DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS SERIES S
Volume **3**, Number **2**, June **2010**

doi:10.3934/dcdss.2010.3.xx

pp. **1–XX**

LOSS OF SMOOTHNESS AND ENERGY CONSERVING ROUGH WEAK SOLUTIONS FOR THE $3d$ EULER EQUATIONS

Claude Bardos

Laboratory J. L. Lions Université Pierre et Marie Curie, Paris, 75013, France
ALSO Wolfgang Pauli Institute, Vienna, Austria

Edriss S. Titi

Department of Mathematics and Department of Mechanical and Aerospace Engineering
The University of California, Irvine, CA 92697, USA
and
Department of Computer Science and Applied Mathematics
The Weizmann Institute of Science, Rehovot 76100, Israel

Abstract. A basic example of shear flow was introduced by DiPerna and Majda to study the weak limit of oscillatory solutions of the Euler equations of incompressible ideal fluids. In particular, they proved by means of this example that weak limit of solutions of Euler equations may, in some cases, fail to be a solution of Euler equations. We use this shear flow example to provide non-generic, yet nontrivial, examples concerning the loss of smoothness of solutions of the three-dimensional Euler equations, for initial data that do not belong to $C^{1,\alpha}$. Moreover, we show by means of this shear flow example the existence of weak solutions for the three-dimensional Euler equations with vorticity that is having a nontrivial density concentrated on non-smooth surface. This is very different from what has been proven for the two-dimensional Kelvin-Helmholtz problem where a minimal regularity implies the real analyticity of the interface. Eventually, we use this shear flow to provide explicit examples of non-regular solutions of the three-dimensional Euler equations that conserve the energy, an issue which is related to the Onsager conjecture.

This paper is dedicated to Professor V. Solonnikov, on the occasion of his 75th birthday, as token of friendship and admiration for his contributions to research in partial differential equations and fluid mechanics.

1. Introduction.

More than 250 years after the Euler equations have been written our knowledge of their mathematical structure and their relevance to describe the complicated phenomenon of turbulence is still very incomplete, to say the least. Both in two and three dimensions certain challenging problems concerning the Euler equations remain open. In particular, we still have no idea of whether three-dimensional solutions of the Euler equations, which start with smooth initial data, remain smooth all the time or whether they may become singular in finite time. In the case of finite time singularity it would be tempting to rely on weak solution formulation. However, there is almost no construction, so far, of weak solutions for

2000 *Mathematics Subject Classification.* Primary: 76F02, 76B03.
Key words and phrases. Loss of smoothness for the three-dimensional Euler equations, Onsager's conjecture and conservation of energy for Euler equations, vortex sheet, Kelvin-Helmholtz.

a given initial value of the three-dimensional Euler equations. Moreover, defining an optimal functional space in which the three-dimensional problem is *well-posed in the sense of Hadamard* is also an important issue.

Configuration where the vorticity is concentrated, as a measure, on a curve (in $2d$) or on a surface (in $3d$) are called Kelvin-Helmholtz flows. They seem to play a rôle in numerical simulations and in the description of turbulence. However, mathematical analysis and experiments show that in $2d$ these configurations are extremely unstable. The main reason for this instability being that the density of vorticity generates a nonlinear elliptic problem (see, e.g., [11], [14], [29] and references therein).

Let us observe that the conservation of energy in the $3d$ Euler equations is always formally true. However, physical intuition and scaling argument, i.e. the *Kolmogorov Obukhov law*, lead to the idea that non conservation of energy in the three-dimensional Euler equations would be intimately related to the loss of regularity. Therefore, Onsager [22] conjectured the existence of a threshold in the regularity of the $3d$ Euler equations that would distinguish between solutions which conserve energy and solutions which might dissipate energy.

For the above reasons we believe that the detailed study of explicit examples remains extremely insightful and useful. Therefore, this contribution is devoted to new information that can be obtained from the study of the example of shear flow that was introduced by DiPerna and Majda [10].

For simplicity we will consider solutions of Euler equations defined in a domain Ω which will denote either the whole space $\mathbb{R}^3$, or the torus $(\mathbb{R}/\mathbb{Z})^3$ when in the latter case the solutions are subject to periodic boundary conditions of period 1.

Observe that when the functions u_1 and u_3 are smooth the vector field

$$u(x,t) = (u_1(x_2), 0, u_3(x_1 - tu_1(x_2))) \tag{1}$$

is an obvious solution of the $3d$ incompressible Euler equations of inviscid (ideal) fluids:

$$\partial_t u + \nabla \cdot (u \otimes u) = -\nabla p \quad \text{and} \quad \nabla \cdot u = 0\,, \tag{2}$$

with $p = 0$, i.e. this is a pressureless flow. When defined on the torus $(\mathbb{R}/\mathbb{Z})^3$ such solutions have finite time-independent energy, that is

$$\frac{d}{dt}\int_{(\mathbb{R}/\mathbb{Z})^3} |u(x,t)|^2 dx = 0\,. \tag{3}$$

It is worth stressing that the following observation will be essential for the remainder of this paper. Specifically, we observe that the above properties remain true under much weaker assumption on the vector field $u(x,t) = (u_1(x_2), 0, u_3(x_1 - tu_1(x_2)))$, provided the notion of weak solution is used.

Definition 1.1. A vector field $u \in L^2_{\mathrm{loc}}(\Omega \times [0,\infty))$ is a weak solution of the Euler equations (2) with initial data

$$u_0 \in L^2_{\mathrm{loc}}(\Omega)\,, \quad \nabla \cdot u_0 = 0\,,$$

if u is divergence free, in the sense of distributions in $\Omega \times [0,\infty)$, and if for any divergence free vector field of test functions $\phi \in C_c^\infty(\Omega \times [0,\infty))$ one has:

$$\int_{\Omega\times[0,\infty)} [u \cdot \partial_t \phi + \langle u \otimes u, \nabla \phi\rangle] dxdt = \int_\Omega u_0(x) \cdot \phi(x,0) dx. \tag{4}$$

Theorem 1.2. *(i) Let $u_1, u_3 \in L^2_{\text{loc}}(\mathbb{R})$, then the shear flow defined by (1) is a weak solution of the Euler equations, in the sense of Definition 1.1, in $\Omega = \mathbb{R}^3$.*

(ii) Let $u_1, u_3 \in L^2(\mathbb{R}/\mathbb{Z})$ then the shear flow defined by (1) is a weak solution of the Euler equations, in the sense of Definition 1.1, in $\Omega = (\mathbb{R}/\mathbb{Z})^3$. Furthermore, in this case the energy of this solution is constant.

The proof of the above statements follows from a lemma, which is deduced from the Fubini theorem. Below we state, without a proof, the periodic case version of such a Lemma.

Lemma 1.3. *Let $\Omega = ((\mathbb{R}/\mathbb{Z}))^3$, $u_1, u_3 \in L^2((\mathbb{R}/\mathbb{Z}))$, then for every test functions $\phi_i \in C^\infty(\mathbb{R}/\mathbb{Z})$, for $i = 1, 2, 3$, and $\phi_4 \in C^\infty_c([0, \infty))$ the following standard formula*

$$\begin{aligned} &\int_{\Omega\times[0,\infty)} u_3(x_1 - tu_1(x_2))\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(t)dx_1dx_2dx_3dt \\ &= \int_{\Omega\times[0,\infty)} u_3(x_1)\phi_1(x_1 + tu_1(x_2))\phi_2(x_2)\phi_3(x_3)\phi_4(t)dx_1dx_2dx_3dt \end{aligned} \tag{5}$$

is valid.

DiPerna and Majda introduced the shear flow (1) in their seminal paper [10] to construct a family of oscillatory solutions of the $3d$ Euler equations whose weak limit does not satisfy the Euler equations. In this paper we will investigate other properties of this shear flow in order to address issues related to the questions of well-posedness, stability of solutions whose vorticity contains density functions that are concentrated on surfaces (this problem being closely related to the Kelvin-Helmholtz problem), and conservation of energy (Onsager conjecture [22]). It is worth mentioning that this shear flow was also investigated by Yudovich [30] to show that the vorticity grows to infinity, as $t \to \infty$, which he calls gradual loss of smoothness. This is a completely different notion of loss of smoothness than the one presented in Theorem 2.2 below, where we show the instantaneous loss of smoothness of the solutions for certain class of initial data.

2. **Instability of Cauchy problem and loss of smoothness.** Most of the basic existing results for the initial value problem concerning the Euler equations (2) rely on the expression of this solution in term of the vorticity, $\omega = \nabla \wedge u$, which satisfies in $\mathbb{R}^n$, for $n = 2, 3$, the equivalent system (under the appropriate boundary conditions at infinity) of equations:

$$\partial_t \omega + u \cdot \nabla\omega = \omega \cdot \nabla u\,, \tag{6}$$

$$\nabla \cdot u = 0\,, \nabla \wedge u = \omega\,. \tag{7}$$

Equation (7) defines u in term of ω, which is given (in $\mathbb{R}^n$, for $n = 2, 3$) by the Biot-Savart law; that is $u = K(\omega)$ where K is a pseudo-differential operator of order -1. Therefore, in this case the map $\omega \mapsto \nabla u$ is an operator of order 0. As it is well known, equation (6) seems to share some similarity with the Riccati equation

$$y' = Cy^2 \text{ whose solution is } y(t) = \frac{y(0)}{1 - Cty(0)}\,, \tag{8}$$

which blows up in finite time for every $y(0) > 0$. There is not enough justification for this similarity to deduce from (8) some blow up property for the Euler equations.

However, one can deduce some *local in time* existence and stability results in any appropriate norm $\|.\|$ which satisfies the relation:

$$\|\omega \cdot \nabla u\| = \|\omega \cdot \nabla (K(\omega))\| \leq C\|\omega\|^2 \,. \tag{9}$$

On the one hand, the operator K is not continuous from C^0 to C^1, therefore the L^∞ norm is not appropriate for this scenario. On the other hand, the Hölder norms, i.e. $\omega \in C^{0,\alpha}$ or $u \in C^{1,\alpha}$, for $\alpha \in (0,1]$, are convenient. With the standard Sobolev estimates the norm H^s, for $s > \frac{5}{2}$, i.e. $\omega \in H^{s-1}$ or $u \in H^s$, would also be convenient (and leads, by virtue of common functional analysis tools, to slightly simpler proofs, see, e.g. [20]). This is fully consistent with the fact that H^s, for $s > \frac{5}{2}$, is continuously imbedded in $C^{1,s-\frac{5}{2}}$.

With this classical observations in mind we recall the following facts (see also the recent surveys for more details [1] and [6]):

(i) For initial data $u(x,0) = u_0(x)$ in $C^{1,\alpha}$ the Euler equations (2) has a unique local in time solution $u(x,t)$ in $C^{1,\alpha}$ (cf. [16]). The same result is valid for initial data in H^s, for $s > \frac{5}{2}$ (cf. [2],[20]). Moreover, this unique solution conserves the energy. In spite of the fact that the above results imply the short time control of the L^∞ norm of the vorticity (which seems to be the relevant quantity) one has the following complementary statement established in [2] (see also [20]). For every initial data $u(x,0)$ in $C^{1,\alpha}$ or in H^s, for $s > \frac{5}{2}$, the solution of the three-dimensional Euler equations exists and depends continuously on the initial data, for as long as the time integral of the L^∞ norm of the vorticity remains bounded.

(ii) Following [8] one can prove (in any space dimension) the existence of initial data $u_0 \in L^2(\Omega)$ (not explicitly constructed) for which the Cauchy problem has, with the same initial data, an infinite family of weak solutions of the Euler equations: a residual set in the space $C(\mathbb{R}_t; L^2_{\text{weak}}(\Omega))$.

(iii) Eventually one does not know the existence of a $3d$ regular (say in $C^{1,\alpha}$) solution of the Euler equations that becomes singular in a finite time (blow up problem).

The shear flow (1) has also been used by DiPerna and Lions [17], and by [30], as an example to demonstrate some issues related to the instability of the solutions of the three-dimensional Euler equations. In particular, DiPerna and Lions (cf. [17] page 124) have established the following

Theorem 2.1 (DiPerna-Lions)**.** *For every $p \geq 1$, $T > 0$ and $M > 0$ given there exists a smooth shear flow solution of the form (1) for which $\|u(x,0)\|_{W^{1,p}} = 1$ and $\|u(x,T)\|_{W^{1,p}} > M$.*

In fact the proof of this theorem that has been presented in [1] (Proposition 3.1) shows that, for every $p \geq 1$, there exist shear flow solutions of the form (1) with $u(x,0) \in W^{1,p}$ and $u(x,t) \notin W^{1,p}$ for any $t \neq 0$. Here, we show, in addition, the instantaneous loss of smoothness of weak solutions for the $3d$ Euler equations with initial data in the Hölder space $C^{0,\alpha}$, with $\alpha \in (0,1)$. This underlines the rôle of the space C^1 as the *critical space* for short time well-posedness of the $3d$ Euler equations; namely: for initial data *more regular than* C^1, say in $C^{1,\beta}$, with $\beta \in (0,1]$, one has well-posedness for the $3d$ Euler equations, and for *less regular* initial data, specifically initial data in $C^{0,\alpha}$, with $\alpha \in (0,1)$, one has ill-posedness.

Theorem 2.2. *(i) For $u_1(x), u_3(x) \in C^{1,\alpha}$, with $\alpha \in (0,1]$, the shear flow solution (1) is in $C^{1,\alpha}$, for all $t \in \mathbb{R}$.*

(ii) For $u_1(x), u_3(x) \in C^{0,\alpha}$, with $\alpha \in (0,1)$, the shear flow solution (1) is always in C^{0,α^2}.

(iii) There exist shear flow solutions, of the form (1), which for $t = 0$ belong to $C^{0,\alpha}$, for some $\alpha \in (0,1)$, and which for $t \neq 0$ do not belong to $C^{0,\beta}$ for any $\beta > \alpha^2$.

Proof. Observe first that in (i) the evolution of regularity concerns only the component u_3 (u_1 remains t independent). The statement (i) is trivial, but it is worth noticing as it shows that our analysis is in line with the classical results of [16]. To prove (ii) we write

$$\begin{aligned}
&\frac{|u_3(x_1 - tu_1(x_2+h)) - u_3(x_1 - tu_1(x_2))|}{h^{\alpha^2}} \\
&= \frac{|u_3(x_1 - tu_1(x_2+h)) - u_3(x_1 - tu_1(x_2))|}{|tu_1(x_2+h) - tu_1(x_2)|^\alpha} \left(\frac{|tu_1(x_2+h) - tu_1(x_2)|}{h^\alpha} \right)^\alpha \\
&\leq |t|^\alpha \|u_3\|_{0,\alpha} \|u_1\|_{0,\alpha}^\alpha \,.
\end{aligned} \tag{10}$$

For the point (iii) of the statement one introduces two periodic functions $u_1(\xi)$ and $u_3(\xi)$ which near the point $\xi = 0$ coincide with the function $|\xi|^\alpha$. Consequently, for every given t and for x_1 and x_2 small enough, $u_3(x_1 - tu_1(x_2))$ coincides with the function

$$|x_1 - t|x_2|^\alpha|^\alpha.$$

In particular, for t given, and for $(x_1, x_2, x_3) = (0, x_2, x_3)$, with x_2 small enough, one has

$$u_3(x_1 - tu_1(x_2)) = |t|^\alpha |x_2|^{\alpha^2},$$

and the conclusion follows.

Remark 1. P.G. Lemarié-Rieusset observed (private communication) that the criticality aspect of the space C^1, in the above context, can be sharpened by considering the Besov and the Triebel-Lizorkin spaces, $B^s_{p,q}$ and $F^s_{p,q}$, respectively. Indeed, one has, on the one hand, the inclusions (see, e.g., [15] and [28])

$$C^{1,\alpha} = B^{1+\alpha}_{\infty,\infty} \subset B^1_{\infty,1} \subset C^1 \subset F^1_{\infty,2} \subset B^1_{\infty,\infty} \subset B^\beta_{\infty,\infty} = C^{0,\beta}\,, \tag{11}$$

for all $\alpha \in (0,1]$ and $\beta \in (0,1)$. On the one hand, the short time well-posedness of the $3d$ Euler equations has been recently proven in the space $B^1_{\infty,1}$ by Pak and Park [23], and on the other hand, calculations inspired by the above proof lead to the construction of shear flows $u(x,t)$, of the form (1), which satisfy:

$$u(x,0) \in F^1_{\infty,2} \text{ and for all } t \neq 0 \;\; u(x,t) \notin F^1_{\infty,2};$$

$$\text{or } u(x,0) \in B^1_{\infty,\infty} \text{ and for all } t \neq 0 \;\; u(x,t) \notin B^1_{\infty,\infty}\,.$$

The details of this analysis and further applications will be reported in a forthcoming paper.

3. **Vorticity Surface density for shear flow and the Kelvin-Helmholtz problem.** For $2d$ Euler equations existence of a weak solutions, with a given single signed Radon measure initial data for the vorticity density which is supported on a curve, has been established by Delort [9]. The condition on the sign of the vorticity has later been slightly relaxed [19]. There is no such theorem in $3d$ case and the only available result for initial data having a density of vorticity *concentrated* on a surface is a local in time existence and uniqueness result under very restrictive

analyticity hypothesis (see [27]). As a consequence it may be interesting to exhibit two examples of shear flows with nontrivial surface density that would emphasize the difference between the $2d$ and $3d$ situations.

Example 1

To present a vorticity concentrated on a surface for the shear flow (1) then it has to be of the following form

$$u_1(s) = \begin{cases} \alpha_1 & \text{for} \quad s < \xi_2 \\ \beta_1 & \text{for} \quad s > \xi_2 \end{cases} \quad \text{and} \quad u_3(s) = \begin{cases} \alpha_3 & \text{for} \quad s < \xi_1 \\ \beta_3 & \text{for} \quad s > \xi_1 \end{cases},$$

for some fixed real parameters $\alpha_1, \alpha_3, \beta_1, \beta_3, \xi_1, \xi_2$, satisfying $\alpha_1 \geq \beta_1$ and $\alpha_3 \neq \beta_3$. The vorticity is therefore concentrated on the singular surface:

$$\begin{aligned} \Sigma(t) = \{(x_1, x_2, x_3)|\, x_2 = \xi_2\} \cup \{(x_1, x_2, x_3)|\, x_1 \\ = \xi_1 + t\alpha_1,\, x_2 \leq \xi_2\} \cup \{(x_1, x_2, x_3)|\, x_1 = \xi_1 + t\beta_1,\, x_2 \geq \xi_2\}\,. \end{aligned}$$

Example 2

In $3d$ with the following configuration

$$u_3(s) = \begin{cases} 1 & \text{for } x < 0 \\ 0 & \text{for } x > 0 \end{cases}, \tag{12}$$

and $y = u_2(s)$ is a C^1 curve, the shear flow:

$$u(x) = (u_1(x_2), 0, u_3(x_1 - tu_1(x_2)))$$

is a weak solution of the $3d$ Euler equations with a singular vorticity which is concentrated on the surface

$$\Gamma(t) = \{(x_1, x_2, x_3)|\ x_1 = tu_1(x_2)\}$$

and is given by:

$$\omega(x,t) = (-t\frac{\partial_{x_2} u_1}{(|t\partial_{x_2} u_1|^2 + 1)^{\frac{1}{2}}} \otimes \delta_{\Gamma(t)}, \frac{1}{(|t\partial_{x_2} u_1|^2 + 1)^{\frac{1}{2}}} \otimes \delta_{\Gamma(t)}, -\partial_{x_2} u_1(x_2))\,.$$

The discussion below, concerning the difference between the $2d$ and $3d$ Kelvin-Helmholtz problem, is motivated, among other things, by the following remark.

Remark 2. Example 1 is a solution of the $3d$ Euler equations with a density of vorticity concentrated on a surface with corners. It is unknown whether the construction of the same type of configuration is possible in $2d$ case. In Example 2 the function $x_2 \mapsto u_1(x_2)$ does not need to be more regular than C^1 to sustain a C^1 vorticity surface density of the corresponding shear flow solution of the $3d$ Euler equations. Moreover, no matter how regular this surface is initially its regularity will be preserved by the dynamics. Furthermore, and by virtue of Theorem 1.2, both examples are weak solutions of the $3d$ Euler equations, and when considered in torus $(\mathbb{R}/\mathbb{Z})^3)$ they both conserve energy.

In an attempt to understand the effect of the dimension it seems appropriate to compare Example 2 with classical results concerning the Kelvin-Helmholtz problem.

As we have mentioned in the introduction the Kelvin-Helmholtz problem corresponds to the situation where the vorticity is concentrated on a moving orientable curve in $r(t, \lambda)$, in $2d$, parameterized by a parameter $\lambda \in \mathbb{R}$, or on a moving orientable surface $r(t, \lambda, \mu)$, in $3d$, parameterized by the parameters $(\lambda, \mu) \in \mathbb{R}^2$.

We assume that the curves or the surfaces are C^1 orientable manifolds, denoted by $\Gamma(t)$, with unit normal $\vec{n}$. For $x \notin \Gamma(t)$, the velocity u can be expressed explicitly in term of the vorticity by the following Biot-Savart formulas:

$$u(x,t) = \begin{cases} \frac{1}{2\pi} R_{\frac{\pi}{2}} \int \frac{x-r(t,\lambda')}{|x-r(t,\lambda')|^2} \tilde{\omega}(t, r(t,\lambda')) |\partial_\lambda r(t,\lambda')| d\lambda' \ \text{ in } 2d\,, \\ -\frac{1}{4\pi} \int \frac{x-r(t,\lambda',\mu')}{|x-r(t,\lambda',\mu')|^3} \tilde{\omega}(t, r(t,\lambda',\mu')) |\partial_\lambda r(t,\lambda',\mu') \wedge \partial_\mu r(t,\lambda',\mu')| d\lambda' d\mu' \ \text{ in } 3d\,, \end{cases} \tag{13}$$

where $R_{\frac{\pi}{2}}$ is the $\frac{\pi}{2}$ rotation matrix, and $\tilde{\omega}$ is the *vorticity density* on these manifolds.

When x converges to a point $r \in \Gamma(t)$ the velocity $u(x,t)$ converges to two different values, on either side of the manifold, $u_\pm(r,t)$. In particular, and in agreement with the divergence free condition, one has

$$u_+(r,t) \cdot \vec{n} = u_-(r,t) \cdot \vec{n}\ , \quad \omega(x,t) = (u_+(r,t) - u_-(r,t)) \wedge \vec{n} \otimes \delta_{\Gamma(t)}(x)\,, \tag{14}$$

for $r \in \Gamma(t)$ and $x \in \mathbb{R}^d$, $d = 2, 3$.

The vorticity density $\tilde{\omega}$ is a vector valued density. In the $2d$ case this vector is orthogonal to the plane of the flow and therefore is identified with a scalar. Hence, the vorticity density is related to the vorticity by the expressions:

$$\begin{aligned} \omega(x,t) &= (u_+(r,t) - u_-(r,t)) \wedge \vec{n} \otimes \delta_{\Gamma(t)}(x) \\ &= \begin{cases} \tilde{\omega}(t, r(t,\lambda)) |\partial_\lambda r(t,\lambda)| d\lambda \ \text{ in } 2d\,, \\ \tilde{\omega}(t, r(t,\lambda,\mu)) |\partial_\lambda r(t,\lambda,\mu) \wedge \partial_\mu r(t,\lambda,\mu)| d\lambda d\mu \ \text{ in } 3d\,. \end{cases} \end{aligned} \tag{15}$$

Formulas (13) remain valid for $x \in \Gamma(t)$ with the integral taken in the sense of Cauchy principal value and with the left-hand side of (13) replaced by the averaged velocity

$$v = \frac{u_+ + u_-}{2}\,. \tag{16}$$

Therefore, with some hypothesis on the regularity of the solution (cf. [18] for details) the problem can be reduced to equation (13) for v with:

$$(\partial_t r - v) \cdot \vec{n} = 0\,, \tag{17}$$

and in $2d$

$$\partial_t \tilde{\omega} + \frac{\partial}{\partial \lambda} \left(\frac{\tilde{\omega}}{|r_\lambda|^2} (v - r_\lambda) \cdot r_\lambda \right) = 0 \tag{18}$$

or in $3d$ with $N = \partial_\lambda r(t,\lambda,\mu) \wedge \partial_\mu r(t,\lambda,\mu)$

$$\begin{aligned} &\partial_t \tilde{\omega} + \frac{\partial}{\partial \lambda} \left(\frac{\tilde{\omega}}{|N|^2} (((v - \partial_t r) \wedge \partial_\mu r) \cdot N) \right) - \frac{\partial}{\partial \mu} \left(\frac{\tilde{\omega}}{|N|^2} (((v - \partial_t r) \wedge \partial_\lambda r) \cdot N) \right) \\ &= \frac{1}{|N|^2} ((\partial_\mu r \wedge N) \cdot \tilde{\omega}) \partial_\lambda v - \frac{1}{|N|^2} ((\partial_\lambda r \wedge N) \cdot \tilde{\omega})) \partial_\mu v\,. \end{aligned} \tag{19}$$

We recall below some classical results which contribute to the understanding of the basic properties of this problem (see also, e.g., [1]).

(i) The initial value problem is locally, in time, well-posed in both, the $2d$ and the $3d$, cases in the class of analytic data. More precisely, for any initial curve (respectively surface) $\Gamma(0,\lambda)$, (respectively $\Gamma(0,\lambda,\mu)$) and any initial density of vorticity $\tilde{\omega}(0, r(0,\lambda))$ (respectively $\tilde{\omega}(0, r(0,\lambda,\mu))$ which can be extended as analytic functions uniformly bounded in the strip $|\Im\lambda| \le c$, in the complex plane $\lambda \in \mathbb{C}$, for some $c > 0$, (respectively $|\Im\lambda| + |\Im\mu| \le c$, for $(\lambda,\mu) \in \mathbb{C}^2$, and for some $c > 0$) there exists a finite time T and a constant C such that the initial value problem (17) and (18) (respectively, (17) and (19)) has, for $0 \le t < T$, a unique solution which is analytic in the strip $|\Im\lambda| \le C(T-t)$ (respectively $|\Im\lambda| + |\Im\mu| \le C(T-t)$) (cf. [27]).

(ii) There exist in $2d$ (to the best of our knowledge this issue has not been addressed in $3d$) analytic solutions that become singular in finite time. This has been first observed by numerical simulations of Baker, Meiron and Orszag [21], then Duchon and Robert [11] have shown the existence of a very large class of singularities which can be reached in a finite time by analytic solutions. Eventually, Caflisch and Orellanna [3] have constructed analytic solutions, for $0 \le t < T$, which exhibit a cusp as t approaches T. Specifically, with $0 < \nu < 1$ they have shown that their solutions satisfy:

$$\begin{aligned}&\lim_{t\to T}(\Gamma(t,\cdot),\tilde\omega(t,r(t,\cdot)))\\ =&(\Gamma(T,\cdot),\tilde\omega(T,r(T,\cdot)))\begin{cases} \notin C^{1,\nu}\times C^{\nu}, & \\ \in C^{1,\nu'}\times C^{\nu'} & \text{for every } \ \nu'\in(0,\nu)\,.\end{cases}\end{aligned}$$

(iii) In $2d$: If in a (t,λ) neighborhood of a point (t_0,λ_0) the vorticity density, $\tilde\omega(t,r(t,\lambda))$, does not vanish and if the functions $r(t,\lambda),\tilde\omega(t,r(t,\lambda))$ have some *limited regularity* then in fact they are analytic in this neighbourhood. By a limited regularity we mean, for instance, that in this neighborhood

$$(r(t,\cdot),\tilde\omega(t,r(t,\cdot)))\in C^{1,\alpha}\times C^{\alpha} \tag{20}$$

$$|\lambda-\lambda'|\le C|r(t,\lambda)-r(t,\lambda')|,\ \text{ with some constant } C<\infty. \tag{21}$$

The hypothesis (21) is called the *chord-arc property*, and the hypothesis (20) matches perfectly the example studied in [3] . In fact under the chord-arc hypothesis a refined version of this statement has been obtained by Wu [29], which matches some numerical observations made by Krasny [13]. The consequence of this observation is that solutions *with limited regularity* do not exist in $2d$. That is, if at some time t_0 and at some point λ_0 the solution, $(r(t,\lambda),\tilde\omega(t,r(t,\lambda))$, ceases to be analytic then it cannot be of limited regularity at a later time. For instance the solution of [3] is no longer in $C^{1,\nu'}\times C^{\nu'}$, for any $\nu'>0$, for $t>T$.

Remark 3. The hypothesis that $\tilde\omega(t,r(t,\lambda))$ does not vanish is natural. This is because if $\tilde\omega$ vanishes near (t_0,λ_0) then there is no more interface, and the ellipticity as described below is lost. This will appear explicitly in formulas (29) and (30) below.

The clue in the above $2d$ results, which have been described under different forms in [11], [14] and [29], lies in the fact that under the above hypothesis the problem is locally a *small* perturbation of a linear elliptic system. Indeed, since this analysis is local one can assume, without loss of generality, that $\Gamma(t)=(x,\epsilon y(x,t))$ is a graph.

As a result, equations (16) , (17) and (18) are equivalent to the system:

$$\partial_t y - v_2 = (v_1 \partial_x y)\,, \tag{22}$$

$$\partial_t \tilde\omega + \partial_x (v_1 \Omega_0) = -\epsilon \partial_x (v_1 \tilde\omega)\,, \tag{23}$$

$$v_1(x,t) = -\frac{1}{2\pi} P.V. \int \frac{y(x,t)-y(x',t)}{(x-x')^2+\epsilon^2(y(x,t)-y(x',t))^2} (\Omega_0 + \epsilon\tilde\omega) dx'\,, \tag{24}$$

$$v_2(x,t) = \frac{1}{2\pi} P.V. \int \frac{x-x'}{(x-x')^2+\epsilon^2(y(x,t)-y(x',t))^2} (\Omega_0 + \epsilon\tilde\omega) dx'\,. \tag{25}$$

For small values of ϵ, this system describes a small perturbations in $\mathbb{R}^2$ about the stationary solution

$$y(x,0) = 0\,, u_- = \frac{\Omega_0}{2}\,, u_+ = -\frac{\Omega_0}{2}\,.$$

Indeed, for functions f and y in C^1, with $\frac{\partial y}{\partial x}$ bounded, the expansion

$$\begin{aligned}&\frac{1}{\pi} P.V. \int \frac{f(x)-f(x')}{(x-x')^2+\epsilon^2(y(x,t)-y(x',t))^2} dx' = \\ &\frac{1}{\pi} P.V. \int \frac{f(x)-f(x')}{(x-x')^2} \left(1 + \sum_{n\ge 1} (-1)^n \epsilon^{2n} \left(\frac{y(x)-y(x'))}{x-x'}\right)^2\right) dx'\end{aligned} \tag{26}$$

leads to the introduction of the operators (Hilbert transform):

$$Hf(x) = \frac{1}{\pi} P.V. \int \frac{1}{x-x'} f(x') dx' = \mathcal{F}^{-1}(-i\mathrm{sgn}(\xi)\hat f(\xi)) \tag{27}$$

$$|D|f(x) = \frac{1}{\pi} P.V. \int \frac{f(x)-f(x')}{(x-x')^2} = \partial_x (Hf(x)) = \mathcal{F}^{-1}(|\xi|\hat f(\xi))\,. \tag{28}$$

This in turn gives, together with formulas (22)-(26), for the perturbation about the stationary solution the system:

$$\begin{aligned}\partial_t y_x - \Omega_0 |D| \tilde\omega &= \epsilon F(y_x, \tilde\omega)_x \\ \partial_t \tilde\omega - |D| y_x &= \epsilon G(y_x, \tilde\omega)_x\,,\end{aligned}$$

where in the right-hand side F and G are first order operators. Eventually with the introduction of the "Laplacian" one has:

$$\partial_{tt}(y_x) + \Omega_0^2 \partial_{xx}(y_x) = \epsilon(\partial_t(F(y_x,\tilde\omega)_x) + |D|(\epsilon G(y_x,\tilde\omega)_x)\,, \tag{29}$$

$$\partial_{tt}(\tilde\omega) + \Omega_0^2 \partial_{xx}(\tilde\omega) = \epsilon(|D|(F(y_x,\tilde\omega)_x) + \partial_t(\epsilon G(y_x,\tilde\omega)_x)\,. \tag{30}$$

We remark that Example 2 is not an exact solution of the $3d$ Kelvin-Helmholtz problem due to the fact that in this case the function

$$\partial_{x_1} u_3(x_1)$$

is not, as in the Example 1, a Dirac mass. However, we *conjecture*, and that may be the object of future contribution, that a solution of the $2d$ Euler equations with a vorticity of the form

$$\nabla \wedge u(.,t) = \omega_1(t) \otimes \delta_{\Gamma(t)} + \omega_2(t) \tag{31}$$

with $r(t,\lambda), \omega(t, r(t,\lambda))$ having some *limited regularity* in the above sense and $\omega_2 \in C^{1+\alpha}(\Omega \times \mathbb{R}_t)$ will exhibit the same type of smoothing effect as in the case of the $2d$ Kelvin-Helmholtz. For instance under these hypothesis the surface $\Gamma(t)$ should belong to C^∞, or even analytic. The intuition for this conjecture stems from the fact that equation (15) is modified by the addition of lower order terms, hence the conclusions are expected to be similar. Now for the Example 2; this regularity property is not true for the surface

$$\Gamma(t) = \{(x_1, x_2, x_3)|\ x_1 = tu_1(x_2)\}\,.$$

The reason for the difference would be that in $2d$ the smoothing effect is due to the ellipticity of the linearized operator while in $3d$ the situation is different as follows: As it was done in $2d$ case, we consider a local perturbation about the stationary solution. In this situation we assume (following the notation of [27] or [4]) that $\Gamma(t)$ can be parameterized in the form $x_3 = \epsilon x(x_1, x_2, t)$, and reduce the analysis to the properties of the *small* perturbation about the stationary state $x_3 = 0, \tilde{\omega}^0(x_1, x_2) = (\tilde{\omega}^0_1, \tilde{\omega}^0_2, 0)$. The leading part of the perturbed equations (as was done above in the $2d$ case) is the linear operator (written in the $2d$ Fourier variables $k = (k_1, k_2)$, the dual of (x_1, x_2))

$$\partial_t \begin{pmatrix} \hat{x}_3 \\ \hat{\omega}_1 \\ \hat{\omega}_2 \\ \hat{\omega}_3 \end{pmatrix} = \mathcal{A} \begin{pmatrix} \hat{x}_3 \\ \hat{\omega}_1 \\ \hat{\omega}_2 \\ \hat{\omega}_3 \end{pmatrix} \tag{32}$$

where

$$\mathcal{A} = \begin{pmatrix} 0 & \frac{i}{2}\sin\theta & -\frac{i}{2}\cos\theta & 0 \\ -\frac{i}{2}|k|^2|\tilde{\omega}^0|^2\sin\theta & 0 & 0 & \frac{1}{2}(k\cdot\tilde{\omega}^0)\sin\theta \\ \frac{i}{2}|k|^2|\tilde{\omega}^0|^2\cos\theta & 0 & 0 & -\frac{1}{2}(k\cdot\tilde{\omega}^0)\cos\theta \\ 0 & -\frac{1}{2}(k\cdot\tilde{\omega}^0)\sin\theta & \frac{1}{2}(k\cdot\tilde{\omega}^0)\cos\theta & 0 \end{pmatrix}, \tag{33}$$

with $k = (k_1, k_2) = |k|(\cos\theta, \sin\theta)$.

The eigenvalues of the matrix $\mathcal{A}$ are

$$\{0, 0, -\frac{1}{2}|k \wedge \tilde{\omega}^0|, \frac{1}{2}|k \wedge \tilde{\omega}^0|\}\,.$$

Therefore, the first order pseudo-differential operator

$$\partial_t - \mathcal{A}$$

is no longer elliptic, as the situation is in the $2d$ case (see (29)-(30)).

4. **Energy conservation for rough solutions.** It has been conjectured by Onsager [22] that for some weak solutions of the $3d$ Euler equations the decay in energy would be related to some loss of regularity in these solutions. Arguing by some dimensional analysis, the Hölder exponent $1/3$ appears to be a critical value of such regularity.

On the one hand, it has been shown rigorously in [7] that the formal conservation of energy in the $3d$ Euler Equations is in fact true for any weak solution which is slightly more regular than the Besov space $\mathcal{B}^{\frac{1}{3}}_{3,\infty}$ (see also [5] and [12]). On the other hand, the existence of very weak solutions *wild solutions* that become identically 0 after a finite time has been established in [24], [25] and most recently in [8]. Moreover, it is commonly believed that for solutions which are slightly weaker than $\mathcal{B}^{\frac{1}{3}}_{3,\infty}$ there might be no conservation of energy. In fact Eyink [12] has constructed a function $u_0(x) \in C^{0,\frac{1}{3}}$ which cannot be the initial data of any weak solution which conserves the energy. This, however, is not a *complete* counter example because the existence of weak solutions for the $3d$ Euler equations with such initial data is still an open problem.

With the shear flow solution of the $3d$ Euler equations:

$$u(x,t) = (u_1(x_2), 0, u_3(x_1 - tu_1(x_2)))$$

in the torus $(\mathbb{R}/\mathbb{Z})^3$, it follows from Theorem 1.2, with $u_1, u_3 \in L^2(\mathbb{R}/\mathbb{Z})$, that there is no hope for a general theorem stating that the conservation of energy implies some type of regularity.

Observe that the hypothesis on the initial data here are much weaker than those for which the Onsager conjecture is stated in [7], [12] or [26] (see also [5]).

In [26] Shvydkoy considers the energy conservation for weak solutions of the Euler equations with singularities on a curve (in $2d$) and on a surface (in $3d$). This class of solutions includes the Kelvin-Helmholtz problem discussed in section 3. In fact the results in [26] turn out to be more relevant for the Kelvin-Helmholtz problem in the three-dimensional case rather than in two-dimensional one. The reason being, as we have mentioned in section 3 above, that in the $2d$ case a minimal regularity for the Kelvin-Helmholtz problem implies analyticity; and therefore the conservation of energy of the solutions follows, while in the $3d$ case the ellipticity of the linearized operator is no longer true and there is room for less regular (non-analytic), and possibly singular, surface solution of the $3d$ Kelvin-Helmholtz problem. In agreement with this observation we propose the following example. Consider in Theorem 1.2 the shear flow (1) in the torus $(\mathbb{R}/\mathbb{Z})^3$, with $u_1, u_3 \in L^2(\mathbb{R}/\mathbb{Z})$, such that $u_1(x_2)$ coincides, near $x_2 = 0$, with the function $\sin \frac{1}{x_2}$, and $u_3(x_1)$ coincides, near $x_1 = 0$, with the function $\operatorname{sgn}(x_1)$. Then by virtue of Theorem 1.2 the shear flow

$$u(x,t) = (u_1(x_2), 0, u_3(x_1 - tu_1(x_2))$$

is a weak solution of the $3d$ Euler equations which conserves the energy and which does not satisfy the hypothesis that are given in [26].

5. **Conclusion.** We have used the simplest example of a genuinely $3d$ flow to obtain the following observations concerning the Euler equations:

(i) In the class of Hölder spaces the space C^1 is the critical space for the initial value problem of the $3d$ Euler equations to be locally, in time, well-posed *in the sense of Hadamard*. Old and classical results [16] (see also [2] and [20]) have shown that the $3d$ Euler equations are well-posed in $C^{1,\alpha}$, for every $\alpha \in (0,1]$, while we have shown in section 2 that the $3d$ Euler equations are not well-posed in C^β, for any $\beta \in (0,1)$. This observation is also in agreement with the recent result of Pak and Park [23], who have established the local well-posed, of the $3d$ Euler equations, in the Besov space $B^1_{\infty,1}$. The consistency between our result and that of [23] is clear from the inclusion relations $C^{1,\alpha} \subset B^1_{\infty,1} \subset C^1$. Moreover, and as we have noted in Remark 1, the analysis in section 2 can be adapted in more exotic spaces, namely, the shear flow solutions, (1), of the $3d$ Euler equations will provide examples of instabilities (i.e., the Cauchy problem is not well-posed) in the in the Besov space $B^1_{\infty,\infty}$, and in the Triebel-Lizorkin space $F^1_{\infty,2}$.

(ii) The Kelvin-Helmholtz problem refers to a free boundary problem where in the $2d$ case limited regularity implies analyticity. We show that in $3d$, for closely related problems constructed with the shear flow, this property is no more true. We propose an explanation for this striking difference between the $2d$ and $3d$ case. This explanation is based on the fact that the linearized operator of the Kelvin-Helmholtz problem is no longer elliptic in $3d$ as the situation is in the $2d$ case.

(iii) The relation between dissipation of energy and loss of regularity is an essential issue in the statistical theory of turbulence, in relation with the Kolmogorov Obukhov law. It has been shown in the deterministic framework that a regularity of this type implies conservation of energy. With the shear flow example we have shown that there is no hope for a converse statement (even in the case of solutions singular on a *slit* as in [26]). We observe that in [8] De Lellis and Szekelyhidi constructed (see cf. Theorem 1.1 a) an infinite set of weak solutions

$$u \in C(\mathbb{R}_t; L^2(\mathbb{R}^3))$$

which satisfy both the strong and local energy equality (in the sense of Definition 2.4 of [8]) hence conserve energy.

The above observations may not invalidate the common *physical belief* because the Kolmogorov Obukhov law belongs to the statistical theory of turbulence, where statements and results are true in some *averaged sense* . On the other hand, our family of shear flow examples are genuinely laminar and therefore not "turbulent." They are particular enough to be of measure zero with respect to any reasonable *ensemble measure* compatible with the statistical theory of ideal (inviscid) turbulent flows (let us recall that, to the best of our knowledge, no such measure has been constructed, up to now, with full mathematical rigor).

Eventually the construction of [8] involves limit of oscillating solutions and therefore is not explicit but closer to the intuition of turbulence. It also relies on the Baire category theorem. Hence it generates a residual set of solutions which is dense in $C(\mathbb{R}_t; L^2_{\text{weak}}(\mathbb{R}^3))$. A tentative justification of the fact that 'in the statistical theory of turbulence conservation of energy may in general imply some regularity of the underlined solutions' would be similar to the situation in classical analysis theory where a dense set may well be a set of measure zero.

Acknowledgments. Claude Bardos would like to thank the University of California, Irvine and the Weizmann Institute of Science for their warm hospitality where part of this work was completed. The authors wish also to thank the anonymous referee for comments leading to the present improved version and the organizers of the Oberwolfach Workshop ID 0930 on Navier-Stokes equations (July 2009) where the material of this contribution was presented and discussed. This work was supported in part by the NSF grant no. DMS-0708832, and by the ISF grant no. 120/06.

REFERENCES

[1] C. Bardos and E. S. Titi, *Euler equations of incompressible ideal fluids*, Uspekhi Matematicheskikh Nauk, UMN, **62** (2007), 5-46; Also in Russian Mathematical Surveys, **62** (2007), 409–451.

[2] J. T. Beale, T. Kato and A. Majda, *Remarks on the breakdown of smooth solutions for the 3D Euler equations*, Commun. Math. Phys., **94** (1984), 61–66.

[3] R. Caflisch and O. Orellana, *Singular solutions and ill-posedness for the evolution of vortex sheets*, SIAM J. Math. Anal., **20** (1989), 293–307.

[4] S. Chandrasekhar, "Hydrodynamic and Hydromagnetic Stability," Oxford: Clarendon Press 1961.

[5] A. Cheskidov, P. Constantin, S. Friedlander and R. Shvydkoy, *Energy conservation and Onsager's conjecture for the Euler equations*, Nonlinearity, **21** (2008), 1233–1252.

[6] P. Constantin, *On the Euler equations of incompressible fluids*, Bulletin, AMS, **44** (2007), 603–621.

[7] P. Constantin, W. E and E. S. Titi, *Onsager's conjecture on the Energy Conservation for Solutions of Euler's equation*, Comm. Math. Phys., **165** (1994), 207–209.
[8] C. De Lellis and L. Székelyhidi Jr., *On admissibility criteria for weak solutions of the Euler equations*, arXiv:0712.3288.
[9] J.-M. Delort, *Existence de nappes de tourbillon en dimension deux*, J. Amer. Math. Soc., **4** (1991), 553–586.
[10] R. DiPerna and A. Majda, *Oscillations and concentrations in weak solutions of the incompressible fluid equations,* Comm. Math. Phys., **108** (1987), 667–689.
[11] J. Duchon and R. Robert, *Global vortex sheet solutions of Euler equations in the plane*, J. Differential Equations, **73** (1988), 215–224.
[12] G. L. Eyink, *Energy dissipation without viscosity in ideal hydrodynamics,* I. Fourier analysis and local energy transfer, Phys. D, **78** (1994), 222–240.
[13] R. Krasny, *Computation of vortex sheet roll-up in Trefftz plane*, J. Fluid Mech., **184** (1987), 123–155.
[14] G. Lebeau, *Régularité du problème de Kelvin-Helmholtz pour l'équation d'Euler 2d*, ESAIM Control Optim. Calc. Var., **8** (2002), 801–825 (electronic).
[15] P. G. Lemarié-Rieusset, "Recent Developments in the Navier-Stokes Problem," vol. **431** of Chapman & Hall/CRC Research Notes in Mathematics, Chapman & Hall/CRC, Boca Raton, FL, 2002.
[16] L. Lichtenstein, *Über einige existenz problem der hydrodynamik homogener unzusammendrückbarer, reibunglosser Flüssikeiten und die Helmholtzschen wirbelsalitze*, Mat. Zeit. Phys., **23** (1925), 89–154; **26** (1927), 193–323; **32** (1930), 608.
[17] P. L. Lions, "Mathematical Topics in Fluid Mechanics," Volume **1**, "Incompressible Models," Oxford Lecture Series in Mathematics and its Applications, Oxford 1998.
[18] M. C. Lopes Filho, H. J. Nussenzveig Lopes and S. Schochet, *A criterion for the equivalence of the Birkhoff–Rott and Euler description of vortex sheet evolution,* Trans. Amer. Math. Soc., **359** (2007), 4125–4142 (electronic).
[19] M. C. Lopes Filho, H. J. Nussenzveig Lopes and Z. Xin, *Existence of vortex sheets with reflection symmetry in two space dimensions*, Arch. Ration. Mech. Anal., **158** (2001), 235–257.
[20] A. Majda and A. Bertozzi, "Vorticity and Incompressible Flow," Cambridge Texts in Applied Mathematics, **27** Cambridge University Press, Cambridge, 2002.
[21] D. Meiron, G. Baker and S. Orszag, *Analytic structure of vortex sheet dynamics. I. Kelvin Helmholtz instability*, J. Fluid Mech., **114** (1982), 283–298,
[22] L. Onsager, *Statistical hydrodynamics*, Nuovo Cimento (Supplemento), **6** (1949), 279–287.
[23] H. C. Pak and Y. J. Park, *Existence of solution for the Euler equations in a critical Besov space*, Communications PDEs, **29** (2004), 1149–1166.
[24] V. Scheffer, *An inviscid flow with compact support in space-time*, J. of Geom. Anal., **3** (1993), 343–401.
[25] A. Shnirelman, *On the nonuniqueness of weak solutions of the Euler equations*, Comm. Pure Appl. Math., **50** (1997), 1261–1286.
[26] R. Shvydkoy, *On the energy of inviscid singular flows*, J. Math. Anal. Appl., **349** (2009), 583–595.
[27] C. Sulem, P.-L. Sulem, C. Bardos and U. Frisch, *Finite time analyticity for the two- and three-dimensional Kelvin–Helmholtz instability*, Comm. Math. Phys., **80** (1981), 485–516.
[28] H. Triebel, "Theory of Function Spaces," vol. **78** of Monographs in Mathematics, Birkhäuser Verlag, Basel, 1983.
[29] S. Wu, *Mathematical analysis of vortex sheets,* Comm. Pure Appl. Math., **59** (2006), 1065–1206.
[30] V. I. Yudovich, *On the loss of smoothness of the solutions of the Euler equations and the inherent instability of flows of an ideal fluid*, Chaos, **10** (2000), 705–719.

Received February 2009; revised July 2009.

E-mail address: claude.bardos@gmail.com
E-mail address: etiti@math.uci.edu;edriss.titi@weizmann.ac.il