Slow and fast decaying solutions to a critical semilinear parabolic equation

I. Ben Arbi & A. Haraux

Résumé: On étudie l'ordre de convergence vers 0, quand $t \longrightarrow +\infty$ de la solution de l'équation $\psi_t - \Delta\psi - \lambda_1\psi + |\psi|^{p-1}\psi = 0$ avec les conditions aux limites de Dirichlet homogenes dans un ouvert connexe borné de $\mathbb{R}^n$ où $p > 1$. On montre que soit $\psi(t, \cdot)$ converge vers 0 plus vite que toute puissance négative de t, soit $\psi(t, \cdot)$ décroit comme $t^{-\frac{1}{p-1}}$.

Abstract: We study the decay rate to 0, as $t \longrightarrow +\infty$ of the solution of equation $\psi_t - \Delta\psi - \lambda_1\psi + |\psi|^{p-1}\psi = 0$ with homogeneous Dirichlet boundary conditions in a bounded smooth open connected domain of $\mathbb{R}^n$ where $p > 1$. We show that either $\psi(t, \cdot)$ converges to 0 faster than any negative power of t, or $\psi(t, \cdot)$ decreases like $t^{-\frac{1}{p-1}}$.

Keywords: rate of decay, parabolic equation

1 Introduction and main results.

In this paper we consider the following nonlinear parabolic equation

$$\begin{cases} \psi_t - \Delta\psi - \lambda_1\psi + g(\psi) = 0 \quad in \quad \mathbb{R}^+ \times \Omega, \\ \psi = 0 \quad on \quad \mathbb{R}^+ \times \partial\Omega. \end{cases} \tag{1.1}$$

where Ω is a bounded smooth open connected domain of $\mathbb{R}^n$ and $g \in C^1(\mathbb{R})$ satisfies

$$g(0) = 0 \tag{1.2}$$

and for some $p > 1$

$$\exists C > 0, \forall s \in \mathbb{R}, , \qquad 0 \leq g'(s) \leq C|s|^{p-1}. \tag{1.3}$$

$$\exists c > 0, \forall s \in \mathbb{R}, \quad |g(s)| \geq c|s|^p \tag{1.4}$$

In addition λ_1 stands for the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$, which means

$$\lambda_1 = \min\{\int_\Omega |\nabla u(x)|^2 dx, \quad u \in H_0^1(\Omega), \int_\Omega |u(x)|^2 dx = 1\}$$

From (1.2)-(1.3) it follows that $g(s)$ has the sign of s and

$$\forall s \in \mathbb{R}, \quad |g(s)| \leq \frac{C}{p}|s|^p \tag{1.5}$$

In addition from (1.4), we deduce

$$\forall s \in \mathbb{R}, \quad g(s)s = |g(s)||s| \geq c|s|^{p+1} \tag{1.6}$$

We define the operator A by

$$D(A) = \{\psi \in H_0^1(\Omega), \Delta\psi \in L^2(\Omega)\}$$

and

$$\forall \psi \in D(A), \quad A\psi = -\Delta\psi - \lambda_1\psi$$

On the other hand the operator B defined by

$$D(B) = \{\psi \in H_0^1(\Omega), A\psi + g(\psi) \in L^2(\Omega)\}$$

and

$$\forall \psi \in D(B), \quad B\psi = A\psi + g(\psi)$$

is well-known to be maximal monotone in $L^2(\Omega)$. As a consequence of [1, 2] for any $\psi_0 \in L^2(\Omega)$ there exists a unique weak solution of the equation

$$\psi' + B\psi = 0 \quad on \quad \mathbb{R}^+; \quad \psi(0,x) = \psi_0. \tag{1.7}$$

In addition it is well known that if $\psi_0 \in L^\infty(\Omega)$, $\psi(t,\cdot)$ remains in $L^\infty(\Omega)$ for all $t > 0$. Finally [9] contains an estimate of the solution in $C(\bar{\Omega})$ and $C^1(\bar{\Omega})$ for $t > 0$, which is valid for any sufficiently regular domain.

Equation(1.1) is critical in the following sense : if we consider the more general problem

$$\begin{cases} \psi_t - \Delta\psi - \lambda\psi + g(\psi) = 0 \quad in \quad \mathbb{R}^+ \times \Omega, \\ \psi = 0 \quad on \quad \mathbb{R}^+ \times \partial\Omega \end{cases} \tag{1.8}$$

the behaviour of solutions for t large depends on the position of λ : if $\lambda < \lambda_1$, all solutions tend to 0 exponentially and if $\lambda > \lambda_1$, there are non trivial time independent positive solutions.

The proofs of our main results will rely on the following preliminary estimates.

Proposition 1.1. *Let g satisfy* (1.2) - (1.4). *Then any solution ψ of (1.1) satisfies the following properties*

$$\forall t > 0, \quad \|\psi(t,.)\|_2 \le C_1 t^{-\frac{1}{p-1}} \tag{1.9}$$

$$\forall t \ge 1, \quad \|\psi(t,.)\|_\infty \le C_2 t^{-\frac{1}{p-1}} \tag{1.10}$$

where C_1, C_2 are independent of the initial data.

Our main result is the following

Theorem 1.2. *Let g satisfy* (1.2) - (1.4). *Then any solution ψ of (1.1) satisfies the following alternative as $t \to \infty$: either*

$$\forall m > 1, \quad \lim_{t\to\infty} t^m \|\psi(t,\cdot)\|_\infty = 0 \tag{1.11}$$

or

$$\exists \delta > 0, \forall t \ge 1, \quad \left| \int_\Omega \psi(t,x)\phi_1(x)dx \right| \ge \delta t^{-\frac{1}{p-1}}, \tag{1.12}$$

where ϕ_1 is the normalized positive eigenfunction associated to λ_1.

In the following Proposition, we consider two special cases showing that both possibilities in the second result in the Theorem 1.2 can actually happen.

Proposition 1.3. *Let g satisfy* (1.2) *and* (1.3). *Then we have*

- (i) *If Ω is symmetric around* 0, *g is odd and $\psi(0,\cdot)$ is an odd function in Ω, then any solution of* (1.1) *satisfies for some $M > 0$*

$$\|\psi(t,\cdot)\|_\infty \leq Me^{-(\lambda_2-\lambda_1)t} \tag{1.13}$$

  Where λ_2 stands for the second eigenvalue of A.

- (ii) *If $\psi(0,\cdot) \geq 0$ and ψ does not vanish a.e in Ω, then any solution of* (1.1) *satisfies* (1.12).

Remark 1.4. It would be interesting to determine whether any solution satisfying (1.11) is exponentially decaying or even satisfies the stronger property (1.13). Our present method does not allow us to decide about this issue.

2 Proof of Proposition 1.1.

Proof. We denote by $\|.\|_2$ the L^2 norm and by (u, v) the inner product of two functions u, v of $L^2(\Omega)$. Multiplying (1.1) by $\psi(t,\cdot)$ and integrating over Ω we find

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\psi(t)\|_2^2 = (\psi_t,\psi) &= \int_\Omega \psi_t(t,x)\psi(t,x)dx \\ &= \int_\Omega (\Delta\psi(t,x) + \lambda_1\psi(t,x))\psi(t,x)dx - \int_\Omega g(\psi(t,x))\psi(t,x)dx \\ &= -\|\nabla\psi(t)\|_2^2 + \lambda_1\|\psi(t)\|_2^2 - \int_\Omega g(\psi(t,x))\psi(t,x)dx\end{aligned}$$

By using the following inequality $\int_\Omega |\nabla\psi(t,x)|^2 \geq \lambda_1 \int_\Omega |\psi(t,x)|^2$, we obtain:

$$\frac{1}{2}\frac{d}{dt}\|\psi(t)\|_2^2 \leq -\int_\Omega g(\psi(t,x))\psi(t,x)dx.$$

As consequence of (1.6), we deduce

$$\frac{1}{2}\frac{d}{dt}\|\psi(t)\|_2^2 \leq -c\|\psi(t)\|_{p+1}^{p+1} \leq -c'\|\psi(t)\|_2^{p+1}$$

with $c' > 0$. By integrating over $[0, t]$ we obtain

$$\|\psi(t,\cdot)\|_2 \le \left\{ \frac{1}{\|\psi(0,\cdot)\|_2^{1-p} + (p-1)c't} \right\}^{\frac{1}{p-1}}$$

then (1.9) follows with $C_1 = [(p-1)c']^{-\frac{1}{p-1}}$. Using the smoothing effect of the heat operator,we deduce easily (1.10) from (1.9). More precisely, as a consequence of [7], Lemma 3.5 p. 475, we have

$$\forall t \ge 1, \quad \|\psi(t,.)\|_\infty \le (2\pi)^{-N/4} \exp(\frac{1}{2}\lambda_1)\|\psi(t-\frac{1}{2},\cdot)\|_2$$

$$\le (2\pi)^{-N/4} \exp(\frac{1}{2}\lambda_1) C_1 (t-\frac{1}{2})^{-\frac{1}{p-1}} \le (2\pi)^{-N/4} \exp(\frac{1}{2}\lambda_1) C_1 (\frac{t}{2})^{-\frac{1}{p-1}}$$

and the result follows with $C_2 = (2\pi)^{-N/4} \exp(\frac{1}{2}\lambda_1) 2^{\frac{1}{p-1}} C_1$

■

3 Proof of Theorem 1.2.

We define the orthogonal projection $P : H \longrightarrow N$, where

$$H = L^2(\Omega), N = kerA \text{ and } P\psi(t,\cdot) = (\int_\Omega \psi(t,x)\phi_1(x)dx)\phi_1(x).$$

It is well known that P and $I-P$ are contractions on L^2 and bounded operators on L^p for any $p \ge 1$ since ϕ_1 is bounded . Moreover in [6], the following result on the range component of the solution was established

$$\forall t \ge 1, \quad \|(I-P)\psi(t)\|_{H_0^1(\Omega)\cap L^\infty(\Omega)} \le M_0 t^{\frac{p}{p-1}} \tag{3.1}$$

where $M_0 > 0$.
We set

$$\psi = y + w \text{ where } y = P\psi = u\phi_1(x) \text{ and } w = (I-P)\psi.$$

Where $u(t) = \int_\Omega \psi(t,x)\phi_1(x)dx$ and $\phi_1(x)$ is the (positive) normalized eigenfunction associated to λ_1. By projecting (1.1) on N we obtain

$$y' + (\int_\Omega (-\Delta\psi(t,x) - \lambda_1\psi(t,x))\phi_1(x)dx)\phi_1(x) + (\int_\Omega g(\psi(t,x))\phi_1(x)dx)\phi_1(x) = 0, \tag{3.2}$$

since the operator A is selfadjoint, we have

$$\int_\Omega ((-\Delta-\lambda_1)\psi(t,x))\phi_1(x)dx = \int_\Omega ((-\Delta-\lambda_1)\phi_1(x))\psi(t,x))dx = 0$$

then we can rewrite (3.2) as

$$[u'(t) + \int_\Omega g(\psi(t,x))\phi_1(x)dx]\phi_1(x) = 0$$

which is equivalent to

$$u'(t) + \int_\Omega g(y)\phi_1(x)dx = -\int_\Omega (g(\psi(t,x)) - g(y))\phi_1(x)dx.$$

which leads us to study the equation:

$$u' + \gamma(u) = h(t) \quad \text{in} \quad \mathbb{R}^+, \tag{3.3}$$

where

$$h(t) = -\int_\Omega [g(\psi(t,x)) - g(y)]\phi_1(x)dx$$

and γ is defined by

$$\forall s \in \mathbb{R}, \quad \gamma(s) = \int_\Omega g(s\phi_1(x))\phi_1(x)dx. \tag{3.4}$$

We know that ϕ_1 is a bounded function and we set

$$\|\phi_1\|_\infty =: K. \tag{3.5}$$

By the assumption (1.3), h satisfies the following estimate

$$|h(t)| \le K'(\|\psi\||_{2p-2}^{p-1} + \|y\|_{2p-2}^{p-1})\|w\|_2. \tag{3.6}$$

Indeed

$$\begin{aligned}\int_\Omega |g(\psi(t,x)) - g(y)||\phi_1(x)|dx &\le \int_\Omega |g(\psi(t,x)) - g(y)|\|\phi_1(x)\|_\infty dx \\ &\le K\|g(\psi(t,x)) - g(y)\|_1 \\ &\le K'(\|\psi\||_{2p-2}^{p-1} + \|y\|_{2p-2}^{p-1})\|w\|_2\end{aligned}$$

with $K' = KC$.
But ψ and y are uniformly bounded, from Proposition 1.1 we have

$$\|\psi\|_{2p-2}^{p-1} + \|y\|_{2p-2}^{p-1} \le 2C_1^{p-1}t^{-1}$$

and using the estimate (3.1) therefore,

$$|h(t)| \leq K'' t^{-1-\frac{p}{p-1}}$$

with $K'' = 2k'K'C_1^{p-1}$.
Let us examine whether γ inherits the properties of g . First of all $\gamma(0) = 0$. Moreover let $(u, \bar{u}) \in \mathbb{R}^2$ and assume that $\sup\{|u|, |\bar{u}|\} \leq r$, then

$$\begin{aligned}
|\gamma(u) - \gamma(\bar{u})| &\leq \int_\Omega |g(u\phi_1(x)) - g(\bar{u}\phi_1(x))||\phi_1(x)|dx \\
&\leq \|g(u\phi_1(x)) - g(\bar{u}\phi_1(x))\|_1 \|\phi_1(x)\|_\infty \\
&\leq p\|\phi_1(x)\|_\infty^2 \sup_{t\in(0,+\infty)} \{g'(u\phi_1(x)), g(\bar{u}\phi_1(x))\} |u - \bar{u}| \\
&\leq p\|\phi_1(x)\|_\infty^{p+1} C \sup_{t\in(0,+\infty)} (|u|^{p-1}, |\bar{u}|^{p-1})|u - \bar{u}| \\
&\leq p\|\phi_1(x)\|_\infty^{p+1} C r^{p-1} |u - \bar{u}|.
\end{aligned}$$

Hence we obtain

$$\forall (u, \bar{u}) \in \mathbb{R}^2, \quad \sup\{|u|, |\bar{u}|\} \leq r \Rightarrow |\gamma(u) - \gamma(\bar{u})| \leq \Gamma p r^{p-1} |u - \bar{u}| \tag{3.7}$$

with

$$\Gamma := \|\phi_1(x)\|_\infty^{p+1} C$$

On the other hand by integrating h over $[t, +\infty[$, we obtain

$$\int_t^{+\infty} |h(s)|ds \leq K''(\frac{p-1}{p}) t^{-\frac{p}{p-1}}$$

Now we recall a result following from [4], Proposition 2.3 in the case $X = \mathbb{R}$, $L = 0, M = 1, r_0 = +\infty$.

Proposition 3.1. *Assume that* $\gamma \in C^1(\mathbb{R})$ *satisfies*

$$\forall (u, \bar{u}) \in \mathbb{R}^2, \quad \sup\{|u|, |\bar{u}|\} \leq r \Rightarrow |\gamma(u) - \gamma(\bar{u})| \leq \Gamma(\alpha + 1) r^\alpha |u - \bar{u}| \tag{3.8}$$

Let $\lambda > \frac{1}{\alpha}$, $R_T = (\frac{\lambda}{(\alpha+1)\Gamma})^{\frac{1}{\alpha}} T^{\lambda - \frac{1}{\alpha}}$ *and* $H_T = \dfrac{\alpha R_T}{\alpha + 1}$. *Then if*

$$\forall t \geq T, \quad \int_t^{+\infty} |h(s)|ds \leq H t^{-\lambda}$$

for some $H < H_T$, (3.3) *has one and only one solution* v *on* $[T, \infty)$ *such that*

$$\sup_{t \geq T} t^\lambda |v(t)| < +\infty$$

In our case, $H = K''(\frac{p-1}{p}), \lambda = \frac{p}{p-1}, \quad \alpha = p-1$ and

$$H_T = \frac{p-1}{p}\left(\frac{1}{(p-1)\Gamma}\right)^{\frac{1}{p-1}} T.$$

Using the condition $H < H_T$ we obtain that for

$$T > K''((p-1)\Gamma)^{\frac{1}{p-1}}$$

the equation (3.3) has one and only one solution v on $[T, +\infty[$ such that

$$\sup_{t \geq T} t^{\frac{p}{p-1}} |v(t)| < +\infty. \tag{3.9}$$

Setting $z = u - v$, we complete the proof analyzing two cases.

<u>1^{st} case</u>: If $z(T_0) = 0$, then for all $t \geq T_0$, $z(t) = 0$. Hence $u \equiv v$, and since v satisfies (3.9),

$$\|\psi\|_2 \leq \|y\|_2 + \|w\|_2 = \|\varphi_1 v\|_2 + \|w\|_2 \leq M_1 t^{-\frac{p}{p-1}}$$

where $M_1 > 0$. Using as in [7] the smoothing effect of the heat operator, we deduce for t large enough

$$\|\psi(t,\cdot)\|_\infty \leq M_1 t^{-\frac{p}{p-1}}.$$

Lemma 3.2. *Assume that for some $\lambda \geq \frac{p}{p-1}$, we have for all t large enough and some $K \geq 0$*

$$\|\psi(t,\cdot)\|_\infty \leq K t^{-\lambda}.$$

Then for all t large enough and some $K' \geq 0$

$$\|\psi(t,\cdot)\|_\infty \leq K' t^{-p\lambda}.$$

Proof. First $w = (I-P)\psi$ satisfies

$$w' + (-\Delta - \lambda_1)w + (I-P)g(\psi) = 0, \tag{3.10}$$

Then by the hypothesis and by the boundedness property of P in L^∞,

$$\|(I-P)g(\psi)\|_\infty \leq C'\|\psi\|_\infty^p \leq C'' t^{-p\lambda}.$$

Since $A = -\Delta - \lambda_1 I$ generates an exponentially damped semigroup on $N^\perp$ endowed with the L^2 norm we infer

$$\|w(t,.)\|_2 \leq C_1 t^{-p\lambda}$$

Therefore, from (3.6), $|h(t)| \leq C_2 t^{-1-p\lambda}$ then by proposition 3.1 we deduce that there exists a solution v_1 of (3.3) such that $|v_1(t)| \leq C_3 t^{-p\lambda}$ for t large enough. By the uniqueness part of this proposition, since $p\lambda \geq \frac{p}{p-1}$ actually $v = v_1$ on some interval $[T, \infty)$, therefore $|v(t)| \leq C_3 t^{-p\lambda}$ for t large enough and then since $\psi = w + v\phi_1$ we obtain:

$$\|\psi(t,\cdot)\|_2 \leq K_1 t^{-p\lambda}$$

Finally by the method of [7] we conclude that

$$\|\psi(t,\cdot)\|_\infty \leq K_2 t^{-p\lambda}$$

□

To complete the proof in this case we reason by induction: we already know that the hypothesis of Lemma 3.2 is fulfilled for $\lambda = \frac{p}{p-1}$. By applying inductively Lemma 3.2 we obtain for all $k \in \mathbb{N}$ the existence of $K(k)$ that

$$\forall t \geq 1, \quad \|\psi(t,\cdot)\|_\infty \leq K(k) t^{-p^k \lambda}$$

Since $p > 1$, (1.11) follows immediately.

2^{nd} case: If $z(T_0) \neq 0$ then $\forall t \geq T_0, z(t) \neq 0$ and we have

$$z'(t) + \frac{\gamma(u(t)) - \gamma(v(t))}{u(t) - v(t)} z(t) = 0.$$

We set

$$\alpha(t) := \frac{\gamma(u(t)) - \gamma(v(t))}{u(t) - v(t)}$$

$\gamma'(s) = \int_\Omega g'(s\phi_1(x))\phi_1^2(x)dx \geq 0$ then for all $t \geq 0, \alpha$ is a strictly positive function and

$$\forall p > 1, \quad \alpha(t) \leq p\|\phi_1(x)\|_\infty^{p+1} C(|u|^{p-1} + |v|^{p-1}). \tag{3.11}$$

Setting $Y = |z|$, we have

$$Y' + \alpha(t)Y \geq 0.$$

Then the estimate (3.11) implies that

$$\begin{aligned} Y' &\geq -p\|\phi_1(x)\|_\infty^{p+1} C(|u|^{p-1} + |v|^{p-1})Y \\ &\geq -p\|\phi_1(x)\|_\infty^{p+1} C(|z+v|^{p-1} + |v|^{p-1})Y. \end{aligned}$$

Hence there exists some constants $c_1, c_2 > 0$ such that

$$Y' \geq -c_1(|z|^{p-1})Y - c_2|v|^{p-1}Y.$$

and since $Y = |z|$, then

$$Y' \geq -c_1 Y^p - c_2|v|^{p-1}Y.$$

Putting $a(t) = c_2|v|^{p-1}$, we deduce that

$$Y' + a(t)Y \geq -c_1 Y^p. \tag{3.12}$$

We set

$$A(t) = -c_2 \int_t^{+\infty} |v(s)|^{p-1} ds, \qquad \omega(t) = e^{A(t)}Y(t)$$

$\forall m > 1, \forall p > 1, t \longrightarrow A(t)$ is a bounded function and by replacing ω in (3.12), we obtain

$$\omega(t) \geq \frac{1}{\omega(t)^{1-p} + (p-1)c_3 t}$$

with $c_3 > 0$. Then for t large enough we have

$$\omega(t) \geq c_4 t^{-\frac{1}{p-1}}.$$

With $c_4 > 0$.
Since $t \longrightarrow e^{A(t)}$ is a bounded function we conclude (1.12) by observing that $u = z + v$ and v tends to 0 very fast at infinity. The proof is now complete.

4 Proof of Proposition 1.3

(i) If $\psi(0, -x) = -\psi(0, x)$, the odd character of g implies

$$\forall t > 0, \quad \forall x \in \Omega, \quad \psi(t, -x) = -\psi(t, x)$$

and since ϕ_1 is even, we obtain

$$\int_\Omega \psi(t, x)\phi_1(x)dx = 0.$$

Since $\psi(t, .)$ remains in the orthogonal of ϕ_1 in which the first eigenvalue of $-\Delta$ is λ_2, we deduce the following estimate

$$(-\Delta\psi, \psi) \geq \lambda_2 \int_\Omega |\psi(t, x)|^2 dx,$$

then we have

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega |\psi(t,x)|^2 dx &= (\psi_t, \psi) \\
&= -(-\Delta\psi, \psi) + \lambda_1 \int_\Omega |\psi(t,x)|^2 dx - \int_\Omega g(\psi)\psi dx \\
&\leq (-\lambda_2 + \lambda_1) \int_\Omega |\psi(t,x)|^2 dx
\end{aligned}
$$

By integrating over $[0,t]$ we obtain (1.13).

(*ii*) If $\psi(0,x) \geq 0$ and ψ does not vanish a.e in Ω, then for all $t \geq 1$ $\psi(t,x) > 0$, it implies that

$$\int_\Omega \psi(t,x)\phi_1(x)dx > 0. \tag{4.1}$$

We suppose that we have $\|\psi(t,\cdot)\|_\infty \leq K_2 t^{-m}$ and we consider the problem (1.1) then we multiply by the first eigenfunction ϕ_1 and we integrate over Ω, we obtain

$$\int_\Omega \psi_t(t,x)\phi_1(x)dx = -\int_\Omega g(\psi(t,x))\phi_1(x)dx. \tag{4.2}$$

An elementary calculation shows that we have

$$
\begin{aligned}
\int_\Omega g(\psi(t,x))\phi_1(x)dx &\leq \frac{c}{p}\int_\Omega |\psi(t,x)|^p \phi_1(x)dx \\
&\leq \frac{c}{p}\int_\Omega |\psi(t,x)|^{p-1}\psi(t,x)\phi_1(x)dx \\
&\leq \frac{c}{p}\int_\Omega \|\psi(t,x)\|_\infty^{p-1}\psi(t,x)\phi_1(x)dx \\
&\leq \frac{c}{p}K_2^{p-1}t^{-(p-1)m}\int_\Omega \psi(t,x)\phi_1(x)dx
\end{aligned}
$$

From (4.2), we deduce

$$\int_\Omega \psi_t(t,x)\phi_1(x)dx \geq -\frac{c}{p}K_2^{p-1}t^{-(p-1)m}\int_\Omega \psi(t,x)\phi_1(x)dx$$

Since $u(t) = \int_\Omega \psi(t,x)\phi_1(x)dx$, we have

$$u'(t) \geq -Mt^{-q}u(t) \tag{4.3}$$

with $M = \frac{c}{p} K_2^{p-1}$ and $q = (p-1)m$. Now we choose $m > \frac{1}{p-1}$ so that $q > 1$. Since $u(t) > 0$ by (4.1), by integrating on the interval $[1, t]$ we obtain

$$u(t) \geq u(1) \exp\left\{-M \int_1^t s^{-q} ds\right\} \geq u(1) \exp\left\{-\frac{M}{q-1}\right\} > 0. \quad (4.4)$$

Hence u does not tend to 0 for t large, this contradicts our hypothesis and we conclude that u satisfies (1.12).

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