Mortar spectral method in axisymmetric domains

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Abstract

We describe the main characteristics of the mortar element methods when handling three dimension axisymmetric domains by considering the Laplace equation. We use the spectral discretization method and the algorithm of Stang & Fix to improve the accuracy of the discretization.

Keywords

Axisymmetric domains, mortar method, spectral methods, Laplace equation, Strang & Fix algorithm.

Résumé : Nous considérons l'équation de Laplace posée sur un domaine tridimensionnel axisymétrique. Nous réduisons le problème de départ, par un développement en série de Fourier par rapport à la variable angulaire, en une famille dénombrable de problèmes bidimensionnels. Nous partitionnons le domaine méridien, supposé polygonal, en un nombre fini de rectangles et on disrétise par une méthode spectrale non conforme. Nous décrivons alors les principales caractéristiques de la méthode des joints et nous utilisons enfin l'algorithme de Stang & Fix pour améliorer la précision de nôtre discrétisation.

1 Introduction

Many problems resulting from physics and mechanics are formulated like partial differential equations in three dimensional domains, this is sometimes difficult to handle numerically because of the complexity of the partial differential equation and the geometry of the domain. It is classical in mechanics to try to reduce the area of study as much as possible both at the modeling and the numerical analysis. In elasticity for example, two-dimensional models of shells or those of one-dimensional for rods are very popular and model fairly accurately the behavior of the three-dimensional bodies [1], [12]. On the other hand, and with the emergence of computers with parallel architecture, methods known as domain decomposition have blossomed [17]. These methods based on the principle of divide and rule, consist in breaking the original problem into a series of problems of smaller sizes. They partition the global domain into several subdomains on which one solve local problems, possibly on different processors, with a transfer of information between subdomains. This method is especially interesting as that the subdomains may have different geometrical or physical characteristics and thus require different numerical treatments. This is for example the case of interaction problems [18]. The mortar element method is a domain decomposition technique which takes benefit of the presence of nonoverlapping subdomains to choose the best discretization adapted to each local behavior [20], the equality of the solution on the interfaces is enforced in a weak sense by Lagrange multipliers. The domain decomposition methods have seen in recent years new applications in time parallelization problems with Maday and all works [21].

This paper fits precisely within this context of double reduction. We are interested in the Laplace problem posed on a three dimensional domain invariant by rotation around an axis said axisymmetric domain. The Problem can then be reduced, without any approximation and by developments in Fourier coefficients with respect to the angular variable, in a countable family of two-dimensional problems associated with each Fourier coefficient. The simplest axisymmetric

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domain is obviously the cylinder generated by a unique rectangle and is treated in [2]. Though interesting, rather frequent in physics and containing the essential theoretical tools, this case is restrictive for the real problems. This is why we generalize with a polygonal meridian domain that models, for example, cylinders with obstacles so common in the industry. The meridian domain can thus be broken up into a finite number of rectangles and a domain decomposition method is then applied. Nevertheless, the presence of corners in the meridian domain induces severe limitations on the regularity of the solutions [16] and thus a low speed of convergence when we approximate our solution. We break up then the solution into a regular part, with optimal regularity and a linear combination of singular functions [13]. The algorithm of Strang & Fix is then used to improve the accuracy of the discretization [23].

To approximate our solution we use spectral methods, which are fully appropriate handle both the Fourier discretization with respect to the angular variable and the weighted measure in the radial direction together with the use of tensorized bases of orthogonal polynomials and of Gauss-Lobatto type quadrature formulas. An analysis in the general framework of Chebyshev or integrable Jacobi weights is performed in [6], [8] and [11]. The mortar element method is then used.

An outline of the paper is as follows. In the second section, we present the geometry and the continuous three dimensional problem. The weighted Sobolev spaces which will be used are introduced. The third section is interested to the spectral discretization in the axisymmetric case. Only the Fourier coefficient of order $k = 0$ is no null and so only one discrete problem has to be solved. An error estimate is established. Section 4 is devoted to the discretization in the general case. In section 5, we come back to the three dimensional problem and estimate the error between the exact solution and the solution tripling approached by truncation of Fourier series, numerical integration and approximation by the spectral method. The section 6 is devoted to Strang and Fix algorithm. Numerical tests which confirm our theoretical predictions are presented in the last section.

2 The geometry and the continuous problem

2.1 Geometry

For a point of $\mathbb{R}^3$, we use the Cartesian coordinates (x, y, z) or cylindrical ones (r, θ, z) with

$$x = r\cos\theta,\ y = r\sin\theta,\ r \in \mathbb{R}_+,\ \theta \in [-\pi, \pi[\,.$$

We denote by $\mathbb{R}^2_+$ the half space of $\mathbb{R}^2$ defined by $\mathbb{R}^2_+ = \{(r, z) \in \mathbb{R}^2,\ r \geq 0\}$. Let Ω be a polygon in $\mathbb{R}^2_+$ with boundary $\partial\Omega = \overset{n}{\underset{i=1}{\cup}} \Gamma_i$ made of a finite number of segments Γ_i, $1 \leq i \leq n$. The finite endpoints of these segments are known as corners of Ω. We call $c_1, c_2, ...c_p$ the corners of the polygon which are on the axis $r = 0$, and $e_1, e_2, ...e_j$ the other corners of Ω. We denote by Γ_0 the intersection of $\partial\Omega$ with the axis $r = 0$ and $\Gamma = \partial\Omega\backslash\Gamma_0$. Let $\breve{\Omega}$ be the domain of $\mathbb{R}^3$ obtained by rotation of Ω around the axis $r = 0$. The set Ω is called meridian domain and we have

$$\breve{\Omega} = \{(r, \theta, z) \in \mathbb{R}^3,\ \ (r, z) \in \Omega \cup \Gamma_0,\ -\pi \leq \theta \leq \pi\}\,.$$

We break up Ω into L open rectangles Ω_ℓ, $1 \leq \ell \leq L$, such that

$$\bar{\Omega} = \overset{L}{\underset{\ell=1}{\cup}} \bar{\Omega}_\ell \ \text{ and } \ \Omega_\ell \cap \Omega_m = \varnothing, \quad 1 \leq \ell < m \leq L. \tag{1}$$

In Figure 1, we illustrate some examples of domains $\breve{\Omega}$ which we treat numerically.

2.2 Weighted Sobolev spaces

We define the Hilbert spaces $L^2_1(\Omega)$, $L^2_{-1}(\Omega)$ and $H^m_1(\Omega)$, for $m \in \mathbb{N}^*$, by:

$$L^2_{\pm 1}(\Omega) = \{u : \Omega \longrightarrow \mathbb{C} \text{ measurable, } \|u\|_{L^2_{\pm 1}(\Omega)} = (\int_\Omega |u^2\,(r, z)\,| r^{\pm 1} drdz)^{\frac{1}{2}} < +\infty\}$$

$$H^m_1(\Omega) = \{u : \Omega \longrightarrow \mathbb{C} \text{ measurable, } \|u\|_{H^m_1(\Omega)} = (\sum_{k=0}^{m}\sum_{\ell=0}^{k} ||\partial^\ell_r \partial^{k-\ell}_z u||^2_{L^2_1(\Omega)})^{\frac{1}{2}} < +\infty\}.$$

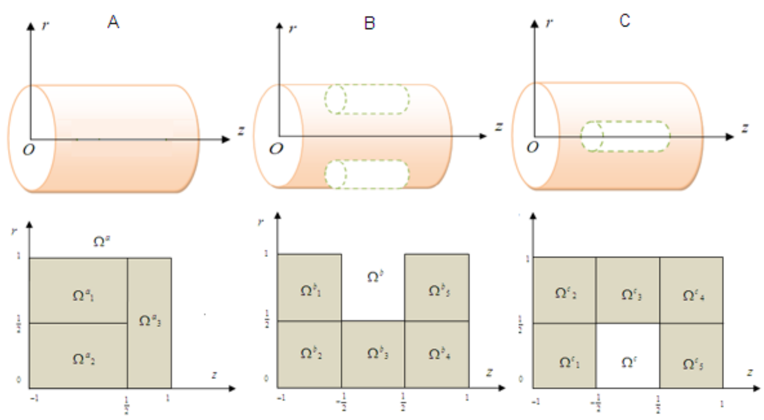


Figure 1: Domains of study

For a positive real number s, the space $H_1^s(\Omega)$ is deduced in a standard way by interpolation between the space $H_1^{[s]}(\Omega)$ and $H_1^{[s]+1}(\Omega)$, where $[s]$ stands for the integral part of s [19]. We define the Hilbert space $V_1^s(\Omega)$ by:

$$V_1^s(\Omega) = \{v \in H_1^s(\Omega), \partial_r^j w_{|\Gamma_0} = 0, 1 \leq j < s-1\} \text{ if } s \text{ is not an integer,}$$
$$V_1^s(\Omega) = \{v \in H_1^s(\Omega), \partial_r^j w_{|\Gamma_0} = 0, 1 \leq j < s-1 \text{ and } \partial_r^{s-1} w \in L_{-1}^2(\Omega)\} \text{ if } s \text{ is an integer}$$

We remark that for $s = 1$, $V_1^1(\Omega)$ coincides with $H_1^1(\Omega) \cap L_{-1}^2(\Omega)$. We provide it with the norm $||w||_{V_1^1(\Omega)} = (||w||^2_{H_1^1(\Omega)} + ||w||^2_{L^2_{-1}(\Omega)})^{\frac{1}{2}}$.
For any $\breve{v}$ defined in $H^s(\breve{\Omega})$, we associate the Fourier coefficients v^k, $k \in \mathbb{Z}$ given by

$$v^k(r,z) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \breve{v}(r,\theta,z) e^{-ik\theta} d\theta. \tag{2}$$

We recall from [2] that the mapping $\breve{v} \longmapsto (v_k)_{k\in\mathbb{Z}}$ is one to one from $H^s(\breve{\Omega})$ onto $\Pi_{k\in Z} H^s_{(k)}(\Omega)$ where $H^s_{(k)}(\Omega)$ is defined by:

$$\text{if } |k| > s-1, \; H^s_{(k)}(\Omega) = V_1^s(\Omega) \text{ endowed with the norm}$$
$$||w||_{H^s_{(k)}(\Omega)} = (||w||^2_{H_1^s(\Omega)} + |k|^{2s} ||r^{-s} w||^2_{L^2_{-1}(\Omega)})^{\frac{1}{2}}$$

and if $|k| \leq s-1$:

$$H^s_{(k)}(\Omega) = \left\{w \in H^s_+(\Omega), \partial_r^j w_{|\Gamma_0} = 0, \; 1 \leq j \leq |k|-1\right\}, \text{ if } k \text{ is even,}$$
$$\text{endowed with the norm } ||.||_{H^s_+(\Omega)},$$
$$H^s_{(k)}(\Omega) = \left\{w \in H^s_-(\Omega), \partial_r^j w_{|\Gamma_0} = 0, \; 1 \leq j \leq |k|-1\right\}, \text{ if } k \text{ is odd,}$$
$$\text{endowed with the norm } ||.||_{H^s_-(\Omega)},$$

The spaces $H^s_\pm(\Omega)$ are defined by:

$$H^s_+(\Omega) = \left\{w \in H_1^s(\Omega), \partial_r^{2j-1} w_{|\Gamma_0} = 0, \; 1 \leq j < s/2\right\}, \tag{3}$$

endowed with the norm $||.||_{H^s_+(\Omega)} = ||.||_{H_1^s(\Omega)}$, if s is not an even integer and

$$H^s_+(\Omega) = \left\{w \in H_1^s(\Omega), \partial_r^{2j-1} w_{|\Gamma_0} = 0, \; 1 \leq j < s/2 \text{ and } \partial_r^{s-1} w \in L^2_{-1}(\Omega)\right\} \tag{4}$$

endowed with the norm $||w||_{H^s_+(\Omega)} = \left(||w||^2_{H_1^s(\Omega)} + ||\partial_r^{s-1} w||^2_{L^2_{-1}(\Omega)}\right)^{1/2}$ if s is an even integer,

$$H^s_-(\Omega) = \left\{w \in H_1^s(\Omega), \partial_r^{2j} w_{|\Gamma_0} = 0, \; 1 \leq j < \frac{s-1}{2}\right\} \tag{5}$$

endowed with the norm $||w||_{H^s_-(\Omega)} = ||w||_{H_1^s(\Omega)}$, if s is not an odd integer and

$$H^s_-(\Omega) = \left\{ w \in H^s_1(\Omega), \partial_r^{2j} w_{|\Gamma_0} = 0,\ 1 \le j < \frac{s-1}{2} \text{ and } \partial_r^{s-1} w \in L^2_{-1}(\Omega) \right\}, \tag{6}$$

endowed with the norm $||w||_{H^s_-(\Omega)} = \left(||w||^2_{H^s_1(\Omega)} + ||\partial_r^{s-1} w||^2_{L^2_{-1}(\Omega)} \right)^{1/2}$ if s is an odd integer. Moreover, the following equivalence of norms holds [2]:

$$c||\breve{v}||_{H^s(\breve{\Omega})} \le \left(\sum_{k\in\mathbb{Z}} ||v^k||^2_{H^s_{(k)}(\Omega)} \right)^{\frac{1}{2}} \le c'||\breve{v}||_{H^s(\breve{\Omega})}. \tag{7}$$

Remark 1 *In the monodimensional case,* $\Lambda = [-1, 1]$*, the spaces* $L^2_{\pm 1}(\Lambda)$*,* $H^s_1(\Lambda)$*,* $V^s_1(\Lambda)$ *and* $H^s_{(k)}(\Lambda)$ *are defined in the same way of the two-dimensional case by using the measure* $d\tau = rdr$ *if* Λ *is pependicular to the axis (Oz) and* $d\tau = dz$ *if not. For more details see [2, Chap. II].*

2.3 The Laplace problem

We are interested by the spectral-Fourier method for the discretization of the Dirichlet Problem

$$\begin{cases} -\Delta \breve{u} = \breve{f} & \text{in } \breve{\Omega}, \\ \breve{u} = 0 & \text{on } \partial\breve{\Omega}, \end{cases} \tag{8}$$

where $\breve{\Omega}$ is an axisymmetric domain for which the meridian domain Ω is a polygon such that the intersection with the rotation axis $r = 0$ is either empty or equal to a finite number of edges. The Fourier coefficient u^k of the solution $\breve{u}$ satisfies for each $k \in \mathbb{Z}$ the equation

$$\begin{cases} -\partial_r^2 u^k - \frac{1}{r}\partial_r u^k - \partial_z^2 u^k + \frac{k^2}{r^2} u^k = f^k & \text{in } \Omega, \\ u^k = 0 & \text{on } \Gamma, \end{cases} \tag{9}$$

where f^k is the kth Fourier coefficients of $\breve{f}$.
The domain Ω being broken as in Figure 1 , singular functions near edges appear in the solution of Problem (9). Let us fix e_i is a vertex in the set $\{e_1, e_2, \dots, e_J\}$ and $\mathcal{V}_{e_i}$ a neighborhood of e_i in Ω such that $\mathcal{V}_{e_i} \cap \mathcal{V}_{e_j} = \varnothing$ if $i \neq j$. And let χ_{e_i} be a smooth cut-off function of r_{e_i}, equal to 1 in a neighborhood of e_i and with a support in $\mathcal{V}_{e_i}$, where r_{e_i} denotes the function distance to e_i. Then for each k, the Fourier coefficient u^k, solution of Problem (9) admits the following splitting [14, Chap.16] and [2, Theo. II.4.11]:

$$u^k = u^k_{reg} + \sum_n \gamma^{(k)n}_{e_i} . \chi_{e_i}(|k| r_{e_i}) S^{(k)n}_{e_i} \text{ in } \mathcal{V}_{e_i} \cap \Omega \tag{10}$$

where u^k_{reg} is the regular part, $(S^{(k)n}_{e_i})_n$ denote a finite number of singular functions related to singularity of the three-dimensional problem with $\gamma^{(k)n}_{e_i}$ as coefficients. The principal part of $S^{(k)n}_{e_i}$ is independent of k.

Let ω_{e_i} denotes the opening angle of Ω at the corner e_i, then for s, such that $0 < \frac{n\pi}{\omega_{e_i}} < s$

$$||u^k_{reg}||_{H^{s+1}_{(k)}(\mathcal{V}_{e_i}\cap\Omega)} + (1 + |k|^{s - \frac{n\pi}{\omega_{e_i}}})|\gamma^{(k)n}_{e_i}| \le ||f^k||_{H^{s-1}_{(k)}(\Omega)}. \tag{11}$$

We remark that for the domains which we consider, the angles in c_i are equal to $\frac{\pi}{2}$ and does not lead to any singularity in the three dimensional domain, so we only consider singularities in the nodes e_i which generate edges.

3 The discretization in the axisymmetric case

We first assume that the datum $\breve{f}$ is axisymmetric, i.e., f is independ of θ, so only its Fourier coefficient of order $k = 0$ is non zero and only one discrete problem, for $k = 0$, has to be solved. The variational formulation of the Problem (9) can be written for $f^0 = f$:

$$\begin{cases} \text{Find } u \in H^1_{1\diamond}(\Omega) \text{, such that} \\ \forall v \in H^1_{1\diamond}(\Omega),\ \ a(u, v) = \sum_{\ell=1}^{L} \int_{\Omega_\ell} fv \; rdrdz, \end{cases} \tag{12}$$

where $a(u,v) = \sum_{\ell=1}^{L} \int_{\Omega_\ell} \nabla u . \nabla v \; rdrdz$, with $\nabla u = (\partial_r u, \partial_z u)$ and

$$H^1_{1\diamond}(\Omega) = \{v \in H^1_1(\Omega);\; v = 0 \text{ on } \Gamma\}.$$

For any datum f in $L^2_1(\Omega)$, Problem (12) admits a unique solution which verifies:

$$\|u\|_{H^1_1(\cup\Omega_\ell)} \leq c\,\|f\|_{L^2_1(\Omega)}$$

where the norm $\|.\|_{H^1_1(\cup\Omega_\ell)}$ and the pseudo-norm $|.|_{H^1_1(\cup\Omega_\ell)}$ are defined by

$$\|.\|_{H^1_1(\cup\Omega_\ell)} = (\sum_{\ell=1}^{L} \|.\|^2_{H^1_1(\Omega_\ell)})^{\frac{1}{2}} \text{ and } |.|_{H^1_1(\cup\Omega_\ell)} = (\sum_{\ell=1}^{L} |.|^2_{H^1_1(\Omega_\ell)})^{\frac{1}{2}}.$$

Indeed, the form a satisfies the following properties of continuity and ellipticity

$$\begin{aligned} \forall(u,v) &\in H^1_{1\diamond}(\Omega)^2, \quad |a(u,v)| \leq \|u\|_{H^1_1(\cup\Omega_\ell)} \|v\|_{H^1_1(\cup\Omega_\ell)} \\ \forall u &\in H^1_{1\diamond}(\Omega), \quad a(u,u) \geq c\,\|u\|^2_{H^1_1(\cup\Omega_\ell)}. \end{aligned} \tag{13}$$

The first inequality is obtained easily by using the inequality of Cauchy-Schwarz on each subdomain Ω_ℓ. The second is a direct consequence of weighted Poincaré-Friedrichs inequality, see [22, Prop. 3] for details. The existence is then a consequence of the Lax-Milgram Lemma.

3.1 The discrete Problem

For any nonnegative integer N, $\mathbb{P}_N(\Omega)$ stands for the space of polynomials on Ω with degree $\leq N$, with respect to each variable r and z. We define the $L-$*tuple* of positive integers $\delta = (N_1, ..., N_L)$ and the skeleton $\mathcal{S}$ of the domain decomposition, equal to $\overset{L}{\underset{\ell=1}{\cup}} \partial\Omega_\ell \backslash \partial\Omega$ which admits a partition without overlap into mortars

$$\bar{\mathcal{S}} = \bigcup_{\mu=1}^{M^+} \gamma^+_\mu \quad \text{with} \quad \gamma^+_\mu \cap \gamma^+_{\mu'} = \varnothing, \quad 1 \leq \mu < \mu' \leq M^+.$$

Above each γ^+_μ is a whole edge of one of Ω_ℓ, which is then denoted by Ω^+_μ. Note that the choice of this decomposition is not unique, however it is decided a priori for all the discretizations we work with. Once it is fixed, we have another partition of the skeleton into non-mortars:

$$\bar{\mathcal{S}} = \bigcup_{\mu=1}^{M^-} \gamma^-_m \quad \text{with} \quad \gamma^-_m \cap \gamma^-_{m'} = \varnothing, \quad 1 \leq m < m' \leq M^-$$

where each γ^-_m is a whole edge of one of $\Omega_\ell \neq \Omega^+_\mu$, which is then denoted by Ω^-_m.
We introduce the discrete space

$$\mathbb{Y}_\delta = \{v_\delta \in L^2_1(\Omega), v_{\delta|\Omega_\ell} = v_\ell \in \mathbb{P}_{N_\ell}(\Omega_\ell), \forall \ell = 1, ...L\}.$$

For all v_δ in $\mathbb{Y}_\delta$, we define the mortar function $\phi_{v_\delta} \in L^2_1(\mathcal{S})$ by $\phi_{v_\delta|\gamma^+_\mu} = (v_{\delta|\Omega^+_\mu})_{|\gamma^+_\mu}$, $1 \leq \mu \leq M^+$.
We define our fundamental discrete space $\mathbb{X}_\delta$ by:

$$\begin{aligned} \mathbb{X}_\delta &= \{v_\delta \in \mathbb{Y}_\delta, \int_{\gamma^-_m} (v_\delta - \phi_{v_\delta})(\tau)\psi(\tau)d\tau = 0 \\ \forall\psi &\in \mathbb{P}_{N_m-2}(\gamma^-_m), \;\; \forall\gamma^-_m, \; 1 \leq m \leq M^-\} \end{aligned} \tag{14}$$

with $d\tau = rdr$ if the non mortar γ^-_m is parallel to the axis $z = 0$ and $d\tau = dz$ if γ^-_m is parallel to the axis $r = 0$.
We also introduce the spaces $\mathbb{X}^\diamond_\delta = \{v_\delta \in \mathbb{X}_\delta, v_\delta = 0 \text{ on } \Gamma\}$ and $\mathbb{X}^\circ_\delta = \{v_\delta \in \mathbb{X}_\delta, v_\delta = 0 \text{ on } \Gamma \cup \Gamma_0\}$ which we endow with the decomposition-dependent norm $\|.\|_{H^1_1(\cup\Omega_\ell)}$.

Remark 2 *We can write the matching condition (14) as follows*

$$\forall\psi \in \mathbb{P}_{N_m-2}(\gamma_m^-), \int_{\gamma_m^-} (v_\delta - v_{\delta|\gamma_m^+})(\tau)\psi(\tau)d\tau = 0 \tag{15}$$

where γ_m^+ is the side γ_m^- seen in the other direction see Figure 2.

The quadrature formulas

The Legendre polynomial L_N is defined by the differential equation:

$$((1-\zeta^2)L_N'(\zeta))' + N(N+1)L_N(\zeta) = 0,\ L_N(1) = 1.$$

We note $\xi_0 = -1$, $\xi_N = 1$ and ξ_i, $1 \le i \le N-1$, the zeros of L_N'.
We introduce the polynomial $M_N = \frac{L_N(\zeta)+L_{N+1}(\zeta)}{1+\zeta}$ and note $\zeta_1 = -1$, $\zeta_{N+1} = 1$ and ζ_j, $2 \le j \le N$, the zeros of M_N'. The polynomials M_N are orthogonal to each other for the measure $(1+\zeta)d\zeta$. The Gauss-Lobatto and weighted Gauss-Lobatto formulas on the square $\Sigma =]-1,1[^2$ are defined by:

$$\begin{aligned}
\forall\phi &\in \mathbb{P}_{2N-1}(\Sigma), \int_\Sigma \phi(\zeta,\xi)(1+\zeta)\,d\zeta d\xi = \sum_{i=1}^{N+1}\sum_{j=0}^{N}\phi(\zeta_i,\xi_j)\,\omega_i\rho_j,\\
\forall\phi &\in \mathbb{P}_{2N-1}(\Sigma), \quad \int_\Sigma \phi(\zeta,\xi)\,d\zeta d\xi = \sum_{i=0}^{N}\sum_{j=0}^{N}\phi(\xi_i,\xi_j)\,\rho_i\rho_j
\end{aligned}$$

where ω_i, respectively ρ_j, are the Gauss-lobatto weights associated with ζ_i, respectively ξ_j.

We note by $(\Omega_\ell)_{1\le\ell\le L_0}$ the rectangles such that $\partial\bar{\Omega}_\ell \cap \Gamma_0 \neq \varnothing$ and by $(\Omega_\ell)_{L_0+1\le\ell\le L}$ those such that $\partial\Omega_\ell \cap \Gamma_0 = \varnothing$. If we note $\Omega_\ell =]0, r_\ell'[\times]z_\ell, z_\ell'[$, for $1 \le \ell \le L_0$ and $\Omega_\ell =]r_\ell, r_\ell'[\times]z_\ell, z_\ell'[$ for $L_0+1 \le \ell \le L$, the images of the M_N' and L_N' zeros by the affine transformation which maps Ω_ℓ onto Σ are given by $\zeta_i^\ell = \frac{r_\ell}{2}(\zeta_i+1)$, $1 \le i \le N_\ell+1$, For $1 \le \ell \le L_0$; $\xi_i^{(r)\ell} = \frac{(r_\ell'-r_\ell)}{2}\xi_i + \frac{(r_\ell'+r_\ell)}{2}$, $0 \le i \le N_\ell$ for $L_0+1 \le \ell \le L$ and $\xi_i^\ell = \frac{(z_\ell'-z_\ell)}{2}\xi_i + \frac{(z_\ell'+z_\ell)}{2}$, $0 \le i \le N_\ell$ for $1 \le \ell \le L$. We can now define for $u, v \in C^0\left(\cup\bar{\Omega}_\ell\right)$ the discrete scalar product

$$\begin{aligned}
(u,v)_\delta &= \sum_{\ell=1}^{L_0}\sum_{i=1}^{N_\ell+1}\sum_{j=0}^{N_\ell} u_\ell(\zeta_j^\ell,\xi_i^\ell)v_\ell(\zeta_j^\ell,\xi_i^\ell)\omega_i^\ell\rho_j^\ell\\
&+ \sum_{\ell=L_0+1}^{L}\sum_{i,j=0}^{N_\ell} u_\ell(\xi_i^{(r)\ell},\xi_j^\ell)v_\ell(\xi_i^{(r)\ell},\xi_j^\ell)\xi_i^{(r)\ell}\rho_i^\ell\rho_j^\ell.
\end{aligned}$$

with obvious choice of the ω_i^ℓ and ρ_j^ℓ for consistence.

Let $\mathcal{I}_N^+$ and $\mathcal{I}_N$ be the interpolation operators from $C^0\left(\bar{\Omega}_\ell\right)$ onto $\mathbb{P}_N(\Omega_\ell)$ which verify $h(\zeta_j^\ell,\xi_i^\ell) = \mathcal{I}_N^+ h(\zeta_j^\ell,\xi_i^\ell)$ and $h(\xi_j^{(r)\ell},\xi_i^\ell) = \mathcal{I}_N h(\xi_j^{(r)\ell},\xi_i^\ell)$ for any h.

If the datum $f \in C^0\left(\cup\bar{\Omega}_\ell\right)$, we construct our discrete problem, associated with (12), by a Galerkin method with numerical integration as

$$\begin{cases} \text{Find } u_\delta \in \mathbb{X}_\delta^\diamond \text{ such that}\\ \forall v_\delta \in \mathbb{X}_\delta^\diamond,\ a_\delta(u_\delta,v_\delta) = (\mathcal{I}_\delta f, v_\delta)_\delta, \end{cases} \tag{16}$$

where $a_\delta(u,v) = (\nabla u, \nabla v)_\delta$ and $\mathcal{I}_{\delta|\Omega_\ell} = \mathcal{I}_{N_\ell}^+$ if Ω_ℓ intersects Γ_0 and $\mathcal{I}_{\delta|\Omega_\ell} = \mathcal{I}_{N_\ell}$ if not.
We want to prove that Problem (16) is well posed. We introduce the number N_a as being the maximum number of corners of $\bar{\Omega}_\ell$ contained in γ_m^-, $1 \le m \le M^-$. For $v \in L_1^2(\Omega)$ with $v_{|\Omega_\ell} \in V_1^1(\Omega_\ell)$, we associate the mortar function ϕ_v and define the space $X(\Omega)$ by:

$$\begin{aligned}
X(\Omega) &= \{v \in L_1^2(\Omega),\ v_{|\Omega_\ell} \in V_1^1(\Omega_\ell) \text{ such that}\\
\forall\, 1 &\le m \le M^-,\ \forall\psi \in \mathbb{P}_{N_a}(\gamma_m^-),\ \int_{\gamma_m^-}(v-\phi_v)\psi d\tau = 0, v = 0 \text{ on } \Gamma\}.
\end{aligned}$$

Proposition 3 *1. There exists a positive constant c depending only on* Ω *such that:*

$$||v||^2_{L^2_1(\Omega)} \leq c|v|^2_{H^1_1(\cup\Omega_\ell)} \; \forall v \in X(\Omega). \tag{17}$$

2. Problem (16) is well posed for any $f \in C^0\left(\cup\bar{\Omega}_\ell\right)$.

Proof. 1. Let $v \in X(\Omega)$ such that $|v|_{H^1_1(\cup\Omega_\ell)} = 0$ and let $v_\ell = v_{|\Omega_\ell}$. Hence v_ℓ is constant on each Ω_ℓ and $v_\ell = 0$ in Ω_ℓ for all ℓ such that $mes\left(\partial\Omega_\ell \cap \Gamma\right) > 0$. We cannot directly conclude that $v_\ell = 0$ for all ℓ, since we have not necessarily $v_\ell = v_m$ on $\gamma^{\ell m}$. So we fix m such that $1 \leq m \leq M^-$ and $mes\left(\partial\Omega^-_m \cap \Gamma\right) > 0$. According to the matching condition (14), we have

$$\int_{\gamma^-_m} \left(v_{\gamma^-_m} - \phi\right)(\tau)\,\psi(\tau)\,d\tau = 0, \forall\psi \in \mathbb{P}_{N^-_m-2}\left(\gamma^-_m\right).$$

Hence we obtain

$$\sum_{j\in J}\int_{\gamma^{jm}} \left(v_{\gamma^-_m} - v_j\right)\psi(\tau)\,d\tau = 0, \forall\psi \in \mathbb{P}_{N^-_m-2}\left(\gamma^-_m\right)$$

where $\gamma^{jm} = \bar{\Omega}_j \cap \bar{\Omega}^-_m$ and $mes(\gamma^{jm} = \bar{\Omega}_j \cap \bar{\Omega}^-_m) > 0$. Since v_ℓ is constant this leads to

$$\sum_{j\in J}\left(v_{\gamma^-_m} - v_j\right)\int_{\gamma^{jm}} \psi(\tau)\,d\tau = 0.$$

We introduce the ends a_{j_0} and a_{j_0-1} of the interface $\gamma^{j_0 m}$ and consider the polynomial χ, of degree $N^-_m - 1$, defined on γ^-_m and such that

$$\chi(a_0) = \chi(a_1) = \cdots = \chi(a_{j_0-1}) = 0 \text{ and } \chi(a_{j_0}) = \chi(a_{j_0+1}) = \cdots = \chi(a_S) = 1.$$

Since $S \leq N_a$, we can choose $\psi_{j_0} = \chi'$ and we have:

$$\int_{\gamma^{jm}} \psi_{j_0}(\tau)\,d\tau = \chi(a_j) - \chi(a_{j-1}) = \delta^j_{j_0}$$

where δ indicate the Kronecker symbol. Consequently, it rests:

$$\sum_{j\in J}\left(v_{\gamma^-_m} - v_j\right)\int_{\gamma^{jm}} \psi_{j_0}(\tau)\,d\tau = v_{\gamma^-_m} - v_{j_0} = 0,$$

and then $v_{\gamma^-_m} = v_{j_0}$. This gives that $v_\ell = 0$ for all ℓ such that Ω_ℓ is adjacent with a rectangle which intersects $\partial\Omega\backslash\Gamma_0$. By extension, we deduce that $v_\ell = 0 \; \forall\ell$. We have then checked that $|.|_{H^1_1(\cup\Omega_\ell)}$ is a norm. By applying the Peetre-Tartar Lemma [15, Chapter 1. Th 2.1] with $E_1 = H^1_1(\Omega)$, $E_2 = E_3 = L^2_1(\Omega)$, $A = \nabla \in \mathcal{L}(E_1, E_2)$ and $B = Id_{E_2}$, we obtain (17).
2. Using the Cauchy-Schwarz inequality and the exactitude of the Gauss-Lobatto formula relatively to each variable r and z, we obtain:

$$\forall u_\delta, v_\delta \in X^\diamond_\delta(\Omega), \;\; |a_\delta(u_\delta, v_\delta)| \leq c|u_\delta|_{H^1_1(\cup\Omega_\ell)}|v_\delta|_{H^1_1(\cup\Omega_\ell)},$$

and since $X^\diamond_\delta(\Omega) \subset X(\Omega)$ for each δ, the following coercivity inequality is true on $X^\diamond_\delta(\Omega)$ with a constant independent of δ.

$$|a_\delta(u_\delta, u_\delta)| \geq c'|u_\delta|^2_{H^1_1(\cup\Omega_\ell)}. \tag{18}$$

Hence, for all $f \in C^0\left(\cup\bar{\Omega}_\ell\right)$, Problem (16) admits a unique solution $u_\delta \in X^\diamond_\delta(\Omega)$ such that

$$||u_\delta||_{H^1_1(\cup\Omega_\ell)} \leq c||\mathcal{I}_\delta f||_{L^2_1(\Omega)}.$$

■

From now on, we suppose that the condition $N_\ell \geq N_a + 2$, $\forall\, 1 \leq \ell \leq L$ holds.

3.2 Error estimates

We study hereafter the error between u, solution of the continuous problem (12) and u_δ, solution of the discrete one (16). Classical techniques in the approximation theory lead to the inequality [23]

$$\begin{aligned} \|u-u_\delta\|_{H^1_1(\cup\Omega_\ell)} \leq\; & c\big(\inf_{v_\delta\in X_\delta^\diamond}\{\|u-v_\delta\|_{H^1_1(\cup\Omega_\ell)} \\ & + \sup_{0\neq w_\delta\in X_\delta^\diamond}\frac{|a(v_\delta,w_\delta)-a_\delta(v_\delta,w_\delta)|}{\|w_\delta\|_{H^1_1(\cup\Omega_\ell)}}\} \\ & + \sup_{0\neq w_\delta\in X_\delta^\diamond}\frac{\left|\sum_{\gamma_m^-\in\mathcal{S}}\int_{\gamma_m^-}(\frac{\partial u}{\partial n_m})[w_\delta]d\tau\right|}{\|w_\delta\|_{H^1_1(\cup\Omega_\ell)}} \\ & + \sup_{0\neq w_\delta\in X_\delta^\diamond}\frac{\left|\int_\Omega f w_\delta r dr dz-(\mathcal{I}_\delta f,w_\delta)_\delta\right|}{\|w_\delta\|_{H^1_1(\cup\Omega_\ell)}}\big). \end{aligned} \tag{19}$$

Above the term $\frac{\partial u}{\partial n_m}$ is the normal derivative of u, and $[w_\delta]$ is the jump of w_δ through γ_m^-. We will consider each term of the right of the inequality above.

Proposition 4 *For any function u such that $u_{|\Omega_\ell}\in H_1^{s_\ell+1}(\Omega_\ell)$, with $s_\ell>\frac{1}{2}$, ($s_\ell>\frac{3}{2}$ if $\ell\leq L_0$), the approximate error verifies:*

$$\inf_{v_\delta\in X_\delta^\diamond}\|u-v_\delta\|_{H^1_1(\cup\Omega_\ell)}\leq c\lambda_\delta^{\frac{1}{2}}\sum_{\ell=1}^{L}N_\ell^{-s_\ell}||u_\ell||_{H_1^{s_\ell+1}(\Omega_\ell)} \tag{20}$$

where

$$\lambda_\delta=\max\{\frac{N_\mu^+}{N_m^-},\frac{N_m^-}{N_\mu^+}\} \tag{21}$$

for all mortars γ_μ^+, $1\leq\mu\leq M^+$ and non-mortars γ_m^-, $1\leq m\leq M^-$ such that $\gamma_\mu^+\cap\gamma_m^-$ has a nonnegative measure.

Before proving the proposition, we recall from [2, Remark IV.3.1, Prop IV.3.4], that there exist projection operators

$$\tilde{\pi}_N^1 : H^1(\Lambda)\longrightarrow\mathbb{P}_N(\Lambda)\text{ and }\tilde{\pi}_N^{+,1} : H_1^1(\Lambda)\longrightarrow\mathbb{P}_N(\Lambda),\ \Lambda=]-1,1[, \tag{22}$$

verifying, for $0\leq t\leq 1\leq s$

$$\|\tilde{\varphi}-\tilde{\pi}_N^1\tilde{\varphi}\|_{H^t(\Lambda)}\leq CN^{t-s}\|\tilde{\varphi}\|_{H^s(\Lambda)}\text{ and }\left\|\tilde{\phi}-\tilde{\pi}_N^{+,1}\tilde{\phi}\right\|_{H_1^t(\Lambda)}\leq CN^{t-s}\left\|\tilde{\phi}\right\|_{H_1^s(\Lambda)} \tag{23}$$

and for which it is easy to verify the following matching conditions:

$$\forall\psi\ \in\ \mathbb{P}_{N-2}(\Lambda),\ \forall\tilde{\varphi}\in H^1(\Lambda),\ \int_{-1}^{1}(\tilde{\varphi}-\tilde{\pi}_N^1\tilde{\varphi})\psi d\tau=0, \tag{24}$$

$$\forall\psi\ \in\ \mathbb{P}_{N-2}(\Lambda),\ \forall\tilde{\phi}\in H_1^1(\Lambda),\ \int_{-1}^{1}(\tilde{\phi}-\tilde{\pi}_N^{+,1}\tilde{\phi})\psi d\tau=0 \tag{25}$$

where $d\tau=dz$ or $d\tau=(1+\zeta)d\zeta$ respectively.
We recall also from [5, Lemma 2.3] the following Lemma.

Lemma 5 *Let a_p, $1\leq p\leq P$, denote P distinct points in Λ. For each $N\geq P+2$, and each p, $1\leq p\leq P$, there exists a polynomial $\eta_p\in\mathbb{P}_N(\Lambda)$ which is equal to 1 in a_p, vanishes in ±1 and $a_{p'\neq p}$ and satisfies:*

$$\left\|\eta_p\right\|_{L_1^2(\Lambda)}\leq cN^{-\frac{1}{2}},\quad\left\|\eta_p'\right\|_{L_1^2(\Lambda)}\leq cN^{\frac{1}{2}}. \tag{26}$$

The constant c depends only on the points a_p.

Proof of Proposition 4. We refer to [22, Prop. 2.3.5] for more details concerning this proof which is divided into three parts.

Part 1. Construction of v_δ^1: We set

$$v_\ell^1 = \mathcal{I}_{N_\ell}^+ u \text{ in } \Omega_\ell \text{ if } 1 \le \ell \le L_0 \text{ and } \mathcal{I}_{N_\ell} u \text{ if } L_0+1 \le \ell \le L.$$

According to [2, (VI.3)], we have for $s_\ell > \frac{1}{2}$ and $s_\ell > \frac{3}{2}$ if $\ell \le L_0$:

$$\left\| u_{|\Omega_\ell} - v_\ell^1 \right\|_{H_1^1(\Omega_\ell)} \le c N_\ell^{-s_\ell} \; \left\| u_{|\Omega_\ell} \right\|_{H_1^{s_\ell+1}(\Omega_\ell)}. \tag{27}$$

And for $1 \le \ell \le L$, we have

$$\left\| u_{|\Omega_\ell} - v_\ell^1 \right\|_{H_1^1(\Gamma)} + N_\ell \left\| u_{|\Omega_\ell} - v_\ell^1 \right\|_{L_1^2(\Gamma)} \le c' N_\ell^{\frac{1}{2}-s_\ell} \; \left\| u_{|\Omega_\ell} \right\|_{H_1^{s_\ell+1}(\Omega_\ell)}. \tag{28}$$

However, v_δ^1 do not verify the mortar matching condition across the interfaces, so we need to modify its values on the non-mortars.

Part 2. Construction of v_δ^2: For $1 \le \mu \le M^+$, let $\mathcal{C}_\mu^+$ be the set of the corners of Ω_ℓ which are inside γ_μ^+. We set

$$v_\delta^2 = \sum_{\mu=1}^{M^+} \sum_{e \in \mathcal{C}_\mu^+} (u - v^1_{\delta|\Omega_\mu^+})(e) \tilde{\Phi}_{\mu,e}, \text{ where } \tilde{\Phi}_{\mu,e} = \begin{cases} \Phi_{\mu,e} & \text{in } \Omega_\mu^+, \\ 0 & \text{in } \Omega \backslash \bar{\Omega}_\mu^+, \end{cases}$$

$\Phi(\zeta,\eta) = \eta_p(\zeta)(\frac{1-\eta}{2})^{N_\mu^+}$ and $\Phi_{\mu,e}$ is obtained from Φ by homothety and translation. It follows that $v_\delta^1 + v_\delta^2 = u$ at all the nodes $e \in \mathcal{C}_\mu^+$. Since $\|\Phi_{\mu,e}\|_{H_1^1(\Omega_\mu^+)}$ is bounded independently of N_μ^+, we have

$$\sum_{\ell=1}^{L} \left\| v_\delta^2 \right\|_{H_1^1(\Omega_\ell)} \le c \sum_{\mu=1}^{M^+} \sum_{e \in \mathcal{C}_\mu} |(u - v^1_{\delta|\Omega_\mu^+})(e)|, \tag{29}$$

with c independent of N. Applying a Galiardo-Niremberg inequality [10] on each γ_μ^+ and using (28), we obtain

$$\left\| u - v^1_{\delta|\Omega_\mu^+} \right\|_{L^\infty(\gamma_\mu^+)} \le c (N_\mu^+)^{-s_\mu^+} \left\| u \right\|_{H_1^{s_\mu^+ + 1}(\Omega_\mu^+)}. \tag{30}$$

We deduce then from (29) and (30) that

$$\sum_{\ell=1}^{L} \left\| v_\delta^2 \right\|_{H_1^1(\Omega_\ell)} \le c \sum_{\mu=1}^{M^+} (N_\mu^+)^{-s_\mu^+} \left\| u \right\|_{H_1^{s_\mu^+ + 1}(\Omega_\mu^+)}.$$

Similarly, we derive from Lemma 5, and (30) :

$$\left\| v_\delta^2 \right\|_{H_1^1(\gamma_\mu^+)} \le c (N_\mu^+)^{\frac{1}{2}-s_\mu^+} \left\| u \right\|_{H_1^{s_\mu^+ + 1}(\Omega_\mu^+)} \text{ and } \left\| v_\delta^2 \right\|_{L_1^2(\gamma_m^-)} \le c (N_\mu^+)^{-\frac{1}{2}-s_\mu^+} \left\| u \right\|_{H_1^{s_\mu^+ + 1}(\Omega_\mu^+)}.$$

Part 3. Construction of v_δ^3 : We set $v_\delta^{12} = v_\delta^1 + v_\delta^2$. According to the first part, the trace $v^{12}_{\delta|\gamma_m^+} - v^{12}_{\delta|\gamma_m^-}$ vanishes at the endpoints of all γ_m^-, $1 \le m \le M^-$. We set respectively $\tilde{\pi}_{N_\ell}^{+,1,(r),\ell}$, $1 \le \ell \le L_0$, $\tilde{\pi}_{N_\ell}^{1,(r),\ell}$, $L_0+1 \le \ell \le L$ and $\tilde{\pi}_{N_\ell}^{1,(z),\ell}$, $1 \le \ell \le L$, the projection corresponding operators respectively with $\tilde{\pi}_N^{+1}$ in the direction r and $\tilde{\pi}_N^1$ in the direction z. We define also the operator $\tilde{\pi}^{\gamma_m^-}$ by:

$$\tilde{\pi}^{\gamma_m^-} = \begin{cases} \tilde{\pi}_m^{+,1,(r)} & \text{if } \gamma_m^- // (Or) \text{ and } \gamma_m^- \cap (Oz) \neq \varnothing, \\ \tilde{\pi}_m^{1,(r)} & \text{if } \gamma_m^- // (Or) \text{ and } \gamma_m^- \cap (Oz) = \varnothing, \\ \tilde{\pi}_m^{1,(z)} & \text{if } \gamma_m^- \; // (Oz) \end{cases}$$

and set

$$v_\delta^3 = \sum_{m=1}^{M^-} [\tilde{\mathcal{R}}_\star^{\gamma_m^-} \circ \tilde{\pi}^{\gamma_m^-} (v^{12}_{\delta|\gamma_m^+} - v^{12}_{\delta|\gamma_m^-})_{|\gamma_m^-}] \tag{31}$$

where γ_m^+ is the side γ_m^- seen in the other direction see Figure 2,

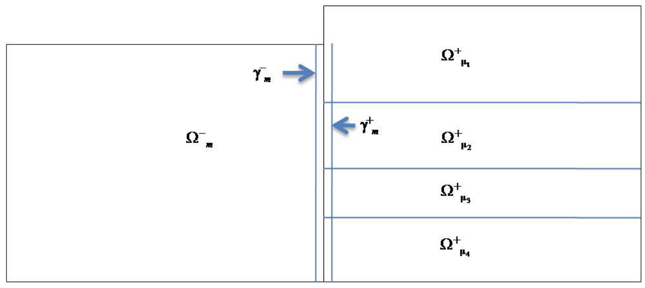


Figure 2:

$$\tilde{\mathcal{R}}^{\gamma}_{\star} = \tilde{\mathcal{R}}^{\gamma}_{-} \text{ if } \gamma//(Or) \text{ and } \gamma \cap (Oz) \neq \varnothing \text{ and } \tilde{\mathcal{R}}^{\gamma} \text{ otherwise,}$$

$\mathcal{R}^{\gamma^-_m}_{-}$ resp. $\mathcal{R}^{\gamma^-_m}$ is the lifting introduced in [3, Prop 4.25], and $\tilde{\mathcal{R}}^{\gamma^-_m}_{-}$ resp. $\tilde{\mathcal{R}}^{\gamma^-_m}$ is the lifting deduced from $\mathcal{R}^{\gamma^-_m}_{-}$ resp. $\mathcal{R}^{\gamma^-_m}$ by dilatation and translation.
We use for each real s, the notation:

$$(H^{s,\gamma}, V^{s,\gamma}, L^{2,\gamma}) = (H^s_1, V^s_1, L^2_1) \text{ if } \gamma//(Or) \text{ and } \gamma \cap (Oz) \neq \varnothing \text{ and } (H^s, V^s, L^2) \text{ otherwise.}$$

Whence, we have

$$\tilde{\mathcal{R}}^{\gamma^-_m}_{\star} \circ \tilde{\pi}^{\gamma^-_m}(v^{12}_{\delta|\gamma^+_m} - v^{12}_{\delta|\gamma^-_m}) = \tilde{\mathcal{R}}^{\gamma^-_m}_{\star}(v^{12}_{\delta|\gamma^+_m} - v^{12}_{\delta|\gamma^-_m}) - \tilde{\mathcal{R}}^{\gamma^-_m}_{\star}(Id - \tilde{\pi}^{\gamma^-_m})(v^{12}_{\delta|\gamma^+_m} - v^{12}_{\delta|\gamma^-_m}).$$

Using the fact that $\left\|\varphi - \tilde{\pi}^{\gamma^-_m}\varphi\right\|_{H^{r,\gamma^-_m}(\Omega^-_m)} \leq c(N^-_m)^{r-s}\left\|\varphi\right\|_{H^{s,\gamma^-_m}(\gamma^-_m)}$, for $0 \leq r \leq 1 \leq s$, we obtain

$$\begin{aligned}\left\|\tilde{\mathcal{R}}^{\gamma^-_m}_{\star} \circ \tilde{\pi}^{\gamma^-_m}(v^{12}_{\delta|\gamma^+_m} - v^{12}_{\delta|\gamma^-_m})\right\|_{H^{1,\gamma^-_m}(\Omega^-_m)} &\leq c\left\|(v^{12}_{\delta|\gamma^+_m} - v^{12}_{\delta|\gamma^-_m})\right\|_{V^{\frac{1}{2},\gamma^-_m}(\gamma^-_m)} \\ &+ N_m^{-\frac{1}{2}}\left\|(v^{12}_{\delta|\gamma^+_m} - v^{12}_{\delta|\gamma^-_m})\right\|_{H^{1,\gamma^-_m}(\gamma^-_m)}\end{aligned}$$

and finally, by summing on m, we obtain

$$\sum_{\ell=1}^{L}\left\|v^3_\delta\right\|_{H^1_1(\Omega_\ell)} \leq c\lambda_\delta^{\frac{1}{2}}\sum_{\ell=1}^{L}N_\ell^{-s_\ell}\left\|u\right\|_{H_1^{s_\ell+1}(\Omega_\ell)}.$$

Finally, the function $v^0_\delta = v^1_\delta + v^2_\delta + v^3_\delta$ satisfies the matching conditions, belongs to $X^\diamond_\delta$, and satisfies the desired estimate since $\inf_{v_\delta \in X^\diamond_\delta}\left\|u - v_\delta\right\|_{H^1_1(\cup\Omega_\ell)} \leq \left\|u - v^0_\delta\right\|_{H^1_1(\cup\Omega_\ell)} \leq \left\|u - v^1_\delta\right\|_{H^1_1(\cup\Omega_\ell)} + \left\|v^2_\delta\right\|_{H^1_1(\cup\Omega_\ell)} + \left\|v^3_\delta\right\|_{H^1_1(\cup\Omega_\ell)}$. ■

Remark 6 *We can replace the total term λ_δ (21) by a local term λ_ℓ :*

$$\lambda_\ell = \max_m \max_{\mu \in \mathcal{K}^-_m}\{\frac{N^+_\mu}{N^-_m}, \frac{N^-_m}{N^+_\mu}\}, \tag{32}$$

where the first max is taken on m of non mortars γ^-_m which is a side of Ω_ℓ and obtain:

$$\inf_{v_\delta \in X^\diamond_\delta}\left\|u - v_\delta\right\|_{H^1_1(\cup\Omega_\ell)} \leq c\sum_{\ell=1}^{L}(1 + \lambda_\ell)^{\frac{1}{2}}N_\ell^{-s_\ell}\left\|u\right\|_{H_1^{s_\ell+1}(\Omega_\ell)}.$$

In the following proposition, we are interested in errors due to non-conformities on interfaces.

Proposition 7 *For any function u solution of Problem (12), such that $u_{|\Omega_\ell} \in H_1^{s_\ell+1}(\Omega_\ell)$, with $s_\ell > \frac{1}{2}$ and $s_\ell > \frac{3}{2}$ if $\ell \leq L_0$ and for all $w_\delta \in X^\diamond_\delta$ the following estimate holds*

$$|\sum_{\gamma^-_m \in \mathcal{S}}\int_{\gamma^-_m}(\frac{\partial u}{\partial n_m})[w_\delta]d\tau| \leq c[\sum_{\ell=1}^{L}N_\ell^{-s_\ell}(\log N_\ell)^{\varrho_\ell}\left\|u\right\|_{H_1^{s_\ell+1}(\Omega_\ell)}]\left\|w_\delta\right\|_{H^1_1(\cup\Omega_\ell)}, \tag{33}$$

where ϱ_ℓ is equal to 1 if one of the side of Ω_ℓ is a γ^-_m and intersects at least two subdomains $\bar{\Omega}_{\ell'}, \ell' \neq \ell$ and 0 otherwise.

Proof. We assume that $\gamma_m^- \subset \cup_{1\le\mu\le I}\,\Omega_\mu$, where I is a nonnegative integer (see Figure 2). In this case, we have $w_{\delta|\Omega_\mu} \in H_1^{\frac{1}{2}-\varepsilon}(\gamma_m^-)$ for any $\varepsilon > 0$ if Ω_μ touches the axis $\{r = 0\}$, then $w_{\delta|\Omega_\mu} \in H_1^{\frac{1}{2}-\varepsilon}(\gamma_m^- \cap \partial\Omega_\mu)$ for any $\varepsilon > 0$, otherwise $w_{\delta|\Omega_\mu} \in H^{\frac{1}{2}-\varepsilon}(\gamma_m^-)$. In an other hand and according to [9, Rem 2.10 page 11], we have for $\varepsilon > 0$ and for any part γ of γ_m^-, the extension by zero is continuous from $H^{\frac{1}{2}-\varepsilon}(\gamma)$ onto $H^{\frac{1}{2}-\varepsilon}(\gamma_m^-)$ and its norm is bounded by $c\varepsilon^{-1}$:

$$\|.\|_{H^{\frac{1}{2}-\varepsilon}(\gamma_m^-)} \le c\varepsilon^{-1}\,\|.\|_{H^{\frac{1}{2}-\varepsilon}(\gamma)}\,. \tag{34}$$

This result remains valid for the space $H_1^{\frac{1}{2}-\varepsilon}(\gamma)$. To unify the two cases $\ell > L_0$ and $\ell \le L_0$, one will use the fact that the norms $\|.\|_{H^{\frac{1}{2}}(\gamma_m^-)}$ and $\|.\|_{H_1^{\frac{1}{2}}(\gamma_m^-)}$ are equivalent if γ_m^- is far from the axis $\{r = 0\}$, with constants depending on the measure of γ_m^-. In the same way if Ω_μ is far from $\{r = 0\}$, the norms $\|.\|_{H^1(\Omega_\mu)}$ and $\|.\|_{H_1^1(\Omega_\mu)}$ are equivalent. In the general case, we can consider that the constants depend on the diameter of Ω. Since $[w_\delta] = w_{\delta|\gamma_m^-} - \Phi_{|\gamma_m^-}$ where $\Phi_{|\gamma_m^-} = \sum_{1\le i\le I} \tilde{w}^i_{\delta|\gamma_m^-}$ and $\tilde{w}^i_{\delta|\gamma_m^-}$ is the extension of $w^i_{\delta|\partial\Omega_\mu\cap\gamma_m^-}$ on γ_m^- and using (14), we obtain

$$\begin{aligned} \left|\int_{\gamma_m^-} \frac{\partial u}{\partial n_m}\,[w_\delta]\,(\tau)\,d\tau\right| &= \int_{\gamma_m^-} \left(\frac{\partial u}{\partial n_m} - \psi^+\right)(\Phi - w_\delta)d\tau \\ &\le c\left\|\frac{\partial u}{\partial n_m}\right\|_{H^{-\frac{1}{2}+\varepsilon}(\gamma_m^-)} \left(\left\|w_{\delta|\Omega_{\gamma_m^-}}\right\|_{H_1^{\frac{1}{2}-\varepsilon}(\gamma_m^-)} + \|\Phi\|_{H_1^{\frac{1}{2}-\varepsilon}(\gamma_m^-)}\right), \end{aligned} \tag{35}$$

where $\psi^+ = \pi^+_{N_m-2}\left(\frac{\partial u}{\partial n_m}\right)$ and π_N^+ is the projection opertor from $L_1^2(\Lambda)$ onto $\mathbb{P}_N(\Lambda)$ defined in [2, § IV.2.b]. Applying the inequalities (34) and

$$\|\Phi\|_{H_1^{\frac{1}{2}-\varepsilon}(\gamma_m^-)} \le \sum_{1\le i\le I} \left\|w^i_{\delta|\Omega_\mu}\right\|_{H_1^{\frac{1}{2}-\varepsilon}(\gamma_m^-)},$$

we obtain :

$$\begin{aligned} \left|\int_{\gamma_m^-} \frac{\partial u}{\partial n_m}\,[w_\delta]\,(\tau)\,d\tau\right| &\le C\left\|\frac{\partial u}{\partial n_m} - \psi^+\right\|_{H_1^{-\frac{1}{2}+\varepsilon}(\gamma_m^-)} \\ &\quad \left(\left\|w_{\delta|\Omega_{\gamma_m^-}}\right\|_{H_1^{\frac{1}{2}-\varepsilon}(\gamma_m^-)} + c\varepsilon^{-1}\sum_{1\le i\le I}\left\|w^i_{\delta|\Omega_\mu}\right\|_{H_1^{\frac{1}{2}-\varepsilon}(\gamma_\mu)}\right) \\ &\le C(1+c\varepsilon^{-1})\left\|\frac{\partial u}{\partial n_m} - \psi^+\right\|_{H_1^{-\frac{1}{2}+\varepsilon}(\gamma_m^-)} \left(\left\|w_{\delta|\Omega_{\gamma_m^-}}\right\|_{H_1^1(\Omega_m)} + \sum_{1\le i\le I}\left\|w^i_{\delta|\Omega_\mu}\right\|_{H_1^1(\Omega_\mu)}\right). \end{aligned}$$

In addition, for $\varepsilon = 1/\log N_m$ we obtain

$$\left\|\frac{\partial u}{\partial n_m} - \psi^+\right\|_{H_1^{-\frac{1}{2}+\varepsilon}(\gamma_m^-)} \le cN_m^{(\varepsilon-\frac{1}{2})-(s_m-\frac{3}{2})}\left\|\frac{\partial u}{\partial n_m}\right\|_{H_1^{s_m-\frac{3}{2}}(\gamma_m^-)} \le ceN_m^{1-s_m}\,\|u\|_{H_1^{s_m}(\Omega_m)}\,.$$

From where it follows that

$$\left|\int_{\gamma_m^-} \frac{\partial u}{\partial n_m}\,[w_\delta]\,(\tau)\,d\tau\right| \le c(1+c\varepsilon^{-1})N_m^{1-s_m}\,\|u\|_{H_1^{s_m}(\Omega_m)}\,\|w_\delta\|_{H_1^1(\cup\Omega_\ell)}\,, \tag{36}$$

and

$$\frac{\left|\int_{\gamma_m^-} \frac{\partial u}{\partial n_m}\,[w_\delta]\,(\tau)\,d\tau\right|}{\|w_\delta\|_{H_1^1(\cup\Omega_\ell)}} \le cN_m^{-s_m}(\log N_m)\,\|u\|_{H_1^{s_m+1}(\Omega_m)}\,.$$

Finally by summing over m we deduce (33). In the conforming case we can eliminate the term $(\log N_m)$ since we have $w_{\delta|\Omega_\mu} \in H_1^{\frac{1}{2}}(\gamma_m^-)$. ■

We are now able to state the following estimate error.

Proposition 8 *Let f such that $f_{|\Omega_\ell} \in H_1^{\sigma_\ell}(\Omega_\ell)$, $\sigma_\ell > 1$ ($\sigma_\ell > \frac{3}{2}$ if $\ell \le L_0$). Let u be a solution of Problem (12) such that $u_{|\Omega_\ell} \in H_1^{s_\ell+1}(\Omega_\ell)$, $s_\ell > \frac{1}{2}$ ($s_\ell > \frac{3}{2}$ if $\ell \le L_0$) and u_δ be the solution of Problem (16), we have:*

$$\|u - u_\delta\|_{H_1^1(\cup\Omega_\ell)} \le c\sum_{\ell=1}^{L}[(1+\lambda_\ell)^{\frac{1}{2}}N_\ell^{-s_\ell}(\log N_\ell)^{\varrho_\ell}\,\|u\|_{H_1^{s_\ell+1}(\Omega_\ell)} + N_\ell^{-\sigma_\ell}\,\|f_{|\Omega_\ell}\|_{H_1^{\sigma_\ell}(\Omega_\ell)}] \tag{37}$$

where c is a nonnegative constant and ϱ_ℓ is defined in the Proposition 7.

Proof. Returning to the inequality (19). Concerning the integration error on the external forces we have [2, Theo VIII.2.6]:

$$\left|\int_\Omega fw_\delta rdrdz - (\mathcal{I}_\delta f, w_\delta)_\delta\right| \leq \sum_{\ell=1}^{L} N_\ell^{-\sigma_\ell} \left\|f_{|\Omega_\ell}\right\|_{H_1^{\sigma_\ell}(\Omega_\ell)}. \tag{38}$$

For the consistence error, we set $\delta - 1 = (N_1 - 1, N_2 - 1, ..., N_L - 1)$ and $x_{\delta-1}$ such that $x_{\delta-1}|_{\Omega_\ell} = \Pi^{+,1}_{N_\ell-1}u$ where $\Pi^{+,1}_{N_\ell-1}$ is the projection operator from $H_1^1(\Omega_\ell)$ into $\mathbb{P}_{N_\ell-1}(\Omega_\ell)$ defined in [2, § V.3.b] and verifying
$\left\|u - \Pi^{+,1}_{N_\ell-1}u\right\|_{H_1^1(\Omega_\ell)} \leq N_\ell^{-s_\ell}||u_\ell||_{H_1^{s_\ell+1}(\Omega_\ell)}$.
We have

$$\begin{aligned}
|-a_\delta(v_\delta, w_\delta) + a(v_\delta, w_\delta)| &= |a(v_\delta - x_{\delta-1}, w_\delta) - a_\delta(v_\delta - x_{\delta-1}, w_\delta)| \\
&\leq c\{\left\|u - \Pi^{+,1}_{N_\ell-1}u\right\|_{H_1^1(\cup\Omega_\ell)} + \|u - v_\delta\|_{H_1^1(\cup\Omega_\ell)}\} \|w_\delta\|_{H_1^1(\cup\Omega_\ell)} \\
&\leq c\{\|u - v_\delta\|_{H_1^1(\cup\Omega_\ell)} + \sum_{\ell=1}^{L} N_\ell^{-s_\ell}||u_\ell||_{H_1^{s_\ell+1}(\Omega_\ell)}\} \|w_\delta\|_{H_1^1(\cup\Omega_\ell)}.
\end{aligned} \tag{39}$$

Finally by combining (19), (20), (33) with (38) and (39), we deduce (37). ■

We propose in the following to obtain an estimate of the error by relaxing the assumption of regularity on the exact solution which depends on the geometry of Ω. We introduce the space defined by

$$\begin{aligned}
\pounds_e^{(0)\lambda,q} &= \{S_e^{(0)},\ S_e^{(0)} = \chi_e(r_e^\lambda) r_e^\lambda \log(r_e)^q \varphi(\theta_e), \text{ where } \varphi \in \mathcal{C}^\infty[0, \omega_e] \\
\text{with } \varphi(0) &= \varphi(\omega_e) = 0\}
\end{aligned} \tag{40}$$

with $\lambda = \frac{n\pi}{\omega_e} + p$ for all integers $n > 0$, $p, q \geq 0$, and recall from, [2, V.7.c], that each function $S_{e_i}^{(0)}$ belongs to $\pounds_{e_i}^{(0)\lambda,q}$.

Proposition 9 *The following estimate holds for any complex number λ with $Re(\lambda) > 0$ and any nonnegative integer q. For any $S_{e_i}^{(0)}$ in $\pounds_{e_i}^{(0)\lambda,q}$*

$$\begin{aligned}
&\inf_{z_\delta \in X_\delta^\diamond} ||S_{e_i}^{(0)} - z_\delta||_{H_1^1(\cup\Omega_\ell)} \leq cN_{e_i}^{-2\lambda}(\log N_{e_i})^{q+\frac{1}{2}}, \\
&N_{e_i} = \min\{N_\ell,\ \Omega_\ell \cap supp(S_{e_i}^{(0)}) \neq \varnothing, 1 \leq \ell \leq L\}.
\end{aligned} \tag{41}$$

Proof. We can adopt the proof of the conforming case [2, V.7.c] since we use a conforming decomposition in the neighborhood of a singular point and the support of a singular function is compact, so it can be chosen sufficiently small to not intersects any other corner. ■

Theorem 10 *For any function $f \in H_+^{s-1}(\Omega)$, $s > \frac{5}{2}$, the following error estimates, respectively in $H_1^1(\cup\Omega_\ell)$ and $L_1^2(\Omega)$, hold between the solution $u \in H_1^1(\Omega)$ of Problem (12) and the solution u_δ of Problem (16)*:

$$1.\ \|u - u_\delta\|_{H_1^1(\cup\Omega_\ell)} \leq c(1+\lambda_\delta)^{\frac{1}{2}} \sup\{N_\delta^{1-s}, E_\delta\} \|f\|_{H_1^{s-1}(\Omega)}, \tag{42}$$

$$2.\ \|u - u_\delta\|_{L_1^2(\Omega)} \leq c(1+\lambda_\delta)^{\frac{1}{2}} \sup\{N_\delta^{1-s}, N_\delta^{-1}(\log N_\delta)^\varrho E_\delta\} \|f\|_{H_1^{s-1}(\Omega)} \tag{43}$$

where $N_\delta = \min\{N_\ell, 1 \leq \ell \leq L\}$, λ_δ is given by (21), $E_\delta = \max\{E_\ell, 1 \leq \ell \leq L\}$,

$$E_\ell = \begin{cases} 0 & \text{if } \bar{\Omega}_\ell \text{ does not contain any } e_i, \\ N_{e_i}^{-4}(\log N_{e_i})^{\frac{3}{2}} & \text{if } \bar{\Omega}_\ell \text{ contains } e_i \text{ with } \omega_{e_i} = \frac{\pi}{2}, \\ N_{e_i}^{-\frac{4}{3}}(\log N_{e_i})^{\frac{1}{2}} & \text{if } \bar{\Omega}_\ell \text{ contains } e_i \text{ with } \omega_{e_i} = \frac{3\pi}{2}, \end{cases}$$

N_{e_i} is given by (41) and *ϱ is zero in conforming decomposition and 1 otherwise.*
Proof. 1. Using the decomposition (10) and writing any $v_\delta \in X_\delta^\diamond$ on the form:

$$v_\delta = w_\delta + \gamma_e^{(0)}\chi_e(r_e)z_\delta + \sum_{n=2} \gamma_e^{(0)n}\chi_e(r_e)z_\delta^n, \tag{44}$$

where w_δ, z_δ and z_δ^ℓ are in $X_\delta^\diamond$, we obtain:

$$\begin{aligned}\inf_{v_\delta\in X_\delta^\diamond}\|u-v_\delta\|_{H_1^1(\cup\Omega_\ell)} \quad &\leq \quad c(\inf_{w_\delta\in X_\delta^\diamond}\|u_{reg}-w_\delta\|_{H_1^1(\cup\Omega_\ell)} \\ &+\inf_{z_\delta\in X_\delta^\diamond}\sum_{\ell=2}\left|\gamma_e^{(0)}\right|\left\|S_e^{(0)}-z_\delta\right\|_{H_1^1(\Omega_\ell)} \\ &+\inf_{z_\delta\in X_\delta^\diamond}\sum_{\ell=2}\sum_{n=2}\left|\gamma_e^{(0)n}\right|\left\|S_e^{(0)}-z_\delta^n\right\|_{H_1^1(\Omega_\ell)}).\end{aligned} \tag{45}$$

Since $S_e^{(0)}=\chi_e(r_e^\lambda)r_e^\lambda(\log r_e)^q\varphi(\theta_e)$ for $\lambda=\frac{n\pi}{\omega_j}+p$, $n>0$, $p\geq 0$, $q\geq 0$, and using the inequality (41) with $q=0$ and $\lambda=\frac{\pi}{\omega_e}$, we have

$$\left\|S_e^{(0)}-z_\delta\right\|_{H_1^1(\cup\Omega_\ell)} \quad \leq \quad N_e^{-4}(\log N_e)^{\frac{3}{2}} \text{ if } \omega_e=\frac{\pi}{2} \tag{46}$$

and

$$\left\|S_e^{(0)}-z_\delta\right\|_{H_1^1(\cup\Omega_\ell)} \quad \leq \quad N_e^{-\frac{4}{3}}(\log N_e)^{\frac{1}{2}} \text{ if } \omega_e=\frac{3\pi}{2}. \tag{47}$$

In addition, we have from 11:

$$\left|\gamma^{(0)n}\right|\leq c\,\|f\|_{H_1^{s-1}(\Omega)} \text{ for } s>2. \tag{48}$$

Finally, combining the inequalities (45), (46), (47), (48) and Proposition 8, we obtain (42).

2. Thanks to the Aubin-Nitsche method of duality, we have

$$\|u-u_\delta\|_{L_1^2(\Omega)}=\sup_{g\in L_1^2(\Omega)}\frac{\int_\Omega (u-u_\delta)(r,z)g(r,z)rdrdz}{\|g\|_{L_1^2(\Omega)}}.$$

For any function g in $L_1^2(\Omega)$ and ℓ, $1\leq\ell\leq L$, we note χ_ℓ the solution in $H_{1\diamond}^1(\Omega_\ell)$ of the variational formulation associated to the problem $-\Delta\chi_\ell=g_\ell$ in Ω_ℓ with $\chi_\ell=0$ on $\partial\Omega_\ell$ if $\ell\geq L_0$, and $\chi_\ell=0$ on Γ_ℓ if $\ell\leq L_0$. Since Ω_ℓ is convex, $\chi_\ell\in H_1^2(\Omega_\ell)$ and verifies $\|\chi_\ell\|_{H_1^2(\Omega_\ell)}\leq c\,\|g_\ell\|_{L_1^2(\Omega_\ell)}$. We set χ such that $\chi_{|\Omega_\ell}=\chi_\ell$ and notice that $\chi\in H_{1\diamond}^1(\Omega)$. We define $\chi_{\delta-1}\in H_{1\diamond}^1(\cup\Omega_\ell)$ by

$\chi_{\delta-1|\Omega_\ell}=\tilde{\Pi}_{N_\ell-1}^{+,1,\diamond}\chi_\ell$ if $1\leq\ell\leq L_0$ and $\chi_{\delta-1|\Omega_\ell}=\tilde{\Pi}_{N_\ell-1}^{-,1,\diamond}\chi_\ell$ if $L_0\leq\ell\leq L$

where $\tilde{\Pi}_{N_\ell-1}^{+,1,\diamond}: H_{1\diamond}^1(\Omega_\ell)\to\mathbb{P}_{N_\ell-1}^{\diamond}(\Omega_\ell)=\{v\in\mathbb{P}_{N_\ell-1}(\Omega_\ell), v=0 \text{ on } \partial\Omega_\ell\backslash(Oz)\}$ and $\tilde{\Pi}_{N_\ell-1}^{-,1,\diamond}: V_{1\diamond}^1(\Omega_\ell)\to\mathbb{P}_{N_\ell-1}^{0}(\Omega_\ell)=\{v\in\mathbb{P}_{N_\ell-1}(\Omega_\ell), v=0 \text{ on } \partial\Omega_\ell\}$ are the projection operators defined in [2, Chapitre V] and which verify:

$$\|\chi-\chi_{\delta-1}\|_{H_1^1(\cup\Omega_\ell)}\leq cN_\delta^{-1}\,\|\chi\|_{H_1^2(\cup\Omega_\ell)}\leq c'N_\delta^{-1}\,\|g\|_{L_1^2(\Omega)}. \tag{49}$$

Such construction leads to

$$\begin{aligned}\int_\Omega (u-u_\delta)\,grdrdz &= \sum_{\ell=1}^{L}\int_{\Omega_\ell}\nabla\chi_\ell\nabla(u-u_\delta)d\tau-\sum_{\gamma_m^-\in\mathcal{S}}\int_{\gamma_m^-}(\frac{\partial\chi}{\partial n_m})[u-u_\delta]d\tau \\ &= a(\chi-\chi_{\delta-1},u-u_\delta)+\int_\Omega f\chi_{\delta-1}d\tau-(\mathcal{I}_\delta f,\chi_{\delta-1})_\delta-\sum_{\gamma_m^-\in\mathcal{S}}\int_{\gamma_m^-}(\frac{\partial\chi}{\partial n_m})[u-u_\delta]d\tau.\end{aligned}$$

By combining the continuity of a, the estimates (33), (38) and (49), we deduce (43). ■

4 The discretization in the general case

For each $k\neq 0$, the variational formulation of Problem (9) is written

$$\left\{\begin{array}{c}\text{Find } u^k\in V_{1\diamond}^1(\Omega) \text{ such that} \\ \forall v\in V_{1\diamond}^1(\Omega),\ a_k(u^k,v)=\sum_{\ell=1}^{L}\int_{\Omega_\ell}f^k\bar{v}\ rdrdz,\end{array}\right. \tag{50}$$

where

$$a_k(u^k,v)=\sum_{\ell=1}^{L}\int_{\Omega_\ell}\{\nabla_r u^k.\nabla_r\bar{v}\ rdrdz+k^2u^k\bar{v}\ r^{-1}\}drdz \text{ and } V_{1\diamond}^1(\Omega)=\{v\in V_1^1(\Omega);\ v=0 \text{ on } \Gamma\}. \tag{51}$$

We can easily prove that the bilinear form $a_k(.,.)$ is continuous and coercive on $V^1_{1\diamond}(\Omega)$ endowed with the norm $\|.\|_{H^1_{(k)}(\cup\Omega_\ell)} = (\sum_{\ell=1}^{L}\|.\|^2_{H^1_{(k)}(\Omega_\ell)})^{\frac{1}{2}}$. Then Problem (50) admits an unique solution u^k such that:

$$\left\|u^k\right\|_{H^1_{(k)}(\cup\Omega_\ell)} \leq c\left\|f^k\right\|_{L^2_1(\Omega)}.$$

The discrete problem associated to Problem (50) is :

$$\left\{\begin{array}{l}\text{Find } u^k_\delta \text{ in } X^\circ_\delta(\Omega) \text{ such that}\\ \forall v_\delta \in X^\circ_\delta(\Omega), \quad a_{k,\delta}\left(u^k_\delta, v_\delta\right) = \left(\mathcal{I}_\delta f^k, v_\delta\right)_\delta,\end{array}\right. \tag{52}$$

where the form $a_{k,\delta}(.,.)$ is defined by:

$$a_{k,\delta}(u_\delta, v_\delta) = a_\delta(u_\delta, v_\delta) + k^2\left(\frac{u_\delta}{r}, \frac{v_\delta}{r}\right)_\delta.$$

Problem (52) is well posed. Indeed, we prove that there exist constants c and C independent of k such that [22, Prop. 2.4.2]:

$$\begin{aligned} a_{k,\delta}(u_\delta, v_\delta) &\leq \sum_{\ell=1}^{L}(|u_\ell|_{H^1_1(\Omega_\ell)} + k\|u_\ell\|_{L^2_{-1}(\Omega_\ell)})(|v_\ell|_{H^1_1(\Omega_\ell)} + k\|v_\ell\|_{L^2_{-1}(\Omega_\ell)})\\ &\leq C\|u_\delta\|_{H^1_{(k)}(\cup\Omega_\ell)}\|v_\delta\|_{H^1_{(k)}(\cup\Omega_\ell)}\end{aligned}$$

and

$$a_{k,\delta}(u_\delta, u_\delta) \geq c(\|u_\delta\|^2_{H^1_1(\cup\Omega_\ell)} + k^2\sum_{\ell=1}^{L}\|u_\ell\|^2_{L^2_{-1}(\Omega_\ell)}) \geq c\|u_\delta\|^2_{H^1_{(k)}(\cup\Omega_\ell)}.$$

Proposition 11 *Let u^k be the solution of Problem (50). We assume that $u^k_{|\Omega_\ell} \in H^{s_\ell+1}_1(\Omega_\ell)$ with $s_\ell > \frac{1}{2}$ ($s_\ell > \frac{5}{2}$ if $\ell \leq L_0$). Then there exists a constant c independent of k such that:*

$$\inf_{v_\delta\in X^\circ_\delta}\left\|u^k - v_\delta\right\|_{H^1_{(k)}(\cup\Omega_\ell)} \leq c\lambda_\delta^{\frac{1}{2}}\sum_{\ell=1}^{L}N_\ell^{-s_\ell}||u^k||_{H^{s_\ell+1}_{(k)}(\Omega_\ell)} \tag{53}$$

where λ_δ is defined in (21) for all mortar γ^+_μ, $1 \leq \mu \leq M^+$ and non-mortar γ^-_m, $1 \leq m \leq M^-$ such that $\gamma^+_\mu \cap \gamma^-_m$ has a nonnegative measure.

Proof. We recall, from [2, Chapter IV.4-V.4], that there exist a projection operator

$$\tilde{\pi}^{(k),1}_N : V^1_1(\Lambda) \longrightarrow \mathbb{P}^*_N(\Lambda), \quad \mathbb{P}^*_N(\Lambda) = \{v \in \mathbb{P}_N(\Lambda), v(-1) = 0\}, \ \Lambda =]-1,1[$$

such that:

$$\int_{-1}^{1}(\tilde{\varphi} - \tilde{\pi}^{(k),1}_N\tilde{\varphi})\psi d\tau = 0, \ \forall\psi \in \mathbb{P}_{N-2}(\Lambda) \text{ and } \left\|\tilde{\varphi} - \tilde{\pi}^{(k),1}_N\tilde{\varphi}\right\|_{H^1_{(k)}(\Lambda)} \leq CN^{1-s}\|\tilde{\varphi}\|_{H^s_{(k)}(\Lambda)} \tag{54}$$

and an interpolation operator $\mathcal{I}^{(k)}_{N_\ell}$ which verifies for $s_\ell > \frac{1}{2}$: ■

$$\left\|\varphi - \mathcal{I}^{(k)}_{N_\ell}\varphi\right\|_{H^1_{(k)}(\Omega_\ell)} \leq cN_\ell^{-s_\ell}\|\varphi\|_{H^{s_\ell+1}_{(k)}(\Omega_\ell)}$$

and

$$\left\|\varphi - \mathcal{I}^{(k)}_{N_\ell}\varphi\right\|_{H^1_{(k)}(\Gamma)} + N_\ell\left\|\varphi - \mathcal{I}^{(k)}_{N_\ell}\varphi\right\|_{L^2_1(\Gamma)} \leq c'N_\ell^{\frac{1}{2}-s_\ell}\ \|\varphi\|_{H^{s_\ell+1}_{(k)}(\Omega_\ell)} \tag{55}$$

where Γ is a side of Ω_ℓ.

We set $v^1_\ell = \mathcal{I}^{(k)}_{N_\ell}u^k_{|\Omega_\ell}$ in Ω_ℓ, v^1_δ such that $v^1_{\delta|\Omega_\ell} = v^1_\ell$ and $v^2_\delta = \sum_{\mu=1}^{M^+}\sum_{e\in\mathcal{C}_\mu}(u^k - v^1_{\delta|\Omega^+_\mu})(e)\tilde{\Phi}_{\mu,e}$ where $\tilde{\Phi}_{\mu,e}$ is defined in the proof of Proposition 4. We have

$$|k|\left\|\tilde{\Phi}_{\mu,e}\right\|_{L^2_{-1}(\Omega)} \leq c|k|N_\delta^{-1}.$$

By using the fact that $|k|N_\delta^{-1} \le 1$ and $\left\|\tilde\Phi_{\mu,e}\right\|_{H^1_1(\Omega)} \le c'$, we deduce that

$$\left\|v_\delta^2\right\|_{H^1_{(k)}(\cup\Omega_\ell)} \le c\sum_{\mu=1}^{M^+}(N_\mu^+)^{-s_\mu^+}\left\|u^k\right\|_{H^{s_\mu^++1}_{(k)}(\Omega_\mu^+)}.$$

In the same way, we have

$$\begin{aligned}\left\|v_\delta^2\right\|_{H^1_1(\gamma_\mu^+)} &\le c(N_\mu^+)^{\frac12-s_\mu^+}\left\|u^k\right\|_{H^{s_\mu^++1}_{(k)}(\Omega_\mu^+)},\\ \left\|v_\delta^2\right\|_{L^2_1(\gamma_m^-)} &\le c(N_\mu^+)^{-\frac12-s_\mu^+}\left\|u^k\right\|_{H^{s_\mu^++1}_{(k)}(\Omega_\mu^+)}.\end{aligned} \tag{56}$$

We set $\tilde\pi_\delta^{(k),\gamma_m^-} = \tilde\pi_{\delta,m}^{(k),1,(r)}$ if γ_m^- is parallel to (Or) and $\tilde\pi_\delta^{(k),\gamma_m^-} = \tilde\pi_{\delta,m}^{(k),1,(z)}$ if γ_m^- is parallel to (Oz),

$$\begin{aligned}v_\delta^{12} &= v_\delta^1 + v_\delta^2,\\ v_\delta^{3*} &= \tilde\pi_\delta^{(k),\gamma_m^-}(v^{12}_{\delta|\gamma_m^+} - v^{12}_{\delta|\gamma_m^-})(\tau)\tilde\chi_{N_m^-}(\sigma)] \text{ in } \bar\Omega_m^-, \quad v_\delta^{3*} = 0 \text{ in } \Omega\backslash\bar\Omega_m^-,\end{aligned}$$

and $v_\delta^3 = \sum_{m=1}^{M^-} v_\delta^{3*}$ where τ resp. σ is the tangential resp. normal variables on γ_m^- and $\tilde\chi_{N_m^-}$ is obtained from $\chi_{N_m^-}$ by homothety and translation ($\chi_{N_m^-}(\sigma) = (\frac{1-\sigma}{2})^{N_m^-}$).

If γ_m^- is parallel to (Or), let $z^{12} = v^{12}_{\delta|\gamma_m^+} - v^{12}_{\delta|\gamma_m^-}$. We have

$$\begin{aligned}&\left\|\tilde\pi_\delta^{(k),\gamma_m^-}(z^{12})(\tau)\tilde\chi_{N_m^-}(\sigma)\right\|_{H^{1,\gamma_m^-}_{(k)}(\Omega_m^-)} \le c(\left\|\tilde\pi_\delta^{(k),\gamma_m^-}(z^{12})\right\|_{H^1_1(\gamma_m^-)}\left\|\tilde\chi_{N_m^-}\right\|_{L^2_1(\Lambda_m^-)} +\\ &\left\|\tilde\pi_\delta^{(k),\gamma_m^-}(z^{12})\right\|_{L^2_1(\gamma_m^-)}\left\|\tilde\chi_{N_m^-}\right\|_{H^1_1(\Lambda_m^-)} + |k|\left\|\tilde\pi_\delta^{(k),\gamma_m^-}(z^{12})\right\|_{L^2_{-1}(\gamma_m^-)}\left\|\tilde\chi_{N_m^-}\right\|_{L^2_1(\Lambda_m^-)}).\end{aligned}$$

From where we deduce that

$$\begin{aligned}\left\|\tilde\pi_\delta^{(k),\gamma_m^-}(z^{12})(\tau)\tilde\chi_{N_m^-}(\sigma)\right\|_{H^{1,\gamma_m^-}_{(k)}(\Omega_m^-)} &\le c(\left\|\tilde\pi_\delta^{(k),\gamma_m^-}(z^{12})\right\|_{L^2_1(\gamma_m^-)}\left\|\tilde\chi_{N_m^-}\right\|_{H^1_1(\Lambda_m^-)}\\ &+\left\|\tilde\pi_\delta^{(k),\gamma_m^-}(z^{12})\right\|_{H^1_{(k)}(\gamma_m^-)}\left\|\tilde\chi_{N_m^-}\right\|_{L^2_1(\Lambda_m^-)}).\end{aligned} \tag{57}$$

In addition, we have

$$\begin{aligned}\left\|\tilde\pi_\delta^{(k),\gamma_m^-}(z^{12})\right\|_{L^2_1(\gamma_m^-)} &\le \left\|v^{12}_{\delta|\gamma_m^+} - v^{12}_{\delta|\gamma_m^-}\right\|_{L^2_1(\gamma_m^-)} \le c[(N_m^-)^{-\frac12-s_m^-}\left\|u^k\right\|_{H^{s_m^-+1}_{(k)}(\Omega_m^-)}\\ &+\sum_{\mu\in\mathcal{K}_m^-}(N_\mu^+)^{-\frac12-s_\mu^+}\left\|u^k\right\|_{H^{s_\mu^++1}_{(k)}(\Omega_\mu^+)}].\end{aligned} \tag{58}$$

We deduce that

$$\begin{aligned}\left\|v^{12}_{\delta|\gamma_m^+} - v^{12}_{\delta|\gamma_m^-}\right\|_{L^2_1(\gamma_m^-)}\left\|\chi_{N_m^-}\right\|_{H^1_1(\Lambda_m^-)} &\le c(N_m^-)^{-s_m^-}\left\|u^k\right\|_{H^{s_m^-+1}_{(k)}(\Omega_m^-)}\\ &+c_{1\gamma_m^-}\sum_{\mu\in\mathcal{K}_m^-}(N_\mu^+)^{-s_\mu^+}\left\|u^k\right\|_{H^{s_\mu^++1}_{(k)}(\Omega_\mu^+)}\end{aligned}$$

with $c_{1\gamma_m^-} = (N_m^-)^{\frac12}/\min_{\mu\in\mathcal{K}_m^-}(N_\mu^+)^{\frac12}$ and $c_{1\gamma_m^-} \le \lambda_\delta^{\frac12}$. According to (55) and (56) we have

$$\begin{aligned}&\left\|\tilde\pi_\delta^{(k),\gamma_m^-}(z^{12})\right\|_{H^1_{(k)}(\gamma_m^-)}\left\|\tilde\chi_{N_m^-}\right\|_{L^2_1(\Lambda_m^-)} \le c\left\|v^{12}_{\delta|\gamma_m^+} - v^{12}_{\delta|\gamma_m^-}\right\|_{H^1_{(k)}(\gamma_m^-)}\left\|\tilde\chi_{N_m^-}\right\|_{L^2_1(\Lambda_m^-)}\\ &\le c(N_m^-)^{-s_m^-}\left\|u^k\right\|_{H^{s_m^-+1}_{(k)}(\Omega_m^-)} + c_{2\gamma_m^-}\sum_{\mu\in\mathcal{K}_m^-}(N_\mu^+)^{-s_\mu^+}\left\|u^k\right\|_{H^{s_\mu^++1}_{(k)}(\Omega_\mu^+)}\end{aligned}$$

with $c_{2\gamma_m^-} = \max_{\mu\in\mathcal{K}_m^-}(N_\mu^+)^{\frac12}/(N_m^-)^{\frac12}$ and $c_{2\gamma_m^-} \le \lambda_\delta^{\frac12}$. By summing on m, we obtain

$$\left\|v_\delta^3\right\|_{H^1_{(k)}(\cup\Omega_\ell)} \le c\lambda_\delta^{\frac12}\sum_{\ell=1}^{L} N_\ell^{-s_\ell}||u^k||_{H^{s_\ell+1}_{(k)}(\Omega_\ell)}.$$

The case where γ_m^- is parallel to (Oz) is treated similarly.
Finally, the function $v_\delta = v_\delta^1 + v_\delta^2 + v_\delta^3$ belongs to discrete space X_δ° and verify the inequality (53).

Remark 12 *Note, that in the case of a conforming decomposition, we obtain the same estimation but with k and N_δ, chosen arbitrarily.*

In the same way that the axisymmetric case, we can prove the following error estimates.

Proposition 13 *Let f^k be a function such that $f^k_{|\Omega_\ell} \in H_1^{\sigma_\ell}(\Omega_\ell)$ with $\sigma_\ell > 1$ ($\sigma_\ell > \frac{3}{2}$ if $\ell \leq L_0$). The following error estimate holds between the solution u^k of problem (50) such that $u^k_{|\Omega_\ell} \in H_1^{s_\ell+1}(\Omega_\ell)$, $s_\ell > \frac{1}{2}$ ($s_\ell > \frac{5}{2}$ if $\ell \leq L_0$), and the solution u_δ^k of Problem (52):*

$$\left\|u^k - u_\delta^k\right\|_{H^1_{(k)}(\cup\Omega_\ell)} \leq c(1+\lambda_\delta)^{\frac{1}{2}} \sum_{\ell=1}^{L} N_\ell^{-s_\ell} (\log N_\ell)^{\varrho_\ell} \left\|u^k\right\|_{H^{s_\ell+1}_{(k)}(\Omega_\ell)} + c \sum_{\ell=1}^{L} N_\ell^{-\sigma_\ell} \left\|f^k\right\|_{H^{\sigma_\ell}_{(k)}(\Omega_\ell)}.$$

where ϱ_ℓ is defined in the Proposition 7 and c is a constant independent of k.

Theorem 14 *For any function $f^k \in H_-^{s-1}(\Omega)$, with $s > \frac{5}{2}$ the following error estimates hold between u^k the solution of Problem (50), and u_δ^k the solution of Problem (52):*

1. $\left\|u^k - u_\delta^k\right\|_{H^1_{(k)}(\cup\Omega_\ell)} \leq c(1+\lambda_\delta)^{\frac{1}{2}} \sup\{N_\delta^{1-s}, E_\delta\} \left\|f^k\right\|_{H^{s-1}_{(k)}(\Omega)}$,
2. $\left\|u^k - u_\delta^k\right\|_{L^2_1(\cup\Omega_\ell)} \leq c(1+\lambda_\delta)^{\frac{1}{2}} \sup\{N_\delta^{1-s}, N_\delta^{-1}\log(N_\delta)^{\varrho} E_\delta\} \left\|f^k\right\|_{H_1^{s-1}(\Omega)}$

where ϱ and E_δ are defined in theorem 10 and c is a constant independent of k.

5 Tridimensional Problem, Fourier truncature

Back to the original three-dimensional problem (8) and let $\breve{u}$ its solution and, for an integer $K > 0$, let $\breve{u}_K$ be a truncation of its Fourier series. We intend to approximate $\breve{u}_K$ by a truncated Fourier series whose coefficients are polynomials. We define $\breve{u}_K$ and $\breve{u}_{K,\delta}$ by:

$$\breve{u}_K(x,y,z) = \frac{1}{\sqrt{2\pi}} \sum_{|k|\leq K} u^k(r,z)\, e^{ik\theta},\ \breve{u}_{K,\delta}(x,y,z) = \frac{1}{\sqrt{2\pi}} \sum_{|k|\leq K} u_\delta^k(r,z)\, e^{ik\theta}, \tag{59}$$

where $u_\delta^0(r,z)$ is solution of Problem (16) for datum f^0 and $u_\delta^k(r,z)$, $k \neq 0$, is solution of Problem (52) for datum f^k, f^k being the Fourier coefficients of $\breve{f}$. But these Fourier coefficients on the data are generally not known accurately and are calculated using the quadrature formula. Then, we define their interpolate by:

$$f_K^k(r,z) = \frac{\sqrt{2\pi}}{2K+1} \sum_{|m|\leq K} \breve{f}(r,\theta_m,z) e^{-ik\theta_m}\ ,\ \ \theta_m = \frac{2m\pi}{2K+1}.$$

The function $\breve{f}_K(r,z) = \frac{1}{\sqrt{2\pi}} \sum_{|k|\leq K} f_K^k(r,z) e^{ik\theta}$ coincides with $\breve{f}$ on all the θ_m and can then be seen as an interpolate of $\breve{f}$. We define then the approximate $\breve{u}^*_{K,\delta}$ by

$$\breve{u}^*_{K,\delta}(x,y,z) = \frac{1}{\sqrt{2\pi}} \sum_{|k|\leq K} u^k_{K,\delta}(r,z)\, e^{ik\theta} \tag{60}$$

where $u^0_{K,\delta}(r,z)$ is solution of Problem (16) for datum f_K^0 and $u^k_{K,\delta}(r,z)$, $k \neq 0$, is solution of Problem (52) for datum f_K^k. We will estimate, in the following the error between the exact solution $\breve{u}$ and the solution approached tripling $\breve{u}^*_{K,\delta}$, by truncation of Fourier series, numerical integration and approximation by the spectral method. The basic formulas for this are the two-dimensional error estimates of the preceding paragraph and the following formula of truncation on the exact solution [2, (VII.1.3) et (II.1.8)]:

$$||\breve{u} - \breve{u}_K||_{H^t(\breve{\Omega})} \leq cK^{t-s} ||\breve{u}||_{H^s(\breve{\Omega})}. \tag{61}$$

According to the definitions of $\breve{u}_K$ and $\breve{u}^*_{K,\delta}$, we note that $\breve{u}^*_{K,\delta}$ does not belong to $H^1(\breve{\Omega})$, so we define the new norm

$$\|.\|_{H^1(\cup\breve{\Omega}_\ell)} = \sum_{\ell=1}^{L} \|.\|_{H^1(\breve{\Omega}_\ell)}.$$

Theorem 15 *We assume that* $\breve{f} \in H^{s-1}(\breve{\Omega})$, $s > \frac{5}{2}$. *We have*

$$1.\ \left\|\breve{u} - \breve{u}^*_{K,\delta}\right\|_{H^1(\cup\breve{\Omega}_\ell)} \leq c(1+\lambda_\delta)^{\frac{1}{2}}\{\sup(N_\delta^{1-s}, E_\delta) + K^{1-s}\} \left\|\breve{f}\right\|_{H^{s-1}(\breve{\Omega})} \quad (62)$$

$$2.\ \left\|\breve{u} - \breve{u}^*_{K,\delta}\right\|_{L^2(\breve{\Omega})} \leq c(1+\lambda_\delta)^{\frac{1}{2}}\{\sup(N_\delta^{1-s}, N_\delta^{-1}\log(N_\delta)^{\varrho}E_\delta) + K^{1-s}\} \left\|\breve{f}\right\|_{H^{s-1}(\breve{\Omega})} \quad (63)$$

where ϱ *and* E_δ *are defined in theorem 10.*

Proof. Error processing is similar that of [2] associated with a conform breakdown with special care on the analysis of two-dimensional non-conformities. We recall here the main steps.

1. The triangle inequality yields:

$$\left\|\breve{u} - \breve{u}^*_{K,\delta}\right\|_{H^1(\cup\breve{\Omega}_\ell)} \leq \left\|\breve{u} - \breve{u}_{K,\delta}\right\|_{H^1(\cup\breve{\Omega}_\ell)} + \left\|\breve{u}_{K,\delta} - \breve{u}^*_{K,\delta}\right\|_{H^1(\cup\breve{\Omega}_\ell)}. \quad (64)$$

We will treat these two terms one by one. For the first one, note that we have

$$\left\|\breve{u}_K - \breve{u}_{K,\delta}\right\|_{H^1(\cup\breve{\Omega}_\ell)} \leq c \sum_{|k|\leq K} \left\|u^k - u^k_\delta\right\|_{H^1_{(k)}(\cup\Omega_\ell)}.$$

So by using the triangular inequality and (61), we obtain

$$\begin{aligned}\left\|\breve{u} - \breve{u}_{K,\delta}\right\|_{H^1(\cup\breve{\Omega}_\ell)} &\leq \left\|\breve{u} - \breve{u}_K\right\|_{H^1(\cup\breve{\Omega}_\ell)} + \left\|\breve{u}_K - \breve{u}_{K,\delta}\right\|_{H^1(\cup\breve{\Omega}_\ell)} \\ &\leq c\{K^{-s}||\breve{u}||_{H^{s+1}(\breve{\Omega})} + \sum_{|k|\leq K} \left\|u^k - u^k_\delta\right\|_{H^1_{(k)}(\cup\Omega_\ell)}\}.\end{aligned}$$

According to Theorem 10, Theorem 14 and the fact that

$$\sum_{k\in\mathbb{Z}} \left\|f^k\right\|_{H^{s-1}_{(k)}(\Omega)} \simeq \left\|\breve{f}\right\|_{H^{s-1}(\breve{\Omega})} \text{ and } \|.\|_{H^s_1(\Omega)} = \|.\|_{H^s_{(0)}(\Omega)} \quad (65)$$

we have

$$\left\|\breve{u} - \breve{u}_{K,\delta}\right\|_{H^1(\cup\breve{\Omega}_\ell)} \leq c(1+\lambda_\delta)^{\frac{1}{2}}\{\sup(N_\delta^{1-s}, E_\delta) + K^{-s}\} \left\|\breve{f}\right\|_{H^{s-1}(\breve{\Omega})}. \quad (66)$$

For the second term of (64), we use the uniform ellipticity and continuity of $a_\delta(.,.)$ and $a_{k,\delta}(.,.)$ to write for all $k \in \mathbb{Z}$:

$$\begin{aligned}\left\|u^k_\delta - u^k_{K,\delta}\right\|_{H^1_{(k)}(\cup\Omega_\ell)} &\leq c||\mathcal{I}_\delta(f^k - f^k_K)||_{L^2_1(\cup\Omega_\ell)} \quad (67)\\ &\leq c\{||f^k - \mathcal{I}_\delta f^k||_{L^2_1(\cup\Omega_\ell)} + ||f^k_K - \mathcal{I}_\delta f^k_K||_{L^2_1(\cup\Omega_\ell)} + ||f^k - f^k_K||_{L^2_1(\cup\Omega_\ell)}\}.\end{aligned}$$

Thanks to (65) and (38), we have

$$\sum_{k\in\mathbb{Z}} ||f^k - \mathcal{I}_\delta f^k||_{L^2_1(\cup\Omega_\ell)} \leq cN_\delta^{1-s}||\breve{f}||_{H^{s-1}(\breve{\Omega})}, \quad (68)$$

and

$$||f^k_K - \mathcal{I}_\delta f^k_K)||_{L^2_1(\cup\Omega_\ell)} \leq cN_\ell^{1-s}||f^k_K||_{H^{s-1}_{(k)}(\Omega_\ell)} \leq cN_\ell^{1-s}(||f^k - f^k_K||_{H^{s-1}_{(k)}(\Omega_\ell)} + ||f^k||_{H^{s-1}_{(k)}(\Omega_\ell)}).$$

Using (61) and (7) we deduce that:

$$\sum_{k\in\mathbb{Z}} ||f^k - f^k_K||_{H^{s-1}_{(k)}(\cup\Omega_\ell)} \leq c||\breve{f} - \breve{f}_K||_{H^{s-1}(\breve{\Omega})} \leq c||\breve{f}||_{H^{s-1}(\breve{\Omega})}$$

and consequently that

$$\sum_{k\in\mathbb{Z}} ||f^k_K - \mathcal{I}_\delta f^k_K)||_{L^2_1(\cup\Omega_\ell)} \leq cN_\delta^{1-s}||\breve{f}||_{H^{s-1}(\breve{\Omega})}. \quad (69)$$

Finally, using

$$\sum_{k\in\mathbb{Z}} ||f^k - f^k_K||_{L^2_1(\cup\Omega_\ell)} \leq c||\breve{f} - \breve{f}_K||_{L^2(\breve{\Omega})} \leq cK^{1-s}||\breve{f}||_{H^{s-1}(\breve{\Omega})} \quad (70)$$

and combining with (67)-(69), we obtain:

$$\left\|\breve{u}_{K,\delta} - \breve{u}^*_{K,\delta}\right\|_{H^1(\cup\breve{\Omega}_\ell)} \leq c\sum_{k\in\mathbb{Z}} \left\|u^k_\delta - u^k_{K,\delta}\right\|_{H^1_{(k)}(\cup\Omega_\ell)} \leq c(K^{1-s} + N_\delta^{1-s})||\breve{f}||_{H^{s-1}(\breve{\Omega})} \quad (71)$$

and (62) is obtained by combining (66) and (71).

2. The estimation in the L^2 is obtained similarly. ∎

6 Strang & Fix algorithm

6.1 Axisymmetric case

We raise to study the case of a singularity due to a convex and a nonconvex corner. We note S_1 the first singular function appearing in the solution of Problem (12) and consider the Hilbert space $\mathring{X}_\delta = X_\delta^\diamond + \mathbb{R}S_1$. We note $\mathring{u}_\delta = u_\delta + \lambda S_1$ and $\mathring{v}_\delta = v_\delta + \mu S_1$. The space $\mathring{X}_\delta$ is provided with the norm

$$\|\mathring{v}_\delta\|_\circ = \sum_{\ell=1}^{L}(\|v_\ell\|^2_{H^1_1(\Omega_\ell)} + |\mu|^2 \|S_1\|^2_{H^1_1(\Omega_\ell)})^{\frac{1}{2}} \tag{72}$$

We define the discrete bilinear form on $\mathring{X}_\delta(\Omega)$ by:

$$\begin{aligned}\mathring{a}_\delta(\mathring{u}_\delta, \mathring{v}_\delta) &= a_\delta(u_\delta, v_\delta) + \sum_{\ell=1}^{L}(\lambda \int_{\Omega_\ell} \nabla S_1 \nabla v_\ell r dr dz \\ &+\mu \int_{\Omega_\ell} \nabla u_\ell \nabla S_1 r dr dz + \lambda\mu \int_{\Omega_\ell} (\nabla S_1)^2 r dr dz).\end{aligned} \tag{73}$$

Taking account of the singularities, Problem (16) becomes :

$$\begin{cases} \text{Find } \mathring{u}_\delta \in \mathring{X}_\delta(\Omega) \text{ such that} \\ \forall \mathring{u}_\delta \in \mathring{X}_\delta(\Omega)\ \mathring{a}_\delta(\mathring{u}_\delta, \mathring{v}_\delta) = (\mathcal{I}_\delta f, \mathring{v}_\delta)_\delta . \end{cases} \tag{74}$$

Theorem 16 *There exist positive constants α and β independent of δ such that:*

$$|\mathring{a}_\delta(\mathring{u}_\delta, \mathring{v}_\delta)| \leq \alpha \|\mathring{u}_\delta\|_\circ \|\mathring{v}_\delta\|_\circ\ \forall (\mathring{u}_\delta, \mathring{v}_\delta) \in \mathring{X}_\delta(\Omega) \times \mathring{X}_\delta(\Omega), \tag{75}$$

$$\forall \mathring{u}_\delta \in \mathring{X}_\delta(\Omega)\ \ \mathring{a}_\delta(\mathring{u}_\delta, \mathring{u}_\delta) \geq \beta \|\mathring{u}_\delta\|^2_\circ \tag{76}$$

and then for any $f \in C^0(\cup\bar{\Omega}_\ell)$, Problem (74) has an unique solution $\mathring{u}_\delta$ in $\mathring{X}_\delta$ such that:

$$\|\mathring{u}_\delta\|_\circ \leq c \|\mathcal{I}_\delta f\|_{L^2_1(\Omega)} .$$

Proof. We use the Cauchy Schwarz and the inequality $(a+b) \leq c(a^2+b^2)^{\frac{1}{2}}$ to deduce (75). For the proof of (76), begin by showing that for any $S \in L^2_1(\Omega)\backslash\mathbb{P}_N(\Omega)$, there exists $\rho_N < 1$ which depends only on N and S, such that:

$$\forall u_N \in \mathbb{P}_N(\Omega), \quad \int_\Omega S u_N r dr dz \leq \rho_N \|S\|_{L^2_1(\Omega)} \|u_N\|_{L^2_1(\Omega)} . \tag{77}$$

Indeed, if one of the two terms $\|S\|_{L^2_1(\Omega)}$ or $\|u_N\|_{L^2_1(\Omega)}$ is zero, then u_N or S is zero almost everywhere and the inequality (77) is checked. We assume that S isn't zero and we set $\Pi_N S$ the orthogonal projection of S on $\mathbb{P}_N(\Omega)$. We have $\int_\Omega (S - \Pi_N S) u_N r dr dz = 0$ and $\|S - \Pi_N S\|^2_{L^2_1(\Omega)} + \|\Pi_N S\|^2_{L^2_1(\Omega)} = \|S\|^2_{L^2_1(\Omega)}$. Whence, the term $S - \Pi_N S$ is never zero and we have

$$\frac{|\int_\Omega S u_N r dr dz|}{\|S\|_{L^2_1(\Omega)} \|u_N\|_{L^2_1(\Omega)}} =\leq \sqrt{1 - \frac{\|S - \Pi_N S\|^2_{L^2_1(\Omega)}}{\|S\|^2_{L^2_1(\Omega)}}} = \rho_N < 1.$$

Now, we use (77) to deduce that:

$$\mathring{a}_\delta(\mathring{u}_\delta, \mathring{u}_\delta) \geq (1-\rho) \sum_{\ell=1}^{L} \int_{\Omega_\ell} [(\nabla u_\ell)^2 + \lambda^2 (\nabla S_1)^2] r dr dz, \ (\rho = \sup\{\rho_{N_\ell}, 1 \leq \ell \leq L\}).$$

From where (76) is proven. ■

Theorem 17 *We assume that $f \in H^{s-1}_+(\Omega)$, $s > \frac{5}{2}$, then the following error estimate holds between the solution u of Problem (12) and the solution $\mathring{u}_\delta$ of Problem (74):*

$$\|u - \mathring{u}_\delta\|_\circ \leq c(1+\lambda_\delta)^{\frac{1}{2}} \sup\{N_\delta^{1-s}, \mathring{E}_\delta\} \|f\|_{H^{s-1}_1(\Omega)} \tag{78}$$

where $N_\delta = \min\{N_\ell, 1 \le \ell \le L\}$, $\mathring{E}_\delta = \max\left\{\mathring{E}_\ell, 1 \le \ell \le L\right\}$, N_{e_i} *is defined in the Proposition 9 and*

$$\mathring{E}_\ell = \begin{cases} 0 & \text{if } \bar{\Omega}_\ell \text{ does not contain any } e_i, \\ N_{e_i}^{-8}(\log N_{e_i})^{\frac{3}{2}} & \text{if } \bar{\Omega}_\ell \text{ contains } e_i \text{ with } \omega_j = \frac{\pi}{2}, \\ N_{e_i}^{\frac{-8}{3}}(\log N_{e_i})^{\frac{1}{2}} & \text{if } \bar{\Omega}_\ell \text{ contains } e_i \text{ with } \omega_j = \frac{3\pi}{2}. \end{cases} \quad (79)$$

Proof. We notice that $\mathring{a}_\delta(\mathring{v}_\delta, \mathring{w}_\delta) - \mathring{a}(\mathring{v}_\delta, \mathring{w}_\delta) = a_\delta(v_\delta, w_\delta) - a(v_\delta, w_\delta)$ and set $u = u_{reg} + \lambda S_1 + \mu S_2$ and $\mathring{v}_\delta = z_\delta + \lambda S_1 + \mu w_\delta$. This leads to

$$\inf_{\mathring{w}_\delta \in \mathring{X}_\delta} \|u - \mathring{v}_\delta\|_\circ \le \inf_{z_\delta \in X_\delta^\circ} \|u_{reg} - z_\delta\|_\circ + \inf_{w_\delta \in X_\delta^\circ} |\mu| \, \|S_2 - w_\delta\|_\circ + ...$$

The first term on the right is estimated in Proposition 4. For the second term, we use the definition $S_e^{(0)} = \chi_e(r_e^\lambda) r_e^\lambda (\log r_e)^q \varphi(\theta_e)$ and the inequality (41) with $\lambda = \frac{2\pi}{\omega_{e_j}}$. The case $\omega_{e_j} = \frac{\pi}{2}, q = 0$, gives

$$\inf_{w_\delta \in X_\delta^\circ} \|S_2 - w_\delta\|_\circ \le N_\ell^{-8} (\log N_\ell)^{\frac{3}{2}}. \quad (80)$$

And the case $\omega_{e_j} = \frac{3\pi}{2}, q = 0$, yields to

$$\inf_{w_\delta \in X_\delta^\circ} \|S_2 - w_\delta\|_\circ \le N_\ell^{-\frac{8}{3}} (\log N_\ell)^{\frac{1}{2}}. \quad (81)$$

Using (11) we obtain $\sup(|\lambda|, |\mu|) \le c \|f\|_{H_1^{s-1}(\Omega)}$ for $s > 2$. ■

6.2 General case

As in the axisymmetric case, we define the space $\ddot{X}_\delta = X_\delta^\circ + \mathbb{R} S_1$ and set for u_δ^k and $v_\delta^k \in \ddot{X}_\delta$:

$$\mathring{u}_\delta^k = u_\delta^k + \lambda S_1 \text{ and } \mathring{v}_\delta = v_\delta + \mu S_1.$$

We note that the first singularity is independent of k and that the singularity S_1 is the same of the axisymmetric case. We define the discrete bilinear form on $\ddot{X}_\delta$ by:

$$\begin{aligned} \mathring{a}_{k,\delta}\left(\mathring{u}_\delta^k, \mathring{v}_\delta\right) &= a_{k,\delta}\left(u_\delta^k, v_\delta\right) \\ &+ \sum_{\ell=1}^{L} (\lambda \int_{\Omega_\ell} \nabla S_1 \nabla v_\ell r dr dz + \mu \int_{\Omega_\ell} \nabla u_\ell^k \nabla S_1 r dr dz \\ &+ \lambda\mu \int_{\Omega_\ell} (\nabla S_1^2) r dr dz + \lambda k^2 \int_{\Omega_\ell} (S_1 v_\ell) r^{-1} dr dz \\ &+ \mu k^2 \int_{\Omega_\ell} (S_1 u_\ell^k) r^{-1} dr dz + \lambda\mu k^2 \int_{\Omega_\ell} (S_1^2) r^{-1} dr dz) \end{aligned} \quad (82)$$

and endow $\ddot{X}_\delta$ with the norm: $\|\mathring{v}_\delta\|_{\circ k} = \sum_{\ell=1}^{L} (\|v_{\delta|\Omega_\ell}\|^2_{H^1_{(k)}(\Omega_\ell)} + |\lambda|^2 \|S_{1|\Omega_\ell}\|^2_{H^1_{(k)}(\Omega_\ell)})^{\frac{1}{2}}$.

The discrete problem writes:

$$\begin{cases} \text{Find } \mathring{u}_\delta \in \ddot{X}_\delta \text{ such that} \\ \forall \mathring{v}_\delta \in \ddot{X}_\delta, \; \mathring{a}_{k,\delta}\left(\mathring{u}_\delta^k, \mathring{v}_\delta\right) = \left(\mathcal{I}_\delta f^k, \mathring{v}_\delta\right)_\delta. \end{cases} \quad (83)$$

Theorem 18 *For all* $f^k \in C^0\left(\cup \bar{\Omega}_\ell\right)$, *Problem (83) has an unique solution* $\mathring{u}_\delta^k$ *in* $\ddot{X}_\delta$ *such that there exists* C *independent of* k *verifying:* $\|\mathring{u}_\delta^k\|_{\circ k} \le C \|\mathcal{I}_\delta f^k\|_{L_1^2(\Omega)}$.

Proof. The Cauchy Schwarz inequality and the inequality $(a+b) \le \sqrt{2}(a^2+b^2)^{\frac{1}{2}}$ lead to the continuity of $\mathring{a}_{k,\delta}$. For the coercivity, we have from (77):

$$\mathring{a}_\delta\left(u_\delta^k + \lambda S_1, u_\delta^k + \lambda S_1\right) \ge (1-\rho) \sum_{\ell=1}^{L} \|\nabla u_\ell^k\|^2_{L_1^2(\Omega_\ell)} + \lambda^2 \|\nabla S_1\|^2_{L_1^2(\Omega_\ell)}].$$

We take $A = k^2[(\frac{u_\delta^k}{r}, \frac{u_\delta^k}{r})_\delta + 2\lambda \sum_{\ell=1}^{L} \int_{\Omega_\ell} (S_1 u_\delta^k) r^{-1} dr dz + \lambda^2 \sum_{\ell=1}^{L} \int_{\Omega_\ell} (S_1^2) r^{-1} dr dz]$ and we apply again (77) for the space L^2_{-1} and the scalar product $(S, u_N)_{-1} = \int_\Omega (S \; u_N) r^{-1} dr dz$ to deduce that

$$A \ge (1-\rho') \sum_{\ell=1}^{L} [\|u_\ell^k\|^2_{L^2_{-1}(\Omega_\ell)} + \lambda^2 \|S_1\|^2_{L^2_{-1}(\Omega_\ell)}] \text{ (since } |k| \ge 1).$$

As $\|.\|^2_{H^1_1(\Omega_\ell)}$ and $|.|^2_{H^1_1(\Omega_\ell)}$ are equivalent, we conclude while writing:

$$\mathring{a}_{k,\delta}\left(u^k_\delta + \lambda S_1, u^k_\delta + \lambda S_1\right) \geq c\sum_{\ell=1}^{L}[\left\|u^k_\ell\right\|^2_{H^1_{(k)}(\Omega_\ell)} + \lambda^2 \left\|S_1\right\|^2_{H^1_{(k)}(\Omega_\ell)}],$$

the constant c does not depend on k. The continuity and coercivity of $\mathring{a}_{k,\delta}$ allows to the result. ∎
Following the steps of Theorem 17, we have the below estimates.

Theorem 19 *Let u^k be the solution of Problem (50) and $\mathring{u}^k_\delta$ the solution of Problem (83). We assume that $f^k \in H^{s-1}_-(\Omega)$ with $s > \frac{5}{2}$ then we have*

$$\left\|u^k - \mathring{u}^k_\delta\right\|_{\circ k} \leq C(1+\lambda_\delta)^{\frac{1}{2}} \sup\{N^{1-s}_\delta, \mathring{E}_\delta\} \left\|f^k\right\|_{H^{s-1}_1(\Omega)}, \tag{84}$$

where N_δ and $\mathring{E}_\delta = \max\left\{\mathring{E}_\ell, 1 \leq \ell \leq L\right\}$ are defined in theorem 17.

6.3 Return to the tridimensional Problem

We consider $\mathring{u}_{K,\delta}(x,y,z) = \frac{1}{\sqrt{2\pi}} \sum_{|k|\leq K} \mathring{u}^k_\delta(r,z)\, e^{ik\theta}$ where $\mathring{u}^0_\delta(r,z)$ is the solution of Problem (74) for datum f^0 and $\mathring{u}^k_\delta(r,z)$, $(k \neq 0)$ is the solution of Problem (83) for datum f^k. And we consider $\mathring{u}^*_{K,\delta}(x,y,z) = \frac{1}{\sqrt{2\pi}} \sum_{|k|\leq K} \mathring{u}^k_{K,\delta}(r,z)\, e^{ik\theta}$ where $\mathring{u}^0_{K,\delta}(r,z)$ is the solution of Problem (74) for datum f^0_K and $\mathring{u}^k_{K,\delta}(r,z)$, $(k \neq 0)$ is the solution of Problem (83) for dataum f^k_K. We define also $\mathring{u}^*_{K,\delta}$ by

$$\mathring{u}^*_{K,\delta}(x,y,z) = \frac{1}{\sqrt{2\pi}} \sum_{|k|\leq K} \mathring{u}^k_{K,\delta}(r,z)\, e^{ik\theta}. \tag{85}$$

Then following the steps of Theorem 15 and using (78) and (84), we have for $\breve{f} \in H^{s-1}(\breve{\Omega})$, $s > \frac{5}{2}$

$$\left\|\breve{u} - \mathring{u}^*_{K,\delta}\right\|_{H^1(\cup\breve{\Omega}_\ell)} \leq c(1+\lambda_\delta)^{\frac{1}{2}}\{\sup(N^{1-s}_\delta, \mathring{E}_\delta) + K^{1-s}\} \left\|\breve{f}\right\|_{H^{s-1}(\breve{\Omega})} \tag{86}$$

where $\mathring{E}_\delta$ is defined in theorem 17

7 Numerical results

In this section, we will present numerical tests which would confirm our theoretical predictions in the axisymmetric and general cases. These tests will be made on three types of domains convex Ω^a or non-convex Ω^b, Ω^c see Figure 1. Each domain is broken up into convex subdomains what will enable us to highlight the convergence of the mortar method.

7.1 Axisymmetric case

Convex domain Ω^a : We consider the rectangle of Figure 1.A broken up into three subdomains Ω^a_1, Ω^a_2, Ω^a_3. For the first series of tests, we assume that $f = 0$ in Ω^a, $g = r^{5/2}$ if $z = \pm 1$ and 1 if $r = 1$. We present in Figures 3.(A-B) the layout of u in the parts Ω^a_1 with $N = 24$, resp. Ω^a_2 with $N = 28$ and Ω^a_3 with $N = 24$ and the layout of u without decomposition of domains. The Figure 3.C represents the isovalues of the solution u with $N = 24$ in the three subdomains.
We note that for N rather large, the three parts in Figure 3.A stick perfectly and that lines of the isovalues in Figure 3.C are continuous through the interfaces.

Nonconvex domain Ω^b: We consider now the domain of Figure 1.B broken up into 5 subdomains. In a first test, we consider the functions $f = r^{1/2}(z^2 + \frac{8}{25}r^2 - 1)$ in Ω^b, $g = 0$ if $z = \pm 1$ and $\frac{4}{25}r^{5/2}(1 - z^2)$ if not. We present in Figure 4.A the layout of u in parts Ω^b_1 with $N = 24$, Ω^b_2 with $N = 28$, Ω^b_3 with $N = 24$, Ω^b_4 with $N = 24$, and Ω^b_5 with $N = 24$. The zoom of the encircled part of Figure 4.A is presented in Figure 4.B.
We make a second test with $f = r^{1/2}z$ in Ω^b, $g = 0$ on the edge $\partial\Omega^b$. We present in Figure 5.A the layout of u in parts Ω^b_1 with $N = 28$, Ω^b_2 with $N = 30$, Ω^b_3 with $N = 30$, Ω^b_4 with $N = 30$, and Ω^b_5 with $N = 24$.

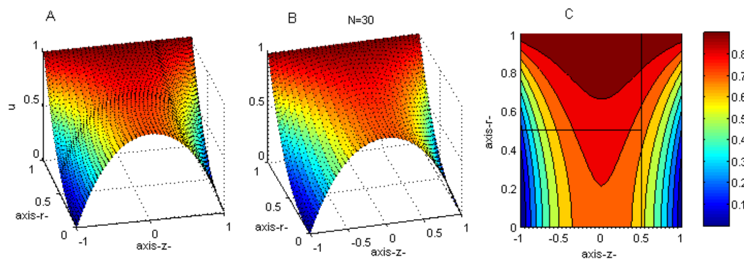


Figure 3:

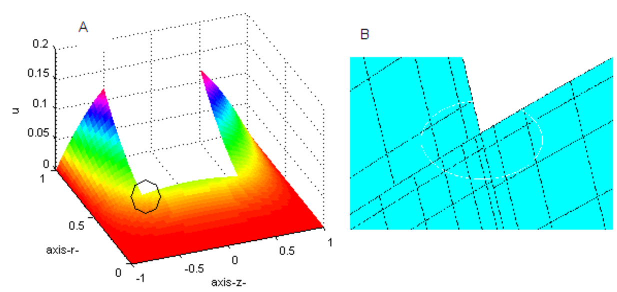


Figure 4:

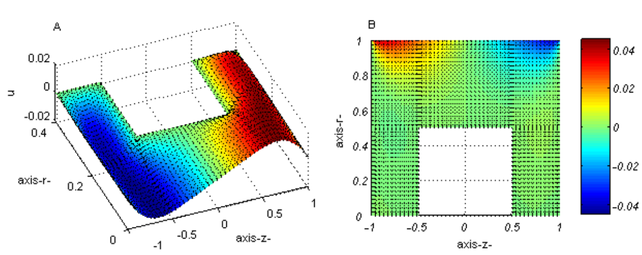
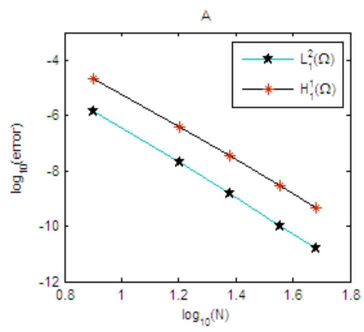


Figure 5:

It is noticed that as long as N_i are chosen in each subdomain of such kind, λ_δ defined in (21) is not large than the layouts in the various parts of the domain stick perfectly and Figure 5.A show continuity in the distribution of the colors through the interfaces.

Nonconvex domain Ω^c : We consider the nonconvex domain of Figure 1.C broken up into five subdomains Ω_1^c– Ω_5^c. We consider the following functions $f = r^{1/3}(r - 0.245)z$ in Ω^c, $g = 0.045z$ if $r = 1$ and 0 if not. In Figure 5.B we illustrate the isovalues of u, with $N = 24$ on Ω_1^c, Ω_3^c and Ω_5^c and $N = 20$ in Ω_2^c, Ω_4^c.

The error measure: We consider, in a first test, the singular function $u = r^{10/3}(z-1)$. In Figure 6.A, we give the curves $\log_{10} \|u - u_\delta\|_{L^2(\breve{\Omega})}$ and $\log_{10} \|u - u_\delta\|_{H^1(\cup\breve{\Omega}_\ell)}$ as functions of $\log_{10}(N)$.



B

N	Norme $L^2(\breve{\Omega})$	Norme $H^1(\cup\breve{\Omega}_\ell)$
8	$2,7348 \times 10^{-7}$	$5,667 \times 10^{-6}$
16	$3,1957 \times 10^{-9}$	$8,9067 \times 10^{-8}$
24	$2,1294 \times 10^{-10}$	$7,2774 \times 10^{-9}$
36	$1,3589 \times 10^{-11}$	$5,6555 \times 10^{-10}$
48	$1,9073 \times 10^{-12}$	$9,0036 \times 10^{-11}$

Figure 6:

We remark that as soon as N reached 48, one has an error of u of the order 10^{-11} in norm L^2 and of order of 4.10^{-10}. This confirms the fact that the spectral method is known by its precision. In Figure 6.A, the slope of the error is independent of N, this is in agreement with the estimates (42) and (43).

In a second test, we take the solution $u = r^{7/2}z$, we measure in Figure 6.B the errors $\|u - u_\delta\|_{L^2(\breve{\Omega})}$, $\|u - u_\delta\|_{H^1(\cup\breve{\Omega}_\ell)}$ on the domain of reference $\breve{\Omega}$. This table shows that the rate of convergence behaves like cN^{-7}, which gives us a doubling about convergence. This result is in conformity with the theoretical predictions. We calculate in Figure 7 the errors $\|u - u_\delta\|_{L^2(\breve{\Omega}_i^a)}$ and $\|u - u_\delta\|_{H^1(\breve{\Omega}_i^a)}$ $(i = 1, \cdots, 3)$ on the domain $\breve{\Omega}^a$. We give the estimates on each subdomain, and remark that these estimates depend on dimensions of each domain in addition to N.

	Ω_1^a		Ω_2^a		Ω_3^a	
N	$L^2(\Omega_1^a)$	$H^1(\Omega_1^a)$	$L^2(\Omega_2^a)$	$H^1(\Omega_2^a)$	$L^2(\Omega_3^a)$	$H^1(\Omega_3^a)$
8	$9,01 \times 10^{-8}$	$1,03 \times 10^{-7}$	$1,25 \times 10^{-7}$	$2,20 \times 10^{-6}$	$2,01 \times 10^{-7}$	$5,21 \times 10^{-6}$
16	$5,35 \times 10^{-10}$	$2,01 \times 10^{-9}$	$1,18 \times 10^{-9}$	$3,36 \times 10^{-8}$	$2,36 \times 10^{-9}$	$2,03 \times 10^{-7}$
24	$7,05 \times 10^{-12}$	$4,20 \times 10^{-10}$	$7,71 \times 10^{-11}$	$3,01 \times 10^{-9}$	$7,03 \times 10^{-11}$	$3,12 \times 10^{-9}$
36	$5,15 \times 10^{-13}$	$3,35 \times 10^{-12}$	$7,52 \times 10^{-13}$	$4,35 \times 10^{-11}$	$4,13 \times 10^{-12}$	$4,36 \times 10^{-10}$
48	2.01×10^{-14}	4.20×10^{-13}	$1,057 \times 10^{-13}$	$6,88 \times 10^{-12}$	2.89×10^{-13}	2.45×10^{-11}

Figure 7:

7.2 General case

Domain Ω^a: We consider the domain Ω^a of Figure 1.A. We take $f = r^{7/2}z\sin(x+y)$ in Ω^a and $g = 0$ on Γ. We represent the isovalues of u^0 in Figure 8.B with $N = 24$ and $K = 6$ and in Figure 8.A the layout of $\mathrm{Re}(u^1)$ in the parts Ω_1^a with $N = 24$ and $K = 6$, Ω_2^a with $N = 28$ and $K = 6$, Ω_3^a with $N = 28$ and $K = 6$.

We remark that the solution $u^{(k)}$, $k \neq 0$ being complex, we specify in Figures $\mathrm{Re}(u^{(k)})$ or $\mathrm{Im}(u^{(k)})$.

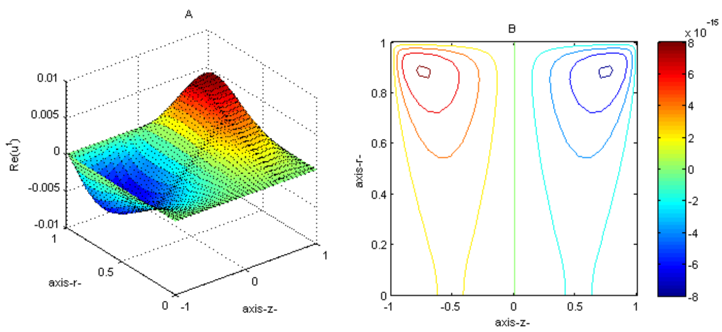


Figure 8:

Domain Ω^b: We consider the functions $f = cos(x+y+z)$ in Ω^b and $g = 0$ on Γ. We represent in Figure 9.A, the isovalues of u^0 with $N = 24$. For $K = 4$, we represent in Figure 9.B the layout of $\mathrm{Re}(u^1)$ in the parts Ω_1^b with $N = 24$, Ω_5^b with $N = 20$ and Ω_2^b, Ω_3^b, Ω_4^b with $N = 30$. We represent in Figure 9.C, the isovalues of $\mathrm{Re}(u^1)$ with $N = 24$.

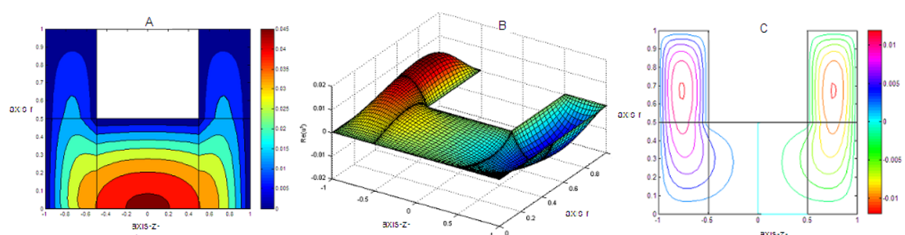


Figure 9:

Measure of the error

Domain $\breve{\Omega}^a$: We set the function $u = r^{7/3}(\cos^2\theta + \sin\theta)$, we fix $K = 3$. The calculus of norms $\|u - u_\delta\|_{L^2(\breve{\Omega}_i^a)}$, $\|u - u_\delta\|_{H^1(\breve{\Omega}_i^a)}$ $(i = 1, \cdots, 3)$, in the domain $\breve{\Omega}^a$ give, see Figure table3:

	$\breve{\Omega}_1^a$		$\breve{\Omega}_2^a$		$\breve{\Omega}_3^a$	
N	$L^2(\breve{\Omega}_1^a)$	$H^1(\breve{\Omega}_1^a)$	$L^2(\breve{\Omega}_2^a)$	$H^1(\breve{\Omega}_2^a)$	$L^2(\breve{\Omega}_3^a)$	$H^1(\breve{\Omega}_3^a)$
8	$2,01 \times 10^{-7}$	$2,80 \times 10^{-5}$	$1,79 \times 10^{-6}$	$2,80 \times 10^{-5}$	$1,09 \times 10^{-6}$	$2,12 \times 10^{-5}$
16	$3,12 \times 10^{-8}$	$1,23 \times 10^{-6}$	$8,10 \times 10^{-8}$	$1,9875 \times 10^{-6}$	$3,26 \times 10^{-8}$	$1,58 \times 10^{-6}$
24	$1,20 \times 10^{-9}$	$4,09 \times 10^{-7}$	$1,20 \times 10^{-8}$	$4,09 \times 10^{-7}$	$4,31 \times 10^{-9}$	$2,99 \times 10^{-7}$
36	$1,42 \times 10^{-10}$	$5,14 \times 10^{-8}$	$1,75 \times 10^{-9}$	$8,39 \times 10^{-8}$	$5,83 \times 10^{-10}$	$5,31 \times 10^{-8}$
48	$1,01 \times 10^{-11}$	$2,45 \times 10^{-9}$	$1,20 \times 10^{-10}$	$3,21 \times 10^{-9}$	$2,10 \times 10^{-11}$	$2,15 \times 10^{-9}$

Figure 10:

We remark that for a choice K which is not very high we find a very good precision for $N = 48$. Since s in the estimation (62), depends only on the regularity on u. Our choice to take $K \leq N_\delta$ in the estimates does not harm the numerical results.

Domain $\breve{\Omega}^b$: For the singular function $u = (x - y)\left(x^2 + y^2\right)^{\frac{5}{4}}$, we fix $K = 4$ and we give the curves $\log_{10}\|u - u_\delta\|_{L^2(\breve{\Omega}^b)}$ and $\log_{10}\|u - u_\delta\|_{H^1(\cup\breve{\Omega}_\ell^b)}$ as functions of $\log_{10}(N)$ in Figure 11.A. In these estimates of error, the slope is independent of N, and this is in perfect with agreement the theoretical estimates (62) and (63).

7.3 Strang and Fix algorithm

Finally we consider the function $u = r^{2,1}z$ and we use the algorithm of Strang and Fix on the domain Ω^b. We then notice a clear improvement of the errors on the curves $\log_{10}\|u - u_\delta\|_{L^2(\breve{\Omega}^b)}$

resp. $\log_{10} \|u - u_\delta\|_{H^1(\cup \breve{\Omega}^b_\ell)}$, as functions of $\log_{10}(N)$ shown in Figures 11.B resp. 11.C.

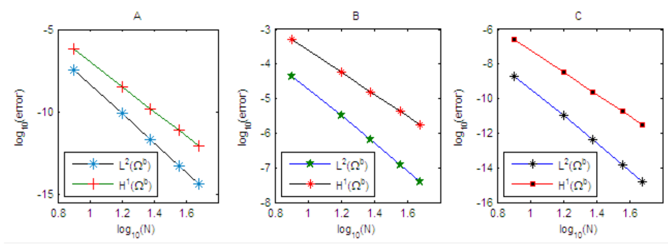


Figure 11:

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