Asymptotic preserving scheme for Euler system with large friction

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Abstract: We construct a 'well-balanced' and 'asymptotic preserving' scheme for the approximation of the model problem of gas dynamics equations with gravity and friction. The friction terms we consider are quite general. We interpret our simple Riemann solver in such a way that the expected properties are directly inherited from the properties of the system of PDEs which is approximated.

1 Introduction

We are interested in deriving schemes having some 'well-balanced' and 'asymptotic preserving' properties for the model problem of gas dynamics equations with gravity and friction. This is a system of balance laws which writes in Eulerian coordinates

$$\left\{\begin{array}{l} \partial_t \varrho + \partial_x(\varrho u) = 0, \\ \\ \partial_t(\varrho u) + \partial_x(\varrho u^2 + p) = \varrho(g - \alpha\varphi(u)), \\ \\ \partial_t(\varrho e) + \partial_x((\varrho e + p)u) = \varrho(gu - \alpha\psi(u)). \end{array}\right. \tag{1}$$

The left-hand side is the classical Euler system, with ϱ the density, the energy e satisfies $e = \varepsilon + u^2/2$, ε is the internal energy and the pressure law $p = p(\varrho, \varepsilon) = \tilde{p}(\tau, \eta)$ is a given function expressed either in terms of (ϱ, ε) or in terms of (τ, η) (with $\tau = 1/\varrho, \eta$ mathematical entropy), satisfying some usual assumptions. In the right-hand side source terms, $\varphi(u)$ and $\psi(u)$ model friction terms and $\alpha > 0$ is some coefficient which can become very large; g is a gravity constant. The functions $\varphi(u)$ and $\psi(u)$ satisfy $\varphi(0) = \psi(0) = 0, \psi'(0) = 0$, φ increasing. In a previous paper ([7]), we have mainly considered the commonly used friction terms

$$\left\{\begin{array}{l} \varphi(u) = |u|^\chi u, \\ \psi(u) = a|u|^{\chi+2}, \quad 0 \leq a \leq 1, \end{array}\right. \tag{2}$$

with $\chi \geq 0$, $\chi = 0$ for a linear friction or $\chi = 1$ for a quadratic friction term and $a = 1$, in which case $\psi(u) = \varphi(u)u$. Here, we will extend this work to the case $\psi(u) = a\varphi(u)u$ with a constant $a < 1$, and surprisingly, we will

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see that the consequences are not negligible. The first consequence is an entropy dissipation since we notice that smooth solutions of (1) satisfy

$$\partial_t \varrho\eta + \partial_x \varrho\eta u = -\alpha \frac{\varrho}{T}(u\varphi(u) - \psi(u)), \tag{3}$$

where $T = -\partial_\eta \tilde{\varepsilon}(\tau, \eta)$ is the temperature.

Well-balanced schemes have been introduced in [12]; we precise the useful notions for our model example. The problem of deriving schemes that work uniformly well with respect to a parameter which can become large has also often been addressed to. Let us emphasize that the question is not only linked to the presence of stiff source terms since, as we will see, a scaling wrt. the time variable is also introduced, so that the limit behavior is governed by a reduced system (see [19], [13], [14]). We mention several recent works related to the subject of preserving equilibrium and asymptotic properties in the case of a diffusive limit system: first we mention the pioneer papers [15] and [17], which underline the importance of the asymptotics and analyze the problem in the semi-discrete setting, and [16], [18], [20]. More recently, [3] considers the same model problem as we do, but in the barotropic case and without gravity. We also mention some contributions for deriving asymptotic preserving and well balanced schemes for other model equations such as [16] for multiscale kinetic equations, [10] for discrete kinetic models, for radiative transfer models: [11], [5] [4], [1].

In [7], we have developed a Godunov-type scheme having these 'well-balanced' and 'asymptotic preserving' properties for (1). If the presentation was quite general, some proofs assumed specific friction terms ($a = 1$) involving no entropy dissipation. The simple Riemann solver, constructed in the spirit of [9], can also be interpreted using a relaxation approach (cf. [8], and see [2]) together with the idea in [6] where the source term is transformed into a differential one. This is the way we will present the scheme below.

First, in section 2, we will recall what we mean by *well-balanced* and *asympotic preserving* property for the approximation of a nonlinear hyperbolic system of balance laws

$$\partial_t \mathbf{U} + \partial_x \mathbf{F}(\mathbf{U}) = \mathbf{S}(\mathbf{U}, \alpha), \quad x \in \mathbb{R}, t > 0, \tag{4}$$

$\mathbf{u}(x,t) \in \Omega \subset \mathbb{R}^n$, with a source term depending on some parameter α (associated in our case to the friction coefficient). Then in section 3, we present the Riemann solver and its extension to a general friction term, using an operator splitting technique. Some numerical illustrations follow.

2 Equilibria. Asymptotic behavior

First, we say that a scheme is *well-balanced* if it preserves, in some sense which has to be precised, *stationary* solutions of (4) which by definition satisfy

$$\partial_t \mathbf{U} = 0 \Rightarrow \partial_x \mathbf{F}(\mathbf{U}) = \mathbf{S}(\mathbf{U}). \tag{5}$$

It results from the detailed study in [7], that the classical equilibria for (1) solve

$$\begin{cases} u = 0, \\ \frac{dp}{dx} = \varrho g, \end{cases} \tag{6}$$

and the gravity term balances the pressure gradient. In the case $a < 1$, we can also find stationary solutions satisfying $u = \overline{u} \neq 0$ constant (thus $\varrho = \overline{\varrho}$) but with ε varying in the domain (see [7]). If we can derive a method preserving these equilibria, we have not been able to prove that the resulting scheme satisfies good stability properties (such as entropy dissipation). We will thus focus in the next section on the natural equilibria (6), which our method will indeed preserve: if $u = 0$ and $\frac{dp}{dx} = \varrho g$ is discretized in a consistent precise way, then the discrete solution (say $\mathbf{u}_i^n$, notation introduced below) is stationary ($\mathbf{u}_i^n = \mathbf{u}_i^0, \forall n$).

Up to what concerns the asymptotic property, we study the long time asymptotic behavior of solutions of (4) as the coefficient α goes to infinity. Assuming that after some scaling of $\mathbf{U} \to \mathbf{V}$ and $t \to s$ wrt to α, the scaled variable $\mathbf{V}(s, x)$ satisfies as $\alpha \to \infty$ some reduced system of PDE, we expect the numerical solution to micmic this behavior. Concerning (1), performing first a formal asymptotic expansion in powers of $\frac{1}{\alpha}$ yields that it is natural to assume that $u \to 0$ as $\alpha \to \infty$. Thus the scaling affects both the velocity and the time variable. Setting $\beta = \alpha^{\frac{1}{\chi+1}}$, we define

$$t = \beta s, \quad v_\beta(x, s) = \beta u(x, t), \ \varrho_\beta(x, s) = \varrho(x, t), \varepsilon_\beta(x, s) = \varepsilon(x, t). \tag{7}$$

Plugging these new set of variables into (1), we get that (ϱ, ε) satisfies as $\beta \to \infty$ (at first order wrt. $1/\beta$)

$$\begin{cases} \partial_s \varrho + \partial_x(\varrho v) = 0, \\ \\ \partial_s(\varrho\varepsilon) + \partial_x((\varrho\varepsilon + p)v) = \varrho(gv - \psi(v)) \end{cases} \tag{8}$$

where $v = \varphi^{-1}(g - \frac{1}{\varrho}\partial_x p)$, $p = p(\varrho, \varepsilon)$ (we have omitted the index $\beta = \infty$). We will not extend over the theoretical issues. In the barotropic case, there are several existing theoretical results [19], [13], [14] justifying the formal analysis above. We will focus on the counterpart of the previous property at the numerical level, which means that we want the scheme after the same scaling to be consistent with (8).

3 Scheme preserving the discrete properties

3.1 Treatment of the source term

In order to ensure the well-balanced property, we follow an idea of [6] and first introduce a so called 'potential' q such that

$$\begin{cases} \partial_x q = \varrho \\ \partial_t q = -\varrho u. \end{cases} \tag{9}$$

Then we can write (1) in an augmented non-conservative *homogeneous* form

$$\begin{cases} \partial_t \varrho + \partial_x(\varrho u) = 0, \\ \partial_t(\varrho u) + \partial_x(\varrho u^2 + p) - (g - \alpha\varphi(u))\partial_x q = 0, \\ \partial_t(\varrho e) + \partial_x((\varrho e + p)u) - (gu - \alpha\psi(u))\partial_x q = 0, \\ \partial_t(\varrho q) + \partial_x(\varrho u q) = 0. \end{cases} \tag{10}$$

The set of states for (10) is $\Omega_q^{Euler} = \{(\varrho, \varrho u, E = \varrho e, q); \varrho > 0, u \in \mathbb{R}, e - u^2/2 > 0, q \in \mathbb{R}\}$. Let us first recall one important property of system (10).

Lemma 3.1 System (10) has four real eigenvalues $u - c, u, u, u + c$, where c is the usual sound speed ($c^2 = \frac{\partial p}{\partial \varrho}(\varrho, \eta)$) and it has a basis of eigenvectors if and only if

$$\psi(u) = u\varphi(u) \tag{11}$$

or for the state $u = 0$. The first and last characteristic fields associated to $u \pm c$ are genuinely nonlinear, while the characteristic field associated to u is linearly degenerate.

The proof is detailed in [7], the result shows that we cannot apply directly the technique in the case $a < 1$. Thus, we will split the source terms in two parts, following the decomposition $\psi = u\varphi + (a-1)u\varphi$, and apply the strategy of [7] in a first time step to (1) with a classical friction term (i.e. where only ψ changes, replaced by $u\varphi$). In a second step, we are left with the need of approximating the (simple) system of ODE

$$\begin{cases} \partial_t \varrho = 0, \\ \partial_t(\varrho u) = 0 \\ \partial_t(\varrho e) = \alpha\varrho(1-a)\varphi(u). \end{cases} \tag{12}$$

We first recall the scheme constructed for the first step. Note that (9) is invariant under the scaling (7) so that the long time asymptotic behavior of the solutions of (10) and (1) are alike, moreover, except for what concerns the potential q, the two systems have the same solutions (see again [7]).

3.2 Scheme for classical friction

We recall briefly the construction of the approximate Riemann solver associated to the formulation with potential (10) to define a well balanced scheme in the case $\psi = u\varphi$. Our approximate solver involves an *exact* explicit Riemann solver for a larger relaxation system built from (10), with only linearly degenerate (LD) fields. At least from a heuristic point of view, the well balanced and asymptotic preserving property are directly derived from this construction since we use exact solutions, respecting Rankine-Hugoniot conditions, and whose long time asymptotic behavior is known, together with a relaxation procedure which preserves the good properties. Indeed, we introduce a larger 5×5 system with a relaxation term in the right-hand side depending on a 'relaxation parameter' ν, which is meant to become arbitrarily large (so that the 'relaxation time' $\frac{1}{\nu}$ goes to 0)

$$\left\{\begin{array}{ll} \partial_t \varrho + \partial_x(\varrho u) & = 0, \\ \partial_t(\varrho u) + \partial_x(\varrho u^2 + \Pi) - (g - \alpha\varphi(u))\partial_x q & = 0, \\ \partial_t(\varrho\eta) + \partial_x(\varrho\eta u) & = 0, \\ \partial_t(\varrho\mathcal{T}) + \partial_x(\varrho\mathcal{T} u) & = \nu\varrho(\tau - \mathcal{T}), \\ \partial_t(\varrho q) + \partial_x(\varrho u q) & = 0. \end{array}\right. \tag{13}$$

Following [8], the closure relation for Π is taken in the form

$$\Pi = \tilde{\Pi}(\tau, \eta, \mathcal{T}) \equiv \tilde{p}(\mathcal{T}, \eta) + \overline{c}^2(\mathcal{T} - \tau), \tag{14}$$

where $\overline{c}$ is a positive constant satisifying the Whitham (or subcharacteristic) condition

$$\overline{c}^2 > -\partial_\tau \tilde{p}(\tau, \eta), \tag{15}$$

for all the values τ, η under consideration. Formally, as the relaxation parameter $\nu \to \infty$, $\mathcal{T} \to \tau$, and $\tilde{\Pi}(\tau, \eta, \mathcal{T}) \to \tilde{p}(\tau, \eta) = p$ and the system relaxes to Euler system with friction written in a formulation with a potential, and where the energy equation has been replaced by the entropy one $\partial_t \varrho\eta + \partial_x(\varrho\eta u) = 0$ (see (3)). Given constant data $\mathbf{U}_{\ell,r} = (\varrho, \varrho u, \varrho\eta, \varrho q)^T_{\ell,r}$, the solution of the Riemann problem for the homogeneous system (13) with $\nu = 0$, say $\mathbf{W}_q(\frac{x}{t}; \mathbf{U}_\ell, \mathbf{U}_r)$ is known, with explicit formulae.

Following the standard finite volume approach, starting from data in variables $\mathbf{u} = (\varrho, \varrho u, \varrho e)^T$, they are first discretized on the intervals $C_j = (x_{j-1/2}, x_{j+1/2})$, which gives $\mathbf{u}_j^0$ then updated following several steps. The first step of the relaxation scheme reconstructs piecewise constant data $(\varrho, u, \eta, \mathcal{T})$ at *equilibrium* (relative to the relaxation procedure), then follow an exact evolution step and a projection step back on the *equilibrium* variety ($\mathcal{T} = \tau$). As for Godunov's method, we use an exact Riemann solver for the evolution step on a time interval with length Δt. It results in

a consistent scheme for $\mathbf{v} = (\varrho, \varrho u, \varrho\eta)^T$ which will not necessarily approximate a weak solution of (1) since the formulation of system (13) preserves the entropy $\varrho\eta$ while the total energy plays the role of a convex entropy. If the formula for the density and energy fluxes is obtained just as for Godunov's method from the integration on half cells of the exact Riemann solution, an additional step is required, involving the definition of a numerical energy flux. It also comes from the integration on half cells of the energy of the exact Riemann solution $\mathbf{W}_q(\frac{x}{\Delta t}; \mathbf{U}^n_{j-1}, \mathbf{U}^n_j)$. Eventually we get a simple finite volume scheme for updating the conservative variables $\mathbf{u} = (\varrho, \varrho u, \varrho e)^T$, consistent with (1), which can be written in the usual form

$$\mathbf{u}^{n+1}_j = \mathbf{u}^n_j - \mu \left(\mathbf{g}^n_{j+1/2-} - \mathbf{g}^n_{j-1/2+} \right), \quad j \in \mathbb{Z}, n \geq 0, \tag{16}$$

where $\mu = \frac{\Delta t}{\Delta x}$ and the $\pm$ sign in the numerical flux $\mathbf{g}^n_\pm$ accounts for the source terms. Moreover a discrete entropy inequality is satisfied

$$(\varrho\eta)^{n+1-}_j \leq (\varrho\eta)^n_j - \mu\big(\mathcal{G}^n_{\varrho\eta,j+1/2} - \mathcal{G}^n_{\varrho\eta,j-1/2}\big), \tag{17}$$

(where $\mathcal{G}^n_{\varrho\eta,j+1/2}$ is simply the third component of the exact Godunov flux, which is conservative in entropy). We skip the technical details here.

3.3 Splitting scheme for general friction

As already said, we use a splitting method with a specific splitting of the source. In a first step, the scheme updates $\mathbf{u}$, according to the modified source term $\psi = u\varphi$. In the second step, we solve (12) and because of the specific form of the system, only the internal energy will change and needs updating. Noting $(\varrho\varepsilon)^{n+1-}_j$ its value at the end of the first step, we set

$$(\varrho\varepsilon)^{n+1}_j = (\varrho\varepsilon)^{n+1-}_j + \alpha\frac{\Delta t}{2}(1-a)\varrho^n_j(\varphi(u^*_{j+1/2})1_{u^*_{j+1/2}<0} + \varphi(u^*_{j-1/2})1_{u^*_{j-1/2}>0}) \tag{18}$$

where u^* is the velocity of the intermediate state of the Riemann solution $\mathbf{W}_q$ defined at the first step, by a formula which involves φ. As expected, the internal energy increases, and when the energy increases, the entropy decreases (because $\partial_\eta\tilde{\varepsilon}(\tau,\eta) = -T > 0$ and $\tau = \varrho^{-1}$ has not changed). Thus, denoting $(\varrho\eta)^{n+1-}_j$ the value at the end of the first step, since $(\varrho\eta)^{n+1}_j \leq (\varrho\eta)^{n+1-}_j$, an entropy inequality still holds, we again skip the details. The resulting scheme exactly coincides with that described in section 4.3 of [7].

The scheme is *well balanced* in the sense that *classical equilibria* are exactly preserved: they are preserved by the first step (see Proposition 10 in section 5.3.4 of [7]), and because $u = 0$, the second step does not change anything.

Because of the specific choice of u^*, the scheme is *asymptotic preserving*. Roughly speaking, the first step preserves (after scaling $t = \beta s, v = \beta u$) the asymptotic behavior of solutions of (1) with classical friction because it uses exact solutions of the Riemann problem of a system that has the same asymptotic behavor as (1). The second step gives by the scaling at the continuous level $\partial_s(\varrho\varepsilon) = \varrho(1-a)v\varphi(v)$, and at the discrete level, the scheme appears as a splitting scheme for (8) when the soure is split according to $\varrho(gv - \psi(v) = \varrho v(g - \varphi(v)) + \varrho(1-a)v\varphi(v)$. The computations are detailed in the appendix of [7].

4 Illustration

We give some numerical illustrations of the relevance of the well-balanced scheme we have proposed in the previous sections. Its capabilities are grounded when comparing its performances to the ones of a classical splitting approach which fails to be asymptotic preserving. The latter consists in two steps, respectively made of the homogeneous Euler equation in a first step and the following ODE system dealing with all the source terms:

$$\left\{ \begin{array}{l} \partial_t \varrho = 0, \\ \partial_t(\varrho u) = \varrho(g - \alpha\varphi(u)), \\ \partial_t(\varrho e) = \varrho(gu - \alpha\psi(u)), \end{array} \right. \tag{19}$$

in a second step. The first step is solved thanks to the homogeneous form of the relaxation model (13). Then the second step is given the following Euler backward approximation in order to avoid any CFL like restriction:

$$\rho_j^{n+1} = \rho_j^{n+1-}, \ (\rho u)_j^{n+1} = \frac{(\rho u)_j^{n+1,-} + \Delta t \rho_j^{n+1} g}{1 + \alpha \Delta t}, \tag{20}$$

and

$$(\rho\varepsilon)_j^{n+1} = (\rho\varepsilon)_j^{n+1,-} + \alpha(1-a)\Delta t \frac{\left((\rho u)_j^{n+1}\right)^2}{\rho_j^{n+1}}, \tag{21}$$

so as to define $(\rho e)_j^{n+1} = (\rho\varepsilon)_j^{n+1} + \left((\rho u)_j^{n+1}\right)^2/(2\rho_j^{n+1})$. Formula (21) comes naturally from the discretization of the equation satisfied by ε: $\partial_t \varrho\varepsilon = \alpha(1-a)\frac{(\varrho u)^2}{\varrho}$ which is easily deduced from (19). Moreover, formula (21) ensures that the positivity of the internal energy is preserved.
The Cauchy problem we consider for the system (1) makes use of a polytropic gamma law with $\gamma = 1.4$. We consider 1-periodic solutions over the interval $[0, 1]$. All the quantities of interest hereafter are given dimensionless forms. The initial data is made of three constant states we express in

the primitive variable $\mathbf{v} = (\varrho,\ u,\ p)^T$ by:

$$\mathbf{v}_0(x) = \begin{cases} (1,\ 0,\ 2), & 0 < x < 1/3, \text{ and } 2/3 < x < 1, \\ (1,\ 0,\ 1), & 1/3 < x < 2/3. \end{cases} \tag{22}$$

The constant coefficients χ and a entering the friction law (2) are respectively set to $\chi = 0$ and $a = 1/2$, the gravity coefficient reads $g = -0.981$ while the friction coefficient α is given the large value $\alpha = 1000$.
For the computations, we use 4 grids built from a constant space step given respectively by $\Delta x = 1/100, 1/200, 1/400, 1/800$. The CFL number is set to the constant value 0.5. The approximate solutions, performed thanks to our well-balanced and asymptotic preserving scheme (noted AP in the figures) and the classical splitting approach (noted Sp), are systematically compared to a reference solution (noted Ref) obtained by the AP-algorithm on an uniform mesh using $\Delta x = 1/1600$. The density profiles are displayed at the physical time $T = 2$ and compared in Figures 1. The AP scheme behaves as expected: the results are already fairly satisfactory using $\Delta x = 1/100$ and the agreement in between the solution with $\Delta x = 1/400$ and the reference one is excellent. By contrast, the convergence rate of the solutions obtained via the classical splitting approach is rather poor, essentially due to a numerical diffusion badly shaped wrt. α. The solution obtained using $\Delta x = 1/800$ is indeed closer to the reference one but mesh refinement is still needed to reach a fully convincing agreement. Similarly, in Figures 2 and 4 we compare the schemes on the velocity and pressure profiles, on the finer and coarser grids.

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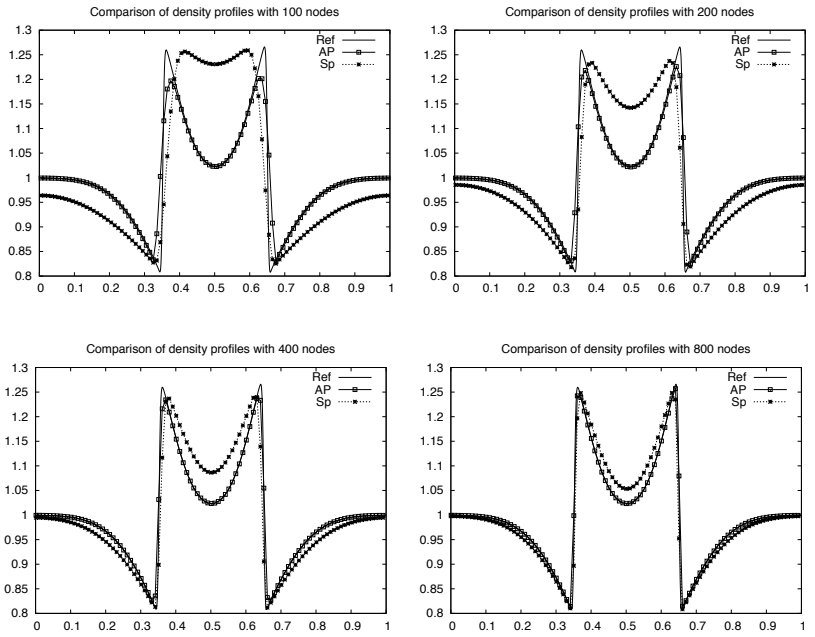

Figure 1: Comparison of the AP and Sp schemes (with 100, 200, 400, 800 nodes) on the density profiles

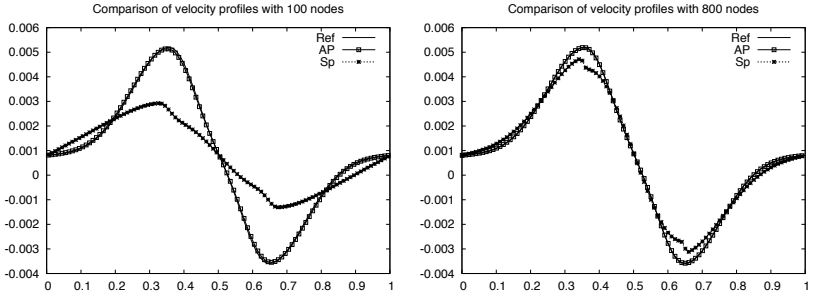

Figure 2: Comparison of the AP and Sp schemes (with 100 and 800 nodes) on the velocity profiles

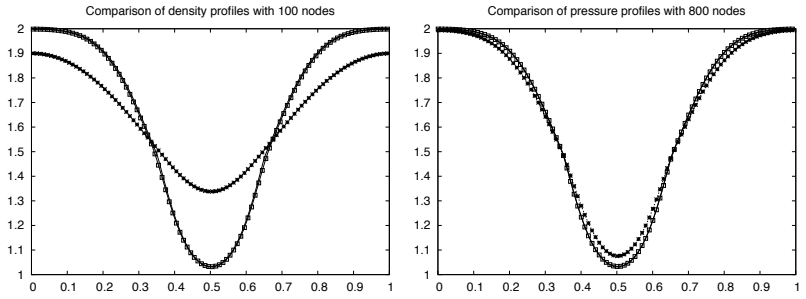

Figure 3: Comparison of the AP and Sp schemes (with 100 and 800 nodes) on the pressure profiles

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