Design of asymptotic preserving schemes for the hyperbolic heat equation on unstructured meshes

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Abstract

The transport equation, in highly scattering regimes, has a limit in which the dominant behavior is given by the solution of a diffusion equation. Angular discretization like the discrete ordinate method (S_N), the truncated spherical harmonic expansion (P_N) or also nonlinear moment models have the same property. For such systems it would be interesting to construct finite volume schemes on unstructured meshes which have the same dominant behavior even if the meshes are coarse. Such schemes are generally called diffusion asymptotic preserving (AP) schemes and are designed presently at most on Cartesian meshes.

In this work we give some answers for unstructured meshes, when considering the lowest order possible angular discretization of the transport equation that is the P_1 model also refereed to as the hyperbolic heat equation, the Cattaneo's equation or the first order formulation of the telegraph equation. We start from the modified upwind AP scheme proposed by Jin and Levermore [JL96] for this equation in 1-D. We show that extended in 2-D on unstructured meshes, the classical edge formulation of this scheme (and also for other AP schemes) is no longer asymptotic preserving. To solve this problem, we propose new schemes built on a node formulation of the Jin and Levermore's scheme which use the analogy between P_1 model and acoustic equations for which schemes with corner's fluxes have been built in the context of gas dynamics [Maz07, MABO07].

Contents

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1 Introduction

The hyperbolic heat equation in dimension one $x \in \mathbb{R}$ is

$$\left\{ \begin{array}{l} \partial_t p + \dfrac{\partial_x u}{\varepsilon} = 0, \\[2ex] \partial_t u + \dfrac{\partial_x p}{\varepsilon} + \dfrac{\sigma}{\varepsilon^2} u = 0. \end{array} \right. \tag{1}$$

Throughout this work we will assume that $0 < \varepsilon \leq 1$ and $\sigma > 0$. We will also assume that the solution (p, u) of (1) is such that for all $(m_1, m_2) \in \mathbb{N}^2$, there exists $C_{m_1,m_2} > 0$ such that

$$\|\partial_x^{m_1} \partial_t^{m_2} p\|_\infty \leq C_{m_1,m_2} \text{ and } \|\partial_x^{m_1} \partial_t^{m_2} u\|_\infty \leq \varepsilon C_{m_1,m_2}.$$

These estimates can be proved by standard means for the solution of the Cauchy problem for (1). The estimate for u is equivalent to say that $v = \frac{u}{\varepsilon}$ satisfies $\|\partial_x^{m_1} \partial_t^{m_2} v\|_\infty \leq C_{m_1,m_2}$. It is well known that if ε goes to 0, the system (1) can by approximated by the diffusion equation

$$\partial_t p - \partial_x \left(\frac{1}{\sigma} \partial_x p \right) = 0. \tag{2}$$

For simplicity σ is a constant in space. In this work we study asymptotic preserving numerical methods, following the seminal work [JL96], that is numerical methods such that the compatibility between (1) and (2) is true also at the discrete level. After a review of some asymptotic methods in 1D, we will consider the telegraph equation in dimension two

$$\left\{ \begin{array}{l} \partial_t p + \dfrac{1}{\varepsilon} \nabla \cdot \mathbf{u} = 0, \\[2ex] \partial_t \mathbf{u} + \dfrac{1}{\varepsilon} \nabla p = -\dfrac{\sigma}{\varepsilon^2} \mathbf{u}. \end{array} \right. \tag{3}$$

The original contributions of our work concern the design and analysis of new asymptotic preserving schemes on unstructured meshes. Our motivation is to extend the domain of application of some asymptotic preserving schemes that have been published in the literature [JL96, GT01, BD06, LM07, BCLM02, DDSV09] after the seminal work of [GL96] on well-balanced schemes, which are restricted to Cartesian meshes.

2 A review of asymptotic preserving cell centered schemes in 1D

For the three different schemes considered in this section, it is a simple task to check if the scheme is asymptotic preserving or not. In order to pave the way for future theoretical developments, we use a more rigorous approach where we analyze the dependence with respect to ε and the mesh size Δx of the consistency error and the stability CFL condition. If the consistency error tends to zero with respect to Δx, independently of ε the scheme is asymptotic preserving.

2.1 The classical finite volume scheme

In finite volume form, this method writes

$$\left\{\begin{array}{l} \dfrac{p_j^{n+1}-p_j^n}{\Delta t}+\dfrac{u_{j+\frac12}^n-u_{j-\frac12}^n}{\varepsilon\Delta x}=0,\\ \dfrac{u_j^{n+1}-u_j^n}{\Delta t}+\dfrac{p_{j+\frac12}^n-p_{j-\frac12}^n}{\varepsilon\Delta x}+\dfrac{\sigma}{\varepsilon^2}u_j^n=0. \end{array}\right. \tag{4}$$

The fluxes are given by the solution of the following linear system which is equivalent to writing the Riemann solver for the linear wave equation

$$\left\{\begin{array}{l} u_j^n+p_j^n=u_{j+\frac12}^n+p_{j+\frac12}^n,\\ u_{j+1}^n-p_{j+1}^n=u_{j+\frac12}^n-p_{j+\frac12}^n, \end{array}\right. \Longleftrightarrow \left\{\begin{array}{l} u_{j+\frac12}^n=\frac12\left(u_j^n+u_{j+1}^n+p_j^n-p_{j+1}^n\right),\\ p_{j+\frac12}^n=\frac12\left(p_j^n+p_{j+1}^n+u_j^n-u_{j+1}^n\right). \end{array}\right. \tag{5}$$

Plugging these fluxes in (4) we obtain an explicit formulation

$$\left\{\begin{array}{l} \dfrac{p_j^{n+1}-p_j^n}{\Delta t}+\dfrac{u_{j+1}^n-u_{j-1}^n}{2\varepsilon\Delta x}-\dfrac{p_{j+1}^n-2p_j^n+p_{j-1}^n}{2\varepsilon\Delta x}=0,\\ \dfrac{u_j^{n+1}-u_j^n}{\Delta t}+\dfrac{p_{j+1}^n-p_{j-1}^n}{2\varepsilon\Delta x}-\dfrac{u_{j+1}^n-2u_j^n+u_{j-1}^n}{2\varepsilon\Delta x}+\dfrac{\sigma}{\varepsilon^2}u_j^n=0. \end{array}\right. \tag{6}$$

Lemma 1. *The classical finite volume scheme (4-6) satisfies the maximum principle for the Riemann invariants $w=p+u$ and $z=p-u$ under CFL*

$$\frac{\Delta t}{\varepsilon\Delta x}+\frac{\sigma\Delta t}{2\varepsilon^2}\le 1.$$

The consistency error of both equations in (4) is $O\left(\frac{\Delta x}{\varepsilon}+\Delta t\right)$.

Proof. Let us define the Riemann invariants $w=p+u$ and $z=p-u$, so that (4-5) rewrites (just compute the sum and the difference of the equations (4))

$$\left\{\begin{array}{l} \dfrac{w_j^{n+1}-w_j^n}{\Delta t}+\dfrac{w_j^n-w_{j-1}^n}{\varepsilon\Delta x}+\dfrac{\sigma}{2\varepsilon^2}\left(w_j^n-z_j^n\right)=0,\\ \dfrac{z_j^{n+1}-z_j^n}{\Delta t}-\dfrac{z_{j+1}^n-z_j^n}{\varepsilon\Delta x}+\dfrac{\sigma}{2\varepsilon^2}\left(-w_j^n+z_j^n\right)=0, \end{array}\right.$$

that is

$$\left\{\begin{array}{l} w_j^{n+1}=\left(1-\dfrac{\Delta t}{\varepsilon\Delta x}-\dfrac{\sigma\Delta t}{2\varepsilon^2}\right)w_j^n+\dfrac{\Delta t}{\varepsilon\Delta x}w_{j-1}^n+\dfrac{\sigma\Delta t}{2\varepsilon^2}z_j^n,\\ z_j^{n+1}=\left(1-\dfrac{\Delta t}{\varepsilon\Delta x}-\dfrac{\sigma\Delta t}{2\varepsilon^2}\right)z_j^n+\dfrac{\Delta t}{\varepsilon\Delta x}z_{j+1}^n+\dfrac{\sigma\Delta t}{2\varepsilon^2}w_j^n. \end{array}\right. \tag{7}$$

Under this form the stability in the maximum norm is a consequence of the non negativity of the diagonal coefficient, that is $1-\frac{\Delta t}{\varepsilon\Delta x}-\frac{\sigma\Delta t}{2\varepsilon^2}\ge 0$.

The consistency error is studied with basic Taylor expansions. We use the form (6) which is well adapted for this purpose. We note $x_j = j\Delta x$ and $t_n = n\Delta t$. The consistency error cp for the first equation is

$$cp_j^n = \frac{p(x_j,t_{n+1}) - p(x_j,t_n)}{\Delta t} + \frac{u(x_{j+1},t_n) - u(x_{j-1},t_n)}{2\varepsilon\Delta x} - \frac{p(x_{j+1},t_n) - 2p(x_j,t_n) + p(x_{j-1},t_n)}{2\varepsilon\Delta x}$$

$$= (\partial_t p(x_j,t_n) + O(\Delta t)) + \left(\frac{\partial_x u(x_j,t_n)}{\varepsilon} + O(\Delta x^2)\right) + O\left(\frac{\Delta x}{\varepsilon}\right) = O\left(\frac{\Delta x}{\varepsilon} + \Delta t\right).$$

The consistency error cu for the second equation is

$$cu_j^n = \frac{u(x_j,t_{n+1}) - u(x_j,t_n)}{\Delta t} + \frac{p(x_{j+1},t_n) - p(x_{j-1},t_n)}{2\varepsilon\Delta x}$$

$$-\frac{u(x_{j+1},t_n) - 2u(x_j,t_n) + u(x_j,t_n)}{2\varepsilon\Delta x} + \frac{\sigma}{\varepsilon^2}u(x_j,t_n)$$

$$= (\partial_t u(x_j,t_n) + O(\Delta t)) + \left(\frac{\partial_x p(x_j,t_n)}{\varepsilon} + O(\frac{\Delta x^2}{\varepsilon})\right) + O\left(\frac{\Delta x}{\varepsilon}\right) + \frac{\sigma}{\varepsilon^2}u(x_j,t_n) = O\left(\frac{\Delta x}{\varepsilon} + \Delta t\right)$$

□

2.2 Jin-Levermore scheme

The Jin-Levermore scheme [JL96] is a modification of the fluxes used in the standard scheme (4) such that the first equation for the unknown p is asymptotic preserving. We will prove that it is equivalent to say that the first equation in (4) is consistent with a consistency error $O(\Delta x)$ independent of ε. This is of course much better than the error $O\left(\frac{\Delta x}{\varepsilon} + \Delta t\right)$ for the classical finite volume scheme when ε is small with respect to Δx. The idea is to modify the fluxes and to consider

$$\left\{\begin{array}{l} u_j^n + p_j^n = u_{j+\frac12}^n + p_{j+\frac12}^n + \dfrac{\sigma\Delta x}{2\varepsilon}u_{j+\frac12}^n, \\ u_{j+1}^n - p_{j+1}^n = u_{j+\frac12}^n - p_{j+\frac12}^n + \dfrac{\sigma\Delta x}{2\varepsilon}u_{j+\frac12}^n, \end{array}\right. \iff \left\{\begin{array}{l} u_{j+\frac12}^n = \frac{1}{2(1+a)}\left(u_j^n + u_{j+1}^n + p_j^n - p_{j+1}^n\right), \\ p_{j+\frac12}^n = \frac12\left(p_j^n + p_{j+1}^n + u_j^n - u_{j+1}^n\right), \end{array}\right. \tag{8}$$

where the coefficient a is

$$a = \frac{\sigma\Delta x}{2\varepsilon}.$$

Lemma 2. *The Jin-Levermore scheme (4)-(8) does not satisfy the maximum principle for the Riemann invariants $w = p + u$ and $z = p - u$. The consistency error of the second equation is $O\left(\frac{\Delta x}{\varepsilon} + \Delta t\right)$, and the consistency error of the first equation is $O\left(\Delta x^2 + \varepsilon\Delta x + \Delta t\right)$.*

Proof. Let us first analyze the stability in the maximum norm. Let us sum the two equations. We get

$$\frac{w_j^{n+1} - w_j^n}{\Delta t} + \frac{\left(w_j^n - au_{j+\frac12}^n\right) - \left(w_{j-1}^n - au_{j-\frac12}^n\right)}{\varepsilon\Delta x} + \frac{\sigma}{2\varepsilon^2}\left(w_j^n - z_j^n\right) = 0.$$

Since $u_{j+\frac12}^n = \dfrac{w_j^n - z_{j+1}^n}{2(1+a)}$, we obtain

$$\frac{w_j^{n+1} - w_j^n}{\Delta t} + \left(1 - \frac{a}{2(1+a)}\right)\frac{w_j^n - w_{j-1}^n}{\varepsilon\Delta x} + \frac{a}{2(1+a)\varepsilon\Delta x}\left(z_{j+1}^n - z_j^n\right) + \frac{\sigma}{2\varepsilon^2}\left(w_j^n - z_j^n\right) = 0$$

that is

$$w_j^{n+1} = \left(1 - \frac{(2+a)\Delta t}{2(1+a)\varepsilon\Delta x} - \frac{\sigma\Delta t}{\varepsilon^2}\right)w_j^n + \frac{\sigma\Delta t}{4(1+a)\varepsilon^2}w_{j-1}^n + \left(\frac{a\Delta t}{2(1+a)\varepsilon\Delta x} + \frac{\sigma\Delta t}{2\varepsilon^2}\right)z_j^n - \frac{a\Delta t}{\varepsilon\Delta x}z_{j+1}^n.$$

under a convenient CFL condition the diagonal coefficient is positive. However it is not possible to guarantee that all non diagonal off coefficients are positive. In particular $\frac{\sigma\Delta t}{2\varepsilon^2} > 0$ and $-\frac{a\Delta t}{\varepsilon\Delta x} < 0$. This is why the maximum principle does not hold.

Let us turn to the analysis of the consistency. Plugging the explicit form of the fluxes (8) in (4) we see that the second equation is not modified therefore its consistency error is still $O\left(\frac{\Delta x}{\varepsilon} + \Delta t\right)$. The first equation writes in a explicit form

$$\frac{p_j^{n+1} - p_j^n}{\Delta t} + \frac{u_{j+1}^n - u_{j-1}^n}{2(1+a)\varepsilon\Delta x} - \frac{p_{j+1}^n - 2p_j^n + p_{j-1}^n}{2(1+a)\varepsilon\Delta x} = 0. \tag{9}$$

So its consistency error is

$$\begin{aligned} cp_j^n &= \frac{p(x_j,t_{n+1}) - p(x_j,t_n)}{\Delta t} + \frac{u(x_{j+1},t_n) - u(x_{j-1},t_n)}{2(1+a)\varepsilon\Delta x} - \frac{p(x_{j+1},t_n) - 2p(x_j,t_n) + p(x_{j-1},t_n)}{2(1+a)\varepsilon\Delta x} \\ &= \partial_t p(x_j,t_n) + O(\Delta t) + \frac{2\Delta x \partial_x u(x_j,t_n) + O(\varepsilon\Delta x^3)}{2(1+a)\varepsilon\Delta x} - \frac{\Delta x^2 \partial_{xx} p(x_j,t_n) + O(\Delta x^4)}{2(1+a)\varepsilon\Delta x} \\ &= \left(\partial_t p(x_j,t_n) + \frac{1}{(1+a)\varepsilon}\partial_x u(x_j,t_n) - \frac{\Delta x}{2(1+a)\varepsilon}\partial_{xx} p(x_j,t_n)\right) + O\left(\Delta x^2 + \frac{\Delta x^3}{(1+a)\varepsilon} + \Delta t\right). \end{aligned}$$

Notice that $(1+a)\varepsilon = \varepsilon + \frac{\sigma\Delta x}{2}$ so that $\frac{\Delta x^3}{(1+a)\varepsilon} = O\left(\Delta x^2\right)$. Since (p,u) is the exact solution then

$$\partial_{xx} p = -\frac{\sigma}{\varepsilon}\partial_x u - \varepsilon\partial_{xt} u = -\frac{\sigma}{\varepsilon}\partial_x u + O(\varepsilon^2).$$

So the term in the first parenthesis is also

$$\begin{aligned} (\cdots) &= \partial_t p(x_j,t_n) + \frac{1}{(1+a)\varepsilon}\partial_x u(x_j,t_n) + \frac{\Delta x}{2(1+a)\varepsilon}\frac{\sigma}{\varepsilon}\partial_x u(x_j,t_n) + O\left(\frac{\varepsilon^2\Delta x}{2(1+a)\varepsilon}\right) \\ &= \frac{1}{\varepsilon}\left(-1 + \frac{1}{(1+a)} + \frac{\Delta x}{2(1+a)}\frac{\sigma}{\varepsilon}\right)\partial_x u(x_j,t_n) + O\left(\frac{\varepsilon^2\Delta x}{2(1+a)\varepsilon}\right). \end{aligned}$$

By definition of a the term between parenthesis is $-1 + \frac{1+a}{1+a} = 0$, and the second term is at least $O(\varepsilon\Delta x)$. Therefore $cp = O(\Delta x^2 + \varepsilon\Delta x + \Delta t)$. □

2.3 Modified Jin-Levermore scheme and Gosse-Toscani scheme

Let us introduce what we call the modified Jin-Levermore scheme with the same fluxes (8), but the sink is modified

$$\begin{cases} \dfrac{p_j^{n+1} - p_j^n}{\Delta t} + \dfrac{u_{j+\frac12}^n - u_{j-\frac12}^n}{\varepsilon\Delta x} = 0, \\ \dfrac{u_j^{n+1} - u_j^n}{\Delta t} + \dfrac{p_{j+\frac12}^n - p_{j-\frac12}^n}{\varepsilon\Delta x} + \dfrac{\sigma}{2\varepsilon^2}\left(u_{j+\frac12}^n + u_{j-\frac12}^n\right) = 0. \end{cases} \tag{10}$$

Lemma 3. *The modified Jin-Levermore scheme is equal to the Gosse-Toscani scheme [GT01].*

Proof. It is a matter of elementary manipulations. We plug the explicit fluxes (8) in (10). We already know (9) that the first equation can be rewritten as the first equation of (3). For the second equation we check that

$$\begin{aligned} \frac{p_{j+\frac12}^n - p_{j-\frac12}^n}{\varepsilon\Delta x} + \frac{\sigma}{2\varepsilon^2}\left(u_{j+\frac12}^n + u_{j-\frac12}^n\right) &= \frac{p_{j+1}^n - p_{j-1}^n}{2\varepsilon\Delta x} - \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{2\varepsilon\Delta x} \\ &+ \frac{\sigma}{\varepsilon^2 4(1+a)}\left(\left(u_j^n + u_{j+1}^n + p_j^n - p_{j+1}^n\right) + \left(u_{j-1}^n + u_j^n + p_{j-1}^n - p_j^n\right)\right) \end{aligned}$$

$$= \left(\frac{1}{2\varepsilon\Delta x} - \frac{\sigma}{\varepsilon^2 4(1+a)}\right)\left(\left(p^n_{j+\frac12} - p^n_{j-\frac12}\right) - \left(u^n_{j+1} - 2u^n_j + u^n_{j-1}\right)\right) + \frac{\sigma}{\varepsilon^2(1+a)} u^n_j.$$

Let us define

$$M = \frac{1}{1+a} = \frac{2\varepsilon}{2\varepsilon + \sigma\Delta x}.$$

One can check that

$$\frac{1}{2\varepsilon\Delta x} - \frac{\sigma}{\varepsilon^2 4(1+a)} = \frac{1}{1\varepsilon\Delta x}\left(1 - \frac{a}{1+a}\right) = \frac{1}{2\varepsilon\Delta x} \times \frac{1}{1+a} = \frac{M}{2\varepsilon\Delta x}.$$

So we obtain the important relation

$$\frac{p^n_{j+\frac12} - p^n_{j-\frac12}}{\varepsilon\Delta x} + \frac{\sigma}{2\varepsilon^2}\left(u^n_{j+\frac12} + u^n_{j-\frac12}\right) = M\left(\frac{p^n_{j+1} - p^n_{j-1}}{2\varepsilon\Delta x} - \frac{u^n_{j+1} - 2u^n_j + u^n_{j-1}}{2\varepsilon\Delta x} + \frac{\sigma}{\varepsilon^2} u^n_j.\right)$$

Therefore the modified Jin-Levermore scheme admit the explicit formulation

$$\begin{cases} \dfrac{p^{n+1}_j - p^n_j}{\Delta t} + M\dfrac{u^n_{j+1} - u^n_{j-1}}{2\varepsilon\Delta x} - M\dfrac{p^n_{j+1} - 2p^n_j + p^n_{j-1}}{2\varepsilon\Delta x} = 0, \\ \dfrac{u^{n+1}_j - u^n_j}{\Delta t} + M\dfrac{p^n_{j+1} - p^n_{j-1}}{2\varepsilon\Delta x} - M\dfrac{u^n_{j+1} - 2u^n_j + u^n_{j-1}}{2\varepsilon\Delta x} + M\dfrac{\sigma}{\varepsilon^2} u^n_j = 0. \end{cases} \quad (11)$$

□

We recognize the Gosse-Toscani scheme [GT01]. Notice also that this scheme is equal to an elementary modification of the classical explicit scheme where one has multiplied all space derivatives and sink by M.

Lemma 4. *The modified Jin-Levermore scheme (equal to the Gosse-Toscani scheme) satisfies the maximum principle for the Riemann invariants under the CFL*

$$\frac{\Delta t}{\varepsilon\Delta x} \le 1.$$

The consistency error of both equations is bounded by $O(\Delta x + \Delta t)$.

Proof. The stability of classical rewrites $\frac{\Delta t}{\varepsilon\Delta x}(1+a) \le 1$. By construction the modified Jin-Levermore scheme is a modification of all fluxes and sink by $M = \frac{1}{1+a}$. So the CFL condition is indeed $\frac{\Delta t}{\varepsilon\Delta x} \le 1$. The consistency of the first equation has already been established for the Jin-Levermore scheme (4)-(8). So it remains to study the consistency of the second equation. One has

$$cu^n_j = \frac{u(x_j, t_{n+1}) - u(x_j, t_n)}{\Delta t} + M\left(\frac{p(x_{j+1}, t_n) - p(x_{j-1}, t_n)}{2\varepsilon\Delta x}\right.$$
$$\left. - \frac{u(x_{j+1}, t_n) - 2u(x_j, t_n) + u(x_j, t_n)}{2\varepsilon\Delta x} + \frac{\sigma}{\varepsilon^2} u(x_j, t_n)\right)$$
$$= \partial_t u(x_j, t_n) + O(\Delta t) + M\left(\frac{2\Delta x \partial_x p(x_j, t_n) + O(\Delta x^3)}{2\varepsilon\Delta x} + \frac{\Delta x}{\varepsilon}\partial_{xx} u(x_j, t_n) + \frac{\sigma}{\varepsilon^2} u(x_j, t_n)\right)$$

We known that $\partial_{xx} u = -\varepsilon\partial_{tx} p$ and $\|\partial_x^{m_1}\partial_t^{m_2} p\|_\infty \le C_{m_1,m_2}$ consequently $\partial_{xx} u = O(\varepsilon)$

$$cu^n_j = (1-M)\partial_t u(x_j, t_n) + O\left(\Delta t + \frac{M\Delta x^2}{\varepsilon} + \Delta x\right).$$

By definition the coefficient M satisfies the following properties: first $M \le 1$, second

$$\frac{M\Delta x}{\varepsilon} = \frac{2\varepsilon\Delta x}{(2\varepsilon + \sigma\Delta x)\varepsilon} = \frac{2\Delta x}{(2\varepsilon + \sigma\Delta x)} \le \frac{2}{\sigma} \Longrightarrow O\left(\frac{M\Delta x^2}{\varepsilon}\right) = O(\Delta x),$$

and third

$$1 - M = \frac{\sigma \Delta x}{2\varepsilon + \sigma \Delta x} \Longrightarrow (1-M) O\left(\frac{\varepsilon \Delta x}{\varepsilon + \Delta x}\right) = O(\Delta x).$$

because $\partial_t u(x_j, t_n) = O(\varepsilon)$. It ends the proof. □

Theorem 5. *The modified Jin-Levermore scheme (equal to the Gosse-Toscani scheme) is convergent in the maximum norm under CFL with an error $O(\Delta x + \Delta t)$. Therefore this scheme is AP.*

Since the modified Jin-Levermore scheme satisfies the maximum principle for $w = p + u$ and $z = p - u$, then it is stable in the maximum norm. For the unknowns (p, u) the stability constant is bounded by 2. So the result of the theorem is an immediate application of the Lax theorem.

3 Design principle in dimension two

In this section we extend in dimension two on unstructured meshes the Jin-Levermore scheme and modified Jin-Levermore scheme. We will first show that a naive use of the Jin-Levermore procedure does not generate an asymptotic preserving scheme on unstructured meshes. Consequently we propose an alternative approach where the Jin-Levermore procedure is incorporated in a cell-centered finite volume scheme with a nodal evaluation of the fluxes. The theoretical analysis of the next section will show that the resulting scheme is indeed an asymptotic preserving scheme on unstructured meshes.

3.1 Definitions

Let us consider a unstructured mesh in dimension 2. The mesh is defined by a finite number of vertices $\mathbf{x}_r$ and cells Ω_j. We denote $\mathbf{x}_j$ a point arbitrarily chosen inside Ω_j. For simplicity we will call this point the center of the cell.

The common edge between the cell j and k is $\partial\Omega_{jk}$ and $\mathbf{x}_{jk}$ is the middle of $\partial\Omega_{jk}$. The area of Ω_j is $|\Omega_j|$. In the case of the edge formulation we define the normal $\mathbf{n}_{jk}$ and the length l_{jk} associate to the edge $\partial\Omega_{jk}$.

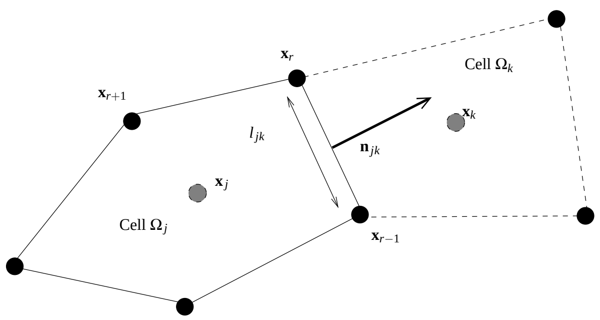

Figure 1: Notation for edge formulation. The interface between cell Ω_j and cell Ω_k has a normal $\mathbf{n}_{jk} = -\mathbf{n}_{kj}$ and a length $l_{jk} = l_{kj}$. The vertices are denoted $\mathbf{x}_{\ldots,r-1,r,r+1,\ldots}$. The center of the cell is an arbitrary point inside the cell.

For the node formulation the definitions are less natural. By convention the vertices are listed counterclockwise $\mathbf{x}_{r-1}, \mathbf{x}_r, \mathbf{x}_{r+1}$ with coordinates $\mathbf{x}_r = (x_r, y_r)$.We also define

$$l_{jr} = \frac{1}{2} \parallel \mathbf{x}_{r+1} - \mathbf{x}_{r-1} \parallel \text{ and } \mathbf{n}_{jr} = \frac{1}{2l_{jr}} \begin{pmatrix} -y_{r-1} + y_{r+1} \\ x_{r-1} - x_{r+1} \end{pmatrix}. \quad (12)$$

The convention is that the length of a vector $\mathbf{x} \in \mathbb{R}^2$ is denoted as $\parallel \mathbf{x} \parallel$. The scalar product of two vectors is $(\mathbf{x}, \mathbf{y})$.

Remark 6. *In all this work, the letters j and k denote cells. The letter r is always the index on a node. This is why l_{jk} and l_{jr} denote different quantities. This abuse of notations allow a easy comparison of edge and node formulations.*

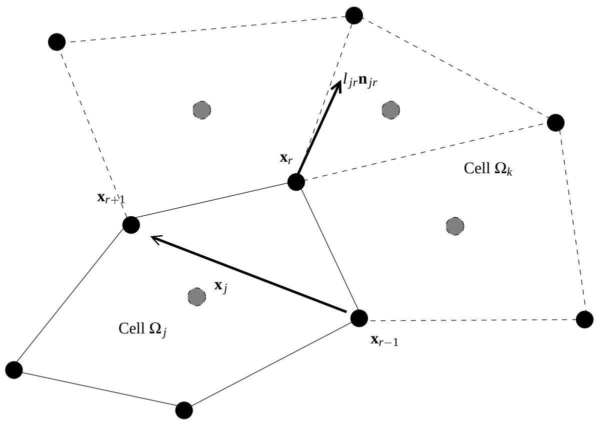

Figure 2: Notation for node formulation. The corner length l_{jr} and the corner normal $\mathbf{n}_{jr}$ are defined in equation (12). Notice that $l_{jr}\mathbf{n}_{jr}$ is equal to the the half of the vector that starts at $\mathbf{x}_{r-1}$ and finish at $\mathbf{x}_{r+1}$. The center of the cell is an arbitrary point inside the cell.

3.2 Edge formulation

We consider the telegraph model in dimension two (3). The purpose of this subsection is to incorporate the Jin-Levermore procedure in a standard finite volume discretization of (3). The starting point is the standard finite volume scheme

$$\left\{\begin{array}{l} |\Omega_j|\,\partial_t p_j(t) + \dfrac{1}{\varepsilon}\displaystyle\sum_k l_{jk}(\mathbf{u}_{jk}, \mathbf{n}_{jk}) = 0, \\ |\Omega_j|\,\partial_t \mathbf{u}_j(t) + \dfrac{1}{\varepsilon}\displaystyle\sum_k l_{jk} p_{jk}\mathbf{n}_{jk} = -\,|\Omega_j|\,\dfrac{\sigma}{\varepsilon^2}\mathbf{u}_j^n, \end{array}\right. \tag{13}$$

where the fluxes associated to the edge $\partial\Omega_{jk}$ common to the cells j and k are defined by

$$\left\{\begin{array}{l} p_{jk} = \dfrac{1}{2}(p_j + p_k) + \dfrac{1}{2}(\mathbf{u_j} - \mathbf{u}_k, \mathbf{n_{jk}}) \\ (\mathbf{u_{jk}}, \mathbf{n_{jk}}) = \dfrac{1}{2}(\mathbf{u_j} + \mathbf{u}_k, \mathbf{n_{jk}}) + \frac{1}{2}(p_j - p_k) \end{array}\right.$$

It will appear that it is more convenient to rewrite these formulas as

$$\left\{\begin{array}{l} p_{jk} - p_j + (\mathbf{u}_{jk} - \mathbf{u}_j, \mathbf{n}_{jk}) = 0, \\ p_{jk} - p_k - (\mathbf{u}_{jk} - \mathbf{u}_k, \mathbf{n}_{jk}) = 0. \end{array}\right. \tag{14}$$

The Jin-Levermore method in dimension one results in the incorporation of the source term in the fluxes, in order to have a more accurate approximation of stationary solution. It is easy to use the same method in dimension two. Consider a stationary solution such that

$$\nabla p = -\frac{\sigma}{\varepsilon}\mathbf{u}.$$

A Taylor expansion shows that

$$\left\{\begin{array}{l} p(\mathbf{x}_{jk}) - p(\mathbf{x}_j) = \frac{\sigma}{\varepsilon}(\mathbf{u}(\mathbf{x}_{jk}), \mathbf{x_j} - \mathbf{x_{jk}}) + O(h^2), \\ p(\mathbf{x}_{jk}) - p(\mathbf{x}_k) = \frac{\sigma}{\varepsilon}(\mathbf{u}(\mathbf{x}_{jk}), \mathbf{x_k} - \mathbf{x_{jk}}) + O(h^2) \end{array}\right. \tag{15}$$

with $\mathbf{x_j}$, (resp. $\mathbf{x_k}$) the center of the cell j and (resp. k), $\mathbf{x}_{jk}$ the middle and h the local characteristic length of the mesh.

Following the interpretation of Jin-Levermore we interpret (14) and (15) as linear relations between differences. The idea is to mix these relations and to get

$$\left\{ \begin{array}{l} p_{jk} - p_j + (\mathbf{u}_{jk} - \mathbf{u}_j, \mathbf{n}_{jk}) = \frac{\sigma}{\varepsilon}(\mathbf{u}_{jk}, \mathbf{x_j} - \mathbf{x_{jk}}), \\ p_{jk} - p_k - (\mathbf{u}_{jk} - \mathbf{u}_k, \mathbf{n}_{jk}) = \frac{\sigma}{\varepsilon}(\mathbf{u}_{jk}, \mathbf{x_k} - \mathbf{x_{jk}}). \end{array} \right. \tag{16}$$

This is a linear system of two equations and three unknowns p_{jk} and $\mathbf{u}_{jk}$ which is not solvable in general. In dimension one, there is no such problem. At this point it is natural but restrictive to assume that the mesh satisfies the Delaunay condition which is equivalent to $(\mathbf{x_{jk}} - \mathbf{x_j}) = d_{jk}\mathbf{n_{jk}}$ and $(\mathbf{x_{jk}} - \mathbf{x_j}) = -d_{kj}\mathbf{n_{jk}}$, with $d_{jk} = d(\mathbf{x_j}, \mathbf{x_{jk}}) > 0$ and $d_{kj} = d(\mathbf{x_k}, \mathbf{x_{jk}}) > 0$. The linear system becomes

$$\left\{ \begin{array}{l} p_{jk} - p_j + (\mathbf{u}_{jk} - \mathbf{u}_j, \mathbf{n}_{jk}) = -\frac{\sigma}{\varepsilon} d_{jk}(\mathbf{u}_{jk}, \mathbf{n_{jk}}), \\ p_{jk} - p_k - (\mathbf{u}_{jk} - \mathbf{u}_k, \mathbf{n}_{jk}) = \frac{\sigma}{\varepsilon} d_{kj}(\mathbf{u}_{jk}, \mathbf{n_{jk}}). \end{array} \right. \tag{17}$$

This is a linear system of two equations and two unknowns p_{jk} and $(\mathbf{u}_{jk}, \mathbf{n}_{jk})$. The solution is

$$\left\{ \begin{array}{l} (\mathbf{u}_{jk}, \mathbf{n}_{jk}) = \dfrac{(\mathbf{u_j} + \mathbf{u}_k).\mathbf{n}_{jk} + (p_j - p_k)}{2 + (\sigma/\varepsilon)(d_{jk} + d_{kj})} \\ p_{jk} = \dfrac{((\mathbf{u_j}, \mathbf{n}_{jk}) + p_j)(1 + d_{kj}(\sigma/\varepsilon)) - ((\mathbf{u}_k, \mathbf{n}_{jk}) - p_k)(1 + d_{jk}(\sigma/\varepsilon))}{2 + (\sigma/\varepsilon)(d_{jk} + d_{kj})} \end{array} \right. \tag{18}$$

The result of this construction is the scheme (13) with the fluxes (18).

Proposition 7. *If the mesh satisfy the Delaunay condition, the asymptotic limit of the scheme (13) with the flux (18) is the diffusion scheme VF4*

$$|\Omega_j|\,\partial_t p_j(t) - \sum_k l_{jk} \frac{p_k - p_j}{d_{jk} + d_{kj}} = 0, \quad \textit{with } d_{jk} + d_{kj} = d(\mathbf{x}_j, \mathbf{x}_k). \tag{19}$$

Proof. We first multiply the second equation by ε^2

$$\varepsilon^2 \,|\Omega_j|\, \partial_t \mathbf{u}_j(t) + \varepsilon \sum_k l_{jk} p_{jk} \mathbf{n}_{jk} = -\,|\Omega_j|\, \sigma \mathbf{u}_j^n \tag{20}$$

By factoring the denominator by $1/\varepsilon$ we obtain the following formulation

$$\varepsilon p_{jk} = \frac{((\mathbf{u}_j, \mathbf{n}_{jk}) + p_j)(\varepsilon^2 + d_{kj}(\sigma\varepsilon)) - ((\mathbf{u}_k, \mathbf{n}_{jk}) - p_k)(\varepsilon^2 + d_{jk}(\sigma\varepsilon))}{(2\varepsilon + \sigma(d_{jk} + d_{kj}))} \tag{21}$$

When ε tends to zero, the right inside of (21) tends to zero. Plugging in (20) it shows that $\mathbf{u}_j^n = 0$. We plug this result in the formula of $(\mathbf{u}_{jk}, \mathbf{n}_{jk})$ (18) and obtain at the limit the scheme (19).

Since the mesh satisfies the Delaunay condition then $(\mathbf{x_j} - \mathbf{x_k}) \perp \partial\Omega_{jk}$ and $d_{jk} + d_{kj} = d(\mathbf{x_j}, \mathbf{x_k})$. we obtain the VF4 scheme. □

Even if the mesh does not satisfy the Delaunay condition the scheme (13)-(18) is well defined. Further it is known that the VF4 scheme doesn't converge for meshes that do not fulfilled the Delaunay condition. The explanation is that the reconstruction of the gradient in the normal direction imposes a strong geometrical restriction. Consequently the scheme (13)-(18) is not asymptotic preserving on all unstructured meshes.

3.3 Nodal formulation

In this section we propose a new scheme for the telegraph equation which is based on a nodal formulation in order to solve the problem perceived in the previous subsection. We use the analogy between the telegraph equation and the linearized Euler equations. B. Després and C. Mazeran in [Maz07] constructed a nodal scheme named GLACE for Euler equation [Des09, CDDL09]. Consequently we want to use this scheme with the Jin-Levermore method. We begin by writing the GLACE scheme for the telegraph equation (3)

$$\begin{cases} |\Omega_j|\,\partial_t p_j(t) + \dfrac{1}{\varepsilon}\displaystyle\sum_r l_{jr}(\mathbf{u}_r.\mathbf{n}_{jr}) = 0 \\ |\Omega_j|\,\partial_t \mathbf{u}_j(t) + \dfrac{1}{\varepsilon}\displaystyle\sum_r l_{jr}p_{jr}\mathbf{n}_{jr} = -\,|\Omega_j|\,\dfrac{\sigma}{\varepsilon^2}\mathbf{u}_j \end{cases} \tag{22}$$

with the fluxes defined by

$$\begin{cases} p_{jr} = p_j + (\mathbf{u_j} - \mathbf{u_r}, \mathbf{n}_{jr}), \\ \displaystyle\sum_j l_{jr}p_{jr}\mathbf{n}_{jr} = 0. \end{cases} \tag{23}$$

The formulas (23) are a nodal generalization of the edge formulas (14). The solution of this linear system is easily computed by elimination of p_{jr}

$$\left(\sum_j (l_{jr}\mathbf{n}_{jr} \otimes \mathbf{n}_{jr})\right)\mathbf{u}_r = \sum_j l_{jr}(p_j\mathbf{n}_{jr} + \mathbf{n}_{jr}\otimes\mathbf{n}_{jr}\mathbf{u}_j). \tag{24}$$

The matrix of the left hand side is always invertible for non-degenerate unstructured meshes.

As before we incorporate the source term in the fluxes. Plugging the source term in the Riemann invariant as (17) we obtain

$$\begin{cases} p_{jr} = p_j + (\mathbf{u}_j - \mathbf{u}_r, \mathbf{n}_{jr}) - \dfrac{\sigma}{\varepsilon}(\mathbf{u}_r, \mathbf{x}_r - \mathbf{x}_j), \\ \left(\displaystyle\sum_j l_{jr}(\mathbf{n}_{jr}\otimes\mathbf{n}_{jr} + \dfrac{\sigma}{\varepsilon}\mathbf{n}_{jr}\otimes(\mathbf{x}_r - \mathbf{x}_j))\right)\mathbf{u}_r = \displaystyle\sum_j l_{jr}(p_j\mathbf{n}_{jr} + \mathbf{n}_{jr}\otimes\mathbf{n}_{jr}\mathbf{u}_j). \end{cases} \tag{25}$$

Definition 8. *The scheme (22)-(25) will be referred to as the JL-(a) scheme.*

This scheme is a valid one provided the linear system in the second equation of (25) is invertible. We will show that this scheme is asymptotic preserving on unstructured meshes.

Another problem that needs to be studied is the well posedness of the asymptotic limit of the JL-(a) scheme.

Proposition 9. *If the matrix of (25) is invertible ,the asymptotic limit of the JL-(a) scheme is*

$$\begin{cases} |\Omega_j|\,\partial_t p_j(t) + \displaystyle\sum_r l_{jr}(\mathbf{u}_r, \mathbf{n}_{jr}) = 0, \\ \sigma\left(\displaystyle\sum_j l_{jr}\mathbf{n}_{jr}\otimes(\mathbf{x}_r - \mathbf{x}_j)\right)\mathbf{u_r} = \displaystyle\sum_j l_{jr}p_j\mathbf{n}_{jr}. \end{cases} \tag{26}$$

Proof. : To demonstrate this result we use a Hilbert expansion as in [BCLM02]. We write the following expansion

$$p_j = p_{j,0} + \varepsilon p_{j,1} + \varepsilon^2 p_{j,2} + ...$$
$$p_{jr} = p_{jr,0} + \varepsilon p_{jr,1} + \varepsilon^2 p_{jr,2} + ...$$

We do the same for all other variables. Plugging this expansion in the first equation of (22) one gets

$$\partial_t p_{j,0}(t) + \frac{1}{\varepsilon}\sum_r l_{jr}(\mathbf{u}_{r,1}, \mathbf{n}_{jr}) + \sum_r l_{jr}(\mathbf{u}_{r,0}, \mathbf{n}_{jr}) = O(\varepsilon).$$

The term proportional to $\frac{1}{\varepsilon}$ is

$$\sum_r l_{jr}(\mathbf{u}_{r,0}, \mathbf{n}_{jr}) = 0. \tag{27}$$

The term proportional to $O(1)$ is

$$\partial_t p_{j,0}(t) + \sum_r l_{jr}(\mathbf{u}_{r,1}, \mathbf{n}_{jr}) = 0. \tag{28}$$

Using the same method for the second equation we obtain three terms

$$\frac{1}{\varepsilon^2}: \qquad \mathbf{u}_{j,0} = 0 \tag{29}$$

$$\frac{1}{\varepsilon}: \qquad \sum_r l_{jr} p_{jr,0} \mathbf{n}_{jr} = -\sigma \mathbf{u}_{j,1} \tag{30}$$

$$\frac{1}{\varepsilon^0}: \qquad \partial_t \mathbf{u}_{j,0}(t) + \sum_r l_{jr} p_{jr,1} \mathbf{n}_{jr} = -\sigma \mathbf{u}_{j,2} \tag{31}$$

Doing the same in (25) we obtain

$$\frac{1}{\varepsilon}: \qquad \sigma\left(\sum_j l_{jr}\mathbf{n}_{jr} \otimes (\mathbf{x}_r - \mathbf{x}_j))\right)\mathbf{u}_{r,0} = 0 \tag{32}$$

and

$$\frac{1}{\varepsilon^0}: \qquad \left(\sum_j l_{jr}(\mathbf{n}_{jr} \otimes \mathbf{n}_{jr})\right)\mathbf{u}_{r,0} + \sigma\left(\sum_j l_{jr}\mathbf{n}_{jr} \otimes (\mathbf{x}_r - \mathbf{x}_j)\right)\mathbf{u}_{r,1} \tag{33}$$

$$= \sum_j l_{jr} p_{j,0} \mathbf{n}_{jr} + l_{jr}(\mathbf{n}_{jr} \otimes \mathbf{n}_{jr})\mathbf{u}_{j,0}.$$

Assuming that the matrix in (32) is invertible, (32) implies that $\mathbf{u}_{r,0} = 0$. Since we have (29), (33) yields

$$\sigma\left(\sum_j l_{jr}\mathbf{n}_{jr} \otimes (\mathbf{x}_r - \mathbf{x}_j)\right)\mathbf{u}_{r,1} = \sum_j p_{j,0}\mathbf{n}_{jr}. \tag{34}$$

This equality and (28) give the result (26). □

Now we propose a variant of (22)-(25) based on another discretization of the source term. This scheme is the dimension two extension of the modified Jin-Levermore scheme (10)-(8)

$$\begin{cases} |\Omega_j|\,\partial_t p_j(t) + \dfrac{1}{\varepsilon}\displaystyle\sum_r l_{jr}(\mathbf{u}_r, \mathbf{n_{jr}}) = 0, \\ |\Omega_j|\,\partial_t \mathbf{u}_j(t) + \dfrac{1}{\varepsilon}\displaystyle\sum_r l_{jr} p_{jr} \mathbf{n_{jr}} = -\dfrac{\sigma}{\varepsilon^2}\displaystyle\sum_r l_{jr}\mathbf{n}_{jr} \otimes (\mathbf{x_r} - \mathbf{x}_j)\mathbf{u}_r. \end{cases} \tag{35}$$

This variant can also be seen as the two dimensional extension of the Gosse-Toscani scheme.

Definition 10. *The scheme (35)-(25) will be referred to as the JL-(b) scheme.*

Proposition 11. *If the matrix of (25) is invertible, the schemes JL-(b) and JL-(a) have the same asymptotic limit (26).*

Proof. The scheme (35) can be written as

$$\begin{cases} |\Omega_j|\,\partial_t p_j(t) + \dfrac{1}{\varepsilon}\displaystyle\sum_r l_{jr}(\mathbf{u}_r, \mathbf{n_{jr}}) = 0, \\ |\Omega_j|\,\partial_t \mathbf{u}_j(t) + \dfrac{1}{\varepsilon}\displaystyle\sum_r l_{jr} p^*_{jr}\mathbf{n}_{jr} = 0, \end{cases} \tag{36}$$

with the flux

$$\begin{cases} p^*_{jr} = p_j + (\mathbf{u_j} - \mathbf{u_r}, \mathbf{n_{jr}}), \\ \left(\displaystyle\sum_j l_{jr}(\mathbf{n_{jr}} \otimes \mathbf{n_{jr}} + \dfrac{\sigma}{\varepsilon}\mathbf{n_{jr}} \otimes (\mathbf{x_r} - \mathbf{x_j}))\right)\mathbf{u_r} = \displaystyle\sum_j l_{jr}(p_j \mathbf{n}_{jr} + \mathbf{n_{jr}} \otimes \mathbf{n_{jr}}\mathbf{u_j}). \end{cases} \tag{37}$$

Using the method of the Hilbert expansion one gets

$$\frac{1}{\varepsilon}: \qquad \sum_r l_{jr}(\mathbf{u}_{r,0}, \mathbf{n}_{jr}) = 0, \tag{38}$$

$$\frac{1}{\varepsilon^0}: \qquad \partial_t p_{j,0}(t) + \sum_r l_{jr}(\mathbf{u}_{r,0}, \mathbf{n}_{jr}) = 0. \tag{39}$$

For the second equation we obtain

$$\frac{1}{\varepsilon}: \qquad \sum_r l_{jr} p^*_{jr,0}\mathbf{n}_{jr} = 0 \tag{40}$$

$$\frac{1}{\varepsilon^0}: \qquad \partial_t \mathbf{u}_{j,0}(t) + \sum_r l_{jr} p^*_{jr,1}\mathbf{n}_{jr} = 0. \tag{41}$$

Now we use the definition of the flux p^*_{jr}

$$\sum_r l_{jr} p^*_{jr,0}\mathbf{n}_{jr} = \sum_r l_{jr} p_{j,0}\mathbf{n}_{jr} + \sum_r l_{jr}\mathbf{n}_{jr} \otimes \mathbf{n}_{jr}(\mathbf{u}_{j,0} - \mathbf{u}_{r,0}) = 0. \tag{42}$$

The first term is zero because we have $\sum_r l_{jr}\mathbf{n}_{jr} = 0$, consequently when we use (42) we obtain

$$\sum_r l_{jr}\mathbf{n}_{jr} \otimes \mathbf{n}_{jr}(\mathbf{u}_{j,0} - \mathbf{u}_{r,0}) = 0. \tag{43}$$

Using the Hilbert expansion in the nodal solver and assuming the same invertibility condition on the matrix $\sum_j \widehat{\beta}_{jr}$, we obtain the two following relations

$$\left(\sum_j l_{jr}\mathbf{n}_{jr} \otimes \mathbf{n}_{jr}\right)\mathbf{u}_{r,0} + \sigma\left(\sum_j l_{jr}\mathbf{n}_{jr} \otimes (\mathbf{x}_r - \mathbf{x}_j)\right)\mathbf{u}_{r,1} = \sum_j l_{jr} p_{j,0}\mathbf{n_{jr}} + \sum_j l_{jr}\mathbf{n}_{jr} \otimes \mathbf{n}_{jr}\mathbf{u}_{j,0} \tag{44}$$

and $\mathbf{u}_{r,0} = 0$.

As $\mathbf{u}_{r,0} = 0$, so (43) is equivalent to $\sum_r l_{jr}(\mathbf{n}_{jr} \otimes \mathbf{n}_{jr})\mathbf{u}_{j,0} = 0$. It is immediate to show that $\sum_r l_{jr}\mathbf{n}_{jr} \otimes \mathbf{n}_{jr}$ is symmetric positive for a non-degenerated mesh, see [Maz07]. So $\mathbf{u}_{j,0} = 0$. This result associates with (43) and (39) ends the demonstration. □

The previous demonstrations require that the matrix $\sum_j l_{jr}\mathbf{n}_{jr} \otimes (\mathbf{x}_r - \mathbf{x}_j)$ is invertible. This is the subject of the next section.

3.4 Study of the nodal solver invertibility

In this section we study the invertibility of the matrix

$$\sum_j l_{jr}(\mathbf{n_{jr}} \otimes \mathbf{n_{jr}} + \frac{\sigma}{\varepsilon}\mathbf{n_{jr}} \otimes (\mathbf{x_r} - \mathbf{x_j})). \tag{45}$$

This is essential to guarantee the well posedness of the nodal flux solver. This matrix is the sum of two matrices. The first one

$$\sum_j l_{jr}(\mathbf{n_{jr}} \otimes \mathbf{n_{jr}}) \tag{46}$$

is positive definite for all non-degenerated meshes [Maz07]. The second matrix is

$$A_r = \sum_j l_{jr}\mathbf{n}_{jr} \otimes (\mathbf{x}_r - \mathbf{x}_j). \tag{47}$$

Notice has A_r has no reason to be symmetric. We will study conditions such that A_r is positive, that is

$$(\mathbf{y}, A_r\mathbf{y}) > 0 \quad \forall \mathbf{y} \in \mathbb{R}^2,\ \mathbf{y} \neq 0.$$

Remark 12. *For triangular meshes, there is an elegant possibility to guarantee that $A_r = A_r^t > 0$.Let us assume that the orthocenter of the triangle is inside the triangle. We choose $\mathbf{x}$ equal to the orthocenter. In this case $\mathbf{x}_r - \mathbf{x}_j = d_{jr}\mathbf{n}_{jr}$ with $d_{jr} > 0$.Then*

$$A_r = \sum_j l_{jr}d_{jr}(\mathbf{n_{jr}} \otimes \mathbf{n_{jr}}) = A_r^t.$$

For a non-degenerated mesh, the sum over j contains more than two independent directions. So The matrix is positive. There is an important restriction. All angles must be strictly less than $\pi/2$, in order that the orthocenters are inside the cells.

Currently we fail to prove a complete result such as that of C. Mazeran. However we can show that if we do not distort the mesh too much, then the matrix is invertible. The idea is to note $Tr(A_r) = 2V_r$ with V_r the control volume around the vertices $\mathbf{x}_r$.

Definition 13. *With a slight abuse of notation, the control volume V_r is defined by the closed loop*

$$\dots, \mathbf{x}_{j-\frac{1}{2}}, \mathbf{x}_j, \mathbf{x}_{j+\frac{1}{2}}, \dots.$$

Here the $\mathbf{x}_j$'s are the center of the cells, and the $\mathbf{x}_{j+\frac{1}{2}}$'s are the middle of the edges around the vertices $\mathbf{x}_r$. A typical example is depicted in figure 3.

The idea is to compare A_r with $V_r\widehat{Id}$ and look which are the conditions for the matrix stay positive definite.We introduce the following definitions.

Proposition 14. *A_r satisfies:*

$$A_r = V_r\widehat{Id} + \frac{1}{2}\sum_j(\mathbf{w}_{j+1/2}^{\perp} \otimes \mathbf{w}_{j+1/2} - \mathbf{v}_{j+1/2}^{\perp} \otimes \mathbf{v}_{j+1/2}) = V_r\widehat{Id} + P \tag{48}$$

with $\mathbf{w}_{j+1/2} = (\mathbf{x}_{j+1} - \mathbf{x}_{j+1/2})$ and $\mathbf{v}_{j+1/2} = (\mathbf{x}_{j+1/2} - \mathbf{x}_j)$.

$$Tr(A_r) = 2V_r. \tag{49}$$

Proof. For the polygon defined by all the points $\mathbf{x}_j$ and $\mathbf{x}_{j+1/2}$ around the vertex $\mathbf{x}_r$, we have the identity which is a consequence of the Stokes theorem

$$V_r\widehat{Id} = \sum_j(\mathbf{x}_{j+1/2} - \mathbf{x}_j)^{\perp} \otimes (\frac{1}{2}(\mathbf{x}_{j+1/2} + \mathbf{x}_j) - \mathbf{x}_r) + (\mathbf{x}_j - \mathbf{x}_{j-1/2})^{\perp} \otimes (\frac{1}{2}(\mathbf{x}_{j-1/2} + \mathbf{x}_j) - \mathbf{x}_r) \tag{50}$$

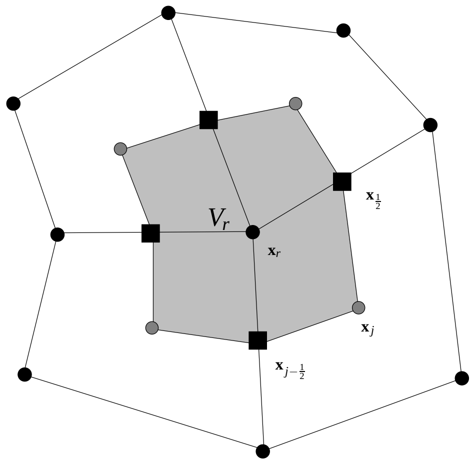

Figure 3: Definition of the control volume V_r around vertex $\mathbf{x}_r$. The control volume is defined by the close loop that joins the center of the cells ($\mathbf{x}_r$'s) and the middle of the edges ($\mathbf{x}_{j+\frac{1}{2}}$'s) around the vertex.

$$= \sum_j (\mathbf{x}_{j+1/2}-\mathbf{x}_j)^\perp \otimes (\frac{1}{2}(\mathbf{x}_{j+1/2}-\mathbf{x}_j)+\mathbf{x}_j-\mathbf{x}_r)+(\mathbf{x}_j-\mathbf{x}_{j-1/2})^\perp \otimes (\frac{1}{2}(\mathbf{x}_{j-1/2}-\mathbf{x}_j)+\mathbf{x}_j-\mathbf{x}_r) \quad (51)$$

Since $l_{jr}\mathbf{n}_{jr} = -(\mathbf{x}_{j+1/2}-\mathbf{x}_j)^\perp - (\mathbf{x}_j-\mathbf{x}_{j-1/2})^\perp$ we have

$$V_r\widehat{Id} = A_r + \frac{1}{2}\sum_j (\mathbf{x}_{j+1/2}-\mathbf{x}_j)^\perp \otimes (\mathbf{x}_{j+1/2}-\mathbf{x}_j) - (\mathbf{x}_j-\mathbf{x}_{j-1/2})^\perp \otimes (\mathbf{x}_j-\mathbf{x}_{j-1/2}) \quad (52)$$

which reads also, using the definition of the matrix P, $A_r = V_r\widehat{Id} + P$. The second point is then evident since for any vector v we have $Tr(\mathbf{v}^\perp \otimes \mathbf{v}) = 0$. □

Introducing the polygon T_{jr} defined by $(\mathbf{x}_r, \mathbf{x}_j, \mathbf{x}_{j+1/2}, \mathbf{x}_{j+1})$ we can also write

$$A_r = \sum_j \left(|T_{jr}|\widehat{Id} + \frac{1}{2}(\mathbf{w}^\perp_{j+1/2} \otimes \mathbf{w}_{j+1/2} - \mathbf{v}^\perp_{j+1/2} \otimes \mathbf{v}_{j+1/2}) \right) = \sum_j \left(|T_{jr}|\widehat{Id} + P_j \right) \quad (53)$$

or if $\widetilde{T}_{jr}$ is defined by $(\mathbf{x}_r, \mathbf{x}_{j-1/2}, \mathbf{x}_j, \mathbf{x}_{j+1/2})$

$$A_r = \sum_j \left(|\widetilde{T}_{jr}|\widehat{Id} + \frac{1}{2}(\mathbf{w}^\perp_{j-1/2} \otimes \mathbf{w}_{j-1/2} - \mathbf{v}^\perp_{j+1/2} \otimes \mathbf{v}_{j+1/2}) \right) = \sum_j \left(|\widetilde{T}_{jr}|\widehat{Id} + P_j \right). \quad (54)$$

Using these decompositions we have the following result

Proposition 15. *The matrix A_r is positive under the sufficient condition that for all j*

$$|T_{jr}| > \frac{1}{2} \parallel \mathbf{x}_{j+1} - \mathbf{x}_j \parallel \parallel \mathbf{x}_{j+1/2} - \frac{1}{2}(\mathbf{x}_{j+1} + \mathbf{x}_j) \parallel \quad (55)$$

or that for all j

$$|\widetilde{T}_{jr}| > \frac{1}{2} \parallel \mathbf{x}_{j+1/2} - \mathbf{x}_{j-1/2} \parallel \parallel \mathbf{x}_j - \frac{1}{2}(\mathbf{x}_{j+1/2} + \mathbf{x}_{j-1/2}) \parallel \quad (56)$$

Proof. For the two decompositions (53) and (54) we have

$$x^t A_r x = \sum_j |T_{jr}| ||x||^2 + x^t P^s_j x \geq \sum_j (|T_{jr}| - \rho(P^s_j)||x||^2$$

where $P_j^s = \frac{1}{2}(P_j + P_j^t)$ is the symmetric part of P_j. For any matrix M, $\rho(M)$ stands for its spectral radius. Thus if for any j $|T_{jr}| - \rho(P_j^s) > 0$ then A_r is positive. Set now $C = \mathbf{w}^\perp \otimes \mathbf{w} - \mathbf{v}^\perp \otimes \mathbf{v}$ with $\mathbf{v} = (a, b) \in \mathbb{R}^2$ and f and $\mathbf{w} = (\tilde{a}, \tilde{b}) \in \mathbb{R}^2$ then

$$C^s = \begin{pmatrix} ab - \tilde{a}\tilde{b} & \dfrac{-a^2 + \tilde{a}^2 + b^2 - \tilde{b}^2}{2} \\ \dfrac{-a^2 + \tilde{a}^2 + b^2 - \tilde{b}^2}{2} & \tilde{a}\tilde{b} - ab \end{pmatrix}$$

We note $x = (a - \tilde{a})$, $y = (b - \tilde{b})$, $\alpha = \frac{1}{2}(b + \tilde{b})$ and $\beta = \frac{1}{2}(a + \tilde{a})$. We have evidently $Tr(C^s) = 0$. Using the factorization $ab - \tilde{a}\tilde{b} = x\alpha + y\beta$, we obtain easily

$$Det(C^s) = -(x^2 + y^2)(\alpha^2 + \beta^2) = - \parallel \mathbf{v} - \mathbf{w} \parallel^2 \parallel \frac{1}{2}(\mathbf{v} + \mathbf{w}) \parallel^2 .$$

Since C^s is a traceless two dimensional matrix one gets

$$\rho(C^s) = \lambda_{max} = \frac{1}{2}(Tr(C^s) + \sqrt{Tr(C^s)^2 - 4Det(C^s)}) = \parallel \mathbf{w} - \mathbf{v} \parallel \parallel \frac{1}{2}(\mathbf{v} + \mathbf{w}) \parallel .$$

Specifying $\mathbf{v}$ and $\mathbf{w}$ for the decompositions (53) and (54) of A_r gives us (55) and (56). $\square$

Remark 16. *For meshes made with equilateral triangles or made with squares, it is easy to check that $A_r > 0$ because $\mathbf{x}_{j+1/2} = \frac{1}{2}(\mathbf{x}_{j+1} + \mathbf{x}_j)$ so that (55) is verified.*

3.5 Invertibility of the nodal solver in triangular case

In this section we study the invertibility of the nodal solver in the case of a general triangular mesh and $\mathbf{x}_j$ is equal to the barycenter of the cell. The case of equilateral triangles has been treated in remark 16. The case where $\mathbf{x}_j$ is the orthocenter of the triangle has been treated in remark 12 with the restriction that the angles of the triangles must be strictly less than $\pi/2$.

The method consists in the characterization of the three local inequalities (56) (one per corner). for the generic triangle depicted in figure 4

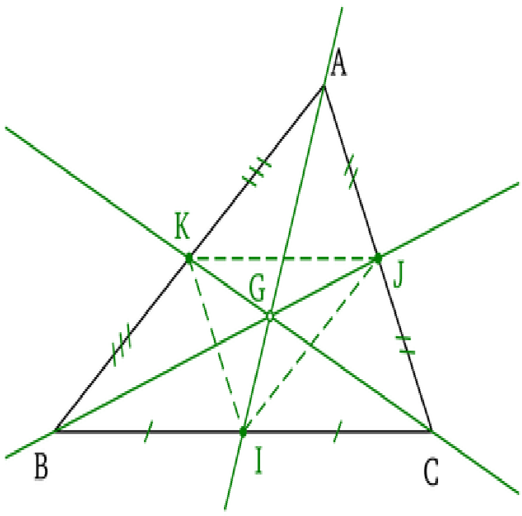

Figure 4: A generic triangle. The barycenter is $G = \frac{1}{3}(A + B + C)$. The middle of the edge BC is $I = \frac{1}{2}(B + C)$, and so on.

We define three angles. The first one is θ_A the angle between AG and KJ, the second one is θ_B the angle between BG and KI and the third one is θ_C the angle between CG and IJ.

Proposition 17. *The three local inequalities (56) are verified if*

$$| \sin(\theta_{A,B,C}) | > \frac{1}{4}. \tag{57}$$

Proof. By symmetry it is sufficient to study the corner condition for vertex A which correspond to

$$\mathbf{x}_r = A, \quad \mathbf{x}_{j+1/2} = J = \frac{A+C}{2}, \quad \mathbf{x}_{j-1/2} = K = \frac{A+B}{2}, \quad \mathbf{x}_j = G = \frac{A+B+C}{3}.$$

The quadrangle $\widetilde{T}_{jr}$ in inequality (56) corresponds to $\widetilde{T}_{jr} = (A, K, G, J)$. One as

$$| \widetilde{T}_{jr} | = \frac{1}{2} \parallel AG \parallel \parallel JK \parallel | \sin(\theta_A) | \,.$$

We see that $\parallel \mathbf{x}_{j+1/2} - \mathbf{x}_{x-1/2} \parallel = \parallel JK \parallel$. One also has

$$\parallel \mathbf{x}_j - \frac{1}{2}(\mathbf{x}_{j+1/2} + \mathbf{x}_{j-1/2}) \parallel = \left\| \frac{A+B+C}{3} - \frac{2A+B+C}{4} \right\|$$

$$= \parallel -\frac{-2A+B+C}{12} \parallel = \parallel \frac{B+C}{2} - A \parallel .$$

As G is the barycenter we can say $\parallel AG \parallel = \frac{2}{3} \parallel AI \parallel$. We will also need $D = \frac{K+J}{2}$, so that

$$\parallel AD \parallel = \parallel A - \frac{2A+B+C}{4} \parallel = \frac{1}{2} \parallel AI \parallel$$

$$\text{So } \parallel AD \parallel = \frac{1}{2} \parallel AI \parallel \text{ and } \parallel DG \parallel = \frac{1}{6} \parallel AI \parallel.$$

Consequently (56) is equivalent to

$$\frac{1}{2} \parallel AG \parallel \parallel JK \parallel | \sin(\theta_A) | > \frac{1}{2} \parallel JK \parallel \parallel DG \parallel .$$

Using the previous results we obtain $| \sin(\theta_A) | > \frac{1}{4}$. ☐

In order to get a more useful interpretation of the previous proposition, we define $\hat{A}$, $\hat{B}$ and $\hat{C}$ the angles associate to the vertex A,B and C.

Proposition 18. *One has the correspondence*

$$| \sin(\theta_A) | = \frac{1}{\sqrt{1 + \frac{\sin^2(\hat{B}-\hat{C})}{4\sin^2\hat{B}\sin^2\hat{C}}}}. \tag{58}$$

Similar formulas hold for the two other angles.

Proof. To simplify the proof we choose the vertex : $A = (x, y)$, $B = (0, 0)$, $C = (1, 0)$ and $D = (0.5, 0)$. We known that $x = \cot(\hat{B})y$ and $1 - x = \cot(\hat{C})y$. Consequently

$$y = \frac{1}{\cot(\hat{B}) + \cot(\hat{C})} \text{ and } y = \frac{\cot(\hat{B})}{\cot(\hat{B}) + \cot(\hat{C})}$$

Now we remark that $x - 1/2 = \cot(\theta_A)y$, so $\cot(\theta_A) = \frac{1}{2}(\cot(\hat{B}) - \cot(\hat{C}))$. This is also equivalent to

$$\cot(\theta_A) = \frac{\sin(\hat{C} - \hat{B})}{2\sin(\hat{C})\sin(\hat{B})}$$

To conclude we use $1 + \cot^2(\theta_A) = \frac{1}{\sin^2(\theta_A)}$. ☐

Finally we desire to characterize all triangles such that the sufficient conditions of positivity (57) with (58) are satisfied. We define the function g such that

$$g(\hat{B},\hat{C}) = \frac{1}{\sqrt{1+\frac{\sin^2(\hat{B}-\hat{C})}{4\sin^2\hat{B}\sin^2\hat{C}}}}.$$

Since $\widehat{A}+\widehat{B}+\widehat{C}=\pi$, the sufficient conditions of positivity rewrites

$$f(\hat{B},\hat{C}) > \frac{1}{4}, \quad f(\hat{B},\hat{C}) = \min(g(\hat{B},\hat{C}), g(\pi-\hat{C}-\hat{B},\hat{C}), g(\hat{B},\pi-\hat{C}-\hat{B})).$$

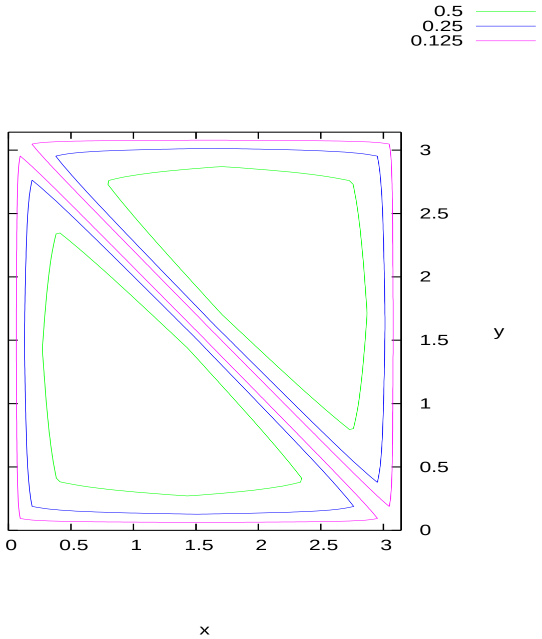

Figure 5: Plot of the function f in function of $x\in[0,\pi]$ and $y\in[0,\pi]$.. The interesting part corresponds to $x\geq 0$, $y\geq 0$ and $x+y\leq\pi$. The positivity domain $f(x,y)>\frac{1}{4}$ is delimited by the blue isoline between the two others.

We observe on figure 5 that if all angles are strictly greater than 0.2 radian,that is 11 degrees, then the positivity criterion $f(x,y)>\frac{1}{4}$ is fulfilled. In consequence the angle restriction is much weaker than with the orthocenter. At least angles more than $\frac{\pi}{2}$ are possible in the mesh, provided there is no angle less than 0.2 **radian**. We believe that sharper positivity estimates are possible. Since the analysis is tricky we leave this issue for further studies.

3.6 Others variants

In this section we define others variants. These variants are based on a tensor formulation that we borrow from [Klu08]. It allows to propose other discretization for the acoustic part and for the part associate to the source term in the fluxes. We rewrite the scheme in the following form

$$\begin{cases} |\Omega_j|\, \partial_t p_j(t) + \dfrac{1}{\varepsilon} \displaystyle\sum_r l_{jr}(\mathbf{u}_r, \mathbf{n}_{jr}) = 0 \\ |\Omega_j|\, \partial_t \mathbf{u}_j(t) + \dfrac{1}{\varepsilon} \displaystyle\sum_r \mathbf{F}_{jr} = -\,|\Omega_j|\, \dfrac{\sigma}{\varepsilon^2} \mathbf{u}_j^n \end{cases} \tag{59}$$

$$\begin{cases} \mathbf{F}_{jr} = l_{jr} p_j \mathbf{n}_{jr} + \widehat{\alpha}_{jr}(\mathbf{u}_j - \mathbf{u}_r) - \dfrac{\sigma}{\varepsilon} \widehat{\beta}_{jr} \mathbf{u}_r \\ \left(\displaystyle\sum_j \widehat{\alpha}_{jr} + \dfrac{\sigma}{\varepsilon} \widehat{\beta}_{jr} \right) \mathbf{u_r} = \displaystyle\sum_j l_{jr} p_j \mathbf{n}_{jr} + \widehat{\alpha}_{jr} \mathbf{u_j} \end{cases} \tag{60}$$

The scheme (22-25) corresponds to the scheme (59-60) with $\widehat{\beta}^I_{jr} = l_{jr}\mathbf{n}_{jr} \otimes (\mathbf{x}_r - \mathbf{x}_j)$. We propose a second possibility $\widehat{\beta}^{II}_{jr} = T_{jr}\widehat{Id}$ with T_{jr} a control volume associated to the node r and the cell j. Consequently $\sum_j \widehat{\beta}_{jr} = V_r \widehat{Id}$, which is always invertible on non-degenerate meshes. To finish we define the variants associated to the acoustic term. The tensor based on the GLACE scheme [Klu08] is

$$\widehat{\alpha}^I_{jr} = l_{jr}\mathbf{n}_{jr} \otimes \mathbf{n}_{jr}. \tag{61}$$

The tensor based on the CHIC scheme write

$$\widehat{\alpha}^{II}_{jr} = \frac{1}{2 l_{jr}} (l_{jr-1,r}\mathbf{n}_{jr-1,r} \otimes \mathbf{n}_{jr-1,r} + l_{jr,r+1}\mathbf{n}_{jr,r+1} \otimes \mathbf{n}_{jr,r+1}) \tag{62}$$

with $l_{jr\pm}$, $\mathbf{n}_{jr\pm}$ normals and length associated to the edges $[\mathbf{x}_{r-1}, \mathbf{x}_r]$ and $[\mathbf{x}_r, \mathbf{x}_{r+1}]$.

Definition 19. *The scheme (59-60) will be called a general JL-(a) scheme. The same scheme (59-60) but with the source term* $-\frac{\sigma}{\varepsilon^2}\sum_r \widehat{\beta}_{jr}\mathbf{u}_r$ *in (59) will be called a general JL-(b) scheme.*

Proposition 20. *The general JL-(a) and JL-(b) schemes have the same asymptotic limit whatever the choice of tensor* $\widehat{\alpha}_{jr}$ *is. In this formulation the limit scheme becomes*

$$\begin{cases} |\Omega_j|\, \partial_t p_j(t) + \displaystyle\sum_r l_{jr}(\mathbf{u}_r, \mathbf{n}_{jr}) = 0, \\ \sigma \left(\displaystyle\sum_j \widehat{\beta}_{jr} \right) \mathbf{u}_r = \displaystyle\sum_j l_{jr} p_j \mathbf{n_{jr}}. \end{cases}$$

3.7 L^2 stability

In this section we prove the L^2 stability of some the schemes that we has define previously. We suppose for simplify that the domain is a torus. We will consider the semi-discrete and implicit schemes.

Proposition 21. *The semi-discrete general JL-(a) scheme with* $\widehat{\beta}_{jr} = T_{jr}\widehat{Id}$ *and* $\widehat{\alpha}_{jr}$ *given by (61) or (62) is stable in* L^2 *norm.*

Proof. The semi-discrete scheme is

$$\begin{cases} p_j^{'}(t) + \dfrac{1}{\varepsilon} \displaystyle\sum_r l_{jr}(\mathbf{u}_r, \mathbf{n}_{jr}) = 0 \\ \mathbf{u}_j^{'}(t) + \dfrac{1}{\varepsilon} \displaystyle\sum_r \mathbf{F}_{jr} = -\,|\Omega_j|\, \dfrac{\sigma}{\varepsilon^2} \mathbf{u}_j^n \end{cases} \tag{63}$$

We define

$$E(t) = \frac{1}{2} \int_\Omega \left(|\; p_h(t) \;|^2 + (\mathbf{u_h}, \mathbf{u_h}) \right) d\mathbf{x}$$

Our goal is to show that $E^{'}(t) \leq 0$. One has

$$E^{'}(t) = \frac{1}{2}\int_{\Omega} \frac{d}{dt}(\mid p_h(t) \mid^2 + (\mathbf{u}_h(t), \mathbf{u}_h(t))) = \int_{\Omega} p_h(t)p_h^{'}(t) + (\mathbf{u}_h(t), \mathbf{u}_h^{'}(t))$$

$$\sum_j |\Omega_j|\, p_j(t)p_j^{'}(t) + (\mathbf{u}_j(t), \mathbf{u}_j^{'}(t)).$$

Using the definition of scheme

$$E^{'}(t) = -\frac{1}{\varepsilon}\sum_j\sum_r l_{jr}p_j(\mathbf{u}_r, \mathbf{n}_{jr}) - \frac{1}{\varepsilon}\sum_j\sum_r(\mathbf{F}_{jr}, \mathbf{u}_j) - \frac{\sigma}{\varepsilon^2}\sum_j |\Omega_j|\,(\mathbf{u}_j, \mathbf{u}_j). \tag{64}$$

We expand the second term of the previous equation

$$\sum_j\sum_r(\mathbf{F}_{jr}, \mathbf{u}_j) = -\sum_j\sum_r l_{jr}p_j(\mathbf{u}_j, \mathbf{n}_{jr}) + \sum_j\sum_r(\widehat{\alpha}_{jr}(\mathbf{u}_j - \mathbf{u}_r), \mathbf{u}_j) - \frac{\sigma}{\varepsilon}\sum_j\sum_r(\widehat{\beta}_{jr}\mathbf{u}_r, \mathbf{u}_j). \tag{65}$$

Since $\sum_r l_{jr}\mathbf{n}_{jr} = 0$ the first term of (65) is zero. Summing on r the second equation of (60) and permuting the sums, we show that

$$\sum_j\sum_r l_{jr}p_j(\mathbf{u}_r, \mathbf{n}_{jr}) = \sum_j\sum_r((\widehat{\alpha}_{jr} + \frac{\sigma}{\varepsilon}\widehat{\beta}_{jr})\mathbf{u}_r, \mathbf{u}_r) - \sum_j\sum_r(\widehat{\alpha}_{jr}\mathbf{u}_j, \mathbf{u}_r). \tag{66}$$

Plugging (65) and (66) in (64) and permuting the sums in $E'(t)$ gives

$$E^{'}(t) = -\frac{1}{\varepsilon}\sum_j\sum_r(\widehat{\alpha}_{jr}(\mathbf{u}_j - \mathbf{u}_r), \mathbf{u}_j - \mathbf{u}_r) - \frac{\sigma}{\varepsilon^2}\sum_r\sum_j(\widehat{\beta}_{jr}\mathbf{u}_r, \mathbf{u}_r - \mathbf{u}_j) - \frac{\sigma}{\varepsilon^2}\sum_j |\Omega_j|\,(\mathbf{u}_j, \mathbf{u}_j) \tag{67}$$

In the following, we consider the $\widehat{\alpha}^I_{jr}$ (61), but the demonstration is exactly equivalent if we use the tensor $\widehat{\alpha}^{II}_{jr}$ (62). We obtain

$$E^{'}(t) = -\frac{1}{\varepsilon}\sum_r\sum_j(l_{jr}, \mathbf{n}_{jr}(\mathbf{u}_j - \mathbf{u}_r))^2 - \frac{\sigma}{\varepsilon^2}\sum_r\sum_j(\widehat{\beta}_{jr}\mathbf{u}_r, \mathbf{u}_r - \mathbf{u}_j) - \frac{\sigma}{\varepsilon^2}\sum_j |\Omega_j|\,(\mathbf{u}_j, \mathbf{u}_j). \tag{68}$$

To finish, we use that $\widehat{\beta}_{jr} =\mid T_{jr} \mid Id$ and $\sum_r \mid V_{jr} \mid = |\Omega_j|$. Therefore we can rewrite the second term and third term of (68) in the following form

$$\frac{\sigma}{\varepsilon^2}\sum_j\sum_r \mid T_{jr} \mid ((\mathbf{u}_r, \mathbf{u}_r) + (\mathbf{u}_j, \mathbf{u}_j) - (\mathbf{u}_r, \mathbf{u}_j))$$

$$= \frac{\sigma}{2\varepsilon^2}\sum_j\sum_r \mid T_{jr} \mid (\|\mathbf{u}_r\|^2 + \|\mathbf{u}_j\|^2 + \|\mathbf{u}_r - \mathbf{u}_j\|^2) \geq 0.$$

□

Proposition 22. *The semi-discrete general JL-(b) scheme with $\widehat{\beta}_{jr} = T_{jr}\widehat{Id}$ or $\widehat{\beta}_{jr} = l_{jr}\mathbf{n}_{jr} \otimes (\mathbf{x}_r - \mathbf{x}_j)$ and $\widehat{\alpha}_{jr}$ given by (61) or (62) is stable in L^2 norm.*

In the case where $\widehat{\beta}_{jr} = l_{jr}\mathbf{n}_{jr}\otimes(\mathbf{x}_r - \mathbf{x}_j)$ we make the hypothesis that $\sum_j \widehat{\beta}_{jr}$ is positive. Sufficient conditions such that this hypothesis holds true are provided in the section 3.4 and 3.5.

Proof. The first part the of proof is exactly the same as previously. One has

$$E^{'}(t) = -\frac{1}{\varepsilon}\sum_j\sum_r(l_{jr}, \mathbf{n}_{jr}(\mathbf{u}_j - \mathbf{u}_r))^2 - \frac{\sigma}{\varepsilon^2}\sum_j\sum_r(\widehat{\beta}_{jr}\mathbf{u}_r, \mathbf{u}_r - \mathbf{u}_j) - \frac{\sigma}{\varepsilon^2}\sum_j |\Omega_j|\,(\mathbf{u}_r, \mathbf{u}_j) \tag{69}$$

One has $|\Omega_j| = \sum_r \widehat{\beta}_{jr}$. Indeed, if $\widehat{\beta}_{jr} = Id \mid T_{jr} \mid$ it's immediate. If $\widehat{\beta}_{jr} = l_{jr}\mathbf{n}_{jr} \otimes (\mathbf{x}_r - \mathbf{x}_j)$ we use the following identity

$$\sum_r l_{jr}\mathbf{n}_{jr} \otimes \mathbf{x}_r = Id\,|\Omega_j|$$

and $\sum_r l_{jr}\mathbf{n}_{jr} = 0$. So

$$E^{'}(t) = -\frac{1}{\varepsilon}\sum_j\sum_r (l_{jr}, \mathbf{n}_{jr}(\mathbf{u}_j - \mathbf{u}_r))^2 - \frac{\sigma}{\varepsilon^2}\sum_j\sum_r (\widehat{\beta}_{jr}\mathbf{u}_r, \mathbf{u}_r) \tag{70}$$

To conclude we permute the sums. Since $\sum_j \widehat{\beta}_{jr}$ is positive then $E^{'}(t) \leq 0$. It ends the proof. □

Remark 23. *The L^2 stability of the semi-discrete scheme imply naturally the L^2 stability of the implicit scheme. This is a classical result therefore we does not detail the proof .*

Remark 24. *Using the same principle of proof we prove that the diffusion scheme is also L^2 stable.*

4 Asymptotic preserving analysis in dimension two

The main result of this section is that the general JL-(a) and JL-(b) schemes are asymptotic preserving for the choice

$$\widehat{\beta}_{jr} = l_{jr}\mathbf{n}_{jr} \otimes (\mathbf{x}_r - \mathbf{x}_j),$$

and for the two choices of $\widehat{\alpha}_{jr}$ considered in this work.

4.1 Preliminary considerations about the convergence analysis

In the 1D case we were able to obtain estimate of convergence with respect to ε and Δx for the telegraph equation. The generalization of this method of analysis in 2D is difficult for two reasons. On the one hand for getting estimates which are uniform with respect to the parameter ε for small values of this parameter, and on the other hand in mastering the finite volume techniques for establishing optimal error estimates in the case $\varepsilon \approx 1$.

In the limit $\varepsilon \to 0$, the scheme that we obtain is supposed to be an accurate discretization of the diffusion equation on unstructured meshes. This is a research topic by itself, and we refer to [BM06, LSSV07, DlP09, DEGH08, AE06] and references therein for a description of the up-to-date theory of discretization for the diffusion scheme on general unstructured meshes. Most of these schemes can be written in a mixed framework, so very close to a Finite Element Method.

The case $\varepsilon \approx 1$ is not so easy also. In our case one has to use a technique of proof well adapted for the specific kind of finite volume developed in [Maz07]. The L^2 stability of semi-discrete scheme is the property needed to demonstrate the convergence.

Even if technical difficulties still exist for the second point, we consider that the main difficulty is the limit diffusion scheme. The reason is that if ever the limit diffusion scheme is non convergent it is not possible to obtain a ε-uniform estimate of convergence. And the resulting scheme may be non convergent as well in numerical experiments. Our numerical results show that it is indeed the case for some of the schemes developed above.

Therefore we will use a different method to establish the asymptotic preserving character of the schemes. We consider the asymptotic limit of the scheme and we will prove that the solution of the semi-discrete diffusion scheme converges to the solution of the correct diffusion equation. It will provide a sufficient criterion to state that any scheme for the 2-D telegraph equation is AP (or not).

4.2 Convergence of the limit semi-discrete scheme

We consider the semi-discrete scheme

$$\begin{cases} p_j'(t) - \dfrac{1}{|\Omega_j|} \displaystyle\sum_r l_{jr} \left(\mathbf{n}_{jr}, \mathbf{u}_r \right) = 0, \\ A_r \mathbf{u}_r = \displaystyle\sum_j l_{jr} \mathbf{n}_{jr} p_j, \qquad\qquad \text{with } A_r = -\left(\sum_j l_{jr} \mathbf{n}_{jr} \otimes (\mathbf{x}_r - \mathbf{x}_j) \right). \end{cases} \tag{71}$$

Before going further, we begin by some remarks or comments. First remark, the sign before the discrete operators has been changed with respect to (26), so that $\mathbf{u}_r$ is a discrete approximation of $\nabla p(\mathbf{x}_r)$. Notice that we took $\sigma = 1$. Second remark, the domain $\Omega = (\mathcal{T})$ of computation is a torus. It eliminates the boundary condition and simplifies the analysis. Third remark, we will assume that the exact solution is smooth

$$p \in W^{3,\infty}(\Omega). \tag{72}$$

We define the consistency error of both equations in (71)

$$\begin{cases} a_j(t) = \partial_t p(\mathbf{x}_j, t) - \dfrac{1}{|\Omega_j|} \displaystyle\sum_r l_{jr} \left(\mathbf{n}_{jr}, \nabla p(\mathbf{x}_r, t) \right), \\ \mathbf{b}_r(t) = \dfrac{1}{V_r} \left(A_r \nabla p(\mathbf{x}_r, t) - \displaystyle\sum_j l_{jr} \mathbf{n}_{jr} p(\mathbf{x}_j, t) \right). \end{cases} \tag{73}$$

Let us assume the characteristic length of the mesh

$$h = \max_j \left(\text{diam}[(\Omega_j)] \right). \tag{74}$$

By construction there exists $C > 0$ such that

$$l_{jr} \le Ch, \qquad \forall j, r. \tag{75}$$

We will make the assumption that the mesh is regular in the sense that there exists two constants C_1 and C_2 such that

$$C_1 h^2 \le |\Omega_j| \le C_2 h^2, \quad \forall r \qquad \text{uniformly with respect to } h. \tag{76}$$

and that

$$C_1 h^2 \le V_r \le C_2 h^2, \quad \forall r \qquad \text{uniformly with respect to } h. \tag{77}$$

Lemma 25. *There exists a constant $C > 0$ such that the following estimates hold*

$$|a_j| \le Ch \quad \textit{for all } j, \tag{78}$$

and

$$\| \mathbf{b}_r \| \le Ch, \quad \textit{for all } r. \tag{79}$$

Proof. By construction

$$\partial_t p(\mathbf{x}_j, t) = \frac{\int_{\Omega_j} \partial_t p(x,t) dx}{|\Omega_j|} + O(h) = \frac{\int_{\Omega_j} \Delta p dx}{|\Omega_j|} + O(h) = \frac{1}{|\Omega_j|} \int_{\partial\Omega_j} \partial_n p d\sigma + O(h).$$

By definition of $l_{jr}\mathbf{n}_{jr}$ one has

$$\sum_r l_{jr} \left(\mathbf{n}_{jr}, \nabla p(\mathbf{x}_r, t) \right) = \sum_k \int_{\Sigma_{jk}} \left(\frac{\nabla p \left(x_{jk}^+ \right) + \nabla p \left(x_{jk}^- \right)}{2}, \mathbf{n}_j \right) d\sigma$$

where the nodes x_{jk}^+ and x_{jk}^- are the extremities of the segment Σ_{jk}. Therefore

$$a_j = O(h) + \frac{1}{|\Omega_j|}\sum_k \int_{\Sigma_{jk}} \left(\nabla p - \frac{\nabla p\left(x_{jk}^+\right) + \nabla p\left(x_{jk}^-\right)}{2}, \mathbf{n}_j \right) d\sigma$$

Since the function under the integral is approximated by the trapezoidal rule, so the error of integration is $O(h^2)$

$$\left| \int_{\Sigma_{jk}} \left(\nabla p - \frac{\nabla p\left(x_{jk}^+\right) + \nabla p\left(x_{jk}^-\right)}{2}, \mathbf{n}_j \right) d\sigma \right| \le Ch^2 \left|\Sigma_{jk}\right| \le Ch^3.$$

After division by s_j and using the lower bound of the regularity hypothesis (76), one gets that $a_j = O(h)$. The other term is

$$\mathbf{b}_r = \frac{1}{V_r}\left(\left(\sum_j l_{jr}\mathbf{n}_{jr} \otimes (\mathbf{x}_r - \mathbf{x}_j) \right) \nabla p(\mathbf{x}_r, t) + \sum_j l_{jr}\mathbf{n}_{jr} p(\mathbf{x}_j, t) \right)$$

$$= \frac{1}{V_r}\sum_j \left((\mathbf{x}_r - \mathbf{x}_j, \nabla p(\mathbf{x}_r, t)) + p(\mathbf{x}_j, t) \right) l_{jr}\mathbf{n}_{jr}$$

A simple Taylor expansion shows that

$$p(\mathbf{x}_j, t) + (\mathbf{x}_r - \mathbf{x}_j, \nabla p(\mathbf{x}_r, t)) = p(\mathbf{x}_r, t) + O_{jr}(h^2).$$

So

$$\mathbf{b}_r = \frac{1}{V_r}\sum_j O_{jr}(h^2) l_{jr}\mathbf{n}_{jr} = O(h)$$

due to (75) and (77). It ends the proof of the claim.

Let us define two error variables

$$e_j(t) = p_j(t) - p(\mathbf{x}_j, t) \text{ and } \mathbf{f}_r(t) = \mathbf{u}_r(t) - \nabla p(\mathbf{x}_r, t).$$

The error e_j defines an error function

$$e(\mathbf{x}, t) = e_j(t) \quad \forall \mathbf{x} \in \Omega_j.$$

The error $\mathbf{f}_r$ defines another error function

$$\mathbf{f}(\mathbf{x}, t) = \mathbf{f}_r(t) \quad \forall \mathbf{x} \in V_r.$$

The numerical error is

$$E(t) = \|e(t)\|^2_{L^2(\Omega)} = \left(\sum_j |\Omega_j| \left(p_j(t) - p(\mathbf{x}_j, t)\right)^2 \right).$$

We will also consider

$$F(t) = \|\mathbf{f}\|^2_{L^2([0,t]\times\Omega)} = \left(\int_0^t \sum_r V_r \left|\mathbf{u}_r(t) - \nabla p(\mathbf{x}_r, t)\right|^2 \right).$$

Theorem 26. *Assume $p \in W^{3,\infty}(\Omega)$. Assume there exists a constant $\alpha > 0$ such that for all r*

$$A_r^s \ge \alpha V_r \quad \text{where } A_r^s = \frac{A_r + A_r^t}{2} \text{ is the symmetric part of } A_r. \tag{80}$$

The inequality is taken in the sense of symmetric matrices. Then the limit diffusion is convergent for all time $T > 0$. There exists a constant $C(T) > 0$ such that

$$\|e(t)\|_{L^2(\Omega)} + \|\mathbf{f}\|_{L^2([0,t]\times\Omega)} \le C(T)h, \qquad 0 < t \le T. \tag{81}$$

By construction

$$\begin{cases} \partial_t e(\mathbf{x}_j,t) - \dfrac{1}{s_j}\displaystyle\sum_r l_{jr}\left(\mathbf{n}_{jr},\mathbf{f}_r(t)\right) = a_j(t), \\ A_r\mathbf{f}_r(t) + \displaystyle\sum_j l_{jr}\mathbf{n}_{jr}e_j(t) = V_r\mathbf{b}_r(t). \end{cases} \tag{82}$$

One has the identity

$$E'(t) = 2\sum_j e_j(t)e_j'(t) = \sum_j e_j(t)\left(\left(\sum_r l_{jr}\left(\mathbf{n}_{jr},\mathbf{f}_r(t)\right)\right) + |\Omega_j|\,a_j(t)\right)$$

$$= 2\sum_j |\Omega_j|\,e_j(t)a_j(t) + 2\sum_j e_j(t)\left(\sum_r l_{jr}\left(\mathbf{n}_{jr},\mathbf{f}_r(t)\right)\right)$$

$$= 2\left(e(t),a(t)\right)_{L^2(\Omega)} + 2\sum_r\left(\sum_j l_{jr}\mathbf{n}_{jr}e_j(t),\mathbf{f}_r(t)\right)$$

$$= 2\left(e(t),a(t)\right)_{L^2(\Omega)} + 2\left(\mathbf{f}(t),\mathbf{b}(t)\right)_{L^2(\Omega)} - 2\sum_r\left(A_r\mathbf{f}_r(t),\mathbf{f}_r(t)\right)$$

$$= 2\left(e(t),a(t)\right)_{L^2(\Omega)} + 2\left(\mathbf{f}(t),\mathbf{b}(t)\right)_{L^2(\Omega)} - 2\sum_r\left(A_r^s\mathbf{f}_r(t),\mathbf{f}_r(t)\right).$$

The assumption of the theorem shows that $\sum_r\left(A_r^s\mathbf{f}_r(t),\mathbf{f}_r(t)\right) \geq \alpha\|\mathbf{f}(t)\ \|^2$. Therefore

$$E'(t) \leq ||e(t)||^2 + ||a(t)||^2 + 2\left(\frac{\mu||\mathbf{f}(t)||^2 + \mu^{-1}||\mathbf{b}(t)||^2}{2}\right) - 2\alpha\|\mathbf{f}(t)\ \|^2, \qquad \forall\mu > 0.$$

Let us choose $\mu = \alpha$ so that

$$E'(t) \leq \frac{1}{2}E(t) - \alpha F(t) + ||a(t)||^2 + \frac{1}{\alpha}||\mathbf{b}(t)||^2 \leq \frac{1}{2}E(t) + Ch^2 \tag{83}$$

since the consistency estimates (78-79) hold and the domain Ω is bounded. By construction $E(0) = O(h^2)$, so by the Gronwall lemma $E(t) \leq Cth^2$ which proves that $||a(t)|| \leq C'th$ for some constant C'. Let us finally integrate (83) in the time interval $[0,T]$. We obtain

$$E(T) + \alpha F(T) \leq E(0) + \int_0^T\left(\frac{1}{2}E(t) + Ch^2\right)dt \leq C''Th^2$$

from which we deduce the final part of the convergence estimate (81). The proof is ended. □

4.3 Asymptotic preserving schemes

In the previous subsection, we have shown that the limit scheme is consistent with the heat equation on the condition that $\widehat{\beta_{jr}}$ is given by the tensor formulation

$$\widehat{\beta_{jr}} = l_{jr}\mathbf{n}_{jr}\otimes(\mathbf{x}_r - \mathbf{x}_j).$$

In the proof this condition appears to be only a sufficient condition. However all our numerical tests with the other choice of $\widehat{\beta}_{jr}$ show that it is also a necessary condition (if the choice has to be done only between these two formulas). In summary one has

Proposition 27. *The scheme JL-(a)*

$$\begin{cases} |\Omega_j|\,\partial_t p_j(t) + \dfrac{1}{\varepsilon}\sum_r l_{jr}(\mathbf{u}_r, \mathbf{n}_{jr}) = 0 \\ |\Omega_j|\,\partial_t \mathbf{u}_j(t) + \dfrac{1}{\varepsilon}\sum_r F_{jr} = -\,|\Omega_j|\,\dfrac{\sigma}{\varepsilon^2}\mathbf{u}_j \end{cases}$$

and the scheme JL-(b)

$$\begin{cases} |\Omega_j|\,\partial_t p_j(t) + \dfrac{1}{\varepsilon}\sum_r l_{jr}(\mathbf{u}_r, \mathbf{n}_{jr}) = 0 \\ |\Omega_j|\,\partial_t \mathbf{u}_j(t) + \dfrac{1}{\varepsilon}\sum_r F_{jr} = -\dfrac{\sigma}{\varepsilon^2}\sum_r \mathbf{n}_{jr}\otimes(\mathbf{x}_r - \mathbf{x}_j)\mathbf{u}_r \end{cases}$$

with the fluxes

$$\begin{cases} F_{jr} = l_{jr}p_j\mathbf{n}_{jr} + \widehat{\alpha}_{jr}(\mathbf{u}_j - \mathbf{u}_r) - \dfrac{\sigma}{\varepsilon}\mathbf{n}_{jr}\otimes(\mathbf{x}_r - \mathbf{x}_j)\mathbf{u}_r \\ \left(\sum_j \widehat{\alpha}_{jr} + \dfrac{\sigma}{\varepsilon}(l_{jr}\mathbf{n}_{jr}\otimes(\mathbf{x}_r - \mathbf{x}_j))\right)\mathbf{u_r} = \sum_j l_{jr}p_j\mathbf{n}_{jr} + \widehat{\alpha}_{jr}\mathbf{u_j} \end{cases}$$

are asymptotic preserving.

Remark 28. *In 1D JL-(b) is equal to the modified Jin-Levermore scheme, equal to the Gosse-Toscani scheme, for which we are able to prove optimal convergence estimates.*

5 Details of discretization for the nodal schemes

5.1 Boundary conditions

In this section we detail the implementation of the Neumann boundary condition. We do not detail the implementation of the Dirichlet boundary condition. It can be easily done by analogy with the given pressure condition stud ied in [CDDL09, Maz07].

5.1.1 Neumann condition for diffusion scheme

We consider the homogeneous Neumann boundary condition

$$(\nabla p(t, \mathbf{x}).\mathbf{n}_{ext}) = 0$$

where $\mathbf{n}_{ext} = (n^1_{ext}, n^2_{ext})$ is the exterior normal. The result (79) shows that $\mathbf{u}_r$ is consistent with $\nabla p(t, \mathbf{x}_r)$. Consequently the Neumann condition is equivalent to $(\mathbf{u}_r, \mathbf{n}_{ext}) = 0$. However the matrix A_r is not always invertible on the boundary in this case. To solve this problem we use a method described in [Klu08, CDDL09, Maz07], where the nodal solver is provided by the solution of a modified linear system

$$\begin{cases} (\mathbf{u}_r, \mathbf{n}_{ext}) = 0 \\ (A_r\mathbf{u}_r, \mathbf{t}_{ext}) = (\sum_j l_{jr}\mathbf{n}_{jr}p_j, \mathbf{t}_{ext}), \end{cases} \tag{84}$$

and $\mathbf{t}_{ext} = (-\mathbf{n}^2_{ext}, \mathbf{n}^1_{ext})$ is tangent to the boundary. For the corner of the domain we take $\mathbf{u}_r = 0$.

5.1.2 Neumann condition for AP schemes applied to the telegraph equation

The method is the same. For the schemes JL-(a) and JL-(b) we use the boundary condition $(\mathbf{u}_r, \mathbf{n}_{ext}) = 0$ which corresponds to the Neumann condition for the scalar telegraph equation.

5.2 Time discretization

Any explicit diffusion scheme has a CFL condition like $\Delta t \le Ch^2$, which is already restrictive. Moreover Lemma (4) states that in 1D, for an explicit time discretization, the modified Jin and Levermore scheme (equivalent to the Gosse-Toscani scheme) is positive if the CFL condition

$$\Delta t \le \varepsilon h$$

is verified. This is more restrictive on coarse grids than that for a diffusion equation. For the original Jin and Levermore scheme the CFL condition, considering L^2 stability only (this scheme does not verify a maximum principle for the Riemann invariants), should be even more restrictive. Consequently explicit time discretization is not useful in the practice. Instead we will use implicit schemes for all numerical results. However the condition number of the matrix corresponding to the global linear system to solve may increase dramatically when ε is small.

5.3 The limit diffusion scheme on Cartesian mesh and spurious modes

Let us study the limit diffusion scheme on Cartesian mesh to show the problem associated to this case. Consider a Cartesian mesh with h the length of one side of a square. In this case $l_{jr} = \frac{h}{\sqrt{2}}$, and the four vector $\mathbf{n}_{jr}$ associate to a cell j are

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \quad \begin{pmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \quad \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix}, \quad \begin{pmatrix} \frac{-1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix}$$

After evident but lengthy calculations one gets that

$$A_r = \begin{pmatrix} h^2 & 0 \\ 0 & h^2 \end{pmatrix},$$

and that the scheme (26) is equivalent to

$$\frac{p^{n+1}_{(i,j)} - p^n_{(i,j)}}{\Delta t} - \frac{p_{(i-1,j-1)} + p_{(i+1,j-1)} + p_{(i-1,j+1)} + p_{(i+1,j+1)} - 4p_{(i,j)}}{2h^2} = 0. \tag{85}$$

This scheme has a well known problem [WH07]. Indeed when the initial condition is a Dirac mass, this scheme does not converge. The reason is that there is a decoupling between two meshes because the cells $(i+1,j)$, $(i-1,j)$, $(i,j+1)$ and $(i,j-1)$ are never used by the scheme. That is the numerical solution of this scheme may be plagued with spurious modes.

Notice however that the Dirac mass is not a regular function so that the hypotheses of the convergence theorem are not fulfilled.

6 Numerical results

6.1 Numerical results for the diffusion scheme

In this section we give numerical results for the limit diffusion scheme (26) on some structured and unstructured meshes. We first compare our scheme with the VF4 scheme on a Cartesian mesh. Then we look at the behavior of schemes on the meshes described in figures 6, 7 and 8.

6.1.1 Test case: Fundamental solution

At first, we study a test case with a smooth solution. The initial condition is the fundamental solution of the heat equation at t=0.001. The final time is $T_f = 0.011$. In this test case we ignore the boundary condition because the solution is equal to zero near to $\partial\Omega$.

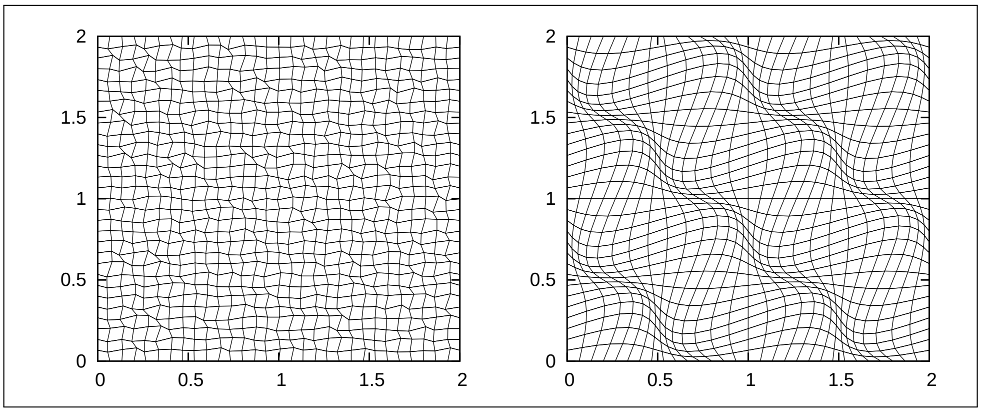

Figure 6: Unstructured quadrangular meshes

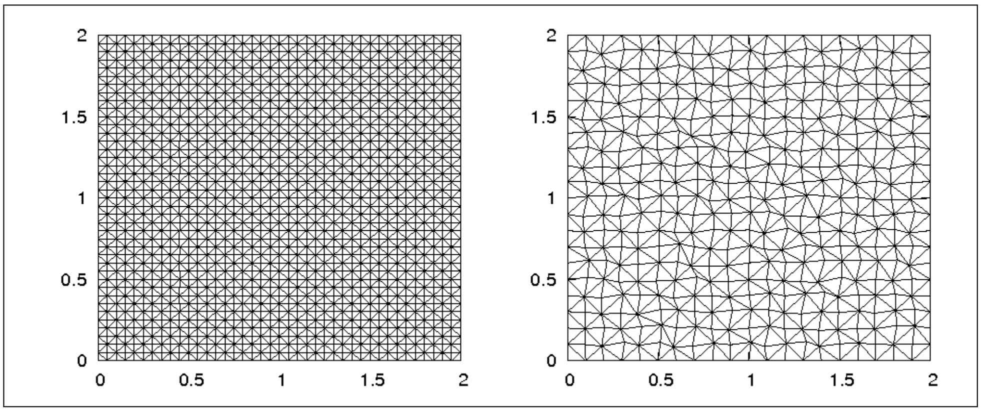

Figure 7: Triangular meshes

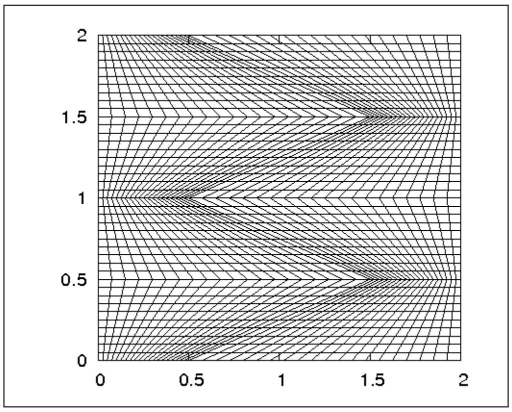

Figure 8: Kershaw mesh mesh

We remark on 9 that, on a Cartesian mesh the VF4 scheme and for the two variants $\widehat{\beta}_{jr}$ to the scheme defined by (26). (which are identical on a Cartesian mesh) are **convergent at order 2** in the L^1 and L^2 norm. The VF4 scheme has a better constant that the nodal scheme.

For the unstructured meshes we observe that the VF4 scheme doesn't converge, figure 10.This

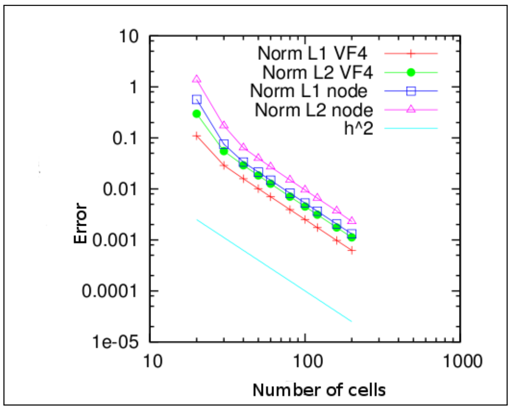

Figure 9: Error for the schemes VF4 and nodal scheme (26) on a Cartesian mesh.

phenomenon is a classical one for VF4 on random meshes. It is the reason why the Jin-Levermore procedure introduced in a standard finite volume scheme with fluxes evaluated on the edges does not generate a asymptotic preserving scheme. It justifies the use of the nodal scheme.

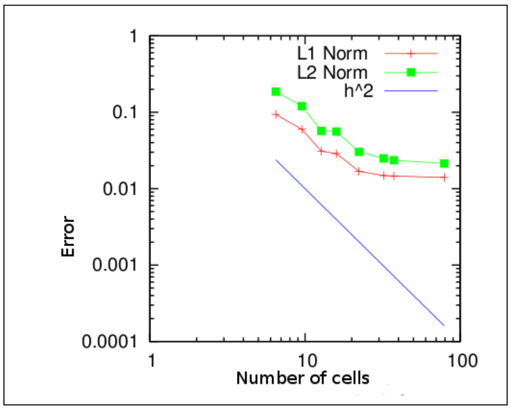

Figure 10: Error for the scheme VF4 on a random mesh of figure 6.

For the nodal scheme the convergence and the order depend on the mesh. On the figure 11 we note that the first variant $\widehat{\beta}^{I}_{jr}$ **converges with order 2** on the random quadrangular mesh. The second variant $\widehat{\beta}^{II}_{jr}$ does not converge on this mesh.

In the tables (1) and (2) we give the order of convergence in L^1 norm for $\widehat{\beta}_{jr} = \mathbf{n}_{jr} \otimes (\mathbf{x}_r - \mathbf{x}_j)$ on different meshes. To evaluate the order we use the following formula. Let h_1, h_2 be two characteristic length of the mesh and let e_{h_1}, e_{h_2} be the corresponding errors. The numerical order of convergence is defined by the formula

$$q = \frac{\log\left(e_{h_1}/e_{h_2}\right)}{\log\left(h_1/h_2\right)},$$

where $h_1 = 100$ and $h_2 = 200$ in practice. The results are given in tables 1, 2. We will use the same method for the other test problem in tables 3 and 4.

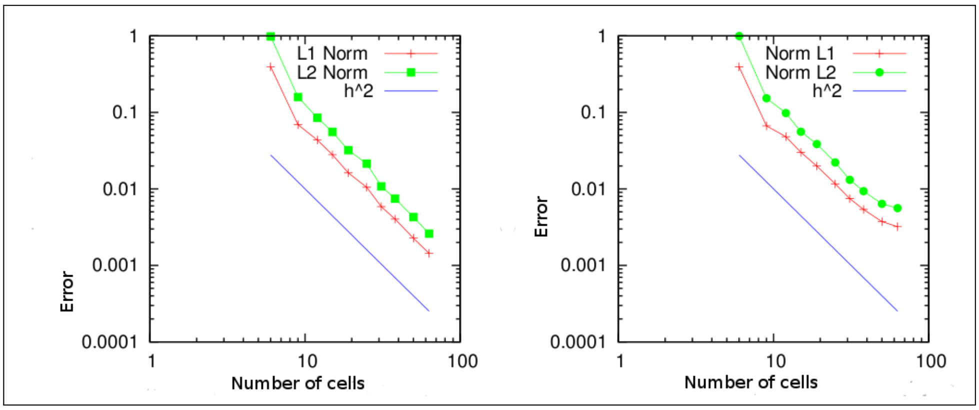

Figure 11: Error for the nodal scheme with $\widehat{\beta}^I_{jr}$ on the left and $\widehat{\beta}^I_{jr}I$ in the right on deformed Cartesian mesh (1)

mesh	Cartesian mesh	random quad. mesh	smooth quad. mesh	Kershaw mesh
q	2.00	1.98	2.01	2.00

Table 1: Order of error of diffusion schemes on quadrangular meshes.

mesh	regular triang. mesh	modified regular triang. mesh	random triang. mesh
q	2.00	0.98	1.32

Table 2: Order of error of diffusion schemes on triangular meshes. The regular triangular mesh is displayed on the left of figure 7. The modified regular triangular mesh is constructed from a Cartesian mesh where all squares are subdivided in the same direction contrary to the regular triangular mesh.

6.1.2 Test case with Neumann condition

Now we study the following test case. $u_0(x, y) = \cos(\pi x)\cos(\pi y)$, $\Omega =]0, 2[\times]0, 2[$. The solution with Neumann boundary condition is

$$u(t, x, y) = \exp(-2\pi^2 t)\cos(\pi x)\cos(\pi y)$$

in the tables 3 and 4 we present the order of convergence for this test case on different meshes.

mesh	Cartesian mesh	random quad. mesh	smooth quad. mesh	Kershaw mesh
q	2.08	2.07	2.8	2.85

Table 3: Order of error of diffusion schemes on quadrangular meshes.

mesh	regular triang. mesh	modified regular triang. mesh	random triang. mesh
q	2.23	1.02	1.25

Table 4: Order of error of diffusion schemes on triangular meshes.

6.2 Numerical results in the transport regime

In this section we check numerically that JL-(a) and JL-(b) are accurate in the case $\varepsilon = 1$. The system (3) is equivalent to the following scalar equation.

$$\begin{cases} \partial_{t^2} f + \partial_t f - \triangle f = 0, \\ f(t=0) = f_0(x,y), \qquad \partial_t f(t=0) = f_1(x,y). \end{cases} \tag{86}$$

So we consider the initial data

$$f_0(x,y) = \cos(\pi x)\cos(\pi y) \qquad f_1(x,y) = 0.$$

The solution is

$$f(t,x,y) = e^{-\frac{1}{2}t}\left(\cos\left(\frac{\sqrt{8\pi^2-1}}{2}t\right) - \frac{1}{\sqrt{8\pi^2-1}}\sin\left(\frac{\sqrt{8\pi^2-1}}{2}t\right)\right)\cos(\pi x)\cos(\pi y).$$

For the implementation we define $p = \partial_t f + f$ and $\mathbf{u} = -\nabla f$, so that :

$$\begin{cases} \partial_t p + \frac{1}{\varepsilon}\nabla.\mathbf{u} = 0 \\ \partial_t \mathbf{u} + \frac{1}{\varepsilon}\nabla p = -\frac{\sigma}{\varepsilon^2}\mathbf{u} \end{cases} \tag{87}$$

The boundary condition of the problem writes $\mathbf{u}.\mathbf{n} = 0$ for (87). The boundary condition for the GLACE scheme are detailed in [Klu08].

For this test case we use all the previous meshes. We plot the error for the scheme with different choice of the tensors $\widehat{\alpha}_{jr}$ in the figure 12. For each mesh we compare the schemes JL-(a), and JL-(b) and the edge Jin-Levermore scheme (13-18).

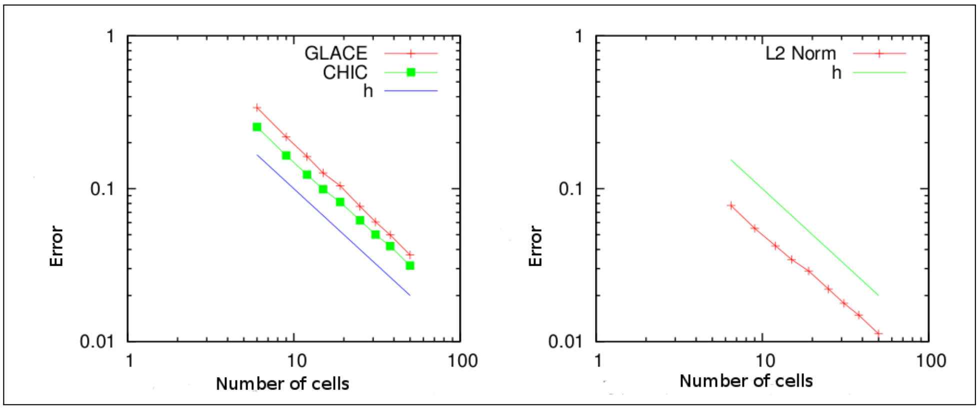

Figure 12: Error L^2 for the nodal scheme in the left and the edge scheme in the right, on random quadrangular mesh.

For all the variants we note that the schemes **converge at the order 1** on the various meshes as is expected for smooth solutions.

6.3 The asymptotic regime

For the numerical test of this section, we use an implicit scheme to avoid any dependence of the CFL condition with respect to ε. We consider that $\varepsilon = O(10^{-3})$ corresponds to the asymptotic regime. We compare the numerical solution with the solution of the two previously diffusion test cases. The numerical observance is the following: **the schemes JL-(a) and JL-(b) converge at an order which is close to 2 when ε tends to 0** (figure 13).

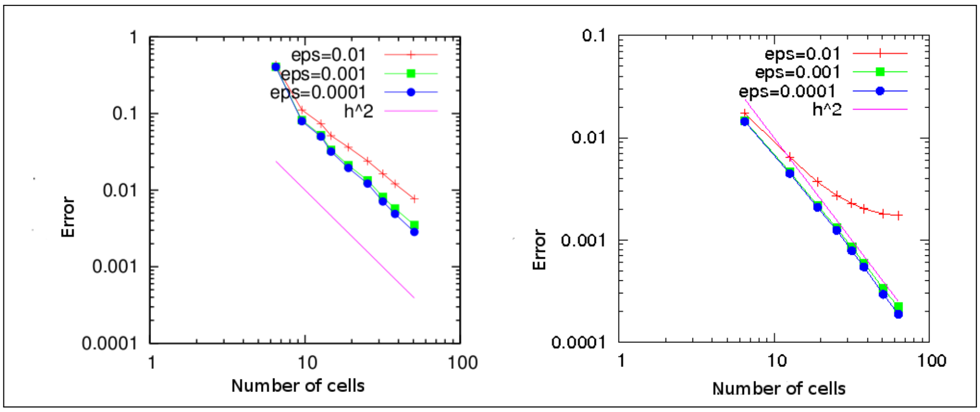

Figure 13: Error between the exact solution of the diffusion equation and the numerical solution of the telegraph equation for various $\varepsilon > 0$. We use the JL-(a) scheme on random quadrangular mesh. The results on the left correspond to the test case described in subsection 6.1.1 for a time such that all solutions are negligeable at the boundary. The results on the right correspond to the test case described in subsection 6.1.2, which is non zero at the boundary. We observe that the error is very small for sufficiently small ε.

We obtain qualitatively the same results for all meshes. In the diffusion regime JL-(a) and JL-(b) are equivalent, for this reason, we only show the results only for the scheme JL-(a).
An other interesting study, is the case of h fixed and ε tends to 0. We choose a random quadrangular mesh with 100 cells in each direction. The results are displayed in table 5.

ε	0.01	0.001	0.0001	0.00001	diffusion
error L1	0.016401	0.00813369	0.00710269	0.00699941	0.00698787
error L2	0.0313253	0.01511494	0.0132377	0.0130463	0.0130249

Table 5: Error between the exact solution of the diffusion equation and the numerical solution of the telegraph equation for a given mesh size $\Delta x \approx \frac{1}{100}$. The scheme is the Jl-(a) one. We also plot in the last column the error between the exact and the numerical solution of the diffusion equation on the same mesh. Since the two last columns are almost identical, we infer that the error is principaly numerical.

7 Conclusion and perspectives

We began this work with the analysis of two asymptotic preserving cell centered schemes in dimension 1 for the telegraph equation: the Jin-Levermore scheme and the Gosse-Toscani scheme. We showed that by modfying the discretization of the source term in the Jin-Levermore scheme we recover the Gosse-Toscani scheme and that in terms of stability and convergence they are suitable schemes.

Then we turn to the 2D case which is our main purpose. We observed that the Jin-Levermore technique may generate schemes which cannot be asymptotic preserving on unstructured meshes. This behavior is a new phenomenon in 2D. We proposed to modify the underlying Finite Volume scheme and to use a corner base Finite Volume Scheme. It solves the obstruction problem characteristic of standard edge Finite Volume methods. The numerical tests, the study of invertibility of A_r and the proof of convergence for the limit diffusion scheme show that the scheme is effective on a lot of unstructured meshes. The numerical tests also show that the schemes JL-(a) and

JL-(b) are asymptotic preserving on unstructured meshes for the telegraph equation. A natural extension of this work would be to construct an asymptotic preserving scheme for the M^1 model and to study and control the spurious modes that may appear on Cartesian meshes.

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