A new exceptional points method with application to cell-centered Lagrangian schemes and curved meshes

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Abstract

The aim of this paper is the numerical simulation of compressible hydrodynamic strong implosions, which take place for instance in Inertial Confinement Fusion. It focuses in particular on two crucial issues, for such calculations: very low CFL number at the implosion center and approximation error on the initial geometry. Thus, we propose an exceptional points method, which is translation invariant and suitable to curved meshes. This method is designed for cell-centered Lagrangian schemes (GLACE [8, 13, 14], CHIC [24, 25, 26]). Several numerical examples on significant test cases are presented to show the relevance of our approach.

Keywords: Lagrangian hydrodynamic, Compressible gas dynamics, Exceptional points, Non conformal meshes, Godunov conservative scheme, Translation invariance, Symmetry preservation, Curved meshes

1. Introduction

In this work, we deal with the numerical simulation of compressible hydrodynamics phenomena using Lagrangian discrete methods, see for instance [2] for a review. In integral form, Euler’s equations are:

\[
\begin{align*}
\frac{d}{dt} \int_{V(t)} 1 \, dV &= \int_{V(t)} \nabla \cdot \mathbf{u} \, dV, & \text{volume conservation} \\
\frac{d}{dt} \int_{V(t)} \rho \, dV &= 0, & \text{mass conservation} \\
\frac{d}{dt} \int_{V(t)} \rho \mathbf{u} \, dV + \int_{V(t)} \nabla p \, dV &= 0, & \text{momentum conservation} \\
\frac{d}{dt} \int_{V(t)} \rho e \, dV + \int_{V(t)} \nabla \cdot (\rho \mathbf{u}) \, dV &= 0, & \text{total energy conservation}
\end{align*}
\]

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where $V(t)$ is a moving Lagrangian volume. Here Lagrangian refers to the mass conservation equation. The physical variables are density $\rho$, velocity $\mathbf{u}$, total energy $e$ and pressure $p$.

Most of the traditional approaches [9, 22, 27, 29], with Lagrangian methods, use conformal meshes. In a conformal Lagrangian mesh, all the points are dynamic points: they move accordingly to the velocity applied to them. We will focus in this work on polar meshes which are very often encountered in the numerical simulation of Inertial Confinement Fusion (ICF) flows, where preserving symmetries is essential. Among all the numerous difficulties that one has to tackle in order to enhance the Lagrangian computation quality, we single out two main issues encountered with polar meshes.

- First problem is the bad aspect ratio of cells near the center of polar meshes. The number of degrees of freedom is important in this location. It involves very short time steps in order to satisfy the restrictive CFL condition, and therefore drastically increases the computational burden.

A solution consists in using the non conformal approach, which allows us to simplify the mesh step and to optimize numerical criteria. In a non conformal 2D Lagrangian mesh, some points, called exceptional points, are unslaved to their neighbors, see the situation described in Figure 3 for instance. Using non conformal approach makes possible to coarsen the mesh near the center of the simulation domain, without losing accuracy in the remainder of the domain. Thus, important calculation gains are possible without even changing the solution quality in good cases. However, this is quite challenging because Lagrangian schemes are known to be very sensitive to mesh perturbations. Therefore, the issue is to allow non conformity as much as possible, while maintaining (enhancing in good cases) stability and accuracy of the scheme. Similar approaches have been developed in the context of staggered Lagrangian scheme, see for instance [7, 21].

- Second problem is the standard meshing procedure used to approximate the initial geometry. The approximation error may be important for polar geometries, just because standard meshes have straight lines. Curved meshes are attractive to reduce the approximation error of the initial geometry. In particular, we will see that our numerical results for non equisectorial meshes show that error is indeed less important with curved meshes than with standard meshes with straight edges. In some cases, this interpolation error at time $t = 0$ may dominate the total numerical error at the final time of the simulation. This is particularly true for multimaterial simulations [23].

Concerning this second problem, we notice that it has been the subject of many researches [9, 11, 15]. However no solution has really emerged, so that to our knowledge Lagrangian calculations with curved meshes are performed only for research studies. The most inspiring work for research purposes is [23].

Our concern in this work is to address these exceptional points and the curved meshes in the context of the cell-centered GLACE scheme [8, 13, 14]. We will present a new formulation
of exceptional points, that is more general than previous ones [7, 21]: it can be used in any dimension and applied to curved meshes. This formulation is essentially based on the definition of some vectors $C_{j,r}$ (see Definition 2.1 page 4) which contains the needed geometric information. Briefly, those vectors are defined at each point indexed by $r$ of a cell with index $j$. They represent gradients of the volume of cell $j$ with respect to points $r \in j$. Geometric considerations that we will develop are based on the role of translation invariance in (2) (see page 4). Therefore they are, with some respect, independent of any specific Lagrangian scheme. In our case, we consider the GLACE scheme which is a cell-centered Lagrangian scheme of Godunov type (that is fluxes are based on a generalized notion of Riemann invariant), based upon the knowledge of vectors $C_{j,r}$ at each time step. These geometric quantities are also used in staggered discretization of Lagrangian equations. Staggered discretization should benefit quite naturally of the concepts developed in this work. In particular, we think about the compatible-hydro scheme for which we refer to a series of works [4, 6]. Even the compatible-hydro scheme needs the definition of corner masses which are not needed for GLACE. The CHIC scheme [24, 25, 26] and the CAVEAT scheme [1] are other cell-centered schemes which do not rely exactly on the same geometric quantities as GLACE or compatible-hydro. In CHIC, the geometric vectors at corners are not defined as the gradient of some potential as in (2). Finally we would like to mention the works of Scovazzi and co-authors [30, 31, 32, 33, 34, 35]. Their formulation is Finite Element based, so in theory, curved meshes can be defined as special isoparametric elements. The topic of the definition of Lagrangian methods for curved meshes, is also explored in [11, 20].

One of the most original benefits of our work is depicted on Figure 1. That is we will describe an automatic method to construct exceptional points on curved meshes. Our numerical results confirm the theoretical analysis and show that the proposed definition provides a stable and accurate procedure for the discretization of a challenging perturbation problem.

![Figure 1: A curved mesh where the exceptional point $M$ is located on an arc of the circle passing throw A and B](image)

An outline of this work is as follows. In a first Section, we present an overview of the GLACE scheme in conformal case. In a second Section, we expand the GLACE scheme to handle non conformal meshes and exceptional points. In a third Section, we define new
geometric vectors $D_{j,r}$ for meshes with straight edges. In a fourth Section, we present the definitions of $C_{j,r}$ and $D_{j,r}$ vectors for meshes with curved edges. In a fifth Section, we propose a variant of the GLACE scheme to preserve the symmetry of our meshes and in a sixth Section, we present some numerical results on significant test cases to show the relevance of our approach.

2. Overview of the GLACE scheme for conformal meshes

The GLACE scheme [8, 13, 14] is a Godunov method\(^1\), a LAgrangian scheme and is Conservative for the total Energy variable. The method is cell-centered as any Godunov type scheme. The \textit{a priori} good properties of Godunov cell-centered methods are:

- a natural conservation of physical quantities,
- no need for artificial viscosity techniques,
- easy implementation of fully conservative remapping and remeshing techniques.

2.1. Introduction of geometric vectors $C_{j,r}$

Before going further, we need some notations that will be used through this text. Let us consider a computational domain $\Omega \in \mathbb{R}^d$ composed of control volumes indexed by $j$ and points indexed by $r$. In practice, the dimension of the problem can be $d = 1, 2$ or $3$. In the sequel we refer systematically to the volume of the cell (this is a convenient abuse of notation). The volume of a cell with index $j$ at time $t_k = t_{k-1} + \Delta t$, with $k$ the time step, is denoted as $V_j^k$ such that:

$$\bigcup_j V_j^k = \Omega \quad \text{and} \quad \dim(V_i^k \cap V_j^k) = d \quad \text{if} \quad i \neq j, \quad \forall k.$$ 

We denote by $x^k = (x^k_1, \cdots, x^k_r, \cdots)$, with $x^k_r \in \mathbb{R}^d$, the collection of all points.

Definition 2.1. Let us assume that at each time step the volume $V_j$ is defined as a function of the points, i.e. $x \mapsto V_j(x)$. We define the gradient of the volume $V_j$ with respect to the nodal positions $x_r$,

$$C_{j,r} = \nabla_{x_r} V_j \in \mathbb{R}^d.$$ 

We will refer to these vectors as corner vectors.

Starting from (2) one has by construction

$$\frac{d}{dt} V_j = (\nabla_x V_j \cdot x') = \sum_{r \in B_j} (C_{j,r} \cdot u_r).$$  \(3\)

\(^1\)In dimension one the scheme is equal to the acoustic Godunov scheme [16, 17].
This equation expresses that the change of volume is due to a discrete divergence. For all point with index \( r \), \( B_r \) is the set of indexes of cells which contain the point \( x_r \). As well, for all cell with index \( j \), \( B_j \) is the set of indexes of the points in cell indexed by \( j \). The volume is homogeneous of degree \( d \) with respect to the points that is \( V_j(\lambda x) = \lambda^d V_j(x) \), where \( d \) is the dimension. This implies the following property

\[
V_j = \frac{1}{d} (\nabla x V_j \cdot x) = \frac{1}{d} \sum_{r \in B_j} (C_{j,r} \cdot x_r).
\]

Two additional properties can be deduced (original proof can be found in [8]).

**Proposition 2.1.** For every cell, one has \( \sum_{r \in B_j} C_{j,r} = 0 \). For every interior point (i.e. point that does not belong to the boundary), one has \( \sum_{j \in B_r} C_{j,r} = 0 \).

We provide another proof of this property with Proposition 2.2. This is fundamentally related to translation invariance.

2.2. Computation of \( C_{j,r} \) (in 2D)

![Figure 2: Geometric definition of vector \( C_{j,r} \) in 2D](image)

Using the 2D notations of Figure 2, let \( x_{r-1} = \left( \begin{array}{c} x_{r-1} \\ y_{r-1} \end{array} \right) \), \( x_r = \left( \begin{array}{c} x_r \\ y_r \end{array} \right) \) and \( x_{r+1} = \left( \begin{array}{c} x_{r+1} \\ y_{r+1} \end{array} \right) \) be three points of the cell indexed by \( j \), its volume is

\[
V_j = \sum_{r \in B_j} \frac{1}{2} (x_r y_{r+1} - y_r x_{r+1}),
\]

where \( \frac{1}{2} (x_r y_{r+1} - y_r x_{r+1}) \) is the oriented area of triangle \( x_r x_{r+1} O \), where \( O \) is the mass center of the cell. Then, vectors \( C_{j,r} = \nabla x_r V_j \) are expressed by:

\[
C_{j,r} = \frac{1}{2} \left( \begin{array}{c} -y_{r-1} + y_{r+1} \\ x_{r-1} - x_{r+1} \end{array} \right).
\]
Geometrically, vector $\mathbf{C}_{j,r}$ is orthogonal to the segment joining the middles of $[\mathbf{x}_r, \mathbf{x}_{r-1}]$ and $[\mathbf{x}_r, \mathbf{x}_{r+1}]$. We refer to [8] for the equivalent definitions in 3D.

2.3. Different steps of the GLACE scheme

The GLACE scheme is a one step forward Euler method that discretizes Lagrangian Equations (1). At the beginning of time step $k$, we must know each points $\mathbf{x}_k^r$, vectors $\mathbf{C}_{j,r}^k$ and the values of the physical unknowns inside the cell with index $j$: $\rho_j^k$, $\mathbf{u}_j^k$, $e_j^k$. All the unknowns are constant in each cell, that is the scheme is cell-centered. Fluxes are corner based. We define a point velocity $\mathbf{u}_j^k$ and corner pressures $p_{j,r}^k$ at each time step in function of the mesh geometry and of the numerical value of the unknowns.

Consider for a while that point velocity $\mathbf{u}_j^k$ and corner pressures $p_{j,r}^k$ have been computed. For sake of simplicity, we keep with continuous in time notations. In Lagrangian schemes, the formula for the variation of the volume implicitly contains a discretization of the divergence operator. Using Equation (3), we can see that $\sum_{r \in B_j} (\mathbf{C}_{j,r} \cdot \mathbf{u}_r)$ is an approximation in the finite volume sense of $\nabla \cdot \mathbf{u}$ over the moving cell indexed by $j$, i.e.

$$\int_{V_j} \nabla \cdot \mathbf{u} \, dx = \int_{\partial V_j} (\mathbf{u} \cdot \mathbf{n}) \, d\sigma \approx \sum_{r \in B_j} (\mathbf{C}_{j,r} \cdot \mathbf{u}_r).$$

In the same way, we have

$$\int_{V_j} \nabla \cdot (p\mathbf{u}) \, dx = \int_{\partial V_j} (\mathbf{u} \cdot \mathbf{n}) \, p \, d\sigma \approx \sum_{r \in B_j} (\mathbf{C}_{j,r} \cdot \mathbf{u}_r) \, p_{j,r} = \sum_{r \in B_j} (\mathbf{u}_r \cdot \mathbf{C}_{j,r} \, p_{j,r}),$$

where $p_{j,r}$ is the pressure at point $\mathbf{x}_r$ seen from the cell indexed by $j$. It yields

$$\int_{V_j} \nabla p \, dx = \int_{\partial V_j} \mathbf{n} p \, d\sigma \approx \sum_{r \in B_j} \mathbf{C}_{j,r} p_{j,r}.$$

At this point, we obtain a semi-discrete numerical approximation of the continuous system (1):

$$\begin{cases}
M_j \tau_j'(t) = \sum_{r \in B_j} (\mathbf{C}_{j,r} \cdot \mathbf{u}_r) \\
M_j \mathbf{u}_j'(t) = - \sum_{r \in B_j} \mathbf{C}_{j,r} p_{j,r} \\
M_j e_j'(t) = - \sum_{r \in B_j} (\mathbf{C}_{j,r} \cdot \mathbf{u}_r) p_{j,r},
\end{cases} \quad (5)$$

where $M_j = V_j(\mathbf{x}(t))\rho_j(t)$ is the mass of a Lagrangian cell (which is constant in time) and $\tau_j = \frac{1}{\rho_j}$ is the specific volume of the cell indexed by $j$.

First equation is the discretization of the mass conservation relation and the last two equations are the semi-discrete versions of momentum and total energy equations.

6
In order to determine velocity \( u_r \) and nodal pressures \( p_{j,r} \), we use a nodal solver based on two different formulas. First formula is a multidimensional generalization of a first-order Riemann solver, but in the direction parallel to vector \( \mathbf{C}_{j,r} \):

\[
p_{j,r} - p_j + \alpha_j ((u_r - u_j) \cdot \mathbf{n}_{j,r}) = 0,
\]

where \( \mathbf{n}_{j,r} = \frac{\mathbf{C}_{j,r}}{|\mathbf{C}_{j,r}|} \) and \( \alpha_j = \rho_j c_j \) is the acoustic impedance with \( c_j \) the sound speed in cell with index \( j \). This relation is an approximation of the Rankine Hugoniot relations for shock hydrodynamics in dimension one, and is commonly used for designing Godunov solvers.

Second formula, required to construct the nodal solver for the integration of (5), expresses that the sum of all forces around the point \( x_r \) is zero:

\[
\sum_{j \in \mathcal{B}_r} \mathbf{C}_{j,r} p_{j,r} = 0.
\]

This formula is natural in finite volume methods and enforces the conservation of momentum. Solution of (6)-(7) is easy to compute. Using (6), we can eliminate the pressures terms in (7). We have then to solve the following linear system:

\[
\mathbf{A}_r \mathbf{u}_r = \mathbf{b}_r,
\]

where

\[
\mathbf{A}_r = \sum_{j \in \mathcal{B}_r} \alpha_j \frac{\mathbf{C}_{j,r} \otimes \mathbf{C}_{j,r}}{|\mathbf{C}_{j,r}|} \in \mathbb{R}^{d \times d} \quad \text{and} \quad \mathbf{b}_r = \sum_{j \in \mathcal{B}_r} \mathbf{C}_{j,r} p_j + \sum_{j \in \mathcal{B}_r} \alpha_j \frac{\mathbf{C}_{j,r} \otimes \mathbf{C}_{j,r}}{|\mathbf{C}_{j,r}|} \mathbf{u}_j \in \mathbb{R}^d.
\]

By construction, \( \mathbf{A}_r \) is symmetric if the mesh is non degenerate. Then, \( \mathbf{A}_r > 0 \) and \( \mathbf{A}_r \) is non-singular [8]. Thus, the solution of the linear system is

\[
\mathbf{u}_r = \mathbf{A}_r^{-1} \mathbf{b}_r.
\]

Once nodal velocities calculated, we can compute nodal pressures using (6).

The first-order GLACE scheme steps are then the followings:

1. At the beginning of the time step \( k \), we compute vectors \( \mathbf{C}_{j,r}^k \) for all cells indexed by \( j \) and for all points with index \( r \) in \( \mathcal{B}_j \).

2. We determine \( \mathbf{u}_r^k \) and \( p_{j,r}^k \) using (10) and (6).

3. Then, we update the total momentum and the total energy using:

\[
M_j \frac{\mathbf{u}_r^{k+1} - \mathbf{u}_r^k}{\Delta t} = - \sum_{r \in \mathcal{B}_j} \mathbf{C}_{j,r}^k p_{j,r}^k \quad \text{for the momentum,}
\]

and

\[
M_j \frac{e_{j}^{k+1} - e_{j}^k}{\Delta t} = - \sum_{r \in \mathcal{B}_j} (\mathbf{C}_{j,r}^k \cdot \mathbf{u}_r^k)p_{j,r}^k, \quad \text{for the total energy.}
\]


4. The points are then moved using:
\[ x_{r}^{k+1} = x_{r}^{k} + \Delta t \, u_{r}^{k}, \]
and we can compute the new volume \( V_{j}^{k+1} \).

5. Finally, we calculate the new density in the cell:
\[ \rho_{j}^{k+1} = \frac{M_{j}}{V_{j}^{k+1}}. \]

Basic properties of this scheme are presented in [8]: in a nutshell, the GLACE scheme preserves the local mass, the total impulse, the total energy and the semi-discrete scheme is entropy consistent. In this work, we insist on the following additional property.

**Proposition 2.2.** The GLACE scheme is translation invariant.

It means that if an arbitrary velocity \( a \in \mathbb{R}^{2} \) is added to the velocity at time \( t = 0 \), that is \( u_{j}^{0} \leftarrow u_{j}^{0} + a \), then the numerical solution is also translated \( u_{j}^{k} \leftarrow u_{j}^{k} + a \) with a similar property for the velocity at points \( u_{r}^{k} \leftarrow u_{r}^{k} + a \). It is merely a consequence of (3). Indeed, if one sets \( u_{r} = a \) for any point with index \( r \) in (3), then translation invariance implies that
\[ 0 = \frac{d}{dt} V_{j} = \left( \sum_{r \in B_{j}} C_{j,r} \cdot a \right) \text{ for all } a, \]
that is
\[ \sum_{r \in B_{j}} C_{j,r} = 0. \] (11)

At this point it is easy to verify that the translation invariance, as stated in Proposition 2.2, is a simple consequence of this property.

3. Extension of the GLACE scheme to handle non conformal meshes

In this Section, we present a general definition of exceptional points and we extend the GLACE scheme to the non conformal case introducing new geometric vectors \( D_{j,r} \).

![Figure 3: Non conformal mesh with two exceptional points N and M](image)
An exceptial point is usually understood as a point where the mesh is non conformal. This point is different from the other points because it is attached to its neighbors, i.e. it is written as a function of the two points at each extremity of the segment where it is located. For instance, using the notations of Figure 3, the point $M$ (resp. $N$) is an exceptional point. We use the formula $M = \frac{1}{2}H + \frac{1}{2}I$ (resp. $N = \frac{1}{2}G + \frac{1}{2}H$) to move it. We have now to define the GLACE scheme in such situation, and also to extend it to handle more complicated geometric situations.

3.1. Representation formula

We propose to rely on a more general formulation of what an exceptional point is.

**Definition 3.1.** The point $M$ is said to be an exceptional point if there is a smooth function $\phi : \mathbb{R}^d \times \mathbb{R}^d \mapsto \mathbb{R}^d$ such that $M = \phi(H, I)$, and such that translation invariance holds

\[
\phi(x + a, y + a) = \phi(x, y) + a, \quad \forall a \in \mathbb{R}^d.
\]

Another natural condition is:

\[
\phi(x, x) = x, \quad \text{for all } x \in \mathbb{R}^d.
\]

A point which is not exceptional is said to be free.

Let us define the two constant matrices $P_1 = \nabla_x \phi(0, 0) \in \mathbb{R}^{d \times d}$ and $P_2 = \nabla_y \phi(0, 0) \in \mathbb{R}^{d \times d}$. Taking the derivative of

\[
\phi(a, a) = \phi(0, 0) + a,
\]

with respect to $a$, one gets

\[
P_1 + P_2 = I_d \in \mathbb{R}^{d \times d}.
\]

**Proposition 3.1.** All solutions of (12) can be expressed as:

\[
\phi(x, y) = P_1 x + P_2 y + \psi(x - y),
\]

with $\psi(0) = 0$ and $\nabla \psi(0) = 0$.

**Proof.** We define $\hat{\phi}(x, y) = \phi(x, y) - P_1 x - P_2 y$. By construction, one has $\hat{\phi}(x + a, y + a) = \hat{\phi}(x, y), \forall a \in \mathbb{R}$. Then taking $a = -y$, we obtain $\hat{\phi}(x, y) = \hat{\phi}(x - y, 0)$. We can then define $\psi(z) = \hat{\phi}(z, 0)$, thus (15) holds.

Taking $x = y$ in (15), one obtains $\phi(x, x) = P_1 x + P_2 x + \psi(0)$, that is, using (13) and (14):

\[
\psi(0) = \phi(x, x) - (P_1 + P_2)x = x - x = 0.
\]

Taking $y = 0$ in (15), we get: $\psi(x) = \phi(x, 0) - P_1 x$. Thus, the gradient at the origin is $\nabla \psi(0) = \nabla_x \phi(0, 0) - P_1 = 0$. \qed
Remark When \(x\) and \(y\) are close to each other, \(\psi(x - y) = O(|x - y|^2)\). Therefore, this term is a second order correction of the principal linear contribution \(P_1x + P_2y\).

In this work, we disregard the possibly non linear second order contribution \(\psi\). Thus, the general representation formula of exceptional points that we will consider corresponds to linear dependance

\[
\phi(x, y) = P_1x + P_2y,
\]

with constraint (14).

3.2. Definition of geometric vectors \(D_{j,r}\)

Let us consider the example depicted in Figure 3. The volume \(V_{j3}\) of the cell indexed by \(j_3\), for instance, is computed using points \(I, D, J\) and \(M\), but \(M = \phi(H, I)\) is an exceptional point. Therefore \(V_{j3}\) is truly computed in function of the free points \(I, D, J\) and \(H\), i.e.

\[
V_{j3} = V(I, D, J, \phi(H, I)).
\]  

(17)

We can now extend the definition of corner vectors (2) to non conformal meshes.

**Definition 3.2.** The derivatives of the volume of a cell with respect to free points is denoted

\[
D_{j,r} = \nabla_{x_r} V_j.
\]

In all cases the chain rule provides an easy way to compute \(D_{j,r}\). Consider once again the example depicted in Figure 3. Indeed, using (3) implies

\[
\frac{dV_{j3}}{dt} = C_{j3,I} \cdot u_I + C_{j3,D} \cdot u_D + C_{j3,J} \cdot u_J + C_{j3,M} \cdot u_M.
\]

Point \(M\) is exceptional that is, \(M = P_1H + P_2I\), which turns into \(u_M = P_1u_H + P_2u_I\). Using the definition of function \(\phi\), we obtain:

\[
\frac{dV_{j3}}{dt} = (C_{j3,I} + P_2^t C_{j3,M}) \cdot u_I + C_{j3,D} \cdot u_D + C_{j3,J} \cdot u_J + P_1^t C_{j3,M} \cdot u_H.
\]  

(18)

By identification, we obtain expressions:

\[
D_{j3,I} = C_{j3,I} + P_2^t C_{j3,M}, \quad D_{j3,D} = C_{j3,D}, \quad D_{j3,J} = C_{j3,J}, \quad D_{j3,H} = P_1^t C_{j3,M}.
\]  

(19)

A fundamental advantage of this procedure it that is preserves by construction the translation invariance property (11), which becomes (20).

**Proposition 3.2.** By definition one has

\[
\sum_{r \in B_j} D_{j,r} = 0, \quad \forall j \quad \text{and} \quad \sum_{j \in B_r} D_{j,r} = 0 \quad \forall r.
\]  

(20)
Proof. For pedagogical purposes, we consider the example of Figure 3. We prove the identity (20) for the cell indexed by \( j_3 \). Exceptional point \( M \) is defined as \( M = P_1 H + P_2 I \), with \( P_1 + P_2 = I_2 \). We have

\[
\sum_{r \in B_{j_3}} D_{j_3,r} = D_{j_3,1} + D_{j_3,D} + D_{j_3,J} + D_{j_3,H} = C_{j_3,1} + P_2^{t} C_{j_3,M} + C_{j_3,D} + C_{j_3,J} + P_1^{t} C_{j_3,M} = \sum_{r \in B_{j_3}} C_{j_3,r} = 0 \text{ (using Proposition 2.1)}.
\]

We prove the second identity only for the point \( H \), that is

\[
\sum_{j \in B_{H}} D_{j,H} = 0.
\]

The total volume around the point \( H \) is \( V = V_{j_1} + V_{j_2} + V_{j_3} + V_{j_4} + V_{j_5} + V_{j_6} \). Then,

\[
\frac{dV}{dt} = \frac{dV_{j_1}}{dt} + \frac{dV_{j_2}}{dt} + \frac{dV_{j_3}}{dt} + \frac{dV_{j_4}}{dt} + \frac{dV_{j_5}}{dt} + \frac{dV_{j_6}}{dt} = 0,
\]

because the total volume is fixed and it is only the point \( H \) that can move. This in turn gives us

\[
0 = C_{j_1,H} \cdot u_{H} + C_{j_2,H} \cdot u_{H} + D_{j_3,H} \cdot u_{H} + D_{j_4,H} \cdot u_{H} + D_{j_5,H} \cdot u_{H} + D_{j_6,H} \cdot u_{H}, \quad \forall u_{H}.
\]

Therefore \( C_{j_1,H} + C_{j_2,H} + D_{j_3,H} + D_{j_4,H} + D_{j_5,H} + D_{j_6,H} = 0 \). \( \square \)

3.3. GLACE scheme with exceptional points

In this context, the numerical semi-discrete approximation of (1) to meshes with exceptional points is easily defined. It is sufficient to write (5) with \( D_{j,r} \) instead of \( C_{j,r} \). One obtains

\[
\begin{align*}
M_j r_j' (t) &= \sum_{r \in B_j} (D_{j,r} \cdot u_r) \\
M_j u_j' (t) &= -\sum_{r \in B_j} D_{j,r} p_{j,r} \\
M_j e_j' (t) &= -\sum_{r \in B_j} (D_{j,r} \cdot u_r) p_{j,r}.
\end{align*}
\]

(21)

To compute the nodal velocities \( u_r \), we use now:

\[
\begin{align*}
p_{j,r} - p_j + \alpha_j ((u_r - u_j) \cdot n_{j,r}) &= 0, \quad \text{with } n_{j,r} = \frac{D_{j,r}}{|D_{j,r}|}, \\
\sum_{j \in B_r} D_{j,r} p_{j,r} &= 0.
\end{align*}
\]

(22)

And of course, we do the same for the fully discrete scheme described at the end of Section 2.3. Since the fundamental relation (20) holds true by construction, semi and fully discrete schemes inherit all theoretical properties: the new scheme preserves the local mass, the total momentum, the total energy, is entropy consistent, and is translation invariant.
4. Two specific definitions of $D_{j,r}$ vectors for meshes with straight edges

In this Section, we use the previous theory to construct two types of exceptional points on a mesh where all the edges are straight lines. First case corresponds to the usual definition of exceptional points, that is an exceptional point is the half sum of its neighbors. Second case is more original. We will consider exceptional points located on arcs of circle.

4.1. Exceptional point in the middle of a segment

This definition of exceptional points is the usual definition used in previous works [7, 21]. Using the notations of Figure 4, the exceptional point $M$ is defined using Definition 3.1 with $\phi$ such that:

$$M = \phi(A, B) = \frac{1}{2} I_2 A + \frac{1}{2} I_2 B = \frac{1}{2} A + \frac{1}{2} B.$$  

(23)

Applying formula (18), we have:

$$D_{j,A} = C_{j,A} + \frac{1}{2} C_{j,M}, \quad \text{and} \quad D_{j,B} = \frac{1}{2} C_{j,M}.$$  

4.2. Exceptional point on an arc of a circle

In the previous Section, we looked at the case where exceptional points are in the middle of a segment. Our idea is that it will help to reach better accuracy and to preserve the sphericity in the 2D polar case. At least, all points are on circles at the initial time. In our mind, this is a first step towards an improvement of mesh definition at time $t = 0$. We will couple this method to curved cells in the next Section. An illustration is depicted in Figure 5.

Using the notations of Figure 5, let $E$ be the middle of segment $[AB]$ and $M$ be the exceptional point, for which the location must be defined by a smooth function of its neighborhood. At this point, the benefits of the theoretical analysis are evident. Indeed the translation invariance property that we desire to maintain implies that the definition of $M$
must be independent of the center point \( O \). This is mandatory. Otherwise, the exceptional point \( M \) would depend on three points, \( A, B \) and \( O \). In this case, maintaining all the theoretical properties of the GLACE scheme would have the unfortunate consequence of a long distance numerical interaction between the current cell and \( O \), because of the vector \( C_{j,O} = \nabla_O V_j \). Since we want to avoid such an unphysical behavior, we decide to reject the possibility of a dependence between \( M \) and \( O \). Thus, we make the assumption that the simulation "approximatively" preserves the angle \( \theta = \overrightarrow{AOB} \). This angle is computed at time \( t = 0 \), and remain unchanged during all the process. The definition of the exceptional points will depend of this frozen angle. We seek to express \( M \) in function of \( A \) and \( B \). To this end, we start by defining \( M \) such that:

\[
M = E + EM = \frac{1}{2}(A + B) + \alpha AB^\perp, \quad \text{with } \alpha \in \mathbb{R}.
\]

We have now to determine the coefficient \( \alpha \). We know that:

\[
A = R \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, \quad B = R \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \text{and} \quad M = R \begin{pmatrix} \cos \left( \frac{\theta}{2} \right) \\ \sin \left( \frac{\theta}{2} \right) \end{pmatrix},
\]

where \( R \) is the radius value. Since \( \cos \theta = 2 \cos^2 \left( \frac{\theta}{2} \right) - 1 \) and \( \sin \theta = 2 \sin \left( \frac{\theta}{2} \right) \cos \left( \frac{\theta}{2} \right) \), we have:

\[
E = R \cos \left( \frac{\theta}{2} \right) \begin{pmatrix} \cos \left( \frac{\theta}{2} \right) \\ \sin \left( \frac{\theta}{2} \right) \end{pmatrix}.
\]

Then we can write: \( EM = R \left( 1 - \cos \left( \frac{\theta}{2} \right) \right) \begin{pmatrix} \cos \left( \frac{\theta}{2} \right) \\ \sin \left( \frac{\theta}{2} \right) \end{pmatrix} \).

Moreover,

\[
AB^\perp = R \begin{pmatrix} \sin \theta \\ 1 - \cos \theta \end{pmatrix} = 2R \sin \left( \frac{\theta}{2} \right) \begin{pmatrix} \cos \left( \frac{\theta}{2} \right) \\ \sin \left( \frac{\theta}{2} \right) \end{pmatrix},
\]

since \( 1 - \cos \theta = 2 \sin^2 \left( \frac{\theta}{2} \right) \). We can then deduce the value of \( \alpha \):

\[
\alpha = \frac{1 - \cos \left( \frac{\theta}{2} \right)}{2 \sin \left( \frac{\theta}{2} \right)} = \frac{2 \sin^2 \left( \frac{\theta}{4} \right)}{4 \sin \left( \frac{\theta}{4} \right) \cos \left( \frac{\theta}{4} \right)} = \frac{1}{2} \tan \left( \frac{\theta}{4} \right).
\]
Then the following relation yields:

\[ M = \frac{1}{2}(A + B) + \frac{1}{2} \tan \left( \frac{\theta}{4} \right) AB^\perp. \]  

(24)

We can write (24) with a matricial notation:

\[ M = P_1 A + P_2 B, \]  

(25)

with

\[ P_1 = \frac{1}{2} \begin{pmatrix} \tan \left( \frac{\theta}{4} \right) & 1 \\ -1 & \tan \left( \frac{\theta}{4} \right) \end{pmatrix} \quad \text{and} \quad P_2 = \frac{1}{2} \begin{pmatrix} 1 & -\tan \left( \frac{\theta}{4} \right) \\ \tan \left( \frac{\theta}{4} \right) & 1 \end{pmatrix}. \]  

(26)

We can notice that \( P_1 + P_2 = I_2 \). With the definition (25) of exceptional points, we have also a new value of the function \( \phi \) in (17) and a new way of defining the geometric vectors \( D_{j,r} \). In this procedure the angle \( \theta \) is frozen. That is, we will use this frozen parameter even if the angular position of the cell changes due to multidimensional velocity field. This is necessary to preserve the theoretical properties of the scheme in terms on stability.

In the configuration of Figure 5, we have the following relations:

\[ D_{j,\mathbf{A}} = C_{j,\mathbf{A}} + P_1^t C_{j,\mathbf{M}}, \quad \text{and} \quad D_{j,\mathbf{B}} = P_2^t C_{j,\mathbf{M}}. \]  

(27)

5. Definitions of \( C_{j,r} \) and \( D_{j,r} \) vectors for meshes with curved edges

Placing exceptional points on circle naturally enhances the mesh quality at time \( t = 0 \), but the geometric error remains important if we still consider straight edges (see Figure 6). To overcome this, we present in this Section a new definition of geometric vectors adapted to curved meshes (without or with exceptional points).

5.1. Curved meshes without exceptional points

![Figure 6: Illustration of the volume difference between the straight mesh and the mesh defined using arcs of circle](image)

We consider cells defined as on Figure 6. In order to have the translation invariance principle, the definition of the volume with respect to the points must be independent of the center

14
point \( O \). Once again this is mandatory. So we have to define each point of the curved mesh without considering \( O \). Here, we present a possibility for which the knowledge of the angle \( \theta = \overline{AOB} \) is sufficient. Indeed,

\[
V = V_{ABCD} + V_{AB} - V_{CD},
\]

where \( V_{ABCD} \) is the volume of quadrilateral \( ABCD \) with straight boundaries and \( V_{AB} \) and \( V_{CD} \) are the hatched volumes on Figure 6. We have:

\[
V_{AB} = \frac{\theta}{2} - \cos \left( \frac{\theta}{2} \right) |AB|^2 \quad \text{and} \quad V_{CD} = \frac{\theta}{2} - \cos \left( \frac{\theta}{2} \right) |CD|^2.
\]

Let \( \alpha = \frac{\theta}{2} - \cos \left( \frac{\theta}{2} \right) > 0 \), then \( V = V_{ABCD} + \alpha(|AB|^2 - |CD|^2) \). The knowledge of point \( O \) is then not required. We compute \( \alpha \) at time \( t = 0 \) and we use this value during all the calculations. We have now a new definition of the geometric vectors \( C_{j,r,\text{curved}} \):

\[
\begin{align*}
C_{j,A,\text{curved}} &= \nabla A V = C_{j,A} + \alpha \nabla A |AB|^2 = C_{j,A} + 2\alpha BA, \\
C_{j,B,\text{curved}} &= \nabla B V = C_{j,B} + \alpha \nabla B |AB|^2 = C_{j,B} + 2\alpha AB, \\
C_{j,C,\text{curved}} &= \nabla C V = C_{j,C} - \alpha \nabla C |CD|^2 = C_{j,C} - 2\alpha DC, \\
C_{j,D,\text{curved}} &= \nabla D V = C_{j,D} - \alpha \nabla D |CD|^2 = C_{j,D} - 2\alpha CD,
\end{align*}
\]

where vectors \( C_{j,r} \) are those defined in (4). A difference with respect to [23] is that we do not use any quadrature formula. Corner vectors are the exact gradients of the exact volumes with curved boundaries.

5.2. Curved meshes with exceptional points

![Figure 7: Case where the exceptional point M is on an arc of the circle passing through A and B and with curved mesh](image)

The introduction of exceptional points (see Figure 7) is done as in the previous Section, that is, we define \( M \) such that:

\[
M = P_1 A + P_2 B,
\]

with \( P_1 \) and \( P_2 \) the matrices defined by (26). In this configuration, the final definition of corner vectors takes into account:

- the exceptional character of point \( M \) thanks to \( P_1 \) and \( P_2 \), and
- the curved geometry of the cells thanks to the use of

\[
\begin{align*}
D_{j,A} &= C_{j,A, \text{curved}} + P_1^t C_{j,M, \text{curved}} \quad \text{and} \quad D_{j,B} = P_2^t C_{j,M, \text{curved}},
\end{align*}
\]

(28)
6. Scheme variants and symmetry issue

The scheme proposed, in the previous Section, is the formal extension of the GLACE scheme (refer to [8]) but same arguments could be used to build the formal extension of the CHIC scheme (see for instance [25]), leading to a very close formulation. The main difference between the first-order version of the two schemes is the calculation of the nodal velocity using the acoustic Riemann solver (Equation (6)). In the CHIC scheme, vectors $C_{j,r}$ are decomposed into the basis defined by the vectors normal to the edges of the cell indexed by $j$, intersecting at the node with index $r$. Let note $(\mathbf{e}_{j,r,1}, \mathbf{e}_{j,r,2})$ this basis (see Figure 8).

![Figure 8: Graphical definition of vector $C_{j,r}$ for the CHIC scheme](image)

The new Riemann invariants, equivalent to those defined by Equation (6) are

$$\begin{align*}
p_{j,r,1} &= p_j + \alpha_j (\mathbf{e}_{j,r,1} \cdot (\mathbf{u}_j - \mathbf{u}_r)) \\
p_{j,r,2} &= p_j + \alpha_j (\mathbf{e}_{j,r,2} \cdot (\mathbf{u}_j - \mathbf{u}_r)),
\end{align*}$$

(29)

where subscripts 1 and 2 account for the corresponding basis vectors. The multidimensional Riemann problem is then solved using the same conservation argument as beforehand, i.e.,

$$\sum_{j \in B_r} C_{j,r,1} p_{j,r,1} + C_{j,r,2} p_{j,r,2} = 0,$$

(30)

where $C_{j,r,1} = (C_{j,r} \cdot \mathbf{e}_{j,r,1}) \mathbf{e}_{j,r,1}$ and $C_{j,r,2} = (C_{j,r} \cdot \mathbf{e}_{j,r,2}) \mathbf{e}_{j,r,2}$.

This leads to a slightly modified definition of matrix $A_r$ and vector $b_r$ with respect to Equation (9):

$$\begin{align*}
A_r &= \sum_{j \in B_r} \alpha_j \left( \frac{C_{j,r,1} \otimes C_{j,r,1}}{|C_{j,r,1}|} + \frac{C_{j,r,2} \otimes C_{j,r,2}}{|C_{j,r,2}|} \right), \\
b_r &= \sum_{j \in B_r} C_{j,r} p_j + \sum_{j \in B_r} \alpha_j \left( \frac{C_{j,r,1} \otimes C_{j,r,1}}{|C_{j,r,1}|} + \frac{C_{j,r,2} \otimes C_{j,r,2}}{|C_{j,r,2}|} \right) \mathbf{u}_j.
\end{align*}$$

(31)
Then $u_r = A_r^{-1}$ is computed as for the GLACE scheme, $p_{j,r,1}$ and $p_{j,r,2}$ are deduced from Equation (29), and momentum and energy are updated using these pressures and velocity:

$$
\begin{align*}
M_j u'_j(t) &= -\sum_{r \in B_j} (C_{j,r,1} p_{j,r,1} + C_{j,r,2} p_{j,r,2}) \\
M_j e'_j(t) &= -\sum_{r \in B_j} ((C_{j,r,1} \cdot u_r) p_{j,r,1} + (C_{j,r,2} \cdot u_r) p_{j,r,2})
\end{align*}
$$

(32)

Whereas formally close, the two schemes have very different behaviors. The GLACE scheme exhibits a low diffusion but in counterpart hourglass modes. The scheme CHIC is very robust but diffusive. This can be seen in the construction of matrix $A_r$. Let recast Equations (9) and (31) in the following form:

$$
A_r = \sum_j A_{j,r}, \quad \text{and} \quad b_r = \sum_j C_{j,r} p_j + \sum_j A_{j,r} u_j,
$$

(33)

where $A_{j,r}$ is:

$$
\begin{align*}
A_{j,r} &= \alpha_j \frac{C_{j,r} \otimes C_{j,r}}{|C_{j,r}|} \quad \text{for the GLACE scheme,} \\
A_{j,r} &= \alpha_j \left( \frac{C_{j,r,1} \otimes C_{j,r,1}}{|C_{j,r,1}|} + \frac{C_{j,r,2} \otimes C_{j,r,2}}{|C_{j,r,2}|} \right) \quad \text{for the CHIC scheme.}
\end{align*}
$$

(34)

Whatever the scheme, matrix $A_r$ is symmetric and positive. It is also always definite for the CHIC scheme, but not for the GLACE scheme. It is easy to check that in this case, the subspace generated by the normal vectors to $C_{j,r}$ is the null space of $A_{j,r}$. It means that, for a given point with index $r$, if you split the cell velocity $u_j$ into $u_{j,r}^{pf}$ and $u_{j,r}^{rn}$, respectively the parallel vector to $C_{j,r}$ and the normal vector to $C_{j,r}$, only the first one is locally seen by the solver. Consequently, no entropy is generated by the second term, but unphysical modes can occur.

In the following, we use this analysis to enforce symmetry preservation, but a lot of other applications are possible. We have noticed that the scheme obtained using definitions (23) or (24) of the exceptional point does not exactly respect symmetry. This is related to the non-symmetry of the calculation stencil, involving a local node (for instance $A$ for the cell with index 1 on Figure 9) and a distant node (for instance the node $E$ for the same cell). To prove it, let us consider the discrete momentum conservation law in the cell indexed by $1$:

$$
M_1 u_1^{n+1} = M_1 u_1^n - \Delta t \sum_{r \in B_1} D_{1,r} p_{1,r}.
$$

(35)

We assume the velocity field and the mesh to be symmetric with respect to the radial direction at time $t^n$. We call $e_{j,r,1}$ and $e_{j,r,2}$ the radial and orthoradial vectors in the center $x_{gj}$ of the cell indexed by $j$. We consider now the cell with index 1 (the subscript 1 is omitted in
Figure 9: Symmetry study

the following since only one cell is needed for this demonstration). The geometric symmetry implies:

\[
\begin{align*}
((D_A + D_E) \cdot e_{r,2}) &= 0 \\
((D_C + D_D) \cdot e_{r,2}) &= 0.
\end{align*}
\]

(36)

**Remark** Equation (36) does not hold in when the exceptional point is defined as located in the middle of a segment (see Figure 4). Indeed, in this case, vectors \(BC\) and \(AD\) are not symmetric with respect to the radial direction. The flow symmetry is then immediately broken.

Noting \(p_A, p_E, p_C, p_D\) the pressures at the corresponding points, we can demonstrate that

\[
\begin{align*}
p_A &= p_E \\
p_C &= p_D
\end{align*}
\]

is a sufficient condition to have \(u^{n+1}\) in the radial direction (using (35) and (36)). It is also a necessary condition (consider for instance all cell-centered velocities surrounding the nodes A and E to be zero at time \(t^n\) and apply (35)).

Nodal pressures are calculated using Equation (6). By symmetry considerations, \(p_C = p_D\). Unfortunately \(p_A \neq p_E\). At these locations, the Riemann solver gives:

\[
\begin{align*}
p_A &= p - \alpha((u_A - u) \cdot n_A) \\
p_E &= p - \alpha((u_E - u) \cdot n_E).
\end{align*}
\]

(38)

In the sequel, we call \(e_A = x_A/||x_A||, e_E = x_E/||x_E||\) and \(e_1 = x_g/||x_g||\), where the subscript \(g\) accounts for the center of the cell. Radial symmetry implies \(u_A = ||u_A||e_A, u_E = ||u_E||e_E\) and \(u = ||u||e_1\) are in the radial direction, and \(||u_A|| = ||u_E||\). But since \(n_A\) and \(n_E\) have not a symmetrical definition (see Equation (27)), \(p_A \neq p_E\).

To solve this problem, we choose to split vectors \(D_{j,r}\) in the local basis \((e_{j,r,1}, e_{j,r,2})\) with \(e_{j,r,1}\) and \(e_{j,r,2}\) the radial and orthonormal vectors in the center \(x_{gj}\) of the cell with index \(j\). We also replace \(\alpha_j\) by \(\alpha_{j,r,1}\) and \(\alpha_{j,r,2}\) into Equation (29). This trick has been already used in [8] for other purposes, and only slightly modifies the diffusivity of the scheme. Then,
Proposition 6.1. Taking

\[
\begin{align*}
\alpha_{j,r,1} &= \rho_j c_j \\
\alpha_{j,r,2} &= \frac{\rho_j c_j}{(e_r \cdot e_{j,r,2})},
\end{align*}
\]  

with \( e_r = x_r/||x_r|| \), gives the symmetry of the scheme for cylindrical flows.

Proof. We consider the previous example illustrated by Figure 9. Dropping index 1 and splitting vectors \( D_r \) into the basis \( e_{r_1} \) and \( e_{r_2} \), Equation (35) recasts:

\[
M u^{n+1} = M u^n - \Delta t \sum_{r \in \mathcal{B}} (D_{r,1} p_{r,1} + D_{r,2} p_{r,2}).
\]  

(40)

Since \( D_{r,1} \) is radial, we have only to ensure \( p_{A,2} = p_{E,2} \). These pressures are defined by:

\[
\begin{align*}
p_{A,2} &= p - \alpha_{A,2}((u_A - u) \cdot e_{A,2}) \\
p_{E,2} &= p - \alpha_{E,2}((u_E - u) \cdot e_{E,2}).
\end{align*}
\]  

(41)

With our choice, \( e_{r,2} \) does not depend on \( r \), and \( e_{A,2} = e_{E,2} = e_2 \).

Symmetry at time \( n \) implies: \( (u \cdot e_2) = 0 \), \( u_A = ||u_A|| e_A \), \( u_E = ||u_E|| e_E \). Then \( p_{A,2} = p_{E,2} \) is equivalent to:

\[
\alpha_{A,2}(e_A \cdot e_2) = \alpha_{E,2}(e_E \cdot e_2).
\]  

(42)

Consequently, \( \alpha_{A,2} = \frac{\rho c}{e_A \cdot e_2} \) and \( \alpha_{E,2} = \frac{\rho c}{e_E \cdot e_2} \) give the symmetry of the flow. \( \square \)

Remark The values of \( \alpha_{j,r,1} \) and \( \alpha_{j,r,2} \) in (39) are not unique. This is the ratio between \( \alpha_{j,r,1} \) and \( \alpha_{j,r,2} \) which is important. These values can then be multiplied by a constant.

Proposition 6.1 can be generalized in the following way:

Proposition 6.2. If the flow and the mesh have an homogeneous direction defined by a normalized vector field \( h(x) \), the symmetry in this direction can be enforced in decomposing vectors \( D_{j,r} \) for each cell of index \( j \) into the basis \( (h(x_{gj}), h^\perp(x_{gj})) \), with \( x_{gj} \) the cell center coordinates. Coefficients \( \alpha_{j,r} \) are given by Equation (39), with \( e_r = h(x_r) \), and \( e_{j,r,2} = h^\perp(x_{gj}) \).

Proof. The arguments are exactly the same as for the previous proof. \( \square \)

Remark Since mesh symmetry is mandatory for Proposition 6.2, it is only verified for the definition of the exceptional point on an arc of a circle, but not in the middle of a segment. This is illustrated on Figure 24.

Using a cylindrical basis to decompose vectors \( D_{j,r} \) can break the Galilean invariance. To avoid this, the basis can be chosen and fixed at the beginning of the simulation. Another interesting alternative is to consider the directions given by the local pressure gradient into each cell. This choice can be related to the construction of tensorial viscosities (see for instance [5, 10]). Since this basis is closely related to the entropy deposit directions, it is a way to optimize the scheme diffusion.

In the sequel, we always use one of these decompositions of \( D_{j,r} \) to preserve the flow symmetry.
7. Numerical results

In this Section, we present several implementations issues and some numerical results for different test cases:

- the Sod shock tube in 1D, in 2D with different definitions of exceptional points on meshes with straight and curved edges,
- the Kidder isentropic compression of a shell without and with perturbation,
- the Sedov shock wave propagation, and,
- the Noh shock tube problem.

For all these results, we have used the $D_{j,r}$ vectors and the variant of GLACE scheme presented in the Section 6.

7.1. Implementation issues

7.1.1. GMDS

To run all these simulations, we have used the mesh data structure GMDS [19] which provides a good flexibility during the implementation stage. The mesh data structure GMDS provides a generic framework allowing the developer to instantiate the type of meshes he needs to work with. More precisely, a mesh data structure can be defined by the type of cells it handles (nodes, edges, faces, regions) and the type of connections it stores. For instance, only the knowledge of nodes, faces and the connection from faces to nodes is sufficient for some algorithms like geometric smoothing. For our studies, we use the GMDS mesh model $\text{DIM2|N|F|N2F|F2N}$. It means that we use a 2D mesh where nodes and faces are stored and each face knows its adjacent nodes, and reciprocally, each node knows its adjacent faces. Using such a mesh data structure allows us to reuse existing algorithms and will be beneficial to extend our implementation to the 3D case and to deal with huge meshes in memory distributed context.

7.1.2. Calculation meshes

The computation of vectors $D_{j,r}$ requires to get data that is not local to a cell. Indeed, it is necessary to take into account some points that are not in the same cell. To avoid tricky connectivity computation during the algorithm process, we do not work on the geometric mesh directly, we work on a calculation mesh where each cell is defined using the points involved in the $D_{j,r}$ computation. The calculation meshes we use are described below.

- Case where the exceptional point is in the middle of a segment

In the previous calculations, we have seen that any exceptional point depends on two free points. For instance, using notations of Figure 10, point $N$ is expressed using $E$ and $F$. If we consider the calculations in the cell with index $j_1$, point $F$ has a contribution in the expression of vectors $D_{j_1,r}$ but this point does not belong to the geometric
cell indexed by $j_1$ (i.e. the quadrilateral cell DENG). Numerically, the data structure GMDS allows us to access to the points in a cell, but as the point F is not in the cell $j_1$, we use a calculation mesh, which fixes this connectivity. Here again, using notations of Figure 10, we can see that the cell with index $j_1$ becomes in the calculation mesh DGFE. In the same way, the cell with index $j_2$ becomes EFGC. Let us note that some cells overlap in the calculation mesh. Numerically, geometric vectors $D_{j,r}$, velocities and pressures are computed on the calculation mesh. In Figure 10, we have represented the new cells of the calculation mesh in blue and green. Cell $j_3$ stays unchanged, i.e. EFBA.

- Case where the exceptional point is on an arc of a circle
  - Meshes with straight edges

This case is quite different from the precedent. The initial mesh contains pentagons (for the cells underneath the exceptional points) as the exceptional point is not in an existent segment. But point M (using notations of Figure 12) is not a free point and B is not in the cell indexed by $j_1$. Thus, we have to change this initial mesh into a new computational mesh with only quadrilaterals.
– Meshes with curved edges

![Calculation mesh when the exceptional point M is defined on an arc of the circle passing throw A and B and with curved mesh](image)

Figure 12: Calculation mesh when the exceptional point M is defined on an arc of the circle passing throw A and B and with curved mesh

Even if exceptional points are defined on arcs of circles, this case is very close to the first one (with exceptional points in the middle of segments). Topology of the geometric mesh is the same (no pentagon), but some edges are defined as arcs of circles.

7.2. The 1D Sod test case

Before starting with the 2D case, we considered the very classical 1D Sod shock problem [36]. It involves a perfect gas with an adiabatic constant $\gamma = 1.4$. The computational domain is $[0, 1] \times [0, 0.1]$.

![Configuration of the 1D Sod test case](image)

Figure 13: Configuration of the 1D Sod test case

Initial conditions, representend on Figure 13, are:

- $p_j = 1$, $\rho_j = 1$ and $\mathbf{u}_j = 0$ for all the points $\mathbf{x}(x, y)$ such that $x \leq 0.5$,
- $p_j = 0.1$, $\rho_j = 0.125$ and $\mathbf{u}_j = 0$ for all the points $\mathbf{x}(x, y)$ such that $x > 0.5$.

Boundary conditions are:

- $\mathbf{u}_r = 0$ for the points $\mathbf{x}(x, y)$ in the axes $x = 0$ and $x = 1$, 
• \( \mathbf{u} \cdot \mathbf{n} = 0 \) for the points \( \mathbf{x}(x,y) \) in the axes \( y = 0 \) and \( y = 0.1 \), where \( \mathbf{n} \) is the external normal to the considered boundary.

We consider three types of meshes with exceptional points:

• a mesh \( \mathcal{M}_1 \) with one line of exceptional points at \( x = 0.5 \),

• a mesh \( \mathcal{M}_2 \) with two lines of exceptional points at \( x = 0.5 \) and \( x = 0.75 \),

• a mesh \( \mathcal{M}_3 \) with two lines of exceptional points in "stairs" (cf. Figure 14 third line on the left).

![Figure 14: 1D Sod test case: initial (left) and final (right) meshes containing exceptional points](image)

On Figure 14 are represented initial meshes (\( \mathcal{M}_1 \), \( \mathcal{M}_2 \) and \( \mathcal{M}_3 \)) and final meshes obtained at time \( t = 0.2 \). We can see that our method is stable whatever the type of mesh is, and that the final shock line is at the same place for the three meshes.

7.3. The 2D Sod test case

We are now considering the 2D Sod test case [36]. The computational domain is a quarter of the disc centered in \((0,0)\) and with a radius equal to 1.

Initial conditions are:

• \( p_j = 1, \rho_j = 1 \) and \( \mathbf{u}_j = 0 \), for cells which have their mass center with a radius \( R > 0.5 \),

• \( p_j = 0.1, \rho_j = 0.125 \) and \( \mathbf{u}_j = 0 \), for cells where \( R < 0.5 \).

Boundary conditions are:

• \( \mathbf{u} \cdot \mathbf{n} = 0 \) for points 1, 2 and 3 and for the points in the arc of the circle centered in \((0,0)\) and with radius equal to 1,

• \( \mathbf{u} \cdot \mathbf{n} = 0 \) for the points on the axes \( x = 0 \) and \( y = 0 \), where \( \mathbf{n} \) is the external normal to the considered boundary.

In the numerical experiments reported in the following, we use three different meshes to test the sensitivity of the numerical solution with respect the presence of the exceptional points:
Figure 15: Configuration of the 2D Sod test case

- a mesh $M_1$ containing exceptional points, in the middle of segments or on an arc of circle, with 40 layers, 10 sectors under the exceptional points, 20 sectors from the exceptional points on,
- a mesh $M_2$ without exceptional point and with 40 layers and 10 sectors,
- a mesh $M_3$ without exceptional point and with 40 layers and 20 sectors.

These meshes have the same number of layers (40) so the numerical solution is under resolved. Our goal is to test the influence of the exceptional points. One clearly sees on Figure 16 that the three numerical solutions are almost undistinguishable, which indicates the stability of the calculation with the exceptional points.

7.3.1. Exceptional points in the middle of segments

We consider exceptional points defined using the function $\phi$ defined as:

$$\phi(X, Y) = \frac{1}{2} I_d X + \frac{1}{2} I_d Y = \frac{1}{2} X + \frac{1}{2} Y.$$ 

On the left of Figure 16, we can see the final mesh obtained with exceptional points in the middle of segments, at time $t = 0.2$ as usually for this test case. On the right of Figure 16, we have the density profiles for meshes $M_1$, $M_2$ and $M_3$ at final time. For $M_1$, we obtain the final situation after 117 time steps, for $M_2$ after 117 time steps too and for $M_3$ after 225 iterations. We can see that we obtain the same results as for the case without exceptional points and 20 sectors and for a smaller number of iterations. We do not change then the quality of the solution, while reducing the number of degrees of freedom and improving the CFL condition. Comparing to mesh containing 10 sectors, we can conclude that we obtain the solution in the same time but with more degrees of freedom, then our approach is more accurate. This shows the pertinence of using exceptional points.
7.3.2. Exceptional points on arcs of circles

In the 2D polar case, in order to have a good accuracy and to preserve the sphericity of the problem, it seems better to consider that the exceptional points are in the arc of a circle. We recall that we define these points using the function $\phi$ such that:

$$\phi(X,Y) = P_1 X + P_2 Y.$$

Like previously, for Figure 17 (bottom), we use meshes $M_1$, $M_2$ and $M_3$ to perform simulations. The final situation is obtained in 116 iterations for $M_1$. As previously, we get the same solution as for the mesh $M_3$ without exceptional point. We do not change the quality of the solution but we reduce the number of degrees of freedom and we improve the CFL condition.
7.3.3. Exceptional points on an arc of circle and curved meshes

We consider here the case of curved meshes (as presented in Figure 7). The exceptional point location is given using:

$$\phi(X, Y) = P_1 X + P_2 Y.$$ 

On Figure 18, are represented the final mesh and the density profiles for meshes $M_1$, $M_2$ and $M_3$.

![Figure 18: 2D Sod test case: final mesh and density profiles for curved mesh with exceptional points](image)

We see that we obtain the same results with or without exceptional points. Like previously, we do not change the quality of the solution.

7.3.4. Mesh containing several layers of exceptional points

We also performed calculation using meshes where:

- several layers of exceptional points are defined in the initial mesh,
- exceptional points are located onto arcs of circle.

On Figure 19, are represented the final mesh and the density profile obtained at final time $t = 0.2$, compared to the density profile obtained with a mesh without exceptional point and 10 sectors. The results were obtained in 48 iterations.

![Figure 19: Final mesh and density profile with exceptional points](image)

7.3.5. Two exceptional points on each arc of circle

In this subsection, we treat the case with two exceptional points on each arc of circle. We consider cells defined as on Figure 20, where $M_1$ and $M_2$ are defined as:

$$M_1 = P_1 A + P_2 B \quad \text{and} \quad M_2 = P_3 A + P_4 B,$$
where the matrices $P_1$, $P_2$, $P_3$ and $P_4$ are defined in the Appendix A such that:

$$P_1 + P_2 = P_3 + P_4 = I_2.$$ 

Note that expressions (A.1) and (A.2) can be extended to handle more exceptional points.

We have performed the same calculations as previously and we represent on Figure 21 the mesh and the density profile obtained at final time.

We remark one artefact on the density profile corresponding to the layer of exceptional points. This is due to the numerical diffusion which is different in cells containing exceptional points. These effects should be attenuated using an order 2 scheme.

7.3.6. Comparison of time steps

To show the relevance of using exceptional points in order to reduce the restrictive CFL conditions, we compare the evolution of the time steps $\Delta t$ for three meshes having the same number of layers: without exceptional points, with one layer of exceptional points and with
six layers of exceptional points. On Figure 22, we see that adding one layer of exceptional points increases the time step by 2.

7.3.7. Translation invariance

With the definition of exceptional points using matrices $P_1$ and $P_2$ and vectors $C_{j,r \text{curved}}$, we have the translation invariance. We performed the same calculations as previously, imposing for all cells with index $j$, $u_j = (1, 0)$, for a curved mesh. On Figure 23, we can see the initial (in blue) and final (in red) meshes. Note that the geometry of the mesh is preserved.

On Figure 23, we can observe that the density profile obtained at final time is the same as taking $u_j = 0$. After proving the invariance principle theoretically, we prove it here numerically with these results.
7.3.8. Symmetry preservation

In this part, we compare the errors of alignment with respect to the arc of the circle for the layer containing the exceptional points. This error is computed as: \( \delta R = R_{\text{max}} - R_{\text{min}} \), where \( R_{\text{max}} \) (resp. \( R_{\text{min}} \)) is the maximal (resp. minimal) radius computed for all the points of the layer. On Figure 24, we compare, on the left, the different methods to define exceptional points (points in the middle, points on an arc of a circle and points on an arc of a circle and curved mesh). On the right of this Figure, we compare these errors for meshes without exceptional points, with one exceptional point on each arc of circle and with two exceptional points on each arc.

We see that it is better to locate the exceptional points on the arc of a circle rather than in the middle of a segment. The Figure 24 shows the non symmetry preservation when exceptional points are in the middles of segments, as we explained in Section 6, and shows that
the number of exceptional points on each arc does not affect the sphericity of the problem.

In order to show an example where the symmetry preservation is difficult but essential, we consider a curved mesh, without exceptional points, where all sectors do not have the same angular difference, as in Figure 25 (left). We performed calculations for the Sod test case on this mesh using the variant of the GLACE scheme explained in Section 6.

![Figure 25: 2D Sod test case and symmetry preservation: mesh in which all sectors do not have the same angular difference and density profiles at final time](image)

On Figure 25 (right), we represent the density profile at final time and we compare this curve to the density profile obtained in Figure 18. We observe that the solution is very similar to the one obtained in the equisectorial case. Moreover, we compare on Figure 26 (left), the errors of alignment with respect to the arc of circle in the 2 cases:

- the classical GLACE scheme,
- the variant of the GLACE scheme described in Section 6.

Using the variant presented in Section 6 clearly provides better results than the classical GLACE scheme. Nevertheless, errors obtained are not so good but if we want a better result, we have to change the nodal solver and then we will loose the translation invariance.

On Figure 26, we compare the errors of alignment with respect to the arc of circle obtained on a mesh with straight edges to the ones obtained on a mesh with curved edges. As expected, we have smaller errors in the curved case.

7.4. The 2D Kidder test case

This test case is not really adapted to show the utility of exceptional points but, since there exists an analytical solution it allows to demonstrate the stability and robustness of
Figure 26: 2D Sod test case and symmetry preservation: comparison of errors of alignment with respect to the arc of circle for scheme with and without variant of Section 6 (left) and between curved and straight meshes.

In [18], Kidder highlighted the analytical solution to an isentropic compression of a shell filled with a perfect gas. We recall the characteristics of this solution to define our test case. Dimensions of the shell are $r_1 = 0.9 \leq r \leq 1 = r_2$, where $r$ is the radius of each node in the shell and $r_1$ (resp. $r_2$) is the radius of the internal (resp. external) interface. The adiabatic constant of the gas is $\gamma = 2$, [18].

Figure 27: Configuration of the Kidder test case

Initial conditions at $t = 0$ are:
\[ \rho_0(r) = \left( \frac{r^2 - r_2^2}{r_2^2 - r_1^2} \rho_1^{\gamma - 1} + \frac{r^2 - r_1^2}{r_2^2 - r_1^2} \rho_2^{\gamma - 1} \right)^{\frac{1}{\gamma - 1}}, \text{ with } \rho_1 = 10^{-3} \text{ and } \rho_2 = 10^{-2}, \]

\[ p_0(r) = s(\rho_0(r)), \text{ the isentropic character of the compression gives us } s = \frac{P_2}{\rho_2^{\gamma}}, \text{ where } P_2 \text{ is an imposed pressure value}, \]

\[ u_j = 0. \]

Let \( R(r, t) \) be the Eulerian radius at time \( t > 0 \) of a fluid particle initially located at radius \( r \). Boundary conditions are:

\[ \text{• sliding walls on the lateral faces,} \]

\[ \text{• a given exterior pressure at the internal frontier } p(R(r_1, t)) = P_1 h(t)^{\frac{2\gamma}{\gamma - 1}}, \]

\[ \text{• a given exterior pressure at the external frontier } p(R(r_2, t)) = P_2 h(t)^{\frac{2\gamma}{\gamma - 1}}, \]

where \( R(r, t) = rh(t) \), \( h(t) \) is the homothety rate \( h(t) = \sqrt{1 - \frac{t^2}{\tau^2}} \), with \( \tau \) the focalization time: \( \tau = \sqrt{\frac{(\gamma - 1)(r_2^2 - r_1^2)}{2(c_1^2 - c_2^2)}} \) for \( c \) the sound speed.

The analytical solution is then, for \( t \in [0, \tau] \):

\[
\begin{cases}
\rho(R(r, t), t) = h(t)^{\frac{\gamma}{\gamma - 1}} \rho_0 \left( \frac{R(r, t)}{h(t)} \right) \\
u(R(r, t), t) = \frac{dh(t)}{dt} \frac{R(r, t)}{h(t)} \\
p(R(r, t), t) = h(t)^{\frac{2\gamma}{\gamma - 1}} p_0 \left( \frac{R(r, t)}{h(t)} \right).
\end{cases}
\]

For this test case, we have used \( P_1 = 0.1, P_2 = 10, s = 10^5 \) and \( \tau = 7.265 \times 10^{-3} \). The time step is computed as in the previous Sod test case.

The implementation of boundary conditions is detailed again in [8].

7.4.1. Exceptional points on arcs of circles

We present here some results for five different meshes:

- mesh \( \mathcal{M}_1 \): (5 layers and 5 sectors) under the exceptional points and (5 layers and 10 sectors) from the exceptional points on,
- mesh \( \mathcal{M}_2 \): (10 layers and 10 sectors) and (10 layers and 20 sectors),
- mesh \( \mathcal{M}_3 \): (20 layers and 20 sectors) and (20 layers and 40 sectors),

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Figure 28: Kidder test case: mesh $\mathcal{M}_1$ at initial and final stages (left), zoom around exceptional points in an arc of a circle for mesh $\mathcal{M}_1$ with straight edges (center), and pressure at initial and final time steps (right)

- mesh $\mathcal{M}_4$: (40 layers and 40 sectors) and (40 layers and 80 sectors),
- mesh $\mathcal{M}_5$: (80 layers and 80 sectors) and (80 layers and 160 sectors).

On Figure 28, we present initial and final meshes (on the left), a zoom on the mesh around exceptional points (on the center), and the value of $p_j$ at times $t = 0$ and $t = 0.85\tau$ for mesh $\mathcal{M}_1$ (on the right). We can remark that, since the exceptional point are on arc of circle, the geometrical mesh contains pentagons.

Figure 29: Kidder test case: convergence curve for density (left) and evolution of the internal and external radius (right) in the case where the exceptional point is on an arc of a circle

As the Kidder test case has an analytical solution, on Figure 29, we represent the curve of the $L^2$-error for the density in function of the number of layers for meshes $\mathcal{M}_1, \cdots, \mathcal{M}_5$. This error is calculated as:

$$err = \frac{\sqrt{\sum_j (\rho_j - \rho_{ref})^2}}{N},$$
where $N$ is the number of cells and $\rho_{\text{ref}} = h(t)^{-2} \rho_0(r)$, with $r$ is the initial radius. We compare this $L^2$-error curve to the function $f(x) = \frac{1}{x}$, and conclude that we get the order 1. On Figure 29, are represented the evolution of the external and the internal radius, compared to the analytical value $R_1 = 0.9 h(t)$ and $R_2 = h(t)$.

7.4.2. Exceptional points on arcs of circles and curved meshes

We present here some results in the case where exceptional points are on an arc of a circle and where the meshes are curved. Meshes $M_1, \cdots, M_5$ are still used but using curved edges.

On Figure 30, is represented the curve of the $L^2$-error for the density in function of the number of layers for meshes $M_1, \cdots, M_5$. We compare this curve to the function $f(x) = \frac{1}{x}$ to obtain again the order 1. Moreover, we can note that error values are slightly inferior (but not that much) in this case than in the case of mesh with straight edges. Here, the numerical error is dominating on the geometric error.

7.5. Perturbation of the Kidder isentropic compression

In this subsection, we study the capability of the scheme in the hydrodynamics instabilities domain. We consider then the implosion of a Kidder’s shell where we perturb the external interface (for the radius $r_2$) and the internal interface (for the radius $r_1$).

We start by defining the perturbation of the initial mesh. The expression of the cartesian coordinates of the points in the perturbed mesh is:

$$
\begin{align*}
x^p &= \left(1 + \frac{a_0}{r} \xi_r\right) \left(x \cos(a_0 \xi_\theta) - y \sin(a_0 \xi_\theta)\right) \\
y^p &= \left(1 + \frac{d_0}{r} \xi_r\right) \left(x \sin(a_0 \xi_\theta) + y \cos(a_0 \xi_\theta)\right),
\end{align*}
$$

Figure 30: Kidder test case: convergence curve for density (left) and evolution of the internal and external radius (right) in the case where the exceptional points are on an arc of a circle and where the mesh is curved
where \((x, y)\) are the initial cartesian coordinates of the considered point, \((r, \theta)\) are the initial polar coordinates of the point, \(a_0\) is the initial amplitude of the perturbation and \(\xi = \begin{pmatrix} \xi_r \\ \xi_\theta \end{pmatrix}\) is the displacement vector of the perturbation. This vector \(\xi\) is defined as:

\[
\begin{align*}
\xi_r(r, \theta) &= n (A_1 r^{-n-1} + A_2 r^{n-1}) \cos(n\theta) \\
\xi_\theta(r, \theta) &= -n (A_1 r^{-n-1} + A_2 r^{n-1}) \sin(n\theta),
\end{align*}
\]

where \(n\) is the perturbation mode and \(A_1\) and \(A_2\) are some constants which can be determined using the data \(\xi_1 = \xi_r(r_1, 0)\) and \(\xi_2 = \xi_r(r_2, 0)\):

\[
\begin{align*}
A_1 &= \frac{1}{n} \frac{\xi_1 r_1^{n-1} - \xi_2 r_1^{n-1}}{r_1^{n-1} r_2^{n-1} - r_1^{n-1} r_2^{n-1}} \\
A_2 &= -\frac{1}{n} \frac{\xi_1 r_2^{n-1} - \xi_2 r_2^{n-1}}{r_1^{n-1} r_2^{n-1} - r_1^{n-1} r_2^{n-1}}.
\end{align*}
\]

Figure 32 gives examples of:

- mesh without perturbation (on left),
- mesh with a perturbed external interface (\(\xi_1 = 0\) and \(\xi_2 = 1\)), with an initial amplitude \(a_0 = 0.02\), and with \(n = 8\) (on center),
- mesh with perturbed external and internal interfaces (\(\xi_1 = 1\) and \(\xi_2 = 1\)), with an initial amplitude \(a_0 = 0.02\), and with \(n = 8\) (on right).

The calculations were conducted with the same initial and boundary conditions as in the non perturbed case. To show the efficiency of our method, we compare the analytical amplification given by (43) with the numerical amplification given by (44).
In [3], it is shown that the amplification \( a(r, \theta, t) \) of the perturbation, at the external interface, is given using the formula:

\[
a(r, \theta, t) = n(A_1 r^{-n^{-1}} G_1(t) + A_2 r^{n^{-1}} G_2(t)) \cos(n\theta),
\]

where \( t \) is the time normalized by the focalization time \( \tau \) and

\[
\begin{align*}
G_1(t) &= \sqrt{1-t^2} \cos \left( \frac{\sqrt{n}}{2} \ln \left( \frac{1-t}{1+t} \right) \right), \\
G_2(t) &= \frac{1}{2} \sqrt{1-t^2} \left( \left( \frac{1-t}{1+t} \right)^{\frac{\sqrt{n}}{2}} + \left( \frac{1+t}{1-t} \right)^{\frac{\sqrt{n}}{2}} \right).
\end{align*}
\]

The results presented here after are obtained with the mode \( n = 4 \), and with a perturbation of the external interface (\( \xi_1 = 0 \) and \( \xi_2 = 1 \)). We start by conducting the calculation of the non perturbed implosion in order to get the location of the point \( R(r_2, t) \), in the external interface with \( \theta = 0 \). Then, we initialize the pressure, the velocity and the density values corresponding to the solution onto the initial non perturbed grid, we perturb the mesh and we conduct the calculation of the perturbed implosion to have the position \( R^p(r_2, t) \) of the same point as in the non perturbed case. The numerical amplification is then given by:

\[
a_{\text{num}}(r_2, t) = \frac{R^p(r_2, t) - R(r_2, t)}{a_0}.
\]

For these calculations, we used \( a_0 = 10^{-6} \). To compare (43) and (44), we run the calculations on the following four meshes:

- a mesh \( \mathbb{M}_1 \) with (12 layers and 22 sectors) below exceptional points, and (13 layers and 44 sectors) above,
• a mesh $\mathbb{M}_2$ with (25 layers and 44 sectors) below exceptional points, and (25 layers and 88 sectors) above,

• a mesh $\mathbb{M}_3$ with (50 layers and 88 sectors) below exceptional points, and (50 layers and 176 sectors) above,

• a mesh $\mathbb{M}_4$ with (100 layers and 176 sectors) below exceptional points, and (100 layers and 352 sectors) above.

On Figure 33, we represent the analytical amplification (43) and the numerical amplification (44) obtained onto the four previous meshes $\mathbb{M}_i$ ($i = 1, \cdots, 4$). We used a logarithmic scale i.e. $\log \left( \frac{a(r,t)}{a_0} \right)$, in function of the time $t$ normalized by the focalization time $\tau$. We can see the convergence of the algorithm to the analytical solution.

7.6. The Sedov test case

This test case is a propagation of a highly powerful cylindrical shock wave produced by a large explosion, that is by the release of an important energy quantity in a restricted volume. Boundary conditions are $\mathbf{u}_r \cdot \mathbf{n} = 0$. We consider a perfect gas with adiabatic constant $\gamma = \frac{5}{3}$, with initial conditions $\rho = 1, p_j = 0, \mathbf{u}_j = 0$ and with an energy peak near the center of a polar mesh.

We conducted the calculations on three curved meshes:

• a mesh $\mathbb{M}_1$ with (10 layers and 12 sectors) below the exceptional points, and (40 layers and 24 sectors) above,

• a mesh $\mathbb{M}_2$ with (20 layers and 25 sectors) below the exceptional points, and (80 layers and 50 sectors) above,
• a mesh $M_3$ with (40 layers and 50 sectors) below the exceptional points, and (160 layers and 100 sectors) above.

For mesh $M_1$, we initialize the specific internal energy $\varepsilon$ to $5000 \frac{4}{\pi}$ in the central cells, i.e., in the triangles of the area $(r, \theta) \in \left[0, \frac{1}{50}\right] \times \left[0, \frac{\pi}{2}\right]$. For other meshes, the same initialization is performed but for several layers of cells.

On Figure 35, we represent the final mesh (at $t = 0.1$) for $M_1$ and the density profiles for the three meshes.
As for the Sod test case, we conducted these calculations on the mesh of Figure 19 with 6 layers of exceptional points. On Figure 36, we represent the final mesh and the density profile compared to the case where the mesh has straight edges.

![Figure 36: Sedov test case and comparison between straight and curved edges: final mesh and density profiles for a mesh with 6 layers of exceptional points](image)

Once again, we can observe that the curved approach gives better results than the straight one. On the density profiles of the straight case, we can remark some artifacts corresponding to each layer of exceptional points and due to the numerical diffusion, which is different in cells containing exceptional points. We remark that these effects is attenuated using curved meshes. We obtain good results with the chock in $r = 0.6$ as in analytical solution.

7.7. The Noh test case

In this subsection, we consider the Noh shock tube problem [28]. The cylinder is fulfilled by a perfect gaz ($\gamma = \frac{5}{3}$). Initial conditions are: $\rho_j = 1, p_j = 0$ and $u_j = -e_r$ where $e_r$ is the unit inward radial vector at each cell. A circular shock wave is generated which propagates at speed $D = \frac{1}{3}$.

We conducted the calculations on three meshes:

- a mesh $M_1$ with (25 layers and 5 sectors) below the exceptional points, and (25 layers and 10 sectors) above,
- a mesh $M_2$ with (50 layers and 10 sectors) below the exceptional points, and (50 layers and 20 sectors) above,
- a mesh $M_3$ with (100 layers and 20 sectors) below the exceptional points, and (100 layers and 40 sectors) above.

On Figure 38, we represent the final mesh (on the left) for the $M_1$ case and the density profiles obtained at final time $t = 0.6$. 

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On the density profiles, we can remark artifacts corresponding to the layer of exceptional points. This is due here again to the numerical diffusion, which is different in cells containing exceptional points. These effects should be attenuated using an order 2 scheme. Nevertheless, we obtain good results for a one order scheme.

On Figure 39, we compared results obtained for meshes with straight edges to results obtained for meshes with curved edges. We remark that we have a better solution in the case of curved meshes. The artefact due to the exceptional points is reduced.
8. Conclusion and perspectives

In this paper, we proposed a new formulation of exceptional points which relies on a matricial expression. The definition of the matrices is compatible with the general principle of the translation invariance, which is a basic property of the fluid mechanics equations. This matricial combination gives us a large range of possibilities for locating exceptional points. We focus in this work on two definitions:

- exceptional points in the middle of the edge (classical definition), and
- exceptional points on the arc of circle (for polar meshes), which is demonstrated to give the best results.

This formalism extends to conformal and non conformal curved meshes. Natural extension of the GLACE scheme is also proposed and is used to ensure the perfect symmetry preservation. Some 1D, cylindrical 1D and 2D test cases show the numerical enhancements provided by this approach. In particular, our numerical results show the pertinence of using exceptional points which allow to reduce the number of degrees of freedom and improve the CFL condition without changing the solution quality. Moreover, for curved meshes, we show in this paper that the geometrical error is lower than for straight meshes.

In the future, we will propose a second order version of this scheme. The method should also be easily extended to 3D. In particular, the matricial formulation can be used to define exceptional points on portions of sphere in 3D. Dynamical mesh adaptation using this formulation could also be an interesting field of future researches.

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Appendix A. Expression of the matrices $P_1$, $P_2$, $P_3$ and $P_4$ in the case of two exceptional points on each arc of circle

We recall that

$$M_1 = P_1 A + P_2 B \quad \text{and} \quad M_2 = P_3 A + P_4 B.$$ 

Using the same type of calculations than for the case with only one exceptional point on each arc of circle, we found:

$$P_1 = \begin{pmatrix}
1 - \frac{(2-\alpha) \sin \frac{\theta}{2}}{2 \sin \frac{\theta}{2}} & \frac{\alpha}{2 \sin \frac{\theta}{2}} \\
-\alpha \frac{\cos \frac{\theta}{2}}{2 \sin \frac{\theta}{2}} & 1 - \frac{(2-\alpha) \sin \frac{\theta}{2}}{2 \sin \frac{\theta}{2}}
\end{pmatrix} \quad \text{and} \quad P_2 = \begin{pmatrix}
\frac{(2-\alpha) \sin \frac{\theta}{2}}{2 \sin \frac{\theta}{2}} & -\alpha \frac{\cos \frac{\theta}{2}}{2 \sin \frac{\theta}{2}} \\
\frac{\alpha}{2 \sin \frac{\theta}{2}} & \frac{(2-\alpha) \sin \frac{\theta}{2}}{2 \sin \frac{\theta}{2}}
\end{pmatrix}, \quad (A.1)$$

$$P_3 = \begin{pmatrix}
\frac{(2-\alpha) \sin \frac{\theta}{2}}{2 \sin \frac{\theta}{2}} & \frac{\alpha}{2 \sin \frac{\theta}{2}} \\
-\alpha \frac{\cos \frac{\theta}{2}}{2 \sin \frac{\theta}{2}} & \frac{(2-\alpha) \sin \frac{\theta}{2}}{2 \sin \frac{\theta}{2}}
\end{pmatrix} \quad \text{and} \quad P_4 = \begin{pmatrix}
1 - \frac{(2-\alpha) \sin \frac{\theta}{2}}{2 \sin \frac{\theta}{2}} & -\alpha \frac{\cos \frac{\theta}{2}}{2 \sin \frac{\theta}{2}} \\
\frac{\alpha}{2 \sin \frac{\theta}{2}} & 1 - \frac{(2-\alpha) \sin \frac{\theta}{2}}{2 \sin \frac{\theta}{2}}
\end{pmatrix}, \quad (A.2)$$

where $\alpha = 1 - \cos \frac{\theta}{2} \sqrt{1 + \tan^2 \frac{\theta}{2}}$.

We can notice that $P_1 + P_2 = P_3 + P_4 = I_2$.

References


