DIMENSIONAL REDUCTION FOR SUPREMAL FUNCTIONALS

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Abstract. A 3D-2D dimensional reduction analysis for supremal functionals is performed in the realm of Γ^*-convergence. We show that the limit functional still admits a supremal representation, and we provide a precise identification of its density in some particular cases. Our results rely on an abstract representation theorem for the Γ^*-limit of a family of supremal functionals.

Mathematics Subject Classification (2010): 49J45, 74K99, 74Q99.

Keywords: Supremal functionals, level convexity, Γ-convergence, dimension reduction, homogenization.

Contents

1. Introduction

Dimension reduction problems consist in studying the asymptotic behavior of the solutions of a partial differential equation (or a minimization problem) stated on a domain where one of the dimensions is much smaller than the others. For instance, in 3D-2D dimensional reduction, the goal is to understand the asymptotics, as $\varepsilon \to 0$, of such solutions defined on thin domains of the form $\Omega_\varepsilon := \omega \times (-\varepsilon, \varepsilon)$, where $\omega \subset \mathbf{R}^2$ is usually a bounded open set, and $0 < \varepsilon << 1$.

These kind of problems have been widely studied in the framework of integral functionals by means of Γ-convergence analysis. Indeed Γ-convergence, which has been introduced in [31] (see also [28, 17, 18] for detailed discussions on that subject), turns out to be well adapted for studying the asymptotic behavior of variational problems depending on a parameter because it gives good informations on the asymptotics of minimizers and of the minimal value.

Definition 1.1. *Let X be a metric space, and $F_n : X \to (-\infty, +\infty]$ be a sequence of functions. We denote by*

$$F'(x) := \inf \left\{ \liminf_{n\to\infty} F_n(x_n) : x_n \to x \text{ in } X \right\}$$

the Γ-lower limit*, or more shortly the* Γ-liminf *of the sequence (F_n). Similarly, we denote by*

$$F''(x) := \inf \left\{ \limsup_{n\to\infty} F_n(x_n) : x_n \to x \text{ in } X \right\}$$

the Γ-upper limit*, or more shortly the* Γ-limsup *of the sequence (F_n). When $F' = F'' = F$, we say that F is the Γ-limit of the sequence (F_n), and it is characterized by the following properties:*

(i) *for every $x \in X$ and for every sequence (x_n) converging to x in X, then*

$$F(x) \le \liminf_{n\to\infty} F_n(x_n);$$

(ii) *for every $x \in X$ there exists a sequence $(\overline{x}_n)$ (called a* recovering sequence*) converging to x in X such that*

$$F(x) = \lim_{n\to\infty} F_n(\overline{x}_n).$$

The motivation for dealing with this variational convergence is explained by the next theorem. Under the assumption of equicoercivity for the sequence (F_n), there holds the fundamental property of convergence of the minimum values and infimizers (see Theorem 7.8 and Corollary 7.17 in [28]).

Theorem 1.2. *Suppose that the sequence (F_n) is equi-coercive in X, i.e., for every $t \in \mathbf{R}$ there exists a fixed compact subset K_t of X such that $\{F_n \le t\} \subset K_t$ for every $n \in \mathbf{N}$. If (F_n) Γ-converges to F in X, then*

$$\min_X F = \lim_{n\to\infty} \inf_X F_n.$$

Moreover if x_n is such that $F_n(x_n) \le \inf_X F_n + \varepsilon_n$, for some sequence $\varepsilon_n \to 0$, and $x_{n_k} \to x$ for some subsequence $(x_{n_k})_k$ of $(x_n)_n$, then $F(x) = \min_X F$.

The integral case has been widely studied in the literature starting from the seminal paper [1]. In [34] the authors derived the Γ-limit (for a suitable topology) of integral functionals of the form

$$\frac{1}{\varepsilon} \int_{\Omega_\varepsilon} W(Du)\, dx,$$

where $W : \mathbf{R}^{3\times 3} \to \mathbf{R}$ is a continuous integrand satisfying standard p-growth and p-coercivity conditions (with $1 < p < \infty$), and $u : \Omega_\varepsilon \to \mathbf{R}^3$ belongs to the Sobolev space $W^{1,p}(\Omega_\varepsilon; \mathbf{R}^3)$. They actually proved that the Γ-limit is finite if and only if the limit fields u are independent of the last variable x_3, and that on its domain, $W^{1,p}(\omega; \mathbf{R}^3)$, it still has an integral form with an explicit density

$$\int_\omega QW_0(D_\alpha u(x_\alpha))\, dx_\alpha.$$

In the previous formula, we have denoted by $x_\alpha := (x_1, x_2) \in \omega$ the in-plane variable (D_α is the derivative with respect to x_α), $W_0(\overline{\xi}) := \inf_{c\in\mathbf{R}^3} W(\overline{\xi}|c)$, and QW_0 is the quasiconvex envelope of W_0.

Later on, a general integral representation result has been proved in [20] in the spirit of [23] (see also [24]). Indeed, the authors showed that integral functionals of the form

$$W^{1,p}(\Omega_\varepsilon; \mathbf{R}^3) \ni u \mapsto \frac{1}{\varepsilon} \int_{\Omega_\varepsilon} W_\varepsilon(x, Du)\, dx$$

always admit a Γ-convergent subsequence, and that the Γ-limit remains of integral type, *i.e.*,

$$W^{1,p}(\omega; \mathbf{R}^3) \ni u \mapsto \int_\omega W^*(x_\alpha, D_\alpha u(x_\alpha))\, dx_\alpha,$$

for some universal function W^*. Then a series of papers have been devoted to the identification of the abstract density W^* in some particular cases (see *e.g.* [8, 9, 10]). Several works have been performed in the case of the critical exponent $p = 1$ (see [19, 11]) where the analysis takes place in BV spaces instead of Sobolev spaces.

In this paper we are interested in studying some dimension reduction problems within the framework of the so-called L^∞ (or supremal) functionals, *i.e.* functionals which are represented as

$$F(u) = \operatorname*{ess\,sup}_{x\in\Omega} f(x, u(x), Du(x)) \tag{1.1}$$

where Ω is a bounded open set of $\mathbf{R}^N$ and $u \in W^{1,\infty}(\Omega)$. We refer to the function f, which represents F, as an admissible *supremand*. The study of this class of functionals was originally motivated by the problem of finding the best Lipschitz extension in Ω of a function φ defined on $\partial\Omega$ (see [7]). The introduction of such functionals becomes useful and essential in order to give a mathematical model for many physical problems as, for example, the problem of modelling the dielectric breakdown for a composite conductor (see [33]). In [14] it is possible to find a list of other applications. A lot of recent papers have been devoted to study the properties of this class of functionals (see [2, 3, 4, 5, 6, 7, 13, 14, 15]). When f is globally continuous, it has been proved in ([16]) that (1.1) is sequential weakly* lower semicontinuous in $W^{1,\infty}(\Omega)$ if and only if f is level convex in its last variable, *i.e.*, for every $\lambda \in (0, 1)$, ξ_1 and $\xi_2 \in \mathbf{R}^N$,

$$f(x, \lambda\xi_1 + (1-\lambda)\xi_2) \leq f(x, \xi_1) \vee f(x, \xi_2),$$

for all $x \in \Omega$. Without having a continuity property on $f(\cdot, \xi)$, one cannot expect that any admissible supremand of a weakly* lower semicontinuous supremal functional is a level convex function (see Remark 3.1 in [32]). However, in [36] it is shown that when F is weakly* lower semicontinuous, then it can be represented through a level convex function. The relaxation of supremal functionals is quite well understood in the case $N = 1$ (see [16] and [2]) and in the case $N > 1$ when f is a globally continuous function (see [36]). In these cases the relaxed functional is still supremal and represented through the level convex envelope of f. Unfortunately, the theory is much less understood when u is vector valued.

In order to study a 3D-2D dimension reduction problem for supremal functionals the first question to be solved is if this class of functionals is stable under Γ-convergence in L^∞. Unfortunately, this is not the case as shows the one-dimensional Example 4.1. Among the contributions given to this problem, we recall [21] in which the authors study the problem of representing the Γ-limit of sequences of supremal functionals in the case $N = 1$; later in [22] the authors study the case of periodic homogenization by showing that the homogenized problem is still supremal. Moreover they prove that the energy density of the homogenized functional can be represented by means of a cell-problem formula. The particular case of the 1-homogeneous supremal functionals is considered in [32] where the authors show that the closure of the class of 1-homogeneous supremal functionals with respect to Γ-convergence is a larger class of functionals (given by the so called difference quotients associated to geodesic distances). By analogy with the integral representation result in [23], in [25] the authors characterize the class of the functionals which can represented in a supremal form, but this result does not easily apply in practice. One reason is that the notion of Γ-convergence is not so well suited for supremal functionals since it may not always be possible to use an argument as the fundamental estimate in the integral case.

To overcome this difficulty it has been convenient to use a generalized notion called Γ^*-convergence (see [28]), which is, roughly speaking, the Γ-convergence on a suitable 'rich' family of open sets. Thanks to this observation it has been proved in the unpublished work [26] how a Γ^*-limit can be represented in a supremal form (see Theorem 4.2).

Having in hand all this theory on Γ^*-convergence of supremal functionals, we propose to apply it to 3D-2D dimension reduction problems by first giving an abstract supremal representation result for the Γ^*-limit (an analogous result to that of [20] in the integral case), and then to identify the Γ^*-limit in some particular cases, as in the case where dimension reduction is coupled to periodic homogenization.

The paper is organized as follows: Section 2 is devoted to introduce notations and it gives basic results concerning supremal functionals. In section 3, we provide some definitions and results necessary to introduce the notion of Γ^*-convergence. In section 4 we state and prove the supremal representation result for Γ^*-limits which has been obtained in [26]. We stress that since this result has nowhere been published, we decided to include the proof for the reader's convenience, and with the agreement of both authors. In section 5, we apply all these concepts to 3D-2D dimension reduction: we first prove an abstract Γ^*-convergence result, and then we precise the specific form of the Γ^*-limit when dimension reduction is coupled to periodic homogenization. The particular case of homogeneous supremand is treated in section 6, where an alternative proof is given without appealing to the general representation result. Finally, in section 7, we state a parametrized homogenization result by Γ^*-convergence for supremal functionals.

2. Preliminaries on supremal functionals

Throughout the paper, we assume that Ω is an open bounded domain of $\mathbf{R}^N$. We denote by $\mathcal{A}$ the family of all open subsets of Ω, and by $\mathcal{B}_N$ the Borel σ-algebra of $\mathbf{R}^N$ (when $N = 1$, we simply write $\mathcal{B}$). Moreover we denote by $\|\cdot\|$ the euclidean norm on $\mathbf{R}^N$, by $B_r(x)$ the open ball $\{y \in \mathbf{R}^N \ : \|x-y\| < r\}$, and by $\mathcal{L}^N$ the Lebesgue measure in $\mathbf{R}^N$. By *supremal* (localized) functional on $W^{1,\infty}(\Omega)$ we mean a functional of the form

$$F(u, A) = \operatorname*{ess\,sup}_{x \in A} f(x, u(x), Du(x)), \tag{2.1}$$

where $u \in W^{1,\infty}(\Omega)$ and $A \in \mathcal{A}$. The function f which represents the functional is called *supremand*. We now give the following precise definitions.

Definition 2.1. *A function $f : \Omega \times \mathbf{R} \times \mathbf{R}^N \to (-\infty, +\infty]$ is said to be*

(a) *a* supremand *if f is $\mathcal{B}_N \otimes \mathcal{B} \otimes \mathcal{B}_N$-measurable;*
(b) *a* normal supremand *if f is a supremand, and there exists a $\mathcal{L}^N$-negligible set $Z \subset \Omega$ such that $f(x, \cdot, \cdot)$ is lower semicontinuous on $\mathbf{R} \times \mathbf{R}^N$ for every $x \in \Omega \setminus Z$;*
(c) *a* Carathéodory supremand *if:*
  (i) *for every $(z, \xi) \in \mathbf{R} \times \mathbf{R}^N$, the function $x \mapsto f(x, z, \xi)$ is measurable in Ω;*
  (ii) *for a.e. $x \in \Omega$, the function $(z, \xi) \mapsto f(x, z, \xi)$ is continuous in $\mathbf{R} \times \mathbf{R}^N$.*

A sufficient condition for the lower semicontinuity of a supremal functional with respect to the weak* topology of $W^{1,\infty}(\Omega)$ has been shown in [16]. It requires that $f(x, z, \cdot)$ is *level convex*, that is for every $t \in \mathbf{R}$ the level set $\left\{\xi \in \mathbf{R}^N : f(x, z, \xi) \leq t\right\}$ is convex for a.e. $x \in \Omega$ and for every $z \in \mathbf{R}$. It can be equivalently stated as follows: for each $\lambda \in (0, 1)$, ξ_1 and $\xi_2 \in \mathbf{R}^N$,

$$f(x, z, \lambda\xi_1 + (1 - \lambda)\xi_2) \leq f(x, z, \xi_2) \vee f(x, z, \xi_2)$$

for a.e. $x \in \Omega$ and all $z \in \mathbf{R}$. Moreover, they showed that if f is uniformly continuous in all variables, this condition is also necessary (see [14, Theorem 2.7]).

In the sequel we will make use of the following Jensen's inequality for level convex functions.

Theorem 2.2. *Let $f : \mathbf{R}^N \to (-\infty, +\infty]$ be a lower semicontinuous and level convex function, and let μ be a probability measure on $\mathbf{R}^N$. Then for every function $u \in L^1_\mu(\mathbf{R}^N)$, we have*

$$f\left(\int_{\mathbf{R}^N} u(x)d\mu(x)\right) \leq \mu\text{-}\operatorname*{ess\,sup}_{x \in \mathbf{R}^N}(f \circ u)(x). \tag{2.2}$$

We recall the following relaxation theorem shown in [36, Theorem 2.6]. If the functional (2.1) is represented by a continuous and coercive function f, then

$$\overline{F}(u) := \sup\left\{G(u) : \ G : W^{1,\infty}(\Omega) \to (-\infty, +\infty] \ \text{ weakly* lower semicontinuous, } G \leq F\right\}$$

is a supremal functional represented by the level convex envelope f^{lc} of f defined by

$$f^{\text{lc}}(x, z, \cdot) := \sup\{h : \mathbf{R}^N \to (-\infty, +\infty] \text{ l.s.c., level convex and } h(\cdot) \leq f(x, z, \cdot)\} \tag{2.3}$$

for every $x \in \Omega$ and $z \in \mathbf{R}$.

Theorem 2.3. *Let $f : \Omega \times \mathbf{R} \times \mathbf{R}^N \to \mathbf{R}$ be a continuous function. Assume that there exists an increasing continuous function $\Psi : [0, +\infty) \to [0, +\infty)$ such that $\Psi(t) \to +\infty$ as $t \to +\infty$, and $f(x, z, \cdot) \geq \Psi(\| \cdot \|)$ for every $x \in \Omega$ and every $z \in \mathbf{R}$. Let $F : W^{1,\infty}(\Omega) \to \mathbf{R}$ be the functional defined by (2.1), then*

$$\overline{F}(u) = \operatorname*{ess\,sup}_{x\in\Omega} f^{\mathrm{lc}}(x, u(x), Du(x))$$

for every $u \in W^{1,\infty}(\Omega)$.

In the sequel we will make use of the following result proved in [27, Theorem 3.1]. It states that under a coercivity condition, a supremal functional can be approximated by a sequence of power law integral functionals.

Theorem 2.4. *Let $f : \Omega \times \mathbf{R}^N \to [0, +\infty)$ be a Carathéodory supremand satisfying*

(i) *$f(x, \cdot)$ is level convex for a.e. $x \in \Omega$;*

(ii) *there exists a constant $C > 0$ such that*

$$f(x, \xi) \geq C\|\xi\|$$

for every $\xi \in \mathbf{R}^N$ and a.e. $x \in \Omega$.

For any $p \geq 1$, let $\mathcal{F}_p : L^\infty(\Omega) \to [0, +\infty]$ be defined by

$$\mathcal{F}_p(u) := \begin{cases} \left(\displaystyle\int_\Omega f^p(x, Du(x))\, dx \right)^{1/p} & \text{if } u \in W^{1,p}(\Omega), \\ +\infty & \text{otherwise,} \end{cases}$$

and let $\mathcal{F} : L^\infty(\Omega) \to [0, +\infty]$ be given by

$$\mathcal{F}(u) := \begin{cases} \operatorname*{ess\,sup}_{x\in\Omega} f(x, Du(x)) & \text{if } u \in W^{1,\infty}(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

Then, the family $(\mathcal{F}_p)_{p\geq 1}$ Γ-converges to $\mathcal{F}$ as $p \to +\infty$ with respect to the topology of the uniform convergence.

3. Inner regular envelope and Γ^*-convergence.

The aim of this section is to recall the notion of Γ^*-convergence. Indeed as already observed in the introduction, the notion of Γ-convergence is not well adapted to supremal functionals.

In particular, we show below (see Example 4.1) that the class of supremal functionals is not necessarily closed with respect to Γ-convergence. To overcome this difficulty, we have to use a generalized notion called Γ^*-convergence. To this purpose, we now recall the concept of inner regular envelope of an increasing functional introduced in [37] (see also [30] for further properties and [32] for an application in the supremal case).

Definition 3.1. *Let $F : \mathcal{C}(\overline{\Omega}) \times \mathcal{A} \to (-\infty, +\infty]$ be a functional. We say that*

(i) *F is a local functional if $F(u, A) = F(v, B)$ for every A, $B \in \mathcal{A}$ with $\mathcal{L}^N(A \triangle B) = 0$, and every u, $v \in \mathcal{C}(\overline{\Omega})$ such that $u(x) = v(x)$ for a.e. $x \in A \cup B$;*

(ii) *F is an increasing functional if $F(u, A) \leq F(u, B)$ for every $u \in \mathcal{C}(\overline{\Omega})$ and for every A, $B \in \mathcal{A}$ such that $A \subset B$;*

(iii) *F is a inner regular functional if F is local and $F(u, A) = \sup\{F(u, B) : \ B \in \mathcal{A}, \ B \subset\subset A\}$;*

(iv) *F is a regular functional if F is lower semicontinuous and inner regular.*

The inner regular envelope of an increasing functional F is defined by

$$F_-(u, A) := \sup\{F(u, B) : \ B \in \mathcal{A}, \ B \subset\subset A\}.$$

The next result has been proved in [30, Proposition 1.6].

Proposition 3.2. *Let $F : \mathcal{C}(\overline{\Omega}) \times \mathcal{A} \to (-\infty, +\infty]$ be an increasing local functional. Then the functional F_- is the greatest regular, increasing local functional less than or equal to F.*

The introduction of F_- is justified by the following property: if F is lower semicontinuous then it coincides with F_- *"for almost"* every open set, and thus a representation formula of F_- gives a representation formula of F on a wide class of open sets. We now make precise this expression.

Definition 3.3. *Let $\mathcal{R}$ be a subfamily of $\mathcal{A}$. We say that $\mathcal{R}$ is rich in $\mathcal{A}$ if for every family $\{A_t\}_{t\in\mathbf{R}} \subset \mathcal{A}$ with $A_t \subset\subset A_s$ whenever $t < s$, the set $\{t :\ A_t \notin \mathcal{R}\}$ is at most countable.*

In particular if $\mathcal{R}$ is rich in $\mathcal{A}$, then for every A, $B \in \mathcal{A}$ such that $A \subset\subset B \subset\subset \Omega$ there exists $C \in \mathcal{R}$ such that $A \subset\subset C \subset\subset B$.

The following proposition establishes a precise relation between F and F_-. Note that the proof of this result relies on the fact that $\mathcal{C}(\overline{\Omega})$ is separable with respect to the uniform convergence (see Proposition 16.4 in [29]).

Proposition 3.4. *Let $F : \mathcal{C}(\overline{\Omega}) \times \mathcal{A} \to (-\infty, +\infty]$ be a lower semicontinuous, increasing and local functional. Then the family $\{A \in \mathcal{A} :\ F_-(u, A) = F(u, A) \quad \forall u \in \mathcal{C}(\overline{\Omega})\}$ is rich in $\mathcal{A}$.*

Following [30], we define the Γ^*-convergence for a sequence of local, increasing functionals as the Γ-convergence on a suitable rich family of open sets.

Definition 3.5. *Let $F_n : \mathcal{C}(\overline{\Omega}) \times \mathcal{A} \to (-\infty, +\infty]$ be a sequence of increasing functionals, define*

$$F'(u, A) := \inf_{(u_n)} \{\liminf_{n\to\infty} F_n(u_n, A) : u_n \to u \text{ uniformly on } \overline{\Omega}\}$$

and

$$F''(u, A) := \inf_{(u_n)} \{\limsup_{n\to\infty} F_n(u_n, A) : u_n \to u \text{ uniformly on } \overline{\Omega}\}.$$

We say that (F_n) Γ^*-converges to F if F coincides with the inner regular envelopes of both functionals F' and F''.

Note that the functionals F' and F'' are increasing and lower semicontinuous but in general they are not inner regular. However, if (F_n) Γ^*-converges to F then F is increasing, lower semicontinuous and inner regular.

The next proposition easily follows from Proposition 3.4.

Proposition 3.6. *Let $F : \mathcal{C}(\overline{\Omega}) \times \mathcal{A} \to (-\infty, +\infty]$ be an increasing, lower semicontinuous, inner regular functional. Then (F_n) Γ^*-converges to F if and only if the family*

$$\{A \in \mathcal{A} :\ F(u, A) = F'(u, A) = F''(u, A)\ \forall u \in \mathcal{C}(\overline{\Omega})\}$$

is rich in $\mathcal{A}$.

Thanks to Proposition 3.6, it follows that the Γ^*-limit of a sequence of functionals is equal to its Γ-limit in a rich class of open sets. Thus, even if this notion is weaker than Γ-convergence, it still gives good informations in the study of the asympotic behavior of the infimum value.

Finally we have the following compactness result (see [30] and also [28, Theorem 16.9]).

Theorem 3.7. *Then every sequence (F_n) of increasing functionals from $\mathcal{C}(\overline{\Omega}) \times \mathcal{A}$ to $(-\infty, +\infty]$ has a Γ^*-convergent subsequence.*

4. A Γ^*-stability result for supremal functionals

The aim of this section is to prove that, under suitable assumptions, the class of supremal functionals is stable under Γ^*-convergence with respect to the uniform convergence. It turns out that the class of supremal functionals on $W^{1,\infty}(\Omega)$ is not closed with respect to Γ-convergence. Indeed, it is easy to see that a supremal functional satisfies the following properties:

(i) *(locality) $F(u, A) = F(v, B)$ for every A, $B \in \mathcal{A}$ with $\mathcal{L}^N(A \triangle B) = 0$, and for every u, $v \in W^{1,\infty}(\Omega)$ such that $u(x) = v(x)$ for a.e. $x \in A \cup B$;*

(ii) *(countable supremality) for every $A_i \in \mathcal{A}$ and $u \in W^{1,\infty}(\Omega)$*

$$F\left(u, \bigcup_{i=1}^{\infty} A_i\right) = \bigvee_{i=1}^{\infty} F(u, A_i).$$

In general the Γ-limit of a sequence of supremal functionals satisfies the locality property, but it does not satisfy the countable supremality, and thus it cannot be represented in the supremal form (2.1) for every open set $A \subset \Omega$. In fact, we can produce the following counterexample (see [35]).

Example 4.1. Let $\Omega := (0,1)$. Let us define $f_n(x,z) := x^n + |z|$. Setting

$$F_n(u,A) := \operatorname*{ess\,sup}_{x\in A} f_n(x,u'(x))$$

for every open set $A \subset (0,1)$ and for every $u \in W^{1,\infty}(0,1)$, it is easy to prove that the Γ-limit of the sequence (F_n) (with respect to the uniform convergence) is given by

$$F(u,(a,b)) = \begin{cases} \operatorname*{ess\,sup}_{x\in(a,b)} |u'(x)| & \text{if } b<1, \\ \operatorname*{ess\,sup}_{x\in(a,b)} |u'(x)| \vee 1 & \text{if } b=1, \end{cases}$$

which is not a supremal functional.

We now state a stability result for supremal functionals with respect to Γ^*-convergence under some suitable assumptions. This result has been obtained by Cardialaguet and Prinari in [26] in an unpublished work. With the agreement of both authors, we reproduce here the proof for the convenience of the reader.

Theorem 4.2. *Let $\Omega \subset \mathbf{R}^N$ be a bounded open set, and $f_n : \Omega \times \mathbf{R} \times \mathbf{R}^N \to \mathbf{R}$ be a sequence of normal supremands. Assume that*

(a) *for every $M > 0$ there exists a modulus of continuity $\omega_M : [0,+\infty) \to [0,+\infty)$ such that*

$$|f_n(x,z_1,\xi_1) - f_n(x,z_2,\xi_2)| \le \omega_M(|z_1 - z_2| + \|\xi_1 - \xi_2\|)$$

for every $n \in \mathbf{N}$, for a.e. $x \in \Omega$ and for every $\xi_1,\ \xi_2 \in \mathbf{R}^N$ and $z_1,\ z_2 \in \mathbf{R}$ with $\|\xi_1\|, \|\xi_2\|, |z_1|, |z_2| \le M$;

(b) *there exists an increasing continuous function $\Psi : [0,+\infty) \to [0,+\infty)$ such that $\Psi(t) \to +\infty$ as $t \to +\infty$ and $f_n(x,z,\cdot) \ge \Psi(\|\cdot\|)$ for every $n \in \mathbf{N}$, for a.e. $x \in \Omega$ and for every $z \in \mathbf{R}$;*

(c) *$f_n(x,z,\cdot)$ is level convex for every $n \in \mathbf{N}$, for a.e. $x \in \Omega$ and for every $z \in \mathbf{R}$.*

Let us suppose that the sequence of supremal functionals $F_n : W^{1,\infty}(\Omega) \times \mathcal{A} \to \mathbf{R}$ defined by

$$F_n(u,A) := \operatorname*{ess\,sup}_{x\in A} f_n(x,u(x),Du(x))$$

Γ^*-converges (with respect to the uniform convergence) to some functional $F : W^{1,\infty}(\Omega) \times \mathcal{A} \to \mathbf{R}$. For any $(x,z,\xi) \in \Omega \times \mathbf{R} \times \mathbf{R}^N$, we define

$$f(x,z,\xi) := \inf\Big\{F(u,B_r(x)) : r>0,\ u \in W^{1,\infty}(\Omega) \text{ s.t. } x \in \widehat{u},\ \text{with } u(x)=z,\ Du(x)=\xi\Big\} \tag{4.1}$$

where

$$\widehat{u} := \{x \in \Omega : x \text{ is a differentiability point of } u \text{ and a Lebesgue point of } Du\}\,.$$

If $f(\cdot,z,\xi)$ is continuous for every $(z,\xi) \in \mathbf{R} \times \mathbf{R}^N$, then

$$F(u,A) = \operatorname*{ess\,sup}_{x\in A} f(x,u(x),Du(x)) \tag{4.2}$$

for any $u \in W^{1,\infty}(\Omega)$ and any $A \in \mathcal{A}$. Moreover, f is level convex with respect to the last variable.

Remark 4.3. Note that if the functions f_n are continuous on $\Omega \times \mathbf{R} \times \mathbf{R}^N$, we can remove assumption (c). Indeed, according to Theorem 2.3, we can compute the Γ^*-limit of the sequence (F_n) by computing the Γ^*-limit of the sequence of the relaxed functionals

$$\overline{F}_n(u,A) := \operatorname*{ess\,sup}_{x\in A} f_n^{\mathrm{lc}}(x,u(x),Du(x))$$

(see also [28, Proposition 16.7]). Moreover, if the functions f_n are equicontinous with respect to x, then f is continuous.

As the next example shows, the representation result may fail if we drop the continuity in x of the function f defined by (4.1). In fact, under the assumptions (a), (b) of Theorem 4.2 we will show that the Γ^*-limit satisfies the countable supremality property, but in general it will satisfy a locality property only with respect to the variable u, *i.e.*,

$$F(u,A) = F(v,A) \text{ for every } u,v \in W^{1,\infty}(\Omega) \text{ such that } u(x) = v(x) \text{ for a.e. } x \in A,$$

losing consequently the property of neglecting sets of zero measure.

Example 4.4. We shall give an example of a sequence (F_n) where the function f defined by (4.1) is not continuous in x, and the Γ^*-limit F does not admit any supremal representation. We consider $\Omega = (0,2)$ and

$$f_n(x,z) := \begin{cases} x^{2n} + |z| & \text{if } x \leq 1, \\ (x-2)^{2n} + |z| & \text{if } x > 1. \end{cases}$$

Setting

$$F_n(u,A) := \operatorname*{ess\,sup}_{x\in A} f_n(x,u'(x))$$

for every open set $A \subset (0,2)$ and for every $u \in W^{1,\infty}(0,2)$, it is easy to prove that the Γ-limit of (F_n) with respect to the uniform convergence is given by

$$G(u,(a,b)) = \begin{cases} \operatorname*{ess\,sup}_{x\in(a,b)} |u'(x)| & \text{if } b < 1 \text{ or } a > 1, \\ \operatorname*{ess\,sup}_{x\in(a,b)} |u'(x)| \vee 1 & \text{if } b = 1 \text{ or } a = 1, \\ \operatorname*{ess\,sup}_{x\in(a,b)} |u'(x)| \vee 1 & \text{if } a < 1 < b, \end{cases}$$

and thus the Γ^*-limit of (F_n) is given by

$$F(u,(a,b)) = \begin{cases} \operatorname*{ess\,sup}_{x\in(a,b)} |u'(x)| & \text{if } b \leq 1 \text{ or } a \geq 1, \\ \operatorname*{ess\,sup}_{x\in(a,b)} |u'(x)| \vee 1 & \text{if } a < 1 < b. \end{cases}$$

Note that

(i) even if all functions f_n are continuous, f is not continuous in $x = 1$ since

$$f(x,\xi) = \begin{cases} |\xi| & \text{if } x < 1 \text{ or } x > 1 \\ |\xi| \vee 1 & \text{if } x = 1. \end{cases}$$

(ii) F does not satisfy the property of *set-locality*: indeed, $F(0,(\frac{1}{2},\frac{3}{2})) = 1$ and $F(0,(\frac{1}{2},1) \cup (1,\frac{3}{2})) = 0$ whereas $\mathcal{L}^1\big((\frac{1}{2},\frac{3}{2}) \setminus (\frac{1}{2},1) \cup (1,\frac{3}{2})\big) = 0$.

The proof of Theorem 4.2 is quite intricated and shall be achieved through several intermediate steps. We will give the proof of Theorem 4.2 when $\Psi(t) = t$, since it is always possible to reduce the problem to this case.

We need some preliminary results. The following lemma gives some properties of the functional F and of the function f.

Lemma 4.5. *Under assumptions (a) and (b) of Theorem 4.2, the functional F satisfies the following properties:*

(i) *(countable supremality) for every $A_i \in \mathcal{A}$ and $u \in W^{1,\infty}(\Omega)$*

$$F\left(u, \bigcup_{i=1}^{\infty} A_i\right) = \bigvee_{i=1}^{\infty} F(u,A_i); \tag{4.3}$$

(ii) *(strong continuity) for every $M > 0$, there is a modulus of continuity $\tilde{\omega}_M$ such that, for every $A \in \mathcal{A}$, and every u, $v \in W^{1,\infty}(\Omega)$ such that $\|u\|_{W^{1,\infty}(\Omega)} \leq M$ and $\|v\|_{W^{1,\infty}(\Omega)} \leq M$, then*

$$|F(u,A) - F(v,A)| \leq \tilde{\omega}_M(\|u-v\|_{W^{1,\infty}(A)});$$

(iii) *(coercivity): for every $A \in \mathcal{A}$, $F(\cdot,A) \geq \Psi(\|\cdot\|_{W^{1,\infty}(A)})$;*

(iv) *for every $A \in \mathcal{A}$ and every u, $v \in W^{1,\infty}(\Omega)$, we have*

$$F(u \vee v, A) \leq F(u, A) \vee F(v, A). \tag{4.4}$$

Proof. **Step 1.** In order to prove that F satisfies the countable supremality property (4.3), we will first show that for every $u \in W^{1,\infty}(\Omega)$, the set function $F(u,\cdot)$ is *finitely sub-supremal* which means that for every A, $B \in \mathcal{A}$

$$F(u, A \cup B) \leq F(u, A) \vee F(u, B)\,. \tag{4.5}$$

Let $u \in W^{1,\infty}(\Omega)$, A, $B \in \mathcal{A}$, and consider $A' \subset\subset A$ and $B' \subset\subset B$. Let $\varphi \in \mathcal{C}_c^\infty(\mathbf{R}^N;[0,1])$ be a cut-off function such that $\varphi = 1$ in A' and $\varphi = 0$ in $\mathbf{R}^N \backslash A$. Let us define the set $A'' = \{\varphi > \frac{1}{3}\}$ so that $A' \subset\subset A'' \subset\subset A$. Let (u_n) and $(v_n) \subset W^{1,\infty}(\Omega)$ be two recovering sequences converging uniformly to u in $\overline{\Omega}$, and

$$\limsup_{n\to\infty} F_n(u_n, A'') = F''(u, A''), \quad \limsup_{n\to\infty} F_n(v_n, B') = F''(u, B').$$

Note that thanks to the coercivity property (b), the sequences (u_n) and (v_n) are bounded by some constant M in $W^{1,\infty}(A'')$ and $W^{1,\infty}(B')$ respectively.

For any $\delta \in (0,1)$ we consider

$$w_n^\delta := \inf\{u_n + \sigma_n + \delta(1-\varphi), v_n + \delta\varphi\},$$

where $\sigma_n \in (0, 1/n)$ is any real number such that $\mathcal{L}^N(\{u_n + \sigma_n + \delta(1-\varphi) = v_n + \delta\varphi\}) = 0$. Such a number exists since the family of sets $\{\Lambda_\sigma\}_{0<\sigma<1/n}$, with $\Lambda_\sigma := \{u_n + \sigma + \delta(1-\varphi) = v_n + \delta\varphi\}$, is made of pairwise disjoint sets whose union has finite Lebesgue measure. Observe that (w_n^δ) uniformly converges to $w_\delta := u + \delta \inf\{1-\varphi, \varphi\}$ in $\overline{\Omega}$. Moreover $w_n^\delta = u_n + \sigma_n + \delta(1-\varphi)$ and $Dw_n^\delta = D(u_n + \sigma_n + \delta(1-\varphi))$ a.e. in $(A' \cup B') \cap \{u_n + \sigma_n + \delta(1-\varphi) < v_n + \delta\varphi\}$ while $w_n^\delta = v_n + \delta\varphi$ and $Dw_n^\delta = D(v_n + \delta\varphi)$ a.e. in $(A' \cup B') \cap \{u_n + \sigma_n + \delta(1-\varphi) > v_n + \delta\varphi\}$. Since the set $\{u_n + \sigma_n + \delta(1-\varphi) = v_n + \delta\varphi\}$ has zero measure from the choice of σ_n, and since the functionals F_n are supremal, we have

$$\begin{aligned} F_n(w_n^\delta, A' \cup B') = {} & F_n(u_n + \sigma_n + \delta(1-\varphi), (A' \cup B') \cap \{u_n + \sigma_n + \delta(1-\varphi) < v_n + \delta\varphi\}) \\ & \vee F_n(v_n + \delta\varphi, (A' \cup B') \cap \{u_n + \sigma_n + \delta(1-\varphi) > v_n + \delta\varphi\}). \end{aligned}$$

Since the sequences (u_n) and (v_n) are uniformly converging to u in $\overline{\Omega}$, then $\{u_n + \sigma_n + \delta(1-\varphi) < v_n + \delta\varphi\} \subset \{\varphi > 1/3\} = A''$ and $\{u_n + \sigma_n + \delta(1-\varphi) > v_n + \delta\varphi\} \subset \{\varphi < 2/3\} \subset (\mathbf{R}^N \backslash A')$, for any n large enough (depending on δ). Therefore, for such n's, we have

$$F_n(w_n^\delta, A' \cup B') = F_n(u_n + \sigma_n + \delta(1-\varphi), A''\,) \vee F_n(v_n + \delta\varphi, B'\,),$$

and letting $n \to +\infty$, we get

$$\begin{aligned} F''(w_\delta, A' \cup B') &\leq \limsup_{n\to\infty} F_n(w_n^\delta, A' \cup B') \\ &\leq \limsup_{n\to\infty} F_n(u_n + \sigma_n + \delta(1-\varphi), A''\,) \vee F_n(v_n + \delta\varphi, B'\,) \\ &\leq \limsup_{n\to\infty} \{F_n(u_n, A''\,) \vee F_n(v_n, B'\,) + \omega_{M+1}(\sigma_n + \delta)\} \\ &\leq F''(u, A'') \vee F''(u, B') + \omega_{M+1}(\delta), \end{aligned}$$

where we used the continuity property (a) of f_n together with the fact that $\|u_n\|_{W^{1,\infty}(A'')} \leq M$ and $\|v_n\|_{W^{1,\infty}(B')} \leq M$. Finally, letting $\delta \to 0^+$ and using the lower semicontinuity of F'' we get that

$$F''(u, A' \cup B') \leq F''(u, A'') \vee F''(u, B') \leq F(u, A) \vee F(u, B).$$

Now let $C \in \mathcal{A}$ be such that $C \subset\subset A \cup B$. Then by [28, Lemma 14.20], there exist A', $B' \in \mathcal{A}$ such that $C \subset A' \cup B'$, $A' \subset\subset A$ and $B' \subset\subset B$. Hence since $F''(u,\cdot)$ is an increasing set function, we infer that

$$F''(u, C) \leq F''(u, A' \cup B') \leq F(u, A) \vee F(u, B),$$

and taking the supremum over all such C, since $F(u, A \cup B) = \sup\{F''(u, C) : C \subset\subset A \cup B\}$, we get (4.5).

We now prove that F actually satisfies (4.3). Indeed, if $A := \bigcup_{i=1}^{\infty} A_i$, by the inner regularity of F, there exists $V_\varepsilon \subset\subset A$ such that $F(u,A) \leq F''(u,V_\varepsilon) + \varepsilon$. Then there exists a finite subset $J \subset \mathbf{N}$ such that $V_\varepsilon \subset\subset \bigcup_{j\in J} A_j$. Therefore $F(u,A) \leq \bigvee_{j\in J} F(u,A_j) + \varepsilon \leq \bigvee_{i=1}^{\infty} F(u,A_i) + \varepsilon$, which concludes the proof of this step since ε is arbitrary, and the other inequality is obvious.

Step 2. To show that F is continuous with some modulus of continuity $\tilde{\omega}_M$, we first prove that F'' is locally bounded. Namely,

$$\forall\ M > 0,\ \exists\ K_M > 0 \text{ such that } \forall u \in W^{1,\infty}(\Omega) \text{ with } \|u\|_{W^{1,\infty}(\Omega)} \leq M, \text{ then } F''(u,\Omega) \leq K_M. \tag{4.6}$$

Indeed, let $u_n \to 0$ uniformly in $\overline{\Omega}$ be such that

$$\limsup_{n\to\infty} F_n(u_n,\Omega) = F''(0,\Omega).$$

Let us set $C = F''(0,\Omega)+1$. From the coercivity assumption (b) and the fact that $\|u_n\|_{L^\infty(\Omega)} \to 0$,

$$\limsup_{n\to\infty} \|u_n\|_{W^{1,\infty}(\Omega)} = \limsup_{n\to\infty} \|Du_n\|_{L^\infty(\Omega;\mathbf{R}^N)} \leq \limsup_{n\to\infty} F_n(u_n,\Omega) = F''(0,\Omega)\ .$$

Therefore, for n large enough, we have $\|u_n\|_{W^{1,\infty}(\Omega)} \leq C$. Now if $u \in W^{1,\infty}(\Omega)$ is such that $\|u\|_{W^{1,\infty}(\Omega)} \leq M$, then by the continuity property (a) and the fact that $\|u_n + u\|_{W^{1,\infty}(\Omega)} \leq M + C$, we get that

$$F''(u,\Omega) \leq \limsup_{n\to\infty} F_n(u+u_n,\Omega) \leq \limsup_{n\to\infty} \big(F_n(u_n,\Omega) + \omega_{M+C}(\|u\|_{W^{1,\infty}(\Omega)})\big) \leq C + \omega_{M+C}(M)\ .$$

So we have proved that (4.6) holds with $K_M = C + \omega_{M+C}(M)$.

We next show that F is locally uniformly continuous. Let $M > 0$, $A \in \mathcal{A}$, and u, v such that $\|u\|_{W^{1,\infty}(\Omega)} \leq M$ and $\|v\|_{W^{1,\infty}(\Omega)} \leq M$. Consider $A' \subset\subset A$, and let (u_n) be a recovering sequence for u such that $u_n \to u$ uniformly in $\overline{\Omega}$, and

$$\limsup_{n\to\infty} F_n(u_n,A') = F''(u,A').$$

By (4.6) and the coercivity property (b), we have that $\|Du_n\|_{L^\infty(A';\mathbf{R}^N)} \leq K_M + 1$ for n large enough, and thus we deduce that $\|u_n\|_{W^{1,\infty}(A')} \leq \max\{K_M+1, M+1\}$ for n large enough. Then, defining $M' = M'(M) = \max\{K_M+1, M+1\} + M$, we have that

$$\begin{aligned} F''(v,A') &\leq \limsup_{n\to\infty} F_n(v + u_n - u, A') \\ &\leq \limsup_{n\to\infty} \big(F_n(u_n,A') + \omega_{M'}(\|v-u\|_{W^{1,\infty}(A')}\big) \\ &\leq F(u,A) + \omega_{M'}(\|v-u\|_{W^{1,\infty}(A')})\ . \end{aligned}$$

Finally, $A' \subset\subset A$ being arbitrary, we can conclude that

$$F(v,A) \leq F(u,A) + \tilde{\omega}_M(\|v-u\|_{W^{1,\infty}(A)})$$

where we have set $\tilde{\omega}_M = \omega_{M'(M)}$.

Step 3. We now prove the coercivity of F. Let $A \in \mathcal{A}$, $u \in W^{1,\infty}(\Omega)$ and $A' \subset\subset A$. Consider a recovering sequence (u_n) such that $u_n \to u$ uniformly in $\overline{\Omega}$, and

$$\limsup_{n\to\infty} F_n(u_n,A') = F''(u,A').$$

Then, from the coercivity property (b) of F_n, the sequence (u_n) actually converges weakly* to u in $W^{1,\infty}(A')$, and thus

$$\|Du\|_{L^\infty(A';\mathbf{R}^N)} \leq \liminf_{n\to\infty} \|Du_n\|_{L^\infty(A';\mathbf{R}^N)} \leq \limsup_{n\to\infty} F_n(u_n,A') = F''(u,A') \leq F(u,A)\ .$$

Letting $A' \to A$ gives the desired result.

Step 4. To show (4.4) let $B \subset\subset A$ be an open set such that $F(u \vee v, A) \leq F''(u \vee v, B) + \varepsilon$. Let (u_n) and (v_n) be such that u_n and v_n uniformly converge in $\overline{\Omega}$ to u and v respectively, and such that

$$F''(u,B) = \limsup_{n\to\infty} F_n(u_n,B), \quad F''(v,B) = \limsup_{n\to\infty} F_n(v_n,B).$$

Then

$$\begin{aligned} F(u\vee v, A) \le F''(u\vee v, B)+\varepsilon &\le \limsup_{n\to\infty} F_n(u_n\vee v_n, B)+\varepsilon \\ &\le \limsup_{n\to\infty} F_n(u_n, B)\vee \limsup_{n\to\infty} F_n(v_n, B)+\varepsilon \\ &= F''(u,B)\vee F''(v,B) \le F(u,A)\vee F(v,A)+\varepsilon, \end{aligned}$$

and the proof of the lemma is complete. □

The next lemma summarizes the properties of the function f given by (4.1) (see Lemmas 3.2, 3.3 and 3.4 in [25]).

Lemma 4.6. *Under assumptions (a) and (b) of Theorem 4.2, the function f defined by (4.1) is a Carathéodory supremand satisfying*

(a) *for every $M>0$ there exists a constant $K=K(M)$ such that, for any $(x,z,\xi)\in\Omega\times\mathbf{R}\times\mathbf{R}^N$ with $|z|+\|\xi\|\le M$, for any $r>0$ with $B_r(x)\subset\Omega$, and any $v\in W^{1,\infty}(\Omega)$,*

$$[\ x\in\hat v,\ v(x)=z,\ Dv(x)=\xi \text{ and } F(v,B_r(x))\le f(x,z,\xi)+1\ ] \quad\Rightarrow\quad \|v\|_{W^{1,\infty}(B_r(x))}\le K\ .$$

(b) *$f(x,\cdot,\cdot)$ is bounded on bounded sets of $\mathbf{R}\times\mathbf{R}^N$, uniformly with respect to x.*

Finally, we will need the following result (see [25, Lemma 3.1]).

Lemma 4.7. *Let $u,\ v\in W^{1,\infty}(\Omega)$, let $x\in\Omega$ be a point of differentiability of u and v, and suppose that $u(x)=v(x)$ and $Du(x)=Dv(x)$. Then, for any $\varepsilon>0$ and any $r>0$, there is some $r'\in(0,r)$, some open set $A\in\mathcal{A}$, with $B_{r'/2}(x)\subset A\subset B_{r'}(x)$ and $\mathcal{L}^N(\partial A)=0$, and some $\alpha\in(0,\varepsilon)$, $\beta\in(0,\varepsilon)$ such that*

$$u(y)=v(y)+\alpha-\beta|y-x| \quad \forall y\in\partial A \quad\text{and}\quad u(y)<v(y)+\alpha-\beta|y-x| \quad \forall y\in A\ .$$

We are now in position to prove Theorem 4.2.

Proof of Theorem 4.2. **Step 1.** First of all we note that for every $u\in W^{1,\infty}(\Omega)$ and for all $A\in\mathcal{A}$

$$F(u,A)\ge \operatorname*{ess\,sup}_{x\in A} f(x,u(x),Du(x)). \tag{4.7}$$

Indeed, let us denote by $L(u)$ the set of points which are at the same time points of differentiability of u and Lebesgue points of $f(x,u(x),Du(x))$. Then for any $A\in\mathcal{A}$, any $x\in L(u)\cap A$, and any $r>0$ with $B_r(x)\subset A$, we have that $f(x,u(x),Du(x))\le F(u,B_r(x))\le F(u,A)$, from which we deduce (4.7).

Step 2. In order to prove the converse inequality, we first consider the case of $\mathcal{C}^1$ functions. Let $u\in\mathcal{C}^1(\Omega)\cap W^{1,\infty}(\Omega)$, and define $M:=\|u\|_{W^{1,\infty}(\Omega)}$. Let us now fix $\varepsilon\in(0,1)$, from the definition of f, for any $x\in A$ there is some $r_x\in(0,\varepsilon)$ with $B_{r_x}(x)\subset A$, some $v_x\in W^{1,\infty}(\Omega)$ with $v_x(x)=u(x)$, $Dv_x(x)=Du(x)$ and

$$f(x,u(x),Du(x))\ge F(v_x,B_{r_x}(x))-\varepsilon\ . \tag{4.8}$$

By Lemma 4.6 there exists a constant $K=K(M)$ such that

$$\|v_x\|_{W^{1,\infty}(B_{r_x}(x))}\le K. \tag{4.9}$$

According to Lemma 4.7 we can find some $r'_x\in(0,r_x)$, some open set A_x with $B_{r'_x/2}(x)\subset A_x\subset B_{r'_x}(x)$ and $\mathcal{L}^N(\partial A_x)=0$, and some constants $\alpha_x\in(0,\varepsilon)$, $\beta_x\in(0,\varepsilon)$ with

$$u(y)=v_x(y)+\alpha_x-\beta_x|y-x| \quad \text{on } \partial A_x \quad\text{and}\quad u(y)<v_x(y)+\alpha_x-\beta_x|y-x| \quad \text{in } A_x\ .$$

Let us set $\tilde v_x(y)=v_x(y)+\alpha_x-\beta_x|y-x|$ for all $y\in A_x$. For simplicity, we extend $\tilde v_x$ to $\overline{\Omega}$ by setting $\tilde v_x=u$ in $\overline{\Omega}\setminus A_x$. According to Lindelöf's Theorem, we can find a sequence (x_n) such that the family (A_{x_n}) is a locally finite covering of A.

Let us set

$$w_\varepsilon(x)=\sup\{\tilde v_{x_n}(x) : n\in\mathbf{N} \text{ such that } x\in A_{x_n}\}\ .$$

Note that, since the family (A_{x_n}) is a locally finite covering of A, then above supremum can actually be replaced by a maximum since the set of indexes n such that $x\in A_{x_n}$ is finite. Moreover, we have that $w_\varepsilon > u$ on A. We claim that w_ε belongs to $W^{1,\infty}(\Omega)$, with a Lipschitz constant independent of ε, that w_ε converges uniformly to u in $\overline{\Omega}$, and that

$$\operatorname*{ess\,sup}_{x\in A} f(x,u(x),Du(x)) \geq F(w_\varepsilon, A) - \omega_M(2\varepsilon) - \varepsilon. \tag{4.10}$$

Note that this statement completes the proof of the representation formula (4.2) on $\mathcal{C}^1(\Omega)\cap W^{1,\infty}(\Omega)$ because, from the lower semicontinuity of F, letting $\varepsilon\to 0^+$ in (4.10) gives

$$\operatorname*{ess\,sup}_{x\in A} f(x,u(x),Du(x)) \geq \liminf_{\varepsilon\to 0^+} F(w_\varepsilon, A) \geq F(u,A)\ .$$

Let us now show that $w_\varepsilon \in W^{1,\infty}(\Omega)$. For this we note that w_ε is the pointwise limit of the Lipschitz maps v_n defined inductively by $v_0 = u$, and

$$v_{n+1}(x) = v_n(x) \vee \tilde v_{x_{n+1}}(x) \text{ if } x \text{ belongs to } A_{x_{n+1}} \text{ and } v_{n+1}(x) = v_n(x) \text{ otherwise.}$$

The maps v_n are Lipschitz continuous, with a Lipschitz constant independent of ε, because $\tilde v_{x_n}$ are equilipschitz continuous on A_{x_n} from (4.9) and because $\alpha_{x_n}\in(0,\varepsilon)$ and $\beta_{x_n}\in(0,\varepsilon)$. Hence w_ε belongs to $W^{1,\infty}(\Omega)$ with a norm which does not depend on ε.

In order to show that (w_ε) uniformly converges to u in $\overline{\Omega}$ as $\varepsilon\to 0^+$, let us consider $x\in A$. Since (A_{x_n}) is a locally finite covering of A, there exists A_{x_n} such that $x\in A_{x_n}$ and $w_\varepsilon(x) = \tilde v_{x_n}(x)$ for some $n\in\mathbf{N}$. If $y\in\partial A_{x_n}$, then $|x-y|\leq 2r_{x_n}\leq 2\varepsilon$ because $A_{x_n}\subset B_{r_{x_n}}(x_n)$ and $r_{x_n}\leq\varepsilon$. Hence by (4.9) and the definition of M,

$$|w_\varepsilon(x) - u(x)| \leq |\tilde v_{x_n}(x) - \tilde v_{x_n}(y)| + |\tilde v_{x_n}(y) - u(y)| + |u(y)-u(x)| \leq 2K\varepsilon + 0 + 2M\varepsilon$$

since $\tilde v_{x_n}(y) = u(y)$. Consequently, $|w_\varepsilon(x)-u(x)|\leq 2(M+K)\varepsilon$ for any $x\in A$ and also for any $x\in\overline{\Omega}$ because $w_\varepsilon = u$ in $\overline{\Omega}\backslash A$. So we have proved that w_ε uniformly converges to u in $\overline{\Omega}$.

For proving (4.10), we first show that

$$F(w_\varepsilon, A) \leq \bigvee_n F(\tilde v_{x_n}, A_{x_n}). \tag{4.11}$$

For this, let us set, for any $x\in A$, $I(x) = \{n\in\mathbf{N} : x\in A_{x_n}\}$. Note that $I(x)$ is finite (because the covering is locally finite), and that $w_\varepsilon(x) = \sup_{n\in I(x)} \tilde v_{x_n}(x)$. We claim that for any fixed $x\in A$, there exists some $\varrho_x > 0$ such that $\tilde v_{x_p}(z) < w_\varepsilon(z)$ for any $p\notin I(x)$ and $z\in B_{\varrho_x}(x)$. Indeed, fix $x\in A$ and let U be a neighborhood of x. Since the covering is locally finite, then $F := \{q\in\mathbf{N} : U\cap A_{x_q}\neq\emptyset\}$ is finite. If $q\in F\setminus I(x)$ then $x\notin A_{x_q}$ and, by definition, $\tilde v_{x_q}(x) = u(x) < w_\varepsilon(x)$. Since the functions $\tilde v_{x_q}$ and w_ε are continuous and F is finite, there exists some $B_{\varrho_x}(x)\subset U$ such that $\tilde v_{x_q}(z) < w_\varepsilon(z)$ for every $z\in B_{\varrho_x}(x)$ and for every $q\in F\setminus I(x)$. On the other hand, if $p\notin I(x)$ and $p\notin F$, then $U\cap A_{x_p}=\emptyset$. Thus for any $z\in B_{\varrho_x}(x)$, we have that $z\notin A_{x_p}$ which implies, by definition, that $\tilde v_{x_p}(z) = u(z) < w_\varepsilon(z)$.

Now, for any $x\in A$ define $B_x := \bigcap_{n\in I(x)} A_{x_n}\cap B_{\varrho_x}(x)$ and let us now fix a new locally finite covering (B_{y_i}) of A. Let us point out that $w_\varepsilon(z) = \bigvee_{n\in I(y_i)} \tilde v_{x_n}(z)$ on B_{y_i}, from the very definition of B_{y_i}. Hence, using property (4.4), we have that

$$F(w_\varepsilon, B_{y_i}) = F\left(\bigvee_{n\in I(y_i)} \tilde v_{x_n}, B_{y_i}\right) \leq \bigvee_{n\in I(y_i)} F(\tilde v_{x_n}, B_{y_i}) \leq \bigvee_{n\in\mathbf{N}} F(\tilde v_{x_n}, A_{x_n})$$

since $B_{y_i}\subset A_{x_n}$ for any $n\in I(y_i)$. Next we use the supremality of F to get

$$F(w_\varepsilon, A) = F\left(w_\varepsilon, \bigcup_i B_{y_i}\right) = \bigvee_i F(w_\varepsilon, B_{y_i}) \leq \bigvee_{n\in\mathbf{N}} F(\tilde v_{x_n}, A_{x_n}),$$

which completes the proof of (4.11).

Using (4.8) and the continuity property for F (Lemma 4.5 (ii)), we get that

$$f(x_n,u(x_n),Du(x_n)) \geq F(v_{x_n}, A_{x_n}) - \varepsilon \geq F(\tilde v_{x_n}, A_{x_n}) - \omega_{M'}(\|\alpha_{x_n} - \beta_{x_n}|\cdot - x_n|\,\|_{W^{1,\infty}(A_{x_n})}) - \varepsilon$$

where $M' = K + 2\varepsilon$. Since $\alpha_{x_n}\in(0,\varepsilon)$, $\beta_{x_n}\in(0,\varepsilon)$ and $A_{x_n}\subset B_\varepsilon(x_n)$, we have

$$f(x_n,u(x_n),Du(x_n)) \geq F(\tilde v_{x_n}, A_{x_n}) - \omega_{M'}(2\varepsilon) - \varepsilon\ .$$

Therefore, by using the continuity of $f(\cdot, u(\cdot), Du(\cdot))$ together with (4.11) we have

$$\begin{aligned}\operatorname*{ess\,sup}_{x\in A} f(x,u(x),Du(x)) &= \sup_{x\in A} f(x,u(x),Du(x)) \geq \sup_{n\in\mathbf{N}} f(x_n,u(x_n),Du(x_n))\\ &\geq \sup_{n\in\mathbf{N}} F(\tilde{v}_{x_n},A_{x_n}) - \omega_{M'}(2\varepsilon) - \varepsilon \geq F(w_\varepsilon, A) - \omega_{M'}(2\varepsilon) - \varepsilon,\end{aligned}$$

and it yields inequality (4.10).

Step 3. In order to extend the representation result on $W^{1,\infty}(\Omega)$, we first show that f is level convex with respect to the last variable. Fix $(x_0, z) \in \Omega\times\mathbf{R}$, ξ_1, $\xi_2 \in \mathbf{R}^N$ and $\lambda \in (0,1)$. By definition there exist a ball $B_r(x_0)$, and functions u and $v \in W^{1,\infty}(\Omega)$, differentiable at x_0 such that $u(x_0) = v(x_0) = z$, $Du(x_0) = \xi_1$, $Dv(x_0) = \xi_2$ and $f(x_0, z, \xi_1) \geq F(u, B_r(x_0)) - \varepsilon$ and $f(x_0, z, \xi_2) \geq F(v, B_r(x_0)) - \varepsilon$. Fix $0 < s < r$, and let (u_n) and $(v_n) \subset W^{1,\infty}(B_s(x_0))$ such that u_n and v_n uniformly converge in $\overline{\Omega}$ to u and v respectively, and

$$F''(u, B_s(x_0)) = \limsup_{n\to\infty} \operatorname*{ess\,sup}_{x\in B_s(x_0)} f_n(x, u_n(x), Du_n(x))$$

and

$$F''(v, B_s(x_0)) = \limsup_{n\to\infty} \operatorname*{ess\,sup}_{x\in B_s(x_0)} f_n(x, v_n(x), Dv_n(x)).$$

Then, by using the level convexity of f_n, we have

$$\begin{aligned}f(x_0,z,\lambda\xi_1+(1-\lambda)\xi_2) &\leq F(\lambda u+(1-\lambda)v, B_s(x_0)) \leq F''(\lambda u+(1-\lambda)v, B_s(x_0))\\ &\leq \limsup_{n\to\infty} F_n(\lambda u_n + (1-\lambda)v_n, B_s(x_0))\\ &\leq \limsup_{n\to\infty} F_n(u_n, B_s(x_0)) \vee F_n(v_n, B_s(x_0))\\ &= F''(u, B_s(x_0)) \vee F''(v, B_s(x_0)) \leq F(u, B_r(x_0)) \vee F(v, B_r(x_0))\\ &\leq (f(x_0,z,\xi_1)-\varepsilon) \vee (f(x_0,z,\xi_2)-\varepsilon).\end{aligned}$$

By letting $\varepsilon \to 0$ we get the thesis.

Step 4. Finally we prove that

$$F(u, A) = \operatorname*{ess\,sup}_{x\in A} f(x, u(x), Du(x))$$

for all $(u, A) \in W^{1,\infty}(\Omega) \times \mathcal{A}$. Since F is inner regular, for any $\varepsilon > 0$, we can find $A' \subset\subset A$ such that

$$F(u, A) \leq F(u, A') + \varepsilon\ .$$

Set $A_\rho := \{x \in A : \operatorname{dist}(x, \partial A) > \rho\}$. Then there exists $\rho_0 > 0$ such that $A' \subset\subset A_\rho$ for every $\rho \leq \rho_0$. Let ϕ_ρ be a standard mollifier and define $u_\rho := u * \phi_\rho$. Since u_ρ is regular, we have that

$$\operatorname*{ess\,sup}_{x\in A'} f(x, u_\rho(x), Du_\rho(x)) = F(u_\rho, A')\ .$$

Now, since $u_\rho \to u$ uniformly and f is uniformly continuous on A', there exists $0 < \rho_1 = \rho_1(\varepsilon) < \rho_0$ such that

$$|f(x, u_\rho(x), \xi) - f(y, u_\rho(y), \xi)| \leq \varepsilon \tag{4.12}$$

for every $x \in A'$, every $y \in B_\rho(x)$ (with $\rho < \rho_1$), and every $\xi \in \mathbf{R}^N$. Since f is level convex, by using Jensen inequality (see Theorem 2.2 with $\mu = \phi_\rho\,\mathcal{L}^N$), we have that for every $x \in A'$

$$f(x, u_\rho(x), Du_\rho(x)) \leq \operatorname*{ess\,sup}_{y\in B_\rho(x)} f(x, u_\rho(x), Du(y))\ ,$$

and thus, by (4.12) it follows that

$$\operatorname*{ess\,sup}_{x\in A'} f(x, u_\rho(x), Du_\rho(x)) \leq \operatorname*{ess\,sup}_{y\in A} f(y, u_\rho(y), Du(y)) + \varepsilon\ .$$

Therefore

$$F(u, A') \leq \liminf_{\rho\to 0^+} F(u_\rho, A') \leq \operatorname*{ess\,sup}_{x\in A} f(x, u(x), Du(x)) + \varepsilon\ ,$$

which implies

$$F(u, A) \leq \operatorname*{ess\,sup}_{x\in A} f(x, u(x), Du(x)) + 2\varepsilon,$$

and it completes the proof of the theorem since $\varepsilon > 0$ is arbitrary. □

5. Application to dimension reduction

5.1. **Abstract representation result.** Let ω be a bounded open subset of $\mathbf{R}^2$. We are interested in studying the asymptotic behavior of a family of supremal functionals in thin domains $\Omega_\varepsilon := \omega \times (-\varepsilon, \varepsilon)$, of the form

$$\operatorname*{ess\,sup}_{y \in \Omega_\varepsilon} f_\varepsilon(y, Dv(y)),$$

where $f_\varepsilon : \Omega_\varepsilon \times \mathbf{R}^3 \to [0, +\infty)$ is a supremand whose precise properties will be stated later. As usual in dimensional reduction, we rescale the problem in order to work on a fixed domain $\Omega := \Omega_1$. To do that, we perform the change of variables $(x_1, x_2, x_3) = (y_1, y_2, y_3/\varepsilon)$ and $u(x) = v(y)$ for $x = (x_1, x_2, x_3) \in \Omega$. Then

$$f_\varepsilon(y, Dv(y)) = W_\varepsilon\left(x, D_\alpha u(x) \Big| \frac{1}{\varepsilon} D_3 u(x)\right),$$

where we have denoted $W_\varepsilon(x, \xi) := f_\varepsilon(x_\alpha, \varepsilon x_3, \xi)$. In the sequel the variable x_α will stand for the in-plane variable (x_1, x_2), and D_α (resp. D_3) will denote the derivative with respect to x_α (resp. x_3).

Let $\mathcal{A}$ be the family of all open subsets of ω. We define the supremal functional $F_\varepsilon : \mathcal{C}(\overline{\Omega}) \times \mathcal{A} \to [0, +\infty]$ by

$$F_\varepsilon(u, A) := \begin{cases} \operatorname*{ess\,sup}_{x \in A \times I} W_\varepsilon\left(x, D_\alpha u(x) \Big| \frac{1}{\varepsilon} D_3 u(x)\right) & \text{if } u \in W^{1,\infty}(A \times I), \\ +\infty & \text{otherwise,} \end{cases}$$

where $I := (-1, 1)$. We next assume that $W_\varepsilon : \Omega \times \mathbf{R}^3 \to [0, +\infty)$ is a Carathéodory supremand satisfying

- (H_1) for each $M > 0$, there exists a modulus of continuity $\omega_M : [0, +\infty) \to [0, +\infty)$ such that
$$|W_\varepsilon(x, \xi) - W_\varepsilon(x, \xi')| \le \omega_M(\|\xi - \xi'\|),$$
for any $\varepsilon > 0$, for a.e. $x \in \Omega$, and for every ξ, $\xi' \in \mathbf{R}^3$ such that $\|\xi\| \le M$, $\|\xi'\| \le M$;
- (H_2) the functions $W_\varepsilon(x, \cdot)$ are level convex for any $\varepsilon > 0$ and for a.e. $x \in \Omega$;
- (H_3) there exists a continuous and increasing function $\Psi : [0, +\infty) \to [0, +\infty)$ such that $\Psi(t) \to +\infty$ as $t \to +\infty$, with the property that $W_\varepsilon(x, \xi) \ge \Psi(\|\xi\|)$ for any $\varepsilon > 0$, for every $\xi \in \mathbf{R}^3$ and a.e. $x \in \Omega$.

Theorem 5.1. *Under assumptions (H_1)-(H_3), there exists a subsequence $(\varepsilon_n) \searrow 0^+$ such that (F_{ε_n}) Γ^*-converges to some functional $F : \mathcal{C}(\overline{\Omega}) \times \mathcal{A} \to [0, +\infty]$. Moreover, let $\overline{W}$ be defined by*

$$\overline{W}(x_0, \overline{\xi}) := \inf \left\{ F(u, B_r(x_0)) : r > 0,\, u \in W^{1,\infty}(\omega),\, x_0 \in \widehat{u}, D_\alpha u(x_0) = \overline{\xi} \right\}, \tag{5.1}$$

for all $(x_0, \overline{\xi}) \in \omega \times \mathbf{R}^2$, where

$$\widehat{u} := \{x_\alpha \in \omega : x_\alpha \text{ is a differentiability point of } u \text{ and a Lebesgue point of } D_\alpha u\}.$$

If $\overline{W}(\cdot, \overline{\xi})$ is continuous for all $\overline{\xi} \in \mathbf{R}^2$, then

$$F(u, A) = \begin{cases} \operatorname*{ess\,sup}_{x_\alpha \in A} \overline{W}(x_\alpha, D_\alpha u(x_\alpha)) & \textit{if } u \in W^{1,\infty}(A), \\ +\infty & \textit{otherwise.} \end{cases}$$

Proof. According to Theorem 3.7, we get the existence of a subsequence $(\varepsilon_n) \searrow 0^+$ such that (F_{ε_n}) Γ^*-converges to some functional $F : \mathcal{C}(\overline{\Omega}) \times \mathcal{A} \to [0, +\infty]$. Moreover, if $F(u, A) < +\infty$, then taken $A' \subset\subset A$, we can consider a recovering sequence (u_n) which converges uniformly to u in $\overline{\Omega}$, and such that

$$\limsup_{n \to \infty} F_{\varepsilon_n}(u_n, A') = F''(u, A') \le F(u, A).$$

As a consequence, $u_n \in W^{1,\infty}(A' \times I)$ for n large enough, and by the coercivity assumption (H_3), we infer that

$$\left\| \left(D_\alpha u_n \Big| \frac{1}{\varepsilon_n} D_3 u_n \right) \right\|_{L^\infty(A' \times I; \mathbf{R}^3)} \le M,$$

for some constant $M > 0$ independent of n. Thus u_n weakly* converges to u in $W^{1,\infty}(A' \times I)$, and $\|D_3 u_n\|_{L^\infty(A'\times I)} \le M\varepsilon_n \to 0$. Hence $D_3 u = 0$ in $\mathcal{D}'(A' \times I)$, and since this property holds for any $A' \subset\subset A$, we deduce that $D_3 u = 0$ in $\mathcal{D}'(A \times I)$ so that $u \in W^{1,\infty}(A)$. Moreover, thanks to the continuity property (H_1), one can show that (4.6) holds, and consequently, the domain of the Γ^*-limit is $W^{1,\infty}(A)$. It remains to identify F on $W^{1,\infty}(A)$.

The rest of the proof follows that of Theorem 4.2. The main difference relies in proving the countable supremality property of F as in Lemma 4.5 (i). Indeed, as usual in dimension reduction problems (see [20, 10, 8]) one must take a cut-off function φ which only depends on the in-plane variable x_α. $\square$

5.2. **Homogenization of thin structures.** Let $W : \Omega \times \mathbf{R}^2 \times \mathbf{R}^3 \to [0, +\infty)$ be a function such that

- (A_1) the function $W(x, y_\alpha, \cdot)$ is level convex for a.e. $(x, y_\alpha) \in \Omega \times \mathbf{R}^2$;
- (A_2) for each $M > 0$, there exists a modulus of continuity $\omega_M : [0, +\infty) \to [0, +\infty)$ satisfying
$$|W(x_\alpha, x_3, y_\alpha, \xi) - W(x'_\alpha, x_3, y_\alpha, \xi')| \le \omega_M(\|x_\alpha - x'_\alpha\| + \|\xi - \xi'\|),$$
for a.e. $(x_3, y_\alpha) \in I \times \mathbf{R}^2$, and for every x_α, $x'_\alpha \in \omega$ and ξ, $\xi' \in \mathbf{R}^3$ such that $\|\xi\|, \|\xi'\| \le M$;
- (A_3) the function $(x_3, y_\alpha) \mapsto W(x_\alpha, x_3, y_\alpha, \xi)$ is measurable for all $x_\alpha \in \omega$ and all $\xi \in \mathbf{R}^3$;
- (A_4) the function $W(x, \cdot, \xi)$ is 1-periodic for a.e. $x \in \Omega$ and all $\xi \in \mathbf{R}^3$;
- (A_5) there exists a continuous and increasing function $\Psi : [0, +\infty) \to [0, +\infty)$ such that $\Psi(t) \to +\infty$ as $t \to +\infty$, with the property that $W(x, y_\alpha, \xi) \ge \Psi(\|\xi\|)$ for every $\xi \in \mathbf{R}^3$, and for a.e. $(x, y_\alpha) \in \Omega \times \mathbf{R}^2$;
- (A_6) there exists a locally bounded function $\beta : [0, +\infty) \to [0, +\infty)$ such that $W(x, y_\alpha, \xi) \le \beta(\|\xi\|)$ for every $\xi \in \mathbf{R}^3$, for a.e. $(x, y_\alpha) \in \Omega \times \mathbf{R}^2$.

Let us define $F_\varepsilon : \mathcal{C}(\overline{\Omega}) \times \mathcal{A} \to [0, +\infty]$ by

$$F_\varepsilon(u, A) := \begin{cases} \operatorname*{ess\,sup}_{x \in A \times I} W\left(x, \dfrac{x_\alpha}{\varepsilon}, D_\alpha u(x) \Big| \dfrac{1}{\varepsilon} D_3 u(x)\right) & \text{if } u \in W^{1,\infty}(A \times I), \\ +\infty & \text{otherwise.} \end{cases}$$

The main result of this section is the following representation theorem.

Theorem 5.2. *Under assumptions (A_1)-(A_6), then the family $(F_\varepsilon)_{\varepsilon>0}$ Γ^*-converges to the functional*

$$F_{\rm hom}(u, A) = \begin{cases} \operatorname*{ess\,sup}_{x_\alpha \in A} W_{\rm hom}(x_\alpha, D_\alpha u(x_\alpha)) & \text{if } u \in W^{1,\infty}(A), \\ +\infty & \text{otherwise,} \end{cases} \tag{5.2}$$

where $W_{\rm hom}$ is given by

$$W_{\rm hom}(x_0, \overline{\xi}) := \inf_{\varphi \in W^{1,\infty}(Q' \times I)} \left\{ \operatorname*{ess\,sup}_{y \in Q' \times I} W(x_0, y_3, y_\alpha, \overline{\xi} + D_\alpha \varphi(y) | D_3 \varphi(y)) : \varphi(\cdot, y_3) \text{ is 1-periodic for all } y_3 \in I \right\}, \tag{5.3}$$

for every $(x_0, \overline{\xi}) \in \omega \times \mathbf{R}^2$, where Q' stands for the unit square $(0, 1)^2$ of $\mathbf{R}^2$.

Clearly if W satisfies (A_1)-(A_6), then the function $W_\varepsilon(x, \xi) = W(x, x_\alpha/\varepsilon, \xi)$ fulfills assumptions (H_1)-(H_3). Hence Theorem 5.1 shows the existence of a subsequence (ε_n) such that F_{ε_n} Γ^*-converges to some functional $F : \mathcal{C}(\overline{\Omega}) \times \mathcal{A} \to [0, +\infty]$. Moreover, in order to ensure that F is representable by the function $\overline{W}$ defined by (5.1) (with $W_\varepsilon(x, \xi) = W(x, x_\alpha/\varepsilon, \xi)$) one needs to ensure that $\overline{W}$ is continuous in its first variable. This the object of the next lemma.

Lemma 5.3. *Assume that W satisfies (A_1)-(A_5), and let $\overline{W}$ be defined by (5.1) with $W_\varepsilon(x, \xi) := W(x, x_\alpha/\varepsilon, \xi)$. Then $\overline{W}(\cdot, \overline{\xi})$ is continuous for every $\overline{\xi} \in \mathbf{R}^2$.*

Proof. Let x_0, $y_0 \in \omega$ and $\overline{\xi} \in \mathbf{R}^2$, define $M := \|\overline{\xi}\|$. By definition (5.1) of $\overline{W}$, for any $\eta \in (0,1)$, there exist $r > 0$, $u \in W^{1,\infty}(\omega)$ such that x_0 is a point of differentiability of u and a Lebesgue point of $D_\alpha u$, $D_\alpha u(x_0) = \overline{\xi}$, and

$$F(u, B_r(x_0)) \leq \overline{W}(x_0, \overline{\xi}) + \eta. \tag{5.4}$$

By applying Lemma 4.6 and thanks to (A_2) and (A_5), we can find a constant $K = K(M)$ (independent of x_0) such that $\|u\|_{W^{1,\infty}(B_r(x_0))} \leq K$. Let $r'' < r' < r$, and consider a recovering sequence (u_n) uniformly converging to u in $\overline{\Omega}$, and satisfying

$$\limsup_{n\to\infty} F_{\varepsilon_n}(u_n, B_{r'}(x_0)) = F''(u, B_{r'}(x_0)). \tag{5.5}$$

Without loss of generality, one can assume that $u_n \in \mathcal{C}(\mathbf{R}^N)$ and that $u_n \to u$ uniformly in $\mathbf{R}^N$. Indeed, if it is not the case, using a cut-off function $\chi \in \mathcal{C}_c^\infty(\mathbf{R}^N; [0,1])$ such that $\chi = 1$ in $B_{r'}(x_0)$ and $\chi = 0$ outside $\overline{\Omega}$, it follows that the sequence $\tilde{u}_n = \chi u_n + (1-\chi)u \in \mathcal{C}(\mathbf{R}^N)$ is such that $\tilde{u}_n = u_n$ in $B_{r'}(x_0)$, $\tilde{u}_n = u$ outside $\overline{\Omega}$ and $\tilde{u}_n \to u$ uniformly in $\mathbf{R}^N$.

Thanks again to (A_2) and (4.6), there exists a constant $C = C(M)$ (independent of x_0) such that $F''(u, B_r(x_0)) \leq C$. By using also the coercivity property (A_5), we can find a constant $M' = M'(M)$ (independent of x_0) such that

$$\left\| \left(D_\alpha u_n \Big| \frac{1}{\varepsilon_n} D_3 u_n \right) \right\|_{L^\infty(B_{r'}(x_0)\times I;\mathbf{R}^3)} \leq M'. \tag{5.6}$$

Let $m_n \in \mathbf{Z}^2$ and $\theta_n \in [0,1)^2$ be such that

$$\frac{x_0 - y_0}{\varepsilon_n} = m_n + \theta_n,$$

and define $s_n := \varepsilon_n \theta_n \to 0$. Let n large enough so that $\|s_n\| < r' - r''$, then we set $v_n(y_\alpha, y_3) := u_n(y_\alpha - y_0 + x_0 - s_n, y_3)$ and $v(y_\alpha, y_3) := u(y_\alpha - y_0 + x_0, y_3)$ for all $y \in B_{r''}(y_0) \times I$. Clearly, $v_n \to v$ uniformly in $\overline{\Omega}$, and thus

$$\begin{aligned} F''(v, B_{r''}(y_0)) &\leq \limsup_{n\to\infty} \operatorname*{ess\,sup}_{x\in B_{r''}(y_0)\times I} W\left(x_\alpha, x_3, \frac{x_\alpha}{\varepsilon_n}, D_\alpha v_n(x) \Big| \frac{1}{\varepsilon_n} D_3 v_n(x)\right) \\ &= \limsup_{n\to\infty} \operatorname*{ess\,sup}_{y\in B_{r''}(x_0-s_n)\times I} W\left(y_\alpha + y_0 - x_0 + s_n, y_3, \frac{y_\alpha + y_0 - x_0 + s_n}{\varepsilon_n}, D_\alpha u_n(y) \Big| \frac{1}{\varepsilon_n} D_3 u_n(y)\right) \\ &\leq \limsup_{n\to\infty} \operatorname*{ess\,sup}_{y\in B_{r'}(x_0)\times I} W\left(y_\alpha + y_0 - x_0 + s_n, y_3, \frac{y_\alpha}{\varepsilon_n}, D_\alpha u_n(y) \Big| \frac{1}{\varepsilon_n} D_3 u_n(y)\right), \end{aligned}$$

where we used the periodicity property (A_4) of W and the fact that $m_n \in \mathbf{Z}^2$. Thus according to (5.6), (A_2), (5.5) and (5.4) we deduce that

$$\begin{aligned} F''(v, B_{r'}(y_0)) &\leq \limsup_{n\to\infty} \operatorname*{ess\,sup}_{y\in B_{r'}(x_0)\times I} W\left(y_\alpha, y_3, \frac{y_\alpha}{\varepsilon_n}, D_\alpha u_n(y) \Big| \frac{1}{\varepsilon_n} D_3 u_n(y)\right) + \omega_{M'}(\|y_0 - x_0\|) \\ &\leq F''(u, B_{r'}(x_0)) + \omega_{M'}(\|y_0 - x_0\|) \leq \overline{W}(x_0, \overline{\xi}) + \eta + \omega_{M'}(\|y_0 - x_0\|). \end{aligned}$$

Finally, since v is admissible for $\overline{W}(y_0, \overline{\xi})$, we conclude, after letting $\eta \to 0$, that

$$\overline{W}(y_0, \overline{\xi}) \leq \overline{W}(x_0, \overline{\xi}) + \omega_{M'}(\|y_0 - x_0\|).$$

By exchanging the roles of x_0 and y_0 we get the desired continuity property. □

In the next lemma, under the same set of assumptions we provide an upper bound for $\overline{W}$, namely we show that $\overline{W} \leq W_{\text{hom}}$.

Lemma 5.4. *Let W satisfying (A_1)-(A_5). For every $x_0 \in \omega$ and every $\overline{\xi} \in \mathbf{R}^2$, we have $W_{\text{hom}}(x_0, \overline{\xi}) \geq \overline{W}(x_0, \overline{\xi})$.*

Proof. By definition (5.3) of W_{hom}, for any $\eta > 0$ there exists $\varphi \in W^{1,\infty}(\mathbf{R}^2 \times I)$ such that $\varphi(\cdot, y_3)$ is 1-periodic for all $y_3 \in I$, and

$$\operatorname*{ess\,sup}_{y\in Q'\times I} W(x_0, y_3, y_\alpha, \overline{\xi} + D_\alpha \varphi(y) | D_3 \varphi(y)) \leq W_{\text{hom}}(x_0, \overline{\xi}) + \eta.$$

Now define $u(x) := \overline{\xi} x_\alpha$ and $u_n(x) := \overline{\xi} x_\alpha + \varepsilon_n \varphi(x_\alpha/\varepsilon_n, x_3)$. Then clearly $u_n \to u$ uniformly in $\overline{\Omega}$ and thus

$$F''(u, B_r(x_0)) \leq \limsup_{n\to\infty} \operatorname*{ess\,sup}_{x\in B_r(x_0)\times I} W\left(x_\alpha, x_3, \frac{x_\alpha}{\varepsilon_n}, D_\alpha u_n(x)\Big|\frac{1}{\varepsilon_n}D_3 u_n(x)\right)$$

$$= \limsup_{n\to\infty} \operatorname*{ess\,sup}_{x\in B_r(x_0)\times I} W\left(x_\alpha, x_3, \frac{x_\alpha}{\varepsilon_n}, \overline{\xi} + D_\alpha\varphi\left(\frac{x_\alpha}{\varepsilon_n}, x_3\right)\Big|D_3\varphi\left(\frac{x_\alpha}{\varepsilon_n}, x_3\right)\right)$$

$$\leq \operatorname*{ess\,sup}_{x_\alpha\in B_r(x_0)} \operatorname*{ess\,sup}_{y\in Q'\times I} W\left(x_\alpha, y_3, y_\alpha, \overline{\xi} + D_\alpha\varphi(y)|D_3\varphi(y)\right).$$

We next use the uniform continuity assumption (A_2) with $M := \|(\overline{\xi} + D_\alpha\varphi|D_3\varphi)\|_{L^\infty(Q'\times I;\mathbf{R}^3)}$ to get that

$$\begin{aligned} F''(u, B_r(x_0)) &\leq \operatorname*{ess\,sup}_{y\in Q'\times I} W(x_0, y_3, y_\alpha, \overline{\xi} + D_\alpha\varphi(y)|D_3\varphi(y)) + \omega_M(r) \\ &\leq W_{\rm hom}(x_0, \overline{\xi}) + \eta + \omega_M(r). \end{aligned}$$

By Proposition 3.4, we can choose a radius $r > 0$ such that $F(u, B_r(x_0)) = F''(u, B_r(x_0))$. By applying the representation formula provided by Lemma 5.3 and the fact that $\overline{W}$ is continuous in its first variable, we get that

$$\overline{W}(x_0, \overline{\xi}) \leq \operatorname*{ess\,sup}_{x_\alpha\in B_r(x_0)} \overline{W}(x_\alpha, \overline{\xi}) \leq W_{\rm hom}(x_0, \overline{\xi}) + \eta + \omega_M(r).$$

Letting $r \to 0^+$ and $\eta \to 0^+$, we get that $\overline{W}(x_0, \overline{\xi}) \leq W_{\rm hom}(x_0, \overline{\xi})$. □

Now our aim is to prove that, under the further assumption (A_6), the supremand $\overline{W}$, which represents the Γ^*-limit of a suitable subsequence F_{ε_n}, actually coincides with the function $W_{\rm hom}$ defined by (5.3).

To this end we need to recall the analogous results in the integral setting (see [20, 8, 9]) that will be exploited in the sequel. We introduce, for every $p > 1$, the family of integral functionals $F^p_\varepsilon : L^\infty(\Omega)\times\mathcal{A} \to [0, +\infty]$ defined by

$$F^p_\varepsilon(u, A) := \begin{cases} \left(\displaystyle\int_A W^p\left(x, \frac{x_\alpha}{\varepsilon}, D_\alpha u(x)\Big|\frac{1}{\varepsilon}D_3 u(x)\right)dx\right)^{1/p} & \text{if } u \in W^{1,p}(A\times I), \\ +\infty & \text{otherwise.} \end{cases}$$

Following [8], we have the following Γ-convergence result:

Theorem 5.5. *Assume that (A_2)-(A_6) hold with $\Psi(t) = C_1 t$ and $\beta(t) = C_2(t+1)$ for some positive constants C_1 and C_2. Then for each $p > 1$, the family $(F^p_\varepsilon)_{\varepsilon>0}$ Γ-converges, with respect to the strong $L^p(\Omega)$-convergence, to the functional $F^p_{\rm hom} : L^p(\Omega)\times\mathcal{A} \to [0, +\infty]$, defined by*

$$F^p_{\rm hom}(u, A) = \begin{cases} \left(\displaystyle\int_A W^p_{\rm hom}(x_\alpha, D_\alpha u(x_\alpha))\,dx_\alpha\right)^{1/p} & \textit{if } u \in W^{1,p}(A), \\ +\infty & \textit{otherwise,} \end{cases} \tag{5.7}$$

where the density $W^p_{\rm hom} : \omega\times\mathbf{R}^2 \to [0, +\infty)$ is defined by

$$\begin{aligned} W^p_{\rm hom}(x_0, \overline{\xi}) \;:=\; \lim_{T\to\infty} \inf\Big\{ \frac{1}{2T^2}\int_{(0,T)^2\times I} W^p(x_0, y_3, y_\alpha, \overline{\xi} + D_\alpha\varphi(y)|D_3\varphi(y))\,dy : \\ \varphi \in W^{1,p}((0,T)^2\times I),\ \varphi = 0 \text{ on } \partial(0,T)^2\times I \Big\}. \end{aligned} \tag{5.8}$$

Note that in [8], the regularity assumptions on W were different to ours. Indeed, in that paper, it is assumed that $W(\cdot, y_\alpha, \xi)$ is measurable for all $(y_\alpha, \xi) \in \mathbf{R}^2\times\mathbf{R}^3$, and $W(x, \cdot, \cdot)$ is continuous for a.e. $x \in \Omega$. However, the Γ-convergence result still holds true with our new set of hypotheses.

We will observe in the next Lemma that formula (5.8) can be specialized into a single cell formula as in (5.9). This is due to the fact that in constrast to [8] where vector valued functions are considered, we are here dealing with scalar valued functions, and thus the Γ-limit remains unchanged if we replace W^p by its convex envelope. This is a well known fact that in the convex case, the homogenization formula reduces to a single cell formula (see [18, Section 14.3]).

Lemma 5.6. *Under assumptions* (A_2)*-*(A_6) *with* $\Psi(t) = C_1 t$ *and* $\beta(t) = C_2(t+1)$ *for some positive constants* C_1 *and* C_2*, then for every* $(x_0, \overline{\xi}) \in \omega \times \mathbf{R}^2$ *we have*

$$W^p_{\text{hom}}(x_0,\overline{\xi}) = \inf\Big\{\frac{1}{2}\int_{Q'\times I} W^p(x_0,y_3,y_\alpha,\overline{\xi}+D_\alpha\varphi(y)|D_3\varphi(y))\,dy : \varphi \in W^{1,p}(Q'\times I),\ \varphi(\cdot,y_3) \text{ is 1-periodic for a.e. } y_3\in I\Big\}. \tag{5.9}$$

Proof. Let us define for every $(x_0,\overline{\xi}) \in \omega\times\mathbf{R}^2$,

$$W^*(x_0,\overline{\xi}) = \inf\Big\{\frac{1}{2}\int_{Q'\times I} W^p(x_0,y_3,y_\alpha,\overline{\xi}+D_\alpha\varphi(y)|D_3\varphi(y))\,dy : \varphi \in W^{1,p}(Q'\times I),\ \varphi(\cdot,y_3) \text{ is 1-periodic for a.e. } y_3\in I\Big\}.$$

According to [20, Remark 4.1] (see also [9, Lemma 2.2]), the limit as $T\to\infty$ in formula (5.8) can also be replaced by an infimum, and consequently $W^p_{\text{hom}}(x_0,\overline{\xi}) \le W^*(x_0,\overline{\xi})$.

In order to prove the opposite inequality we argue as in [18, Theorem 14.7]. Let $k\in\mathbf{N}$ and $\varphi\in W^{1,p}((0,k)^2\times I)$ such that $\varphi = 0$ on $\partial(0,k)^2\times I$, let $i = (i_1,i_2)\in\{0,1,\dots,k-1\}^2$. Define for every $y = (y_\alpha,y_3)\in Q'\times I$

$$\psi(y_\alpha,y_3) := \frac{1}{k^2}\sum_i \varphi(y_\alpha+i,y_3).$$

Clearly $\psi\in W^{1,p}(Q'\times I)$ and $\psi(\cdot,x_3)$ is 1-periodic for a.e. $x_3\in I$. On the other hand, since the minimization defining the function W^* is stated over scalar valued functions, a well known relaxation result of integral functionals in the scalar case guarantees that

$$W^*(x_0,\overline{\xi}) = \inf\Big\{\frac{1}{2}\int_{Q'\times I} C(W^p)(x_0,y_3,y_\alpha,\overline{\xi}+D_\alpha\varphi(y)|D_3\varphi(y))\,dy : \varphi \in W^{1,p}(Q'\times I),\ \varphi(\cdot,y_3) \text{ is 1-periodic for a.e. } y_3\in I\Big\},$$

where $C(W^p)$ stands for the convex envelope of W^p. Consequently, by virtue of the periodicity of ψ and $C(W^p)$, it results

$$\begin{aligned}W^*(x_0,\overline{\xi}) &\le \int_{Q'\times I} C(W^p)(x_0,y_3,y_\alpha,\overline{\xi}+D_\alpha\psi(y)|D_3\psi(y))\,dy\\ &= \frac{1}{k^2}\int_{(0,k)^2\times I} C(W^p)(x_0,y_3,y_\alpha,\overline{\xi}+D_\alpha\psi(y)|D_3\psi(y))\,dy\\ &= \frac{1}{k^2}\int_{(0,k)^2\times I} C(W^p)\left(x_0,y_3,y_\alpha,\frac{1}{k^2}\sum_i\left(\overline{\xi}+D_\alpha\varphi(y_\alpha+i,y_3)|D_3\varphi(y_\alpha+i,y_3)\right)\right)dy.\end{aligned}$$

Using the convexity of $C(W^p)$ and changing variable yields

$$\begin{aligned}W^*(x_0,\overline{\xi}) &\le \frac{1}{k^4}\sum_i\int_{(0,k)^2\times I} C(W^p)(x_0,y_3,y_\alpha,\overline{\xi}+D_\alpha\varphi(y_\alpha+i,y_3)|D_3\varphi(y_\alpha+i,y_3))\,dy\\ &= \frac{1}{k^2}\int_{(0,k)^2\times I} C(W^p)(x_0,y_3,y_\alpha,\overline{\xi}+D_\alpha\varphi(y)|D_3\varphi(y))\,dy\\ &\le \frac{1}{k^2}\int_{(0,k)^2\times I} W^p(x_0,y_3,y_\alpha,\overline{\xi}+D_\alpha\varphi(y)|D_3\varphi(y))\,dy.\end{aligned}$$

Since the previous inequality holds for any arbitrary function φ, taking the infimum with respect to φ and the limit as $k\to\infty$ leads to $W^p_{\text{hom}}(x_0,\overline{\xi}) \ge W^*(x_0,\overline{\xi})$. □

We next introduce the power law approximation of the supremal functional (5.2) by (5.7). We omit the proof of the following result since it is sufficient to repeat that of [22, Lemma 3.2] with some suitable changes.

Lemma 5.7. *Let* $W : \Omega\times\mathbf{R}^2\times\mathbf{R}^3 \to [0,+\infty)$ *be a supremand satisfying* (A_1)*-*(A_6) *with* $\Psi(t) = C_1 t$ *and* $\beta(t) = C_2(t+1)$ *for some positive constants* C_1 *and* C_2*. Then*

(i) *for every* $(x_0, \overline{\xi}) \in \omega \times \mathbf{R}^2$

$$\lim_{p \to +\infty} (W^p_{\mathrm{hom}}(x_0, \overline{\xi}))^{1/p} = W_{\mathrm{hom}}(x_0, \overline{\xi});$$

(ii) *for all* $A \in \mathcal{A}$ *and* $u \in W^{1,\infty}(A)$,

$$\lim_{p \to +\infty} \left(\int_A W^p_{\mathrm{hom}}(x_\alpha, D_\alpha u(x_\alpha))\, dx_\alpha \right)^{1/p} = \operatorname*{ess\,sup}_{x_\alpha \in A} W_{\mathrm{hom}}(x_\alpha, D_\alpha u(x_\alpha)).$$

We are now in position to prove the lower bound. The argument employed in the following result is very close to that of [22], and uses a power law approximation together with the analogous integral result (see [8]).

Lemma 5.8. *For every* $x_0 \in \omega$ *and every* $\overline{\xi} \in \mathbf{R}^2$, *we have* $W_{\mathrm{hom}}(x_0, \overline{\xi}) \le \overline{W}(x_0, \overline{\xi})$.

Proof. We report only the sketch of the proof of this lemma since it is analogous to [22, Proposition 4.2].

Step 1. We first assume that f satisfies standard coercivity and growth conditions (A_5) and (A_6), namely that there exist two positive constants C_1 and C_2 such that

$$C_1 \|\xi\| \le W(x_\alpha, y_3, y_\alpha, \xi) \le C_2(\|\xi\| + 1)$$

for every $(x_\alpha, \xi) \in \omega \times \mathbf{R}^2$ and for a.e. $(y_\alpha, y_3) \in \mathbf{R}^2 \times I$. Let $u(x_\alpha) := \overline{\xi} \cdot x_\alpha$ and $r > 0$ be such that $F(u, B_r(x_0)) = F''(u, B_r(x_0))$. Consider a recovering sequence (u_n), converging uniformly to u in $\overline{\Omega}$, and such that $F_{\varepsilon_n}(u_n, B_r(x_0)) \to F(u, B_r(x_0))$. By applying Theorem 5.5, we have the following chain of inequalities:

$$\begin{aligned}
&\left(\int_{B_r(x_0)} W^p_{\mathrm{hom}}(x_\alpha, \overline{\xi})\, dx_\alpha \right)^{1/p} \\
&\qquad \le \liminf_{n \to \infty} \left(\int_{B_r(x_0) \times I} W^p \left(x_\alpha, x_3, \frac{x_\alpha}{\varepsilon_n}, D_\alpha u_n(x) \Big| \frac{1}{\varepsilon_n} D_3 u_n(x) \right) dx \right)^{1/p} \\
&\qquad\qquad \le (2\pi r^2)^{1/p} \liminf_{n \to \infty} \operatorname*{ess\,sup}_{x \in B_r(x_0) \times I} W \left(x_\alpha, x_3, \frac{x_\alpha}{\varepsilon_n}, D_\alpha u_n(x) \Big| \frac{1}{\varepsilon} D_3 u_n(x) \right)
\end{aligned}$$

for each $p > 1$. Invoking Lemma 5.7 and Theorem 5.1 and passing to the limit as $p \to +\infty$ we obtain

$$\begin{aligned}
W_{\mathrm{hom}}(x_0, \overline{\xi}) \le \operatorname*{ess\,sup}_{x_\alpha \in B_r(x_0)} W_{\mathrm{hom}}(x_\alpha, \overline{\xi}) &= \lim_{p \to +\infty} \left(\int_{B_r(x_0)} W^p_{\mathrm{hom}}(x_\alpha, \overline{\xi})\, dx_\alpha \right)^{1/p} \\
&\le \operatorname*{ess\,sup}_{x_\alpha \in B_r(x_0)} \overline{W}(x_\alpha, \overline{\xi}).
\end{aligned}$$

Finally, since by Lemma 5.3, $\overline{W}$ is continuous in the first variable, we deduce by letting $r \to 0$ that $W_{\mathrm{hom}}(x_0, \overline{\xi}) \le \overline{W}(x_0, \overline{\xi})$.

Step 2. Assume that $W(x_\alpha, y_3, y_\alpha, \cdot)$ satisfies (A_5) with $\Psi(t) = t$ and (A_6) in its general form. for every $M > 0$ one can define

$$W_M(x_\alpha, y_3, y_\alpha, \xi) = W(x_\alpha, y_3, y_\alpha, \xi) \wedge \left[M \vee \frac{1}{2}(1 + \|\xi\|) \right].$$

The function W_M clearly fulfills all the assumptions of Step 1, and thus since $W_M \le W$, we deduce that

$$(W_M)_{\mathrm{hom}}(x_0, \overline{\xi}) \le \overline{W}(x_0, \overline{\xi}),$$

where

$$(W_M)_{\rm hom}(x_0,\overline{\xi}) \quad := \quad \inf\left\{ \operatorname*{ess\,sup}_{y\in Q'\times I} W_M(x_0,y_3,y_\alpha,\overline{\xi}+D_\alpha\varphi(y)|D_3\varphi(y)) : \right.$$

$$\left. \varphi\in W^{1,\infty}(Q'\times I),\ \varphi(\cdot,y_3) \text{ is 1-periodic for all } y_3\in I \right\}.$$

The proof of this case can be easily completed by showing that for M large enough,

$$(W_M)_{\rm hom}(x_0,\overline{\xi}) = W_{\rm hom}(x_0,\overline{\xi}).$$

Step 3. The general case follows by applying the previous step to the function $\Psi^{-1}(W)$. □

Proof of Theorem 5.2. By Theorem 5.1 and Lemmas 5.3, 5.4 and 5.8, we infer that $F = F_{\rm hom}$. Moreover since the Γ^*-limit is independent of the extracted subsequence (ε_n), there is actually no need to extract a subsequence thanks to [28, Proposition 16.8]. □

Remark 5.9. As a consequence of the above results we obtain that the homogenization and the power-law approximation commute as summarized by the following diagram:

$$\begin{array}{ccc}
F^p_\varepsilon & \xrightarrow[p\,\longrightarrow\,\infty]{\Gamma(L^\infty)} & F_\varepsilon \\
\Gamma(L^p)\ \Big\downarrow\ {\scriptstyle \varepsilon\to 0} & & {\scriptstyle \varepsilon\to 0}\ \Big\downarrow\ \Gamma^*(L^\infty) \\
F^p_{\rm hom} & \xrightarrow[p\,\longrightarrow\,\infty]{\Gamma(L^\infty)} & F_{\rm hom}
\end{array}$$

Indeed, the left vertical arrow has been shown in Theorem 5.5, the above horizontal arrow has been proved in Theorem 2.4, the right vertical arrow follows from Theorem 5.2, while the down arrow is established in the following Theorem 5.10.

Theorem 5.10. *Let* $W : \Omega\times\mathbf{R}^2\times\mathbf{R}^3 \to [0,+\infty)$ *be a supremand satisfying* (A_1)-(A_6) *with* $\Psi(t) = C_1 t$ *and* $\beta(t) = C_2(t+1)$ *for some positive constants* C_1 *and* C_2. *Then the family* $(F^p_{\rm hom})_{p>1}$ *defined in* (5.7) Γ-*converges, as* $p\to+\infty$, *with respect to the uniform convergence to* $F_{\rm hom}$ *defined by* (5.2).

Proof. By virtue of Theorem 5.5, the functional $F^p_{\rm hom}$ is a Γ-limit, with respect to the $L^p(\Omega)$-topology. Consequently it is lower semicontinuous on $W^{1,p}(\omega)$ with respect to the $L^p(\omega)$-topology, and so, in particular, it is lower semicontinuous on $W^{1,\infty}(\omega)$ with respect to the $L^\infty(\omega)$ topology. Moreover the family $(F^p_{\rm hom})_{p>1}$ is increasing in p and, by virtue of Lemma 5.7, it pointwise converges to $F_{\rm hom}$, as $p\to+\infty$. As a consequence, thanks to Proposition 5.4 and Remark 5.5 in [28], $(F^p_{\rm hom})_{p>1}$ Γ-converges, as $p\to+\infty$, with respect to the uniform convergence to $F_{\rm hom}$. □

6. The homogeneous case

In this section we treat the particular case where $W_\varepsilon(x,\xi) = W(\xi)$. Without assuming that the function W is level convex, and without appealing to the general representation result Theorem 5.1, we will provide a representation theorem analogous to that shown in the integral case (see [34, Theorem 2]). On the other hand, for technical reasons, we will replace coercivity condition (A_5) by a linear standard coerciviness as in (6.1).

Theorem 6.1. *Let* $W : \mathbf{R}^3 \to [0,+\infty)$ *be a continuous function, and assume that*

$$W(\xi) \geq C\|\xi\| \text{ for every } \xi\in\mathbf{R}^3. \tag{6.1}$$

For each $\varepsilon > 0$, define $F_\varepsilon : \mathcal{C}(\overline{\Omega}) \times \mathcal{A} \to [0, +\infty]$ by

$$F_\varepsilon(u, A) := \begin{cases} \operatorname*{ess\,sup}_{x \in A \times I} W\left(D_\alpha u(x)\Big|\frac{1}{\varepsilon}D_3u(x)\right) & \text{if } u \in W^{1,\infty}(A \times I), \\ +\infty & \text{otherwise.} \end{cases} \tag{6.2}$$

Then the family $(F_\varepsilon)_{\varepsilon>0}$ Γ^*-converges to the functional $F_0 : \mathcal{C}(\overline{\Omega}) \times \mathcal{A} \to [0, +\infty]$ given by

$$F_0(u, A) := \begin{cases} \operatorname*{ess\,sup}_{x_\alpha \in A}(W_0)^{\rm lc}\left(D_\alpha u(x_\alpha)\right) & \text{if } u \in W^{1,\infty}(A), \\ +\infty & \text{otherwise,} \end{cases} \tag{6.3}$$

where for every $\overline{\xi} \in \mathbf{R}^2$,

$$W_0(\overline{\xi}) = \inf_{c \in \mathbf{R}} W(\overline{\xi}|c), \tag{6.4}$$

and $(W_0)^{\rm lc}$ is the level convex envelope of W_0, (see (2.3)). Moreover for any bounded open set $\Omega \subset \mathbf{R}^N$ satisfying one of the following properties

(C2) *Ω is of class $\mathcal{C}^2$;*

(S) *Ω is strongly star-shaped;*

the family $(F_\varepsilon(\cdot, \Omega))_{\varepsilon>0}$ Γ-converges to $F_0(\cdot, \Omega)$.

For the proof we will follow an approach closer to that of [34] for the lower bound, and to that of [22] for the upper bound, which do not rest on an abstract Γ-convergence result.

Before going into the proof of Theorem 6.1, we state technical results which precise again formula (6.3) when W is level convex. Indeed, we first observe that if W is already level convex, the same property is inherited by W_0.

Proposition 6.2. *Let $W : \mathbf{R}^3 \to [0, +\infty)$ be a continuous and level convex function. Then W_0 defined in (6.4) is level convex as well.*

Proof. Let $\overline{\xi}_1$, $\overline{\xi}_2 \in \mathbf{R}^2$, $\lambda \in [0, 1]$ and $\eta > 0$. There exist c_1, $c_2 \in \mathbf{R}$ such that $W_0(\overline{\xi}_1) + \eta \geq W(\overline{\xi}_1|c_1)$ and $W_0(\overline{\xi}_2) + \eta \geq W(\overline{\xi}_2|c_2)$. Then by definition of W_0 and since W is level convex, we have that

$$\begin{aligned} W_0(\lambda\overline{\xi}_1 + (1-\lambda)\overline{\xi}_2) &\leq W(\lambda\overline{\xi}_1 + (1-\lambda)\overline{\xi}_2|\lambda c_1 + (1-\lambda)c_2) \\ &\leq W(\overline{\xi}_1|c_1) \vee W(\overline{\xi}_2|c_2) \\ &\leq (W_0(\overline{\xi}_1) + \eta) \vee (W_0(\overline{\xi}_2) + \eta). \end{aligned}$$

The arbitrariness of η allows us to conclude the proof. □

In the following proposition we will show that the level convex envelope and the minimum with respect to the third variable commute.

Proposition 6.3. *Let $W : \mathbf{R}^3 \to [0, +\infty)$ be a continuous function, then $(W_0)^{\rm lc}(\overline{\xi}) = (W^{\rm lc})_0(\overline{\xi})$ for every $\overline{\xi} \in \mathbf{R}^2$.*

Proof. It is easily observed that for every $\overline{\xi} \in \mathbf{R}^2$ and every $c \in \mathbf{R}$,

$$(W_0)^{\rm lc}(\overline{\xi}) \leq W_0(\overline{\xi}) \leq W(\xi|c).$$

Thus, since $(W_0)^{\rm lc}$ is level convex, we have that

$$(W_0)^{\rm lc}(\overline{\xi}) \leq W^{\rm lc}(\overline{\xi}|c),$$

and taking the infimum with respect to $c \in \mathbf{R}$ on the right hand side we obtain

$$(W_0)^{\rm lc}(\overline{\xi}) \leq (W^{\rm lc})_0(\overline{\xi}).$$

In order to prove the opposite inequality, we clearly have for every $\overline{\xi} \in \mathbf{R}^2$ and every $c \in \mathbf{R}$,

$$(W^{\rm lc})_0(\overline{\xi}) \leq W^{\rm lc}(\overline{\xi}|c) \leq W(\overline{\xi}|c).$$

Taking the infimum with respect to $c \in \mathbf{R}$ in the right hand side of the previous inequality, yields

$$(W^{\rm lc})_0(\overline{\xi}) \leq W_0(\overline{\xi}).$$

By virtue of Proposition 6.2, the function $(W^{\rm lc})_0$ is level convex and thus we obtain

$$(W^{\rm lc})_0(\overline{\xi}) \le (W_0)^{\rm lc}(\overline{\xi}),$$

which completes the proof of the proposition. □

We now show that when $W = W(\xi)$ the function $(W_0)^{\rm lc}$ coincides with the function defined by (5.3).

Lemma 6.4. *Under assumption* (6.1) $(W_0)^{\rm lc}(\overline{\xi}) = W_{\rm hom}(\overline{\xi})$*, namely for every* $\overline{\xi} \in \mathbf{R}^2$*,*

$$(W_0)^{\rm lc}(\overline{\xi}) \;=\; \inf\left\{ \operatorname*{ess\,sup}_{y\in Q'\times I} W(\overline{\xi} + D_\alpha\varphi(y)|D_3\varphi(y)) : \varphi \in W^{1,\infty}(Q'\times I),\ \varphi(\cdot, y_3) \text{ is 1-periodic for all } y_3 \in I \right\}.$$

Proof. Assume that W is a level convex function so that W_0 is level convex as well by Proposition 6.2. According to definition formula (5.3), we denote

$$W_{\rm hom}(\overline{\xi}) \;:=\; \inf\left\{ \operatorname*{ess\,sup}_{y\in Q'\times I} W(\overline{\xi} + D_\alpha\varphi(y)|D_3\varphi(y)) : \varphi \in W^{1,\infty}(Q'\times I),\ \varphi(\cdot, y_3) \text{ is 1-periodic for all } y_3 \in I \right\}$$

for every $\xi \in \mathbf{R}^2$. Then, by definition of W_0 and by applying Jensen's inequality (2.2) to the level convex function W_0, we have that

$$\begin{aligned} W_{\rm hom}(\overline{\xi}) \;&\ge\; \inf\left\{ \operatorname*{ess\,sup}_{y\in Q'\times I} W_0(\overline{\xi} + D_\alpha\varphi(y)) : \varphi \in W^{1,\infty}(Q'\times I),\ \varphi(\cdot, y_3) \text{ is 1-periodic for all } y_3 \in I \right\} \\ &\ge\; \inf\left\{ \operatorname*{ess\,sup}_{y_3\in I} W_0\left(\int_{Q'} (\overline{\xi} + D_\alpha\varphi(y_\alpha, y_3))\,dy_\alpha\right) : \varphi \in W^{1,\infty}(Q'\times I),\ \varphi(\cdot, y_3) \text{ is 1-periodic for all } y_3 \in I \right\} \\ &=\; W_0(\overline{\xi}). \end{aligned}$$

In order to prove the opposite inequality, for every $p > 1$, let us define $G_p : W^{1,\infty}(Q'\times I) \to [0, +\infty)$ by

$$G_p(\varphi) := \left(\int_{Q'\times I} W^p(\overline{\xi} + D_\alpha\varphi(y)|D_3\varphi(y))\,dy\right)^{1/p}.$$

Thanks to Theorem 2.4 (see [27, Theorem 3.1]), the family $(F_p)_{p>1}$ Γ-converges, with respect to the uniform convergence to

$$\overline{G}(\varphi) := \operatorname*{ess\,sup}_{y\in Q'\times I} W(\overline{\xi} + D_\alpha\varphi(y)|D_3\varphi(y)),$$

as $p \to +\infty$. Consequently, by the convergence of minimizers and by using [20, Remark 3.3], we have that

$$\begin{aligned} W_{\rm hom}(\overline{\xi}) \;&=\; \lim_{p\to+\infty} \inf\left\{ \left(\int_{Q'\times I} W^p(\overline{\xi} + D_\alpha\varphi(y)|D_3\varphi(y))\,dy\right)^{1/p} : \varphi \in W^{1,\infty}(Q'\times I),\ \varphi(\cdot, y_3) \text{ is 1-periodic for all } y_3 \in I \right\} \\ &=\; \lim_{p\to\infty} [C((W^p)_0)(\overline{\xi})]^{1/p}, \end{aligned}$$

where we have denoted by $C((W^p)_0)$ the convex envelope of $(W^p)_0$, and $(W^p)_0(\overline{\xi}) := \inf_{c\in\mathbf{R}} W^p(\overline{\xi}|c)$. Finally, since $[C(W^p)_0(\overline{\xi})]^{1/p} \leq W_0(\overline{\xi})$, we obtain $W_{\text{hom}}(\overline{\xi}) \leq W_0(\overline{\xi})$.

If W is not level convex, by applying the first part of this proof to the level convex function W^{lc} we obtain that

$$(W^{\text{lc}})_0(\overline{\xi}) = \inf\Bigg\{ \operatorname*{ess\,sup}_{y\in Q'\times I} W^{\text{lc}}(\overline{\xi} + D_\alpha\varphi(y)|D_3\varphi(y)) : \varphi \in W^{1,\infty}(Q'\times I),\ \varphi(\cdot, y_3) \text{ is 1-periodic for all } y_3 \in I \Bigg\}.$$

Therefore, by applying Proposition 6.3 and the relaxation theorem 2.3, we can conclude

$$(W_0)^{\text{lc}}(\overline{\xi}) = (W^{\text{lc}})_0(\overline{\xi}) = \inf\Bigg\{ \operatorname*{ess\,sup}_{y\in Q'\times I} W(\overline{\xi} + D_\alpha\varphi(y)|D_3\varphi(y)) : \varphi \in W^{1,\infty}(Q'\times I),\ \varphi(\cdot, y_3) \text{ is 1-periodic for all } y_3 \in I \Bigg\}.$$

□

We now are in position to prove Theorem 6.1.

Proof of Theorem 6.1. First of all we prove that for every $(u, A) \in \mathcal{C}(\overline{\Omega}) \times \mathcal{A}$ and every sequence $(u_\varepsilon) \subset \mathcal{C}(\overline{\Omega})$ uniformly converging to u in $\overline{\Omega}$, one has

$$F_0(u, A) \leq \liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, A).$$

In fact, if $F_0(u, A) = +\infty$, there is nothing to prove. Otherwise, if $F_0(u, A) < +\infty$, then we can consider a subsequence $(\varepsilon_n) \searrow 0^+$ such that

$$\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, A) = \lim_{n\to\infty} F_{\varepsilon_n}(u_{\varepsilon_n}, A),$$

and thanks to (6.2), the sequence $(u_{\varepsilon_n}) \subset W^{1,\infty}(A\times I)$. According to the coercivity condition (6.1), we have

$$\left\| \left(D_\alpha u_{\varepsilon_n} \Big| \frac{1}{\varepsilon_n} D_3 u_{\varepsilon_n} \right) \right\|_{L^\infty(A\times I;\mathbf{R}^3)} \leq M,$$

for some constant $M > 0$ independent of n. Hence the sequence (u_{ε_n}) weakly* converges to u in $W^{1,\infty}(A \times I)$ and $u \in W^{1,\infty}(A)$. By definition of $(W_0)^{\text{lc}}$, we have

$$\operatorname*{ess\,sup}_{x\in A\times I} W\left(D_\alpha u_{\varepsilon_n}(x) \Big| \frac{1}{\varepsilon_n} D_3 u_{\varepsilon_n}(x) \right) \geq \operatorname*{ess\,sup}_{x\in A\times I} (W_0)^{\text{lc}}(D_\alpha u_{\varepsilon_n}(x)).$$

But since $(W_0)^{\text{lc}}$ is level convex, the supremal functional

$$v \mapsto \operatorname*{ess\,sup}_{x\in A\times I} (W_0)^{\text{lc}}(D_\alpha v(x))$$

is sequentially weakly* lower semicontinuous in $W^{1,\infty}(A \times I)$, hence

$$\begin{aligned}\lim_{n\to\infty} \operatorname*{ess\,sup}_{x\in A\times I} W\left(D_\alpha u_{\varepsilon_n}(x) \Big| \frac{1}{\varepsilon_n} D_3 u_{\varepsilon_n}(x) \right) &\geq \liminf_{n\to\infty} \operatorname*{ess\,sup}_{x\in A\times I} (W_0)^{\text{lc}}(D_\alpha u_{\varepsilon_n}(x)) \\ &\geq \operatorname*{ess\,sup}_{x_\alpha\in A} (W_0)^{\text{lc}}(D_\alpha u(x_\alpha)),\end{aligned}$$

since u is independent of x_3. Finally, we have that

$$\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, A) \geq F_0(u, A).$$

For the Γ^*-limsup, the proof develops as in [22, Theorem 5.2]. Thus we present here just the main steps. First, using Lemma 6.4, and arguing as in the proof of Lemma 5.4, we get the limsup inequality on affine functions. The second step consists in a fundamental estimate, and it is proved exactly as [22, Step 2 in Theorem 5.2], with the only difference that the domain A is now a subset of ω and the polyhedral partition is of the type A_1 an A_2, with cut-off function φ_δ which just depends on the planar variables x_α. In this way it is created a partition of the set $A \times I$. The other steps are also similar, *i.e.*, next one can provide the upper bound on $\mathcal{C}^1$ functions and on regular open sets. Finally the Γ-lim sup is obtained on $W^{1,\infty}$ functions, giving the representation on $\mathcal{C}^2$ and star-shaped domains. Finally one may proceed obtaining the sub-supremality of the inner regular envelope, obtaining the Γ-lim sup inequality on a rich family of open sets and for every $u \in W^{1,\infty}(\Omega)$. □

7. Application to parametrized homogenization

We conclude this paper with a further application to Theorem 4.2 to the case of parametrized homogenization. Let $\Omega \subset \mathbf{R}^N$ be a bounded open set, and $W_\varepsilon(x,\xi) := W\left(x, \frac{x}{\varepsilon}, \xi\right)$. We are interested in the homogenization (by Γ-convergence with respect to the uniform convergence) of the following supremal functionals $G_\varepsilon : \mathcal{C}(\overline{\Omega}) \times \mathcal{A} \to [0,+\infty]$ defined by

$$G_\varepsilon(u,A) := \begin{cases} \operatorname*{ess\,sup}_{x\in A} W\left(x, \dfrac{x}{\varepsilon}, Du(x)\right) & \text{if } u \in W^{1,\infty}(A), \\ +\infty & \text{otherwise,} \end{cases} \tag{7.1}$$

where $W : \Omega \times \mathbf{R}^N \times \mathbf{R}^N \to [0,+\infty)$ is a function such that

- (B_1) the function $W(x,y,\cdot)$ is level convex for all $x \in \Omega$ and a.e. $y \in \mathbf{R}^N$;
- (B_2) for each $M > 0$ there exists a modulus of continuity $\omega_M : [0,+\infty) \to [0,+\infty)$ satisying
$$|W(x,y,\xi) - W(x',y,\xi')| \le \omega_M(\|x - x'\| + \|\xi - \xi'\|)$$
for a.e. $y \in \mathbf{R}^N$ and for every x, $x' \in \Omega$, ξ, $\xi' \in \mathbf{R}^N$ with $\|\xi\|$, $\|\xi'\| \le M$;
- (B_3) the function $y \mapsto W(x,y,\xi)$ is measurable for all $x \in \Omega$ and all $\xi \in \mathbf{R}^N$;
- (B_4) the function $W(x,\cdot,\xi)$ is 1-periodic for all $x \in \Omega$ and $\xi \in \mathbf{R}^N$;
- (B_5) there exists a continuous increasing function $\Psi : [0,+\infty) \to [0,+\infty)$ such that $\Psi(t) \to +\infty$ as $t \to +\infty$, with the property that $W(x,y,\xi) \ge \Psi(\|\xi\|)$ for every $\xi \in \mathbf{R}^N$, all $x \in \Omega$, and a.e. $y \in \mathbf{R}^N$;
- (B_6) there exists a locally bounded function $\beta : [0,+\infty) \to [0,+\infty)$ such that $W(x,y,\xi) \le \beta(\|\xi\|)$ for every $\xi \in \mathbf{R}^N$, all $x \in \Omega$ and a.e. $y \in \mathbf{R}^N$.

Theorem 7.1. *Under assumptions (B_1)-(B_6), the familly $(G_\varepsilon)_{\varepsilon>0}$ defined by (7.1) Γ^*-converges to the functional $G_{\mathrm{hom}} : \mathcal{C}(\overline{\Omega}) \times \mathcal{A} \to [0,+\infty]$ defined by*

$$G_{\mathrm{hom}}(u,A) = \begin{cases} \operatorname*{ess\,sup}_{x\in A} W_{\mathrm{hom}}(x, Du(x)) & \text{if } u \in W^{1,\infty}(A), \\ +\infty & \text{otherwise,} \end{cases}$$

where W_{hom} is the continuous function given by

$$W_{\mathrm{hom}}(x_0,\xi) := \inf_{\varphi \in W^{1,\infty}(Q)} \left\{ \operatorname*{ess\,sup}_{y\in Q} W(x_0, y, \xi + D\varphi(y)) : \varphi \text{ is } Q\text{-periodic} \right\},$$

for all $(x_0,\xi) \in \Omega \times \mathbf{R}^N$, and Q is the unit square $(0,1)^N$.

We do not report the proof whose scheme follows the lines of Theorem 5.2 with some suitable changes:

1. by applying Theorem 3.7, we get the existence of a subsequence $(\varepsilon_n) \searrow 0^+$ such that G_{ε_n} Γ^*-converges to some functional $G : \mathcal{C}(\overline{\Omega}) \times \mathcal{A} \to [0,+\infty]$. Thanks to Theorem 4.2, in order to represent G in a supremal form, one checks that the function $\overline{W}$ is continuous as done in Lemma 5.3;
2. by proceeding as in Lemma 5.4 one shows that $W_{\mathrm{hom}} \ge \overline{W}$;
3. by applying an approximation argument by integral functionals (as in Lemma 5.8) one shows that $W_{\mathrm{hom}} \le \overline{W}$. We only remark that, in this last step, instead of applying Theorem 5.5, it is necessary to refer to [12] (see also [18, Proposition 2.23] and [9]).

We also observe that a commutative diagram such as presented in Remark 5.9 may be reproduced in the framework of parametrized homogenization when dealing with the integral counterpart of functionals G_ε in (7.1), namely when considering the Γ-limit as $\varepsilon \to 0^+$ and $p \to +\infty$ of

$$u \mapsto \left(\int_\Omega W^p \left(x, \frac{x}{\varepsilon}, Du \right) dx \right)^{1/p}.$$

Remark 7.2. We observe that Theorem 7.1 could be obtained as a particular case of dimensional reduction, having in mind that the result proven in Theorem 5.2 hold true also for the passage $(N+1)D - ND$ (for any $N \geq 1$). For the readers' convenience, the subsequent considerations will be made in the case $N = 2$.

To deduce Theorem 7.1 from Theorem 5.2 it suffices to assume that the energy density $W_\varepsilon(x, \xi) = W(x_\alpha, \frac{x_\alpha}{\varepsilon}, \overline{\xi})$ has no dependence on the transverse variable x_3 and on the last variable of the gradient. Indeed, the proof of the lower bound is straightforward since – with the notations of section 5 – we clearly have for every $(u, A) \in W^{1,\infty}(\omega) \times \mathcal{A}$, and every $A' \subset\subset A$,

$$\begin{aligned} G'(u, A') &:= \inf_{(u_\varepsilon) \subset W^{1,\infty}(\omega)} \left\{ \liminf_{\varepsilon \to 0} \operatorname*{ess\,sup}_{x_\alpha \in A'} W \left(x_\alpha, \frac{x_\alpha}{\varepsilon}, D_\alpha u_\varepsilon(x_\alpha) \right) : u_\varepsilon \to u \text{ uniformly in } \overline{\omega} \right\} \\ &\geq \inf_{(u_\varepsilon) \subset W^{1,\infty}(\Omega)} \left\{ \liminf_{\varepsilon \to 0} \operatorname*{ess\,sup}_{(x_\alpha, x_3) \in A' \times I} W \left(x_\alpha, \frac{x_\alpha}{\varepsilon}, D_\alpha u_\varepsilon(x_\alpha, x_3) \right) : u_\varepsilon \to u \text{ uniformly in } \overline{\Omega} \right\}. \end{aligned}$$

Hence taking the supremum with respect to all $A' \subset\subset A$ yields

$$G'_-(u, A) \geq \operatorname*{ess\,sup}_{x_\alpha \in A} W_{\text{hom}}(x_\alpha, D_\alpha u(x_\alpha)),$$

where according to Theorem 5.2,

$$\begin{aligned} W_{\text{hom}}(x_0, \overline{\xi}) \quad &:= \inf_{\varphi \in W^{1,\infty}(Q' \times I)} \left\{ \operatorname*{ess\,sup}_{y \in Q' \times I} W(x_0, y_\alpha, \overline{\xi} + D_\alpha \varphi(y)) : \right. \\ &\qquad\qquad \left. \varphi(\cdot, y_3) \text{ is 1-periodic for all } y_3 \in I \right\} \\ &= \inf_{\varphi \in W^{1,\infty}_{\text{per}}(Q')} \left\{ \operatorname*{ess\,sup}_{y \in Q'} W(x_0, y_\alpha, \overline{\xi} + D_\alpha \varphi(y_\alpha)) \right\} \quad \text{for all } (x_0, \overline{\xi}) \in \omega \times \mathbf{R}^2. \end{aligned}$$

For what concerns the upper bound, thanks again to Theorem 5.2, we infer the existence of a sequence $(u_\varepsilon) \subset W^{1,\infty}(\Omega)$ uniformly converging to u in $\overline{\Omega}$, and such that

$$\begin{aligned} &\limsup_{\varepsilon \to 0} \operatorname*{ess\,sup}_{(x_\alpha, x_3) \in A' \times I} W \left(x_\alpha, \frac{x_\alpha}{\varepsilon}, D_\alpha u_\varepsilon(x_\alpha, x_3) \right) \\ &= \inf_{(w_\varepsilon) \subset W^{1,\infty}(\Omega)} \left\{ \limsup_{\varepsilon \to 0} \operatorname*{ess\,sup}_{(x_\alpha, x_3) \in A' \times I} W \left(x_\alpha, \frac{x_\alpha}{\varepsilon}, D_\alpha w_\varepsilon(x_\alpha, x_3) \right) : w_\varepsilon \to u \text{ uniformly in } \overline{\Omega} \right\}. \end{aligned}$$

Hence for a.e. $s \in I$, $v_\varepsilon := u_\varepsilon(\cdot, s) \in W^{1,\infty}(\omega)$, $v_\varepsilon \to u$ uniformly in $\overline{\omega}$ and

$$\begin{aligned} &\limsup_{\varepsilon \to 0} \operatorname*{ess\,sup}_{x_\alpha \in A'} W \left(x_\alpha, \frac{x_\alpha}{\varepsilon}, D_\alpha v_\varepsilon(x_\alpha) \right) \\ &\leq \inf_{(w_\varepsilon) \subset W^{1,\infty}(\Omega)} \left\{ \limsup_{\varepsilon \to 0} \operatorname*{ess\,sup}_{(x_\alpha, x_3) \in A' \times I} W \left(x_\alpha, \frac{x_\alpha}{\varepsilon}, D_\alpha w_\varepsilon(x_\alpha, x_3) \right) : w_\varepsilon \to u \text{ uniformly in } \overline{\Omega} \right\}. \end{aligned}$$

so that

$$\begin{aligned} G''(u, A') &:= \inf_{(v_\varepsilon) \subset W^{1,\infty}(\omega)} \left\{ \limsup_{\varepsilon \to 0} \operatorname*{ess\,sup}_{x_\alpha \in A'} W \left(x_\alpha, \frac{x_\alpha}{\varepsilon}, D_\alpha v_\varepsilon(x_\alpha) \right) : v_\varepsilon \to u \text{ uniformly in } \overline{\omega} \right\}. \\ &\leq \inf_{(w_\varepsilon) \subset W^{1,\infty}(\Omega)} \left\{ \limsup_{\varepsilon \to 0} \operatorname*{ess\,sup}_{(x_\alpha, x_3) \in A' \times I} W \left(x_\alpha, \frac{x_\alpha}{\varepsilon}, D_\alpha w_\varepsilon(x_\alpha, x_3) \right) : w_\varepsilon \to u \text{ uniformly in } \overline{\Omega} \right\}. \end{aligned}$$

Finally, taking the supremum with respect to all $A' \subset\subset A$ yields

$$G''_-(u, A) \leq \operatorname*{ess\,sup}_{x_\alpha \in A} W_{\text{hom}}(x_\alpha, D_\alpha u(x_\alpha)).$$

8. Acknowledgements

This research was supported through GNAMPA with project 'Problemi variazionali per funzionali supremali ed applicazioni', 2009. The authors gratefully acknowledge the hospitality of Universitá del Salento, Universitá degli Studi di Salerno and Universitá degli Studi di Pisa where a big part of these results have been achieved. Moreover they wish to thank Prof. P. Cardaliaguet for several helpful ideas.

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