On the Dirichlet Boundary Controllability of the 1D-Heat Equation. Semi-Analytical Calculations and Ill-posedness Degree.

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December 23, 2010

Abstract

The ill-posedness degree for the controlability of the one-dimensional heat equation by a Dirichlet boundary control is the purpose of this work. This problem is severely (or exponentially) ill-posed. We intend to shed more light on this assertion and the underlying mathematics. We start by discussing the framework liable to fit an efficient numerical implementation without introducing further complications in the theoretical analysis. We expose afterward the Fourier calculations that transform the ill-posedness issue to a one related to the linear algebra. This consists in investigating the singular values of some infinite structured matrices that are obtained as sums of Cauchy matrices. Calling for the Gershgorin-Hadamard theorem and the Collatz-Wielandt formula, we are able to provide lower and upper bounds for the largest singular value of these matrices. After checking out that they are also solutions of some symmetric Lyapunov (or Sylvester) equations with a very low displacement rank, we use an estimate that improves former Penzl's result to bound the ratio smaller/largest singular values of these matrices. Accordingly, the controllability problem is confirmed to be severely ill-posed. The bounds proved here will be supported by computations made by means of Matlab procedures. At last, the well known explicit inverse of Cauchy's type matrices allows to provide a formal exponential series representation of the Dirichlet control in a long horizon controllability. That series has to be checked afterward whether it is convergent or not to find out whether the desired state is reachable or not. Here again, some examples run within Matlab will be discussed and commented.

KEYWORDS: Controllability, HUM control, Dirichlet control, Cauchy matrices, Löewner matrices, Lyapunov equation, Penzl's type bounds, ill-posedness degree.

1 Introduction

Let I be the segment $(0, \pi)$ of the real axis. Let T be a positive real number. We set $Q = I\times]0, T[$. The generic point in I is denoted by x and the generic time is t. The one dimensional controllability problem we deal with reads as follows. Let y_T be a given function that represents a desired state we hope to reach at a time T. We aim at: *finding a Dirichlet data u enforced at $x = \pi$ satisfying*

$$y_u(x, T) = y_T(x), \qquad \forall x \in I, \tag{1}$$

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where the symbol y_v denotes the solution of the heat equation with a Dirichlet boundary condition:

$$\partial_t y_v - \partial_{xx}^2 y_v = 0 \qquad \text{in } Q, \tag{2}$$

$$y_v(0,\cdot) = 0, \qquad y_v(\pi,\cdot) = v \qquad \text{on } (0,T), \tag{3}$$

$$y_v(\cdot,0) = 0 \qquad \text{on } I. \tag{4}$$

The fact that the initial condition is fixed to zero is not really a restriction. It is adopted only for simplification and the linear superposition principle restore that generality.

Knowledge on the controllability of the heat equation is substantially less achieved than for the wave equation. The difficulty arisen in the heat equation is caused by the dissipation which is mainly irreversible in opposition to the propagation which is a reversible phenomenon. Nevertheless, some theoretical results are already stated. May be, the main reference on the topic together with the popular HUM method is the book by J.-L. Lions [15]. Some others are necessary surveys for who are involved in the subject, such as [16, 12, 33] without being exhaustive. This problem is not exactly controllable. Some states y_T are not reachable. As a matter of fact, an arbitrary y_T should be considered as non-reachable. Admittedly, the problem is ill-posed in any reasonable 'Sobolev' framework. In contrast, controls do exist that steer the trajectory arbitrarily close to any given state y_T after a duration T. We have therefore approximate controllability (see [15, 24, 33]). Notice that whenever, y_T comes from a solution of the heat equation, the controllability is equivalent to the null-controllability that is when the initial condition is given, $y_v(\cdot,0) = y_0$ and the final state is $y_T = 0$. This last pertains, in the practice, to the stabilization of dynamical systems. Results on the null-controllability are stated by H. O. Fattorini and D. Russel (see [6]) and also by G. Lebeau and L. Robbiano (see [13]), in one dimension for the firsts and in the general dimension for the seconds. The major result established in there is the observability estimate that yields the well-posedness of the null-controllability problem. The numerical counterpart of this problem has been considered by S. Micu and E. Zuazua (see [19]) where they show the severe ill-posedness of the problems they have to cope with computationally. Some issues treated in [19] are similar to those we are involved in here. The methodology is however very different. Their approach, mainly based on the analysis of some Bi-Orthogonal sequences, is the one proposed earlier in [6] and surveyed in [24]. Here, as will be detailed later on, we rather take profit of the spectral estimates for some known structured matrices such as Cauchy and Löwner type matrices to establish our results.

As indicated and discussed in [2] for the optimal control problem, the point is to consider a subspace of admissible Dirichlet controls that brings facilities in the computations. That should be a subspace of $L^2(0,T)$ ([1]) that allows to define the final observation $y_v(\cdot,T) \in L^2(I)$. It can not coincide with the full Lebesgue space $L^2(0,T)$, since the final observation $y_v(\cdot,T)$ may not belong to $L^2(I)$ for some Dirichlet data in $L^2(0,T)$ (see [15, Chap. 9]). We describe the framework we use that turns out to be advantageous when a regularization strategy is aimed which is mandatory to handle safely the computations. Next, comes the main concern related to the incorrect-posedness of the controllability problem by Dirichlet controls. To state its severe ill-posedness according to

[1] $L^2(0,T)$ is the Lebesgue space of square integrable functions.

the classification provided in [32], we perform here semi-analytical computations that transfer the difficulty to the study of the spectrum of some Cauchy type matrices. Connecting them to Lyapunov equations, and using the important results on the spectrum of their solution, we are able to obtain the desired ill-posedness results. A predicted effect of it is that when a reachable y_T is perturbed by noise, the control we compute may be drastically changed, it may even blow up exponentially fast. Additionally, owing to the explicit inverse of Cauchy matrices we are enabled to provide the exact formal expression of the control in the case of infinite horizon controllability which demonstrates, if needed, the exponential instability of the controllability problem. Notice that the algebraic tools on which the analysis conducted here relies, are mainly developed by the control community.

An outline of the paper is as follows. We describe in Section 2 the variational formulation of the heat equation (2)-(4) derived by duality. This approach enables to account for the lack of smoothness of Dirichlet's data (see [12]). As a consequence, we obtain a mathematical setting to the controllability problem in the natural space $H^{-1}(I)$ ([2]). Engineers prefer the $L^2(I)$-setting. Despite the practicability and effectiveness of that framework from the numerical implementation view, particular caution should be paid to the analysis. In Section 3, we provide and discuss the suitable tools to successfully deal with that issue. We put then the controllability problem under an abstract form. The HUM method fails and the problem is ill-posed. This is the main subject of Section 4. Using Fourier calculations, we check out how the controllability problem may be put under an integral equation defined by a kernel operator. The ill-posedness degree of this equation is therefore tightly connected to asymptotics of the singular values of the related integral operator. This enables us to construct an infinite countable symmetric and positive definite matrix of a Cauchy type whose eigenvalues are exactly the squares of the singular values of the operator under investigation. Their distribution along the positive semi-axis is obtained after writing this matrix as the solution of a Lyapunov equation with a low rank displacement. Then, using estimates established by many authors (see [25, 9] and wide bibliographies therein) produces the desired information on the behavior on the singular values. Their asymptotics establish the severe ill-posedness of the controllability problem. Some Matlab computations allow to check numerically these theoretical predictions. Owing to the inversion formula of Cauchy matrices, we provide in Section 5 an infinite sum representation of the Dirichlet control that solves the controllability problem for a given target y_T and we check why it mostly blows up exponentially fast. Lastly, Section 6 is dedicated to the discussion of some numerical examples to check out the theoretical predictions.

Notation — Let X be a Banach space endowed with its norm $\|\cdot\|_X$. We denote by $L^2(0,T;X)$ the space of measurable functions v from $(0,T)$ in X such that

$$\|v\|_{L^2(0,T;X)} = \Big(\int_0^T \|v(\cdot,s)\|_X^2 \, ds\Big)^{1/2} < +\infty. \tag{5}$$

[2]The dual space of $H_0^1(I)$, the Sobolev space that contains all functions that belongs to $L^2(I)$ so as their derivatives and vanish at $x = 0$ and π.

For any positive integer m, we introduce the space $H^m(0,T;X)$ of functions in $L^2(0,T;X)$ such that all their time derivatives up to the order m belong to $L^2(0,T;X)$. We also use the space $\mathscr{C}(0,T;X)$ of continuous functions v from $[0,T]$ in X. Finally, we consider the Sobolev space $H^1(I)$ of all the functions that belong to $L^2(I)$ together with their first derivatives. The space $H_0^1(I)$ defined in footnote (2), is then the closure in $H^1(I)$ of the space $\mathscr{D}(I)$ of infinitely differentiable functions with a compact support in I, and by $H^{-1}(I)$ its dual space.

2 Controllability in $H^{-1}(I)$

Let us first recall the variational formulation of the parabolic system (2)-(4). As indicated in the introduction, our willing is to consider controls in the Lebesgue space $L^2(0,T)$. Because of the lack of regularity on the Dirichlet datum v, the variational form is proceeded by a duality argument. We follow the transposition method detailed in [15, Th. 9.1]. Let then f be arbitrarily given in $L^2(Q)$. Consider r_f as the solution of the backward heat equation

$$\begin{aligned} -\partial_t r_f - \partial^2_{xx} r_f &= f \qquad && \text{in } Q,\\ r_f(0,\cdot)=0, \qquad r_f(\pi,\cdot) &= 0 && \text{on } \partial(0,T),\\ r_f(\cdot,T) &= 0 && \text{in } I. \end{aligned}$$

The solution r_f does exist and is unique in $L^2(0,T;H_0^1(I))\cap\mathscr{C}(0,T;L^2(I))$ (see [17, Chap. 4, Th. 1.1]). Provided that the boundary is regular enough which is assumed from now on, the normal derivative $\partial_x r_f(\pi,\cdot)$ belongs to $L^2(0,T)$ and is subjected to the stability

$$\|\partial_x r_f(\pi,\cdot)\|_{L^2(0,T)} \le C\,\|f\|_{L^2(Q)}. \tag{6}$$

Applying the transposition method to problem (2)-(4), we come up with the following variational equation: *find y_v in $L^2(Q)$ such that*

$$\int_Q y_v(x,t)f(x,t)\,dx\,dt = -\int_{(0,T)} v(t)\big(\partial_x r_f\big)(\pi,t)\ dt, \qquad \forall f\in L^2(Q). \tag{7}$$

The equivalence between (7) and problem (2)-(4) has been checked for instance in [2, Proposition 2.2]. The existence and uniqueness for y_v are direct consequences of (6) and Riesz' Theorem (see [12, 14]). Then, for any v in $L^2(0,T)$, problem (7) has a unique solution y_v in $L^2(Q)$ which belongs to $\mathscr{C}([0,T];H^{-1}(I))$.

Now, we are in position to rewrite problem (1) in a rigorous mathematical words before adding some modifications to the functional setting for practical reasons that will be commented later on. The controllability problem consists then in: *finding $u\in L^2(0,T)$ such that*

$$y_u(\cdot,T) = y_T(\cdot), \qquad \text{in } H^{-1}(I). \tag{8}$$

When the data y_T suffers from some perturbations, existence for problem (8) may fail. Noisy data are unavoidable, this may be caused by erroneous measurement or preliminary processes to prepare the data before feeding any computational program. More, even when y_T is an ideal mathematical fixed state, when switching to computation, the numerical treatment of that y_T generates with certainty some disturbances. The consequence is that in simulations users can not spare using a regularization strategy necessary for obtaining pertinent results. (see [29, 5]). In crude computations, the noise generates oscillations on the approximated control. Their intensity depends on the bad conditioning of the discrete controllability problem. The indicator to quantify how it is badly conditioned is the ill-posedness degree that has been introduced by G. Wahba (see [32]). This is the main subject here. We intend to demonstrate the severe ill-posedness of the controllability problem.

The choice of the framework where to proceed requires some comments at this stage. Most users interested in prefer to consider an $L^2(I)$ setting rather than $H^{-1}(I)$ which seems to be the natural space. Why? The answer is strongly related to the numerical implementation and then to practical engineering issues. As a matter of fact, computing the L^2-norm of $(y_v(\cdot,T))$, necessary when regularization is required, is an easy matter while the evaluation of H^{-1}-norm arises some difficulties practitioners prefer to avoid. Moreover, a comprehensive analysis of the regularization strategy may also be achieved in the modified framework. This is the content of the forthcoming work together with the study of the effect of the penalization of the Dirichlet control by a Fourier boundary condition.

3 Controllability in $L^2(I)$

The key of the analysis in the $L^2(I)$–framework is a Green formula written for non-regular functions. It has been stated in [2]. Nevertheless, to be self contained and in order to introduce some technical tools related to, we need to provide that result.

Let y_v be the solution of problem (7). We have already mentioned that $y_v(\cdot,T)$ fails to belong to $L^2(I)$ for some v. We define then the unbounded operator B in $L^2(0,T)$,

$$Bv = y_v(\cdot,T).$$

The domain $\mathbb{D}(B)$ is hence composed of functions $v \in L^2(0,T)$ for which $y_v(\cdot,T)$ lies in $L^2(I)$,

$$\mathbb{D}(B) = \Big\{ v \in L^2(0,T); \quad y_v(\cdot,T) \in L^2(I) \Big\}.$$

Following [2, Lemma 2.4] it can be checked that B is closed.

Lemma 3.1 *The operator B is closed. As a consequence, $\mathbb{D}(B)$ is dense in $L^2(0,T)$ and the graph norm*

$$\|v\|_{\mathbb{D}(B)} = \Big(\|v\|^2_{L^2(0,T)} + \|y_v(\cdot,T)\|^2_{L^2(I)} \Big)^{1/2},$$

determines a Hilbert structure on $\mathbb{D}(B)$.

Proof: Let $(v_m, Bv_m)_m$ be a convergent sequence in $L^2(0,T) \times L^2(I)$ and (v,ψ) be the limit. That y_{v_m} belongs to $\mathscr{C}(0,T;H^{-1}(I))$ implies that $y_{v_m}(\cdot,T)$ converges towards $y_v(\cdot,T)$ in $H^{-1}(I)$.

Now, since $L^2(I)$ is continuously embedded in $H^{-1}(I)$ we derive that $\psi = y_v(\cdot, T)$. Hence, not onlu $y_v(\cdot, T) \in L^2(I)$ but also $Bv = \psi$. The proof is complete. ∎

A further result is that the adjoint operator B^* is well defined with a dense domain $\mathbb{D}(B^*)$ in $L^2(I)$. It is specified by the identity

$$(Bv, \psi)_{L^2(I)} = (v, B^*\psi)_{L^2(0,T)}, \qquad \forall (v, \psi) \in \mathbb{D}(B) \times \mathbb{D}(B^*). \tag{9}$$

A closed form of it may be provided (see for instance [12, 2]). Let ψ be given in $L^2(I)$ and denote by q_ψ the unique solution of the problem

$$-\partial_t q_\psi - \partial_{xx}^2 q_\psi = 0 \qquad \text{in } Q, \tag{10}$$

$$q_\psi(0,\cdot), \qquad q_\psi(\pi,\cdot) = 0 \qquad \text{on } (0,T), \tag{11}$$

$$q_\psi(\cdot,T) = \psi \qquad \text{on } I. \tag{12}$$

The function q_ψ does exit in $L^2(0,T; H^1(I)) \cap \mathscr{C}(0,T; L^2(I))$ (see [17, Chap. 4]). The following result is stated for instance in [2, Lemma 2.5]. Nevertheless, we choose to check it again to be self-contained. Besides, the proof is really short.

Lemma 3.2 *The domain of B^* is given by*

$$\mathbb{D}(B^*) = \Big\{\psi \in L^2(I); \quad \partial_x q_\psi(\pi,\cdot) \in L^2(0,T)\Big\},$$

and B^* is defined as follows

$$B^*\psi = -\partial_x q_\psi(\pi,\cdot).$$

Proof: For a given function v in $\mathscr{D}(0,T)$, multiply (10) by y_v and integrate by part on Q. This is legitimate since we dispose of enough smoothness. We derive that

$$(Bv, \psi)_{L^2(I)} = \int_I y_v(x,T)\psi \, dx = \langle v, (-\partial_x q_\psi(\pi,\cdot))\rangle.$$

Using (9) and a density argument yields that $B^*\psi = -\partial_x q_\psi(\pi,\cdot)$ in $\mathscr{D}'(0,T)$. In view of the fact that $B^*\psi$ lies in $L^2(0,T)$, we deduce that $\partial_x q_\psi(\pi,\cdot)$ belongs also to $L^2(0,T)$. This concludes the proof. ∎

Remark 3.1 *The space $\mathbb{D}(B^*)$ is a Hilbert space when endowed with the graph norm*

$$\|\psi\|_{\mathbb{D}(B^*)} = \Big(\|\psi\|^2_{L^2(I)} + \|\partial_x q_\psi(\pi,\cdot)\|^2_{L^2(0,T)}\Big)^{1/2}, \qquad \forall \psi \in \mathbb{D}(B^*).$$

Given that B^* is injective, the mapping $\psi \mapsto \partial_x q_\psi(\pi,\cdot)$ is a norm in $\mathbb{D}(B^*)$. The Hilbert space $\mathcal{H}$ obtained by the completion of $\mathbb{D}(B^*)$ with respect to this norm can not contain $L^2(I)$. Otherwise, B^* would be bounded. However, as already noticed in [19], *the regularity of $(\partial_x q_\psi(\pi,\cdot))$ is not affected by the* (non-)*smoothness of ψ away from $x = \pi$. As a result, $\mathcal{H}$ may contain non-regular final states ψ provided that their singularities or high oscillations are located in $[0,\pi[$.*

Remark 3.2 *The invertibility of B^* pertains to the Cauchy problem for the backward heat equation where the final condition is to be reconstructed from over-specified Cauchy boundary conditions at $x = \pi$. This is known in the literature as the sideways problem. That problem has at most one solution by the unique continuation theorem* [26]*. This means that B^* is injective and therefore*

$$Ker\ B^* = \{0\}. \tag{13}$$

Homogeneous Cauchy data on $\{\pi\} \times (0,T)$ for the problem satisfied by q_ψ yields that $q_\psi = 0$ and thus $\psi = 0$. As a consequence, the range $\mathcal{R}(B)$ is dense in $L^2(I)$. The approximate controllability holds (see [12, 33])*. Moreover, it will be checked that it has a non-closed range and can not be continuously invertible.*

A Green formula is the by-product of the construction of B and B^*. We aim a general version for it. We then assume given ψ in $\mathbb{D}(B^*)$ and f in $L^2(Q)$. We introduce the solution $q_{(\psi,f)}$ in $L^2(0,T;H_0^1(I)) \cap \mathscr{C}(0,T;L^2(I))$ of the problem

$$\begin{aligned} -\partial_t q_{(\psi,f)} - \partial_{xx} q_{(\psi,f)} = f \qquad & \text{in } Q,\\ q_{(\psi,f)}(0,\cdot) = 0, \qquad q_{(\psi,f)}(\pi,\cdot) = 0 \qquad & \text{on } (0,T),\\ q_{(\psi,f)}(\cdot,T) = \psi \qquad & \text{on } I. \end{aligned}$$

We have the following result

Lemma 3.3 *Let f be in $L^2(Q)$, v in $\mathbb{D}(B)$ and ψ in $\mathbb{D}(B^*)$. The following Green formula holds*

$$\int_Q f(x,t) y_v(x,t)\,dxdt + \int_I \psi(x) y_v(x,T)\,dx + \int_{(0,T)} v(t)\,\partial_x q_{(\psi,f)}(\pi,t)\,dt = 0.$$

Proof: It may be found in [2, Proposition 2.5 and Corollary 2.6]. ∎

That the operator B is defined in the L^2-framework, the controllability problem (8) can be reworded as an equation involving B. Given a data $y_T \in L^2(I)$. The problem consists in: *finding $u \in \mathbb{D}(B)$ such that*

$$Bu = y_T, \qquad \text{in } L^2(I). \tag{14}$$

This is the very problem we deal with in the subsequent, unless explicitly contradicted. Whenever we talk about the controllability problem we mean this one and not (8) (in $H^{-1}(I)$).

It is worth to notice that the range subspace $\mathcal{R}(B)$ is dense in $L^2(I)$ as made Remark 3.2. However, it will be more instructive to know that the complementary of $\mathcal{R}(B)$, composed of the non-reachable states y_T, is also dense in $L^2(I)$. This means that any perturbation of any reachable state $y_T = y_v(\cdot,T)$, for some v, may produce a non-reachable perturbed final state $\tilde{y}_T \notin \mathcal{R}(B)$. This result is important for the analysis of regularization strategies which are mandatory for the computational treatment of the problem given its severe ill-posedness.

Lemma 3.4 *The set of non-reachable states $L^2(I) \setminus \mathcal{R}(B)$ is dense in $L^2(I)$.*

Proof: Let us proceed by contradiction. Assume then that $L^2(I) \setminus \mathcal{R}(B)$ is not dense in $L^2(I)$. Hence, $\mathcal{R}(B)$ has a non-empty interior. As it is a vector subspace, there is no other possibility than it coincides with the entire space $L^2(I)$. The operator B will be therefore surjective. Now, suppose that $\mathbb{D}(B)$ is endowed with the graph norm $\|\cdot\|_{\mathbb{D}(B)}$ provided in Lemma 3.1, it is a Hilbert space. Consider hence B as an operator mapping $\mathbb{D}(B)$ in $L^2(I)$. It is obviously injective and continuous. In addition, it is surjective and then bijective. Owing to the open map theorem it defines an isomorphism. In particular, we obtain that

$$\|v\|_{L^2(0,T)} \leq C\|Bv\|_{L^2(I)}, \qquad \forall v \in \mathbb{D}(B).$$

As a result, B^{-1} will be a bounded operator from $L^2(I)$ into $L^2(0,T)$ which is necessarily false. $L^2(I) \setminus \mathcal{R}(B)$ is then dense in $L^2(I)$. The proof is complete. ∎

Remark 3.3 *As an alternative to* (14)*, duality for the convex optimization suggests to look at the minimization problem*

$$\min_{\chi\in\mathbb{D}(B^*)} J^*(\chi) = \min_{\chi\in\mathbb{D}(B^*)} \frac{1}{2}\|\partial_x q_\chi\|^2_{L^2(0,T)} - (y_T, \chi)_{L^2(I)}.$$

Whenever the minimum is reached, let us say for $\psi \in \mathbb{D}(B^*)$, using Green's formula of Lemma 3.3 *we state that the control $u^\dagger = B^*\psi = -\partial_x q_\psi$. It is the HUM control of problem* (14)*. As well known, and will be checked again below, the controllability problem has an infinite number of solutions. The solution $u^\dagger$, issued from the duality is the one, among all the possible solutions, that has the minimum norm in $L^2(0,T)$.*

Remark 3.4 *Users may be tempted to apply HUM theory to the controllability problem* (8)*, set in $H^{-1}(I)$. Standard Green's formula* ([3]) *results in the reduced problem that reads as follows:* find $u \in L^2(0,T)$ such that

$$\int_{(0,T)} u(t)\, \partial_x q_\psi(\pi, t)\, dt = \langle y_T, \psi\rangle_{H^{-1},H^1_0}, \qquad \forall \psi \in H^1_0(I).$$

The symbol q_ψ is for the solution of (10)-(12)*. The well posedness requires the observability estimate that is*

$$\|\psi\|_{H^1(I)} \leq C\|\partial_x q_\psi(\pi,\cdot)\|_{L^2(0,T)}, \qquad \forall \psi \in H^1_0(I).$$

Unfortunately, this does not hold. Switching to the space $L^2(I)$ in (8)*, instead of $H^{-1}(I)$, does not change that much the situation. Indeed, although we need some weaker observability inequality*

$$\|\psi\|_{L^2(I)} \leq C\|\partial_x q_\psi(\pi,\cdot)\|_{L^2(0,T)}, \qquad \forall \psi \in \mathbb{D}(B^*),$$

this is false too. The best inequality of similar kind we know of so far is a Carleman estimate established by M. Klibanov in [11, 2006].

[3]Green's formula in Lemma 3.3 serves when the controllability equation is written in $L^2(I)$ and the control is sought for in $\mathbb{D}(B)$.

4 The Severe Ill-posedness

Needless to say that the ill-posedness of problem (14) is governed by the properties of the operator B. On account of its unboundedness, the degree of ill-posedness of the controllability problem, in the classification of G. Wahba (see [32]), is inferred from the speed at which the ratio (larger/lower) singular values of B blows up to infinity. We mainly pursue the behavior of the singular values of B. The chief tool of our study is the Fourier analysis and the spectra of some structured matrices.

4.1 B and B^* as Kernel Operators

Since B and B^* have the same non-vanishing singular values we choose to focus rather on B^* and then infer the expression of B by duality. After all, the equation defined by B^* is interesting in itself since it is the mathematical modeling of the inverse problem, well known as the sideways equation, that consists in reconstructing the initial data from the knowledge of the Cauchy boundary conditions (see [1]).
Notice that the construction of B^* is based on the backward heat equation. For easiness, especially when the Laplace transform is used, we prefer to handle the direct heat equation. All the computations and constructions are also valid modulo some slight modifications for the backward heat problem. In particular the structured matrices we have to deal with are not affected by progressive or backward problems. We consider then the problem

$$\partial_t q - \partial^2_{xx} q = 0 \qquad \text{in } I \times (0,T), \tag{15}$$

$$q(0,\cdot) = 0, \qquad q(\pi,\cdot) = 0 \qquad \text{on } (0,T), \tag{16}$$

$$q(\cdot,0) = \psi \qquad \text{on } I. \tag{17}$$

The 'adjoint' operator B^* is still expressed as follows

$$B^*\psi = -\partial_x q(\pi,t), \qquad \text{in } (0,T).$$

Remark 4.1 *In this context, the controllability problem related to* (15)-(17) *is therefore governed by the backward heat equation as that state equation*

$$\begin{aligned} -\partial_t y_v - \partial^2_{xx} y_v &= 0 \qquad && \text{in } Q,\\ y_v(0,\cdot) = 0, \qquad y_v(\pi,\cdot) &= v && \text{on } (0,T),\\ y_v(\cdot,T) &= 0 && \text{on } I. \end{aligned}$$

The control is the Dirichlet boundary condition at point π, still denoted u, that realizes $y(\cdot,0) = y_0$, where $y_0 \in L^2(I)$ is the desired state. Of course, the overall results we state hereafter are valid as well for problem (14).

The methodology followed here has been used earlier. We refer for instance to [6, 22, 19, 33]. Given that $(\sin(kx))_{k\geq 1}$ is an orthogonal basis in $L^2(I)$, we may expand ψ, as follows

$$\psi(x) = \sum_{k\geq 1} \psi_k \sin(kx) \qquad \text{in } I. \tag{18}$$

Plug it into problem (15)-(17) and all computations conducted, B^* may put under a kernel operator form

$$(B^*\psi)(t) = \sum_{k\geq 1}(-1)^{k+1}ke^{-k^2t}\psi_k = \int_I K(t,x)\psi(x)\,dx.$$

The kernel $K(\cdot,\cdot)$ is defined to be

$$K(t,x) = \frac{2}{\pi}\sum_{k\geq 1}(-1)^{k+1}ke^{-k^2t}\sin(kx), \qquad \text{in } I\times(0,T).$$

It can be checked directly that

$$\mathbb{D}(B^*) = \Big\{\psi\in L^2(I); \qquad \sum_{k\geq 1}\sum_{m\geq 1}(-1)^{k+m}\frac{km}{k^2+m^2}(1-e^{-(k^2+m^2)T})\psi_k\psi_m<\infty\Big\},$$

and that $\text{Ker } B^* = \{0\}$. Things are a little bit more subtle when we are involved in the range $\mathcal{R}(B^*)$ which is an important issue upon which depends the (non)uniqueness for the controllability problem (14). In fact, it allows for the determination of $\text{Ker } B$. In view of the expression of $B^*\psi$ the sequence of functions $(e^{-k^2t})_{k\geq 1}$ is total in $\mathcal{R}(B^*)$. Since the $(k^{-2})_{k\geq 1}$ is summable, then according to a variant of the Müntz theorem (see [28]) it comes out that $(e^{-k^2t})_{k\geq 1}$ can not be dense in $L^2(0,T)$, neither is $\mathcal{R}(B^*)$ in $L^2(0,T)$.

Remark 4.2 *The same Müntz theorem enables one to check that the co-dimension of $\mathcal{R}(B^*)$ in $L^2(0,T)$ is infinite. Actually, a more accurate theorem by Borwein-Erdélyi in* [3, Theorem 6.4] *implies that any function v in the closure $\overline{\mathcal{R}(B^*)}$ is analytic in $]0,T]$ and admits the following representation*

$$v(t) = \sum_{k\geq 1}v_ke^{-k^2t}, \qquad \forall t\in]0,T].$$

That v belongs to $L^2(0,T)$ yields the condition

$$\sum_{k\geq 1}\sum_{m\geq 1}\frac{1-e^{-(k^2+m^2)T}}{k^2+m^2}v_kv_m<\infty.$$

Be aware that these results are concerned with the controllability of the backward heat equation in Remark 4.1. *When the progressive equation is considered $u^\dagger$ will be analytic on $[0,T[$ except probably at $t=T$. The important observation to retain here is that any reachable state y_T may be realized by an analytic control on $[0,T[$. Recall that the reachable states y_T are all indefinitely smooth in $[0,\pi[$. They may contain a singularity at $x=\pi$. That singularity contaminates the 'minimal' control only at the final time $t=T$. At last, notice that this holds true for any initial condition $y_0\in L^2(I)$ and not only when $y=0$. A suitable superposition principle applied to the problem allows such a generalization.*

Now, let $v\in\mathbb{D}(B)$ be given and denote by $v_{(T)}$ its trivial extension to the semi-axis $\mathbb{R}_+$. The operator B can also be expressed as a kernel operator that is

$$(Bv)(x) = \int_{\mathbb{R}_+}K(t,x)v_{(T)}(t)\,dt = \frac{2}{\pi}\sum_{k\geq 1}(-1)^{k+1}\left[k\widehat{v_{(T)}}(k^2)\right]\sin(kx), \qquad \text{in } I. \tag{19}$$

The symbol $\widehat{v_{(T)}}$ is used for the Laplace transform of the extended function $v_{(T)} \in L^2(\mathbb{R}_+)$. We deduce immediately an explicit characterization of $\mathbb{D}(B)$ that is

$$\mathbb{D}(B) = \Big\{ v \in L^2(0,T); \qquad \sum_{k\geq 1} \big[k\widehat{v_{(T)}}(k^2) \big]^2 < \infty \Big\}.$$

Moreover the kernel of B is specified as follows

$$\text{Ker } B = \Big\{ v \in \mathbb{D}(B); \qquad \widehat{v_{(T)}}(k^2) = 0, \quad \forall k \geq 1 \Big\}.$$

Remark 4.3 *To have a deeper insight about what really happens, let us consider the particular case $T = \infty$. One can construct a collection of functions $(v_a)_{a>1}$ that belong to Ker B. Indeed, it can be checked that the function*

$$v_a(t) = \sqrt{\pi}\ t^{-3/2} \exp(-\frac{(a-1)\pi^2}{4t}) \Big[\sin(\frac{\pi^2\sqrt{a}}{2t}) + \sqrt{a}\cos(\frac{\pi^2\sqrt{a}}{2t}) \Big]$$

lies in $L^2(\mathbb{R}_+)$ and its Laplace transform coincides with (see [21, example 7.13, p. 279])

$$\widehat{v}_a(p) = 2\exp(-\pi\sqrt{ap}) \sin(\pi\sqrt{p})$$

Then we have that $v_a \in$ Ker B for any $a > 1$ and the dimension of Ker B *is thus infinity.*

4.2 Infinite Matrices of BB^* and BB^*

Now, the computation of the singular values of B^* requires the specification of the kernel function, we denote by $G(\cdot,\cdot)$, of the operator BB^*. Recall that BB^* is the HUM operator. The HUM control, whenever its existence is ensured, is given by $u^\dagger = B^*(BB^*)^{-1}y_T$, for a reachable state y_T. A closed form of the kernel $G(\cdot,\cdot)$ is provided by

$$\begin{aligned} G(x,y) &= \int_{(0,T)} K(t,x)K(t,y)\ dt \\ &= \frac{4}{\pi^2} \sum_{k\geq 1}\sum_{m\geq 1} (-1)^{k+m} \frac{km}{k^2+m^2}(1-e^{-(k^2+m^2)T}) \sin(kx)\sin(my), \qquad x,y \in I. \end{aligned}$$

Let then ψ be given in $\mathbb{D}(BB^*)$. Using the expansion in (18) there holds that

$$\begin{aligned} (BB^*\psi)(x) &= \int_I G(x,y)\psi(y)\ dy \\ &= \frac{2}{\pi} \sum_{k\geq 1}\sum_{m\geq 1} (-1)^{k+m} \frac{km}{k^2+m^2}(1-e^{-(k^2+m^2)T})\psi_m \sin(kx), \qquad \forall x \in I. \end{aligned}$$

The eigenvalues of the operator BB^* are exactly those of the infinite-dimensional matrix $\mathcal{B}_\infty = (b_{km})_{k,m\geq 1}$ where the entries a provided by

$$b_{km} = \frac{2}{\pi}(-1)^{k+m} \frac{km}{k^2+m^2}(1-e^{-(k^2+m^2)T}), \qquad \forall k,m \geq 1. \tag{20}$$

The matrix $\mathcal{B}_\infty$ may be viewed as an unbounded operator in $\ell^2(\mathbb{R})$(4) which is a representation of BB^* on the Fourier basis. It is symmetric and non-negative definite.

[4] $\ell^2(\mathbb{R})$ is the space the numerical sequences that are square summable.

Remark 4.4 *Conducting similar computations for the operator B^*B is of course possible. It may be defined by the kernel operator*

$$H(t,\tau) = \sum_{k\geq 1} k^2 e^{-k^2(t+\tau)}, \qquad \forall t,\tau \in (0,T).$$

In addition, the eigenvectors should be sought for in $\mathcal{R}(B^*)$ and yield to the spectral decomposition of the matrix $(\mathcal{B}_\infty)' = (b'_{km})_{k,m\geq 1}$ where

$$b'_{km} = \frac{2}{\pi}\frac{k^2}{k^2+m^2}(1 - e^{(k^2+m^2)T}), \qquad \forall k,m \geq 1.$$

It can be checked that $\mathcal{B}_\infty$ and $(\mathcal{B}_\infty)'$ are equivalent infinite-dimensional matrices. Indeed, there hold that

$$(\mathcal{B}_\infty)' = \mathcal{Q}\mathcal{B}_\infty\mathcal{Q}^{-1}.$$

The matrix $\mathcal{Q}$ is diagonal with $q_{kk} = (-1)^k k$. Of course, a particular caution should be paid to the domain in $\ell^2(\mathbb{R})$ of each of them.

4.3 Asymptotics of Singular Values. Severe Ill-posedness

The subsequent is to exhibit asymptotics of the eigenvalues of the infinite-dimensional Gram matrix $\mathcal{B}_\infty = (b_{km})_{k,m\geq 1}$. From now to the end the index ∞ will be dropped off which does not modify at all the results we have in mind. The entries of $\mathcal{B}$ are re-transcribed as follows

$$b_{km} = (-1)^{k+m}\frac{km}{k^2+m^2}(1 - e^{-(k^2+m^2)T}), \qquad \forall k,m \geq 1.$$

The constant term $2/\pi$ involved in the expression of b_{km} given in (20) is removed. This has no real incidence on the analysis we undertake here. Define the new matrix $\mathcal{Z} = (z_{km})_{k,m\geq 1}$ where

$$z_{km} = \frac{km}{k^2+m^2}(1 - e^{-(k^2+m^2)T}), \qquad \forall k,m \geq 1.$$

Obviously $\mathcal{B}$ and $\mathcal{Z}$ are equivalent and have the same spectrum. Studying $\mathcal{Z}$ (actually the truncated form of it) is advantageous because it has positive entries and the Perron-Frobenius theory allows to obtain further informations about the largest eigenvalue (of the truncated $\mathcal{Z}$ introduced here below) called the Perron-Frobenius root (see [30]). The matrix $\mathcal{B}$ and consequently $\mathcal{Z}$ inherit the properties of the operator BB^* and are therefore both non-negative definite. As a result, their eigenvalues which are common are all non-negative. The matrix $\mathcal{Z}$ may be written as the difference of two Cauchy matrices. The entries of those two Cauchy matrices are respectively

$$\left(\frac{km}{k^2+m^2}\right)_{k,m\geq 1}, \qquad \left(\frac{(ke^{-k^2T})(me^{-m^2T})}{k^2+m^2}\right)_{k,m\geq 1}.$$

Next, let us introduce the infinite dimensional diagonal matrix $\mathcal{D} = \text{diag}\,\{k^2\}_{k\geq 1}$ and vectors $\ell = (k)_{k\geq 1}$ and $\ell' = (ke^{-k^2T})_{k\geq 1}$. Set $\mathcal{L} = \ell\ell^* - (\ell')(\ell')^*$. It is readily checked that the matrix $\mathcal{Z}$ satisfies the Lyapunov equation

$$\mathcal{D}\mathcal{Z} + \mathcal{Z}\mathcal{D} = \mathcal{L}. \tag{21}$$

Now, to find out what happens for the eigenvalues of $\mathcal{Z}$, we focus on the truncated matrix $\mathcal{Z}_N = (z_{km})_{1\leq k,m\leq N}$. It is the solution of the finite dimensional Lyapunov equation that is

$$\mathcal{D}_N \mathcal{Z}_N + \mathcal{Z}_N \mathcal{D}_N = \mathcal{L}_N. \tag{22}$$

All vectors and matrices are here truncated in an obvious way. Matrix equations (22) and (21) are also called symmetric Sylvester equations. They currently arise in the controllability/observability theory of Linear Time-Invariant systems.

The asymptotic expansion of the eigenvalues of $\mathcal{Z}$ are strongly connected to the properties of the diagonal matrix $\mathcal{D}_N$ and to the rank p of $\mathcal{L}_N$, called the displacement rank of equation (22). Solving it pertains to the third Zolotarev problem. Bounds will be derived after applying results assembled, sorted, commented and improved in two interesting Ph.D. dissertations, J. Sabino [25, 2006] and A. Gryson [9, 2009]. The authors propose also a wide bibliography on the subject. Let us first denote by $((\mu_N)_k)_{1\leq k\leq N}$ the eigenvalues and $((\sigma_N)_k)_{1\leq k\leq N}$ the singular values of $\mathcal{Z}_N$ ordered decreasingly. The following result holds

Proposition 4.1 *We have that*

$$\sqrt{\frac{\log N}{2}} \leq (\sigma_N)_1 \leq \sqrt{\frac{N}{2}}. \tag{23}$$

The singular values $((\sigma_N)_k)_{1\leq k\leq N}$ satisfy the bound

$$(\sigma_N)_{2k+1} \leq C\sqrt{N} \exp\left(-\frac{\pi^2 k}{4\log(2N)}\right), \qquad 1 \leq k \leq [(N-1)/2]. \tag{24}$$

Proof: The bound on the Perron-Frobenius root of $\mathcal{Z}_N$ which is $(\mu_N)_1$ may be established owing to the double formula (see [18, Chapter 8]),

$$\min_{1\leq k\leq N} \sum_{1\leq m\leq N} \frac{km}{k^2+m^2}(1-e^{-(k^2+m^2)T}) \leq (\mu_N)_1 \leq \max_{1\leq k\leq N} \sum_{1\leq m\leq N} \frac{km}{k^2+m^2}(1-e^{-(k^2+m^2)T}).$$

The right bound comes from the Gershgorin-Hadamard circle theorem while the left one is directly issued from Collatz-Wielandt formula. We deduce that

$$\frac{\log N}{2} \leq (\mu_N)_1 \leq \frac{N}{2}. \tag{25}$$

The bound (23) is established. The first thing we learn from these bounds is that the large eigenvalues blows up with N. This is due to (and expresses) the unboundedness of the operator $\mathcal{Z}$. Before stating (24), observe first that the displacement rank of equation (22) is two, $p = 2$. Calling for Theorem 2.1.1 of [25, pp 39-40] (see also [25, Formula (2.14) in p. 46]) yields the following bound

$$\frac{(\mu_N)_{pk+1}}{(\mu_N)_1} = \frac{(\mu_N)_{2k+1}}{(\mu_N)_1} \leq C \exp\left(-\frac{\pi^2 k}{\log(4\kappa(\mathcal{D}_N))}\right). \tag{26}$$

The constant C is independent of N. The symbol $\kappa(\mathcal{D}_N)$ is for the condition number of the matrix $\mathcal{D}_N$. This bound valid for general Lyapunov equations is related to the matrix $\mathcal{D}_N$ while the

eigenvalue $(\mu_N)_1$ is connected to the right hand side of the equation. Notice how the displacement rank influences the counting of the eigenvalues.
Given that the condition number $\kappa(\mathcal{D}_N)$ is easily computed for diagonal matrices, it coincides with the ratio of the maximal and the minimal diagonal terms and equals then to N^2. Hence we obtain that

$$(\mu_N)_{2k+1} \leq C(\mu_N)_1 \exp\left(-\frac{\pi^2 k}{2\log(2N)}\right).$$

Using the right bound in (25) we obtain that

$$(\mu_N)_{2k+1} \leq CN \exp\left(-\frac{\pi^2 k}{2\log(2N)}\right).$$

The proof is complete after recalling that $(\mu_N)_k = [(\sigma_N)_k]^2$. ■

Remark 4.5 *When $j = 2k+1 \approx \theta N$ with $0 < \theta \leq 1$ we have that*

$$(\sigma_N)_j \leq C\sqrt{N} \exp\left(-\frac{\pi^2 \theta N}{8\log(2N)}\right)$$

These singular values decrease toward zero exponentially fast. To summarize, the largest singular value grows up with N, at least like a constant times $\sqrt{\log(N)}$([5]), while the smallest eigenvalues decay exponentially fast. The bound in (26) *yields that the ratio between the extreme singular values $(\sigma_N)_1$ and $(\sigma_N)_N$ is bounded from below,*

$$\frac{(\sigma_N)_1}{(\sigma_N)_N} \geq C \exp\left(\frac{\pi^2 N}{8\log(2N)}\right)$$

This lower bound confirms the severe ill-posedness of the controllability problem.

Remark 4.6 *We emphasize here on the fact that the controllability problem is intrinsically severely ill-posed, no matter the support of the control coincides with the whole boundary of the domain or not. This is in contrast with the distributed control* (see [20])*, where the problem becomes mildly ill-posed if the control is operating on the whole domain.*

Remark 4.7 *Users may be interested in the behavior of the singular values $((\sigma_N)_k)_{1\leq k\leq N}$ with respect to the final time T. It may hence be indicated that the constant C in* (24) *is uniformly bounded with respect to T. However, in case where a sharp study is intended, the dependence of the bounds on $((\sigma_N)_k)_{1\leq k\leq N}$ may be inferred from formula* (26) *where $(\sigma_N)_1$ should be accurately estimated with respect to T. A simple look at the Gershgorin-Hadamard bound on $(\mu_N)_1$ shows that it decays toward zero when T comes close to zero. This means that the exact singular values $(\sigma_k)_{k\in\mathbb{Z}}$ is shifted toward zero for small T. This fact is expected as it indicates the higher difficulty to realize the desired state y_T in a shorter time T.*

All the ingredients required to provide the main result involved with the ill-posedness degree of the controllability of the one dimensional equation. We have the following

Theorem 4.2 *The controllability problem* (14) *is severely ill posed.*

[5]It will be checked out numerically that it actually grows like $\sqrt{N}$.

5 Solution of the Controllability Problem

The aim we are assigned here is to solve the controllability problem (14) in the context of Fourier computations. Here we need to express the operator B in respect with the progressive heat equation. Let then v be in $L^2(0,T)$, we set $\underline{v}(\cdot) = v(T-\cdot)$. It comes out that

$$(Bv)(x) = \frac{2}{\pi}\sum_{k\geq 1}(-1)^{k+1}\big[k\widehat{\underline{v}_{(T)}}(k^2)\big]\sin(kx), \qquad \text{in } I.$$

We look for a Dirichlet control u that drives the solution y_u from 0 at time $t=0$ to a fixed state $y_T \in L^2(I)$ at time $t=T$. Let us decompose the data $y_T \in L^2(I)$ as follows

$$y_T(x) = \sum_{k\geq 1}(y_T)_k \sin(kx) \qquad \text{in } I.$$

On account of the expression (19), we obtain the infinite family of equations on u that is

$$\widehat{\underline{u}_{(T)}}(k^2) = (-1)^{k+1}\frac{\pi}{2}\frac{(y_T)_k}{k}, \qquad \forall k \geq 1. \tag{27}$$

Recall that, in general, we look for the control $u^\dagger$ that belongs to $\overline{\mathcal{R}(B^*)}$. Owing to the representation (4.2) of the functions in $\overline{\mathcal{R}(B^*)}$, we consider the following expansion

$$u^\dagger(t) = \sum_{m\geq 1}(u^\dagger)_m\, e^{-m^2(T-t)}, \qquad \forall t \in (0,T).$$

Here, it was necessary to change t into $(T-t)$ to take into account that we are controlling the progressive equation. Plugging it into equation (27) yields that

$$\sum_{m\geq 1}\frac{1-e^{-(k^2+m^2)T}}{k^2+m^2}(u^\dagger)_m = (-1)^{k+1}\frac{\pi}{2}\frac{(y_T)_k}{k}, \qquad \forall k \geq 1. \tag{28}$$

The problem in the form (27) looks like the inversion of some discrete Laplace Transform or a moment problem where the solution is compactly supported. It may be formally solved by means of the biorthogonal basis related to $(e^{-k^2 t})_{k\geq 1}$ (see [24]). The point is that the formal expansion we obtain for the solution $u^\dagger$ will blow up exponentially fast. Here, in order to have a better insight about what really happens we conduct explicit computations and use the inversion of some Cauchy type matrices. This turn out to be possible when the controllability is aimed at a sufficiently long horizon. We mean that T is large enough to justify the neglecting of the exponential terms in the coefficients of (28). We are then left with the equation to solve which is somehow simpler

$$\sum_{m\geq 1}\frac{1}{k^2+m^2}(u^\dagger)_m = (-1)^{k+1}\frac{\pi}{2}\frac{(y_T)_k}{k}, \qquad \forall k \geq 1. \tag{29}$$

The methodology we follow consists first in realizing a truncation to a large cut-off N, solve the finite dimensional square system and finally to pass formally to the limit. As things stand, the finite dimensional system to tackle reads as

$$\sum_{1\leq m\leq N}\frac{1}{k^2+m^2}(u^\dagger)_m = (-1)^{k+1}\frac{\pi}{2}\frac{(y_T)_k}{k}, \qquad \forall 1 \leq k \leq N. \tag{30}$$

The symmetric matrix involved in this system, denoted by $\mathcal{C}_N = (c_{km} = 1/(k^2 + m^2))_{1\leq k,m\leq N}$, is a Cauchy matrix. The inverse $(\mathcal{C}_N)^{-1}$ may be achieved explicitly. Indeed, we have that (see [27])

$$(\mathcal{C}_N)^{-1} = \mathcal{D}_N \mathcal{C}_N \mathcal{D}_N.$$

The matrix $\mathcal{D}_N = \text{ diag } (d_k)_{1\leq k\leq N}$ is diagonal and the entries are given by

$$d_k = \frac{k^2 \prod_{m=1}^N (1 + \dfrac{k^2}{m^2})}{\prod_{m=1, m\neq k}^N (1 - \dfrac{k^2}{m^2})}, \qquad \forall k (1 \leq k \leq N).$$

Now, for a large cut-off $N \approx \infty$, we switch to infinite products. All calculations achieved([6]), we obtain the simplified closed form of those diagonal cœfficients,

$$d_k = \frac{2}{\pi}(-1)^k k \sinh(k\pi), \qquad \forall k (1 \leq k \leq N).$$

Returning to the inverse matrix $\mathcal{C}_N = (\tilde{c}_{km})_{1\leq k,m\leq N}$, we obtain that

$$\tilde{c}_{km} = d_k c_{km} d_m = \frac{4}{\pi^2}(-1)^{k+m} \frac{km}{k^2 + m^2} \sinh(k\pi) \sinh(m\pi), \qquad k, m \geq 1.$$

All the ingredient are thus available to derive the coefficients $(u^\dagger)_k$ of the control $u^\dagger$

$$(u^\dagger)_k = \sum_{m\geq 1} \frac{(-1)^{m+1}}{m} \tilde{c}_{km} (y_T)_m = \frac{4}{\pi^2}(-1)^{k+1} k \sinh(k\pi) \sum_{m\geq 1} \frac{\sinh(m\pi)}{k^2 + m^2} (y_T)_m.$$

For $u^\dagger$ to lie in $L^2(I)$, one has to check the bound given in the end of Remark 4.1, skipped over because the expression is pretty long. We limit ourselves to the following observation, that condition requires a high regularity on that data y_T. Intuitively, the subspace of reachable states appears quiete shrunk (although dense) in $L^2(I)$. Moreover, using this formula, we see that a small y_T, assimilated to a noise, may produce a solution $u^\dagger$ that blows up exponentially fast.

Remark 5.1 *In a short horizon controllability, we observe that the matrix related to the truncated system of* (28) *whose entries are*

$$\left(\frac{1 - e^{-(k^2+m^2)T}}{k^2 + m^2}\right)_{1\leq k,m\leq N}$$

may be decomposed as the product of two matrices defined by

$$\text{Diag } (e^{-k^2 T})_{1\leq k\leq N}, \qquad \left(\frac{e^{k^2 T} - e^{-m^2 T}}{k^2 + m^2}\right)_{1\leq k,m\leq N}.$$

[6]We use the well known infinite product formulae

$$\frac{\sin(\pi\zeta)}{\pi\zeta} = \prod_{m=1}^{\infty} (1 - \frac{\zeta^2}{m^2}), \qquad \forall \zeta \in \mathbb{R},$$
$$\frac{\sinh(\pi\zeta)}{\pi\zeta} = \prod_{m=1}^{\infty} (1 + \frac{\zeta^2}{m^2}), \qquad \forall \zeta \in \mathbb{R}.$$

The first is diagonal and then easy to invert while the second is again a particular structured matrix of Löewner type. Different processes do exist to compute its inverse either analytically or numerically using some particular algorithms. Given that it would be long to expose either of both methods we simply refer to [31].

Remark 5.2 *Realizing similar calculations for the null-controllability yields a problem that looks like equation* (28). *Recall that for the null-controllability the initial data is fixed to y_0 while the final target is $y_T = 0$. The linear problem to solve on $u^\dagger$ may be written as*

$$\sum_{m\geq 1} \frac{1 - e^{-(k^2+m^2)T}}{k^2 + m^2}(u^\dagger)_m = (-1)^{k+1} e^{-k^2 T} \frac{(y_0)_k}{k}, \qquad \forall k \geq 1.$$

It may be put under the following form

$$\sum_{m\geq 1} \frac{e^{k^2 T} - e^{-m^2 T}}{k^2 + m^2}(u^\dagger)_m = (-1)^{k+1} \frac{(y_0)_k}{k}, \qquad \forall k \geq 1.$$

In a compact form, it may be written as

$$\mathcal{Q}\mathcal{L}_\infty u^\dagger = -(\mathcal{Y}_0)_\infty.$$

The matrix $\mathcal{Q}$ is the diagonal one defined in Remark 4.4 *and $\mathcal{L}_\infty$ is the infinite Löewner matrix provided in* Remark 5.1. *The null-controllability problem has been studied in details in* [19], *where the well posedness and the observability estimate are stated. This result is to relate to the invertibility of the infinite Löewner matrix* (see [7]).

6 Numerical Realizations

In order to check these theoretical predictions, we run first some computational investigations of the singular values $(\sigma_k)_{k\in\mathbb{N}}$ of the operator B. We examine the truncated matrix $\mathcal{B}_N = (b_{km})_{1\leq k,m\leq N}$ for large values of N, corresponding to the restriction of BB^* to the N first Fourier modes. Their eigenvalues are the squares of the singular values of the truncation of B. Calculations are realized in Matlab. Next, we consider the controllability problem where the target y_T is a parabolic profile. The control $u^\dagger$ is obtained by the inversion of the Cauchy system addressed in Section 5 after regularization.

The first computations are accomplished when the cut-off N ranges from 5 to 24. The singular values $(\sigma_N)_{1\leq k\leq N}$ are plotted in Figure 1 with different final times $T = 10,\ 1,\ 0.1$ and 0.01. Most of them seem to spread towards zero when N grows higher. Although it is hardly perceptible on these plots, the largest singular-values grow slowly towards infinity. In contrast, the smallest ones decreases visually fast towards zero. Notice that when N exceeds 20, the lowest ones are smaller than 10^{-8} and the accuracy suffers at that level. This might explain the clustering effect observed on those singular values. Indeed, Matlab fails to approximate them and sends back

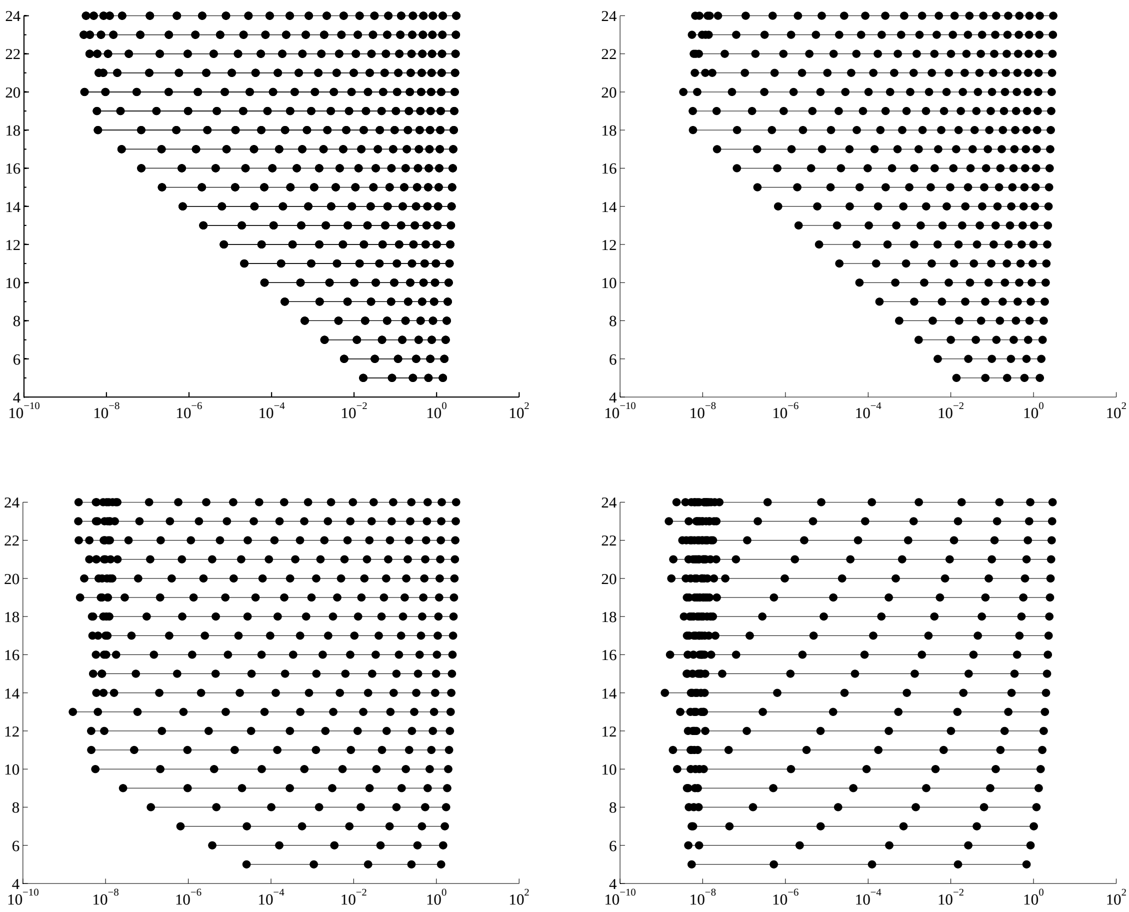

Figure 1: Singular values of B_N for various cut-offs N. The up-left diagram is for $T = 10$, the up-right for $T = 1$, the down-left for $T = 0.1$ and the down-right for $T = 0.01$.

inaccurate values. To confirm these first trends, we depict in the right diagram of Figure 2, the behavior of the largest singular value $(\sigma_N)_1$ with respect to N for the two final times $T = 1$ and $T = 0.01$. The shape of the curves and the (non-linear) regression suggests that $(\sigma_N)_1$ behaves rather like the upper bound predicted in (25) It grows up like $\sqrt{N}$. In the left panel we represent in a linear-logarithmic scale, the singular values $(\sigma_N)_N, (\sigma_N)_{N-2}$ and $(\sigma_N)_{N-4}$ where $T = 1$. The curves are close to straight lines which suggests that those singular values decay exponentially fast with respect to N. A careful look at the plots yields that for a fixed k, the sequences $((\sigma_N)_{N-k})_N$ increases like $\sqrt{N}$ while the sequence $((\sigma_N)_k)_N$ decays exponentially fast toward zero for growing N. Another result confirmed by by numerical experimentation, we claim that an the fraction of the singular values decays to zero is substantially more important than the fraction that grows to infinity. Computations show that for $k_N = 0.1 \times N$, we have that $(\sigma_N)_{k_N}$ decreases toward zero. This indicates that more than nine-tenth of the singular values are small. Another feature may be noted from those plots. When the final time T is smaller the set of the singular values is noticeably shifted towards zero. For instance, if $T = 0.01$, even for $N = 5$, the couple of smaller singular values $((\sigma_5)_5, (\sigma_5)_5)$ are lower than 10^{-6}. This is in agreement with the intuition one has on the higher difficulty to reach the target y_T in a shorter time T.

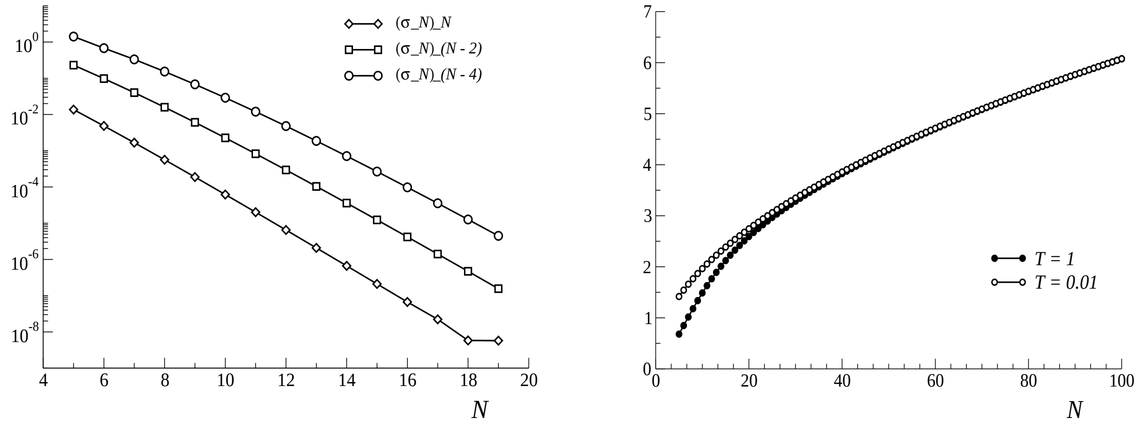

Figure 2: The left panel depicts the lowest singular values $((\sigma_2, (\sigma_N)_{N-4})$ of B_N for various cut-offs N. Observe that the lowest $(\sigma_N)_N$ becomes erratic with MATLAB at $N = 19$. The largest singular value $(\sigma_N)_1$ to the right.

We turn now to the effective solving of the controllability problem (28). Because of the exponential ill-conditioning of the truncated matrix, a crude inverting of it provides expectedly irrelevant results. Indeed, any small (numerical) perturbations on the data y_T or on the matrix entries may have dramatic effects on the computations. Hence, to capture a reasonable control, a regularization of the system is mandatory. We use the Tikhonov method by adding to the system a stabilizing term so that we have to calculate the solution $u_\varrho^\dagger$ of the following regularized equation

$$\varrho(u_\varrho^\dagger)_k + \sum_{m\geq 1} \frac{1 - e^{-(k^2+m^2)T}}{k^2 + m^2}(u_\varrho^\dagger)_m = (-1)^{k+1}\frac{\pi}{2}\frac{(y_T)_k}{k}, \qquad \forall k \geq 1. \tag{31}$$

The real-number $\varrho > 0$ has to be selected suitably and automatically. Some popular rules can be used such as the Discrepancy Principle, the General Cross Validation or the L-Curve (see [5, 29]). This issue is familiar in ill-posed problems and still feeds a lot of scientific work. Addressing it with the necessary care is beyond our scope. Only, notice that the ideal choice is the one that realizes a balancing between the accuracy and the stabilization of the numerical computation.
The plots of Figure 3 correspond to the computations obtained by attempting to realize a parabolic profile. The target is then fixed to

$$y_T(x) = x(\pi - x), \qquad \forall x \in [0, \pi].$$

The panel to the le left depicts the approximated states realized by three numerical simulations where the regularization parameter ϱ is equal to $10^{-2}, 10^{-4}$ or 10^{-6}. The second choice $\varrho = 10^{-4}$ seems the best as it results on a profile that is close to the very one we want to reach, even at the vicinity of $x = \pi$. The first $\varrho = 10^{-2}$ results in an over-regularization where the shape of the computed profile suffer form lesser accuracy. Even if the profile realized by under regularization

$\varrho = 10^{-6}$ is closer to y_T within a part of the domain I, it suffers clearly from an unacceptable inaccuracy at the vicinity $x = \pi$. A look to the right panel where the controls are represented confirms these observations. A under-regularization provokes useless energy expenses while an over-regularization yields a decreasing in the amount of the energy to reach the target. To close, we draw the attention to the fact that the smoothness of the control $u^\dagger$ for $t < T$. The oscillations around $t = T$ suggest the presence of singularity at the final time. This is in accordance with Remark 4.2.

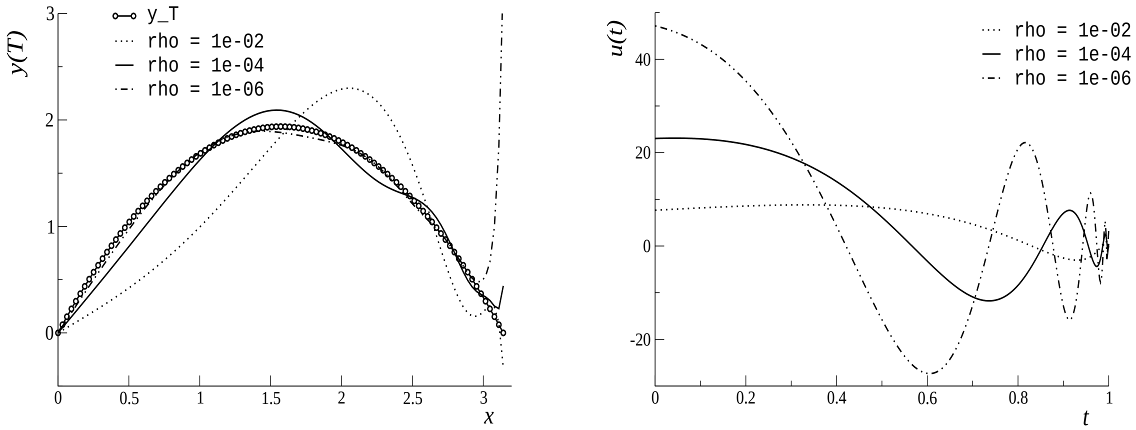

Figure 3: The left panel depicts the exact final state $y_T = x(\pi - x)$ and those computed using Tikhonov regularization with the parameter ϱ. The right panel draw the corresponding controls.

7 Conclusion

The issue of ill-posedness of the Dirichlet boundary controllability of the heat equation arises technical difficulties because the linear control operator has two properties that are somehow “opposite” to each other. One is the unboundedness that is responsible of the fact that a fraction of its singular values grows to infinity. The other is connected with the compactness and produces small singular values. The spectrum of that operator is therefore spread from zero to infinity. Conducting Fourier computations reduces the point to the study of some particularly structured matrices. The analysis of their singular values concludes to the severe ill-posedness of the problem. In fact, things happens as if the control operator is a direct sum of two sub-operators, one has a compact resolvent and the other is compact with an exponential compactness degree that is the corresponding singular values decrease exponentially fast towards zero.

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