbol{d}}} (\nu_f\, \partial_n \boldsymbol{u} - p\boldsymbol{n})(\boldsymbol{\tau})\cdot(\boldsymbol{e}\circ(\boldsymbol{\varphi}+\boldsymbol{d})^{-1})(\boldsymbol{\tau})\, d\boldsymbol{\tau} = \int_\omega (\nu_f\, \partial_{a_3}\breve{\boldsymbol{u}} - \breve{p}\boldsymbol{a}_3)(\boldsymbol{x})\cdot\boldsymbol{e}(\boldsymbol{x})\, J_{\boldsymbol{d}}\, d\boldsymbol{x}, \tag{2.18}$$

where $J_{\boldsymbol{d}}$ denotes the Jacobian of $\boldsymbol{\varphi}+\boldsymbol{d}$. Moreover, the quantity $\boldsymbol{k}$ only depends on the gravity, and no traction nor moment is applied to the shell.

In view of the discretization, we prefer to handle the tangency constraint $\boldsymbol{s}\cdot\boldsymbol{a}_3 = 0$ in the definition of $\mathbb{V}(\omega)$ via the introduction of a Lagrange multiplier. Indeed, when setting

$$\mathbb{X}(\omega) = H^1_\diamond(\omega)^3 \times H^1_\diamond(\omega)^3, \qquad \mathbb{M}(\omega) = H^1_\diamond(\omega),$$

it is readily checked that problem (2.15) is equivalent to the next one:

Find $(\boldsymbol{D},\psi)$ *in* $\mathbb{X}(\omega)\times\mathbb{M}(\omega)$ *such that*

$$\begin{aligned} &\forall \boldsymbol{E}\in\mathbb{X}(\omega), \quad a_s(\boldsymbol{D},\boldsymbol{E}) + b_s(\boldsymbol{E},\psi) = \mathcal{L}(\boldsymbol{E}), \\ &\forall \chi\in\mathbb{M}(\omega), \quad b_s(\boldsymbol{D},\chi) = 0, \end{aligned} \tag{2.19}$$

where the bilinear form $b_s(\cdot,\cdot)$ is defined by

$$b_s(\boldsymbol{E},\chi) = \int_\omega \partial_\alpha(\boldsymbol{s}\cdot\boldsymbol{a}_3)\partial_\alpha\chi\, d\boldsymbol{x}. \tag{2.20}$$

From now on, we are interested in the full coupled system (2.5) – (2.6) – (2.19). However it does not seem useless to recall the main properties of problems (2.5) – (2.6) and (2.19) separately.

About the fluid. The forms $a_{f,\boldsymbol{d}}(\cdot,\cdot)$, $b_{f,\boldsymbol{d}}(\cdot,\cdot)$ and $c_{f,\boldsymbol{d}}(\cdot;\cdot,\cdot)$ are continuous on the spaces $H^1(\Omega_{\boldsymbol{d}})^3\times H^1(\Omega_{\boldsymbol{d}})^3$, $H^1(\Omega_{\boldsymbol{d}})^3\times L^2_\circ(\Omega_{\boldsymbol{d}})$, and $H^1(\Omega_{\boldsymbol{d}})^3\times H^1(\Omega_{\boldsymbol{d}})^3\times H^1(\Omega_{\boldsymbol{d}})^3$, respectively. Moreover, the form $a_{f,\boldsymbol{d}}(\cdot,\cdot)$ satisfies the following ellipticity property, for a positive constant $\alpha_{f,\boldsymbol{d}}$,

$$\forall \boldsymbol{v}\in H^1_0(\Omega_{\boldsymbol{d}})^3, \quad a_{f,\boldsymbol{d}}(\boldsymbol{v},\boldsymbol{v}) \geq \alpha_{f,\boldsymbol{d}}\, \|\boldsymbol{v}\|^2_{H^1(\Omega_{\boldsymbol{d}})^3}; \tag{2.21}$$

the form $b_{f,\boldsymbol{d}}(\cdot,\cdot)$ satisfies the following inf-sup condition, for a positive constant $\beta_{f,\boldsymbol{d}}$,

$$\forall q \in L^2_\circ(\Omega_{\boldsymbol{d}}), \qquad \sup_{v \in H^1_0(\Omega_{\boldsymbol{d}})^3} \frac{b_{f,\boldsymbol{d}}(\boldsymbol{v},q)}{\|\boldsymbol{v}\|_{H^1(\Omega_{\boldsymbol{d}})^3}} \geq \beta_{f,\boldsymbol{d}} \, \|q\|_{L^2(\Omega_{\boldsymbol{d}})}; \tag{2.22}$$

the form $c_{f,\boldsymbol{d}}(\cdot;\cdot,\cdot)$ satisfies the following antisymmetry property, for any divergence-free function $\boldsymbol{w}$ in $H^1(\Omega_{\boldsymbol{d}})^3$,

$$\forall \boldsymbol{v} \in H^1_0(\Omega_{\boldsymbol{d}})^3, \quad c_{f,\boldsymbol{d}}(\boldsymbol{w};\boldsymbol{v},\boldsymbol{v}) = 0. \tag{2.23}$$

Note however that the constants $\alpha_{f,\boldsymbol{d}}$ and specially $\beta_{f,\boldsymbol{d}}$ depend on the geometry of $\Omega_{\boldsymbol{d}}$. By combining the Hopf's lemma and the Brouwer's fixed point theorem (see [11, Chapter IV, Section 2] for instance), we easily derive that, for a given domain $\Omega_{\boldsymbol{d}}$ and for any data $(\boldsymbol{f}, \boldsymbol{g}_{\text{in}}, \boldsymbol{g}_{\text{out}})$ in $H^{-1}(\Omega_{\boldsymbol{d}})^3 \times H^{\frac{1}{2}}_{00}(\Gamma_{\text{in}})^3 \times H^{\frac{1}{2}}_{00}(\Gamma_{\text{out}})^3$, problem (2.5) – (2.6) admits a solution. Unfortunately, the uniqueness of this solution is only proved when the ratio of the norms of the data to ν_f^2 is small enough, which is rather restrictive.

About the shell. The forms $a_s(\cdot,\cdot)$ and $b_s(\cdot,\cdot)$ are continuous on the spaces $\mathbb{X}(\omega)\times\mathbb{X}(\omega)$ and $\mathbb{X}(\omega)\times\mathbb{M}(\omega)$, respectively. Moreover, the form $a_s(\cdot,\cdot)$ satisfies the following ellipticity property, for a positive constant α_s and with obvious definition of the norm $\|\cdot\|_{\mathbb{X}(\omega)}$,

$$\forall \boldsymbol{E} \in \mathbb{V}(\omega), \quad a_s(\boldsymbol{E},\boldsymbol{E}) \geq \alpha_s \, \|\boldsymbol{E}\|^2_{\mathbb{X}(\omega)}, \tag{2.24}$$

(we refer to [4] for the proof of this); the form $b_s(\cdot,\cdot)$ satisfies the following inf-sup condition, for a positive constant β_s,

$$\forall \chi \in \mathbb{M}(\omega), \qquad \sup_{\boldsymbol{E} \in \mathbb{X}(\omega)} \frac{b_s(\boldsymbol{E},\chi)}{\|\boldsymbol{E}\|_{\mathbb{X}(\omega)}} \geq \beta_s \, \|\chi\|_{H^1(\omega)}. \tag{2.25}$$

Note also that the composition by $\boldsymbol{\varphi} + \boldsymbol{d}$ maps $H^1(\omega)$ onto $H^1(\Gamma_{\boldsymbol{d}})$ whenever $\boldsymbol{d}$ belongs to $W^{1,\infty}(\omega)^3$. Consequently, for any data $\boldsymbol{k}$ in $L^2(\omega)^3$ and triple $(\boldsymbol{u}, p, \boldsymbol{d}^*)$ such that $(\nu_f \, \partial_{a_3} \breve{\boldsymbol{u}} - \breve{p}\boldsymbol{a}_3)(\boldsymbol{x}) \, J_{\boldsymbol{d}^*}$ belongs to the dual space of $H^1_\diamond(\omega)^3$ (see (2.18)), problem (2.19) admits a unique solution.

A further regularity result for the displacement $\boldsymbol{d}$ has recently been proved in [1] (and requires some further regularity of the domain ω). Here, we are led to make a stronger assumption which is not unlikely, due to the periodicity of $\boldsymbol{d}$ with respect to $\boldsymbol{x}$ (but can require a little more regularity on $\boldsymbol{\varphi}$).

Assumption 2.2. For any data $\boldsymbol{k}$ in $W^{\frac{2}{3},3}(\omega)^3$, the part $\boldsymbol{d}$ of the solution $\boldsymbol{D}$ of the problem: *Find* $\boldsymbol{D} = (\boldsymbol{d}, \boldsymbol{r})$ *in* $\mathbb{V}(\omega)$ *such that*

$$\forall \boldsymbol{E} \in \mathbb{V}(\omega), \quad a_s\big(\boldsymbol{D},\boldsymbol{E}\big) = \int_\omega \boldsymbol{k}\cdot\boldsymbol{e}\,d\boldsymbol{x}, \tag{2.26}$$

belongs to $W^{\frac{8}{3},3}(\omega)^3$ and satisfies

$$\|\boldsymbol{d}\|_{W^{\frac{8}{3},3}(\omega)^3} \leq \alpha \, \|\boldsymbol{k}\|_{W^{\frac{2}{3},3}(\omega)^3}. \tag{2.27}$$

3. Analysis of the model.

Before studying the previous problem we need a basic "geometric" lemma.

Lemma 3.1. *There exists a constant c_φ only depending on $\boldsymbol{\varphi}$ such that, for any $\boldsymbol{d}$ in $W^{1,\infty}(\omega)^3$ satisfying*

$$\|\boldsymbol{d}\|_{W^{1,\infty}(\omega)^3} < c_\varphi, \tag{3.1}$$

the following properties hold:
(i) The mapping $\boldsymbol{\varphi} + \boldsymbol{d}$ is one-to-one from $\overline{\omega}$ onto $\Gamma_{\boldsymbol{d}}$;
(ii) The operator $D(\boldsymbol{\varphi}+\boldsymbol{d})$ is an isomorphism, its norm and that of its inverse are bounded independently of $\boldsymbol{d}$.

Proof: We proceed in two steps.
1) Since $\Gamma_{\boldsymbol{d}}$ is the range of $\boldsymbol{\varphi} + \boldsymbol{d}$, we only have to check that it is one-to-one. Let $\boldsymbol{x}$ and $\boldsymbol{y}$ two points of $\overline{\omega}$ such that

$$(\boldsymbol{\varphi} + \boldsymbol{d})(\boldsymbol{x}) = (\boldsymbol{\varphi} + \boldsymbol{d})(\boldsymbol{y}).$$

It follows from (2.9) that

$$c_\flat\, |\boldsymbol{x} - \boldsymbol{y}| \le |\boldsymbol{\varphi}(\boldsymbol{x}) - \boldsymbol{\varphi}(\boldsymbol{y})| = |\boldsymbol{d}(\boldsymbol{x}) - \boldsymbol{d}(\boldsymbol{y})| \le \|\boldsymbol{d}\|_{W^{1,\infty}(\omega)^3}\, |\boldsymbol{x} - \boldsymbol{y}|.$$

Thus, if $\|\boldsymbol{d}\|_{W^{1,\infty}(\omega)^3} < \boldsymbol{c}_\flat$, $\boldsymbol{x}$ is equal to $\boldsymbol{y}$, which concludes the proof of part (i).
2) For any $\boldsymbol{x}$ in ω, since the vectors $\boldsymbol{a}_\alpha(\boldsymbol{x})$ are linearly independent, the operator $D\boldsymbol{\varphi}(\boldsymbol{x})$ is an isomorphism. Moreover, it follows from the formula

$$D(\boldsymbol{\varphi} + \boldsymbol{d})(\boldsymbol{x}) = D\boldsymbol{\varphi}(\boldsymbol{x})\, \Big(Id + (D\boldsymbol{\varphi})^{-1}(\boldsymbol{x})\, D\boldsymbol{d}(\boldsymbol{x})\Big),$$

that, when $\|\boldsymbol{d}\|_{W^{1,\infty}(\omega)^3} < 1/\|(D\boldsymbol{\varphi})^{-1}\|_{L^\infty(\Gamma)}$, all the assertions of part (ii) hold.

We now prove the existence of a solution to problem (2.5) – (2.6) – (2.19). The key argument relies on the fact that the displacement $\boldsymbol{d}$ of the shell can be assumed to be small enough, in an appropriate sense. From now on, we suppose that the data $\boldsymbol{f}$, $\boldsymbol{g}_{\text{in}}$, $\boldsymbol{g}_{\text{out}}$ and $\boldsymbol{k}$ satisfy

$$\boldsymbol{f} \in L^3(\Omega)^3, \qquad \boldsymbol{g}_{\text{in}} \in W_0^{\frac{5}{3},3}(\Gamma_{\text{in}})^3 \qquad \boldsymbol{g}_{\text{out}} \in W_0^{\frac{5}{3},3}(\Gamma_{\text{out}})^3, \qquad \boldsymbol{k} \in W^{\frac{2}{3},3}(\omega)^3, \tag{3.2}$$

together with the standard compatibility condition

$$\int_{\Gamma_{\text{in}}} \boldsymbol{g}_{\text{in}}(\boldsymbol{\tau}) \cdot \boldsymbol{n}\, d\boldsymbol{\tau} + \int_{\Gamma_{\text{out}}} \boldsymbol{g}_{\text{out}}(\boldsymbol{\tau}) \cdot \boldsymbol{n}\, d\boldsymbol{\tau} = 0, \tag{3.3}$$

where $\boldsymbol{n}$ is the unit outward normal vector to Ω, here equal to $(-1, 0, 0)$ on Γ_{in} and to $(1, 0, 0)$ on Γ_{out}.

Let $\mathcal{S}$ stand for the Stokes operator in Ω, i.e., the operator which associates with any data $(\boldsymbol{f}, \boldsymbol{g}_{\text{in}}, \boldsymbol{g}_{\text{out}})$ in $H^{-1}(\Omega)^3 \times H_{00}^{\frac{1}{2}}(\Gamma_{\text{in}})^3 \times H_{00}^{\frac{1}{2}}(\Gamma_{\text{out}})^3$ the part $\boldsymbol{u}$ of the solution $(\boldsymbol{u}, p)$ of the Stokes problem on Ω: The pair $(\boldsymbol{u}, p)$ belongs to $H^1(\Omega)^3 \times L^2_\circ(\Omega)$, satisfies

$$\boldsymbol{u} = \boldsymbol{g}_{\text{in}} \quad \text{on } \Gamma_{\text{in}}, \qquad \boldsymbol{u} = \boldsymbol{g}_{\text{out}} \quad \text{on } \Gamma_{\text{out}}, \qquad \text{and} \qquad \boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma, \tag{3.4}$$

and

$$\begin{aligned}&\forall \boldsymbol{v} \in H_0^1(\Omega)^3, \qquad a_f(\boldsymbol{u}, \boldsymbol{v}) + b_f(\boldsymbol{v}, p) = \langle \boldsymbol{f}, \boldsymbol{v} \rangle,\\ &\forall q \in L^2_\circ(\Omega), \qquad b_f(\boldsymbol{u}, q) = 0,\end{aligned} \tag{3.5}$$

with obvious definition of the forms $a_f(\cdot,\cdot)$ as $a_{f,\mathbf{0}}(\cdot,\cdot)$ and $b_f(\cdot,\cdot)$ as $b_{f,\mathbf{0}}(\cdot,\cdot)$. Setting $\mathcal{F}(\boldsymbol{u}) = (\boldsymbol{f} - (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}, \boldsymbol{g}_{\text{in}}, \boldsymbol{g}_{\text{out}})$, we observe that a pair $U = (\boldsymbol{u}, p)$ is a solution of problem (2.5) – (2.6) in Ω if and only if

$$\mathcal{H}(\boldsymbol{u}) = \boldsymbol{u} - \mathcal{S}\mathcal{F}(\boldsymbol{u}) = 0.$$

According to the definition in [5] (see also [11, Chap. IV, Section 3.1]), this solution is called nonsingular if $\mathrm{Id} - \mathcal{S}D\mathcal{F}(\boldsymbol{u})$ (where D stands for the differential operator) is an isomorphism of $H_0^1(\Omega)^3$. We now intend to prove the existence of a solution of (2.5) – (2.6) in $\Omega_{\boldsymbol{d}}$ which is in a neighbourhood of a nonsingular solution of this same problem in Ω. Several steps are needed for that.

We first reformulate problem (2.5) – (2.6) on the domain Ω. For this, we introduce a fixed lifting $\boldsymbol{W}$ of the displacement of the shell, more precisely

$$\boldsymbol{W} = \boldsymbol{d} \circ \boldsymbol{\varphi}^{-1} \quad \text{on } \Gamma \qquad \text{and} \qquad \boldsymbol{W} = \mathbf{0} \quad \text{on } \Gamma_{\text{in}} \cup \Gamma_{\text{out}}. \tag{3.6}$$

We make the further assumption that, if $\boldsymbol{d}$ belongs to $W^{\frac{7}{4},4}(\omega)^3$, the function $\boldsymbol{W}$ belongs to $W^{2,4}(\Omega)^3$ and satisfies

$$\|\boldsymbol{W}\|_{W^{2,4}(\Omega)^3} \le c\, \|\boldsymbol{d}\|_{W^{\frac{7}{4},4}(\omega)^3}. \tag{3.7}$$

The simplest example is given by the solution $\boldsymbol{W}$ of the Laplace equation

$$\begin{cases} -\Delta \boldsymbol{W} = \mathbf{0} & \text{in } \Omega,\\ \boldsymbol{W} = \mathbf{0} & \text{on } \Gamma_{\text{in}} \cup \Gamma_{\text{out}},\\ \boldsymbol{W} = \boldsymbol{d} \circ \boldsymbol{\varphi}^{-1} & \text{on } \Gamma.\end{cases}$$

In this case, the regularity property follows from [13, Cor. 2.5.2.2] when the angles beween Γ_{in} and Γ and between Γ_{out} and Γ are zero and from [2, Thm II.4.9] when Ω is a cylinder, but it also holds for more general geometries. However, there exist many other liftings and, from now on, we assume that property (3.7) holds.

Due to the imbedding of $W^{2,4}(\Omega)$ into $W^{1,\infty}(\Omega)$, we also have the following analogue of Lemma 3.1, we skip its proof since it is exactly the same.

Lemma 3.2. *There exists a constant $c^*_{\boldsymbol{\varphi}}$ only depending on $\boldsymbol{\varphi}$ such that, for any $\boldsymbol{d}$ in $W^{\frac{7}{4},4}(\omega)^3$ satisfying*

$$\|\boldsymbol{d}\|_{W^{\frac{7}{4},4}(\omega)^3} < c^*_{\boldsymbol{\varphi}}, \tag{3.8}$$

the following properties hold:
(i) The mapping $\mathrm{Id} + \boldsymbol{W}$ is one-to-one from Ω onto $\Omega_{\boldsymbol{d}}$;
(ii) The operator $D(\mathrm{Id} + \boldsymbol{W})$ is an isomorphism on $\mathbb{R}^3$, its norm and that of its inverse are bounded independently of $\boldsymbol{d}$.

We now apply the Piola transform to the solution of problem (2.5) – (2.6), see [11, Chap. III, Eq. (4.63)] for instance: If $\boldsymbol{W}$ satisfies (3.6) with data $\boldsymbol{d}$, the Piola transform $\mathcal{P}$ associates with any vector field $\boldsymbol{v}$ defined on $\Omega_{\boldsymbol{d}}$ the vector field $\mathcal{P}\boldsymbol{v}$ defined on Ω by

$$(\mathcal{P}\boldsymbol{v}) \circ (\mathrm{Id} + \boldsymbol{W})^{-1} = J(\mathrm{Id} + \boldsymbol{W})\, D(\mathrm{Id} + \boldsymbol{W})^{-1} \boldsymbol{v},$$

where the Jacobian $J(\mathrm{Id}+\boldsymbol{W})$ is the determinant of $D(\mathrm{Id}+\boldsymbol{W})$. Thus and with obvious notation, the pair $(\boldsymbol{u^d}, p^{\boldsymbol{d}})$ is a solution of problem (2.5) – (2.6) on $\Omega_{\boldsymbol{d}}$ if and only if the pair $(\widetilde{\boldsymbol{u}}^{\boldsymbol{d}}, \widetilde{p}^{\boldsymbol{d}})$, with $\widetilde{\boldsymbol{u}}^{\boldsymbol{d}} = \mathcal{P}\boldsymbol{u^d}$ and $\widetilde{p}^{\boldsymbol{d}} = p^{\boldsymbol{d}} \circ (\mathrm{Id}+\boldsymbol{W}) + c(p^{\boldsymbol{d}})$, is a solution of

Find $(\widetilde{\boldsymbol{u}}^{\boldsymbol{d}}, \widetilde{p}^{\boldsymbol{d}})$ in $H^1(\Omega)^3 \times L^2_\circ(\Omega)$ such that

$$\widetilde{\boldsymbol{u}}^{\boldsymbol{d}} = \boldsymbol{g}_{\mathrm{in}} \quad \text{on } \Gamma_{\mathrm{in}}, \qquad \widetilde{\boldsymbol{u}}^{\boldsymbol{d}} = \boldsymbol{g}_{\mathrm{out}} \quad \text{on } \Gamma_{\mathrm{out}}, \qquad \text{and} \qquad \widetilde{\boldsymbol{u}}^{\boldsymbol{d}} = \boldsymbol{0} \quad \text{on } \Gamma, \tag{3.9}$$

and

$$\begin{aligned} &\forall \boldsymbol{v} \in H^1_0(\Omega)^3, \qquad \widetilde{a}_{f,\boldsymbol{d}}(\widetilde{\boldsymbol{u}}^{\boldsymbol{d}}, \boldsymbol{v}) + \widetilde{c}_{f,\boldsymbol{d}}(\widetilde{\boldsymbol{u}}^{\boldsymbol{d}}; \widetilde{\boldsymbol{u}}^{\boldsymbol{d}}, v) + \widetilde{b}_{f,\boldsymbol{d}}(\boldsymbol{v}, \widetilde{p}^{\boldsymbol{d}}) = \langle \widetilde{\boldsymbol{f}}, \boldsymbol{v}\rangle, \\ &\forall q \in L^2_\circ(\Omega), \qquad \widetilde{b}_{f,\boldsymbol{d}}(\widetilde{\boldsymbol{u}}^{\boldsymbol{d}}, q) = 0, \end{aligned} \tag{3.10}$$

with the function $\widetilde{\boldsymbol{f}}$ defined by

$$\langle \widetilde{\boldsymbol{f}}, \boldsymbol{v}\rangle = \langle \boldsymbol{f}, \mathcal{P}^{-1}\boldsymbol{v}\rangle. \tag{3.11}$$

It is readily checked that the form $\widetilde{b}_{f,\boldsymbol{d}}(\cdot,\cdot)$, given by

$$\widetilde{b}_{f,\boldsymbol{d}}(\boldsymbol{v}, \boldsymbol{q}) = -\int_\Omega \big(\mathrm{div}\, \boldsymbol{v}\big)(\boldsymbol{X}) q(\boldsymbol{X})\, d\boldsymbol{X},$$

is equal to $b_f(\cdot,\cdot)$. The forms $\widetilde{a}_{f,\boldsymbol{d}}(\cdot,\cdot)$ and $\widetilde{c}_{f,\boldsymbol{d}}(\cdot;\cdot,\cdot)$ are a little more complex, they are given by

$$\widetilde{a}_{f,\boldsymbol{d}}(\boldsymbol{u}, \boldsymbol{v}) = \nu_f \int_\Omega \sum_{i=1}^3 (A_{\boldsymbol{d}i}\boldsymbol{u})(\boldsymbol{X}) : (A_{\boldsymbol{d}i}\boldsymbol{v})(\boldsymbol{X})\, J(\mathrm{Id}+\boldsymbol{W})(\boldsymbol{X})\, d\boldsymbol{X},$$

and

$$\begin{aligned} &\widetilde{c}_{f,\boldsymbol{d}}(\boldsymbol{w}; \boldsymbol{u}, \boldsymbol{v}) \\ &\quad = \int_\Omega \sum_{i=1}^3 \big(D(\mathrm{Id}+\boldsymbol{W})\boldsymbol{w}\big)_i(\boldsymbol{X}) \cdot (A_{\boldsymbol{d}i}\boldsymbol{u})(\boldsymbol{X}) \big(D(\mathrm{Id}+\boldsymbol{W})\boldsymbol{v}\big)(\boldsymbol{X}) \frac{1}{J(\mathrm{Id}+\boldsymbol{W})(\boldsymbol{X})}\, d\boldsymbol{X}, \end{aligned}$$

with the operator $A_{\boldsymbol{d}i}$ defined by

$$\begin{aligned} A_{\boldsymbol{d}i}\boldsymbol{v} = &\frac{1}{J(\mathrm{Id}+\boldsymbol{W})} \big(D(\mathrm{Id}+\boldsymbol{W})\big)\partial_{x_i}\boldsymbol{v} \\ &+ \frac{1}{J(\mathrm{Id}+\boldsymbol{W})} \big(\partial_{x_i} D(\mathrm{Id}+\boldsymbol{W})\big)\boldsymbol{v} - \frac{\partial_{x_i} J(\mathrm{Id}+\boldsymbol{W})}{J(\mathrm{Id}+\boldsymbol{W})^2} \big(D(\mathrm{Id}+\boldsymbol{W})\big)\boldsymbol{v}. \end{aligned}$$

Note that the addition of the constant $c(p^{\boldsymbol{d}})$ is only made for simplicity, in order that $\widetilde{p}^{\boldsymbol{d}}$ belongs to $L^2_\circ(\Omega)$.

The idea to go further consists in writing

$$\widetilde{a}_{f,\boldsymbol{d}}(\boldsymbol{u}, \boldsymbol{v}) = a_f(\boldsymbol{u}, \boldsymbol{v}) + a^*_{f,\boldsymbol{d}}(\boldsymbol{u}, \boldsymbol{v}), \qquad \widetilde{c}_{f,\boldsymbol{d}}(\boldsymbol{w}; \boldsymbol{u}, \boldsymbol{v}) = c_f(\boldsymbol{w}; \boldsymbol{u}, \boldsymbol{v}) + c^*_{f,\boldsymbol{d}}(\boldsymbol{w}; \boldsymbol{u}, \boldsymbol{v}), \tag{3.12}$$

with

$$a_f(\boldsymbol{u},\boldsymbol{v}) = \nu_f \int_\Omega \big(\mathbf{grad}\,\boldsymbol{u}\big)(\boldsymbol{X}) \,:\, \big(\mathbf{grad}\,\boldsymbol{v}\big)(\boldsymbol{X})\,d\boldsymbol{X},$$

$$c_f(\boldsymbol{w};\boldsymbol{u},\boldsymbol{v}) = \int_\Omega \big((\boldsymbol{w}\cdot\nabla)\boldsymbol{u}\big)(\boldsymbol{X}) \,\cdot\, \boldsymbol{v}(\boldsymbol{X})\,d\boldsymbol{X}.$$

Indeed, the forms $a_f(\cdot,\cdot)$ and $c_f(\cdot;\cdot,\cdot)$ are obviously continuous on $H^1(\Omega)^3 \times H^1(\Omega)^3$ and $H^1(\Omega)^3 \times H^1(\Omega)^3 \times H^1(\Omega)^3$, respectively, while the norms of the forms $a^*_{f,\boldsymbol{d}}(\cdot,\cdot)$ and $c^*_{f,\boldsymbol{d}}(\cdot;\cdot,\cdot)$ are bounded as a function of $\boldsymbol{d}$, as stated in the next lemma.

Lemma 3.3. *If the displacement* $\boldsymbol{d}$ *belongs to* $W^{\frac{7}{4},4}(\omega)^3$*, the following continuity property holds for all* $\boldsymbol{u}$*,* $\boldsymbol{v}$*, and* $\boldsymbol{w}$ *in* $H^1(\Omega_{\boldsymbol{d}})^3$*:*

$$\begin{aligned}
&|a^*_{f,\boldsymbol{d}}(\boldsymbol{u},\boldsymbol{v})| \le c\,\|\boldsymbol{d}\|_{W^{\frac{7}{4},4}(\omega)^3}\,\|\boldsymbol{u}\|_{H^1(\Omega)^3}\|\boldsymbol{v}\|_{H^1(\Omega)^3},\\
&\qquad |c^*_{f,\boldsymbol{d}}(\boldsymbol{w};\boldsymbol{u},\boldsymbol{v})| \le c\,\|\boldsymbol{d}\|_{W^{\frac{7}{4},4}(\omega)^3}\,\|\boldsymbol{w}\|_{H^1(\Omega)^3}\|\boldsymbol{u}\|_{H^1(\Omega)^3}\|\boldsymbol{v}\|_{H^1(\Omega)^3}.
\end{aligned}\tag{3.13}$$

Proof: We establish only the first part of (3.13) since the second one is simpler. It follows from (3.7) that the function $\boldsymbol{W}$ belongs to $W^{2,4}(\Omega)^3$. We also have

$$\begin{aligned}
a^*_{f,\boldsymbol{d}}(\boldsymbol{u},\boldsymbol{v}) &= \nu_f \int_\Omega \mathbf{grad}\,\boldsymbol{u} \,:\, \mathbf{grad}\,\boldsymbol{v}\,\big(J(\mathrm{Id}+\boldsymbol{W})(\boldsymbol{X}) - 1\big)\,d\boldsymbol{X}\\
&\quad + \nu_f \int_\Omega \sum_{i=1}^3 (A_{\boldsymbol{d}i}\boldsymbol{u})(\boldsymbol{X}) \,\cdot\, (A_{\boldsymbol{d}i}\boldsymbol{v} - \partial_{x_i}\boldsymbol{v})(\boldsymbol{X})\,J(\mathrm{Id}+\boldsymbol{W})(\boldsymbol{X})\,d\boldsymbol{X}\\
&\qquad + \nu_f \int_\Omega \sum_{i=1}^3 (A_{\boldsymbol{d}i}\boldsymbol{u} - \partial_{x_i}\boldsymbol{u})(\boldsymbol{X}) \,\cdot\, (\partial_{x_i}\boldsymbol{v})(\boldsymbol{X})\,J(\mathrm{Id}+\boldsymbol{W})(\boldsymbol{X})\,d\boldsymbol{X}.
\end{aligned}$$

To bound the first term, we deduce from the trilinearity of the Jacobian that

$$\|J(\mathrm{Id}+\boldsymbol{W}) - 1\|_{L^\infty(\Omega)} \le c\,\|\boldsymbol{W}\|_{W^{1,\infty}(\Omega)^3} \le c'\,\|\boldsymbol{d}\|_{W^{\frac{7}{4},4}(\omega)^3}.$$

To evaluate the second and third ones it suffices to bound for instance $\|A_{\boldsymbol{d}i}\boldsymbol{v} - \partial_{x_i}\boldsymbol{v}\|_{L^2(\Omega)^3}$, hence to bound (owing to the imbedding of $H^1(\Omega)$ into $L^6(\Omega)$) the three quantities

$$\Big\|\frac{D(\mathrm{Id}+\boldsymbol{W})}{J(\mathrm{Id}+\boldsymbol{W})} - 1\Big\|_{L^\infty(\Omega)^{3\times3}}, \qquad \Big\|\frac{\partial_{x_i} D(\mathrm{Id}+\boldsymbol{W})}{J(\mathrm{Id}+\boldsymbol{W})}\Big\|_{L^3(\Omega)^{3\times3}},$$
$$\Big\|\frac{\partial_{x_i} J(\mathrm{Id}+\boldsymbol{W})}{J(\mathrm{Id}+\boldsymbol{W})^2}\,\big(D(\mathrm{Id}+\boldsymbol{W})\big)\Big\|_{L^3(\Omega)^{3\times3}}.$$

All this follows from (3.6) and (3.7).

Let us now consider a nonsingular solution $(\boldsymbol{u},p)$ of problem (2.5) – (2.6) on Ω. We are interested in proving the existence of a solution of the same problem on $\Omega_{\boldsymbol{d}}$ or equivalently of problem (3.9) – (3.10) when $\boldsymbol{d}$ is small enough. We first consider the modified Stokes problem: The pair $(\widetilde{\boldsymbol{u}},\widetilde{p})$ belongs to $H^1(\Omega)^3 \times L^2_\circ(\Omega)$, satisfies (3.4) and

$$\begin{aligned}
&\forall \boldsymbol{v} \in H^1_0(\Omega)^3, \qquad \widetilde{a}_{f,\boldsymbol{d}}(\widetilde{\boldsymbol{u}},\boldsymbol{v}) + b_f(\boldsymbol{v},\widetilde{p}) = \langle \widetilde{\boldsymbol{f}},\boldsymbol{v}\rangle,\\
&\forall q \in L^2_\circ(\Omega), \qquad b_f(\widetilde{\boldsymbol{u}},q) = 0.
\end{aligned}\tag{3.14}$$

Let $\mathcal{S}_{\boldsymbol{d}}$ denote the modified Stokes operator, i.e., the operator which associates with $(\widetilde{\boldsymbol{f}}, \boldsymbol{g}_{\rm in}, \boldsymbol{g}_{\rm out})$ the part $\widetilde{\boldsymbol{u}}$ of the solution $(\widetilde{\boldsymbol{u}}, \widetilde{p})$ of this problem. Its existence results from the following lemma.

Lemma 3.4. *There exist positive constants α_f and λ_1 such that, if the displacement $\boldsymbol{d}$ belongs to $W^{\frac{7}{4},4}(\omega)^3$ and satisfies*

$$\|\boldsymbol{d}\|_{W^{\frac{7}{4},4}(\omega)^3} \le \lambda_1, \tag{3.15}$$

the following ellipticity property holds

$$\forall \boldsymbol{v} \in H_0^1(\Omega)^3, \quad \widetilde{a}_{f,\boldsymbol{d}}(\boldsymbol{v}, \boldsymbol{v}) \ge \alpha_f \, \|\boldsymbol{v}\|^2_{H^1(\Omega)^3}. \tag{3.16}$$

Proof: By using expansion (3.12) combined with the ellipticity property of the form $a_f(\cdot,\cdot)$ (see (2.21)) and Lemma 3.3, we obtain

$$\widetilde{a}_{f,\boldsymbol{d}}(\boldsymbol{v}, \boldsymbol{v}) \ge \big(\alpha_{f,\boldsymbol{0}} - c\, \|\boldsymbol{d}\|_{W^{\frac{7}{4},4}(\omega)^3}\big) \|\boldsymbol{v}\|^2_{H^1(\Omega)^3},$$

whence the desired result.

An immediate consequence of this lemma is that, if condition (3.15) holds, the operator $\mathcal{S}_{\boldsymbol{d}}$ is continuous from $H^{-1}(\Omega)^3 \times H_{00}^{\frac{1}{2}}(\Gamma_{\rm in})^3 \times H_{00}^{\frac{1}{2}}(\Gamma_{\rm out})^3$ into $H^1(\Omega)^3$, with norm bounded independently of $\boldsymbol{d}$. We can also derive the next lemma.

Lemma 3.5. *The following estimate holds for any $(\boldsymbol{f}, \boldsymbol{g}_{\rm in}, \boldsymbol{g}_{\rm out})$ in $H^{-1}(\Omega)^3 \times H_{00}^{\frac{1}{2}}(\Gamma_{\rm in})^3 \times H_{00}^{\frac{1}{2}}(\Gamma_{\rm out})^3$*

$$\begin{aligned} &\|(\mathcal{S} - \mathcal{S}_{\boldsymbol{d}})(\boldsymbol{f}, \boldsymbol{g}_{\rm in}, \boldsymbol{g}_{\rm out})\|_{H^1(\Omega)^3} \\ &\qquad \le c\,\big(\|\boldsymbol{f}\|_{H^{-1}(\Omega)^3} + \|\boldsymbol{g}_{\rm in}\|_{H_{00}^{\frac{1}{2}}(\Gamma_{\rm in})^3} + \|\boldsymbol{g}_{\rm out}\|_{H_{00}^{\frac{1}{2}}(\Gamma_{\rm out})^3}\big)\, \|\boldsymbol{d}\|_{W^{\frac{7}{4},4}(\omega)^3}. \end{aligned} \tag{3.17}$$

Proof: Setting $\boldsymbol{u} = \mathcal{S}(\boldsymbol{f}, \boldsymbol{g}_{\rm in}, \boldsymbol{g}_{\rm out})$ and $\boldsymbol{u}_{\boldsymbol{d}} = \mathcal{S}_{\boldsymbol{d}}(\boldsymbol{f}, \boldsymbol{g}_{\rm in}, \boldsymbol{g}_{\rm out})$, we observe from expansion (3.12) that

$$\forall \boldsymbol{v} \in \mathbb{K}(\Omega), \quad a_f(\boldsymbol{u} - \boldsymbol{u}_{\boldsymbol{d}}, \boldsymbol{v}) = a^*_{f,\boldsymbol{d}}(\boldsymbol{u}_{\boldsymbol{d}}, \boldsymbol{v}),$$

where $\mathbb{K}(\Omega)$ stands for the kernel of $b_f(\cdot,\cdot)$ in $H_0^1(\Omega)^3$. Thus, since $\boldsymbol{u} - \boldsymbol{u}_{\boldsymbol{d}}$ belongs to $\mathbb{K}(\Omega)$, the desired result follows from Lemma 3.3, combined with the continuity of $\mathcal{S}_d$.

To go further, we introduce the mapping $C_{\boldsymbol{d}}$ defined with values in $H^{-1}(\Omega)^3$ by

$$\forall \boldsymbol{v} \in H_0^1(\Omega)^3, \quad \langle C_{\boldsymbol{d}}(\boldsymbol{u}), \boldsymbol{v}\rangle = \widetilde{c}_{f,\boldsymbol{d}}(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v}).$$

Indeed, a pair $U_{\boldsymbol{d}} = (\widetilde{\boldsymbol{u}}^{\boldsymbol{d}}, \widetilde{p}^{\boldsymbol{d}})$ is a solution of problem (3.9) – (3.10) if and only if it satisfies

$$\mathcal{H}_{\boldsymbol{d}}(\widetilde{\boldsymbol{u}}_{\boldsymbol{d}}) = \widetilde{\boldsymbol{u}}_{\boldsymbol{d}} - \mathcal{S}_{\boldsymbol{d}} \mathcal{F}_{\boldsymbol{d}}(\widetilde{\boldsymbol{u}}_{\boldsymbol{d}}) = 0, \qquad \text{with} \quad \mathcal{F}_{\boldsymbol{d}}(\boldsymbol{u}) = (\widetilde{\boldsymbol{f}} - C_{\boldsymbol{d}}(\boldsymbol{u}), \boldsymbol{g}_{\rm in}, \boldsymbol{g}_{\rm out}). \tag{3.18}$$

Lemma 3.6. *Let $U = (\boldsymbol{u}, p)$ is a nonsingular solution of problem (2.5) – (2.6) in Ω. There exist a neighbourhood $\mathcal{V}$ of $\boldsymbol{u}$ and a positive constant λ_2 such that, if the displacement $\boldsymbol{d}$ belongs to $W^{\frac{7}{4},4}(\omega)^3$ and satisfies*

$$\|\boldsymbol{d}\|_{W^{\frac{7}{4},4}(\omega)^3} \le \lambda_2, \tag{3.19}$$

for all $\boldsymbol{v}$ in $\mathcal{V}$, the operator $D\mathcal{H}_{\boldsymbol{d}}(\boldsymbol{v})$ is an isomorphism of $H_0^1(\Omega)^3$ and the norm of its inverse is bounded independently of $\boldsymbol{d}$.

Proof: We have

$$D\mathcal{H}_{\boldsymbol{d}}(\boldsymbol{v}) = D\mathcal{H}(\boldsymbol{u}) - (\mathcal{S} - \mathcal{S}_{\boldsymbol{d}})\big(D(\boldsymbol{u}.\nabla)\boldsymbol{u}, \mathbf{0}, \mathbf{0}\big) \\ + \mathcal{S}_{\boldsymbol{d}}\big(DC_{\boldsymbol{d}}(\boldsymbol{u}) - D(\boldsymbol{u}.\nabla)\boldsymbol{u}, \mathbf{0}, \mathbf{0}\big) + \mathcal{S}_{\boldsymbol{d}}\big(DC_{\boldsymbol{d}}(\boldsymbol{v}) - DC_{\boldsymbol{d}}(\boldsymbol{u}), \mathbf{0}, \mathbf{0}\big).$$

By definition, $D\mathcal{H}(\boldsymbol{u})$ is an isomorphism of $H_0^1(\Omega)^3$. Next,
1) the second term is bounded from Lemma 3.5 as a function of $\|\boldsymbol{d}\|_{W^{\frac{7}{4},4}(\omega)^3}$;
2) the third term only depends on $c^*_{f,\boldsymbol{d}}(\cdot;\cdot,\cdot)$ and, thanks to Lemma 3.3 and the continuity of $\mathcal{S}_d$, is also bounded as a function of $\|\boldsymbol{d}\|_{W^{\frac{7}{4},4}(\omega)^3}$;
3) due to the continuity of $\mathcal{S}_{\boldsymbol{d}}$ and $DC_{\boldsymbol{d}}$, the fourth term is bounded for $\boldsymbol{v}$ in an appropriate neighbourhood of $\boldsymbol{u}$.
Combining all this yields the lemma.

We skip the proofs of the next two lemmas, which are obvious.

Lemma 3.7. *The mapping: $\boldsymbol{v} \mapsto D\mathcal{H}_{\boldsymbol{d}}(\boldsymbol{v})$ is Lipschitz-continuous on any bounded neighbourhood of $\boldsymbol{u}$, with Lipschitz constant independent of $\boldsymbol{d}$.*

Lemma 3.8. *There exists a constant c such that the following property holds*

$$\|\mathcal{H}_{\boldsymbol{d}}(\boldsymbol{u})\|_{H^1(\Omega)^3} \leq c\,\|\boldsymbol{d}\|_{W^{\frac{7}{4},4}(\omega)^3}. \tag{3.20}$$

Combining all this with a fixed-point theorem (see [11, Chap. IV, Thm 3.1] in a different framework) leads to the following result.

Proposition 3.9. *There exists a positive constant λ such that, for any nonsingular solution U of problem (2.5) – (2.6) in Ω and for all $\boldsymbol{d}$ in $W^{\frac{7}{4},4}(\omega)^3$ satisfying*

$$\|\boldsymbol{d}\|_{W^{\frac{7}{4},4}(\omega)^3} \leq \lambda, \tag{3.21}$$

problem (3.9) – (3.10) has a unique solution in an appropriate neighbourhood of U.

Proof: Let γ be the norm of $D\mathcal{H}_{\boldsymbol{d}}(\boldsymbol{u})^{-1}$ as an isomorphism of $H_0^1(\Omega)^3$, μ be the Lipschitz constant of $D\mathcal{H}_{\boldsymbol{d}}(\cdot)$ in a neighbourhood of $\boldsymbol{u}$ and ε be equal to $\|\mathcal{H}_{\boldsymbol{d}}(\boldsymbol{u})\|_{H^1(\Omega)^3}$. We define the function Φ on $H^1(\Omega)^3$ by

$$\Phi(\boldsymbol{v}) = \boldsymbol{v} - \big(D\mathcal{H}_{\boldsymbol{d}}(\boldsymbol{u})\big)^{-1}\,\mathcal{H}_{\boldsymbol{d}}(\boldsymbol{v}).$$

We choose λ such that $4\gamma^2\,\mu\,\varepsilon < 1$ (this follows from Lemma 3.8, combined with Lemmas 3.6 and 3.7). Thus, it can be checked that
1) Φ maps the sphere with centre $\boldsymbol{u}$ and radius $2\gamma\varepsilon$ into itself: This is easily derived from the formula

$$\Phi(\boldsymbol{v}) - \boldsymbol{u} = -\big(D\mathcal{H}_{\boldsymbol{d}}(\boldsymbol{u})\big)^{-1}\,\mathcal{H}_{\boldsymbol{d}}(\boldsymbol{u}) \\ + \big(D\mathcal{H}_{\boldsymbol{d}}(\boldsymbol{u})\big)^{-1}\Big(\int_0^1 \big(D\mathcal{H}_{\boldsymbol{d}}(\boldsymbol{u}) - D\mathcal{H}_{\boldsymbol{d}}(\boldsymbol{u} + t(\boldsymbol{v}-\boldsymbol{u}))\big)\,dt\Big)\cdot(\boldsymbol{v}-\boldsymbol{u}),$$

which leads to the estimate

$$\|\Phi(\boldsymbol{v})-\boldsymbol{u}\|_{H^1(\Omega)^3} \le \gamma\,\varepsilon + \frac{\gamma\mu}{2}\,\|\boldsymbol{v}-\boldsymbol{u}\|^2_{H^1(\Omega)^3},$$

whence the desired result.
2) Φ is a strict contraction of this same sphere: Indeed, we have

$$\Phi(\boldsymbol{v})-\Phi(\boldsymbol{w}) = \big(D\mathcal{H}_{\boldsymbol{d}}(\boldsymbol{u})\big)^{-1}\Big(\int_0^1 \big(D\mathcal{H}_{\boldsymbol{d}}(\boldsymbol{u}) - D\mathcal{H}_{\boldsymbol{d}}(\boldsymbol{w}+t(\boldsymbol{v}-\boldsymbol{w}))\big)\,dt\Big)\cdot(\boldsymbol{v}-\boldsymbol{w}),$$

which yields the contraction property.
Thus, Φ has a unique fixed point $\widetilde{\boldsymbol{u}}_{\boldsymbol{d}}$ in this sphere. The existence of a $\widetilde{p}_{\boldsymbol{d}}$ such that $(\widetilde{\boldsymbol{u}}_{\boldsymbol{d}},\widetilde{p}_{\boldsymbol{d}})$ is a solution of problem (3.9) – (3.10) is then a direct consequence of the inf-sup condition (2.22) applied on Ω.

Less regularity than stated in Assumption 2.2 was required for the previous results (note that $W^{\frac{8}{3},3}(\omega)$ is imbedded into $W^{\frac{7}{4},4}(\omega)$). However, we need a further regularity property of the solution $(\widetilde{\boldsymbol{u}}_{\boldsymbol{d}},\widetilde{p}_{\boldsymbol{d}})$ which requires the $W^{\frac{8}{3},3}(\omega)^3$ regularity of $\boldsymbol{d}$. From now on, we assume that the Stokes operator $\mathcal{S}$ is continuous from $L^3(\Omega)^3\times W^{\frac{5}{3}}(\Gamma_{\rm in})^3\times W^{\frac{5}{3}}(\Gamma_{\rm out})^3$ into $W^{2,3}(\Omega)^3$.

Lemma 3.10. *For any nonsingular solution U of problem (2.5) – (2.6) in Ω and for all $\boldsymbol{W}$ in $W^{2,\infty}(\omega)^3\cap W^{3,3}(\omega)^3$, the solution $(\widetilde{\boldsymbol{u}}_{\boldsymbol{d}},\widetilde{p}_{\boldsymbol{d}})$ of problem (3.9) – (3.10) exhibited in Proposition 3.9 belongs to $W^{2,3}(\Omega)^3\times W^{1,3}(\Omega)$ and satisfies*

$$\|\widetilde{\boldsymbol{u}}_{\boldsymbol{d}}\|_{W^{2,3}(\Omega)^3} + \|\widetilde{p}_{\boldsymbol{d}}\|_{W^{1,3}(\Omega)} \le c\,\Big(\|\boldsymbol{f}\|_{L^3(\Omega)^3} + \|\boldsymbol{g}_{\rm in}\|_{W^{\frac{5}{3},3}(\Gamma_{\rm in})^3} + \|\boldsymbol{g}_{\rm out}\|_{W^{\frac{5}{3},3}(\Gamma_{\rm out})^3}\Big). \quad (3.22)$$

Proof: We first consider the Stokes problem (3.4) – (3.14). By letting $\boldsymbol{v}$ run through $\mathscr{D}(\Omega)^3$, q run through $\mathscr{D}(\Omega)$ and divide by $J(\mathrm{Id}+\boldsymbol{W})$, we can write it in the distribution sense

$$\begin{cases} -\nu_f\,\Delta\widetilde{\boldsymbol{u}} + \mathbf{grad}\,\widetilde{p} = R(\widetilde{\boldsymbol{u}}) + \widetilde{\boldsymbol{f}}\,J(\mathrm{Id}+\boldsymbol{W})^{-1} & \text{in } \Omega,\\ \mathrm{div}\,\widetilde{\boldsymbol{u}} = 0 & \text{in } \Omega,\end{cases}$$

plus of course the boundary conditions. The term $R(\widetilde{\boldsymbol{u}})$ is rather complex, however, by combining the regularity assumption on $\boldsymbol{W}$ with the imbedding of $H^1(\Omega)$ into $L^6(\Omega)$, it can be checked that R maps $H^1(\Omega)^3$ into $L^2(\Omega)^3$. Thus, the regularity property of the Stokes problem combined with a boot-strap argument, yield the desired property for $\mathcal{S}_{\boldsymbol{d}}$. Finally the same property holds for the Navier–Stokes equations (3.9) – (3.10) thanks to another boot-strap argument.

We are now interested in solving problem (2.15). Assuming that $\boldsymbol{d}$ is known and satisfies (3.21), we denote by $(\widetilde{\boldsymbol{u}}(\boldsymbol{d}),\widetilde{p}(\boldsymbol{d}))$ the unique solution of problem (3.9) – (3.10) exhibited in Proposition 3.9. The right-hand side of problem (2.15) can now be written

$$\mathcal{L}(\boldsymbol{E}) = \int_\omega \boldsymbol{k}\cdot\boldsymbol{e}\,\sqrt{a}\,d\boldsymbol{x} + \mathcal{M}_{\boldsymbol{d}}(\boldsymbol{E}),$$

with

$$\mathcal{M}_{\boldsymbol{d}}(\boldsymbol{E}) = \int_{\Gamma_{\boldsymbol{d}}} (\nu_f\,\partial_n\boldsymbol{u}(\boldsymbol{d}) - p(\boldsymbol{d})\boldsymbol{n})(\boldsymbol{\tau})\cdot(\boldsymbol{e}\circ(\boldsymbol{\varphi}+\boldsymbol{d})^{-1})(\boldsymbol{\tau})\,d\boldsymbol{\tau}. \quad (3.23)$$

We also denote by $M(\boldsymbol{d})$ the element of the dual space of $H^1_\diamond(\omega)^3$ defined by

$$\forall \boldsymbol{e} \in H^1_\diamond(\omega)^3, \quad \langle M(\boldsymbol{d}), \boldsymbol{e}\rangle = \mathcal{M}_{\boldsymbol{d}}(\boldsymbol{E}) \tag{3.24}$$

(indeed, this last quantity does not depend on the second argument of $\boldsymbol{E}$).

Let $\mathcal{T}$ be the linear operator which associates with any $\boldsymbol{k}$ in $L^2(\omega)^3$ the part $\boldsymbol{d}$ of the solution $\boldsymbol{D}$ of problem (2.26). Thus, problem (2.15) can be written equivalently

$$\boldsymbol{d} - \mathcal{T}\big(\boldsymbol{k}\sqrt{a} + M(\boldsymbol{d})\big) = 0. \tag{3.25}$$

We now prove some properties of the mapping M. We are led to make a further assumption, which involves a stronger regularity on $\boldsymbol{\varphi}$ (and is satisfied for instance when $\partial\Omega$ is of class $\mathcal{C}^{2,1}$, see [13, Thm 2.5.1.1]).

Assumption 3.11. The mapping: $\boldsymbol{d} \mapsto \boldsymbol{W}$, where $\boldsymbol{W}$ is a fixed lifting operator satisfying (3.6), is continuous from $W^{2,\infty}(\omega)^3 \cap W^{\frac{8}{3},3}(\omega)^3$ into $W^{2,\infty}(\Omega)^3 \cap W^{3,3}(\Omega)^3$.

Lemma 3.12. *If Assumption* 3.11 *holds, the following property holds for all* $\boldsymbol{d}$ *in the ball of* $W^{\frac{8}{3},3}(\omega)^3$ *with radius* λ,

$$\|M(\boldsymbol{d})\|_{W^{\frac{2}{3},3}(\omega)^3} \le c\,\big(1 + \|\boldsymbol{d}\|_{W^{\frac{8}{3},3}(\omega)^3}\big). \tag{3.26}$$

Proof: Recalling that the transformation $\mathrm{Id} + \boldsymbol{W}$ maps Ω onto Ω_d and Γ onto Γ_d, we have

$$\mathcal{M}_{\boldsymbol{d}}(\boldsymbol{E}) = \int_\Gamma (\nu_f\,\partial_n \widetilde{\boldsymbol{u}}(\boldsymbol{d}) - \widetilde{p}(\boldsymbol{d})\boldsymbol{n})(\boldsymbol{\tau}) \cdot \widetilde{\boldsymbol{e}}(\boldsymbol{\tau})\, J(\mathrm{Id} + \boldsymbol{W})\, d\boldsymbol{x}, \tag{3.27}$$

with appropriate definition for $\widetilde{\boldsymbol{e}}$. Since $J(\mathrm{Id}+\boldsymbol{W})$ is also bounded by $c\,\big(1+\|\boldsymbol{d}\|_{W^{\frac{8}{3},3}(\omega)^3}\big)$, the desired result follows from Lemma 3.10 which implies that $\nu_f\,\partial_n \widetilde{\boldsymbol{u}}(\boldsymbol{d}) - \widetilde{p}(\boldsymbol{d})\boldsymbol{n}$ belongs to $W^{\frac{2}{3},3}(\omega)^3$ and the product theorem, see [13, Thm 1.4.4.2] for instance.

Lemma 3.13. *If Assumption* 3.11 *holds, the following property holds for all* $\boldsymbol{d}_1$ *and* $\boldsymbol{d}_2$ *in the ball of* $W^{\frac{8}{3},3}(\omega)^3$ *with radius* λ,

$$\|M(\boldsymbol{d}_1) - M(\boldsymbol{d}_2)\|_{W^{\frac{2}{3},3}(\omega)^3} \le c\,\|\boldsymbol{d}_1 - \boldsymbol{d}_2\|_{W^{\frac{8}{3},3}(\omega)^3}. \tag{3.28}$$

Proof: Using the same transformation as in the previous proof, we have with obvious notation

$$\begin{aligned}
&\mathcal{M}_{\boldsymbol{d}_1}(\boldsymbol{E}) - \mathcal{M}_{\boldsymbol{d}_2}(\boldsymbol{E}) \\
&\quad = \int_\Gamma (\nu_f\,\partial_n \widetilde{\boldsymbol{u}}_1(\boldsymbol{d}) - \widetilde{p}_1(\boldsymbol{d})\boldsymbol{n})(\boldsymbol{\tau}) \cdot \widetilde{\boldsymbol{e}}(\boldsymbol{\tau})\,\big(J(\mathrm{Id} + \boldsymbol{W}_1) - J(\mathrm{Id} + \boldsymbol{W}_2)\big)\, d\boldsymbol{x} \\
&\quad + \int_\Gamma \big((\nu_f\,\partial_n \widetilde{\boldsymbol{u}}(\boldsymbol{d}_1) - \widetilde{p}(\boldsymbol{d}_1)\boldsymbol{n}) - (\nu_f\,\partial_n \widetilde{\boldsymbol{u}}(\boldsymbol{d}_2) - \widetilde{p}(\boldsymbol{d}_2)\boldsymbol{n})\big)(\boldsymbol{\tau}) \cdot \widetilde{\boldsymbol{e}}(\boldsymbol{\tau})\, J(\mathrm{Id} + \boldsymbol{W}_2)\, d\boldsymbol{x}.
\end{aligned} \tag{3.29}$$

Bounding the first line follows from the trilinearity of the Jacobian. To handle the second line, we derive from (3.18) that $\widetilde{\boldsymbol{u}}(\boldsymbol{d}_1) - \widetilde{\boldsymbol{u}}(\boldsymbol{d}_2)$ satisfies

$$\mathcal{H}_{\boldsymbol{d}_1}(\widetilde{\boldsymbol{u}}(\boldsymbol{d}_1)) - \mathcal{H}_{\boldsymbol{d}_1}(\widetilde{\boldsymbol{u}}(\boldsymbol{d}_2)) = -(\mathcal{S}_{\boldsymbol{d}_2} - \mathcal{S}_{\boldsymbol{d}_1})\mathcal{F}_{\boldsymbol{d}_2}(\widetilde{\boldsymbol{u}}(\boldsymbol{d}_2)) - \mathcal{S}_{\boldsymbol{d}_1}\big(C_{\boldsymbol{d}_1}(\widetilde{\boldsymbol{u}}(\boldsymbol{d}_2)) - C_{\boldsymbol{d}_2}(\widetilde{\boldsymbol{u}}(\boldsymbol{d}_2)), \boldsymbol{0}, \boldsymbol{0}\big).$$

An upper bound for the right-hand side is easily derived from extensions of Lemmas 3.3 and 3.5. On the other hand, we have

$$\mathcal{H}_{\boldsymbol{d}_1}(\widetilde{\boldsymbol{u}}(\boldsymbol{d}_1)) - \mathcal{H}_{\boldsymbol{d}_1}(\widetilde{\boldsymbol{u}}(\boldsymbol{d}_2)) = \Big(D\mathcal{H}_{\boldsymbol{d}_1}(\widetilde{\boldsymbol{u}}(\boldsymbol{d}_1)) + \int_0^1 \Big(D\mathcal{H}_{\boldsymbol{d}_1}\big(\widetilde{\boldsymbol{u}}(\boldsymbol{d}_1) + t(\widetilde{\boldsymbol{u}}(\boldsymbol{d}_2) - \widetilde{\boldsymbol{u}}(\boldsymbol{d}_1))\big) - D\mathcal{H}_{\boldsymbol{d}_1}(\widetilde{\boldsymbol{u}}(\boldsymbol{d}_1))\Big)\,dt \Big) \cdot (\widetilde{\boldsymbol{u}}(\boldsymbol{d}_1) - \widetilde{\boldsymbol{u}}(\boldsymbol{d}_2)).$$

It is readily checked that $D\mathcal{H}_{\boldsymbol{d}_1}(\widetilde{\boldsymbol{u}}(\boldsymbol{d}_1))$ is an isomorphism of $W^{2,3}(\Omega)^3$. Combining all this with an analogue of Lemma 3.7 yields

$$\|\widetilde{\boldsymbol{u}}(\boldsymbol{d}_1) - \widetilde{\boldsymbol{u}}(\boldsymbol{d}_2)\|_{W^{2,3}(\Omega)^3} \le c\, \|\boldsymbol{d}_1 - \boldsymbol{d}_2\|_{W^{\frac{8}{3},3}(\omega)^3}.$$

A similar estimate for $\|\widetilde{p}(\boldsymbol{d}_1) - \widetilde{p}(\boldsymbol{d}_2)\|_{W^{1,3}(\Omega)}$ is then derived by standard arguments. Inserting this into (3.29) gives the desired estimate.

We now prove the existence result in an obvious way.

Proposition 3.14. *There exists a positive real number α_0 such that, if Assumption* 2.2 *holds with $\alpha \le \alpha_0$ and Assumption* 3.11 *also holds, for any nonsingular solution U of problem* (2.5) – (2.6) *in Ω, problem* (2.15) *has a solution $\boldsymbol{D} = (\boldsymbol{d}, r)$ with $\boldsymbol{d}$ in $W^{\frac{8}{3},3}(\omega)^3$.*

Proof: From the previous two lemmas, for $\alpha \le \alpha_0$, there exists a positive λ^* such that the mapping: $\boldsymbol{d} \mapsto \mathcal{T}\big(\boldsymbol{k}\sqrt{a} + M(\boldsymbol{d})\big)$
1) sends the ball of $W^{\frac{8}{3},3}(\omega)^3$ with radius λ^* into itself,
2) is a contraction of this ball.
Thus, there exists a unique solution of problem (3.25) in this ball. Defining $\boldsymbol{r}$ as the part of the solution of problem (3.25) associated with $\boldsymbol{d}$ (see the definition of $\mathcal{T}$), the pair $\boldsymbol{D} = (\boldsymbol{d}, \boldsymbol{r})$ is a solution of (2.15).

Owing to the inf-sup condition (2.25), we are in a position to state the main result of this section.

Theorem 3.15. *If Assumptions* 2.2 *with $\alpha \le \alpha_0$ and* 3.11 *hold, problem* (2.5) – (2.6) – (2.19) *has a solution $(\boldsymbol{u}, p, \boldsymbol{D}, \psi)$ in $H^1(\Omega_{\boldsymbol{d}})^3 \times L^2(\Omega_{\boldsymbol{d}}) \times \mathbb{X}(\omega) \times \mathbb{M}(\omega)$.*

The fact that α is small enough in the regularity property (2.27) only depends on the properties of the shell: For instance, it holds if the Young modulus E is large enough with respect to its thickness t.

4. An algorithm for uncoupling the unknowns.

In view of the previous analysis, we decide to construct a sequence $(\widetilde{\boldsymbol{u}}^n, \widetilde{p}^n, \boldsymbol{D}^n, \psi^n)_n$ which converges to a solution of problem (3.9) – (3.10) – (2.19) of course coupled with the construction of a fixed lifting operator $\boldsymbol{W}$ satisfying (3.6). Such a sequence is constructed by induction in the following way.

Starting from $\boldsymbol{d}^0 = \boldsymbol{0}$ and assuming by induction that $\boldsymbol{d}^n$ is known, we first introduce the solution $\boldsymbol{W}^n$ of the problem

$$\begin{cases} -\Delta \boldsymbol{W}^n = \boldsymbol{0} & \text{in } \Omega, \\ \boldsymbol{W}^n = \boldsymbol{0} & \text{on } \Gamma_{\text{in}} \cup \Gamma_{\text{out}}, \\ \boldsymbol{W}^n = \boldsymbol{d}^n \circ \boldsymbol{\varphi}^{-1} & \text{on } \Gamma, \end{cases} \tag{4.1}$$

and we assume that condition (3.7) holds. Next, we consider the problem

Find $(\widetilde{\boldsymbol{u}}^n, \widetilde{p}^n)$ *in* $H^1(\Omega)^3 \times L^2_\circ(\Omega)$ *such that*

$$\widetilde{\boldsymbol{u}}^n = \boldsymbol{g}_{\text{in}} \quad \text{on } \Gamma_{\text{in}}, \qquad \widetilde{\boldsymbol{u}}^n = \boldsymbol{g}_{\text{out}} \quad \text{on } \Gamma_{\text{out}}, \qquad \text{and} \qquad \widetilde{\boldsymbol{u}}^n = \boldsymbol{0} \quad \text{on } \Gamma, \tag{4.2}$$

and

$$\begin{aligned} &\forall \boldsymbol{v} \in H^1_0(\Omega)^3, \qquad \widetilde{a}_{f,\boldsymbol{d}^n}(\widetilde{\boldsymbol{u}}^n, \boldsymbol{v}) + \widetilde{c}_{f,\boldsymbol{d}^n}(\widetilde{\boldsymbol{u}}^n; \widetilde{\boldsymbol{u}}^n, v) + \widetilde{b}_f(\boldsymbol{v}, \widetilde{p}^n) = \langle \widetilde{\boldsymbol{f}}^n, \boldsymbol{v} \rangle, \\ &\forall q \in L^2_\circ(\Omega), \qquad \widetilde{b}_f(\widetilde{\boldsymbol{u}}^n, q) = 0, \end{aligned} \tag{4.3}$$

with obvious notation for $\widetilde{\boldsymbol{f}}^n$, see (3.11). Exactly the same arguments as for Proposition 3.9 yields the following result.

Lemma 4.1. *There exists a positive constant* λ *such that, for any nonsingular solution* U *of problem* (2.5) – (2.6) *in* Ω *and for all* $\boldsymbol{d}^n$ *in* $W^{\frac{7}{4},4}(\omega)^3$ *satisfying* (3.21), *problem* (4.2) – (4.3) *has a unique solution* $(\widetilde{\boldsymbol{u}}^n, \widetilde{p}^n)$ *in an appropriate neighbourhood of* U.

Next, assuming that $\boldsymbol{d}^n$ and $(\widetilde{\boldsymbol{u}}^n, \widetilde{p}^n)$ are known, we are interested in the problem

Find $(\boldsymbol{D}^{n+1}, \psi^{n+1})$ *in* $\mathbb{X}(\omega) \times \mathbb{M}(\omega)$ *such that*

$$\begin{aligned} &\forall \boldsymbol{E} \in \mathbb{X}(\omega), \quad a_s(\boldsymbol{D}^{n+1}, \boldsymbol{E}) + b_s(\boldsymbol{E}, \psi^{n+1}) = \mathcal{L}_n(\boldsymbol{E}), \\ &\forall \chi \in \mathbb{M}(\omega), \quad b_s(\boldsymbol{D}^{n+1}, \chi) = 0, \end{aligned} \tag{4.4}$$

where the form $\mathcal{L}_n(\cdot)$ is now defined by

$$\mathcal{L}_n(\boldsymbol{E}) = \int_\omega \boldsymbol{k} \cdot \boldsymbol{e} \sqrt{a}\, d\boldsymbol{x} + \int_{\Gamma_{\boldsymbol{d}^n}} (\nu_f\, \partial_n \boldsymbol{u}^n - p^n \boldsymbol{n})(\boldsymbol{\tau}) \cdot (\boldsymbol{e} \circ (\boldsymbol{\varphi} + \boldsymbol{d}^n)^{-1})(\boldsymbol{\tau})\, d\boldsymbol{\tau}. \tag{4.5}$$

The next result is now a direct consequence of the ellipticity property (2.24) of the form $a_s(\cdot, \cdot)$ and of the inf-sup condition (2.25) (and also of Lemma 3.10 for the continuity of the right-hand side).

Lemma 4.2. *If Assumptions* 2.2 *and* 3.11 *hold, problem* (4.4) *has a unique solution* $(\boldsymbol{D}^{n+1}, \psi^{n+1})$ *with* $\boldsymbol{d}^{n+1}$ *in* $W^{\frac{8}{3},3}(\omega)^3$.

The main idea is now to use the arguments in the proof of Proposition 3.14 to prove the convergence of the sequence $(\widetilde{\boldsymbol{u}}^n, \widetilde{p}^n, \boldsymbol{D}^n, \psi^n)_n$ toward a solution $(\widetilde{\boldsymbol{u}}, \widetilde{p}, \boldsymbol{D}, \psi)$ of (3.9) –

(3.10) − (2.19). Indeed, with the notation of Section 3, each $\boldsymbol{d}^{n+1}$ is the solution of the problem

$$\boldsymbol{d}^{n+1} - \mathcal{T}\big(\boldsymbol{k}\,\sqrt{a} + M(\boldsymbol{d}^n)\big) = 0. \tag{4.6}$$

This yields the following result.

Lemma 4.3. *If Assumptions* 2.2 *with* $\alpha \le \alpha_0$ *and* 3.11 *hold, the sequence* $(\boldsymbol{d}^n)_n$ *is bounded in* $W^{\frac{8}{3},3}(\omega)^3$ *and, moreover satisfies for a constant* $\kappa < 1$ *and any integer* $n > 0$

$$\|\boldsymbol{d}^{n+1} - \boldsymbol{d}^n\|_{W^{\frac{8}{3},3}(\omega)^3} \le \kappa\, \|\boldsymbol{d}^n - \boldsymbol{d}^{n-1}\|_{W^{\frac{8}{3},3}(\omega)^3}. \tag{4.7}$$

Proof: An induction argument (we start with $\boldsymbol{d}^0 = \boldsymbol{0}$) combined with Lemma 3.12 yields the boundedness property. Estimate (4.7) thus follows from equation (4.6), Lemma 3.13 and Assumption 2.2.

The previous lemma implies that $(\boldsymbol{d}^n)_n$ is a Cauchy sequence: Indeed, for any positive integers n and m,

$$\|\boldsymbol{d}^{n+m} - \boldsymbol{d}^n\|_{W^{\frac{8}{3},3}(\omega)^3} \le \frac{1-\kappa^m}{1-\kappa}\kappa^n\, \|\boldsymbol{d}^1 - \boldsymbol{d}^0\|_{W^{\frac{8}{3},3}(\omega)^3}. \tag{4.8}$$

We are thus in a position to prove the convergence result.

Theorem 4.4. *If Assumptions* 2.2 *with* $\alpha \le \alpha_0$ *and* 3.11 *hold, the whole sequence* $(\widetilde{\boldsymbol{u}}^n, \widetilde{p}^n, \boldsymbol{D}^n, \psi^n)_n$ *converges toward a solution* $(\widetilde{\boldsymbol{u}}, \widetilde{p}, \boldsymbol{D}, \psi)$ *of problem* (3.9) − (3.10) − (2.19) *in* $H^1(\Omega)^3 \times L^2_\circ(\Omega) \times \mathbb{X}(\omega) \times \mathbb{M}(\omega)$.

Proof: It follows from (4.8) that $(\boldsymbol{d}^n)_n$ converges to a limit $\boldsymbol{d}$ in $W^{\frac{8}{3},3}(\omega)^3$. We proceed in two steps.

1) Equation (4.3) can equivalently be written

$$\begin{aligned}
\forall \boldsymbol{v} \in H^1_0(\Omega)^3, \qquad & \widetilde{a}_{f,\boldsymbol{d}}(\widetilde{\boldsymbol{u}}^n, \boldsymbol{v}) + \widetilde{c}_{f,\boldsymbol{d}}(\widetilde{\boldsymbol{u}}^n; \widetilde{\boldsymbol{u}}^n, v) + \widetilde{b}_f(\boldsymbol{v}, \widetilde{p}^n) = \langle \widetilde{\boldsymbol{f}}^n, \boldsymbol{v}\rangle \\
& \qquad + (\widetilde{a}_{f,\boldsymbol{d}} - \widetilde{a}_{f,\boldsymbol{d}^n})(\widetilde{\boldsymbol{u}}^n, \boldsymbol{v}) + (\widetilde{c}_{f,\boldsymbol{d}} - \widetilde{c}_{f,\boldsymbol{d}^n})(\widetilde{\boldsymbol{u}}^n; \widetilde{\boldsymbol{u}}^n, v), \\
\forall q \in L^2_\circ(\Omega), \qquad & \widetilde{b}_f(\widetilde{\boldsymbol{u}}^n, q) = 0.
\end{aligned}$$

It follows from the convergence of $(\boldsymbol{d}_n)_n$ that the right-hand side tends to $\langle \widetilde{\boldsymbol{f}}, \boldsymbol{v}\rangle$. On the other hand, the sequence $(\boldsymbol{u}_n, p_n)_n$ is bounded in $H^1(\Omega)^3 \times L^2(\Omega)$, and, due to (4.7), it is also a Cauchy sequence. Thus, the whole sequence $(\boldsymbol{u}_n, p_n)_n$ converges to a solution $(\boldsymbol{u}, p)$ of problem (3.9) − (3.10) (this requires the uniqueness of such a solution in a neighbourhood of U).

2) From this convergence, the right-hand side $\mathcal{L}_n(\cdot)$ of problem (4.4) converges to the linear form $\mathcal{L}(\cdot)$ defined in (2.17). Thus, from the ellipticity property (2.24) of $a_s(\cdot,\cdot)$ and the inf-sup condition (2.25) of $b_s(\cdot,\cdot)$, we easily deduce the convergence of $(\boldsymbol{D}^n, \psi^n)_n$ to a solution $(\boldsymbol{D}, \psi)$ of problem (2.19).

This concludes the proof.

Passing to the limit on m in (4.8) yields that the convergence of the sequence $(\boldsymbol{d}^n)_n$ is geometric, indeed

$$\|\boldsymbol{d} - \boldsymbol{d}^n\|_{W^{\frac{8}{3},3}(\omega)^3} \le \frac{1}{1-\kappa}\kappa^n\, \|\boldsymbol{d}^1 - \boldsymbol{d}^0\|_{W^{\frac{8}{3},3}(\omega)^3}. \tag{4.9}$$

Similar results can be derived for the other unknowns in problem (3.9) − (3.10) − (2.19), we do not state them for simplicity. Estimate (4.9) proves that the previous algorithm provides an elegant way to define a discretization, as will be performed in the second part of this work.

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