A TWO-GRADIENT APPROACH FOR PHASE TRANSITIONS IN THIN FILMS

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Abstract. Motivated by solid-solid phase transitions in elastic thin films, we perform a Γ-convergence analysis for a singularly perturbed energy describing second order phase transitions in a domain of vanishing thickness. Under a two-wells assumption, we derive a sharp interface model with an interfacial energy depending on the asymptotic ratio between the characteristic length scale of the phase transition and the thickness of the film. In each case, the interfacial energy is determined by an explicit optimal profile problem. This asymptotic problem entails a nontrivial dependance on the thickness direction when the phase transition is created at the same rate as the thin film, while it shows a separation of scales if the thin film is created at a faster rate than the phase transition. The last regime, when the phase transition is created at a faster rate than the thin film, is more involved. Depending on growth conditions of the potential and the compatibility of the two phases, we either obtain a sharp interface model with scale separation, or a trivial situation driven by rigidity effects.

Keywords: phase transitions, thin films, singular perturbations, double-well potential, Γ-convergence

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1. Introduction

In the last few years, many mathematical efforts have been devoted to variational problems arising in the modelling of phase transitions in solids, see *e.g.* [6,16,17,18]. These problems often involve singularly perturbed functionals of the form

$$\mathbf{E}_\varepsilon(\mathbf{u},\mathscr{B}) = \int_{\mathscr{B}} \frac{1}{\varepsilon}\, W(\nabla \mathbf{u}) + \varepsilon |\nabla^2 \mathbf{u}|^2 \, d\mathbf{x}\,, \tag{1.1}$$

where $\mathbf{u} : \mathscr{B} \subset \mathbb{R}^3 \to \mathbb{R}^3$ represents the displacement of an elastic body $\mathscr{B}$, $\varepsilon > 0$ is a small parameter, and W is a (nonnegative) free energy density with multiple minima corresponding to martensitic materials. Due to the multiple well structure, nucleation of phases in a given configuration may occur without increasing $\int W(\nabla \mathbf{u})$, so that the free energy may admit many (eventually constrained) minimizers. In order to select preferred configurations, the Van der Waals-Cahn-Hilliard theory adds higher order terms leading to functionals of the form (1.1). In such functionals a competition occurs between the two terms: the free energy favors gradients close to a minimum value of W, while $|\nabla^2\mathbf{u}|^2$ penalizes transitions from one minima to another.

The Γ-convergence method provides a suitable framework to study the asymptotic behavior of singularly perturbed energies like $\mathbf{E}_\varepsilon$ (see *e.g.* [12,19] for a more detailed overview of this subject). One of the first applications of Γ-convergence was actually obtained in [30,31,35] in the context of fluid-fluid phase transitions (see *e.g.* [25]). Here the authors deal with energy functionals of the form $\int \frac{1}{\varepsilon}\, W(v) + \varepsilon|\nabla v|^2\, dx$ where the potential W has a double well structure, *i.e.*, $\{W = 0\} = \{\alpha, \beta\}$. It is shown that such family of energies Γ-converges (in a suitable topology) as $\varepsilon \to 0$ to a functional which calculates the area of the interface between the two phases α and β, for limiting BV-functions v with values in $\{\alpha, \beta\}$. Since then this result has been generalized in many different ways (see *e.g.* [1,5,7,10,24,32]), in particular in [23] for an intermediate situation where the singular perturbation $|\nabla v|^2$ is replaced by the higher order term $|\nabla^2 v|^2$.

The first Γ-convergence result for functionals acting on gradient vector fields has been obtained in [16]. Assuming that $\{W = 0\} = \{A, B\}$ for some rank-one connected matrices A and B (and some additional constitutive conditions on W), the authors prove the Γ-convergence of $\mathbf{E}_\varepsilon$ as $\varepsilon \to 0$. Once again the effective functional returns the total area of the interfaces separating the phases A and B, for limiting functions $\mathbf{u}$ satisfying $\nabla\mathbf{u} \in \{A, B\}$ a.e. and $\nabla\mathbf{u} \in BV$. Here the rank-one connection between the wells A and B turns out to be necessary for the existence of non-affine $\mathbf{u}$'s satisfying $\nabla\mathbf{u} \in \{A, B\}$, and the interfaces must be planar and oriented according to the connection, see [6]. However the two-wells assumption on W is quite unrealistic for the modelling of martensitic materials. Indeed, to take into account the frame-indifference, one should assume instead that the zero level set of W is of the form $SO(3)A \cup SO(3)B$. Due in particular to compactness issues, this situation is much more involved, but [16] still represents an important step towards a better understanding of the $SO(3)$-invariant problem. In two dimensions the Γ-convergence of frame-indifferent functionals has been recently solved in [17] (see also [18] in the context of linear elasticity).

Another topic of increasing interest related to solid mechanics concerns thin elastic films, that is the derivation of limiting models starting from 3D nonlinear elasticity in domains of vanishing thickness. Also in this context, the Γ-convergence method provides a suitable framework as it has been shown in [28,13] for the membrane theory, and more recently in [21,22] for nonlinear plate models. In the regime of membranes, several studies have focused their attention on the impact of a higher order perturbation on the behavior of thin films. The first variational approach has been addressed in [9] where the authors add the perturbation $\varepsilon^2 \int |\nabla^2\mathbf{u}|^2$ to the free energy $\int W(\nabla\mathbf{u})$ for a domain of small thickness h. They obtain in the limit $h \to 0$ a 2D energy density which depends on the deformation gradient of the mid-surface, and the Cosserat vector b which gives an asymptotic description of the out-of-plane deformation. An important consequence of the results of [9] is that for many interesting materials the low energy states in the thin film limit are different from the ones in three dimensional samples. However [9] does not treat possible correlations between the thickness h and the parameter ε. This issue was first conducted in [34], and more intensively in [20] to keep track of the Cosserat vector. It is shown in [20] that the limiting model is determined by the asymptotic ratio h/ε as $h \to 0$ and $\varepsilon \to 0$, and it depends whether $h/\varepsilon \sim 0$, $h/\varepsilon \sim \infty$, or $h/\varepsilon \sim 1$.

The general idea of this paper is to study a class of singularly perturbed functionals describing phase transitions in thin films. In this direction, some models have been recently analyzed, see [8], and also [15,26] for models without singular perturbation leading to sharp interfaces. Here we want to carry out an analysis in the spirit of [16] including correlations between the strength of an interfacial energy and the thickness of the film. As in [9,20,34] we consider a *"membrane scaling"*, and we introduce the normalized functional $\mathbf{F}^h_\varepsilon$ defined for $\mathbf{u} \in H^2(\Omega_h; \mathbb{R}^3)$ by

$$\mathbf{F}^h_\varepsilon(\mathbf{u}) := \frac{1}{h}\int_{\Omega_h} W(\nabla\mathbf{u}) + \varepsilon^2|\nabla^2\mathbf{u}|^2\, d\mathbf{x}\,,$$

where $\Omega_h := \omega \times hI \subset \mathbb{R}^3$, $I := (-\frac{1}{2}, \frac{1}{2})$, and the midsurface $\omega \subset \mathbb{R}^2$ is a bounded, connected, Lipschitz open set. Considering configurations $\mathbf{u}$ with energy of order ε, that is $\mathbf{F}^h_\varepsilon(\mathbf{u}) \leqslant O(\varepsilon)$, and renormalizing by $1/\varepsilon$ we are led to the energy $\frac{1}{h}\mathbf{E}_\varepsilon$ in the thin domain Ω_h. We are interested in the variational convergence of $\{\frac{1}{h}\mathbf{E}_\varepsilon(\cdot, \Omega_h)\}$ as $h \to 0$ and $\varepsilon \to 0$. To this aim, we introduce the standard rescaling

$$u(x) = \mathbf{u}(\mathbf{x}) \text{ with } (x_1, x_2, x_3) = (\mathbf{x}_1, \mathbf{x}_2, \frac{\mathbf{x}_3}{h})\,,$$

which yields functionals $\{F^h_\varepsilon\}$ defined for $u \in H^2(\Omega;\mathbb{R}^3)$ by

$$F^h_\varepsilon(u) := \int_\Omega \frac{1}{\varepsilon}\, W(\nabla_h u) + \varepsilon |\nabla_h^2 u|^2\, dx\,,$$

where $\Omega := \Omega_1$, and $\nabla_h := \big(\partial_1, \partial_2, \frac{1}{h}\partial_3\big)$ is the rescaled gradient operator.

The Γ-convergence as $\varepsilon \to 0$ and $h \to 0$ of the family $\{F^h_\varepsilon\}$ is the problem we would like to address. As a first attempt in this direction, we will assume some simplifying conditions. First we assume that

(H_0) ω is a bounded and *convex* open set of $\mathbb{R}^2$.

The other conditions concern the potential W which will be of *double-well* type as in [16], but with a relaxed condition on the connection between the wells similar to [6] saying that the two wells are compatible in the plane but not necessarily in the bulk. More precisely, we assume that $W : \mathbb{R}^{3\times 3} \to [0, +\infty)$ is continuous and satisfies the following assumptions:

(H_1) $\{W = 0\} = \{A, B\}$ where $A = (A', A_3) \in \mathbb{R}^{3\times 2} \times \mathbb{R}^3$ and $B = (B', B_3) \in \mathbb{R}^{3\times 2} \times \mathbb{R}^3$ are distinct matrices satisfying $A' - B' = 2a \otimes \bar\nu$ for some $a \in \mathbb{R}^3$ and $\bar\nu \in \mathbb{S}^1$;

(H_2) $\frac{1}{C_1}\,|\xi|^p - C_1 \leqslant W(\xi) \leqslant C_1\,|\xi|^p + C_1$, for $p \geqslant 2$ and some constant $C_1 > 0$.

Under this first set of assumptions, we shall derive compactness properties for sequences with uniformly bounded energy. In our setting the limiting configurations space turns out to be

$$\mathscr{C} := \big\{(u,b) \in W^{1,\infty}(\Omega;\mathbb{R}^3) \times L^\infty(\Omega;\mathbb{R}^3) : (\nabla' u, b) \in BV\big(\Omega;\{A,B\}\big)\,,\ \partial_3 u = \partial_3 b = 0\big\}\ , \tag{1.2}$$

where we write $\nabla' := (\partial_1, \partial_2)$. Throughout the paper we might identify pairs $(u, b) \in \mathscr{C}$ with functions defined on the mid-surface ω, that is $u(x) = u(x')$, $b(x) = b(x')$ with $x' := (x_1, x_2)$. In particular, for any $(u, b) \in \mathscr{C}$, we can write

$$(\nabla' u, b)(x') = \big(1 - \chi_E(x')\big)A + \chi_E(x')B \quad \text{for } \mathcal{L}^2\text{-a.e. } x' \in \omega\,, \tag{1.3}$$

where $E \subset \omega$ is a set of finite perimeter in ω, and χ_E denotes its characteristic function. For $A' \neq B'$ the (reduced) boundary of E consists of countably many planar interfaces with normal $\bar\nu$, while E is an arbitrary set of finite perimeter in ω if $A' = B'$ (see Theorem 2.1, and [6]).

The general compactness result is formulated in Theorem 1.1 below. As a matter of fact this theorem does not provide optimal compactness in the case $\varepsilon \ll h$. Indeed, in this regime we may expect the film to behave like a three dimensional sample. For instance, if the wells are not compatible in the bulk, *i.e.*, $\operatorname{rank}(A - B) > 1$, it reasonable to believe that sequences with bounded energy converge to trivial limits. This question will be addressed in the last section with some partial results (see Theorems 6.3 and 6.4).

Theorem 1.1 (Compactness). *Assume that ω is a bounded, connected, Lipschitz open set, and that $(H_1) - (H_2)$ hold. Let $h_n \to 0^+$ and $\varepsilon_n \to 0^+$ be arbitrary sequences, and let $\{u_n\} \subset H^2(\Omega;\mathbb{R}^3)$ be such that $\sup_n F^{h_n}_{\varepsilon_n}(u_n) < +\infty$. Then there exist a subsequence (not relabeled) and $(u, b) \in \mathscr{C}$ such that $\tilde u_n := u_n - \fint_\Omega u_n\, dx \to u$ in $W^{1,p}(\Omega;\mathbb{R}^3)$ and $\frac{1}{h_n}\partial_3 u_n \to b$ in $L^p(\Omega;\mathbb{R}^3)$.*

To describe our Γ-convergence results we need some additional assumptions on the internal structure of the potential W (these assumptions will be used only in the construction of recovery sequences). First of all we may assume without loss of generality that $A = -B$ and $\bar\nu = e_1' := (1, 0)$, so that

$$A' = -B' = a \otimes e_1' \quad \text{and} \quad A_3 = -B_3\,. \tag{1.4}$$

Indeed, the general case can be reduced to (1.4) by considering a modified bulk energy density W_{mod} defined by $W_{\text{mod}}(\xi) := W(\xi R + C)$ where $C = 1/2(A + B)$ and $R = \operatorname{diag}(R', 1)$ with $R' \in SO(2)$ satisfying $R'\bar\nu = e_1'$. This new potential W_{mod} obviously satisfies hypotheses (H_1) and (H_2) with (1.4). According to (1.4) we assume that

(H_3) there exist constants $\varrho > 0$ and $C_2 > 0$ such that

$$\frac{1}{C_2}\operatorname{dist}\big(\xi,\{A,B\}\big)^p \leqslant W(\xi) \leqslant C_2 \operatorname{dist}\big(\xi,\{A,B\}\big)^p \text{ if } \operatorname{dist}\big(\xi,\{A,B\}\big) \leqslant \varrho\,;$$

(H_4) $W(\xi_1,0,\xi_3) \leqslant W(\xi)$ for all $\xi = (\xi_1,\xi_2,\xi_3) \in \mathbb{R}^{3\times 3}$;

(H_5) if $A' = B' = 0$, then $W(\xi',\xi_3) = V\big(|\xi'|,\xi_3\big)$ for some $V : [0,+\infty)\times\mathbb{R}^3 \to [0,+\infty)$.

Here $|\cdot|$ stands for the usual Euclidean norm, $\xi' := (\xi_1,\xi_2) \in \mathbb{R}^{3\times 2}$ and $|\xi'|^2 = |\xi_1|^2 + |\xi_2|^2$. We observe that in the case $A' = B'$ (so that (1.4) yields $A' = B' = 0$), assumptions (H_4) and (H_5) require the function $r \mapsto V\big(r,\xi_3\big)$ to be a nondecreasing function of r for every $\xi_3 \in \mathbb{R}^3$. All these assumptions are satisfied by the prototypical potential W given by $W(\xi) = \min\big(|\xi - A|^2, |\xi - B|^2\big)$.

We now consider the family of functionals $\mathcal{F}^h_\varepsilon : [L^1(\Omega;\mathbb{R}^3)]^2 \to [0,+\infty]$ defined by

$$\mathcal{F}^h_\varepsilon(u,b) := \begin{cases} F^h_\varepsilon(u) & \text{if } u \in H^2(\Omega;\mathbb{R}^3) \text{ and } b = \frac{1}{h}\partial_3 u\,,\\ +\infty & \text{otherwise}\,. \end{cases}$$

We will prove that the behavior of $\mathcal{F}^h_\varepsilon$ depends on the asymptotic ratio $\frac{h}{\varepsilon} \to \gamma \in [0,+\infty]$ as h and ε tend to 0, and that the sequence $\{\mathcal{F}^h_\varepsilon\}$ Γ-converges to a functional $\mathscr{F}_\gamma : [L^1(\Omega;\mathbb{R}^3)]^2 \to [0,+\infty]$ given by

$$\mathscr{F}_\gamma(u,b) := \begin{cases} K_\gamma \operatorname{Per}_\omega(E) & \text{if } (u,b) \in \mathscr{C}\,,\\ +\infty & \text{otherwise}\,, \end{cases} \tag{1.5}$$

where $(\nabla' u, b)(x) = (1-\chi_E(x'))A + \chi_E(x')B$ as in (1.3), and $\operatorname{Per}_\omega(E) := \mathcal{H}^1(\partial^* E \cap \omega)$ denotes the perimeter of E in ω. Here the constant $K_\gamma > 0$ is determined by an optimal profile problem depending on the value of γ. By assumption (H_4), the description of K_γ involves in each case the 2D energy density $\mathcal{W} : \mathbb{R}^{3\times 2} \to [0,+\infty)$ defined by

$$\mathcal{W}(\zeta_1,\zeta_2) := W(\zeta_1,0,\zeta_2)\,,$$

and two reference maps $\bar{u}_0 \in W^{1,\infty}_{\text{loc}}(\mathbb{R};\mathbb{R}^3)$ and $\bar{b}_0 \in L^\infty(\mathbb{R};\mathbb{R}^3)$ given by

$$\bar{u}_0(t) := |t|a \quad \text{and} \quad \bar{b}_0(t) := \begin{cases} A_3 & \text{if } t \geqslant 0\,,\\ B_3 & \text{if } t < 0\,. \end{cases} \tag{1.6}$$

Our first convergence result deals with the critical regime where the thickness of the film and the strength of the interfacial energy are of the same order, that is $\gamma \in (0,+\infty)$.

Theorem 1.2 (Critical Regime). *Assume that $(H_0) - (H_5)$ hold with (1.4). Let $h_n \to 0^+$ and $\varepsilon_n \to 0^+$ be arbitrary sequences such that $h_n/\varepsilon_n \to \gamma$ for some $\gamma \in (0,+\infty)$. Then the functionals $\{\mathcal{F}^{h_n}_{\varepsilon_n}\}$ Γ-converge for the strong L^1-topology to the functional $\mathscr{F}_\gamma$ given by (1.5) with*

$$K_\gamma := \inf\bigg\{\frac{1}{\gamma}\int_{\ell I\times\gamma I} \mathcal{W}(\nabla v) + |\nabla^2 v|^2\,dy \;:\; \ell > 0\,,\ v \in C^2(\ell I\times\gamma I;\mathbb{R}^3)\,, \\ \nabla v(y) = \big(\bar{u}_0', \bar{b}_0\big)(y_1) \text{ nearby } \big\{|y_1| = \ell/2\big\}\bigg\}\,. \tag{1.7}$$

We observe that the formula for K_γ (with $\gamma \in (0,+\infty)$) entails a nontrivial dependence on the vertical direction in the asymptotic problem, at least for $A_3 \neq B_3$. Actually, even in the case $A_3 = B_3$, one can find potentials W for which a nontrivial dependance on x_3 still occurs, see [16, Section 8]. Note that in many second order phase transitions problems, optimal profiles usually have an oscillatory behavior along the limiting interface, see [16,27] and references therein (see also Theorem 1.4 below).

In contrast with the critical regime, one may expect the case $\gamma = 0$ (*i.e.*, $h \ll \varepsilon$) to lead to a simpler behavior with respect to the x_3-variable by separation of scales. Indeed, the energies formally behave like two dimensional ones, and optimal transition layers should only depend on the distance to the interface by assumptions $(H_4)-(H_5)$. We will illustrate this facts with more details in Section 5 (see Remark 5.4). Our results for this regime can be summarized in the following theorem.

Theorem 1.3 (Subcritical Regime). *Assume that* $(H_0)-(H_5)$ *hold with* (1.4). *Let* $h_n \to 0^+$ *and* $\varepsilon_n \to 0^+$ *be arbitrary sequences such that* $h_n/\varepsilon_n \to 0$. *Then the functionals* $\{\mathcal{F}^{h_n}_{\varepsilon_n}\}$ Γ*-converge for the strong* L^1*-topology to the functional* $\mathscr{F}_0$ *given by* (1.5) *with*

$$K_0 := \inf\left\{ \int_{-\ell}^{\ell} \mathcal{W}\big(\phi(t)\big) + |\phi_1'(t)|^2 + 2|\phi_2'(t)|^2\,dt \;:\; \ell > 0\,,\right.$$
$$\left.\phi = (\phi_1,\phi_2) \in C^1\big([-\ell,\ell];\mathbb{R}^{3\times 2}\big)\,,\ \phi(\ell) = \big(\bar u_0', \bar b_0\big)(\ell)\ \text{and}\ \phi(-\ell) = \big(\bar u_0', \bar b_0\big)(-\ell)\right\}. \quad (1.8)$$

In the supercritical case $\gamma = +\infty$ (*i.e.*, $\varepsilon \ll h$), one may again expect a separation of scales to hold. In other words, we should be able to recover the limiting functional by taking first the limit $\varepsilon \to 0$, and then operating the thin film limit $h \to 0$. Hence, to obtain a nontrivial Γ-limit, it is natural to ask for A and B to be compatible in the bulk with a non vertical connection. As already mentioned, we have exhibited rigidity effects in the opposite case, at least for some particular potentials (see Theorems 6.3 and 6.4). For this reason we assume in the supercritical regime that $A-B$ is a rank-one matrix, and that $A' \neq B'$. Under the structure (1.4), this assumption is equivalent to the existence of $\lambda \in \mathbb{R}$ such that $A_3 = -B_3 = \lambda a$. Then the wells A and B can be written as

$$A = -B = a \otimes (e_1 + \lambda e_3) \qquad (a \neq 0)\,. \quad (1.9)$$

Unfortunately we have only obtained partial results for this regime through lower and upper bounds for the Γ-lim inf and Γ-lim sup. However, our estimates turn out to be nearly optimal in the sense that upper and lower bounds agree whenever $\lambda = 0$, $p = 2$, and W is symmetric with respect to ξ_3. In this latter case, it follows that the separation of scales is indeed true by [16, Theorem 1.4].

Theorem 1.4 (Supercritical Regime). *Assume that* $(H_0)-(H_4)$ *and* (1.9) *hold for some* $\lambda \in \mathbb{R}$. *Let* $h_n \to 0^+$ *and* $\varepsilon_n \to 0^+$ *be arbitrary sequences such that* $h_n/\varepsilon_n \to +\infty$. *Then,*

$$c\,\mathscr{F}_\infty \leqslant \Gamma\big(L^1\big) - \liminf_{n\to+\infty} \mathcal{F}^{h_n}_{\varepsilon_n} \quad \text{and} \quad \Gamma\big(L^1\big) - \limsup_{n\to+\infty} \mathcal{F}^{h_n}_{\varepsilon_n} \leqslant \mathscr{F}_\infty\,,$$

for a constant $c > 0$, *where the functional* $\mathscr{F}_\infty$ *is given by* (1.5) *with*

$$K_\infty := (1+\lambda^2)^{\frac12} \inf\left\{ \int_{Q'_\lambda} \ell\,\mathcal{W}(\nabla v) + \frac{1}{\ell}|\nabla^2 v|^2\,dy \;:\; \ell > 0\,,\ v \in C^2(\mathbb{R}^2;\mathbb{R}^3)\,,\right.$$
$$\left.\nabla v(y) = \big(\bar u_0', \bar b_0\big)(y_1)\ \text{nearby}\ \big\{|y\cdot\nu_\lambda| = 1/2\big\}\,,\ v\ \text{is 1-periodic in the direction orthogonal to}\ \nu_\lambda\right\},$$

where $\nu_\lambda := \frac{1}{\sqrt{1+\lambda^2}}(1,\lambda) \in \mathbb{S}^1$, *and* Q'_λ *denotes the unit cube of* $\mathbb{R}^2$ *centered at the origin with two faces orthogonal to* ν_λ. *Moreover, if* $p = 2$ *in* (H_1), $\lambda = 0$ *in* (1.9), *and* W *satisfies* $W(\xi',\xi_3) = W(\xi',-\xi_3)$ *for all* $\xi \in \mathbb{R}^{3\times 3}$, *then the functionals* $\{\mathcal{F}^{h_n}_{\varepsilon_n}\}$ Γ*-converge for the strong* L^1*-topology to* $\mathscr{F}_\infty$.

The paper is organized as follows. We start in Section 2 with a structure result for the class $\mathscr{C}$ of limiting maps, and the compactness theorem is proved in Section 3. The proofs of Theorems 1.2, 1.3, and 1.4 are given in Sections 4, 5, and 6 respectively. We complete Section 6 with the aforementioned rigidity results in the supercritical case.

2. Preliminaries

Throughout the paper, Q and Q' denote the standard open unit cubes centered at the origin of $\mathbb{R}^3$ and $\mathbb{R}^2$ respectively, while $I := (-\frac12, \frac12)$. For simplicity, the differential operators $\frac{\partial}{\partial x_i}$ and $\frac{\partial^2}{\partial x_i \partial x_j}$ are written ∂_i and ∂^2_{ij} respectively. The rescaled gradient ∇_h and rescaled Hessian ∇^2_h operators are given by

$$\nabla_h u = \left(\partial_1 u, \partial_2 u, \frac{1}{h}\partial_3 u\right) \quad \text{and} \quad \nabla^2_h u = \begin{pmatrix} \partial^2_{11}u & \partial^2_{12}u & \frac{1}{h}\partial^2_{13}u \\ \partial^2_{12}u & \partial^2_{22}u & \frac{1}{h}\partial^2_{23}u \\ \frac{1}{h}\partial^2_{13}u & \frac{1}{h}\partial^2_{23}u & \frac{1}{h^2}\partial^2_{33}u \end{pmatrix}.$$

For a Borel set $B \subset \mathbb{R}^3$ and an admissible map u we write

$$F^h_\varepsilon(u, B) := \int_B \frac{1}{\varepsilon} W(\nabla_h u) + \varepsilon |\nabla^2_h u|^2 \, dx \,,$$

Throughout the paper we will consider two reference maps, u_0 and b_0, defined for $x \in \mathbb{R}^3$ by

$$u_0(x) := \bar{u}_0(x_1) \quad \text{and} \quad b_0(x) := \bar{b}_0(x_1)\,, \tag{2.1}$$

where $\bar{u}_0$ and $\bar{b}_0$ are given by (1.6).

We follow [4] for the standard results and notations on functions of bounded variation. We only recall that, given an open set $\mathscr{O} \subset \mathbb{R}^N$, a Borel $E \subset \mathscr{O}$ is said to be of finite perimeter in $\mathscr{O}$ if its characteristic function χ_E belongs to $BV(\mathscr{O})$. In such a case, the perimeter of E in $\mathscr{O}$, that we write $\mathrm{Per}_{\mathscr{O}}(E)$, is the total variation $|D\chi_E|(\mathscr{O})$, and it is equal to $\mathcal{H}^{N-1}(\partial^* E \cap \mathscr{O})$ where $\partial^* E$ denotes the reduced boundary of E.

We now state a structure result for the class $\mathscr{C}$ of limiting configurations (see (1.2)). To this purpose, let us define

$$\alpha_{\min} := \inf\big\{x_1 \in \mathbb{R} : x = (x_1, x_2) \in \omega\big\} \quad \text{and} \quad \alpha_{\max} := \sup\big\{x_1 \in \mathbb{R} : x = (x_1, x_2) \in \omega\big\}\,. \tag{2.2}$$

We have the following theorem as a consequence of [6,16].

Theorem 2.1. *Assume that* (H_0) *and* (1.4) *hold. Then for every pair* $(u, b) \in \mathscr{C}$, $(\nabla' u, b)$ *is of the form*

$$\big(\nabla' u(x), b(x)\big) = (1 - \chi_E(x'))A + \chi_E(x')B\,,$$

where $E \subset \omega$ *is a set of finite perimeter in* ω. *Moreover, if* $A' \neq B'$ *then* u *is of the form*

$$u(x) = c_0 + x_1 a - 2\psi(x_1)a\,, \tag{2.3}$$

where $c_0 \in \mathbb{R}^3$, $c_0 \cdot a = 0$, $\psi \in W^{1,\infty}((\alpha_{\min}, \alpha_{\max}); \mathbb{R})$ *and* $\psi' \in BV_{\mathrm{loc}}((\alpha_{\min}, \alpha_{\max}); \{0, 1\})$. *In particular, if* $A' \neq B'$ *then* E *is layered perpendicularly to* e_1', *i.e.,*

$$\partial^* E \cap \omega = \bigcup_{i \in \mathscr{I}} \{\alpha_i\} \times J_i\,, \tag{2.4}$$

where $\mathscr{I} \subset \mathbb{Z}$ *is made by successive integers,* $\{\alpha_i\} \subset (\alpha_{\min}, \alpha_{\max})$ *is locally finite in* $(\alpha_{\min}, \alpha_{\max})$, $\alpha_i < \alpha_{i+1}$, *and the sets* $J_i := \{t \in \mathbb{R} : (\alpha_i, t) \in \omega\}$ *are open bounded intervals.*

Proof. *Step 1.* We start with the case where $A' \neq B'$. Given $(u, b) \in \mathscr{C}$, we can write

$$(\nabla' u, b) = (1 - \chi_F)A + \chi_F B\,, \tag{2.5}$$

for some set $F \subset \Omega$ of finite perimeter in Ω. Since $\partial_3 u = 0$, we have $\nabla u \in BV(\Omega; \{(A', 0), (B', 0)\})$. Then we observe that (1.4) yields $(A', 0) - (B', 0) = a \otimes e_1$. Thanks to the convexity of ω, we can apply [16, Theorem 3.3] to deduce that u is of the form (2.3). From (2.3), (2.5), and the convexity of ω, it readily follows that $\chi_F = \chi_{E \times I}$ a.e. for some set $E \subset \omega$ of finite perimeter in ω satisfying (2.4).

Step 2. We now consider the case $A' = B'$. Given $(u, b) \in \mathscr{C}$, we have $b \in BV(\Omega; \{A_3, B_3\})$, so that $b = \chi_F A_3 + (1 - \chi_F)B_3$ for some set $F \subset \Omega$ of finite perimeter in Ω. By standard slicing results (see[4, Section 3.11]), $b^{x'} := b(x', \cdot)$ belongs to $BV(I; \mathbb{R}^3)$ for $\mathcal{L}^2$-a.e. $x' \in \omega$, and $\mathcal{L}^2 \lfloor \omega \otimes Db^{x'} = \partial_3 b$. Since $\partial_3 b = 0$, we deduce that $Db^{x'} = 0$ for $\mathcal{L}^2$-a.e. $x' \in \omega$. On the other hand, we can find a representative

b^* of b such that $(b^*)^{x'} := b^*(x', \cdot)$ is a good representative of $b^{x'}$ for $\mathcal{L}^2$-a.e. $x' \in \omega$. Since $Db^{x'} = 0$, we conclude that $(b^*)^{x'}$ is constant for $\mathcal{L}^2$-a.e. $x' \in \omega$, that is $b^*(x) = b^*(x')$. Then it follows that $\chi_F = \chi_{E\times I}$ a.e. for some set $E \subset \omega$ of finite perimeter in ω. □

3. Compactness

This section is devoted to the proof of Theorem 1.1, and we assume that (H_1) and (H_2) hold. We consider arbitrary sequences $h_n \to 0^+$, $\varepsilon_n \to 0^+$ as $n \to +\infty$, and $\{u_n\}_{n\in\mathbb{N}} \subset H^2(\Omega;\mathbb{R}^3)$ such that $\sup_n F^{h_n}_{\varepsilon_n}(u_n) < +\infty$. Throughout this section we write $b_n := \frac{1}{h_n}\partial_3 u_n$.

Proof of Theorem 1.1. *Step 1.* We claim that there exist a subsequence $\{\varepsilon_n\}$ (not relabeled), a pair $(u,b) \in W^{1,\infty}(\Omega;\mathbb{R}^3) \times L^\infty(\Omega;\mathbb{R}^3)$ satisfying $\partial_3 u = 0$, and $\theta \in L^\infty(\Omega;[0,1])$ such that

$$u_n - \fint_\Omega u_n \rightharpoonup u \text{ weakly in } W^{1,p}(\Omega;\mathbb{R}^3)\,,\ b_n \rightharpoonup b \text{ weakly in } L^p(\Omega;\mathbb{R}^3) \text{ as } n\to+\infty\,, \tag{3.1}$$

and

$$(\nabla' u, b)(x) = \big(1-\theta(x)\big)A + \theta(x)B \quad \text{for } \mathcal{L}^3\text{-a.e. } x\in\Omega\,. \tag{3.2}$$

Indeed, we first deduce from the growth assumption (H_2) that

$$\int_\Omega \big(|\nabla' u_n|^p + |b_n|^p\big)\,dx \leqslant C\bigg(\int_\Omega W(\nabla' u_n, b_n)\,dx + 1\bigg) \leqslant C\big(\varepsilon_n F^{h_n}_{\varepsilon_n}(u_n)+1\big) \leqslant C\,.$$

Therefore $\{b_n\}$ is uniformly bounded in $L^p(\Omega;\mathbb{R}^3)$, and $\{u_n - \fint_\Omega u_n\}$ is uniformly bounded in $W^{1,p}(\Omega;\mathbb{R}^3)$, thanks to the Poincaré-Wirtinger Inequality. Hence we may extract a subsequence such that (3.1) holds for some pair $(u,b)\in W^{1,p}(\Omega;\mathbb{R}^3)\times L^p(\Omega;\mathbb{R}^3)$. Since $\|\partial_3 u_n\|_{L^p(\Omega)} \leqslant Ch_n$, we deduce that $\partial_3 u \equiv 0$.

Next we observe that the sequence $\{(\nabla' u_n, b_n)\}$ also generates a Young measure $\{\nu_x\}_{x\in\Omega}$. From the fundamental theorem on Young measures (see *e.g.* [33, Theorem 6.11]), we derive

$$\int_\Omega\int_{\mathbb{R}^{3\times3}} W(\xi)\,d\nu_x(\xi)\,dx \leqslant \lim_{n\to+\infty}\int_\Omega W\big(\nabla' u_n, b_n\big)\,dx = 0\,,$$

so that $\operatorname{supp}\nu_x \subset \{A,B\}$ for $\mathcal{L}^3$-a.e. $x\in\Omega$. Hence there exists $\theta \in L^1\big(\Omega;[0,1]\big)$ such that

$$\nu_x = \big(1-\theta(x)\big)\delta_{\xi=A} + \theta(x)\delta_{\xi=B} \quad \text{for } \mathcal{L}^3\text{-a.e. } x\in\Omega\,.$$

Multiplying this last equality by ξ and integrating with respect to ξ yields (3.2), which completes the proof of the claim.

Step 2. We claim that $(u,b)\in\mathscr{C}$. We shall distinguish two distinct cases.

Case a). We first assume that $A' \neq B'$. For $M>0$ and $\xi' \in \mathbb{R}^{3\times2}$, we define

$$\varphi(\xi') := \inf\bigg\{\int_0^1 \min\Big(\sqrt{W_0\big(g(s)\big)}, M\Big)|g'(s)|\,ds \,:\, g\in W^{1,\infty}([0,1];\mathbb{R}^{3\times2})\,,\ g(0)=A' \text{ and } g(1)=\xi'\bigg\}\,,$$

where $W_0(\xi') := \min\{W(\xi',z) : z\in\mathbb{R}^3\}$ is a continuous function of ξ'. One may easily check that φ is Lipchitz continuous, $\varphi(\xi')=0$ if and only if $\xi' = A'$, and that

$$|\nabla\varphi(\xi')| \leqslant \min\big\{\sqrt{W_0(\xi')}, M\big\} \quad \text{for } \mathcal{L}^{3\times2}\text{-a.e. } \xi'\in\mathbb{R}^{3\times2}\,. \tag{3.3}$$

We claim that $\big\{\varphi(\nabla' u_n)\big\}$ is uniformly bounded in $W^{1,1}(\Omega;\mathbb{R})$. Indeed, estimate first

$$\int_\Omega \big|\nabla\big(\varphi(\nabla' u_n)\big)\big|\,dx \leqslant \int_\Omega \sqrt{W_0\big(\nabla' u_n(x)\big)}\,\big|\nabla(\nabla' u_n)\big|\,dx \leqslant \frac{1}{2}F^{h_n}_{\varepsilon_n}(u_n) \leqslant C\,,$$

and by (3.3),

$$\int_\Omega \big|\varphi\big(\nabla' u_n(x)\big)\big| \, dx \leqslant M \int_\Omega |\nabla' u_n(x)| \, dx + \varphi(0)\mathcal{L}^3(\Omega)\,.$$

Hence, up to a further subsequence (not relabeled),

$$\varphi(\nabla' u_n) \to H \quad \text{in } L^1(\Omega) \text{ as } n \to +\infty\,, \tag{3.4}$$

for some $H \in BV(\Omega)$. On the other hand, the Young measure $\{\mu_x\}_{x\in\Omega}$ generated by $\big\{\varphi(\nabla' u_n)\big\}$ is given by

$$\mu_x = \big(1-\theta(x)\big)\delta_{t=\varphi(A')} + \theta(x)\delta_{t=\varphi(B')}\,.$$

Then the strong convergence in (3.4) yields $\mu_x = \delta_{t=H(x)}$, so that

$$\delta_{t=H(x)} = \big(1-\theta(x)\big)\delta_{t=\varphi(A')} + \theta(x)\delta_{t=\varphi(B')}\,.$$

As a consequence $\theta(x) \in \{0,1\}$ for $\mathcal{L}^3$-a.e. $x \in \Omega$, and

$$H(x) = \big(1-\theta(x)\big)\varphi(A') + \theta(x)\varphi(B') = \theta(x)\varphi(B') \quad \text{for } \mathcal{L}^3\text{-a.e. } x \in \Omega\,.$$

Since $\varphi(B') \neq 0$, it yields $\theta \in BV\big(\Omega;\{0,1\}\big)$, and we may now write $\theta = \chi_F$ for some set $F \subset \Omega$ of finite perimeter in Ω. In view of (3.2), we obtain $(\nabla' u, b) = (1-\chi_F)A + \chi_F B$ $\mathcal{L}^3$-a.e. in Ω, and thus $(\nabla' u, b)$ belongs to $BV\big(\Omega;\{A,B\}\big)$. Since $\partial_3 u = 0$, we have $\nabla u \in BV\big(\Omega;\{(A',0),(B',0)\}\big)$ and it follows from [16, Theorem 3.3] that $F = E \times I$ for some set $E \subset \omega$ of finite perimeter in ω, which in turn implies that $\partial_3 b = 0$, and thus $(u,b) \in \mathscr{C}$.

Case b). Let us now assume that $A' = B'$ and $A_3 \neq B_3$. For $M > 0$ and $z \in \mathbb{R}^3$, we define

$$\psi(z) := \inf\left\{ \int_0^1 \min\Big(\sqrt{W_1\big(g(s)\big)}, M\Big)|g'(s)|\, ds \;:\; g \in W^{1,\infty}([0,1];\mathbb{R}^3)\,,\, g(0) = A_3 \text{ and } g(1) = z \right\},$$

where $W_1(z) := \min\{W(\xi', z) : \xi' \in \mathbb{R}^{3\times 2}\}$ is a continuous function of z. As previously ψ is Lipschitz continuous, and $\psi(z) = 0$ if and only if $z = A_3$. Arguing as in Case a), we obtain that $\{\psi(b_n)\}$ is uniformly bounded in $W^{1,1}(\Omega;\mathbb{R})$, and

$$\frac{1}{h_n}\int_\Omega \big|\partial_3\big(\psi(b_n)\big)\big|\, dx \leqslant \frac{1}{h_n}\int_\Omega \sqrt{W_1(b_n)}\,|\partial_3 b_n|\, dx \leqslant \frac{1}{2} F_{\varepsilon_n}^{h_n}(u_n) \leqslant C\,.$$

Therefore, up to a subsequence,

$$\psi(b_n) \to G \quad \text{in } L^1(\Omega) \text{ as } n \to +\infty\,, \tag{3.5}$$

for some $G \in BV(\Omega)$ satisfying $\partial_3 G = 0$. Arguing again as in Case a), we deduce from (3.5) that

$$\delta_{t=G(x)} = (1-\theta(x))\delta_{t=\psi(A_3)} + \theta(x)\delta_{t=\psi(B_3)}\,,$$

which yields $\theta \in BV(\Omega;\{0,1\})$. Since $\partial_3 G = 0$ we can argue as in the proof of Theorem 2.1, Step 2, to deduce that $\theta(x) = \chi_E(x')$ for some set $E \subset \omega$ of finite perimeter in ω. In view of (3.2), we conclude that $(\nabla' u, b) \in BV\big(\Omega;\{A,B\}\big)$ and $\partial_3 b = 0$, and thus $(u,b) \in \mathscr{C}$.

Step 3. In view of the previous steps, we know that $(\nabla u_n, b_n) \rightharpoonup (\nabla u, b)$ weakly in $L^p(\Omega)$, and that $\{(\nabla' u_n, b_n)\}_{n\in\mathbb{N}}$ generates the Young measure $\{\nu_x\}_{x\in\Omega}$ given by

$$\nu_x(\xi) = \chi_E(x')\delta_{\xi=A} + \big(1-\chi_E(x')\big)\delta_{\xi=B} = \delta_{\xi=(\nabla' u, b)}\,.$$

By standard results on Young measures (see *e.g.* [33, Proposition 6.12]), it follows that $(\nabla' u_n, b_n) \to (\nabla' u, b)$ strongly in $L^p(\Omega;\mathbb{R}^{3\times 3})$, and the proof Theorem 1.1 is complete. □

4. Γ-convergence in the critical regime

This section is devoted to the proof of Theorem 1.2. The Γ-liminf and Γ-limsup inequalities are derived in Theorem 4.1 and Theorem 4.2 respectively, while Corollary 4.2 shows that the lower and upper inequalities actually coincide. In proving lower inequalities, we adopt the approach of [16] once adapted to the dimension reduction setting. Throughout this section the parameter $\gamma \in (0, +\infty)$ is given.

4.1. *The* Γ-lim inf *inequality*

We introduce the constant

$$K^\star_\gamma := \inf\Big\{ \liminf_{n\to+\infty} F^{h_n}_{\varepsilon_n}(u_n, Q) \,:\, h_n \to 0^+ \text{ and } \varepsilon_n \to 0^+ \text{ with } h_n/\varepsilon_n \to \gamma\,, \\ \{u_n\} \subset H^2(Q;\mathbb{R}^3),\ (u_n, \frac{1}{h_n}\partial_3 u_n) \to (u_0, b_0) \text{ in } [L^1(Q;\mathbb{R}^3)]^2 \Big\}\,, \quad (4.1)$$

where the functions u_0 and b_0 are given by (2.1). The constant $K^\star_\gamma$ turns out to be finite, as one may check by considering an admissible sequence $\{u_n\}$ made of suitable regularizations of u_0 and b_0 (see also the proof of Theorem 4.2). In this subsection we shall prove that under assumptions $(H_0)-(H_2)$ and (H_5), the lower Γ-limit evaluated at any $(u, b) \in \mathscr{C}$ is bounded from below by $K^\star_\gamma$ times the length of the jump set of $(\nabla' u, b)$. We first prove this statement in the case of an elementary jump set.

Proposition 4.1. *Assume that (H_1), (H_2) and (1.4) hold. Let $h_n \to 0^+$ and $\varepsilon_n \to 0^+$ be arbitrary sequences such that $h_n/\varepsilon_n \to \gamma$. Let $\rho > 0$ and $\alpha \in \mathbb{R}$, let $J \subset \mathbb{R}$ be a bounded open interval, and consider the cylinder $U := (\alpha - \rho, \alpha + \rho) \times J \times I$. Let $(u, b) \in W^{1,\infty}(U;\mathbb{R}^3) \times L^\infty(U;\mathbb{R}^3)$ satisfying $\partial_3 u = \partial_3 b = 0$, and*

$$(\nabla' u, b) = \chi_{\{x_1<\alpha\}}\, B + \chi_{\{x_1\geqslant\alpha\}}\, A \quad \text{or} \quad (\nabla' u, b) = \chi_{\{x_1\leqslant\alpha\}}\, A + \chi_{\{x_1>\alpha\}}\, B\,. \quad (4.2)$$

Then for any sequence $\{u_n\} \subset H^2(U;\mathbb{R}^3)$ satisfying $(u_n, \frac{1}{h_n}\partial_3 u_n) \to (u, b)$ in $[L^1(U;\mathbb{R}^3)]^2$, we have

$$\liminf_{n\to+\infty} F^{h_n}_{\varepsilon_n}(u_n, U) \geqslant K^\star_\gamma \mathcal{H}^1(J)\,.$$

The proof of Proposition 4.1 relies on scaling properties and the translation invariance of the energy functional F^h_ε. To determine the corresponding properties in the limit we introduce the following set function. For an open bounded set $J \subset \mathbb{R}$ and $\rho > 0$, we write $J_\rho := \rho I \times J \times I$, and we define

$$\mathcal{E}_\gamma(J, \rho) := \inf\Big\{ \liminf_{n\to+\infty} F^{h_n}_{\varepsilon_n}(u_n, J_\rho) \,:\, h_n \to 0^+ \text{ and } \varepsilon_n \to 0^+ \text{ with } h_n/\varepsilon_n \to \gamma, \\ \{u_n\} \subset H^2(J_\rho;\mathbb{R}^3),\ (u_n, \frac{1}{h_n}\partial_3 u_n) \to (u_0, b_0) \text{ in } [L^1(J_\rho;\mathbb{R}^3)]^2 \Big\}\,.$$

Noticing that $K^\star_\gamma = \mathcal{E}_\gamma(I, 1)$ we now state the following lemma.

Lemma 4.1. *Assume that (H_1), (H_2) and (1.4) hold. Then*

(i) $\mathcal{E}_\gamma(t + J, \rho) = \mathcal{E}_\gamma(J, \rho)$ *for all* $t \in \mathbb{R}$;

(ii) *if* $J_1 \subset J_2$ *and* $\rho_1 \leqslant \rho_2$, *then* $\mathcal{E}_\gamma(J_1, \rho_1) \leqslant \mathcal{E}_\gamma(J_2, \rho_2)$;

(iii) *if* $J_1 \cap J_2 = \emptyset$, *then* $\mathcal{E}_\gamma(J_1 \cup J_2, \rho) \geqslant \mathcal{E}_\gamma(J_1, \rho) + \mathcal{E}_\gamma(J_2, \rho)$;

(iv) *if* $\alpha > 0$, *then* $\mathcal{E}_\gamma(\alpha J, \alpha\rho) = \alpha\mathcal{E}_\gamma(J, \rho)$;

(v) *if* $0 < \alpha < 1$, *then* $\mathcal{E}_\gamma(\alpha J, \rho) \geqslant \alpha\mathcal{E}_\gamma(J, \rho)$;

(vi) *if J is an interval then* $\mathcal{E}_\gamma(J, \rho) = \mathcal{H}^1(J)\, \mathcal{E}_\gamma(I, \rho)$;

(vii) *if J is an interval then* $\mathcal{E}_\gamma(J, \rho) = \mathcal{E}_\gamma(J, \delta)$ *for all* $\delta > 0$.

Proof. We first observe that *(i)* follows from the translation invariance of the functional F^h_ε. Then *(ii)* is due to the fact that any admissible sequence for $\mathcal{E}_\gamma(J_2,\rho_2)$ yields an admissible sequence for $\mathcal{E}_\gamma(J_1,\rho_1)$ whenever $J_1\subset J_2$ and $\rho_1\leqslant\rho_2$. The proof of claim *(iii)* follows a similar argument.

Proof of (iv). Let $h_n\to 0^+$ and $\varepsilon_n\to 0^+$ be such that $h_n/\varepsilon_n\to\gamma$, and let $\{u_n\}$ be an admissible sequence for $\mathcal{E}_\gamma(\alpha J,\alpha\rho)$. We define for $x\in J_\rho$, $v_n(x):=\frac{1}{\alpha}u_n(\alpha x',x_3)$, $\tilde h_n:=h_n/\alpha$ and $\tilde\varepsilon_n:=\varepsilon_n/\alpha$. By homogeneity of u_0 and b_0, we infer that $v_n(x)\to\frac{1}{\alpha}u_0(\alpha x',x_3)=u_0(x)$ and $\frac{1}{\tilde h_n}\partial_3 v_n\to b_0$ in $L^1(J_\rho;\mathbb{R}^3)$ as $n\to+\infty$. In particular, $\{v_n\}$ with $\{(\tilde h_n,\tilde\varepsilon_n)\}$ is admissible for $\mathcal{E}_\gamma(J,\rho)$. Changing variables then yields $F^{\tilde h_n}_{\tilde\varepsilon_n}(v_n,J_\rho)=\frac{1}{\alpha}F^{h_n}_{\varepsilon_n}\big(u_n,(\alpha J)_{\alpha\rho}\big)$. Letting $n\to+\infty$ we deduce that $\liminf_n F^{h_n}_{\varepsilon_n}\big(u_n,(\alpha J)_{\alpha\rho}\big)\geqslant\alpha\mathcal{E}_\gamma(J,\rho)$. In view of the arbitrariness of $\{(h_n,\varepsilon_n)\}$ and $\{u_n\}$, we conclude that $\mathcal{E}_\gamma(\alpha J,\alpha\rho)\geqslant\alpha\mathcal{E}_\gamma(J,\rho)$. The reverse inequality follows from the arbitrariness of $\alpha>0$.

Proof of (v). If $0<\alpha<1$ then $(\alpha J)_{\alpha\rho}\subset(\alpha J)_\rho$, and we derive from *(ii)* and *(iv)* that $\mathcal{E}_\gamma(\alpha J,\rho)\geqslant\mathcal{E}_\gamma(\alpha J,\alpha\rho)=\alpha\mathcal{E}_\gamma(J,\rho)$.

Proof of (vi). Write $J=Z\cup(\bigcup_{k=1}^N J_k)$ for some finite set Z and some family $\{J_k\}_{k=1}^N$ of mutually disjoint open intervals of the form $J_k=a_k+\alpha_k I$ with $0<\alpha_k<1$. In particular, $\mathcal{H}^1(J)=\sum_{k=1}^N\alpha_k$. We infer from *(i)*, *(iii)* and *(v)* that $\mathcal{E}_\gamma(J,\rho)\geqslant(\sum_k\alpha_k)\mathcal{E}_\gamma(I,\rho)$, leading to $\mathcal{E}_\gamma(J,\rho)\geqslant\mathcal{H}^1(J)\mathcal{E}_\gamma(I,\rho)$. The reverse inequality can be obtained in the same way inverting the roles of I and J.

Proof of (vii). Claim *(vii)* is a straightforward consequence of *(iv)* and *(vi)*. □

Remark 4.1. As a consequence of *(vi)* and *(vii)* in Lemma 4.1, we have $\mathcal{E}_\gamma(J,\rho)=K^\star_\gamma\mathcal{H}^1(J)$ for every $\rho>0$ and every bounded open interval $J\subset\mathbb{R}$.

An important consequence of Lemma 4.1 is that the energy of optimal sequences for $\mathcal{E}_\gamma(J,\rho)$ is concentrated near the limiting interface. We shall make use of Corollary 4.1 in the next subsection in order to compare the constants $K^\star_\gamma$ and K_γ.

Corollary 4.1. *Assume that* (H_1), (H_2) *and* (1.4) *hold. Let* $0<\delta<\rho$ *and let* $J\subset\mathbb{R}$ *be a bounded open interval. For any sequences* $h_n\to0^+$, $\varepsilon_n\to0^+$ *and* $\{u_n\}\subset H^2(J_\rho;\mathbb{R}^3)$ *such that* $h_n/\varepsilon_n\to\gamma$, $(u_n,\frac{1}{h_n}\partial_3u_n)\to(u_0,b_0)$ *in* $[L^1(J_\rho;\mathbb{R}^3)]^2$, *and* $\lim_n F^{h_n}_{\varepsilon_n}(u_n,J_\rho)=K^\star_\gamma\mathcal{H}^1(J)$, *we have*

$$\lim_{n\to+\infty}F^{h_n}_{\varepsilon_n}(u_n,J_\rho\setminus J_\delta)=0\,.$$

Proof. Since $J_\delta\subset J_\rho$, we deduce from Remark 4.1 that $\limsup_n F^{h_n}_{\varepsilon_n}(u_n,J_\delta)\leqslant\mathcal{E}_\gamma(J,\rho)<+\infty$. On the other hand the sequence $\{u_n\}$ is admissible for $\mathcal{E}_\gamma(J,\delta)$. In view of Lemma 4.1, we infer that

$$\liminf_{n\to+\infty}F^{h_n}_{\varepsilon_n}(u_n,J_\delta)\geqslant\mathcal{E}_\gamma(J,\delta)=\mathcal{E}_\gamma(J,\rho)=\lim_{n\to+\infty}F^{h_n}_{\varepsilon_n}(u_n,J_\rho)\,.$$

Therefore $\lim_n F^{h_n}_{\varepsilon_n}(u_n,J_\delta)=\lim_n F^{h_n}_{\varepsilon_n}(u_n,J_\rho)$ which clearly implies the announced result. □

Proof of Proposition 4.1. By the translation invariance of $F^{h_n}_{\varepsilon_n}$, we have $\liminf_n F^{h_n}_{\varepsilon_n}(u_n,U)=\liminf_n F^{h_n}_{\varepsilon_n}(\tau u_n,J_\rho)$, where $\tau u_n(x):=u_n(x_1+\alpha,x_2,x_3)$. Obviously $(\tau u_n,\frac{1}{h_n}\partial_3\tau u_n)\to(\tau u,\tau b)$ in $[L^1(J_\rho;\mathbb{R}^3)]^2$ with $\tau u(x):=u(x_1+\alpha,x_2,x_3)$ and $\tau b(x):=b(x_1+\alpha,x_2,x_3)$. If the first case in (4.2) holds, then $(\tau u,\tau b)=(u_0+c_0,b_0)$ for some constant $c_0\in\mathbb{R}^3$. Subtracting the constant c_0, we derive from the definition of $\mathcal{E}_\gamma$ and Remark 4.1 that

$$\liminf_{n\to+\infty}F^{h_n}_{\varepsilon_n}(u_n,U)=\liminf_{n\to+\infty}F^{h_n}_{\varepsilon_n}(\tau u_n-c_0,J_\rho)\geqslant\mathcal{E}_\gamma(J,\rho)=K^\star_\gamma\,\mathcal{H}^1(J)\,.$$

If the alternative case in (4.2) holds, then $(-\tau u,\tau b)(-x)=(u_0+c_0,b_0)(x)$ for some constant $c_0\in\mathbb{R}^3$. Observe that $F^{h_n}_{\varepsilon_n}(\tau u_n,J_\rho)=F^{h_n}_{\varepsilon_n}(v_n,J_\rho)$ with $v_n(x)=-\tau u_n(-x)-c_0$. Then $(v_n,\frac{1}{h_n}\partial_3v_n)\to(u_0,b_0)$ in $[L^1(J_\rho;\mathbb{R}^3)]^2$. Hence,

$$\liminf_{n\to+\infty}F^{h_n}_{\varepsilon_n}(u_n,J_\rho)=\liminf_{n\to+\infty}F^{h_n}_{\varepsilon_n}(v_n,J_\rho)\geqslant\mathcal{E}_\gamma(J,\rho)=K^\star_\gamma\,\mathcal{H}^1(J)\,,$$

and the proof is complete. □

Remark 4.2. Setting $K^\star_\infty$ to be the constant defined by (4.1) with $\gamma=+\infty$ (see (6.1)), and assuming that $K^\star_\infty<+\infty$, one may check that Proposition 4.1 and Corollary 4.1 still hold in the case $\gamma=+\infty$ with the same proofs.

We are now ready to prove the main result of this subsection which extends Proposition 4.1 to the general case. The proof for $A'\neq B'$ will be a direct consequence of Proposition 4.1, while the case $A'=B'$ will require an additional analysis based on a blow-up argument.

Theorem 4.1. *Assume that* $(H_0)-(H_2)$, (H_5) *and* (1.4) *hold. Let* $h_n\to 0^+$ *and* $\varepsilon_n\to 0^+$ *be arbitrary sequences such that* $h_n/\varepsilon_n\to\gamma$. *Then, for any* $(u,b)\in\mathscr{C}$ *and any sequences* $\{u_n\}\subset H^2(\Omega;\mathbb{R}^3)$ *such that* $(u_n,\frac{1}{h_n}\partial_3 u_n)\to(u,b)$ *in* $[L^1(\Omega;\mathbb{R}^3)]^2$, *we have*

$$\liminf_{n\to+\infty} F^{h_n}_{\varepsilon_n}(u_n)\geqslant K^\star_\gamma\,\mathrm{Per}_\omega(E)\,,$$

where $(\nabla' u,b)(x)=\big(1-\chi_E(x')\big)A+\chi_E(x')B\,.$

Proof. *Step 1.* We first assume that $A'\neq B'$. Assuming that E is non trivial, by Theorem 2.1 we can write $\partial^*E\cap\omega$ as in (2.4). Then we have $\mathcal{H}^1(\partial^*E\cap\omega)=\sum_{i\in\mathscr{I}}\mathcal{H}^1(J_i)<+\infty$. Consider an arbitrarily small $\delta>0$ and choose $k^-=k^-(\delta)\in\mathscr{I}$ and $k^+=k^+(\delta)\in\mathscr{I}$ such that $k^-\leqslant k^+$, and $\mathcal{H}^1(\partial^*E\cap\omega)\leqslant\sum_{i=k^-}^{k^+}\mathcal{H}^1(J_i)+\delta$. For each $i=k^-,\dots,k^+$, let $J_i'\subset\subset J_i$ be an open interval satisfying $\mathcal{H}^1(J_i)\leqslant\mathcal{H}^1(J_i')+\frac{\delta}{k^+-k^-+1}$. Since $\{\alpha_i\}\times J_i'\subset\subset\omega$ and $\alpha_i<\alpha_{i+1}$, we may find $\rho>0$ small in such a way that the sets $(\alpha_i-\rho,\alpha_i+\rho)\times J_i'$ are still compactly contained in ω, and $\alpha_i+\rho<\alpha_{i+1}-\rho$ for $i=k^-,\dots,k^+$. Then we set for each $i=k^-,\dots,k^+$, $U_i:=(\alpha_i-\rho,\alpha_i+\rho)\times J_i'\times I\subset\Omega$. Observe that the U_i's are pairwise disjoint, and that $(\nabla'u,b)$ is of the form (4.2) in each U_i.

Let us now fix an arbitrary sequence $\{u_n\}\subset H^2\big(\Omega;\mathbb{R}^3\big)$ such that $(u_n,\frac{1}{h_n}\partial_3u_n)\to(u,b)$ in $[L^1(\Omega;\mathbb{R}^3)]^2$. Using Proposition 4.1, we estimate

$$\liminf_{n\to+\infty}F^{h_n}_{\varepsilon_n}(u_n)\geqslant\sum_{i=k^-}^{k^+}\liminf_{n\to+\infty}F^{h_n}_{\varepsilon_n}\big(u_n,U_i\big)\geqslant\sum_{i=k^-}^{k^+}K^\star_\gamma\mathcal{H}^1(J_i')\geqslant K^\star_\gamma\mathcal{H}^1(\partial^*E\cap\omega)-2\delta K^\star_\gamma\,,$$

and the conclusion follows letting $\delta\to0$.

Step 2. We now consider the case $A'=B'$ ($=0$ by (1.4)). Then $u_0=0$ and u is constant. Without loss of generality we may assume that $u\equiv0$. In the remaining of this proof, we shall identify any $b\in L^1(Q';\mathbb{R}^3)$ with its extension to Q given by $b(x)=b(x')$. With this convention, we introduce for $b\in L^1(Q';\mathbb{R}^3)$,

$$\mathcal{G}_\gamma(b):=\inf\Big\{\liminf_{n\to+\infty}F^{h_n}_{\varepsilon_n}\big(u_n,Q\big)\;:\;h_n\to0\text{ and }\varepsilon_n\to0\text{ with }h_n/\varepsilon_n\to\gamma,\\ \{u_n\}\subset H^2\big(Q;\mathbb{R}^3\big),\,(u_n,\frac{1}{h_n}\partial_3u_n)\to(0,b)\text{ in }[L^1\big(Q;\mathbb{R}^3\big)]^2\Big\}.$$

Notice that $K^\star_\gamma=\mathcal{G}_\gamma(b_0)$. We shall require the sequential L^1-lower semicontinuity of the functional $\mathcal{G}_\gamma$ stated below. The proof of Lemma 4.2 only involves a standard diagonalization argument, and we shall omit it.

Lemma 4.2. $\mathcal{G}_\gamma$ *is sequentially lower semicontinuous for the strong* L^1*-topology.*

Let $\{u_n\}\subset H^2(\Omega;\mathbb{R}^3)$ be an arbitrary sequence satisfying $(u_n,\frac{1}{h_n}\partial_3u_n)\to(0,b)$ in $[L^1(\Omega;\mathbb{R}^3)]^2$. Without loss of generality, we may assume that $\liminf_n F^{h_n}_{\varepsilon_n}(u_n)=\lim_n F^{h_n}_{\varepsilon_n}(u_n)<+\infty$. By Theorem 2.1 we have $b(x)=b(x')=\big(1-\chi_E(x')\big)A_3+\chi_E(x')B_3$ for a set $E\subset\omega$ of finite perimeter in ω.

Using Fubini's theorem, we define a finite nonnegative Radon measure μ_n on ω by setting

$$\mu_n:=\left(\int_{-\frac12}^{\frac12}\frac{1}{\varepsilon_n}W\left(\nabla_{h_n}u_n\right)+\varepsilon_n\left|\nabla^2_{h_n}u_n\right|^2dx_3\right)\mathcal{L}^2\lfloor\omega\,,$$

and $\mu_n(\omega) = F^{h_n}_{\varepsilon_n}(u_n)$. In particular $\sup_n \mu_n(\omega) < +\infty$, and thus there is a subsequence (not relabeled) such that $\mu_n \rightharpoonup \mu$ weakly* in the sense of measures for some finite nonnegative Radon measure μ on ω. By lower semicontinuity we have $\mu(\omega) \leqslant \lim_n \mu_n(\omega) = \lim_n F^{h_n}_{\varepsilon_n}(u_n)$. It then suffices to prove that $\mu(\omega) \geqslant K^\star_\gamma \mathcal{H}^1(\partial^* E \cap \omega)$. By the Radon-Nikodým Theorem, we can decompose μ as $\mu = \mu_0 + \mu_s$, where μ_0 and μ_s are mutually singular nonnegative Radon measures on ω, and $\mu_0 \ll \mathcal{H}^1 \lfloor \partial^* E \cap \omega$. It is enough to show that $\mu_0(\omega) \geqslant K^\star_\gamma \mathcal{H}^1(\partial^* E \cap \omega)$ which can be obtained by proving that

$$\frac{d\mu_0}{d\mathcal{H}^1 \lfloor \partial^* E \cap \omega}(x_0') \geqslant K^\star_\gamma \quad \text{for } \mathcal{H}^1\text{-a.e. } x_0' \in \partial^* E \cap \omega\,.$$

For $\nu \in \mathbb{S}^1$ and $\delta > 0$, we denote by $Q_\nu' \subset \mathbb{R}^2$ the unit cube centered at the origin with two sides orthogonal to ν, and $Q_\nu'(x_0', \delta) := x_0' + \delta Q_\nu'$. By a generalization of the Besicovitch Differentiation Theorem (see [2, Proposition 2.2]), there exists a Borel set $Z \subset \omega$ such that $\mathcal{H}^1(Z) = 0$, the Radon-Nikodým derivative of μ_0 with respect to $\mathcal{H}^1 \lfloor \partial^* E \cap \omega$ exists and is finite at every $x_0' \in (\partial^* E \cap \omega) \setminus Z$, and

$$\frac{d\mu_0}{d\mathcal{H}^1 \lfloor \partial^* E \cap \omega}(x_0') = \lim_{\delta \to 0^+} \frac{\mu_0\big(Q_\nu'(x_0', \delta)\big)}{\mathcal{H}^1(\partial^* E \cap Q_\nu'(x_0', \delta))} \quad \text{for every } x_0' \in (\partial^* E \cap \omega) \setminus Z \text{ and all } \nu \in \mathbb{S}^1\,.$$

For $x_0' \in \partial^* E \cap \omega$, let us denote by $\nu_0 \in \mathbb{S}^1$ the generalized outer normal to E at x_0'. By Theorem 3.59 and Remark 2.82 in [4], we have

$$\lim_{\delta \to 0^+} \frac{\mathcal{H}^1(\partial^* E \cap Q_{\nu_0}'(x_0', \delta))}{\delta} = 1 \quad \text{for } \mathcal{H}^1\text{-a.e. } x_0' \in \partial^* E \cap \omega\,. \tag{4.3}$$

Moreover, it is well known (see *e.g.* [4, Example 3.68]) that

$$\lim_{\delta \to 0^+} \frac{1}{\delta^2} \int_{Q_{\nu_0}'(x_0', \delta)} |b(x') - \bar{b}_{x_0}(x')|\, dx' = 0 \quad \text{for } \mathcal{H}^1\text{-a.e. } x_0' \in \partial^* E \cap \omega\,, \tag{4.4}$$

where $\bar{b}_{x_0}(x') := \chi_{\{(x' - x_0')\cdot \nu_0 > 0\}}(x') A_3 + \chi_{\{(x' - x_0')\cdot \nu_0 < 0\}}(x') B_3$.

Let us now fix a point $x_0' \in (\partial^* E \cap \omega) \setminus Z$ satisfying (4.3)-(4.4). We choose a sequence $\delta_k \to 0^+$ such that $Q_{\nu_0}'(x_0', \delta_k) \subset \omega$ and $\mu\big(\partial Q_{\nu_0}'(x_0', \delta_k)\big) = 0$ for all $k \in \mathbb{N}$. Then

$$\begin{aligned}\frac{d\mu_0}{d\mathcal{H}^1 \lfloor \partial^* E \cap \omega}(x_0') &= \lim_{k \to +\infty} \frac{1}{\delta_k} \mu\big(Q_{\nu_0}'(x_0', \delta_k)\big) = \lim_{k \to +\infty} \lim_{n \to +\infty} \frac{1}{\delta_k} \mu_n\big(Q_{\nu_0}'(x_0', \delta_k)\big) \\ &= \lim_{k \to +\infty} \lim_{n \to +\infty} \frac{1}{\delta_k} F^{h_n}_{\varepsilon_n}\big(u_n, Q_{\nu_0}'(x_0', \delta_k) \times I\big)\,,\end{aligned}$$

where we have used Fubini's theorem in the last equality. Let $R \in SO(2)$ be such that $Re_1' = \nu_0$, and define for $x \in Q$, $v_{n,k}(x) := \frac{1}{\delta_k} u_n\big(x_0' + \delta_k R x', x_3\big)$. Observe that $\bar{b}_{x_0}(x_0' + Rx') = b_0(x')$. We also have $h_{n,k} := \frac{h_n}{\delta_k} \to 0$, $\varepsilon_{n,k} := \frac{\varepsilon_n}{\delta_k} \to 0$, $h_{n,k}/\varepsilon_{n,k} \to \gamma$, and $(v_{n,k}, \frac{1}{h_{n,k}} \partial_3 v_{n,k}) \to (0, b_k)$ in $[L^1(Q;\mathbb{R}^3)]^2$ where $b_k(x') := b\big(x_0' + \delta_k R' x'\big)$. Changing variables, we derive from assumption (H_5) that

$$F^{h_n}_{\varepsilon_n}\big(u_n, Q_{\nu_0}'(x_0', \delta_k) \times I\big) = \delta_k F^{h_{n,k}}_{\varepsilon_{n,k}}(v_{n,k}, Q)\,.$$

Then it follows from the definition of $\mathcal{G}_\gamma$ that

$$\frac{d\mu_0}{d\mathcal{H}^1 \lfloor \partial^* E \cap \omega}(x_0') = \lim_{k \to +\infty} \lim_{n \to +\infty} F^{h_{n,k}}_{\varepsilon_{n,k}}\big(v_{n,k}, Q\big) \geqslant \liminf_{k \to +\infty} \mathcal{G}_\gamma(b_k)\,.$$

On the other hand, by (4.4) we have $b_k \to b_0$ in $L^1(Q';\mathbb{R}^3)$. In view of Lemma 4.2, we deduce that

$$\frac{d\mu_0}{d\mathcal{H}^1 \lfloor \partial^* E \cap \omega}(x_0') \geqslant \liminf_{k \to +\infty} \mathcal{G}_\gamma(b_k) \geqslant \mathcal{G}_\gamma(b_0) = K^\star_\gamma\,,$$

which completes the proof. $\square$

4.2. *Lower bound on $K^\star_\gamma$*

In order to compare the constant $K^\star_\gamma$ with K_γ, we prove in this subsection that under the additional assumptions $(H_3)-(H_4)$, sequences realizing $K^\star_\gamma$ can be prescribed near the two sides $\{x_1=\pm\frac{1}{2}\}$, and chosen to be independent of the x_2-variable. This is the object of Proposition 4.3 below. First we state some useful facts on potentials W satisfying assumptions $(H_1)-(H_3)$ that we shall use throughout the paper. The proof of Lemma 4.3 is elementary and we omit it.

Lemma 4.3. *Assume that (H_1) holds. Then W satisfies $(H_2)-(H_3)$ if and only if there exists a constant $C_*\geqslant 1$ such that for every $\xi\in\mathbb{R}^{3\times 3}$,*

$$\frac{1}{C_*}\min\left(|\xi-A|^p,|\xi-B|^p\right)\leqslant W(\xi)\leqslant C_*\min\left(|\xi-A|^p,|\xi-B|^p\right). \tag{4.5}$$

In particular, if $(H_1)-(H_3)$ hold, then

$$W(\xi)\leqslant C_*^2 2^{p-1}\big(W(\bar\xi)+|\xi-\bar\xi|^p\big)\qquad\forall\xi,\bar\xi\in\mathbb{R}^{3\times 3}.$$

We now state the pinning condition described above. It parallels [16, Proposition 6.2] in the context of dimension reduction.

Proposition 4.2. *Assume that $(H_1)-(H_4)$ and (1.4) hold. Then there exist sequences $\varepsilon_n\to 0$, $\{c_n\}\subset\mathbb{R}^3$, $\{g_n\}\subset C^2(Q;\mathbb{R}^3)$ such that g_n is independent of x_2 (i.e., $g_n(x)=:\hat g_n(x_1,x_3)$), $c_n\to 0$, $g_n\to u_0$ in $W^{1,p}(Q;\mathbb{R}^3)$, $\frac{1}{\gamma\varepsilon_n}\partial_3 g_n\to b_0$ in $L^p(Q;\mathbb{R}^3)$,*

$$g_n=u_0+\gamma\varepsilon_n x_3 b_0\ \text{ in } Q\cap\{x_1>1/4\}\,,\quad g_n=u_0+\gamma\varepsilon_n x_3 b_0+c_n\ \text{ in } Q\cap\{x_1<-1/4\}\,,$$

and $\lim_n F^{\gamma\varepsilon_n}_{\varepsilon_n}(g_n,Q)=K^\star_\gamma$.

Proof. *Step 1.* Let us consider sequences $h_n\to 0$, $\tilde\varepsilon_n\to 0$ and $\{\tilde u_n\}\subset H^2(Q;\mathbb{R}^3)$ such that $h_n/\tilde\varepsilon_n\to\gamma$, $(\tilde u_n,\frac{1}{h_n}\partial_3\tilde u_n)\to(u_0,b_0)$ in $[L^1(Q;\mathbb{R}^3)]^2$, and $\lim_n F^{h_n}_{\tilde\varepsilon_n}(\tilde u_n,Q)=K^\star_\gamma$. Applying standard regularization techniques if necessary, we may assume that $\tilde u_n\in C^2(Q;\mathbb{R}^3)$. By Theorem 1.1, we have $\tilde u_n\to u_0$ in $W^{1,p}(Q;\mathbb{R}^3)$, and $\frac{1}{h_n}\partial_3\tilde u_n\to b_0$ in $L^p(Q;\mathbb{R}^3)$. Setting $\varepsilon_n:=h_n/\gamma$, we claim that $\lim_n F^{h_n}_{\varepsilon_n}(\tilde u_n,Q)=K^\star_\gamma$. Indeed, it suffices to notice that

$$\Big|F^{h_n}_{\varepsilon_n}(\tilde u_n,Q)-F^{h_n}_{\tilde\varepsilon_n}(\tilde u_n,Q)\Big|\leqslant\bigg(\big|1-\frac{\gamma\tilde\varepsilon_n}{h_n}\big|+\big|1-\frac{h_n}{\gamma\tilde\varepsilon_n}\big|\bigg)F^{h_n}_{\tilde\varepsilon_n}(\tilde u_n,Q)\underset{n\to+\infty}{\longrightarrow}0\,. \tag{4.6}$$

Extracting a further subsequence if necessary, we can find an exceptional set $Z\subset I$ of vanishing $\mathcal{H}^1$-measure such that for every $x_2\in I\setminus Z$, the slices $\tilde u_n(\cdot,x_2,\cdot)$ and $\frac{1}{h_n}\partial_3\tilde u_n(\cdot,x_2,\cdot)$ converge to u_0 in $W^{1,p}(I\times\{x_2\}\times I;\mathbb{R}^3)$ and b_0 in $L^p(I\times\{x_2\}\times I;\mathbb{R}^3)$ respectively. We select a level $s_n\in I\setminus Z$ satisfying

$$\int_{I\times\{s_n\}\times I}\frac{1}{\varepsilon_n}W(\nabla_{h_n}\tilde u_n)+\varepsilon_n|\nabla^2_{h_n}\tilde u_n|^2\,d\mathcal{H}^2\leqslant F^{h_n}_{\varepsilon_n}(\tilde u_n,Q)\,. \tag{4.7}$$

From now on we write

$$u_n(x):=\tilde u_n(x_1,s_n,x_3)\quad\text{and}\quad\hat u_n(x_1,x_3):=\tilde u_n(x_1,s_n,x_3)\,.$$

By our choice of s_n, we have $u_n\to u_0$ in $W^{1,p}(Q;\mathbb{R}^3)$, and $\frac{1}{h_n}\partial_3 u_n\to b_0$ in $L^p(Q;\mathbb{R}^3)$. Since $\partial_2 u_n\equiv 0$, assumption (H_4) together with (4.7) yields

$$\limsup_{n\to+\infty}F^{h_n}_{\varepsilon_n}(u_n,Q)\leqslant\limsup_{n\to+\infty}\int_{I\times\{s_n\}\times I}\frac{1}{\varepsilon_n}W(\nabla_{h_n}\tilde u_n)+\varepsilon_n|\nabla^2_{h_n}\tilde u_n|^2\,d\mathcal{H}^2\leqslant K^\star_\gamma\,.$$

On the other hand, $\liminf_n F^{h_n}_{\varepsilon_n}(u_n,Q)\geqslant K^\star_\gamma$ by definition of $K^\star_\gamma$. Hence $\lim_n F^{h_n}_{\varepsilon_n}(u_n,Q)=K^\star_\gamma$.

Step 2 (first matching). We start partitioning $\left(\frac{1}{12},\frac{1}{6}\right)\times Q'$ into $M_n:=\left[\frac{1}{\varepsilon_n}\right]$ layers of width $\frac{1}{12M_n}$ ($[\cdot]$ denotes the integer part). By Corollary 4.1, the energy concentrates near the interface $\{x_1=0\}$. Therefore we can find a layer $L_n:=\left(\theta_n-\frac{1}{12M_n},\theta_n\right)\times Q'\subset\left(\frac{1}{12},\frac{1}{6}\right)\times Q'$ for which

$$M_n\left(\int_{L_n}|u_n-u_0|^p+\Big|\frac{1}{h_n}\partial_3u_n-b_0\Big|^p+|\nabla u_n-\nabla u_0|^p\,dx+F^{h_n}_{\varepsilon_n}(u_n,L_n)\right)\leqslant$$
$$\int_{\left(\frac{1}{12},\frac{1}{6}\right)\times Q'}|u_n-u_0|^p+\Big|\frac{1}{h_n}\partial_3u_n-b_0\Big|^p+|\nabla u_n-\nabla u_0|^p\,dx+F^{h_n}_{\varepsilon_n}\big(u_n,\left(\tfrac{1}{12},\tfrac{1}{6}\right)\times Q'\big)=:\alpha_n\to0\,. \tag{4.8}$$

Then select a level $t_n\in\left(\theta_n-\frac{1}{12M_n},\theta_n\right)$ satisfying

$$\int_{\{t_n\}\times Q'}|u_n-u_0|^p+\Big|\frac{1}{h_n}\partial_3u_n-b_0\Big|^p+|\nabla u_n-\nabla u_0|^p\,d\mathcal{H}^2$$
$$+\int_{\{t_n\}\times Q'}\frac{1}{\varepsilon_n}W(\nabla_{h_n}u_n)+\varepsilon_n|\nabla^2_{h_n}u_n|^2\,d\mathcal{H}^2\leqslant12\alpha_n\,. \tag{4.9}$$

Let $\varphi_n\in C^\infty(\mathbb{R})$ be a cut-off function satisfying

$$\begin{cases}0\leqslant\varphi_n\leqslant1\,,\\ \varphi_n(t)=1 \text{ for } t\leqslant\theta_n-\frac{1}{12M_n}\,,\\ \varphi_n(t)=0 \text{ for } t\geqslant\theta_n\,,\\ \varepsilon_n|\varphi_n'|+\varepsilon_n^2|\varphi_n''|\leqslant C\,,\end{cases} \tag{4.10}$$

for a constant C independent of n. For $x\in L_n$, we set

$$v_n(x):=\big(1-\varphi_n(x_1)\big)\big(u_0(x)+h_nx_3b_0(x)+\bar{u}_n(x_3)\big)+\varphi_n(x_1)u_n(x)\,,$$

with

$$\bar{u}_n(x_3):=\hat{u}_n(t_n,x_3)-\bar{u}_0(t_n)-x_3\int_I\partial_3\hat{u}_n(t_n,s)\,ds\,.$$

We claim that

$$\int_{L_n}|v_n-u_0|^p\,dx\to0\,, \tag{4.11}$$
$$\frac{1}{\varepsilon_n}\int_{L_n}\Big|\frac{1}{h_n}\partial_3v_n-b_0\Big|^p\,dx\to0\,, \tag{4.12}$$
$$\frac{1}{\varepsilon_n}\int_{L_n}|\nabla'v_n-\nabla'u_0|^p\,dx\to0\,, \tag{4.13}$$
$$\frac{1}{\varepsilon_n}\int_{L_n}W(\nabla_{h_n}v_n)\,dx\to0\,, \tag{4.14}$$
$$\varepsilon_n\int_{L_n}\left|\nabla^2_{h_n}v_n\right|^2\,dx\to0\,. \tag{4.15}$$

Applying Jensen's inequality, we derive from (4.9) that

$$\int_I|\bar{u}_n|^p\,dx_3\leqslant C\int_{\{t_n\}\times Q'}|u_n-u_0|^p+|\partial_3u_n|^p\,d\mathcal{H}^2\leqslant C\alpha_n\to0\,. \tag{4.16}$$

Then (4.11) easily follows from (4.8), (4.9), and (4.16). To prove (4.12), we first estimate for $x_3\in I$,

$$|\bar{u}_n'(x_3)|\leqslant\int_I|\partial^2_{33}\hat{u}_n(t_n,s)|\,ds\leqslant\left(\int_{\{t_n\}\times Q'}|\partial^2_{33}u_n|^2\,d\mathcal{H}^2\right)^{1/2}\leqslant C\alpha_n^{1/2}\varepsilon_n^{3/2}\,, \tag{4.17}$$

where we have used Poincaré's inequality, Hölder's inequality, and (4.9). We may now infer that

$$\frac{1}{\varepsilon_n}\int_{L_n}\big|\frac{1}{h_n}\partial_3 v_n - b_0\big|^p\,dx \leqslant C\left(\frac{1}{\varepsilon_n}\int_{L_n}\big|\frac{1}{h_n}\partial_3 u_n - b_0\big|^p\,dx + \frac{1}{\varepsilon_n^p}\int_I |\bar{u}_n'|^p\,dx_3\right) \leqslant C\alpha_n \to 0\,,$$

thanks to (4.8) and (4.17). Observing that

$$u_n - u_0 - h_n x_3 b_0 - \overline{u}_n = x_3 \int_{\{t_n\}\times Q'} \big(\partial_3 u_n - h_n b_0\big)\,d\mathcal{H}^2 \quad \text{on } \{t_n\}\times Q'\,,$$

we can apply Poincaré's inequality to obtain

$$\int_{L_n} |u_n - u_0 - h_n x_3 b_0 - \bar{u}_n|^p\,dx \leqslant \frac{C}{M_n}\int_{\{t_n\}\times Q'} \big|\partial_3 u_n - h_n b_0\big|^p\,d\mathcal{H}^2 + C\left(\frac{1}{M_n}\right)^p \int_{L_n} |\partial_1 u_n - \partial_1 u_0|^p\,dx \leqslant C\alpha_n \varepsilon_n^{p+1}\,. \tag{4.18}$$

Then, using (4.8), (4.9), (4.10), and (4.18) we derive

$$\frac{1}{\varepsilon_n}\int_{L_n} |\nabla' v_n - \nabla' u_0|^p\,dx \leqslant \frac{C}{\varepsilon_n}\int_{L_n}\left(\frac{1}{\varepsilon_n^p}|u_n - u_0 - h_n x_3 b_0 - \bar{u}_n|^p + |\nabla' u_n - \nabla' u_0|^p\right)dx \leqslant C\alpha_n \to 0\,,$$

and (4.13) is proved. In view of (4.5), estimate (4.14) follows from (4.12) and (4.13), *i.e.*,

$$\begin{aligned}\frac{1}{\varepsilon_n}\int_{L_n} W(\nabla_{h_n} v_n)\,dx &\leqslant \frac{C_\star}{\varepsilon_n}\int_{L_n} \min\left(|\nabla_{h_n} v_n - A|^p, |\nabla_{h_n} v_n - B|^p\right)\,dx \\ &\leqslant \frac{C_\star}{\varepsilon_n}\int_{L_n} |\nabla_{h_n} v_n - (\nabla' u_0, b_0)|^p\,dx \to 0\,.\end{aligned} \tag{4.19}$$

We prove (4.15) in separate parts. In view of (4.10) we have

$$\varepsilon_n \int_{L_n} |(\nabla')^2 v_n|^2\,dx \leqslant C\biggl(\varepsilon_n\int_{L_n} |(\nabla')^2 u_n|^2\,dx + \frac{1}{\varepsilon_n}\int_{L_n}|\nabla' u_n - \nabla' u_0|^2\,dx + \frac{1}{\varepsilon_n^3}\int_{L_n} |u_n - u_0 - h_n x_3 b_0 - \bar{u}_n|^2\,dx\biggr)\,,$$

and we shall estimate each term separately. The first term on the right-hand-side of the inequality converges to 0 by (4.8). For the last two terms, we use (4.8) and (4.18) together with Hölder's inequality to obtain

$$\frac{1}{\varepsilon_n}\int_{L_n} |\nabla' u_n - \nabla' u_0|^2\,dx \leqslant \frac{|L_n|^{\frac{p-2}{p}}}{\varepsilon_n}\left(\int_{L_n} |\nabla' u_n - \nabla' u_0|^p\,dx\right)^{2/p} \leqslant C\alpha_n^{2/p} \to 0\,,$$

and

$$\frac{1}{\varepsilon_n^3}\int_{L_n} |u_n - u_0 - h_n x_3 b_0 - \bar{u}_n|^2\,dx \leqslant \frac{|L_n|^{\frac{p-2}{p}}}{\varepsilon_n^3}\left(\int_{L_n} |u_n - u_0 - h_n x_3 b_0 - \bar{u}_n|^p\,dx\right)^{2/p} \leqslant C\alpha_n^{2/p} \to 0\,,$$

and we conclude that $\varepsilon_n \int_{L_n} |(\nabla')^2 v_n|^2\,dx \to 0$. Finally we estimate

$$\varepsilon_n\int_{L_n} \big|\nabla'\big(\frac{1}{h_n}\partial_3 v_n\big)\big|^2\,dx \leqslant C\biggl(\varepsilon_n\int_{L_n}\big|\nabla'\big(\frac{1}{h_n}\partial_3 u_n\big)\big|^2\,dx + \frac{1}{\varepsilon_n}\int_{L_n}\big|\frac{1}{h_n}\partial_3 u_n - b_0\big|^2\,dx + \frac{1}{\varepsilon_n^2}\int_I |\bar{u}_n'|^2\,dx_3\biggr) \leqslant C\Bigl(\alpha_n + \alpha_n^{2/p}\Bigr) \to 0\,,$$

where we have used again (4.10), Hölder's inequality, (4.8), and (4.17). Since $\bar{u}_n''(x_3) = \partial^2_{33}\hat{u}_n(t_n, x_3)$, we infer from (4.8), (4.9), and (4.10) that

$$\varepsilon_n\int_{L_n}\frac{1}{h_n^4}\big|\partial^2_{33} v_n\big|^2\,dx \leqslant C\biggl(\frac{\varepsilon_n}{h_n^4}\int_{L_n}\big|\partial^2_{33} u_n\big|^2\,dx + \frac{\varepsilon_n^2}{h_n^4}\int_{\{t_n\}\times Q'}\big|\partial^2_{33} u_n\big|^2\,d\mathcal{H}^2\biggr) \leqslant C\varepsilon_n\alpha_n \to 0\,,$$

which ends the proof of (4.15).

Step 3 (second matching). Let $\psi_n \in C^\infty(\mathbb{R})$ be such that $0 \leqslant \psi_n \leqslant 1$, $\psi_n(t) = 1$ if $t \leqslant \theta_n$, $\psi_n(t) = 0$ if $t \geqslant 1/4$, and $|\psi_n'| + |\psi_n''| \leqslant C$ for a constant C independent of n. For $x \in \big(\theta_n, \frac{1}{4}\big) \times Q'$, we set

$$w_n(x) := u_0(x) + h_n x_3 b_0(x) + c_n^+ + \psi_n(x_1)\big(\overline{u}_n(x_3) - c_n^+\big)\,,$$

where $c_n^+ := \int_I \bar{u}_n(x_3)\,dx_3 \to 0$ thanks to (4.16). We claim that

$$\int_{(\theta_n,\frac{1}{4})\times Q'} |w_n - u_0|^p\,dx \to 0\,, \tag{4.20}$$

$$\frac{1}{\varepsilon_n}\int_{(\theta_n,\frac{1}{4})\times Q'} \Big|\frac{1}{h_n}\partial_3 w_n - b_0\Big|^p\,dx \to 0\,, \tag{4.21}$$

$$\frac{1}{\varepsilon_n}\int_{(\theta_n,\frac{1}{4})\times Q'} W\left(\nabla_{h_n} w_n\right)\,dx \to 0\,, \tag{4.22}$$

$$\varepsilon_n\int_{(\theta_n,\frac{1}{4})\times Q'} \left|\nabla^2_{h_n} w_n\right|^2\,dx \to 0\,. \tag{4.23}$$

First, (4.20) and (4.21) are easy consequences of (4.9) and (4.17) respectively. Next we apply Poincaré's inequality and (4.17) to derive that

$$\frac{1}{\varepsilon_n}\int_{\left(\theta_n,\frac{1}{4}\right)\times Q'} |\nabla' w_n - \nabla' u_0|^p\,dx \leqslant \frac{C}{\varepsilon_n}\int_I |\bar{u}_n - c_n^+|^p\,dx_3 \leqslant \frac{C}{\varepsilon_n}\int_I |\bar{u}_n'|^p\,dx_3 \leqslant C\varepsilon_n^{\frac{3p-2}{2}}\alpha_n^{\frac{p}{2}} \to 0\,. \tag{4.24}$$

Then, to prove (4.22) we argue exactly as in (4.19) using (4.21) and (4.24). We finally obtain in a similar way that

$$\varepsilon_n\int_{\left(\theta_n,\frac{1}{4}\right)\times Q'} \left|\nabla^2_{h_n} w_n\right|^2\,dx \leqslant$$
$$C\bigg(\varepsilon_n\int_{\left(\theta_n,\frac{1}{4}\right)\times Q'} |\bar{u}_n - c_n^+|^2\,dx + \frac{1}{\varepsilon_n}\int_I |\bar{u}_n'|^2\,dx_3 + \frac{\varepsilon_n}{h_n^4}\int_{\{t_n\}\times Q'} |\partial^2_{33} u_n|^2\,d\mathcal{H}^2\bigg) \leqslant C\alpha_n \to 0\,,$$

and (4.23) is proved.

Step 4. To conclude the proof, we first set for $x \in Q$,

$$g_n^+(x) := \begin{cases} u_n(x) & \text{for } x_1 < \theta_n - \frac{1}{12M_n}\,, \\ v_n(x) & \text{for } \theta_n - \frac{1}{12M_n} \leqslant x_1 < \theta_n\,, \\ w_n(x) & \text{for } \theta_n \leqslant x_1 < \frac{1}{4}\,, \\ u_0(x) + h_n x_3 b_0(x) + c_n^+ & \text{for } \frac{1}{4} \leqslant x_1 \leqslant \frac{1}{2}\,. \end{cases} \tag{4.25}$$

Recalling that $h_n = \gamma\varepsilon_n$, it follows from the previous steps and Corollary 4.1 that $g_n^+ \in C^2(Q;\mathbb{R}^3)$, $g_n^+ \to u_0$ in $W^{1,p}(Q;\mathbb{R}^3)$, $\frac{1}{\gamma\varepsilon_n}\partial_3 g_n^+ \to b_0$ in $L^p(Q;\mathbb{R}^3)$, and $\lim_n F^{\gamma\varepsilon_n}_{\varepsilon_n}(g_n^+, Q) = \lim_n F^{\gamma\varepsilon_n}_{\varepsilon_n}(u_n, Q) = K^\star_\gamma$. The sequence $\{g_n^+\}$ satisfies the pinning condition $g_n^+ = u_0 + \gamma\varepsilon_n x_3 b_0 + c_n^+$ in $Q \cap \{x_1 > 1/4\}$. Then we repeat construction to modify g_n^+ in $(-\frac{1}{2}, 0) \times Q'$ in order build a new field $g_n^- \in C^2(Q;\mathbb{R}^3)$ satisfying $g_n^- = u_0 + \gamma\varepsilon_n x_3 b_0 + c_n^-$ in $Q \cap \{x_1 < -1/4\}$ for some constants $c_n^- \to 0$. Now it suffices to set $g_n := g_n^- - c_n^+$ and $c_n := c_n^- - c_n^+$. By construction g_n does not dependent on x_2, that is $g_n(x) =: \hat{g}_n(x_1, x_3)$. □

Corollary 4.2. *Assume that* $(H_1) - (H_4)$ *and* (1.4) *hold. Then* $K^\star_\gamma \geqslant K_\gamma$.

Proof. We consider the sequences $\{\varepsilon_n\}$ and $\{g_n\}$ given by Proposition 4.2. Remind that $g_n(x) = \hat{g}_n(x_1, x_3)$. We set $\ell_n := 1/\varepsilon_n$, and for $y = (y_1, y_2) \in \ell_n I \times \gamma I$, $v_n(y) := \frac{1}{\varepsilon_n}\hat{g}_n\big(\varepsilon_n y_1, y_2/\gamma\big)$. Then straightforward computations yield $\nabla v_n(y) = (\bar{u}_0'(y_1), \bar{b}_0(y_1))$ nearby $\{|y_1| = \ell_n/2\}$, and

$$K_\gamma \leqslant \frac{1}{\gamma}\int_{\ell_n I\times\gamma I} \mathcal{W}(\nabla v_n) + |\nabla^2 v_n|^2\,dy = F^{\gamma\varepsilon_n}_{\varepsilon_n}(g_n, Q)\,.$$

By construction of $\{g_n\}$, the conclusion follows letting $n \to +\infty$. $\square$

4.3. *The* Γ-lim sup *inequality*

We conclude this section with the construction of a recovery sequence. Then Theorem 4.2 together with Corollary 4.2 and Theorem 4.1 concludes the proof of Theorem 1.2.

Theorem 4.2. *Assume that* $(H_0) - (H_5)$ *and* (1.4) *hold. Let* $\varepsilon_n \to 0$ *and* $h_n \to 0$ *be arbitrary sequences such that* $h_n/\varepsilon_n \to \gamma$. *Then, for every* $(u, b) \in \mathscr{C}$, *there exists a sequence* $\{u_n\} \subset H^2(\Omega;\mathbb{R}^3)$ *such that* $u_n \to u$ *in* $W^{1,p}(\Omega;\mathbb{R}^3)$, $\frac{1}{h_n}\partial_3 u_n \to b$ *in* $L^p(\Omega;\mathbb{R}^3)$ *and*

$$\lim_{n\to+\infty} F^{h_n}_{\varepsilon_n}(u_n) = K_\gamma \operatorname{Per}_\omega(E)\,, \tag{4.26}$$

where $(\nabla' u, b)(x) = \big(1-\chi_E(x')\big)A + \chi_E(x')B$.

Proof. *Step 1.* We first assume that $A' \neq B'$, so that $\partial^* E \cap \omega$ is of the form (2.4) by Theorem 2.1. We also assume that it is made by finitely many interfaces, *i.e.*, $\mathscr{I} = \{1,\dots,m\}$ in (2.4). In this case, by Theorem 2.1, we have $u(x) = \bar{u}(x_1)$ and $b(x) = \bar{b}(x_1)$ where $\bar{u} \in W^{1,\infty}((\alpha_{\min},\alpha_{\max});\mathbb{R}^3)$, $(\bar{u}', \bar{b}) \in BV((\alpha_{\min},\alpha_{\max}); \{(a, A_3), (-a, -A_3)\})$, and $(\alpha_{\min},\alpha_{\max})$ is defined by (2.2). Without loss of generality we may assume that $\bar{u}'(x_1) = -a$ for $x_1 < \alpha_1$. Then $\bar{u}'(t) = a$ for $\alpha_i < t < \alpha_{i+1}$ if i is odd, $\bar{u}'(t) = -a$ for $\alpha_i < t < \alpha_{i+1}$ if i is even, and $\bar{u}'(t) = a$ or $\bar{u}'(t) = -a$ for $t > \alpha_m$ if m is odd or even respectively.

Let us consider for each $k \in \mathbb{N}$, some $\ell_k > 0$ and $v_k \in C^2(\ell_k I \times \gamma I;\mathbb{R}^3)$ such that $\nabla v_k(y) = (\bar{u}'_0, \bar{b}_0)(y_1)$ nearby $\{|y_1| = \ell_k/2\}$, and

$$\frac{1}{\gamma}\int_{\ell_k I\times\gamma I} \mathcal{W}(\nabla v_k) + |\nabla^2 v_k|^2\, dy \leqslant K_\gamma + 2^{-k}\,. \tag{4.27}$$

Subtracting a constant to v_k if necessary, we may assume that

$$v_k(y) = \begin{cases} ay_1 + A_3 y_2 + c_k & \text{nearby } \{y_1 = \ell_k/2\}\,,\\[2mm] -ay_1 + B_3 y_2 - c_k & \text{nearby } \{y_1 = -\ell_k/2\}\,, \end{cases} \tag{4.28}$$

for some $c_k \in \mathbb{R}^3$.

Let $h_n \to 0$ be an arbitrary sequence, and without loss of generality we can choose $\varepsilon_n := h_n/\gamma$ (see (4.6)). We fix for each $i = 1,\dots,m$, a bounded open interval $J'_i \subset \mathbb{R}$ such that

$$J_i \subset\subset J'_i \quad\text{and}\quad \mathcal{H}^1(J'_i \setminus J_i) \leqslant 2^{-k}\,, \tag{4.29}$$

and we shall consider integers n large enough in such a way that $\alpha_i + \ell_k\varepsilon_n/2 < \alpha_{i+1} - \ell_k\varepsilon_n/2$ for every $i = 1,\dots,m-1$. We write for each $i = 1,\dots,m$,

$$\alpha^n_{i-} := \alpha_i - \frac{\ell_k\varepsilon_n}{2} \quad\text{and}\quad \alpha^n_{i+} := \alpha_i + \frac{\ell_k\varepsilon_n}{2}\,. \tag{4.30}$$

Note that by convexity of ω,

$$\big((\alpha^n_{i-},\alpha^n_{i+})\times\mathbb{R}\big)\cap\omega \subset (\alpha^n_{i-},\alpha^n_{i+})\times J'_i \tag{4.31}$$

whenever n is sufficiently large.

We define the transition layer near each interface as follows: for each $i = 1,\dots,m$, we set for $x \in (\alpha^n_{i-},\alpha^n_{i+})\times J'_i \times I$,

$$w^i_{n,k}(x) := (-1)^{i+1} v_k\left((-1)^{i+1}\frac{x_1-\alpha_i}{\varepsilon_n}, (-1)^{i+1}\gamma x_3\right) + \big(1+(-1)^i\big)\left(a\frac{\ell_k}{2} + c_k\right).$$

Observe that (4.28) yields

$$w^i_{k,n}(\alpha^n_{i-}, x_2, x_3) = v_k\left((-1)^i\,\frac{\ell_k}{2}, \gamma x_3\right), \tag{4.32}$$

and

$$w^i_{k,n}(\alpha^n_{i+},x_2,x_3)=v_k\left((-1)^{i+1}\frac{\ell_k}{2},\gamma x_3\right)+2\left(1+(-1)^i\right)c_k\,. \tag{4.33}$$

Setting

$$\beta^n_i:=\sum_{j=1}^{i}\bar u(\alpha^n_{j+})-\bar u(\alpha^n_{j-})\quad\text{and}\quad \kappa_i:=2\sum_{j=1}^{i}\left(1+(-1)^j\right)\,, \tag{4.34}$$

with $\beta^n_0:=0$, $\kappa_0:=0$, we finally define for n large enough and $x\in\Omega$,

$$u_{n,k}(x):=\begin{cases}\bar u(x_1)+\varepsilon_n v_k\left(-\dfrac{\ell_k}{2},\gamma x_3\right) & \text{for } x_1\leqslant \alpha^n_{1-}\,,\\ \bar u(\alpha^n_{i-})-\beta^n_{i-1}+\varepsilon_n w^i_{k,n}(x)+\varepsilon_n\kappa_{i-1}c_k & \text{for } \alpha^n_{i-}<x_1<\alpha^n_{i+}\,,\\ \bar u(x_1)-\beta^n_i+\varepsilon_n v_k\left((-1)^{i+1}\dfrac{\ell_k}{2},\gamma x_3\right)+\varepsilon_n\kappa_i c_k & \text{for } \alpha^n_{i+}\leqslant x_1\leqslant\alpha^n_{(i+1)-}\,,\\ \bar u(x_1)-\beta^n_m+\varepsilon_n v_k\left((-1)^{m+1}\dfrac{\ell_k}{2},\gamma x_3\right)+\varepsilon_n\kappa_m c_k & \text{for } x_1\geqslant\alpha^n_{m+}\,.\end{cases}$$

In view of (4.32)-(4.33) we have $u_{n,k}\in H^2(\Omega;\mathbb{R}^3)$. Moreover, $u_{n,k}$ does not depend on the x_2-variable, and

$$\left(\partial_1 u_{n,k},\frac{1}{h_n}\partial_3 u_{n,k}\right)(x)=\begin{cases}\nabla v_k\left((-1)^{i+1}\dfrac{x_1-\alpha_i}{\varepsilon_n},(-1)^{i+1}\gamma x_3\right) & \text{if } \alpha^n_{i-}<x_1<\alpha^n_{i+}\,,\\ (\bar u',\bar b)(x_1) & \text{otherwise}\,.\end{cases} \tag{4.35}$$

Since $\bar u$ is Lipschitz continuous, we have $|\beta^n_i|\leqslant C\varepsilon_n$ for a constant C independent of n. In addition, v_k and ∇v_k are bounded, and we infer that $u_{n,k}\to u$ in $W^{1,p}(\Omega;\mathbb{R}^3)$ and $\frac{1}{h_n}\partial_3 u_{n,k}\to b$ in $L^p(\Omega;\mathbb{R}^3)$ as $n\to+\infty$. Using (4.31), (4.35), and changing variables, we estimate

$$\begin{aligned}F^{h_n}_{\varepsilon_n}(u_{n,k})&\leqslant\sum_{i=1}^{m}F^{h_n}_{\varepsilon_n}\left(\varepsilon_n w^i_{n,k},(\alpha^n_{i-},\alpha^n_{i+})\times J'_i\times I\right)\\&\leqslant\sum_{i=1}^{m}\frac{\mathcal{H}^1(J'_i)}{\gamma}\int_{\ell_k I\times\gamma I}\mathcal{W}(\nabla v_k)+|\nabla^2 v_k|^2\,dy\\&\leqslant K_\gamma\operatorname{Per}_\omega(E)+C_0 2^{-k}\,,\end{aligned}$$

for a constant C_0 which only depends on m and $\operatorname{Per}_\omega(E)$.

For each $k\in\mathbb{N}$, we can now find $N_k\in\mathbb{N}$ such that

$$\|u_{n,k}-u\|_{W^{1,p}(\Omega)}\leqslant 2^{-k}\,,\quad \|\frac{1}{h_n}\partial_3 u_{n,k}-b\|_{L^p(\Omega)}\leqslant 2^{-k}\,,\quad F^{h_n}_{\varepsilon_n}(u_{n,k})\leqslant K_\gamma\operatorname{Per}_\omega(E)+C_0 2^{-k}$$

for every $n\geqslant N_k$. Moreover we can assume that the resulting sequence $\{N_k\}$ satisfies $N_k<N_{k+1}$ for every $k\in\mathbb{N}$. Then for every $n\geqslant N_0$, there exists a unique k_n such that $N_{k_n}\leqslant n<N_{k_n+1}$, and $k_n\to+\infty$ as $n\to+\infty$. We define $u_n:=u_{n,k_n}$ and it follows that $u_n\to u$ in $W^{1,p}(\Omega;\mathbb{R}^3)$, $\frac{1}{h_n}\partial_3 u_n\to b$ in $L^p(\Omega;\mathbb{R}^3)$,

$$\limsup_{n\to+\infty}F^{h_n}_{\varepsilon_n}(u_n)\leqslant K_\gamma\operatorname{Per}_\omega(E)\,. \tag{4.36}$$

Finally (4.26) holds by (4.36), Theorem 4.1, and Corollary 4.2.

Step 2. We now consider the case where $A'\neq B'$ and $\partial^*E\cap\omega$ is made by infinitely many interfaces, *i.e.*, $\partial^*E\cap\omega$ is as in (2.4) with $\mathscr{I}\subset\mathbb{Z}$ infinite. We may assume for simplicity that $\mathscr{I}=\mathbb{N}$. The general case can be recovered from the discussion below with the obvious modifications.

By Theorem 2.1, we have $\lim_k\sum_{i=0}^k\mathcal{H}^1(J_i)=\sum_{i\in\mathbb{N}}\mathcal{H}^1(J_i)=\operatorname{Per}_\omega(E)$, and α_k converges to $\alpha_{\max}$. For $k\in\mathbb{N}$ large enough, we define some $u_k\in W^{1,\infty}(\Omega;\mathbb{R}^3)$ in the following way: we set

$u_k(x) := u(x)$ for $x \in \Omega \cap \{x_1 < \alpha_{k+1}\}$, and we extend u_k to be affine in the remaining of Ω in such a way that u_k and ∇u_k are continuous across the interface $\{x_1 = \alpha_{k+1}\}$. Similarly we define for $x \in \Omega \cap \{x_1 < \alpha_{k+1}\}$, $b_k(x) := b(x)$, and we extend b_k by a suitable constant in the remaining of Ω so that it remains continuous across $\{x_1 = \alpha_{k+1}\}$. Then one may check that $(u_k, b_k) \in \mathscr{C}$, and that $(\nabla' u_k, b_k) = \big(1 - \chi_{E_k}(x')\big)A + \chi_{E_k}(x')B$ with $\partial^* E_k \cap \omega = \bigcup_{i=0}^k \{\alpha_i\} \times J_i$. Moreover, using the fact that $\alpha_k \to \alpha_{\max}$, we derive that $u_k \to u$ in $W^{1,p}(\Omega;\mathbb{R}^3)$ and $b_k \to b$ in $L^p(\Omega;\mathbb{R}^3)$.

Let $h_n \to 0$ and $\varepsilon_n \to 0$ be arbitrary sequences such that $h_n/\varepsilon_n \to \gamma$. Since $\partial^* E_k \cap \omega$ is made by finitely many interfaces, by Step 1, we can find $\{u_{n,k}\} \subset H^2(\Omega;\mathbb{R}^3)$ such that $u_{n,k} \to u_k$ in$W^{1,p}(\Omega\mathbb{R}^3)$, $\frac{1}{h_n}\partial_3 u_{n,k} \to b_k$ in $L^p(\Omega;\mathbb{R}^3)$, and $\lim_n F_{\varepsilon_n}^{h_n}(u_{n,k}) = K_\gamma \sum_{i=0}^k \mathcal{H}^1(J_i)$. Then the conclusion follows for a suitable diagonal sequence $u_n := u_{n,k_n}$ as already pursued in Step 1.

Step 3. We finally treat the case $A' = B'$ ($= 0$ by (1.4)). Without loss of generality we may assume that $u = 0$. According to Theorem 2.1, we have $b(x) = (1 - \chi_E(x'))A_3 + \chi_E(x')B_3$ where $E \subset \omega$ is a set of finite perimeter in ω. By Lemma 4.3 in [3], we can find a sequence $\{E_k\}$ of bounded open sets in $\mathbb{R}^2$ with smooth boundary such that $\chi_{E_k} \to \chi_E$ in $L^1(\omega)$, and $\lim_k \mathcal{H}^1(\partial E_k \cap \overline{\omega}) = \mathrm{Per}_\omega(E)$. We define for $x \in \Omega$, $b_k(x) := (1 - \chi_{E_k}(x'))A_3 + \chi_{E_k}(x')B_3$, so that $b_k \to b$ in $L^p(\Omega;\mathbb{R}^3)$. Since $\mathcal{M}^k := \partial E_k$ is a smooth submanifold of $\mathbb{R}^2$, for every $k \in \mathbb{N}$ we can find $\delta_k > 0$ such that the nearest point projection onto $\mathcal{M}^k$ is well defined and smooth in the tubular δ_k-neighborhood

$$U_k := \{x' \in \mathbb{R}^2 : \mathrm{dist}(x', \mathcal{M}^k) < \delta_k\}\,.$$

We define the signed distance to $\mathcal{M}^k$ as the function $d_k : \mathbb{R}^2 \to [0,+\infty)$ given by

$$d_k(x') := \begin{cases} -\mathrm{dist}(x', \mathcal{M}^k) & \text{if } x \in E_k\,, \\ \mathrm{dist}(x', \mathcal{M}^k) & \text{otherwise}\,. \end{cases} \tag{4.37}$$

Then d_k is smooth in U_k, the level sets $\{d_k = t\} =: \mathcal{M}_t^k$ are smooth for all $t \in (-\delta_k, \delta_k)$, and the function $t \in (-\delta_k, \delta_k) \mapsto \mathcal{H}^1(\mathcal{M}_t^k \cap \overline{\omega})$ is upper semicontinuous (see *e.g.* [4, Proposition 1.62]). In particular,

$$\limsup_{t \to 0} \mathcal{H}^1(\mathcal{M}_t^k \cap \overline{\omega}) \leqslant \mathcal{H}^1(\mathcal{M}^k \cap \overline{\omega})\,. \tag{4.38}$$

Next we consider for each $k \in \mathbb{N}$, some $\ell_k > 0$ and $v_k \in C^2(\ell_k I \times \gamma I;\mathbb{R}^3)$ satisfying $\nabla v_k(y) = (0, \bar{b}_0(y_1))$ nearby $\{|y_1| = \ell_k/2\}$, and (4.27).

Let $h_n \to 0$ be an arbitrary sequence. Here again we can choose $\varepsilon_n := h_n/\gamma$. For each $k \in \mathbb{N}$ and $n \in \mathbb{N}$ such that $\varepsilon_n \ell_k < \delta_k$, we define for $x \in \Omega$,

$$u_{n,k}(x) := \begin{cases} \varepsilon_n v_k\left(\dfrac{d_k(x')}{\varepsilon_n}, \gamma x_3\right) & \text{if } |d_k(x')| < \dfrac{\ell_k \varepsilon_n}{2}\,, \\ \varepsilon_n v_k\left(\dfrac{\ell_k}{2}, \gamma x_3\right) & \text{if } d_k(x') \geqslant \dfrac{\ell_k \varepsilon_n}{2}\,, \\ \varepsilon_n v_k\left(-\dfrac{\ell_k}{2}, \gamma x_3\right) & \text{if } d_k(x') \leqslant -\dfrac{\ell_k \varepsilon_n}{2}\,. \end{cases}$$

Then $u_{n,k} \in H^2(\Omega;\mathbb{R}^3)$, and

$$\nabla_{h_n} u_n(x) = \begin{cases} \left(\partial_1 v_k\left(\dfrac{d_k(x')}{\varepsilon_n}, \gamma x_3\right) \otimes \nabla d_k(x'), \partial_2 v_k\left(\dfrac{d_k(x')}{\varepsilon_n}, \gamma x_3\right)\right) & \text{if } |d_k(x')| < \dfrac{\ell_k \varepsilon_n}{2}\,, \\ (0, b_k(x)) & \text{otherwise}\,. \end{cases} \tag{4.39}$$

From the boundedness of v_k and ∇v_k together with the smoothness of d_k in U_k, we infer that $u_{n,k} \to 0$ in $W^{1,p}(\Omega;\mathbb{R}^3)$ and $\frac{1}{h_n}\partial_3 u_{n,k} \to b_k$ in $L^p(\Omega;\mathbb{R}^3)$ as $n \to +\infty$. Now it remains to estimate $F_{\varepsilon_n}^{h_n}(u_{n,k})$. First of all, (4.39) yields $F_{\varepsilon_n}^{h_n}\big(u_{n,k}, \Omega \setminus \{|d_k(x')| < \ell_k\varepsilon_n/2\}\big) = 0$. Using the fact that $|\nabla d_k| = 1$ $\mathcal{L}^2$-a.e. in $\mathbb{R}^2$, we infer from (H_5) that for $x \in \Omega \cap \{|d_k(x')| < \ell_k\varepsilon_n/2\}$,

$$W\big(\nabla_{h_n} u_{n,k}(x)\big) = V\big(\big|\partial_1 v_k(d_k(x')/\varepsilon_n, \gamma x_3)\big|, \partial_2 v_k(d_k(x')/\varepsilon_n, \gamma x_3)\big) = \mathcal{W}\big(\nabla v_k(d_k(x')/\varepsilon_n, \gamma x_3)\big)\,.$$

Next we compute for $x \in \Omega \cap \{|d_k(x')| < \ell_k\varepsilon_n/2\}$,

$$\begin{aligned}\big|\nabla^2_{h_n} u_{n,k}(x)\big|^2 = \frac{1}{\varepsilon_n^2}\big|\nabla^2 v_k(d_k(x')/\varepsilon_n,\gamma x_3)\big|^2 + \big|\partial_1 v_k(d_k(x')/\varepsilon_n,\gamma x_3)\big|^2|\nabla^2 d_k(x')|^2 \\ + \frac{2}{\varepsilon_n}\Big(\partial_1 v_k(d_k(x')/\varepsilon_n,\gamma x_3)\cdot\partial_1^2 v_k(d_k(x')/\varepsilon_n,\gamma x_3)\Big)\Big(\nabla^2 d_k(x')\cdot\big(\nabla d_k(x')\otimes\nabla d_k(x')\big)\Big)\,,\end{aligned}$$

which yields

$$\big|\nabla^2_{h_n} u_{n,k}(x)\big|^2 \leqslant \frac{1+\varepsilon_n}{\varepsilon_n^2}\big|\nabla^2 v_k(d_k(x')/\varepsilon_n,\gamma x_3)\big|^2 + \frac{C_k}{\varepsilon_n}\big|\partial_1 v_k(d_k(x')/\varepsilon_n,\gamma x_3)\big|^2\,,$$

for $x \in \Omega \cap \{|d_k(x')| < \ell_k\varepsilon_n/2\}$ and some constant C_k independent of n. Therefore,

$$F^{h_n}_{\varepsilon_n}(u_{n,k}) = F^{h_n}_{\varepsilon_n}\big(u_{n,k},\Omega\cap\{|d_k(x')|<\ell_k\varepsilon_n/2\}\big) \leqslant I^k_n + II^k_n\,, \tag{4.40}$$

with

$$I^k_n := \frac{1}{\varepsilon_n}\int_{\Omega\cap\{|d_k|<\ell_k\varepsilon_n/2\}} \mathcal{W}\big(\nabla v_k(d_k(x')/\varepsilon_n,\gamma x_3)\big) + \big|\nabla^2 v_k\big(d_k(x')/\varepsilon_n,\gamma x_3\big)\big|^2\,dx\,,$$

and

$$II^k_n := \int_{\Omega\cap\{|d_k|<\ell_k\varepsilon_n/2\}} \big|\nabla^2 v_k\big(d_k(x')/\varepsilon_n,\gamma x_3\big)\big|^2 + C_k\,\big|\partial_1 v_k\big(d_k(x')/\varepsilon_n,\gamma x_3\big)\big|^2\,dx\,.$$

Using Fubini's theorem, the Coarea Formula, the fact that $|\nabla d_k| = 1$, and changing variables we estimate

$$\begin{aligned} I^k_n =& \frac{1}{\varepsilon_n}\int_I\bigg(\int_{\omega\cap\{|d_k|<\ell_k\varepsilon_n/2\}}\mathcal{W}\big(\nabla v_k(d_k(x')/\varepsilon_n,\gamma x_3)\big) + \big|\nabla^2 v_k(d_k(x')/\varepsilon_n,\gamma x_3)\big|^2 dx'\bigg)dx_3 \\ =& \frac{1}{\varepsilon_n}\int_I\bigg(\int_{\ell_k\varepsilon_n I}\big(\mathcal{W}\big(\nabla v_k(t/\varepsilon_n,\gamma x_3)\big) + \big|\nabla^2 v_k(t/\varepsilon_n,\gamma x_3)\big|^2\big)\mathcal{H}^1(\mathcal{M}^k_t\cap\omega)\,dt\bigg)dx_3 \\ =& \frac{1}{\varepsilon_n}\int_{\ell_k\varepsilon_n I\times I}\Big(\mathcal{W}\big(\nabla v_k(t/\varepsilon_n,\gamma x_3)\big) + \big|\nabla^2 v_k(t/\varepsilon_n,\gamma x_3)\big|^2\Big)\,\mathcal{H}^1(\mathcal{M}^k_t\cap\omega)\,dtdx_3 \\ \leqslant& \frac{1}{\gamma}\int_{\ell_k I\times\gamma I}\Big(\mathcal{W}\big(\nabla v_k(y)\big) + \big|\nabla^2 v_k(y)\big|^2\Big)\,\mathcal{H}^1(\mathcal{M}^k_{\varepsilon_n y_1}\cap\overline{\omega})\,dy\,.\end{aligned}$$

Then Fatou's lemma, (4.38), and (4.27) yield

$$\limsup_{n\to+\infty} I^k_n \leqslant (K_\gamma + 2^{-k})\mathcal{H}^1(\mathcal{M}^k\cap\overline{\omega})\,. \tag{4.41}$$

Arguing in the same way we infer that

$$\lim_{n\to+\infty} II^k_n = \lim_{n\to+\infty}\frac{\varepsilon_n}{\gamma}\int_{\ell_k I\times\gamma I}\big(\big|\nabla^2 v_k(y)\big|^2 + C_k\big|\partial_1 v_k(y)\big|^2\big)\mathcal{H}^1(\mathcal{M}_{\varepsilon_n y_1}\cap\overline{\omega})\,dy = 0\,. \tag{4.42}$$

Gathering (4.40), (4.41) and (4.42), we derive

$$\limsup_{k\to+\infty}\limsup_{n\to+\infty} F^{h_n}_{\varepsilon_n}(u_{n,k}) \leqslant K_\gamma \mathrm{Per}_\omega(E)\,.$$

Since $\lim_k\lim_n\|u_{n,k}\|_{W^{1,p}(\Omega)} = 0$, and $\lim_k\lim_n\|\frac{1}{h_n}\partial_3 u_{n,k} - b\|_{L^p(\Omega)} = \lim_k\|b_k - b\|_{L^p(\Omega)} = 0$, the conclusion follows for a suitable diagonal sequence $u_n := u_{n,k_n}$ as in Step 1. □

5. Γ-convergence in the subcritical regime

This section is devoted to the proof of Theorem 1.3. The Γ-liminf inequality is obtained through a slicing argument, and by establishing a lower asymptotic inequality for a reduced 2D functional (see Proposition 5.1) much in the spirit of Section 4.1. The Γ-lim inf and Γ-lim sup inequalities are stated in Theorem 5.1 and Theorem 5.2 respectively, and Corollary 5.1 shows that lower and upper inequalities agree.

5.1. *The* Γ-lim inf *inequality*

For a bounded open set $A \subset \mathbb{R}^2$ and $\varepsilon > 0$, we introduce the localized functional $F^0_\varepsilon(\cdot,\cdot,A)$ defined for a pair $(u,b) \in H^2(A;\mathbb{R}^3) \times H^1(A;\mathbb{R}^3)$ by

$$F^0_\varepsilon(u,b,A) := \int_A \frac{1}{\varepsilon}\, W(\nabla' u, b) + \varepsilon\big(|(\nabla')^2 u|^2 + 2|\nabla' b|^2\big)\, dx' \,. \tag{5.1}$$

Then we consider the constant

$$K^\star_0 := \inf\Big\{ \liminf_{n\to+\infty} F^0_{\varepsilon_n}(u_n,b_n,Q') \;:\; \varepsilon_n \to 0^+\,,\; \{(u_n,b_n)\} \subset H^2(Q';\mathbb{R}^3)\times H^1(Q';\mathbb{R}^3)\,,$$
$$(u_n,b_n) \to (u_0,b_0) \text{ in } [L^1(Q';\mathbb{R}^3)]^2 \Big\}\,. \tag{5.2}$$

Here again the constant $K^\star_0$ is finite, as one may check by considering an admissible sequence $\{(u_n,b_n)\}$ made of suitable (standard) regularizations of u_0 and b_0. As in the previous section, we first provide a lower bound in terms of $K^\star_0$ for the lower Γ-limit of the family $\{F^0_\varepsilon\}$ in case of an elementary jump set.

Proposition 5.1. *Assume that assumptions* (H_1), (H_2) *and* (1.4) *hold. Let* $\varepsilon_n \to 0^+$ *be an arbitrary sequence. Let* $\rho > 0$ *and* $\alpha \in \mathbb{R}$, *let* $J \subset \mathbb{R}$ *be a bounded open interval, and consider the cylinder* $U' := (\alpha-\rho,\alpha+\rho)\times J$. *Let* $(u,b) \in W^{1,\infty}(U';\mathbb{R}^3)\times L^\infty(U';\mathbb{R}^3)$ *satisfying* (4.2). *Then for any sequence* $\{(u_n,b_n)\} \subset H^2(U';\mathbb{R}^3)\times H^1(U';\mathbb{R}^3)$ *such that* $(u_n,b_n) \to (u,b)$ *in* $[L^1(U';\mathbb{R}^3)]^2$, *we have*

$$\liminf_{n\to+\infty} F^0_{\varepsilon_n}(u_n,b_n,U') \geqslant K^\star_0 \mathcal{H}^1(J)\,.$$

Proof. Here the proof closely follows the one of Proposition 4.1. The arguments are essentially the same with the obvious modifications once we consider

$$\mathcal{E}_0(J,\rho) := \inf\Big\{ \liminf_{n\to+\infty} F^0_{\varepsilon_n}(u_n,b_n,J'_\rho) \;:\; \varepsilon_n \to 0^+\,,\; \{(u_n,b_n)\} \subset H^2(J'_\rho;\mathbb{R}^3)\times H^1(J'_\rho;\mathbb{R}^3)\,,$$
$$(u_n,b_n) \to (u_0,b_0) \text{ in } [L^1(J'_\rho;\mathbb{R}^3)]^2 \Big\}$$

in place of $\mathcal{E}_\gamma(J,\rho)$ with $J \subset \mathbb{R}$ a bounded open set, $\rho > 0$, and $J'_\rho := \rho I \times J$. Then one proves the analogue of Lemma 4.1, in particular that $\mathcal{E}_0(J,\rho) = K^\star_0\mathcal{H}^1(J)$. We omit any further details. □

Remark 5.1. As in Corollary 4.1, the energy of optimal sequences for $\mathcal{E}_0(J,\rho)$ is concentrated near the limiting interface, *i.e.*, given $0 < \delta < \rho$, for any sequences $\varepsilon_n \to 0^+$ and $\{(u_n,b_n)\} \subset H^2(J'_\rho;\mathbb{R}^3)\times H^1(J'_\rho;\mathbb{R}^3)$ such that $(u_n,b_n) \to (u_0,b_0)$ in $[L^1(J'_\rho;\mathbb{R}^3)]^2$ and $\lim_n F^0_{\varepsilon_n}(u_n,b_n,J'_\rho) = \mathcal{E}_0(J,\rho)$, we have $\lim_n F^0_{\varepsilon_n}(u_n,b_n,J'_\rho \setminus J'_\delta) = 0$.

We now prove the lower inequality for the Γ-lim inf of $\{F^h_\varepsilon\}$ essentially as in Theorem 4.1 together with a slicing argument involving the functionals $\{F^0_\varepsilon\}$.

Theorem 5.1. *Assume that assumptions* $(H_0)-(H_2)$, (H_5) *and* (1.4) *hold. Let* $h_n \to 0^+$ *and* $\varepsilon_n \to 0^+$ *be arbitrary sequences such that* $h_n/\varepsilon_n \to 0$. *Then, for any* $(u,b) \in \mathscr{C}$ *and any sequence* $\{u_n\} \subset H^2(\Omega;\mathbb{R}^3)$ *such that* $(u_n, \frac{1}{h_n}\partial_3 u_n) \to (u,b)$ *in* $[L^1(\Omega;\mathbb{R}^3)]^2$, *we have*

$$\liminf_{n\to+\infty} F^{h_n}_{\varepsilon_n}(u_n) \geqslant K^\star_0 \operatorname{Per}_\omega(E)\,,$$

where $(\nabla' u, b)(x) = \big(1-\chi_E(x')\big)A + \chi_E(x')B\,.$

Proof. *Step 1.* First we may assume that $\liminf_n F^{h_n}_{\varepsilon_n}(u_n) = \lim_n F^{h_n}_{\varepsilon_n}(u_n) < +\infty$. We set $b_n := \frac{1}{h_n}\partial_3 u_n \in H^1(\Omega;\mathbb{R}^3)$. It is well known that for $\mathcal{L}^1$-a.e. $x_3 \in I$ the slices $u_n^{x_3}(x') := u_n(x',x_3)$

and $b_n^{x_3}(x') := b_n(x', x_3)$ belong to $H^2(\omega;\mathbb{R}^3)$ and $H^1(\omega;\mathbb{R}^3)$ respectively, and horizontal weak derivatives coincide $\mathcal{L}^3$-a.e. in Ω (see *e.g.* [4, p. 204]). Moreover, up to a subsequence, $(u_n^{x_3}, b_n^{x_3}) \to (u, b)$ in $[L^1(\omega;\mathbb{R}^3)]^2$ for $\mathcal{L}^1$-a.e. $x_3 \in I$. Hence, using Fubini's theorem we can estimate

$$F_{\varepsilon_n}^{h_n}(u_n) = \int_I \left(\int_{\omega\times\{x_3\}} \frac{1}{\varepsilon_n} W(\nabla_{h_n} u_n) + \varepsilon_n |\nabla_{h_n}^2 u_n|^2 \, d\mathcal{H}^2 \right) dx_3 \geqslant \int_I F_{\varepsilon_n}^0 \left(u_n^{x_3}, b_n^{x_3}, \omega\right) dx_3 \,,$$

and then infer from Fatou's lemma that

$$\lim_{n\to+\infty} F_{\varepsilon_n}^{h_n}(u_n) \geqslant \int_I \liminf_{n\to+\infty} F_{\varepsilon_n}^0 \left(u_n^{x_3}, b_n^{x_3}, \omega\right) dx_3 \,.$$

Now it remains to prove that for $\mathcal{L}^1$-a.e. $x_3 \in I$,

$$\liminf_{n\to+\infty} F_{\varepsilon_n}^0 \left(u_n^{x_3}, b_n^{x_3}, \omega\right) \geqslant K_0^\star \, \mathrm{Per}_\omega(E) \,. \tag{5.3}$$

The next steps are devoted to the proof of (5.3).

Step 2. First assume that $A' \neq B'$. We obtain estimate (5.3) by applying Proposition 5.1 together with the covering argument used in the proof of Theorem 4.1, Step 1. Further details are left to the reader.

Step 3. We now consider the case $A' = B'$ ($= 0$ by (1.4)), and we may assume that $u \equiv 0$. Then consider an arbitrary sequence $\{(u_n, b_n)\} \subset H^2(\omega;\mathbb{R}^3) \times H^1(\omega;\mathbb{R}^3)$ satisfying $(u_n, b_n) \to (0, b)$ in $[L^1(\omega;\mathbb{R}^3)]^2$. We may also assume that $\liminf_n F_{\varepsilon_n}^0(u_n, b_n) = \lim_n F_{\varepsilon_n}^0(u_n, b_n) < +\infty$. By Theorem 2.1 we have $b(x') = \big(1 - \chi_E(x')\big)A_3 + \chi_E(x')B_3$ for a set $E \subset \omega$ of finite perimeter in ω. We prove the announced result following the blow-up argument in the proof of Theorem 4.1, Step 3. We introduce the finite nonnegative Radon measure μ_n on ω given by

$$\mu_n := \left(\frac{1}{\varepsilon_n} W\left(\nabla' u_n, b_n\right) + \varepsilon_n \left(\left|(\nabla')^2 u_n\right|^2 + 2\left|\nabla' b_n\right|^2 \right) \right) \mathscr{L}^2 \lfloor \omega \,.$$

Then $\mu_n(\omega) = F_{\varepsilon_n}^0(u_n, b_n)$, $\sup_n \mu_n(\omega) < +\infty$, and there is a subsequence (not relabeled) such that $\mu_n \rightharpoonup \mu$ weakly* in the sense of measures for some finite nonnegative Radon measure μ on ω. By lower semicontinuity we have $\mu(\omega) \leqslant \lim_n F_{\varepsilon_n}^0(u_n, b_n)$, and we have to prove that $\mu(\omega) \geqslant K_0^\star \mathcal{H}^1(\partial^* E \cap \omega)$. This estimate can be achieved as in the proof of Theorem 4.1, Step 3, with minor modifications. □

Remark 5.2. Let $\varepsilon_n \to 0^+$ be an arbitrary sequence. By the arguments above, for any $(u, b) \in \mathscr{C}$ and any sequence $\{(u_n, b_n)\} \subset H^2(\omega;\mathbb{R}^3) \times H^1(\omega;\mathbb{R}^3)$ satisfying $(u_n, b_n) \to (u, b)$ in $[L^1(\omega;\mathbb{R}^3)]^2$, we have $\liminf_n F_{\varepsilon_n}^0(u_n, b_n, \omega) \geqslant K_0^\star \, \mathrm{Per}_\omega(E)$ where $(\nabla' u, b) = \big(1 - \chi_E\big)A + \chi_E B$.

5.2. *Lower bound on* $K_0^\star$

As in Proposition 4.3, we now prove that sequences realizing $K_0^\star$ can be prescribed near the two sides $\{x_1 = \pm\frac{1}{2}\}$, and chosen to be independent of the x_2-variable.

Proposition 5.2. *Assume that* $(H_1) - (H_4)$ *and* (1.4) *hold. Then there exist sequences* $\varepsilon_n \to 0^+$, $\{c_n\} \subset \mathbb{R}^3$, *and* $\{(g_n, d_n)\} \subset C^2(Q';\mathbb{R}^3) \times C^1(Q';\mathbb{R}^3)$ *such that* (g_n, d_n) *is independent of* x_2 *(i.e.,* $g_n(x') =: \bar{g}_n(x_1)$ *and* $d_n(x') =: \bar{d}_n(x_1)$*),* $c_n \to 0$, $g_n \to u_0$ *in* $W^{1,p}(Q';\mathbb{R}^3)$, $d_n \to b_0$ *in* $L^p(Q';\mathbb{R}^3)$,

$$(g_n, d_n) = (u_0, b_0) \ \text{ in } Q' \cap \{x_1 > 1/4\}\,, \quad (g_n, d_n) = (u_0 + c_n, b_0) \ \text{ in } Q' \cap \{x_1 < -1/4\}\,,$$

and $\lim_n F_{\varepsilon_n}^0(g_n, d_n, Q') = K_0^\star$.

Proof. *Step 1.* Consider sequences $\varepsilon_n \to 0^+$ and $\{(u_n, b_n)\} \subset H^2(Q';\mathbb{R}^3) \times H^1(Q';\mathbb{R}^3)$ such that $(u_n, b_n) \to (u_0, b_0)$ in $[L^1(Q';\mathbb{R}^3)]^2$, and $\lim_n F_{\varepsilon_n}^0(u_n, b_n, Q') = K_0^\star$. Arguing as in the proof of Proposition 4.2, we may assume that $(u_n, b_n) \in C^2(Q';\mathbb{R}^3) \times C^1(Q';\mathbb{R}^3)$, and that (u_n, b_n) is independent of x_2, *i.e.*, $(u_n, b_n)(x) =: (\bar{u}_n(x_1), \bar{b}_n(x_1))$. Moreover, the arguments used in the proof of Theorem 1.1 (with minor modifications) yield $u_n \to u_0$ in $W^{1,p}(Q';\mathbb{R}^3)$, and $b_n \to b_0$ in $L^p(Q';\mathbb{R}^3)$.

Step 2. Here again we consider a partition of $\left(\frac{1}{12},\frac{1}{6}\right)$ into $M_n := \left[\frac{1}{\varepsilon_n}\right]$ intervals of length $\frac{1}{12M_n}$. By Remark 5.1, the energy concentrates near the interface $\{x_1 = 0\}$, and we can find a suitable interval $I_n := \left(\theta_n - \frac{1}{12M_n}, \theta_n\right) \subset \left(\frac{1}{12},\frac{1}{6}\right)$ for which

$$\begin{aligned} M_n &\left(\int_{I_n} |\bar u_n - \bar u_0|^p + |\bar b_n - \bar b_0|^p + |\bar u_n' - \bar u_0'|^p dx_1 + F^0_{\varepsilon_n}(u_n, b_n, I_n \times I) \right) \\ &\leqslant \int_{\left(\frac{1}{12},\frac{1}{6}\right)} |\bar u_n - \bar u_0|^p + |\bar b_n - \bar b_0|^p + |\bar u_n' - \bar u_0'|^p dx_1 + F^0_{\varepsilon_n}\left(u_n, b_n, \left(\tfrac{1}{12},\tfrac{1}{6}\right) \times I\right) =: \alpha_n \to 0\,. \end{aligned} \tag{5.4}$$

We select a level $t_n \in \left(\theta_n - \frac{1}{12M_n}, \theta_n\right)$ satisfying

$$\begin{aligned} &|\bar u_n(t_n) - \bar u_0(t_n)|^p + |\bar b_n(t_n) - \bar b_0(t_n)|^p + |\bar u_n'(t_n) - \bar u_0'(t_n)|^p \\ &\qquad + \frac{1}{\varepsilon_n} W\big(\bar u_n'(t_n), 0, \bar b_n(t_n)\big) + \varepsilon_n |\bar u_n''(t_n)|^2 + 2\varepsilon_n |\bar b_n'(t_n)|^2 \leqslant 12\alpha_n\,. \end{aligned} \tag{5.5}$$

Let $\varphi_n \in C^\infty(\mathbb{R})$ be a cut-off function as in (4.10). For $x_1 \in I_n$ we set

$$v_n(x_1) := \big(1 - \varphi_n(x_1)\big)\big(\bar u_0(x_1) + c_n^+\big) + \varphi_n(x_1)\bar u_n(x_1)\,,$$

with $c_n^+ := \bar u_n(t_n) - \bar u_0(t_n) \to 0$, and

$$\zeta_n(x_1) := \big(1 - \varphi_n(x_1)\big)\bar b_0(x_1) + \varphi_n(x_1)\bar b_n(x_1)\,,$$

We claim that

$$\int_{I_n} |v_n - \bar u_0|^p dx_1 \to 0\,, \tag{5.6}$$

$$\frac{1}{\varepsilon_n}\int_{I_n} \left|\zeta_n - \bar b_0\right|^p dx_1 \to 0\,, \tag{5.7}$$

$$\frac{1}{\varepsilon_n}\int_{I_n} |v_n' - \bar u_0'|^p dx_1 \to 0\,, \tag{5.8}$$

$$\frac{1}{\varepsilon_n}\int_{I_n} W(v_n', 0, \zeta_n)\, dx_1 \to 0\,, \tag{5.9}$$

$$\varepsilon_n \int_{I_n} |v_n''|^2 + 2|\zeta_n'|^2\, dx_1 \to 0\,. \tag{5.10}$$

Estimates (5.6) and (5.7) come straightforward from (5.4). We apply Poincaré's inequality to obtain

$$\int_{I_n} |\bar u_n - \bar u_0 - c_n^+|^p dx_1 \leqslant C\left(\frac{1}{M_n}\right)^p \int_{I_n} |\bar u_n' - \bar u_0'|^p\, dx_1 \leqslant C\alpha_n \varepsilon_n^{p+1}\,, \tag{5.11}$$

and using (5.4), (4.10), and (5.11), we derive

$$\frac{1}{\varepsilon_n}\int_{I_n} |v_n' - \bar u_0'|^p\, dx_1 \leqslant \frac{C}{\varepsilon_n}\int_{I_n} \left(\frac{1}{\varepsilon_n^p}|\bar u_n - \bar u_0 - c_n^+|^p + |\bar u_n' - \bar u_0'|^p\right) dx_1 \leqslant C\alpha_n \to 0\,,$$

so that (5.8) is proved. Now (5.9) follows from (5.7) and (5.8) exactly as (4.19). Finally we obtain (5.10) arguing as in the proof of Proposition 4.2 with minor modifications. We omit further details.

Step 3. We conclude as in the proof of Proposition 4.2, Step 4. We first define a sequence (g_n^+, d_n^+) by setting for $x' \in Q'$,

$$(g_n^+, d_n^+)(x') := \begin{cases} \big(\bar u_n(x_1), \bar b_n(x_1)\big) & \text{for } x_1 < \theta_n - \frac{1}{12M_n}\,, \\ \big(v_n(x_1), \zeta_n(x_1)\big) & \text{for } \theta_n - \frac{1}{12M_n} \leqslant x_1 < \theta_n\,, \\ \big(\bar u_0(x_1) + c_n^+, \bar b_0(x_1)\big) & \text{for } \theta_n \leqslant x_1 \leqslant \frac{1}{2}\,. \end{cases} \tag{5.12}$$

Then we repeat the procedure above to modify g_n^+ in $(-\frac{1}{2}, 0) \times I$. Again, we omit further details. □

Corollary 5.1. *Assume that* $(H_1)-(H_4)$ *and* (1.4) *hold. Then* $K_0^\star \geqslant K_0$.

Proof. Consider the sequences $\{\varepsilon_n\}$ and $\{(g_n,d_n)\}$ given by Proposition 5.2. We set $\ell_n := \varepsilon_n/2$, and for $t\in[-\ell_n,\ell_n]$, $\phi_n(t) := (\phi_{1,n},\phi_{2,n})(t) := \big(\bar g_n'(t/\varepsilon_n),\bar d_n(t/\varepsilon_n)\big)$. Then straightforward computations yield $\phi_n = (\bar u_0',\bar b_0)$ nearby $\{|t|=\ell_n\}$, and

$$\int_{-\ell_n}^{\ell_n} \mathcal{W}(\phi_{1,n}(t),\phi_{2,n}(t)) + |\phi_{1,n}'(t)|^2 + 2|\phi_{2,n}'(t)|^2\,dt = F^0_{\varepsilon_n}(g_n,d_n,Q')\,.$$

By definition of K_0 and the construction of $\{(g_n,d_n)\}$ we have $K_0 \leqslant F^0_{\varepsilon_n}(g_n,d_n,Q') \to K_0^\star$ as $n\to+\infty$, and the proof is complete. □

5.3. *The* **Γ-lim sup** *inequality*

We now complete the proof of Theorem 1.3 with the construction of recovery sequences.

Theorem 5.2. *Assume that* $(H_0)-(H_5)$ *and* (1.4) *hold. Let* $\varepsilon_n\to 0^+$ *and* $h_n\to 0^+$ *be arbitrary sequences such that* $h_n/\varepsilon_n\to 0$. *Then, for every* $(u,b)\in\mathscr{C}$, *there exists a sequence* $\{u_n\}\subset H^2(\Omega;\mathbb{R}^3)$ *such that* $u_n\to u$ *in* $W^{1,p}(\Omega;\mathbb{R}^3)$, $\frac{1}{h_n}\partial_3 u_n \to b$ *in* $L^p(\Omega;\mathbb{R}^3)$, *and*

$$\lim_{n\to+\infty} F^{h_n}_{\varepsilon_n}(u_n) = K_0\,\mathrm{Per}_\omega(E)\,,$$

where $(\nabla' u,b)(x) = \big(1-\chi_E(x')\big)A + \chi_E(x')B\,.$

Proof. The proof parallels the one of Theorem 4.2, and we shall refer to it for the notation.

Step 1. We first assume that $A'\neq B'$, and that $\partial^* E\cap\omega$ is made by finitely many interfaces. We also assume that the pair (u,b) is given by $(u,b)(x) = (\bar u,\bar b)(x_1)$ as in the proof of Theorem 4.2, Step 1.

For $k\in\mathbb{N}$ arbitrary, we choose some $\ell_k>0$ and $(\phi_{1,k},\phi_{2,k}):\mathbb{R}\to\mathbb{R}^{3\times 2}$ of class C^1 such that $(\phi_{1,k},\phi_{2,k}) = (\bar u_0',\bar b_0)$ in $\{|t|\geqslant \ell_k/2\}$, and

$$\int_{-\ell_k/2}^{\ell_k/2} \mathcal{W}(\phi_{1,k}(t),\phi_{2,k}(t)) + |\phi_{1,k}'(t)|^2 + 2|\phi_{2,k}'(t)|^2\,dt \leqslant K_0 + 2^{-k}\,. \tag{5.13}$$

Without loss of generality, we may assume that $\phi_{2,k}'$ is Lipschitz continuous. In the remaining of this step we shall drop the subscript k for simplicity. For each $i=1,\dots,m$ we fix some bounded open interval $J_i'\subset\mathbb{R}$ satisfying (4.29), and we consider for n large enough the coefficients $\{\alpha^n_{i\pm}\}$ as in (4.30), and such that (4.31) holds.

Let $h_n\to 0^+$ and $\varepsilon_n\to 0^+$ be arbitrary sequences such that $h_n/\varepsilon_n\to 0$, and define

$$\Phi(t) := \int_{-\ell/2}^{t} \phi_1(s)\,ds - c \qquad \text{with } c := \frac{1}{2}\int_{-\ell/2}^{\ell/2}\phi_1(s)\,ds\,. \tag{5.14}$$

We set for $i=1,\dots,m$ and $x\in(\alpha^n_{i-},\alpha^n_{i+})\times\mathbb{R}\times I$,

$$w^i_n(x) := (-1)^{i+1}\Phi\left((-1)^{i+1}\frac{x_1-\alpha_i}{\varepsilon_n}\right) + \frac{h_n}{\varepsilon_n}x_3\phi_2\left((-1)^{i+1}\frac{x_1-\alpha_i}{\varepsilon_n}\right) + \big(1+(-1)^i\big)\,c\,.$$

Then we have

$$w^i_n(\alpha^n_{i-},x_2,x_3) = \begin{cases} \Phi(-\ell/2) + \dfrac{h_n}{\varepsilon_n}x_3\phi_2(-\ell/2) & \text{if } i \text{ is odd}\,,\\[2mm] \Phi(\ell/2) + \dfrac{h_n}{\varepsilon_n}x_3\phi_2(\ell/2) & \text{if } i \text{ is even}\,,\end{cases} \tag{5.15}$$

and

$$w^i_n(\alpha^n_{i+},x_2,x_3) = \begin{cases} \Phi(\ell/2) + \dfrac{h_n}{\varepsilon_n}x_3\phi_2(\ell/2) & \text{if } i \text{ is odd}\,,\\[2mm] \Phi(-\ell/2) + \dfrac{h_n}{\varepsilon_n}x_3\phi_2(-\ell/2) + 4c & \text{if } i \text{ is even}\,.\end{cases} \tag{5.16}$$

Setting β_i^n and κ_i as in (4.34), we define for $x \in \Omega$,

$$u_n(x) := \begin{cases} \overline{u}(x_1) + \varepsilon_n \Phi(\frac{-\ell}{2}) + h_n x_3 \phi_2\big(\frac{-\ell}{2}\big) & \text{for } x_1 \leqslant \alpha_{1-}^n\,, \\ \overline{u}(\alpha_{i-}^n) - \beta_{i-1}^n + \varepsilon_n w_n^i(x) + \varepsilon_n \kappa_{i-1} c & \text{for } \alpha_{i-}^n < x_1 < \alpha_{i+}^n\,, \\ \overline{u}(x_1) - \beta_i^n + \varepsilon_n \Phi\big(\frac{(-1)^{i+1}\ell}{2}\big) + h_n x_3 \phi_2\big(\frac{(-1)^{i+1}\ell}{2}\big) + \varepsilon_n \kappa_i c & \text{for } \alpha_{i+}^n \leqslant x_1 \leqslant \alpha_{(i+1)-}^n\,, \\ \overline{u}(x_1) - \beta_m^n + \varepsilon_n \Phi\big(\frac{(-1)^{m+1}\ell}{2}\big) + h_n x_3 \phi_2\big(\frac{(-1)^{m+1}\ell}{2}\big) + \varepsilon_n \kappa_m c & \text{for } x_1 \geqslant \alpha_{m+}^n\,. \end{cases}$$

In view of (5.15)-(5.16), and since $\phi_2'(\pm\ell/2) = 0$, we have $u_n \in H^2(\Omega;\mathbb{R}^3)$. Moreover u_n does not depend on x_2,

$$\partial_1 u_n(x) = \begin{cases} \phi_1\left((-1)^{i+1}\dfrac{x_1-\alpha_i}{\varepsilon_n}\right) + (-1)^{i+1}\dfrac{h_n}{\varepsilon_n} x_3 \phi_2'\left((-1)^{i+1}\dfrac{x_1-\alpha_i}{\varepsilon_n}\right) & \text{for } \alpha_{i-}^n < x_1 < \alpha_{i+}^n\,, \\ \bar{u}'(x_1) & \text{otherwise}\,, \end{cases}$$

and

$$\frac{1}{h_n}\partial_3 u_n(x) = \begin{cases} \phi_2\left((-1)^{i+1}\dfrac{x_1-\alpha_i}{\varepsilon_n}\right) & \text{for } \alpha_{i-}^n < x_1 < \alpha_{i+}^n\,,\ i = 1,\dots,m\,, \\ \bar{b}(x_1) & \text{otherwise}\,. \end{cases}$$

Arguing as in the proof of Theorem 4.2, Step 1, we derive that $u_n \to u$ in $W^{1,p}(\Omega;\mathbb{R}^3)$, and $\frac{1}{h_n}\partial_3 u_n \to b$ in $L^p(\Omega;\mathbb{R}^3)$. Then we estimate

$$F_{\varepsilon_n}^{h_n}(u_n) \leqslant \sum_{i=1}^m F_{\varepsilon_n}^{h_n}\big(\varepsilon_n w_n^i, (\alpha_{i-}^n, \alpha_{i+}^n) \times J_i' \times I\big)\,.$$

Changing variables and using Fubini's theorem, we obtain

$$\begin{aligned} &F_{\varepsilon_n}^{h_n}(\varepsilon_n w_n^i, (\alpha_{i-}^n, \alpha_{i+}^n) \times J_i' \times I) \\ &\quad = \mathcal{H}^1(J_i') \int_{-\ell/2}^{\ell/2} \left(\int_I \mathcal{W}\big(\phi_1(t) + (-1)^{i+1}\frac{h_n}{\varepsilon_n} x_3 \phi_2'(t), \phi_2(t)\big)\, dx_3 + |\phi_1'(t)|^2 + 2|\phi_2'(t)|^2 \right) dt \\ &\qquad + \frac{\mathcal{H}^1(J_i') h_n^2}{12\varepsilon_n^2} \int_{-\ell/2}^{\ell/2} |\phi_2''(t)|^2\, dt\,. \end{aligned}$$

Since $\mathcal{W}$ is continuous and $h_n/\varepsilon_n \to 0$, we infer that

$$\lim_{n\to+\infty} F_{\varepsilon_n}^{h_n}(\varepsilon_n w_n^i, (\alpha_{i-}^n, \alpha_{i+}^n) \times J_i' \times I) = \mathcal{H}^1(J_i') \int_{-\ell/2}^{\ell/2} \mathcal{W}\left(\phi_1(t), \phi_2(t)\right) + |\phi_1'(t)|^2 + 2|\phi_2'(t)|^2\, dt\,,$$

which leads to

$$\limsup_{n\to+\infty} F_{\varepsilon_n}^{h_n}(u_n) \leqslant K_0 \operatorname{Per}_\omega(E) + C_0 2^{-k}\,,$$

for a constant C_0 which only depends on m and $\operatorname{Per}_\omega(E)$. Then the conclusion follows for a suitable diagonal sequence as already pursued in the proof of Theorem 4.2.

Step 2. In the case where $A' \neq B'$ and $\partial^* E \cap \omega$ is made by infinitely many interfaces, the proof follows from the previous step through a diagonalization argument as in the proof of Theorem 4.2.

Step 3. We now consider the case $A' = B'\,(= 0)$, and we proceed as in the proof of Theorem 4.2 (we refer to it for the notation). We may assume that $u = 0$, and $b(x) = (1-\chi_E(x'))A_3 + \chi_E(x')B_3$ where $E \subset \omega$ has finite perimeter in ω. We consider a sequence $\{E_k\}$ of smooth bounded subset of $\mathbb{R}^2$ such that $\chi_{E_k} \to \chi_E$ in $L^1(\omega)$, and $\lim_k \mathcal{H}^1(\partial E_k \cap \overline{\omega}) = \operatorname{Per}_\omega(E)$. We define $b_k := (1-\chi_{E_k})A_3 + \chi_{E_k}B_3$, and the signed distance d_k to $\mathcal{M}^k := \partial E_k$ as in (4.37). Here again we shall drop the subscript k for simplicity.

For $k \in \mathbb{N}$ arbitrary, we choose $\ell > 0$ and $(\phi_1, \phi_2) : \mathbb{R} \to \mathbb{R}^{3\times 2}$ of class C^1 satisfying $(\phi_1,\phi_2)(t) = (0, \bar{b}_0(t))$ nearby $\{|t| = \ell/2\}$ and (5.13). We may also assume ϕ_2' to be Lipschitz continuous. Defining Φ as in (5.14), we set for $x \in \Omega$,

$$u_n(x) = \begin{cases} \varepsilon_n \Phi\Big(\dfrac{d(x')}{\varepsilon_n}\Big) + h_n x_3 \phi_2\Big(\dfrac{d(x')}{\varepsilon_n}\Big) & \text{if } |d(x')| < \dfrac{\ell\varepsilon_n}{2}\,, \\ \varepsilon_n \Phi\Big(\dfrac{\ell}{2}\Big) + h_n x_3 \phi_2\Big(\dfrac{\ell}{2}\Big) & \text{if } d(x') \geqslant \dfrac{\ell\varepsilon_n}{2}\,, \\ \varepsilon_n \Phi\Big(-\dfrac{\ell}{2}\Big) + h_n x_3 \phi_2\Big(-\dfrac{\ell}{2}\Big) & \text{if } d(x') \leqslant -\dfrac{\ell\varepsilon_n}{2}\,. \end{cases}$$

Then $u_n \in H^2(\Omega;\mathbb{R}^3)$, and we compute

$$\nabla' u_n(x) = \begin{cases} \Big(\phi_1\Big(\dfrac{d(x')}{\varepsilon_n}\Big) + \dfrac{h_n}{\varepsilon_n} x_3 \phi_2'\Big(\dfrac{d(x')}{\varepsilon_n}\Big)\Big) \otimes \nabla d(x') & \text{if } |d(x')| < \dfrac{\ell\varepsilon_n}{2}\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

and

$$\frac{1}{h_n}\partial_3 u_n(x) = \begin{cases} \phi_2\Big(\dfrac{d(x')}{\varepsilon_n}\Big) & \text{if } |d(x')| < \dfrac{\ell\varepsilon_n}{2}\,, \\ b(x) & \text{otherwise}\,. \end{cases}$$

Since $|\nabla d| = 1$ $\mathcal{L}^2$-a.e. in $\mathbb{R}^2$, in the set $\{|d| < \ell\varepsilon_n/2\}$ we have

$$\begin{aligned} \big|\nabla^2_{h_n} u_n\big|^2 = \frac{1}{\varepsilon_n^2}\big|\phi_1'(d/\varepsilon_n) + \frac{h_n}{\varepsilon_n} x_3 \phi_2''(d/\varepsilon_n)\big|^2 + 2\big|\phi_2'(d/\varepsilon_n)\big|^2 \\ + \frac{1}{\varepsilon_n}\Big(\phi_1'(d/\varepsilon_n) + \frac{h_n}{\varepsilon_n} x_3\phi_2''(d/\varepsilon_n)\big) \cdot \big(\phi_1(d/\varepsilon_n) + \frac{h_n}{\varepsilon_n} x_3 \phi_2'(d/\varepsilon_n)\Big)\big(\nabla^2 d \cdot (\nabla d \otimes \nabla d)\big) \\ + \big|\phi_1(d/\varepsilon_n) + \frac{h_n}{\varepsilon_n} x_3 \phi_2'(d/\varepsilon_n)\big|^2 |\nabla^2 d|^2\,. \end{aligned}$$

As in the proof of Theorem 4.2, Step 3, we derive that $u_n \to 0$ in $W^{1,p}(\Omega;\mathbb{R}^3)$, and $\frac{1}{h_n}\partial_3 u_n \to b_k$ in $L^p(\Omega;\mathbb{R}^3)$. Then, using the fact that $|\nabla d| = 1$ and assumption (H_5), we estimate

$$F^{h_n}_{\varepsilon_n}(u_n) = F^{h_n}_{\varepsilon_n}\big(u_n, \Omega \cap \{|d(x')| < \ell\varepsilon_n/2\}\big) \leqslant I_n + C(\varepsilon_n + h_n^2/\varepsilon_n^2)\,, \tag{5.17}$$

with

$$\begin{aligned} I_n := \frac{1}{\varepsilon_n}\int_{\Omega\cap\{|d|<\ell\varepsilon_n/2\}} \mathcal{W}\Big(\phi_1\big(d(x')/\varepsilon_n\big) + \frac{h_n}{\varepsilon_n} x_3 \phi_2'\big(d(x')/\varepsilon_n\big), \phi_2\big(d(x')/\varepsilon_n\big)\Big)\, dx \\ + \frac{1}{\varepsilon_n}\int_{\Omega\cap\{|d|<\ell\varepsilon_n/2\}} \big|\phi_1'\big(d(x')/\varepsilon_n\big)\big|^2 + 2\,\big|\phi_2'\big(d(x')/\varepsilon_n\big)\big|^2\, dx\,, \end{aligned}$$

and a constant C independent of n. Using the Coarea Formula, we derive as in the proof of Theorem 4.2, Step 3, that

$$I_n = \int_{-\ell/2}^{\ell/2}\bigg(\int_I \mathcal{W}\Big(\phi_1(t) + \tfrac{h_n}{\varepsilon_n} x_3 \phi_2'(t), \phi_2(t)\Big)\, dx_3 + |\phi_1'(t)|^2 + 2\,|\phi_2'(t)|^2\bigg)\mathcal{H}^1(\mathcal{M}^k_{\varepsilon_n t} \cap \omega)\, dt\,.$$

Since $\mathcal{W}$ is continuous and $h_n/\varepsilon_n \to 0$, we infer from Fatou's lemma, (4.38), (5.13) and (5.17) that

$$\limsup_{n\to+\infty} F^{h_n}_{\varepsilon_n}(u_n) \leqslant (K_0 + 2^{-k})\mathcal{H}^1(\mathcal{M}^k \cap \overline{\omega})\,.$$

Then the conclusion follows for a suitable diagonal sequence as already pursued in the proof of Theorem 4.2, Step 3. □

Remark 5.3. Given $\varepsilon_n \to 0^+$, a slight modification of the above arguments yields for every $(u,b) \in \mathscr{C}$ a sequence $\{(u_n,b_n)\} \subset H^2(\omega:\mathbb{R}^3)\times H^1(\omega;\mathbb{R}^3)$ such that $(u_n,b_n) \to (u,b)$ in $W^{1,p}(\omega;\mathbb{R}^3)\times L^p(\omega;\mathbb{R}^3)$ and $\lim_n F^0_{\varepsilon_n}(u_n,b_n,\omega) = K_0\,\mathrm{Per}_\omega(E)$ where $(\nabla' u,b) = \big(1-\chi_E\big)A+\chi_E B$, and $F^0_{\varepsilon_n}$ is defined by (5.1).

Remark 5.4. Let us consider an arbitrary sequence $h_n \to 0^+$ and $\varepsilon > 0$ fixed. It is well known (see *e.g.* [9]) that the functionals $\{\mathscr{F}^{h_n}_\varepsilon\}$ Γ-converge for the strong L^1-topology to

$$\mathscr{F}^0_\varepsilon(u,b) := \begin{cases} F^0_\varepsilon(u,b,\omega) & \text{if } (u,b) \in H^2(\Omega:\mathbb{R}^3)\times H^1(\Omega;\mathbb{R}^3) \text{ and } \partial_3 u = \partial_3 b = 0\,,\\ +\infty & \text{otherwise}\,,\end{cases}$$

where $F^0_{\varepsilon_n}$ is defined by (5.1), and we have identified functions (u,b) satisfying $\partial_3 u = \partial_3 b = 0$ with functions defined on the midsurface ω. Let us now consider an arbitrary sequence $\varepsilon_n \to 0^+$. By Remark 5.2 and Remark 5.3, the functionals $\{\mathscr{F}^0_{\varepsilon_n}\}$ in turn Γ-converge for the strong L^1-topology to $\mathscr{F}_0$ (compactness follows as in Theorem 1.1 with minor modifications).

6. Γ-convergence in the supercritical regime

This section is essentially devoted to the proof of Theorem 1.4. The Γ-liminf inequality is a direct consequence of the results in Section 4.1 once we have proved that under assumption (1.9), $K^\star_\infty < +\infty$. In contrast with the lower inequality, the estimate for the Γ-lim sup requires a slightly more sophisticated construction based on an homogenization procedure. The Γ-lim inf and Γ-lim sup inequalities are stated in Theorem 6.1 and Theorem 6.2 respectively, and the conclusion follows from Lemma 6.1. For $p = 2$, $\lambda = 0$, and under the symmetry assumption on W, we obtain the Γ-convergence of the functionals through Corollary 6.1. In a last subsection, we consider the situation where the wells A and B are not compatible, and we illustrate some rigidity phenomena in Theorems 6.3 and 6.4.

6.1. *The* Γ-lim inf *inequality*

We define the constant $K^\star_\infty$ as in (4.1) with $\gamma = +\infty$, *i.e.*,

$$K^\star_\infty := \inf\bigg\{\liminf_{n\to+\infty} F^{h_n}_{\varepsilon_n}(u_n,Q) \,:\, h_n \to 0^+ \text{ and } \varepsilon_n \to 0^+ \text{ with } h_n/\varepsilon_n \to +\infty,\\ \{u_n\} \subset H^2(Q;\mathbb{R}^3),\, (u_n, \frac{1}{h_n}\partial_3 u_n) \to (u_0,b_0) \text{ in } [L^1(Q;\mathbb{R}^3)]^2\bigg\}. \tag{6.1}$$

We start by proving that if (1.9) holds, then $K^\star_\infty$ is finite and strictly positive.

Lemma 6.1. *Assume that* $(H_1)-(H_3)$ *and* (1.9) *hold for some* $\lambda \in \mathbb{R}$. *Then* $0 < K^\star_\infty < +\infty$.

Proof. Let us consider arbitrary sequences $\varepsilon_n \to 0^+$, $h_n \to 0^+$ such that $h_n/\varepsilon_n \to +\infty$. Observe that under assumption (1.9), we have $A = -B = (a,0,\lambda a)$ so that $A-B$ is rank-1 connected. By the results in [16], there exists a sequence $\{w_n\} \subset H^2\big((-1,1);\mathbb{R}^3\big)$ such that $w_n \to \bar{u}_0$ in $W^{1,p}\big((-1,1);\mathbb{R}^3\big)$, and

$$\sup_{n\in\mathbb{N}} \int_{-1}^1 \frac{1}{\varepsilon_n}\min\left\{|w_n'-a|^p, |w_n'+a|^p\right\} + \varepsilon_n |w_n''|^2\,dt < +\infty\,.$$

For n large enough, we consider the sequence $\{u_n\} \subset H^2(Q;\mathbb{R}^3)$ defined by $u_n(x) := w_n(x_1+\lambda h_n x_3)$. Then one may check that $(u_n, \frac{1}{h_n}\partial_3 u_n) \to (u_0,b_0)$ in $[L^1(Q;\mathbb{R}^3)]^2$. Using Lemma 4.3, we estimate

$$F^{h_n}_{\varepsilon_n}(u_n,Q) \leqslant C\int_{-1}^1 \frac{(1+\lambda^2)^{p/2}}{\varepsilon_n}\min\left\{|w_n'-a|^p, |w_n'+a|^p\right\} + \varepsilon_n(1+\lambda^2)^2|w_n''|^2\,dt\,,$$

which shows that $\sup_n F^{h_n}_{\varepsilon_n}(u_n,Q) < +\infty$, and thus $K^*_\infty < +\infty$. On the other hand, we have

$$K^\star_\infty \geqslant \inf\bigg\{\liminf_{n\to+\infty}\int_Q \frac{1}{\varepsilon_n}W(\nabla' u_n,b_n) + \varepsilon_n|(\nabla')^2 u_n|^2 + 2\varepsilon_n|\nabla' b_n|^2\,dx' \,:\, \varepsilon_n \to 0^+\,,\\ \{(u_n,b_n)\} \subset H^2(Q;\mathbb{R}^3)\times H^1(Q;\mathbb{R}^3)\,,\, (u_n,b_n) \to (u_0,b_0) \text{ in } [L^1(Q;\mathbb{R}^3)]^2\bigg\}.$$

In view of Lemma 4.3 and [16], we easily infer that

$$K^\star_\infty \geqslant \inf\Big\{\liminf_{n\to+\infty}\int_I \frac{1}{C_*\varepsilon_n}\min\left\{|v_n'-a|^p,|v_n'+a|^p\right\}+\varepsilon_n|v_n''|^2\,dt \,:\, \varepsilon_n\to 0^+\,,$$
$$\{v_n\}\subset H^2(I;\mathbb{R}^3)\,,\, v_n\to\bar u_0 \text{ in } L^1(I;\mathbb{R}^3)\Big\}$$
$$\geqslant \inf\Big\{\int_{-L}^{L}\frac{1}{C_*}\min\left\{|v(t)-a|^p,|v(t)+a|^p\right\}+|v'|^2\,dt \,:\, L>0\,,\, v \text{ piecewise } C^1\,,$$
$$v(L)=v(-L)=a\Big\}>0\,,$$

and the proof is complete. □

Thanks to Lemma 6.1 and Remark 4.2, we can now reproduce the first step in the proof of Theorem 4.1 to obtain the following result.

Theorem 6.1. *Assume that* $(H_0)-(H_3)$ *and* (1.9) *hold for some* $\lambda\in\mathbb{R}$. *Let* $h_n\to 0^+$ *and* $\varepsilon_n\to 0^+$ *be arbitrary sequences such that* $h_n/\varepsilon_n\to+\infty$. *Then, for any* $(u,b)\in\mathscr{C}$ *and any sequences* $\{u_n\}\subset H^2(\Omega;\mathbb{R}^3)$ *such that* $(u_n,\frac{1}{h_n}\partial_3 u_n)\to(u,b)$ *in* $[L^1(\Omega;\mathbb{R}^3)]^2$, *we have*

$$\liminf_{n\to+\infty} F^{h_n}_{\varepsilon_n}(u_n)\geqslant K^\star_\infty\,\mathrm{Per}_\omega(E)\,,$$

where $(\nabla' u,b)(x)=\big(1-\chi_E(x')\big)A+\chi_E(x')B$.

6.2. *Lower bound on $K^\star_\infty$ in the case $\lambda=0$*

As a direct consequence of Lemma 4.3, we have the following elementary property in the case $\lambda=0$.

Lemma 6.2. *Assume that* $(H_1)-(H_3)$ *and* (1.9) *hold with* $\lambda=0$. *Then there is a constant* $C_W>0$ *such that* $W(\xi)\geqslant C_W|\xi_3|^p$ *for all* $\xi=(\xi_1,\xi_2,\xi_3)\in\mathbb{R}^{3\times 3}$.

In parallel with Proposition 4.2, the next propositions will establish that realizing sequences for $K^\star_\infty$ can be first chosen with lateral boundary conditions, and then periodic in the vertical direction.

Proposition 6.1. *Assume that* $(H_1)-(H_4)$ *and* (1.9) *hold with* $\lambda=0$. *Then there exist sequences* $h_n\to 0^+$, $\varepsilon_n\to 0$, $\{c_n\}\subset\mathbb{R}^3$, *and* $\{g_n\}\subset C^2(Q;\mathbb{R}^3)$ *such that* $h_n/\varepsilon_n\to+\infty$, g_n *is independent of* x_2 *(i.e.,* $g_n(x)=:\hat g_n(x_1,x_3)$*),* $c_n\to 0$, $g_n\to u_0$ *in* $W^{1,p}(Q;\mathbb{R}^3)$, $\frac{1}{h_n}\partial_3 g_n\to 0$ *in* $L^p(Q;\mathbb{R}^3)$,

$$g_n=u_0 \;\text{ in } Q\cap\{x_1<-1/4\}\,,\quad g_n=u_0+c_n \;\text{ in } Q\cap\{x_1<-1/4\}\,,$$

and $\lim_n F^{h_n}_{\varepsilon_n}(g_n,Q)=K^\star_\infty$.

Proof. *Step 1.* Since $\lambda=0$ we have $b_0=0$, and in view of Lemma 6.1, there exist sequences $h_n\to 0^+$, $\varepsilon_n\to 0^+$ and $\{u_n\}\subset H^2(Q;\mathbb{R}^3)$ such that $h_n/\varepsilon_n\to+\infty$, $(u_n,\frac{1}{h_n}\partial_3 u_n)\to(u_0,0)$ in $[L^1(Q;\mathbb{R}^3)]^2$, and $\lim_n F^{h_n}_{\varepsilon_n}(u_n,Q)=K^\star_\infty<+\infty$. Arguing as in the proof of Proposition 4.2, we may assume that $u_n\in C^2(Q;\mathbb{R}^3)$, and that u_n is independent of x_2, *i.e.*, $u_n(x)=:\hat u_n(x_1,x_3)$. By Theorem 1.1, $u_n\to u_0$ in $W^{1,p}(Q;\mathbb{R}^3)$, and $\frac{1}{h_n}\partial_3 u_n\to 0$ in $L^p(Q;\mathbb{R}^3)$.

Step 2 (first matching). As in the proof of Proposition 4.2 we consider a partition of $(\frac{1}{12},\frac16)\times Q'$ into $M_n:=\big[\frac{1}{\varepsilon_n}\big]$ layers along the x_1-direction. By Lemma 6.1 and Remark 4.2, we can find such a layer $L_n:=(\theta_n-\frac{1}{12M_n},\theta_n)\times Q'\subset(\frac{1}{12},\frac16)\times Q'$ such that (4.8) holds (with $b_0=0$). Then select a level $t_n\in\big(\theta_n-\frac{1}{12M_n},\theta_n\big)$ for which (4.9) holds. We consider a cut-off function $\varphi_n\in C^\infty(\mathbb{R})$ satisfying (4.10), and we set for $x\in L_n$,

$$v_n(x):=\big(1-\varphi_n(x_1)\big)\big(\bar u_0(x_1)+\bar u_n(x_3)\big)+\varphi_n(x_1)u_n(x)\,,$$

with $\bar u_n(x_3):=\hat u_n(t_n,x_3)-\bar u_0(t_n)$.

We claim that estimates (4.11), (4.12), (4.13), (4.14), and (4.15) still hold (with $b_0 = 0$). First note that (4.11) is an easy consequence of (4.8) and (4.9). In view of Lemma 6.2, we infer from (4.9) that

$$\frac{1}{\varepsilon_n}\int_I \frac{1}{h_n^p}|\bar{u}_n'(x_3)|^p dx_3 + \varepsilon_n \int_I \frac{1}{h_n^4}|\bar{u}_n''(x_3)|^2 dx_3 \leqslant C\alpha_n\,. \tag{6.2}$$

Combining (4.8) and (6.2) yields (4.12). By construction $u_n - u_0 - \bar{u}_n = 0$ on $\{x_1 = t_n\} \cap Q$, and applying Poincaré's inequality we deduce from (4.8),

$$\int_{L_n} |u_n - u_0 - \bar{u}_n|^p\, dx \leqslant C\left(\frac{1}{M_n}\right)^p \int_{L_n} |\partial_1 u_n - \partial_1 u_0|^p dx \leqslant C\alpha_n\varepsilon_n^{p+1}\,. \tag{6.3}$$

Using (4.10), we may now infer that

$$\frac{1}{\varepsilon_n}\int_{L_n} |\nabla' v_n - \nabla' u_0|^p dx \leqslant \frac{C}{\varepsilon_n}\int_{L_n} |\partial_1 u_n - \partial_1 u_0|^p + \frac{1}{\varepsilon_n^p}|u_n - u_0 - \bar{u}_n|^p dx \leqslant C\alpha_n \to 0\,. \tag{6.4}$$

Estimates (4.12) and (4.13) being proved, (4.14) now follows exactly as in (4.19).

Using again (4.10), we estimate

$$\varepsilon_n \int_{L_n} |\nabla^2_{h_n} v_n|^2\, dx \leqslant C\bigg(\varepsilon_n \int_{L_n} |\nabla^2_{h_n} u_n|^2 dx + \frac{1}{\varepsilon_n}\int_{L_n} |\nabla' u_n - \nabla' u_0|^2 dx$$
$$+ \frac{1}{\varepsilon_n^3}\int_{L_n} |u_n - u_0 - \bar{u}_n|^2 dx + \frac{1}{\varepsilon_n}\int_{L_n} |\frac{1}{h_n}\partial_3 u_n|^2 dx + \int_I \frac{1}{h_n^2}|\bar{u}_n'(x_3)|^2 dx_3 + \varepsilon_n^2 \int_I \frac{1}{h_n^4}|\bar{u}_n''(x_3)|^2 dx_3\bigg)\,,$$

Arguing as in the proof of Proposition 4.2, Step 2, (4.15) now follows from (4.8), (6.2), (6.3) and (6.4) together with Hölder's inequality.

Step 3 (second matching). Let $\psi_n \in C^\infty(\mathbb{R})$ be a cut-off function such that $0 \leqslant \psi_n \leqslant 1$, $\psi_n(t) = 1$ if $t \leqslant \theta_n$, $\psi_n(t) = 0$ if $t \geqslant 1/4$, and satisfying $|\psi_n'| + |\psi_n''| \leqslant C$ for a constant C independent of n. For $x \in \{\theta_n < x_1 < \frac{1}{4}\} \cap Q$, we set

$$w_n(x) := u_0(x) + c_n^+ + \psi_n(x_1)\big(\bar{u}_n(x_3) - c_n^+\big)\,,$$

where $c_n^+ := \int_I \bar{u}_n\, dx_3 \to 0$, thanks to (4.9). We claim that (4.20), (4.21), (4.22), and (4.23) hold.

First (4.20) and (4.21) are direct consequences of (4.9) and (6.2) respectively. Next we apply (6.2) and Poincaré's inequality to derive that

$$\frac{1}{\varepsilon_n}\int_{\{\theta_n < x_1 < \frac{1}{6}\}\cap Q} |\nabla' w_n - \nabla' u_0|^p dx \leqslant \frac{C}{\varepsilon_n}\int_I |\bar{u}_n(x_3) - c_n^+|^p dx_3 \leqslant C h_n^p \alpha_n \to 0\,. \tag{6.5}$$

To prove (4.22), we can argue exactly as in (4.19) using (4.21) and (6.5).

We finally obtain in much similar ways that

$$\varepsilon_n \int_{\left(\theta_n, \frac{1}{4}\right)\times Q'} \left|\nabla^2_{h_n} w_n\right|^2 dx \leqslant C\varepsilon_n \int_I |\bar{u}_n - c_n^+|^2 + \frac{1}{h_n^2}|\bar{u}_n'|^2 + \frac{1}{h_n^4}|\bar{u}_n''|^2\, dx_3 \leqslant C\alpha_n \to 0\,,$$

and (4.23) is proved.

Step 4. We conclude the proof as in Proposition 4.2, Step 4. We first define g_n^+ as in (4.25) (with $b_0 = 0$), and then we repeat the procedure to modify g_n^+ in $(-\frac{1}{2}, 0) \times Q'$. We omit further details. □

We now prove that, in the case where $p = 2$, $\lambda = 0$, and W is symmetric in ξ_3, optimal sequences for $K_\infty^\star$ can be modified into 1-periodic functions in the x_3-variable without increasing the energy.

Proposition 6.2 (vertical periodicity). *Assume that* $(H_1) - (H_4)$ *and* (1.9) *hold with* $p = 2$, $\lambda = 0$, *and that* $W(\xi', \xi_3) = W(\xi', -\xi_3)$ *for every* $(\xi', \xi_3) \in \mathbb{R}^{3\times 2} \times \mathbb{R}^3$. *Then there exist sequences* $h_n \to 0^+$, $\varepsilon_n \to 0^+$, *and* $\{f_n\} \subset C^2(\mathbb{R}^3; \mathbb{R}^3)$ *such that* $h_n/\varepsilon_n \to +\infty$, f_n *is independent of* x_2 *(i.e.,* $f_n(x) = \hat{f}_n(x_1, x_3)$*),* $f_n \to u_0$ *in* $H^1(Q; \mathbb{R}^3)$, $\frac{1}{h_n}\partial_3 f_n \to 0$ *in* $L^2(Q; \mathbb{R}^3)$, f_n *is 1-periodic in the* x_3*-variable,* $\nabla f_n = \nabla u_0$ *in* $\{|x_1| > 1/4\}$, *and*

$$\lim_{n\to+\infty} F_{\varepsilon_n}^{h_n}(f_n, Q) = K_\infty^\star\,.$$

Proof. *Step 1.* We claim that it suffices to find sequences $h_n \to 0^+$, $\varepsilon_n \to 0^+$, and $\{g_n^\sharp\} \subset C^2(\mathbb{R}^3;\mathbb{R}^3)$ such that $h_n/\varepsilon_n \to +\infty$, $g_n^\sharp(x) =: \hat{g}_n^\sharp(x_1,x_3)$, $\nabla g_n^\sharp = \nabla u_0$ in $\{|x_1| > 1/4\}$, $g_n^\sharp$ is 2-periodic in x_3, $g_n^\sharp \to u_0$ in $H^1(Q;\mathbb{R}^3)$, $\frac{1}{h_n}\partial_3 g_n^\sharp \to 0$ in $L^2(Q;\mathbb{R}^3)$, and $\limsup_n F_{\varepsilon_n}^{h_n}(g_n^\sharp, 2Q) \leqslant 4K_\infty^\star$. Indeed, if the claim holds we set $f_n(x) := \frac{1}{2}g_n^\sharp(2x)$ for $x \in \mathbb{R}^3$. Then $f_n \to u_0$ in $H^1(Q;\mathbb{R}^3)$, $\frac{1}{h_n}\partial_3 f_n \to 0$ in $L^2(Q;\mathbb{R}^3)$. By definition of $K_\infty^\star$, a change of variables yields

$$K_\infty^\star \leqslant \liminf_{n\to+\infty} F_{\frac{\varepsilon_n}{2}}^{h_n}(f_n,Q) \leqslant \limsup_{n\to+\infty} F_{\frac{\varepsilon_n}{2}}^{h_n}(f_n,Q) = \limsup_{n\to+\infty} \frac{1}{4} F_{\varepsilon_n}^{h_n}(g_n^\sharp, 2Q) \leqslant K_\infty^\star\,,$$

and thus $\{f_n\}$ satisfies the requirements (with $\varepsilon_n/2$ instead of ε_n).

Step 2. Let $h_n \to 0^+$ and $\varepsilon_n \to 0^+$ satisfying $h_n/\varepsilon_n \to +\infty$. Consider an arbitrary sequence $\{u_n\} \subset H^2(Q;\mathbb{R}^3)$ such that $(u_n, \frac{1}{h_n}\partial_3 u_n) \to (u_0, b_0)$ in $[L^1(Q;\mathbb{R}^3)]^2$, and $\lim_n F_{\varepsilon_n}^{h_n}(u_n,Q) = K_\infty^\star$. We claim that for any $0 < \delta < 1/2$, we have

$$\limsup_{n\to+\infty} F_{\varepsilon_n}^{h_n}\big(u_n, Q' \times \big((1/2-\delta, 1/2) \cup (-1/2, -1/2+\delta)\big)\big) \leqslant 2\delta K_\infty^\star\,.$$

This is of course equivalent to the following inequality,

$$\liminf_{n\to+\infty} F_{\varepsilon_n}^{h_n}\left(u_n, Q' \times (-1/2+\delta, 1/2-\delta)\right) \geqslant (1-2\delta)K_\infty^\star\,, \tag{6.6}$$

that we prove by rescaling. For $x \in Q$, we set $v_n(x) := u_n(x', (1-2\delta)x_3)$ and $\tilde{h}_n := (1-2\delta)h_n$. Then $\tilde{h}_n/\varepsilon_n \to +\infty$ and $(v_n, \frac{1}{\tilde{h}_n}\partial_3 v_n) \to (u_0, b_0)$ in $[L^1(Q;\mathbb{R}^3)]^2$. Therefore,

$$K_\infty^\star \leqslant \liminf_{n\to+\infty} F_{\varepsilon_n}^{h_n}(v_n,Q) = \liminf_{n\to+\infty} \frac{1}{1-2\delta} F_{\varepsilon_n}^{h_n}\left(u_n, Q' \times (-1/2+\delta, 1/2-\delta)\right)\,,$$

and (6.6) follows.

Step 3. Consider the sequences $\{h_n\}$, $\{\varepsilon_n\}$, and $\{g_n\} \subset C^2(Q;\mathbb{R}^3)$ given by Proposition 6.1, and let us fix $m \in \mathbb{N}$ arbitrarily large. We infer from Step 2 (with $\delta = 1/m$) that

$$\limsup_{n\to+\infty} F_{\varepsilon_n}^{h_n}\left(g_n, Q \cap \left\{\frac{1}{2} - \frac{1}{m} < |x_3| < \frac{1}{2}\right\}\right) \leqslant \frac{2}{m}K_\infty^\star\,. \tag{6.7}$$

Next we divide $Q' \times (\frac{1}{2} - \frac{1}{m}, \frac{1}{2})$ into $[\frac{h_n}{\varepsilon_n}]$ thin horizontal strips $R_{m,n,i}^+$ of width $\frac{1}{m}[\frac{h_n}{\varepsilon_n}]^{-1}$, *i.e.*,

$$R_{m,n,i}^+ := Q' \times \left(\frac{1}{2} - \frac{i}{m}\left[\frac{h_n}{\varepsilon_n}\right]^{-1}, \frac{1}{2} - \frac{i-1}{m}\left[\frac{h_n}{\varepsilon_n}\right]^{-1}\right)$$

for $i = 1, \ldots, [\frac{h_n}{\varepsilon_n}]$. We proceed symmetrically in the set $Q' \times (-\frac{1}{2}, -\frac{1}{2} + \frac{1}{m})$, and we denote by $R_{m,n,i}^-$ the resulting strips. Applying Lemma 6.2, we infer from (6.7) that for n large enough,

$$\sum_{i=1}^{[\frac{h_n}{\varepsilon_n}]} \int_{R_{m,n,i}^- \cup R_{m,n,i}^+} \left(\frac{1}{\varepsilon_n}W(\nabla_{h_n} g_n) + \varepsilon_n \left|\nabla_{h_n}^2 g_n\right|^2 + |\nabla' g_n - \nabla' u_0|^2 \right.$$
$$\left. + \frac{C_W}{\varepsilon_n}\left|\frac{1}{h_n}\partial_3 g_n\right|^2 + |g_n - u_0|^2\right) dx \leqslant \frac{4}{m}K_\infty^\star\,.$$

where we also have used the fact that $\|g_n - u_0\|_{H^1(Q)} \to 0$. Now consider a pair of strips $(R_{m,n,i_0}^-, R_{m,n,i_0}^+)$ with $i_0 = i_0(m,n)$ satisfying

$$\int_{R_{m,n,i_0}^- \cup R_{m,n,i_0}^+} \left(\frac{1}{\varepsilon_n}W(\nabla_{h_n} g_n) + \varepsilon_n \left|\nabla_{h_n}^2 g_n\right|^2 + |\nabla' g_n - \nabla' u_0|^2 \right.$$
$$\left. + \frac{C_W}{\varepsilon_n}\left|\frac{1}{h_n}\partial_3 g_n\right|^2 + |g_n - u_0|^2\right) dx \leqslant \frac{4}{m}\left[\frac{h_n}{\varepsilon_n}\right]^{-1} K_\infty^\star\,, \tag{6.8}$$

and we shall write for simplicity $R^\pm_{m,n} := R^\pm_{m,n,i_0}$ (respectively). Then we choose a level

$$t_{m,n} \in \left(\frac{1}{2} - \frac{i_0 - 1/2}{m} \left[\frac{h_n}{\varepsilon_n}\right]^{-1}, \frac{1}{2} - \frac{i_0 - 1}{m} \left[\frac{h_n}{\varepsilon_n}\right]^{-1} \right)$$

for which

$$\int_{Q\cap\{|x_3|=t_{m,n}\}} \left(\frac{1}{\varepsilon_n} W(\nabla_{h_n} g_n) + \varepsilon_n \left|\nabla^2_{h_n} g_n\right|^2 + |\nabla' g_n - \nabla' u_0|^2 \right.$$
$$\left. + \frac{C_W}{\varepsilon_n} \left|\frac{1}{h_n}\partial_3 g_n\right|^2 + |g_n - u_0|^2 \right) d\mathcal{H}^2 \leqslant 8K^\star_\infty \,. \quad (6.9)$$

Let $\varphi_{m,n} : \mathbb{R} \to [0,1]$ be a smooth cut-off function such that $\varphi_{m,n}(t) = 0$ for $t > t_{m,n}$, $\varphi_{m,n}(t) = 1$ for $t < t_{m,n} - \frac{1}{2m}\left[\frac{h_n}{\varepsilon_n}\right]^{-1}$, and

$$\frac{\varepsilon_n}{mh_n}|\varphi'_{m,n}| + \frac{\varepsilon_n^2}{m^2h_n^2}|\varphi''_{m,n}| \leqslant C \,, \quad (6.10)$$

for a constant C independent of m and n. We define for $x \in Q$,

$$w_{m,n}(x) := \varphi_{m,n}(x_3)g_n(x) + \big(1 - \varphi_{m,n}(x_3)\big)\hat{g}_n(x_1, t_{m,n}) \,.$$

We shall prove in Step 4 below that

$$\limsup_{m\to+\infty} \limsup_{n\to+\infty} \|w_{m,n} - u_0\|_{H^1(Q)} + \Big\|\frac{1}{h_n}\partial_3 w_{m,n}\Big\|_{L^2(Q)} = 0 \,, \quad (6.11)$$

and

$$\limsup_{m\to+\infty} \limsup_{n\to+\infty} F^{h_n}_{\varepsilon_n}(w_{m,n}, Q) \leqslant K^\star_\infty \,. \quad (6.12)$$

Assuming for the moment that (6.11) and (6.12) hold, we find a diagonal sequence $n_m \to +\infty$ such that setting $\varepsilon_m := \varepsilon_{n_m}$, $h_m := h_{n_m}$, and $w_m := w_{m,n_m}$, we have $w_m \to u_0$ in $H^1(Q;\mathbb{R}^3)$, $\frac{1}{h_m}\partial_3 w_m \to 0$ in $L^2(Q;\mathbb{R}^3)$, and $\limsup_m F^{h_m}_{\varepsilon_m}(w_m, Q) \leqslant K^\star_\infty$. We now repeat this construction in the strip $R^-_{m,n}$, and we write $\tilde{w}_m$ the resulting function.

Since $\tilde{w}_m$ is independent of x_3 in a neighborhood of $\{|x_3| = 1/2\} \cap Q$, we may first reflect $\tilde{w}_m$ across the hyperplane $\{x_3 = 1/2\}$ setting for $\frac{1}{2} \leqslant x_3 \leqslant \frac{3}{2}$, $\tilde{w}_m(x', x_3) := w_m(x', 1 - x_3)$, and then we extend $\tilde{w}_m$ by periodicity to all values of x_3. The resulting function $\tilde{w}_m$ belongs to $C^2(Q' \times \mathbb{R};\mathbb{R}^3)$. Since $\nabla\tilde{w}_m = \nabla u_0$ in $\{|x_1| > 1/4\}$, we can extend linearly $\tilde{w}_m$ in x_1, and constantly in x_2. We finally set for $x \in \mathbb{R}^3$, $g^\sharp_m(x) := \tilde{w}_m\big(x', x_3 - \frac{1}{2}\big)$. Since $W(\xi', -\xi_3) = W(\xi', \xi_3)$ for all $\xi \in \mathbb{R}^{3\times3}$, we find that

$$F^{h_m}_{\varepsilon_m}(g^\sharp_m, 2Q) = 4F^{h_m}_{\varepsilon_m}(\tilde{w}_m, Q) \,,$$

so that the function $g^\sharp_m$ satisfies all the requirements of Step 1.

Step 4. We now complete the proof by showing that (6.11) and (6.12) do hold. To this purpose we shall write

$$L^+_{m,n} := Q' \times \left(\frac{1}{2} - \frac{i_0}{m}\left[\frac{h_n}{\varepsilon_n}\right]^{-1}, \frac{1}{2} \right) .$$

We first estimate

$$\begin{aligned} F^{h_n}_{\varepsilon_n}(w_{m,n}, Q) = {} & F^{h_n}_{\varepsilon_n}(g_n, Q \setminus L^+_{m,n}) + F^{h_n}_{\varepsilon_n}(w_{m,n}, R^+_{m,n}) \\ & + \int_{L^+_{m,n}\setminus R^+_{m,n}} \frac{1}{\varepsilon_n} W\big(\partial_1\hat{g}_n(x_1, t_{m,n}), 0, 0\big) + \varepsilon_n\big|\partial_1^2\hat{g}_n(x_1, t_{m,n})\big|^2 \, dx \\ \leqslant {} & F^{h_n}_{\varepsilon_n}(g_n, Q) + F^{h_n}_{\varepsilon_n}(w_{m,n}, R^+_{m,n}) \\ & + \frac{1}{m}\int_{-1/2}^{1/2} \frac{1}{\varepsilon_n} W\big(\partial_1\hat{g}_n(x_1, t_{m,n}), 0, 0\big) + \varepsilon_n\big|\partial_1^2\hat{g}_n(x_1, t_{m,n})\big|^2 \, dx_1 \,. \end{aligned} \quad (6.13)$$

By Lemma 4.3 and Lemma 6.2, we have

$$W\big(\partial_1\hat{g}_n(x_1,t_{m,n}),0,0\big) \leqslant C\left(W\big(\nabla_{h_n}g_n(x',t_{m,n})\big)+\big|\frac{1}{h_n}\partial_3 g_n(x',t_{m,n})\big|^2\right) \leqslant CW\big(\nabla_{h_n}g_n(x',t_{m,n})\big)\,,$$

so that (6.9) yields

$$\frac{1}{m}\int_{-1/2}^{1/2}\frac{1}{\varepsilon_n}W\big(\partial_1\hat{g}_n(x_1,t_{m,n}),0,0\big)+\varepsilon_n\big|\partial_1^2\hat{g}_n(x_1,t_{m,n})\big|^2\,dx_1 \leqslant \frac{C}{m}\int_{Q\cap\{x_3=t_{m,n}\}}\frac{1}{\varepsilon_n}W(\nabla_{h_n}g_n)+\varepsilon_n\left|\nabla^2_{h_n}g_n\right|^2\,d\mathcal{H}^2 \leqslant \frac{C}{m}\,. \tag{6.14}$$

Similarly, we infer from (6.9) that

$$\begin{aligned}\int_Q |w_{m,n}-u_0|^2+\big|\frac{1}{h_n}\partial_3 w_{m,n}\big|^2\,dx &\leqslant \int_Q |g_n-u_0|^2+\big|\frac{1}{h_n}\partial_3 g_n\big|^2\,dx+\int_{R^+_{m,n}}|w_{m,n}-u_0|^2+\big|\frac{1}{h_n}\partial_3 w_{m,n}\big|^2\,dx\\ &\qquad+\frac{1}{m}\int_{-1/2}^{1/2}|\hat{g}_n(x_1,t_{m,n})-\bar{u}_0(x_1)|^2\,dx_1\\ &\leqslant \int_Q |g_n-u_0|^2+\big|\frac{1}{h_n}\partial_3 g_n\big|^2\,dx+\int_{R^+_{m,n}}|w_{m,n}-u_0|^2+\big|\frac{1}{h_n}\partial_3 w_{m,n}\big|^2\,dx+\frac{C}{m}\,.\end{aligned} \tag{6.15}$$

In view of (6.13), (6.14), (6.15), and Theorem 1.1, to prove (6.11) and (6.12) it suffices to show that for every $m\in\mathbb{N}$ large enough,

$$\lim_{n\to+\infty}\int_{R^+_{m,n}}|w_{m,n}-u_0|^2+\big|\frac{1}{h_n}\partial_3 w_{m,n}\big|^2\,dx=0\,, \tag{6.16}$$

and

$$\lim_{n\to+\infty}F^{h_n}_{\varepsilon_n}(w_{m,n},R^+_{m,n})=0\,. \tag{6.17}$$

We start with the proof of (6.17). Writing for $x\in R^+_{m,n}$,

$$\partial_1 w_{m,n}(x)=\partial_1 g_n(x)+\big(1-\varphi_{m,n}(x_3)\big)\big(\partial_1\hat{g}_n(x_1,t_{m,n})-\partial_1 g_n(x)\big)\,,$$

and

$$\frac{1}{h_n}\partial_3 w_{m,n}(x)=\frac{1}{h_n}\partial_3 g_n(x)-\big(1-\varphi_{m,n}(x_3)\big)\frac{1}{h_n}\partial_3 g_n(x)+\frac{\varphi'_{m,n}(x_3)}{h_n}\big(g_n(x)-\hat{g}_n(x_1,t_{m,n})\big)\,,$$

we derive from Lemma 4.3, Lemma 6.2, and (6.10) that

$$W\big(\nabla_{h_n}w_{m,n}(x)\big)\leqslant C\bigg(W\big(\nabla_{h_n}g(x)\big)+|\partial_1 g_n(x)-\partial_1\hat{g}_n(x_1,t_{m,n})|^2+\frac{m^2}{\varepsilon_n^2}\big|g_n(x)-\hat{g}_n(x_1,t_{m,n})\big|^2\bigg)\,. \tag{6.18}$$

Using Poincaré's inequality and (6.8), we estimate

$$\frac{m^2}{\varepsilon_n^3}\int_{R^+_{m,n}}\big|g_n(x)-\hat{g}_n(x_1,t_{m,n})\big|^2\,dx\leqslant C\frac{1}{\varepsilon_n}\int_{R^+_{m,n}}\big|\frac{1}{h_n}\partial_3 g_n(x)\big|^2\,dx\leqslant\frac{C}{m}\frac{\varepsilon_n}{h_n}\,, \tag{6.19}$$

and

$$\frac{1}{\varepsilon_n}\int_{R^+_{m,n}}|\partial_1 g(x)-\partial_1\hat{g}_n(x_1,t_{m,n})|^2\,dx\leqslant C\frac{\varepsilon_n}{m^2h_n^2}\int_{R^+_{m,n}}|\partial^2_{13}g(x)|^2\,dx\leqslant\frac{C}{m^3}\frac{\varepsilon_n}{h_n}\,. \tag{6.20}$$

In view of (6.18), we have thus obtained

$$\frac{1}{\varepsilon_n}\int_{R^+_{m,n}}W(\nabla_{h_n}w_{m,n})\,dx\leqslant C\left(\frac{1}{\varepsilon_n}\int_{R^+_{m,n}}W(\nabla_{h_n}g_n)\,dx+\frac{\varepsilon_n}{h_n}\right)\leqslant C\frac{\varepsilon_n}{h_n}\underset{n\to+\infty}{\longrightarrow}0\,.$$

Then, straightforward computations using (6.10) yield

$$\left|\nabla^2_{h_n} w_{m,n}(x)\right|^2 \leqslant C\bigg(\left|\nabla^2_{h_n} g_n(x)\right|^2 + \left|\partial^2_1 \hat{g}_n(x_1, t_{m,n})\right|^2 + \frac{m^2}{\varepsilon_n^2}\left|\partial_1 g_n(x) - \partial_1 \hat{g}_n(x_1, t_{m,n})\right|^2 + \frac{m^2}{\varepsilon_n^2}\left|\frac{1}{h_n}\partial_3 g_n(x)\right|^2 + \frac{m^4}{\varepsilon_n^4}\left|g_n(x) - \hat{g}_n(x_1, t_{m,n})\right|^2\bigg).$$

Combining (6.8), (6.9), (6.19), and (6.20), we deduce that

$$\varepsilon_n \int_{R^+_{m,n}} \left|\nabla^2_{h_n} w_{m,n}\right|^2 dx \leqslant Cm\frac{\varepsilon_n}{h_n} \underset{n\to+\infty}{\longrightarrow} 0\,,$$

which completes the proof of (6.17).

Using (6.8), (6.9), and (6.19), we finally estimate

$$\int_{R^+_{m,n}} |w_{m,n} - u_0|^2 + \left|\frac{1}{h_n}\partial_3 w_{m,n}\right|^2 dx \leqslant C\bigg(\int_{R^+_{m,n}} |g_n - u_0|^2 + \left|\frac{1}{h_n}\partial_3 g_n\right|^2 dx + \frac{m^2}{\varepsilon_n^2}\int_{R^+_{m,n}} \left|g_n(x) - \hat{g}_n(x_1, t_{m,n})\right|^2 dx + \frac{\varepsilon_n}{h_n}\int_{Q\cap\{x_3 = t_{m,n}\}} |g_n - u_0|^2\, d\mathcal{H}^2\bigg) \leqslant C\frac{\varepsilon_n}{h_n}\,,$$

and (6.16) is proved. □

Corollary 6.1. *Assume that* $(H_1) - (H_4)$ *and* (1.9) *hold with* $p = 2$, $\lambda = 0$, *and that* $W(\xi', \xi_3) = W(\xi', -\xi_3)$ *for every* $\xi = (\xi', \xi_3) \in \mathbb{R}^{3\times 3}$. *Then* $K^\star_\infty \geqslant K_\infty$.

Proof. We consider the sequences $h_n \to 0^+$, $\varepsilon_n \to 0^+$, and $\{f_n\} \subset C^2(\mathbb{R}^3;\mathbb{R}^3)$ given by Proposition 6.2. We define $N_n := [\frac{1}{h_n}]$, $\rho_n := \frac{1}{N_n h_n}$, and $\ell_n := \frac{1}{\rho_n \varepsilon_n}$ ($[\cdot]$ still denotes the integer part). Recalling that $f_n(x) = \hat{f}_n(x_1, x_3)$, we define for $y \in \mathbb{R}^2$,

$$v_n(y) := \rho_n \hat{f}_n\left(\frac{y_1}{\rho_n}, \frac{y_2}{\rho_n h_n}\right).$$

Then v_n is $1/N_n$-periodic in the y_2-variable, and $\nabla v_n(y) = (\bar{u}_0(y_1), 0)$ in $\{|y_1| > \frac{\rho_n}{4}\}$. Since v_n is $1/N_n$-periodic in y_2, and N_n being an integer, we deduce that v_n is also 1-periodic in y_2. Moreover, since $\rho_n \to 1$, we have for n large enough

$$\nabla v_n(y) = (\bar{u}_0(y_1), 0) \text{ in } \{|y_1| > 1/3\}\,. \tag{6.21}$$

Hence,

$$\int_{Q'} \ell_n\, \mathcal{W}(\nabla v_n) + \frac{1}{\ell_n}|\nabla^2 v_n|^2\, dy \geqslant K_\infty\,.$$

Changing variables, using (6.21) and the 1-periodicity in x_3 of f_n, we compute

$$\int_{Q'} \ell_n\, \mathcal{W}(\nabla v_n) + \frac{1}{\ell_n}|\nabla^2 v_n|^2\, dy = \rho_n h_n F^{h_n}_{\varepsilon_n}(f_n, Q' \times N_n I) = F^{h_n}_{\varepsilon_n}(f_n, Q) \underset{n\to+\infty}{\longrightarrow} K^\star_\infty\,,$$

which completes the proof. □

6.3. *The* Γ-lim sup *inequality*

The next theorem provides the announced upper bound for the $\Gamma - \limsup$ of the functionals $\{F^h_\varepsilon\}$ when $\varepsilon \ll h$, and thus completing the proof of Theorem 1.4.

Theorem 6.2. *Assume that* $(H_0) - (H_4)$ *and* (1.9) *hold for some* $\lambda \in \mathbb{R}$. *Let* $\varepsilon_n \to 0^+$ *and* $h_n \to 0^+$ *be arbitrary sequences such that* $h_n/\varepsilon_n \to +\infty$. *Then, for every* $(u, b) \in \mathscr{C}$, *there exists a sequence* $\{u_n\} \subset H^2(\Omega;\mathbb{R}^3)$ *such that* $u_n \to u$ *in* $W^{1,p}(\Omega;\mathbb{R}^3)$, $\frac{1}{h_n}\partial_3 u_n \to b$ *in* $L^p(\Omega;\mathbb{R}^3)$, *and*

$$\limsup_{n\to+\infty} F^{h_n}_{\varepsilon_n}(u_n) \leqslant K_\infty \operatorname{Per}_\omega(E)\,, \tag{6.22}$$

where $(\nabla' u, b)(x) = \big(1-\chi_E(x')\big)A + \chi_E(x')B$.

Proof. We first introduce some useful notation. For a given a sequence $h_n \to 0^+$, we define

$$\nu_n := \frac{e_1 + \lambda h_n e_3}{\sqrt{1+\lambda^2 h_n^2}} \in \mathbb{S}^2 \quad \text{and} \quad \nu_n^\perp := \frac{-\lambda h_n e_1 + e_3}{\sqrt{1+\lambda^2 h_n^2}} \in \mathbb{S}^2 .$$

We recall that Q'_λ denotes the unit cube of $\mathbb{R}^2$ centered at the origin with two faces orthogonal to the unit vector $\nu_\lambda = \frac{1}{\sqrt{1+\lambda^2}}(1,\lambda)$.

By Theorem 2.1 and (1.9), $\partial^* E \cap \omega$ is of the form (2.4). We assume that $\partial^* E \cap \omega$ is made by finitely many interfaces, *i.e.*, $\mathscr{I} = \{1,\dots,m\}$ in (2.4). The proof for infinitely many interfaces follows from a diagonalization argument as in the proof of Theorem 4.2. Then $u(x) = \bar{u}(x_1)$ for some function $\bar{u}$ that we may assume to be as in the proof of Theorem 4.2, Step 1 (we refer to it for the notation). Then (1.9) yields $b(x) = \lambda \bar{u}'(x_1)$.

Let us now consider for each $k \in \mathbb{N}$, some $\ell_k > 0$ and some function $v_k \in C^2(\mathbb{R}^2;\mathbb{R}^3)$ 1-periodic in the direction $\nu_\lambda^\perp := \frac{1}{\sqrt{1+\lambda^2}}(-\lambda,1)$, satisfying $\nabla v_k(y) = \pm(a,\lambda a)$ nearby $\{y\cdot\nu_\lambda = \pm 1/2\}$ respectively, and such that

$$\int_{Q'_\lambda} \ell_k \mathcal{W}(\nabla v_k) + \frac{1}{\ell_k}|\nabla^2 v_k|^2\,dy \leqslant \frac{K_\infty + 2^{-k}}{\sqrt{1+\lambda^2}} .$$

Without loss of generality we may assume that

$$v_k(y) = \begin{cases} \sqrt{1+\lambda^2}\,(y\cdot\nu_\lambda)a + c_k & \text{nearby } \{y\cdot\nu_\lambda = 1/2\}\,, \\ -\sqrt{1+\lambda^2}\,(y\cdot\nu_\lambda)a - c_k & \text{nearby } \{y\cdot\nu_\lambda = -1/2\}\,, \end{cases} \tag{6.23}$$

for some constant $c_k \in \mathbb{R}^3$. From now on we drop the subscript k for simplicity.

Let $\varepsilon_n \to 0^+$ and $h_n \to 0^+$ be arbitrary sequences such that $h_n/\varepsilon_n \to +\infty$. Again we choose for each index $i = 1,\dots,m$, an bounded open interval $J'_i \subset \mathbb{R}$ such that $J_i \subset\subset J'_i$ and $\mathcal{H}^1(J'_i \setminus J_i) \leqslant 2^{-k}$. We write

$$\alpha^n_{i\pm} := \frac{1}{\sqrt{1+\lambda^2 h_n^2}}\left(\alpha_i \pm \frac{\ell\varepsilon_n\sqrt{1+\lambda^2}}{2}\right),$$

and we consider integers n large enough in such a way that $\alpha^n_{i+} < \alpha^n_{(i+1)-}$ for every i, and for which (4.31) holds. We define the transition layers as follows: for $i = 1,\dots,m$ and for $x \in \mathbb{R}^3$, we set

$$w^i_n(x) := (-1)^{i+1} v\left((-1)^{i+1}\frac{x_1-\alpha_i}{\ell\varepsilon_n}, (-1)^{i+1}\frac{h_n x_3}{\ell\varepsilon_n}\right) + \big(1+(-1)^i\big)\left(\frac{1}{2}\sqrt{1+\lambda^2}\,a + c\right).$$

Then (6.23) yields

$$w^i_n(x) = \frac{1}{2}\sqrt{1+\lambda^2}\,a - (-1)^{i+1}c \quad \text{on } \left\{x\cdot\nu_n = \alpha^n_{i-}\right\}, \tag{6.24}$$

and

$$w^i_n(x) = \left(\frac{1}{2}\sqrt{1+\lambda^2}\,a + (-1)^{i+1}c\right) + 2(1+(-1)^i)c \quad \text{on } \left\{x\cdot\nu_n = \alpha^n_{i+}\right\}. \tag{6.25}$$

Setting

$$\beta^n_i := \sum_{j=1}^{i} \bar{u}\left(\alpha^n_{j+}\sqrt{1+\lambda^2 h_n^2}\right) - \bar{u}\left(\alpha^n_{j-}\sqrt{1+\lambda^2 h_n^2}\right),$$

with $\beta_0^n := 0$ and κ_i as in (4.34), we define for n large enough and $x \in \Omega$,

$$u_n(x) := \begin{cases} \bar{u}(x_1 + \lambda h_n x_3) + \ell\varepsilon_n\big(\frac{a}{2}\sqrt{1+\lambda^2} - c\big) & \text{for } x\cdot\nu_n \leqslant \alpha^n_{1-}\,, \\ \bar{u}\left(\alpha^n_{i-}\sqrt{1+\lambda^2 h_n^2}\right) - \beta^n_{i-1} + \ell\varepsilon_n\big(w^i_n(x) + \kappa_{i-1}c\big) & \text{for } \alpha^n_{i-} < x\cdot\nu_n < \alpha^n_{i+}\,, \\ \bar{u}(x_1 + \lambda h_n x_3) - \beta^n_i + \ell\varepsilon_n\big(\frac{a}{2}\sqrt{1+\lambda^2} + ((-1)^{i+1} + \kappa_i)c\big) & \text{for } \alpha^n_{i+} \leqslant x\cdot\nu_n \leqslant \alpha^n_{(i+1)-}\,, \\ \bar{u}(x_1 + \lambda h_n x_3) - \beta^n_m + \ell\varepsilon_n\big(\frac{a}{2}\sqrt{1+\lambda^2} + ((-1)^{m+1} + \kappa_m)c\big) & \text{for } x\cdot\nu_n \geqslant \alpha^n_{m+}\,. \end{cases}$$

Using (6.24)-(6.25) one may check that u_n and ∇u_n are continuous across each interface $\{x\cdot\nu_n = \alpha^n_{i\pm}\}$, and thus $u_n \in H^2(\Omega;\mathbb{R}^3)$. In addition $\partial_2 u_n \equiv 0$, and

$$\left(\partial_1 u_n, \frac{1}{h_n}\partial_3 u_n\right)(x) = \begin{cases} \nabla v\left((-1)^{i+1}\dfrac{x_1 - \alpha_i}{\ell\varepsilon_n}\,, (-1)^{i+1}\dfrac{h_n x_3}{\ell\varepsilon_n}\right) & \text{for } \alpha^n_{i-} < x\cdot\nu_n < \alpha^n_{i+}\,, \\ \big(\bar{u}'(x_1 + \lambda h_n x_3), \lambda\bar{u}'(x_1 + \lambda h_n x_3)\big) & \text{otherwise}\,. \end{cases} \tag{6.26}$$

Then one observes that the maps $x \in \Omega \mapsto \bar{u}(x_1 + \lambda h_n x_3)$ and $x \in \Omega \mapsto \bar{u}'(x_1 + \lambda h_n x_3)$ converge to u and b in $W^{1,p}(\Omega;\mathbb{R}^3)$ and in $L^p(\Omega;\mathbb{R}^3)$ respectively as $n \to +\infty$ (here we also use the fact that $b = \lambda\bar{u}'$). On the other hand, v and ∇v are bounded in $\{|y\cdot\nu_\lambda| \leqslant 1/2\}$ by periodicity in the direction $\nu_\lambda^\perp$, and $|\beta^n_i| \leqslant C\varepsilon_n$ for a constant C independent of i and n by the Lipschitz continuity of $\bar{u}$. Hence $u_n \to u$ in $W^{1,p}(\Omega;\mathbb{R}^3)$ and $\frac{1}{h_n}\partial_3 u_n \to b$ in $L^p(\Omega;\mathbb{R}^3)$.

By (4.31) we have for n large,

$$\Omega \cap \big\{\alpha^n_{i-} < x\cdot\nu_n < \alpha^n_{i+}\big\} \subset \big\{x \in \mathbb{R}^3 : \alpha^n_{i-} < x\cdot\nu_n < \alpha^n_{i+}\,, |x_3| < 1/2\,, x_2 \in J_i'\big\} =: \Omega^n_i\,,$$

Using (6.26) we estimate for n large enough,

$$F^{h_n}_{\varepsilon_n}(u_n) \leqslant \sum_{i=1}^m F^{h_n}_{\varepsilon_n}\left(\ell\varepsilon_n w^i_n, \Omega^n_i\right)\,, \tag{6.27}$$

and it remains to estimate each term of the sum in the right-hand side of (6.27).

Changing variables, one obtains

$$F^{h_n}_{\varepsilon_n}\left(\ell\varepsilon_n w^i_n, \Omega^n_i\right) = \frac{\ell\varepsilon_n}{h_n}\mathcal{H}^1(J_i')\int_{\Theta^n_i} \ell\,\mathcal{W}(\nabla v) + \frac{1}{\ell}\left|\nabla^2 v\right|^2\,dy\,, \tag{6.28}$$

where $\Theta^n_i := \{y \in \mathbb{R}^2 : |y\cdot\nu_\lambda| < 1/2\,, |y_2| < h_n/(2\ell\varepsilon_n)\}$. Notice that for every $t \in (-\frac{1}{2}, \frac{1}{2})$, we have

$$\begin{aligned}\Theta^n_i \cap \{y\cdot\nu_\lambda = t\} = \{y\cdot\nu_\lambda = t\} \cap \left\{|y\cdot\nu_\lambda^\perp + \lambda t| < \frac{h_n\sqrt{1+\lambda^2}}{2\ell\varepsilon_n}\right\} \\ \subset \{y\cdot\nu_\lambda = t\} \cap \left\{|y\cdot\nu_\lambda^\perp + \lambda t| < \frac{N_n}{2}\right\}\,,\end{aligned}$$

with $N_n := \left[\frac{h_n\sqrt{1+\lambda^2}}{\ell\varepsilon_n}\right] + 1$. Using Fubini's theorem and the periodicity of v, we estimate

$$\begin{aligned}\int_{\Theta^n_i} \ell\,\mathcal{W}(\nabla v) + \frac{1}{\ell}\left|\nabla^2 v\right|^2\,dy &= \int_{-\frac{1}{2}}^{\frac{1}{2}}\left(\int_{\Theta^n_i\cap\{y\cdot\nu_\lambda = t\}} \ell\,\mathcal{W}(\nabla v) + \frac{1}{\ell}\left|\nabla^2 v\right|^2\,d\mathcal{H}^1\right)dt \\ &\leqslant \int_{-\frac{1}{2}}^{\frac{1}{2}}\left(\int_{\{y\cdot\nu_\lambda = t\}\cap\left\{|y\cdot\nu_\lambda^\perp + \lambda t| < \frac{N_n}{2}\right\}} \ell\,\mathcal{W}(\nabla v) + \frac{1}{\ell}\left|\nabla^2 v\right|^2\,d\mathcal{H}^1\right)dt \\ &\leqslant N_n\int_{Q_\lambda'} \ell\,\mathcal{W}(\nabla v) + \frac{1}{\ell}\left|\nabla^2 v\right|^2\,dy\,.\end{aligned} \tag{6.29}$$

Combining (6.28) with (6.29) yields

$$F^{h_n}_{\varepsilon_n}\left(\ell\varepsilon_n w^i_n, \Omega^n_i\right) \leqslant \frac{\ell\varepsilon_n N_n}{h_n\sqrt{1+\lambda^2}}\left(K_\infty + 2^{-k}\right)\mathcal{H}^1(J_i')\,.$$

Summing up over i this last inequality, and passing to the limit $n \to +\infty$ in (6.27) leads to

$$\limsup_{n\to+\infty} F^{h_n}_{\varepsilon_n}(u_n) \leqslant K_\infty \operatorname{Per}_\omega(E) + C_0 2^{-k}\,,$$

for a constant C_0 independent of k. Then the conclusion follows for a suitable diagonal sequence as already pursued in the proof of Theorem 4.2, Step 3. □

6.4. *Rigidity results*

For $\varepsilon \ll h$ we expect the thin film to behave like a three dimensional sample by separation of scales, so that sequences with uniformly bounded energy should have trivial limits under suitable assumptions on A and B. The first situation we can consider is when $A' = B'$. Indeed, in this case if we first perform the asymptotic $\varepsilon \to 0$, the limiting configurations u with finite energy must satisfy $\nabla u = (1-\chi_K(x_3))A + \chi_K(x_3))B$ for some finite set $S \subset I$, and the Γ-limit is proportional to $\frac{1}{h}\mathrm{Card}(S)\mathcal{L}^2(\omega)$, see [16]. This latter energy can be bounded with respect to h only if $\mathrm{Card}(S) = 0$ for h small, and it formally explain the expected rigidity effect. We have rigorously proved this fact only in the case where ε is sufficiently small relative to h as stated in the following theorem.

Theorem 6.3. *Assume $(H_0)-(H_3)$ and (1.4) hold with $A' = B'$. Let $h_n \to 0^+$ and $\varepsilon_n \to 0^+$ be arbitrary sequences such that $\sup_n \varepsilon_n/h_n^p < +\infty$. Then, for any $\{u_n\} \subset H^2(\Omega;\mathbb{R}^3)$ such that $\sup_n F^{h_n}_{\varepsilon_n}(u_n) < +\infty$, there exist a subsequence (not relabeled) and $\xi_0 \in \{A, B\}$ such that $\nabla_{h_n} u_n \to \xi_0$ in $L^p(\Omega;\mathbb{R}^{3\times3})$.*

Proof. By Theorem 1.1, we can find a subsequence such that $\nabla_{h_n} u \to (\nabla' u, b)$ in $L^p(\Omega;\mathbb{R}^{3\times3})$ for some $(u, b) \in \mathscr{C}$. Since $A' = B'$, $\nabla' u$ is constant, and we only have to prove that b is constant.

By (1.4) we have $A' = B' = 0$, and thus Lemma 4.3 yields

$$W(\xi) \geqslant \frac{1}{C_*}\big(|\xi'|^p + \min\big\{|\xi_3 - A_3|^p, |\xi_3 - B_3|^p\big\}\big)\,, \quad \forall \xi \in \mathbb{R}^{3\times3}\,,$$

Setting $v_n := \frac{1}{h_n} u_n$, we deduce that

$$\sup_{n\in\mathbb{N}} \int_\Omega \frac{h_n^p}{\varepsilon_n}|\nabla' v_n|^p + \frac{1}{\varepsilon_n}\min\big\{|\partial_3 v_n - A_3|^p, |\partial_3 v_n - B_3|^p\big\}\,dx < +\infty\,.$$

Hence $\{\nabla v_n\}$ is bounded in $L^p(\Omega;\mathbb{R}^3)$. By Poincaré's inequality, there exists a further subsequence (not relabeled) such that $v_n - \fint_\Omega v_n \rightharpoonup v$ weakly in $W^{1,p}(\Omega;\mathbb{R}^3)$ for some $v \in W^{1,p}(\Omega;\mathbb{R}^3)$. But since $\partial_3 v_n = \frac{1}{h_n}\partial u_n$, we have $\partial_3 v_n \to b$ strongly in $L^p(\Omega;\mathbb{R}^3)$. Hence $\partial_3\big(v(x) - b(x')x_3\big) = 0$, and we can argue as in Theorem 2.1, Step 2, to prove that $v(x) = b(x')x_3 + w(x')$ for some function $w \in BV(\omega;\mathbb{R}^3)$. Integrating this equality in x_3 over the interval I yields $w(x') = \int_I v(x', x_3)\,dx_3$ a.e. in ω. It obviously implies that $w \in W^{1,p}(\omega;\mathbb{R}^3)$. Since $b(x')x_3 = v(x) - w(x')$, we conclude that $b \in W^{1,p}(\omega;\{A_3, B_3\})$, and thus b must be constant. □

The other case where one can expect rigidity is when A and B are not rank-one connected, and thus not compatible in the bulk [6]. We will show that rigidity occurs at least for some particular potentials W as a consequence of a two-wells rigidity estimate due to Chaudhuri & Müller [14] (see [21] for single well rigidity). The class of double-well potentials we consider is as follows. For simplicity we will assume that

$$A = I_d\,, \quad \text{and} \quad B = \mathrm{diag}(\theta_1, 1, \theta_2)\,, \tag{6.30}$$

for some $\theta_1, \theta_2 \in \mathbb{R}$ satisfying

$$\theta_i > 0 \quad i = 1, 2\,, \quad \text{and} \quad (1-\theta_1)(1-\theta_2) > 0\,. \tag{6.31}$$

Here I_d denotes the 3×3 identity matrix. The second assumption in (6.31) corresponds to the strong incompatibility condition between A and B in the sense of Matos [29] (see also [14,15]). Noticing that

A' and B' are rank-one connected, we shall consider continuous potentials $W : \mathbb{R}^{3\times 3} \to [0, +\infty[$ such that $(H_1) - (H_3)$ hold with $p = 2$.

Using the rigidity estimate of [14] and an argument similar to [15], we have obtained the following result.

Theorem 6.4. *Assume $(H_0) - (H_3)$ hold with $p = 2$, (6.30), and (6.31). Let $h_n \to 0^+$ and $\varepsilon_n \to 0^+$ be arbitrary sequences such that $\varepsilon_n/h_n \to +\infty$. Then, for any sequence $\{u_n\} \subset H^2(\Omega;\mathbb{R}^3)$ such that $\sup_n F^{h_n}_{\varepsilon_n}(u_n) < +\infty$, there exist a subsequence (not relabeled) and $\xi_0 \in \{A, B\}$ such that $\nabla_{h_n} u_n \to \xi_0$ in $L^2(\Omega;\mathbb{R}^{3\times 3})$.*

Proof. By Theorem 1.1, there is a subsequence such that $u_n - \fint_\Omega u_n\, dx \to u$ in $H^1(\Omega;\mathbb{R}^3)$ and $\frac{1}{h_n}\partial_3 u_n \to b$ in $L^2(\Omega;\mathbb{R}^3)$ for some $(u, b) \in \mathscr{C}$. To prove the announced result, it suffices to prove that for an arbitrary open set $A \subset \omega$, $(\nabla' u, b)$ is constant in $A \times I$. Without loss of generality we may assume that $A = Q'$ the unit cube of $\mathbb{R}^2$. We proceed as follows.

Step 1. First we infer from Lemma 4.3 that

$$W(\xi) \geqslant \frac{1}{C_*} \min\left\{ \min_{R\in SO(3)} |\xi - R|^2, \min_{R\in SO(3)} |\xi - RB|^2 \right\} = \frac{1}{C_*}\mathrm{dist}^2\big(\xi, K\big) \qquad \forall \xi \in \mathbb{R}^{3\times 3},$$

where $K := SO(3) \cup SO(3)B$. From this estimate we deduce that

$$\frac{1}{h_n}\int_Q \mathrm{dist}^2\big(\nabla_{h_n} u_n, K\big)\, dx \leqslant C\frac{\varepsilon_n}{h_n}\,. \tag{6.32}$$

Setting $M_n := [\frac{2}{h_n}]$, we now divide Q' into M_n^2 squares $S_{a,n}$ of the form

$$S_{a,n} = a + M_n^{-1}Q' \quad \text{with} \quad a \in \mathscr{A}^n := M_n^{-1}\mathbb{Z}^2 \cap Q'\,,$$

so that $Q' = \cup_{a\in\mathscr{A}^n} S_{a,n}$ up to a set of $\mathcal{L}^2$-measure zero. Then for each $a \in \mathscr{A}^n$, we define the rescaled map $v_n^a : M_n^{-1}Q \to \mathbb{R}^3$ by $v_n^a(y) := u_n\left(a + y', \frac{y_3}{h_n}\right)$. By [14, Theorem 2], there exists a universal constant C_{univ} such that for each $a \in \mathscr{A}^n$ we can find $R_n^a \in K$ satisfying

$$\int_{M_n^{-1}Q} |\nabla v_n^a - R_n^a|^2\, dy \leqslant C_{\mathrm{univ}} \int_{M_n^{-1}Q} \mathrm{dist}^2\big(\nabla v_n^a, K\big)\, dy\,.$$

Scaling back, we derive that

$$\int_{S_{a,n}\times\frac{1}{2}I} |\nabla_{h_n} u_n - R_n^a|^2\, dx \leqslant C_{\mathrm{univ}} \int_{S_{a,n}\times I} \mathrm{dist}^2\big(\nabla_{h_n} u_n, K\big)\, dx \qquad \forall a \in \mathscr{A}^n\,. \tag{6.33}$$

Defining the piecewise constant map $R_n : Q' \to K$ by $R_n(x') := R_n^a$ for $x' \in S_{a,n}$, and adding the previous inequalities in (6.33) leads to

$$\int_{Q'\times\frac{1}{2}I} |\nabla_{h_n} u_n - R_n(x')|^2\, dx \leqslant C_{\mathrm{univ}} \int_Q \mathrm{dist}^2\big(\nabla_{h_n} u_n, K\big)\, dx \leqslant C\varepsilon_n \underset{n\to+\infty}{\longrightarrow} 0\,,$$

thanks to (6.32). Since $\nabla_{h_n} u_n \to (\nabla' u, b)$ in $L^2(\Omega;\mathbb{R}^{3\times 3})$, we conclude that $R_n \to (\nabla' u, b)$ in $L^2(Q';\mathbb{R}^{3\times 3})$.

Step 2. Let $\delta > 0$ be a small parameter to be chosen. We divide $\mathscr{A}^n$ into the following classes,

$$\mathscr{A}_0^n := \left\{ a \in \mathscr{A}^n \,:\, \int_{S_{a,n}\times I} \mathrm{dist}^2\big(\nabla_{h_n} u_n, K\big)\, dx \geqslant \delta h_n^2 \right\},$$

$\mathscr{A}_1^n := \big\{a \in \mathscr{A}^n \setminus \mathscr{A}_0^n \,:\, R_n^a \in SO(3)\big\}$, and $\mathscr{A}_2^n := \big\{a \in \mathscr{A}^n \setminus \mathscr{A}_0^n \,:\, R_n^a \in SO(3)B\big\}$. We observe that (6.32) yields

$$\mathrm{Card}(\mathscr{A}_0^n) \leqslant \frac{1}{\delta h_n^2}\int_Q \mathrm{dist}^2\big(\nabla_{h_n} u_n, K\big)\, dx = o(1/h_n)\,, \tag{6.34}$$

where Card denotes the counting measure. Next we consider the sets

$$G_0^n := \bigcup_{a\in\mathscr{A}_0^n} S_{a,n}\,, \quad G_1^n := \bigcup_{a\in\mathscr{A}_1^n} S_{a,n}\,, \quad G_2^n := \bigcup_{a\in\mathscr{A}_2^n} S_{a,n}\,,$$

so that $Q' = G_0^n \cup G_1^n \cup G_2^n$ up to a set of $\mathcal{L}^2$-measure zero.

Now, we shall enumerate the edges Γ_a^j $(j = 1, 2, 3, 4)$ of a square $S_{a,n}$ according the counterclockwise sense, Γ_a^1 being the bottom edge. We observe that each boundary ∂G_i^n is polyhedral and made by the edges Γ_a^j (of length M_n^{-1}) of some squares $S_{a,n}$ with $a \in \mathscr{A}_i^n$, that we call *boundary squares.* For $i = 0, 1, 2$ and $j = 1, 2, 3, 4$, we set

$$\mathscr{B}_i^n := \big\{a \in \mathscr{A}_i^n : S_{a,n} \text{ is a } \textit{boundary square}\big\} \quad \text{and} \quad \mathscr{E}_{i,j}^n := \big\{a \in \mathscr{B}_i^n\ :\ \Gamma_a^j \subset \partial G_i^n \cap Q'\big\}\,.$$

We claim that if $\delta > 0$ is chosen small enough, then for every $n \in \mathbb{N}$ large enough, and for $i = 1, 2$, $j = 1, 2, 3, 4$,

$$\Gamma_a^j \subset \partial G_0^n \qquad \forall a \in \mathscr{E}_{i,j}^n\,. \tag{6.35}$$

We shall prove (6.35) in the next step. Assuming that (6.35) is true, we estimate for $i = 1, 2$,

$$\mathcal{H}^1(\partial G_i^n \cap Q') = \sum_{j=1}^{4} \sum_{a\in\mathscr{E}_{i,j}^n} \mathcal{H}^1(\Gamma_a^j) \leqslant 4h_n \mathrm{Card}(\mathscr{A}_0^n) \underset{n\to+\infty}{\longrightarrow} 0\,,$$

thanks to (6.34). Therefore, we can extract a subsequence such that for $i = 1, 2$, either $\mathcal{L}^2(Q'\backslash G_i^n) \to 0$ or $\mathcal{L}^2(G_i^n) \to 0$. Since $\mathcal{L}^2(G_0^n) \to 0$ by (6.34), and $Q' = G_0^n \cup G_1^n \cup G_2^n$, we must have $\mathcal{L}^2(Q' \setminus G_1^n) \to 0$ or $\mathcal{L}^2(Q' \setminus G_2^n) \to 0$. Without loss of generality, we may assume that $\mathcal{L}^2(Q' \setminus G_1^n) \to 0$. Then we estimate

$$\int_{Q'} \mathrm{dist}^2\big((\nabla' u, b), SO(3)\big)\, dx' \leqslant \int_{Q'} \big|(\nabla' u, b) - R_n\big|^2 dx + C\mathcal{L}^2(Q' \setminus G_1^n) \underset{n\to+\infty}{\longrightarrow} 0\,,$$

which yields $(\nabla' u, b)(x') \in SO(3) \cap \{I_d, B\}$ for $\mathcal{L}^2$-a.e. $x' \in Q'$. Since $B \notin SO(3)$, we finally conclude that $(\nabla' u, b) \equiv I_d$ in Q'.

Step 3. It remains to prove (6.35). We argue by contradiction. Without loss of generality, we may assume that there exists $a \in \mathscr{E}_{1,1}^n$ such that $\Gamma_a^1 \not\subset \partial G_0^n$. Since $\Gamma_a^1 \subset Q'$, we have $\tilde{a} := a - (0, M_n^{-1}) \in M_n^{-1}\mathbb{Z}^2 \cap Q'$, and $\tilde{a} \notin \mathscr{A}_0^n \cup \mathscr{A}_1^n$. Thus $\tilde{a} \in \mathscr{A}_2^n$. As in Step 1, we can apply [14, Theorem 2] to find $\tilde{R}_n^a \in K$ such that

$$\begin{aligned}\frac{1}{h_n^2}\int_{(S_{a,n}\cup S_{\tilde{a},n})\times\frac{1}{2}I} \big|\nabla_{h_n} u_n - \tilde{R}_n^a\big|^2 dx &\leqslant \frac{\tilde{C}_{\mathrm{univ}}}{h_n^2}\int_{(S_{a,n}\cup S_{\tilde{a},n})\times I} \mathrm{dist}^2\big(\nabla_{h_n} u_n, K\big)\, dx \\ &\leqslant 2\max\{C_{\mathrm{univ}}, \tilde{C}_{\mathrm{univ}}\}\delta\,,\end{aligned}$$

for some universal constant $\tilde{C}_{\mathrm{univ}}$. Then we have

$$|R_n^a - \tilde{R}_n^a|^2 \leqslant \frac{16}{h_n^2}\int_{(S_{a,n})\times\frac{1}{2}I} \big|\nabla_{h_n} u_n - R_n^a\big|^2 + \big|\nabla_{h_n} u_n - \tilde{R}_n^a\big|^2 dx \leqslant 32\max\{C_{\mathrm{univ}}, \tilde{C}_{\mathrm{univ}}\}\delta\,.$$

We proceed similarly to get $|R_n^{\tilde{a}} - \tilde{R}_n^a|^2 \leqslant 32\max\{C_{\mathrm{univ}}, \tilde{C}_{\mathrm{univ}}\}\delta$, and we obtain a contradiction whenever $\delta < [32\max\{C_{\mathrm{univ}}, \tilde{C}_{\mathrm{univ}}\}]^{-1}\mathrm{dist}^2\big(SO(3), SO(3)B\big)$. □

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