Convergence and decay estimates for a class of second order dissipative equations involving a non-negative potential energy

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Abstract. We estimate the rate of decay of the difference between a solution and its limiting equilibrium for the following abstract second order problem

$$\ddot{u}(t) + g(\dot{u}(t)) + \mathcal{M}(u(t)) = 0, \quad t \in \mathbb{R}_+,$$

where $\mathcal{M}$ is the gradient operator of a non-negative functional and g is a nonlinear damping operator, under some conditions relating the Lojasiewicz exponent of the functional and the growth of the damping around the origin.

1 Introduction

The convergence problem for bounded solutions of semilinear dissipative wave equation has been the object of many specialized works in the last 30 years. Assuming Ω to be a bounded open connected domain of $\mathbb{R}^N$, a convergence result has been shown in [6] for the problem

$$\begin{cases} u_{tt} + cu_t - \Delta u + f(u) = 0 \quad \text{in } \mathbb{R}_+ \times \Omega, \\ u(t,x) = 0 \quad \text{on } \mathbb{R}_+ \times \partial\Omega \end{cases} \tag{1.1}$$

when the function $s \to f(s) + \lambda_1 s$ is non-decreasing, relying on the fact that the set of equilibria is then one-dimensional. This result was established under a growth assumption on f and assuming precompactness of the solution curve $(u(t,.), u_t(t,.)$in the energy space. A first generalization allowing some types of nonlinear dampings was done by E. Zuazua in [15]. A much more general theory dealing with one-dimensional sets of equilibria was developped later by J. Hale and G. Raugel [5]. The hypothesis on the dimension of the equilibrium set can be relaxed if f is analytic, and general convergence results as well as rates of convergencence were proved in this direction by M.A. Jendoubi and the second author, c.f. [12, 9, 10] by using the Lojasiewiz gradient inequality , cf. [13, 14]. The case of a genuinely nonlinear damping seems to be more delicate. As a basic example, a convergence result for bounded solutions of the problem

$$\begin{cases} u_{tt} + |u_t|^\alpha u_t - \Delta u + f(u) = 0, \quad \text{in } \mathbb{R}_+ \times \Omega, \\ u(t) = 0, \quad \text{on } \mathbb{R}_+ \times \partial\Omega, \\ u(0,\cdot) = u_0 \in H_0^1(\Omega), u_t(0,\cdot) = u_1 \in L^2(\Omega), \end{cases} \tag{1.2}$$

has been proved by L. Chergui in [4] assuming $0 < \alpha < 1$ and the following conditions

$$f : \mathbb{R} \to \mathbb{R} \text{ is analytic,} \tag{1.3}$$

there exists $C \geq 0$ and $\eta > 0$ whith $(N-2)\eta < 2$ such that :

$$\text{if } \ N \geq 2 \text{ then we have } |f'(s)| \leq C(1 + |s|^\eta) \text{ on } \mathbb{R}. \tag{1.4}$$

There exists $\theta \in \left]\frac{\alpha}{\alpha+1}, \frac{1}{2}\right]$ and $C > 0$ such that for all φ in the equilibrium set

$$\Sigma = \{\psi \in H^2(\Omega) \cap H_0^1(\Omega) \text{ and } \Delta\psi = f(\psi)\}$$

there is σ_φ such that for every $u \in H_0^1(\Omega)$ one has

$$\|u - \varphi\|_{H_0^1(\Omega)} < \sigma_\varphi \Longrightarrow \|\Delta u + f(u)\|_{H^{-1}(\Omega)} \geq C|\mathcal{E}(u) - \mathcal{E}(\varphi)|^{1-\theta}. \tag{1.5}$$

However L. Chergui did not obtain any estimate of the rate of convergence.

In this paper we shall obtain such an estimate for a class of equations which contains (1.2) when f satisfies some additional assumptions. The plan of the paper is as follows: In Section 2 we state and prove our main result. In Section 3 we give simple applications of this result. In Section 4 we show how to check the hypotheses for a more general class of nonlinear operators. Section 5 contains the convergence and decay results in the more elaborate examples related to the results of Section 4. Section 6 is devoted to the case where a rapidly decaying source term appears in the right-hand side of the equation.

2 Main result

2.1 Functional setting

Throughout this article we let H and V be two Hilbert spaces. We assume that V is densely and continuously embedded into H. Identifying H with its dual H', we obtain $V \hookrightarrow H = H' \hookrightarrow V'$. We denote by $\langle \cdot, \cdot \rangle$ scalar products and duality relations; the spaces in question will be specified by subscripts. The notation $\langle f, u \rangle$ without any subscript will be used sometimes to denote $\langle f, u \rangle_{V',V}$. Throughout the text , we let $C_1 \geq 0$ be such that

$$\|v\|_{V'} \leq C_1 \|v\|_H \leq C_1^2 \|v\|_V, \quad v \in V. \tag{2.1}$$

Other constants in the calculations will be denoted by C_i $(i \geq 2)$.

Let $\mathcal{E} \in C^2(V, \mathbb{R})$, and denote by $\mathcal{M} \in C^1(V, V')$ the first derivative of $\mathcal{E}$. Throughout the text we shall assume that $\mathcal{E}$ and $\mathcal{M}$ are bounded on bounded sets of V with values to V and V' respectively. We study the following abstract Cauchy problem :

$$\begin{cases} \ddot{u}(t) + g(\dot{u}(t)) + \mathcal{M}(u(t)) = 0, & t \geq 0, \\ u(0) = u_0, & u_0 \in V, \\ \dot{u}(0) = u_1, & u_1 \in H, \end{cases} \tag{2.2}$$

under the following assumptions on g and $\mathcal{M}$:
1) $g : H \to V'$ is such that there exists $\alpha \in]0,1[$, $\rho_1 > 0$ and $\rho_2 > 0$ for which

$$\forall v \in V,\ \langle g(v), v \rangle_{V',V} \geq \rho_1 \|v\|_H^{\alpha+2}, \tag{2.3}$$

$$\forall v \in H,\ \|g(v)\|_{V'} \leq \rho_2 \|v\|_H^{\alpha+1}. \tag{2.4}$$

2) There exists a real number θ such that

$$\theta \in \left] \frac{\alpha}{\alpha+1}, \frac{1}{2} \right], \tag{2.5}$$

and some constants $c_1 > 0,\ c_2 > 0$ and a bounded subset B of Vsuch that the function $\mathcal{E}$ is nonnegative on B and satisfies the following assumption

$$\forall u \in B, \quad c_1 \mathcal{E}(u)^\gamma \geq \|\mathcal{M}(u)\|_{V'} \geq c_2 \mathcal{E}(u)^{1-\theta} \tag{2.6}$$

where $\gamma > 0$ is such that :

$$\frac{1}{2} - \alpha(1-\theta) \leq \gamma \leq 1 - \theta. \tag{2.7}$$

2.2 The result

The main result of this paper is

Theorem 2.1 *Let $u \in W^{1,1}_{loc}(\mathbb{R}^+, V) \bigcap W^{2,1}_{loc}(\mathbb{R}^+, H)$ be a solution of* (2.2) *such that $u(t) \in B$ for t large where B denotes a closed subset of V. Assume that the hypotheses* (2.3), (2.4) *and* (2.6) *are satisfied. Assume in addition that there exists a constant $C \geq 0$ such that for $u \in B$, the following inequality holds :*

$$\forall v \in V, \quad |\langle \mathcal{M}'(u)v, v\rangle_{V'}| \leq C\|v\|_H^2. \tag{2.8}$$

Let θ be as in (2.5) *and* (2.6)*. Let us introduce*

$$\xi = \frac{1-(\alpha+1)(1-\theta)}{(\alpha+2)(1-\theta)-1}; \quad \lambda = \frac{1}{(\alpha+2)(1-\theta)-1}.$$

Then there exist $a \in B$ and a constant $C > 0$ such that

$$\forall t \geq T, \quad \|u(t) - a\|_H \leq Ct^{-\xi} \tag{2.9}$$

Moreover there are positive constants M_1, M_2 such that

$$\forall t \geq T, \quad \|\dot{u}(t)\|_H \leq M_1 t^{-\frac{\lambda}{2}} \tag{2.10}$$

$$\forall t \geq T, \quad \mathcal{E}(u(t)) \leq M_2 t^{-\lambda} \tag{2.11}$$

Finally, if either u has precompact range in V, or $\mathcal{M} : B \to V'$ is weakly continuous for the topology of V, we have

$$a \in \mathbb{E} := \{y \in V, \mathcal{M}(y) = 0\}$$

Remarks 2.2 1) Let us observe that in the case $V = H = \mathbb{R}^N$, $g(v) = \|\dot{v}\|^\alpha \dot{v}$ and $\mathcal{M}(u(t)) = \nabla F(u(t))$, L. Chergui [3] studied the differential system

$$\ddot{u}(t) + \|\dot{u}(t)\|^\alpha \dot{u}(t) + \nabla F(u(t)) = 0,\ t \in \mathbb{R}_+. \tag{2.12}$$

He proved a convergence result for bounded solutions of (2.12) and he obtained the rate of decay given in Theorem 2.1.

2) The hypothesis (2.6) implies that any equilibrium point $a \in B$ satisfies $\mathcal{E}(a) = 0$. Hence the minimum of $\mathcal{E}$ on B is achieved on equilibrium points. In particular, if either u has precompact range in V, or $\mathcal{M} : B \to V'$ is weakly continuous for the topology of V , the existence of a region B with the above mentioned properties implies the not so trivial conclusion that the minimum of $\mathcal{E}$ on B is achieved and equal to 0 .

2.3 Proof of Theorem 2.1

Let u be a solution of equation (2.2) such that $u \in W^{1,1}_{loc}(\mathbb{R}^+, V) \bigcap W^{2,1}_{loc}(\mathbb{R}^+, H)$ and let $T > 0$ be such that $u(t) \in B$ for $t \geq T$. Let us define the nonnegative function

$$E(t) = \frac{1}{2}\|\dot{u}(t)\|_H^2 + \mathcal{E}(u(t)).$$

Then we have

$$E'(t) = -\langle \dot{u}(t), g(\dot{u}(t))\rangle_{V,V'} \leq 0$$

Therefore E, being nonicreasing and nonnegative, remains bounded. In particular $\dot{u}(t)$ is bounded in H. Now let $0 < \varepsilon \leq 1$ be a real constant. We define the function

$$H(t) = E(t) + \varepsilon E(t)^{\beta}\langle \mathcal{M}(u(t)), \dot{u}(t)\rangle_{V'},$$

where $\beta = \alpha(1-\theta)$, θ is the Lojasiewicz exponent defined in (2.6).
Our bounded solution u being fixed, we can choose ε small enough in order to achieve

$$\frac{E(t)}{2} \leq H(t) \leq 2E(t), \quad \text{for all } t \geq T. \tag{2.13}$$

In fact thanks to the definition of H together with assumption (2.6) we have for all $t \geq T$

$$\begin{aligned} H(t) &= E(t) + \varepsilon E(t)^{\beta}\langle \mathcal{M}(u(t)), \dot{u}(t)\rangle_{V'} \\ &\leq E(t) + \varepsilon E(t)^{\beta}\|\mathcal{M}(u(t))\|_{V'}\|\dot{u}(t)\|_{V'} \\ &\leq E(t) + \varepsilon C_1 c\sqrt{2}E(t)^{\alpha(1-\theta)+\gamma+\frac{1}{2}}. \end{aligned}$$

Now, since E is bounded, it follows thanks to (2.7) that $E(t)^{\alpha(1-\theta)+\gamma+\frac{1}{2}} \leq KE(t)$, for all $t \geq T$, where K is a positive constant. Then by choosing ε small enough we get $H(t) \leq 2E(t)$. The reverse inequality follows similarly .

We now compute

$$H'(t) = E'(t) + \varepsilon\beta E'(t)E(t)^{\beta-1}\langle \mathcal{M}(u(t)), \dot{u}(t)\rangle_{V'} + \varepsilon E(t)^{\beta}\langle \mathcal{M}'(u(t))\dot{u}(t), \dot{u}(t)\rangle_{V'}$$

$$-\varepsilon E(t)^\beta \langle \mathcal{M}(u(t)), g(\dot u(t))\rangle_{V'} - \varepsilon E(t)^\beta \|\mathcal{M}(u(t))\|^2_{V'}.$$

Then we get

$$H'(t) = -\langle \dot u(t), g(\dot u(t))\rangle_{V,V'} - \varepsilon\beta \langle \dot u(t), g(\dot u(t))\rangle_{V,V'} E(t)^{\beta-1}\langle \mathcal{M}(u(t)), \dot u(t)\rangle_{V'}$$
$$+\varepsilon E(t)^\beta \langle \mathcal{M}'(u(t))\dot u(t), \dot u(t)\rangle_{V'} - \varepsilon E(t)^\beta \langle \mathcal{M}(u(t)), g(\dot u(t))\rangle_{V'} - \varepsilon E(t)^\beta \|\mathcal{M}(u(t))\|^2_{V'}.$$

By using (2.3), $\langle \dot u(t), g(\dot u(t))\rangle_{V,V'}$ is nonnegative and then we get thanks to the definition of E together with assumption (2.6)

$$\begin{aligned}
-\langle \dot u(t), g(\dot u(t))\rangle_{V,V'} E(t)^{\beta-1}\langle \mathcal{M}(u(t)), \dot u(t)\rangle_{V'} &\leq C_2 \langle \dot u(t), g(\dot u(t))\rangle_{V,V'} E(t)^{\beta-1}\mathcal{E}(u)^\gamma E(t)^{\frac{1}{2}}\\
&\leq C_2 \langle \dot u(t), g(\dot u(t))\rangle_{V,V'} E(t)^{\beta+\gamma-\frac{1}{2}}\\
&= C_2 \langle \dot u(t), g(\dot u(t))\rangle_{V,V'} E(t)^{\alpha(1-\theta)+\gamma-\frac{1}{2}}.
\end{aligned}$$

Thanks to the last inequality together with assumption (2.7) and since E is bounded we obtain

$$-\langle \dot u(t), g(\dot u(t))\rangle_{V,V'} E(t)^{\beta-1}\langle \mathcal{M}(u(t)), \dot u(t)\rangle_{V'} \leq C_3 \langle \dot u(t), g(\dot u(t))\rangle_{V,V'}. \tag{2.14}$$

Thanks to assumption (2.8) and since $E \geq 0$, by applying Young's inequality we get for all $t \geq T$ large enough

$$\varepsilon E(t)^\beta \langle \mathcal{M}'(u(t))\dot u(t), \dot u(t)\rangle_{V'} \leq \varepsilon C_4 E(t)^\beta \|\dot u(t)\|^2_H \leq \frac{\varepsilon}{4} E(t)^{\beta\frac{(\alpha+2)}{\alpha}} + \varepsilon C_5 \|\dot u(t)\|^{\alpha+2}_H.$$

Now by using the definition of E and β we obtain

$$\begin{aligned}
\varepsilon E(t)^\beta \langle \mathcal{M}'(u(t))\dot u(t), \dot u(t)\rangle_{V'} &\leq \frac{\varepsilon}{4} E(t)^{(1-\theta)(\alpha+2)} + \varepsilon C_5 \|\dot u(t)\|^{\alpha+2}_H\\
&\leq \frac{\varepsilon}{4}\left(\|\dot u(t)\|_H^{2(1-\theta)(\alpha+2)} + \mathcal{E}(u)^{(1-\theta)(\alpha+2)}\right) + \varepsilon C_5 \|\dot u(t)\|^{\alpha+2}_H.
\end{aligned}$$

Since $\dot u$ is bounded we get

$$\varepsilon E(t)^\beta \langle \mathcal{M}'(u(t))\dot u(t), \dot u(t)\rangle_{V'} \leq \varepsilon C_6 \|\dot u(t)\|^{\alpha+2}_H + \frac{\varepsilon}{4}\mathcal{E}(u)^{\alpha(1-\theta)}\mathcal{E}(u)^{2(1-\theta)}.$$

Thanks to assumption (2.6) we get

$$\begin{aligned}
\varepsilon E(t)^\beta \langle \mathcal{M}'(u(t))\dot u(t), \dot u(t)\rangle_{V'} &\leq \varepsilon C_6 \|\dot u(t)\|^{\alpha+2}_H + \frac{\varepsilon}{4}\mathcal{E}(u)^{\alpha(1-\theta)}\|\mathcal{M}(u(t))\|^2_{V'}\\
&\leq \varepsilon C_6 \|\dot u(t)\|^{\alpha+2}_H + \frac{\varepsilon}{4}E(t)^{\alpha(1-\theta)}\|\mathcal{M}(u(t))\|^2_{V'}.
\end{aligned}$$

Therefore we have for all $t \geq T$

$$\varepsilon E(t)^\beta \langle \mathcal{M}'(u(t))\dot{u}(t), \dot{u}(t)\rangle_{V'} \leq \varepsilon C_6 \|\dot{u}(t)\|_H^{\alpha+2} + \frac{\varepsilon}{4} E(t)^\beta \|\mathcal{M}(u(t))\|_{V'}^2. \tag{2.15}$$

We have thanks to Cauchy Schwarz inequality together with assumption (2.4)

$$\varepsilon E(t)^\beta \langle \mathcal{M}(u(t)), g(\dot{u}(t))\rangle_{V'} \leq \varepsilon \rho_2 E(t)^\beta \|\mathcal{M}(u(t))\|_{V'} \|\dot{u}(t)\|_H^{\alpha+1}.$$

Let $C_7 = 1 + \sup_{\mathbb{R}_+} \|\mathcal{M}(u(t))\|_{V'}$, by using Young's inequality there exists $C_8 \geq 0$ such that

$$\varepsilon \|\mathcal{M}(u(t))\|_{V'} \|\dot{u}(t)\|_H^{\alpha+1} \leq \varepsilon \frac{1-\alpha}{4(1+\alpha)C_7^\alpha} \|\mathcal{M}(u(t))\|_{V'}^{\alpha+2} + \varepsilon C_8 \|\dot{u}(t)\|_H^{\alpha+2}.$$

Then since E is bounded we obtain for all $t \geq T$

$$\varepsilon E(t)^\beta \langle \mathcal{M}(u(t)), g(\dot{u}(t))\rangle_{V'} \leq \frac{\varepsilon}{4} E(t)^\beta \|\mathcal{M}(u(t))\|_{V'}^2 + \varepsilon C_9 \|\dot{u}(t)\|_H^{\alpha+2}. \tag{2.16}$$

Thanks to assumptions (2.3), (2.14), (2.15) and (2.16) and by choosing ε small enough it follows that for all $t \geq T$

$$H'(t) \leq -C_{10} \left(\|\dot{u}(t)\|_H^{\alpha+2} + E(t)^\beta \|\mathcal{M}(u(t))\|_{V'}^2 \right). \tag{2.17}$$

Now by using the last inequality together with assumption (2.6) and the definition of E we get

$$\begin{aligned}
-H'(t) &\geq C_{10}(\|\dot{u}(t)\|_H^{\alpha+2} + \mathcal{E}(u)^{(\alpha+2)(1-\theta)}) \\
&\geq C_{10}\mathcal{E}(u)^{(\alpha+2)(1-\theta)} \\
&\geq C_{10}(E(t) - \frac{1}{2}\|\dot{u}(t)\|_H^2)^{(\alpha+2)(1-\theta)} \\
&\geq C_{11}E(t)^{(\alpha+2)(1-\theta)} - C_{12}\|\dot{u}(t)\|_H^{2(\alpha+2)(1-\theta)} \\
&\geq C_{11}E(t)^{(\alpha+2)(1-\theta)} - C_{13}\|\dot{u}(t)\|_H^{\alpha+2} \\
&\geq C_{11}E(t)^{(\alpha+2)(1-\theta)} + C_{14}H'(t).
\end{aligned}$$

Then we obtain

$$-C_{15}H'(t) \geq E(t)^{(\alpha+2)(1-\theta)}. \tag{2.18}$$

By combining (2.18) and (2.13) we obtain the next differential inequality for all $t \geq T$

$$H(t)^{(\alpha+2)(1-\theta)} + C_{16}H'(t) \leq 0.$$

It follows by applying Lemma 2.8 from [1] that for all $t \geq T$

$$H(t) \leq C_{17} t^{-\lambda}, \tag{2.19}$$

where $\lambda = \frac{1}{(\alpha+2)(1-\theta)-1}$. By using (2.17) together with (2.19) we obtain for all $t \geq T$

$$\int_t^{2t} \|\dot{u}(s)\|_H^{\alpha+2}\, ds \leq -\frac{1}{C_{10}} \int_t^{2t} H'(s)\, ds \leq \frac{1}{C_{10}} H(t) \leq C_{15} t^{-\lambda}.$$

Since we have

$$\int_t^{2t} \|\dot{u}(s)\|_H\, ds \leq t^{\frac{\alpha+1}{\alpha+2}} \left(\int_t^{2t} \|\dot{u}(s)\|_H^{\alpha+2}\, ds \right)^{\frac{1}{\alpha+2}}.$$

Therefore we get for all $t \geq T$

$$\int_t^{2t} \|\dot{u}(s)\|_H\, ds \leq C_{18} t^{-\frac{\lambda}{\alpha+2}} t^{\frac{\alpha+1}{\alpha+2}} = C_{18} t^{-\xi},$$

where $\xi = \frac{1-(\alpha+1)(1-\theta)}{(\alpha+2)(1-\theta)-1}$.

Then we obtain for all $t \geq T$

$$\begin{aligned} \int_t^{\infty} \|\dot{u}(s)\|_H\, ds &\leq \sum_{k=0}^{\infty} \int_{2^k t}^{2^{k+1} t} \|\dot{u}(s)\|_H\, ds \\ &\leq C_{19} \sum_{k=0}^{\infty} (2^k t)^{-\xi} \\ &\leq C_{19} t^{-\xi}. \end{aligned}$$

In particular, $\dot{u} \in L^1(T, \infty, H)$. Hence, $u(t)$ has a limit a in H as $t \to \infty$ and

$$\|u(t) - a\|_H \leq C t^{-\xi}.$$

The other estimates follow rather easily from (2.19) and (2.13). Then by (2.6), we see that $\mathcal{M}(u(t))$ tends to 0 as $t \to \infty$ and the last conclusion follows immediately.

3 Direct applications

In this section we apply our main result to various simple example in order to test the sharpness of the estimates given by that theorem.

3.1 A second order ODE

As a first application of the abstract theorem 2.1 let us consider the following second order ODE :

$$u'' + |u'|^{\alpha} u' + f(u) = 0, \tag{3.1}$$

where $\alpha \in (0,1)$ and f is such that for some reals a, b with $a \leq b$

$$f(s) = m(s)[(s-b)^+ - (s-a)^-]$$

where

$$m \in W^{1,\infty}_{loc}(\mathbb{R}), \text{ and } \inf_{s\in\mathbb{R}} m(s) > 0 \tag{3.2}$$

For this example it suffices to choose $V = \mathbb{R}$, then conditions (2.6) and (2.7) are verified with $\theta = \gamma = \dfrac{1}{2}$ for any compact interval B of $\mathbb{R}$. The assumptions (2.3) and (2.4) are clearly true for equation (3.1). As a consequence of (3.2) it is easy to see that all solutions of (3.1) are bounded and globally Lipschitz on $\mathbb{R}_+$. By applying theorem 2.1 we prove that there exists $c \in \mathbb{R}$ such that $|u(t) - c| \leq Ct^{-\frac{1}{\alpha}+1}$, where C is a positive constant. Clearly $c \in [a, b]$. This result is optimal.

3.2 A critical semilinear wave equation

In what follows, Ω is a bounded connected open subset of $\mathbb{R}^N$. As a second application of the abstract theorem 2.1, we let $V = H^1_0(\Omega)$, $H := L^2(\Omega)$, $\mathcal{M}(u) = -\Delta u - \lambda_1 u + |u|^{p-1}u$, where λ_1 is the first eigenvalue of $-\Delta$ and $p > 1$ satisfies $(N-2)p < N+2$,

$$\mathcal{E}(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 - \lambda_1 |u|^2)\, dx + \frac{1}{p+1}\int_\Omega |u|^{p+1}\, dx$$

and we consider the following system

$$\begin{cases} u_{tt} + g(u_t) - \Delta u - \lambda_1 u + |u|^{p-1}u = 0, & \text{in } \mathbb{R}_+ \times \Omega, \\ u(t,x) = 0, & \text{on } \mathbb{R}_+ \times \partial\Omega, \end{cases} \tag{3.3}$$

where $g : H \longrightarrow V'$ satisfies (2.3)-(2.4) with $0 < \alpha < \frac{1}{p}$. It has been established in [9] that under the above conditions, $\mathcal{E} \in C^2(V, V')$ and (2.8) is fulfilled. In order to apply our main result to this example the main assumption remaining to be checked is therefore assumption (2.6). Now it has been proved in [11] that for u small enough in V

$$\|\mathcal{M}(u)\|_{H^{-1}} \geq c_1(\mathcal{E}(u))^{1-\theta}, \text{ where } \theta = \frac{1}{p+1}.$$

This result suffices to study the asymptotics of solutions knowing in advance that they converge to 0 in V. Actually a refinement of the method of [11] allows to verify that for any $R > 0$, there is $c_1(R) > 0$ for which

$$\forall u \in V, \quad \|u\|_V \leq R \Rightarrow \|\mathcal{M}(u)\|_{H^{-1}} \geq c_1(R)(\mathcal{E}(u))^{\frac{p}{p+1}}$$

For the proof, see Section 4, Corollary 4.2 and Remark 4.3, 2). On the other hand it is not difficult to check that (2.6) holds true on any bounded subset of V with $\gamma = \frac{1}{2}$. Indeed

$$\|\mathcal{M}(u)\|_{H^{-1}} \leq \| - \Delta u - \lambda_1 u\|_{H^{-1}} + \||u|^{p-1}u\|_{H^{-1}}.$$

We infer

$$\| - \Delta u - \lambda_1 u\|_{H^{-1}} \leq (2\mathcal{E}(u))^{\frac{1}{2}}$$

Indeed

$$\forall v \in V, \quad \langle -\Delta u - \lambda_1 u, v\rangle = \int_\Omega (\nabla u.\nabla v - \lambda_1 uv)dx$$

By Cauchy-Schwarz inequality we deduce

$$\forall v \in V, \quad \langle -\Delta u - \lambda_1 u, v\rangle \leq (\int_\Omega (\|\nabla u\|^2 - \lambda_1 |u|^2)dx)^{\frac{1}{2}} (\int_\Omega (\|\nabla v\|^2 - \lambda_1 |v|^2)dx)^{\frac{1}{2}}$$

$$\leq (\int_\Omega (\|\nabla u\|^2 - \lambda_1 |u|^2)dx)^{\frac{1}{2}} \|v\|_V$$

And the result follows. Now, if $N \leq 2$ the estimate of the nonlinear part is obvious, while if $N > 2$ since $p < \dfrac{N+2}{N-2}$ then we have $\dfrac{p}{p+1} < \dfrac{1}{2} + \dfrac{1}{N} = \dfrac{1}{(2^*)'}$.
So we obtain

$$\||u|^{p-1}u\|_{L^{(2^*)'}} \leq c_2 \||u|^{p-1}u\|_{L^{\frac{p+1}{p}}} \leq c_3(\mathcal{E}(u))^{\frac{p}{p+1}}.$$

Then we get

$$\||u|^{p-1}u\|_{H^{-1}} \leq c_4(\mathcal{E}(u))^{\frac{p}{p+1}}.$$

In order to apply Theorem 2.1 we first observe that

$$\forall u \in V, \quad \mathcal{E}(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 - \lambda_1 |u|^2)\, dx + \frac{1}{p+1}\int_\Omega |u|^{p+1}\, dx \geq \frac{1}{2}\int_\Omega |\nabla u|^2\, dx - M$$

for some positive constant M (the proof for p = 1 is just slightly more delicate), and as a consequence of the fact that $E(t)$ is non-increasing we deduce that any solution u is such that $u(t)$ remains bounded in V.

Then we compute

$$\xi = \frac{1-(\alpha+1)(1-\theta)}{(\alpha+2)(1-\theta)-1} = \frac{1-\alpha p}{(\alpha+1)p-1}; \quad \lambda = \frac{1}{(\alpha+2)(1-\theta)-1} = \frac{p+1}{(\alpha+1)p-1}.$$

Applying (2.9) we obtain

$$|(u(t) - a)| \leq Ct^{-\frac{1-\alpha p}{(\alpha+1)p-1}}$$

for some a which will turn out to be 0 by the last part of the Theorem. On the other hand, since (cf. Proposition 4.1, formula (4.6)) for some $\eta > 0$ we have $\mathcal{E}(u) \geq \eta \|u\|_V^{p+1}$, applying (2.11) we find

$$\|u(t)\|_V \leq Mt^{-\frac{1}{(\alpha+1)p-1}}$$

which is always sharper. This shows that it is sometimes preferable to apply directly the energy estimate rather than (2.9).

3.3 A semilinear wave equation with Neumann boundary conditions

As a third application of the abstract theorem 2.1, we let $V = H^1(\Omega)$, $H = L^2(\Omega)$, $\mathcal{M}(u) = -\Delta u + |u|^{p-1}u$, where $1 \le p < \frac{N+2}{N-2}$, $\mathcal{E}(u) = \dfrac{1}{2}\displaystyle\int_\Omega |\nabla u|^2\,dx + \dfrac{1}{p+1}\displaystyle\int_\Omega |u|^{p+1}\,dx$ and we consider the following system

$$\begin{cases} u_{tt} + g(u_t) - \Delta u + |u|^{p-1}u = 0, & \text{in } \mathbb{R}_+ \times \Omega, \\ \frac{\partial u}{\partial n} = 0 \ \ \text{on } \mathbb{R}_+ \times \partial\Omega, \end{cases} \tag{3.4}$$

where $g : H \to V'$ satisfies (2.3)-(2.4) with $0 < \alpha < \frac{1}{p}$. As in the previous example the results of [9] show that under the above conditions, $\mathcal{E} \in C^2(V, V')$ and (2.8) is fulfilled. The main assumption to be checked is again assumption (2.6). It is rather easy to verify that for any $R > 0$, there is $c_1(R) > 0$ for which

$$\forall u \in V, \quad \|u\|_V \le R \Rightarrow \|\mathcal{M}(u)\|_{H^{-1}} \ge c_1(R)(\mathcal{E}(u))^{\frac{p}{p+1}}$$

For the proof, see Section 4, Corollary 4.4 and Remark 4.5, 2). Moreover it is not difficult to check that (2.6) holds true on any bounded subset of V with $\gamma = \frac{1}{2}$. Indeed

$$\|\mathcal{M}(u)\|_{V'} \le \|-\Delta u\|_{V'} + \||u|^{p-1}u\|_{V'}.$$

We infer

$$\|-\Delta u\|_{V'} \le (2\mathcal{E}(u))^{\frac{1}{2}}$$

Indeed

$$\forall v \in V, \quad \langle -\Delta u, v\rangle = \int_\Omega \nabla u.\nabla v dx$$

By Cauchy-Schwarz inequality we deduce

$$\forall v \in V, \quad \langle -\Delta u, v\rangle \le (\int_\Omega \|\nabla u\|^2 dx)^{\frac{1}{2}}(\int_\Omega \|\nabla v\|^2 dx)^{\frac{1}{2}}$$

$$\le (\int_\Omega \|\nabla u\|^2 dx)^{\frac{1}{2}}\|v\|_V$$

And the result follows. Then by the same method as in the Dirichlet case we obtain easily

$$\||u|^{p-1}u\|_{V'} \le c_4(\mathcal{E}(u))^{\frac{p}{p+1}}.$$

In order to apply Theorem 2.1 we first observe that

$$\forall u \in V, \quad \mathcal{E}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{1}{p+1}\int_\Omega |u|^{p+1}\,dx \ge \delta \int_\Omega |\nabla u|^2 + u^2\,dx - M$$

for some constants $M \geq 0, \delta > 0$, and as a consequence of the fact that $E(t)$ is non-increasing we deduce that any solution u is such that $u(t)$ remains bounded in V.

By applying Theorem 2.1 and using Proposition 4.1, formula (4.6), we obtain exactly the same estimates as in the previous example. In particular we find

$$\|u(t)\|_V \leq Mt^{-\frac{1}{(\alpha+1)p-1}}$$

The degree of sharpness of this estimate is not clear, cf. remark 3.4 .

3.4 Examples of damping operators and convergence of weak solutions

In the examples of both paragraphs 3.2 and 3.3, the initial value problem can be solved for any initial state in $V \times H$ under relevant conditions on the damping term. In this paragraph we shall consider 2 basic examples

Example 3.1 Let γ be a locally Lipschitz function such that

$$\exists K > 0, \quad \gamma'(s) \geq -K, \text{ a.e. on } \mathbb{R}$$

Assume that γ satisfies the following conditions

$$\forall s \in \mathbb{R}, \quad \gamma(s)s \geq \rho_1 |s|^{\alpha+2} \qquad (C_1)$$

$$\forall s \in \mathbb{R}, \quad |\gamma(s)| \leq \rho_2 |s|^{\alpha+1} \qquad (C_2)$$

The typical case is

$$\gamma(s) = \gamma_1 (s^+)^{\alpha+1} - \gamma_2 (s^-)^{\alpha+1}$$

By setting

$$g(v)(x) = \gamma(v(x)) \text{ a.e. on } \Omega$$

we define an operator $g : H \to V'$ with $V = H^1(\Omega)$ (resp $V = H^1_0(\Omega)$) for any $\alpha \in [0, 1]$ if $N \leq 2$ and for any $\alpha \in [0, \frac{2}{N}]$ if $N \geq 3$. In addition in such a case g satisfies automatically (2.3)-(2.4).

Example 3.2 Let m be a locally Lipschitz function such that

$$\forall s \in \mathbb{R}_+, \quad \sigma_1 s^\alpha \leq m(s) \leq \sigma_2 s^\alpha \qquad (M)$$

and

$$\exists K > 0, \quad m'(s) \geq -K, \text{ a.e. on } \mathbb{R}$$

By setting

$$g(v)(x) = m(\|v\|_H)(v(x)) \text{ a.e. on } \Omega$$

we define an operator $g : H \to H \subset V'$ with $V = H^1(\Omega)$ (resp $V = H^1_0(\Omega)$) for any $\alpha > 0$. In addition in such a case g satisfies automatically (2.3)-(2.4).

By applying the results of [10], It is rather straightforward to see that for any g satisfying the conditions of Example 3.1,the initial value problems associated to both equations (3.3) and (3.4) are well-posed for any initial state in $V \times H$. In addition, for initial data in $D(A) \times V$, the solution has the regularity required for the applicability of Theorem 2.1. In addition the solution depends continuously on the initial state as a map from $V \times H$ to $C([0,T,V] \cap C^1([0,T,H]$ for any $T > 0$. A careful inspection of the results from [10] shows that exactly the same property can be deduced for the same quation when g has the non-local form described in Example 3.2.

By combining these properties with the results previously obtained, we obtain

Corollary 3.3 *For any g satisfying the conditions of Example 3.1 or Example 3.2 , for any $(u_0, u_1) \in V \times H$ there is a unique weak global solution u of* (3.3) *(resp* (3.4)*) in the sense of [7] which satisfies $u(0,.) = u_0$ and $u_t(0,.) = u_1$. In addition if we assume $\alpha < \frac{1}{p}$ and in the case of Example 3.1, for $N > 2$, the additional condition $\alpha \le \frac{2}{N}$, then we have, for some constants $M, M' > 0$*

$$\forall t \ge 1, \quad \|u(t,.)\|_V \le M t^{-\frac{1}{(\alpha+1)p-1}}$$

and

$$\forall t \ge 1, \quad \|u_t(t,.)\|_H \le M' t^{-\frac{p+1}{2[(\alpha+1)p-1]}}$$

Proof. For a strong solution, the result is a direct consequence of the fact that g satisfies (2.3)-(2.4). Moreover it follows obviously from our method of proof that the estimates on u and u_t are uniform when the initial data (u_0, u_1) remain bounded in $V \times H$. Then the result in the general case follows by density. □

Remark 3.4 The Neumann case contains in particular the case of the ODE

$$u'' + g(u') + |u|^{p-1}u = 0$$

which was studied in [8]. It is not difficult to see that the rate of decay given by Corollary 3.3 for $g(s) := |s|^\alpha s$ is optimal for space-independant solutions only if $p = 1$.

4 A gradient inequality for some non analytic functionals.

In this section we shall find the optimal Lojasiewicz exponent for a class of nonnegative potentials associated to semi-linear PDE problems. In what follows, Ω is a bounded connected open subset of $\mathbb{R}^N$.

4.1 A general class of possibly non-analytic functionals. Application to various operators and boundary conditions

In this paragraph, V is a Hilbert space continuously imbedded in $H = L^2(\Omega)$, $A \in \mathcal{L}(V, V')$ is symmetric, such that

$$\forall u \in V, \quad \langle Au, u\rangle \geq 0 \tag{4.1}$$

and we set

$$\mathcal{M}(u) = Au + f(u)$$

where $f : V \to V'$ is the gradient of a functional $\mathcal{F} \in C^1(V)$. The energy functional is

$$\mathcal{E}(u) = \frac{1}{2}\langle Au, u\rangle + \mathcal{F}(u)$$

We assume that $N = \ker A$ is finite dimensional and we denote by $P : H \to N$ the orthogonal projection on N in H.

Proposition 4.1 *Under the following hypotheses*

$$\exists \eta > 0, \quad \forall v \in V \cap N^{\perp}, \quad \langle Av, v\rangle \geq \eta \|v\|_V^2 \tag{4.2}$$

$$\exists \mu > 0, \quad \forall u \in V, \quad \langle f(u), u\rangle \geq \mu \mathcal{F}(u) \tag{4.3}$$

For any $R > 0$ there is a constant $M(R)$ such that

$$\forall u \in V, \ \|u\|_V \leq R \Longrightarrow \|u\|_H^{r+1} \leq M(R)\mathcal{F}(u) \tag{4.4}$$

Then for any $R > 0$

1) there is a constant $C(R)$ such that

$$\forall u \in V, \|u\|_V \leq R \Longrightarrow (\mathcal{E}(u))^{\frac{r}{r+1}} \leq C(R)\|\mathcal{M}(u)\|_{V'} \tag{4.5}$$

2) there is a constant $P(R)$ such that

$$\forall u \in V, \|u\|_V \leq R \leq R \Longrightarrow \|u\|_V^{r+1} \leq P(R)\mathcal{E}(u) \tag{4.6}$$

Proof. We have

$$\forall u \in V, \quad \langle \mathcal{M}u, u\rangle = \langle A(u - Pu), u - Pu\rangle + \langle f(u), u\rangle$$

$$\geq \eta \|u - Pu\|_V^2 + \mu \mathcal{F}(u)$$

On the other hand

$$\forall u \in V, \quad \|u\|_V \leq \|u - Pu\|_V + \|Pu\|_V \leq \|u - Pu\|_V + C_1 \|Pu\|_H$$

$$\forall u \in V, \ \|u\|_V \leq R \Longrightarrow \|u\|_V \leq \|u - Pu\|_V + C_2(R)\mathcal{F}(u)^{\frac{1}{r+1}}$$

Hence
$$\forall u \in V,\ \|u\|_V \le R \Longrightarrow \|u\|_V \le C_3(R)(\|u - Pu\|_V^{\frac{2}{r+1}} + \mathcal{F}(u)^{\frac{1}{r+1}})$$
which implies by using the inequality $a + b \le 2(a^q + b^q)^{1/q}$ applied with $q = r + 1$
$$\forall u \in V,\ \|u\|_V \le R \Longrightarrow \|u\|_V \le 2C_3(R)(\|u - Pu\|_V^2 + \mathcal{F}(u))^{\frac{1}{r+1}}$$
Therefore (4.6) is proved and moreover
$$\forall u \in V,\ \|u\|_V \le R \Longrightarrow \quad \eta\|u - Pu\|_V^2 + \mu\mathcal{F}(u) \le \langle \mathcal{M}u, u\rangle$$
$$\le 2C_3(R)(\|u - Pu\|_V^2 + \mathcal{F}(u))^{\frac{1}{r+1}}\|\mathcal{M}u\|_{V'}$$
Then (4.5) becomes an immediate consequence of the simple inequality
$$\langle Au, u\rangle = \langle A(u - Pu), u - Pu\rangle \le \|A\|_{L(V,V')}\|u - Pu\|_V^2$$
□

We now state 3 simple applications of Proposition 4.1

1) The Dirichlet case. Let $V = H_0^1(\Omega)$, $H := L^2(\Omega)$. We consider
$$\mathcal{M}(u) = -\Delta u - \lambda_1 u + c_1(u^+)^p - c_2(u^-)^q$$
where λ_1 is the first eigenvalue of $-\Delta$ and $c_1, c_2 > 0$, $1 \le \inf\{p, q\}$ and either $N \le 2$, or $\sup\{p, q\} < \frac{N+2}{N-2}$. The energy is given by
$$\mathcal{E}(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 - \lambda_1|u|^2)\,dx + \frac{c_1}{p+1}\int_\Omega (u^+)^{p+1}\,dx + \frac{c_2}{q+1}\int_\Omega (u^-)^{q+1}\,dx$$
Corollary 4.2 *Under the above conditions, we have* (4.5) *with* $r = \sup\{p, q\}$.

Proof. We set
$$Au = -\Delta u - \lambda_1 u; \quad f(u) = c_1(u^+)^p - c_2(u^-)^q$$
and
$$\mathcal{F}(u) : \frac{c_1}{p+1}\int_\Omega (u^+)^{p+1}\,dx + \frac{c_2}{q+1}\int_\Omega (u^-)^{q+1}\,dx$$
Then under the assumptions on p, q, f is a continuous map from V to V' and it is in fact the gradient of $\mathcal{F}$. In addition (4.3) is fulfilled with $\mu = 1 + \inf\{p, q\} \ge 2$ and we also have clearly, using the positive character of both c_1 and c_2
$$\|u^+\|_H^{p+1} \le K_1\mathcal{F}(u)$$

and

$$\|u^-\|_H^{q+1} \le K_2 \mathcal{F}(u)$$

By addition we find

$$\|u^+\|_H^{r+1} + \|u^-\|_H^{r+1} \le K_1 \mathcal{F}(u)\|u\|_H^{r-p} + K_2 \mathcal{F}(u)\|u\|_H^{r-q} \le K\mathcal{F}(u)(\|u\|_H^{r-p} + \|u\|_H^{r-q})$$

and (4.4) follows easily. Finally we observe that since Ω is connected, N is one dimensional and to check (4.2), introducing the second eigenvalue λ_2 of $-\Delta$ on V we find

$$\forall v \in V \cap N^\perp, \quad \langle Av, v\rangle \ge \lambda_2 \|v\|_H^2$$

In addition by definition of the norm in V we have

$$\forall v \in V, \quad \|v\|_V^2 = \langle Av, v\rangle + \lambda_1 \|v\|_H^2$$

Hence

$$\forall v \in V \cap N^\perp, \quad \|v\|_V^2 \le \langle Av, v\rangle + \frac{\lambda_1}{\lambda_2}\langle Av, v\rangle = (1 + \frac{\lambda_1}{\lambda_2})\langle Av, v\rangle$$

which gives (4.2) with $\eta = \dfrac{\lambda_2}{\lambda_1 + \lambda_2}$. Therefore all conditions of Proposition 4.1 are fulfilled and the result follows □

Remarks 4.3 1) If $p \ne q$ or $c_1 \ne c_2$, the functional is not analytic since even its restriction to the subspace of multiples of the first eigenfunction is not analytic.
2) If $p = q$ and $c_1 = c_2$ we recover the example of Section 3.2.

2) The Neuman case. Let $V = H^1(\Omega)$, $H := L^2(\Omega)$. We consider

$$\mathcal{M}(u) = -\Delta u + c_1(u^+)^p - c_2(u^-)^q$$

where and $c_1, c_2 > 0$, $1 \le \inf\{p, q\}$ and either $N \le 2$, or $\sup\{p, q\} < \frac{N+2}{N-2}$. The energy is given by

$$\mathcal{E}(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2\, dx + \frac{c_1}{p+1}\int_\Omega (u^+)^{p+1}\, dx + \frac{c_2}{q+1}\int_\Omega (u^-)^{q+1}\, dx$$

Corollary 4.4 *Under the above conditions, we have* (4.5) *with* $r = \sup\{p, q\}$.

Proof. We set

$$Au = -\Delta u; \quad f(u) = c_1(u^+)^p - c_2(u^-)^q$$

and

$$\mathcal{F}(u) : \frac{c_1}{p+1}\int_\Omega (u^+)^{p+1}\, dx + \frac{c_2}{q+1}\int_\Omega (u^-)^{q+1}\, dx$$

The properties of f and $\mathcal{F}$ are similar to the previous example. Since Ω is connected, N is one dimensional, being reduced to constant functions and and to check (4.2), introducing the second eigenvalue λ_2 of $-\Delta$ on V we find

$$\forall v \in V \cap N^\perp, \quad \langle Av, v\rangle \geq \lambda_2 \|v\|_H^2$$

In addition by definition of the norm in V we have here

$$\forall v \in V, \quad \|v\|_V^2 = \langle Av, v\rangle + \|v\|_H^2$$

Hence

$$\forall v \in V \cap N^\perp, \quad \|v\|_V^2 \leq \langle Av, v\rangle + \frac{1}{\lambda_2}\langle Av, v\rangle = (1 + \frac{1}{\lambda_2})\langle Av, v\rangle$$

which gives (4.2) with $\eta = \dfrac{\lambda_2}{1+\lambda_2}$. Therefore all conditions of Proposition 4.1 are fulfilled and the result follows $\square$

Remarks 4.5 1) If $p \neq q$ or $c_1 \neq c_2$, the functional is not analytic since even its restriction to the subspace of multiples of the first eigenfunction is not analytic.
2) If $p = q$ and $c_1 = c_2$ we recover the example of Section 3.3.

3) A fourth order operator. Let $V = H_0^2(\Omega)$, $H = L^2(\Omega)$ and let λ_1 denote here the first eigenvalue of Δ^2 on $H_0^2(\Omega)$.We assume $c_1 > 0, c_2 > 0$ and $p, q > 1$ with

$$(N-4)\sup\{p, q\} < N + 4$$

We consider

$$\mathcal{M}(u) = \Delta^2 u - \lambda_1 u + c_1(u^+)^p - c_2(u^-)^q$$

The energy is given by

$$\mathcal{E}(u) = \frac{1}{2}\int_\Omega (|\Delta u|^2 - \lambda_1 |u|^2)\, dx + \frac{c_1}{p+1}\int_\Omega (u^+)^{p+1}\, dx + \frac{c_2}{q+1}\int_\Omega (u^-)^{q+1}\, dx$$

Corollary 4.6 *Under the above conditions, we have* (4.5) *with* $r = \sup\{p, q\}$.

Proof. We set

$$Au = \Delta^2 u - \lambda_1 u; \quad f(u) = c_1(u^+)^p - c_2(u^-)^q$$

and

$$\mathcal{F}(u) : \frac{c_1}{p+1}\int_\Omega (u^+)^{p+1}\, dx + \frac{c_2}{q+1}\int_\Omega (u^-)^{q+1}\, dx$$

Then under the assumptions on p, q, f is a continuous map from V to V' and it is in fact the gradient of $\mathcal{F}$. The rest of the proof is identical to the proof of Corollary 4.4 $\square$

4.2 Multiple equilibria under Neumann boundary conditions

In this paragraph we let $V = H^1(\Omega)$, $H = L^2(\Omega)$, and we set

$$\mathcal{M}(u) = -\Delta u + f(u)$$

$$\mathcal{E}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \int_\Omega F(u(x))\,dx$$

The hypotheses that we shall make on f will imply that the energy functional is bounded from below since it will be the case for any primitive of f. We shall choose for F the primitive with minimum equal to 0.

More specifically, let us define for some reals a, b with $a \le b$

$$\forall s \in \mathbb{R}, \quad \rho(s) = (s-b)^+ - (s-a)^-$$

Let $p > 1$ be such that

$$(N-2)p < N+2$$

in order that

$$V \subset L^{p+1}(\Omega)$$

We assume that $f \in W^{1,\infty}_{loc}(\mathbb{R})$ with $f = 0$ on $[a, b]$ and for some $c > 0$

$$\forall s \in \mathbb{R}, \quad f(s)\rho(s) \ge c|\rho(s)|^{p+1} \tag{4.7}$$

We define then

$$F(s) := \int_a^s f(u)du$$

In particular $F = 0$ on $[a, b]$ and it follows clearly from (4.7) that F satisfies

$$\forall s \in \mathbb{R}, \quad F(s) \ge \frac{c}{p+1}|\rho(s)|^{p+1}$$

We assume that F satisfies for some $C > 0$ the upper bound

$$\forall s \in \mathbb{R}, \quad F(s) \le C|\rho(s)|^{p+1} \tag{4.8}$$

Proposition 4.7 *Under the conditions* (4.7), (4.8), *for any* $R > 0$ *there is a constant* $C(R)$ *such that*

$$\forall u \in V, \|u\|_V \le R \Longrightarrow (\mathcal{E}(u))^{\frac{p}{p+1}} \le C(R)\|\mathcal{M}(u)\|_{V'} \tag{4.9}$$

Proof. Setting

$$Pv := \frac{1}{|\Omega|}\int_\Omega v(x)dx$$

we have since f is nondecreasing

$$\forall u \in H^2(\Omega), \quad \langle \mathcal{M}u, u - Pu\rangle = \int_\Omega |\nabla u|^2 + f(u)(u - Pu)dx$$

$$\geq \int_\Omega |\nabla u|^2 + f(Pu)(u - Pu)dx = \int_\Omega |\nabla u|^2 dx$$

similarly

$$\forall u \in H^2(\Omega), \quad \langle \mathcal{M}u, \rho(u)\rangle = \int_\Omega (\rho'(u)|\nabla u|^2 + f(u)\rho(u))dx$$

$$\geq c \int_\Omega |\rho(u)|^{p+1} dx = c\|\rho(u)\|_{p+1}^{p+1}$$

hence by addition

$$\forall u \in H^2(\Omega), \quad \langle \mathcal{M}u, u - Pu + \rho(u)\rangle \geq \|\nabla u\|_2^2 + c\|\rho(u)\|_{p+1}^{p+1}$$

As a consequence of Poincaré's inequality , we have

$$\|u - Pu\|_V \leq K_1 \|\nabla u\|_2$$

In addition

$$\|\rho(u)\|_V \leq \|\nabla(\rho(u))\|_2 + \|\rho(u)\|_2 \leq \|\nabla u\|_2 + K_2\|\rho(u)\|_{p+1}$$

hence

$$\|u - Pu + \rho(u)\|_V \leq K_3(\|\nabla u\|_2 + \|\rho(u)\|_{p+1})$$

As a consequence we find for some fixed $\delta > 0$

$$\|\mathcal{M}u\|_* \geq \delta \frac{\|\nabla u\|_2^2 + \|\rho(u)\|_{p+1}^{p+1}}{\|\nabla u\|_2 + \|\rho(u)\|_{p+1}}$$

by writing

$$\|\nabla u\|_2 \leq \|\nabla u\|_2^{1-\frac{2}{p+1}} \|\nabla u\|_2^{\frac{2}{p+1}} \leq \|u\|_V^{1-\frac{2}{p+1}} \|\nabla u\|_2^{\frac{2}{p+1}}$$

we deduce

$$\|\mathcal{M}u\|_* \geq \delta \frac{\|\nabla u\|_2^2 + \|\rho(u)\|_{p+1}^{p+1}}{(1 + \|u\|_V^{1-\frac{2}{p+1}})(\|\nabla u\|_2^{\frac{2}{p+1}} + \|\rho(u)\|_{p+1})}$$

By using the inequality $a + b \leq 2(a^q + b^q)^{1/q}$ applied with $q = p + 1$ we find

$$\|\mathcal{M}u\|_* \geq \delta \frac{(\|\nabla u\|_2^2 + \|\rho(u)\|_{p+1}^{p+1})^{\frac{p}{p+1}}}{2(1 + \|u\|_V^{1-\frac{2}{p+1}})}$$

and finally

$$\mathcal{E}(u)^{\frac{p}{p+1}} \leq M(1 + \|u\|_V^{1-\frac{2}{p+1}})\|\mathcal{M}u\|_*$$

□

5 More elaborate convergence results.

In this section we state and prove the convergence results corresponding to the specific cases of Section 4. These convergence results will contain as special cases all the examples given in Section 3.

5.1 Preliminary results

In order to apply Theorem 2.1 , we shall need among other things to verify the left hand side of (2.6). For this we shall rely in all cases on the following simple preliminary result

Proposition 5.1 *Let Ω is a bounded open subset of $\mathbb{R}^N$, V a Hilbert space continuously imbedded in $H = L^2(\Omega)$, and $A \in \mathcal{L}(V, V')$ symmetric, satisfying* (4.1) *and* (4.2) *. Let $f : V \to V'$ be the gradient of a nonnegative functional $\mathcal{F} \in C^1(V)$. Assume that for any $R > 0$ there is a constant $K(R)$ such that*

$$\forall u \in V,\ \|u\|_V \le R \Longrightarrow \|f(u)\|_{V'} \le K(R)(\mathcal{F}(u))^{\frac{1}{2}} \tag{5.1}$$

Then for any $R > 0$, there is a constant $M(R)$ such that

$$\forall u \in V, \quad \|u\|_V \le R \Longrightarrow \|Au + f(u)\|_{V'} \le M(R)(\langle Au, u\rangle + 2\mathcal{F}(u))^{\frac{1}{2}} \tag{5.2}$$

Proof. For any $u \in V$ we have

$$\|Au\|_{V'} = \|A(u - Pu)\|_{V'} \le \|A\|\|u - Pu\|_V \le \frac{\|A\|}{\eta^{\frac{1}{2}}}\langle Au, u\rangle^{\frac{1}{2}}$$

The result follows by combining this inequality with (5.1). □

In practice, to treat the nonlinear part $f(u)$, we shall use repeatedly the following simple Lemma

Lemma 5.2 *Let Ω , V, H be as above and $p \ge 1$ be such that $V \subset L^{p+1}(\Omega)$ with continuous and dense imbedding. Then for any $c \in \mathbb{R}$ and any $R > 0$, there is a constant $P(R)$ such that*

$$\forall u \in V, \quad \|u\|_V \le R \Longrightarrow \|\{(u - c)^+\}^p\|_{V'} \le P(R)\|(u - c)^+\|_{p+1}^{\frac{p+1}{2}} \tag{5.3}$$

Proof. By duality we have $L^{\frac{p+1}{p}}(\Omega) \subset V'$ with continuous imbedding. Therefore for any $u \in V$ we have

$$\|\{(u - c)^+\}^p\|_{V'} \le K_1\|\{(u - c)^+\}^p\|_{L^{\frac{p+1}{p}}}\| = K_1\|\{(u - c)^+\}\|_{p+1}^p$$

Since $p \ge \frac{p+1}{2}$, the result follows easily by combining this inequality with the imbedding $V \subset L^{p+1}(\Omega)$. □

Finally, boundedness of the solutions in V will follow in all examples as a consequence of the following lemmas

Lemma 5.3 *Let V, H be as above and $A \in \mathcal{L}(V, V')$ symmetric, satisfying* (4.1). *Assume that there are $\lambda_0 > 0, \gamma_0 > 0$ such that*

$$\forall u \in V, \quad \langle Au, u\rangle + \lambda_0 \|u\|_H^2 \geq \gamma_0 \|u\|_V^2$$

In addition, assume that there are $\sigma_0 > 0, K_0 \geq 0$ such that

$$\forall u \in V, \quad \mathcal{F}(u) \geq \sigma_0 \|u\|_H^2 - K_0 \tag{5.4}$$

Then there exists $\sigma > 0$ such that

$$\forall u \in V, \quad \mathcal{E}(u) \geq \sigma \|u\|_V^2 - K_0 \tag{5.5}$$

Proof. If $\sigma_0 \geq \frac{\lambda_0}{2}$, the result is obvious since then

$$\mathcal{E}(u) = \frac{1}{2}\langle Au, u\rangle + \mathcal{F}(u) \geq \frac{1}{2}\gamma_0 \|u\|_V^2 + (\sigma_0 - \frac{1}{2}\lambda_0)\|u\|_H^2 - K_0 \geq \frac{1}{2}\gamma_0\|u\|_V^2 - K_0$$

If $\sigma_0 < \frac{\lambda_0}{2}$, we write

$$\mathcal{E}(u) = \frac{1}{2}\langle Au, u\rangle + \mathcal{F}(u) \geq \frac{\sigma_0}{\lambda_0}\langle Au, u\rangle + \mathcal{F}(u) \geq \frac{\sigma_0\gamma_0}{\lambda_0}\|u\|_V^2 - K_0$$

□

Lemma 5.4 *Let $f \in W^{1,\infty}_{loc}(\mathbb{R})$ satisfy for some $p > 1, c > 0, A \geq 0$*

$$\forall s \in \mathbb{R}, |s| \geq A \Rightarrow f(s)s \geq c|s|^{p+1}$$

Then for any primitive F of f there is $\delta > 0$ and $C > 0$ for which

$$\forall s \in \mathbb{R}, \quad F(s) \geq \delta|s|^2 - C$$

In particular in this case, the functional

$$\mathcal{F}(u) := \int_\Omega F(u(x))dx$$

satisfies (5.4).

Proof. The first inequality is an obvious consequence of a trivial lower estimate on F and the second one follows by integration since Ω is bounded. Of course without additional conditions on f the functional $\mathcal{F}$ is not necessarily defined on all $u \in V$, it may take infinite values. In the examples we always assume that the functional is defined on V as a consequence of growth conditions on F. □

5.2 Convergence to 0 and rate of decay

Again, in what follows, Ω is a bounded connected open subset of $\mathbb{R}^N$. We consider the following problems

$$\begin{cases} u_{tt} + g(u_t) - \Delta u - \lambda_1 u + c_1(u^+)^p - c_2(u^-)^q = 0, & \text{in } \mathbb{R}_+ \times \Omega, \\ u(t,x) = 0, \quad \text{on } \mathbb{R}_+ \times \partial\Omega, \end{cases} \tag{5.6}$$

and

$$\begin{cases} u_{tt} + g(u_t) - \Delta u + c_1(u^+)^p - c_2(u^-)^q = 0, & \text{in } \mathbb{R}_+ \times \Omega, \\ \frac{\partial u}{\partial n} = 0 \quad \text{on } \mathbb{R}_+ \times \partial\Omega, \end{cases} \tag{5.7}$$

The main result of this paragraph is the following :

Theorem 5.5 *Define V, H as in Section 4.1. Assuming that p, q, c_1, c_2 fulfill the conditions of Section 4.1 and g satisfies (2.3)-(2.4), any solution $u \in W^{1,1}_{loc}(\mathbb{R}^+, V) \bigcap W^{2,1}_{loc}(\mathbb{R}^+, H)$ of one of these systems converges to 0 in V as $t \to \infty$ and we have*

$$\|u(t,.)\|_V \le M t^{-\frac{1}{(\alpha+1)r-1}}; \quad \|u_t(t,.)\|_H \le M' t^{-\frac{r+1}{2[(\alpha+1)r-1]}}$$

where $r = \sup\{p, q\}$ and M, M' are some positive constants depending on the solution.

Proof. The main assumptions to be checked are (2.6) and boundedness of $u(t)$ in V. The relevant Lojasiewicz inequality is uniform on any bounded subset of V and has been proved in Corollary 4.2 as a consequence of Proposition 4.1. We now check that the left-handside of (2.6) holds true on any bounded subset of V with $\gamma = \frac{1}{2}$. This is in fact a simple consequence of Lemma 5.2 applied with $c = 0$. Indeed under the hypothesis of Theorem 5.5, we have

$$\forall u \in V, \quad \|u\|_V \le R \Longrightarrow \|\{u^+\}^p\|_{V'} \le P(R)\|u^+\|_{p+1}^{\frac{p+1}{2}} \le p^{\frac{1}{2}} P(R)(\mathcal{F}(u))^{\frac{1}{2}}$$

Changing u to $-u$ and replacing p by q we find

$$\forall u \in V, \quad \|u\|_V \le R \Longrightarrow \|\{u^-\}^q\|_{V'} \le q^{\frac{1}{2}} Q(R)(\mathcal{F}(u))^{\frac{1}{2}}$$

Then by an obvious combination we find than $f(u) = c_1(u^+)^p - c_2(u^-)^q$ satisfies (5.1). By applying Proposition 5.1 we conclude that the left-handside of (2.6) holds true on any bounded subset of V with $\gamma = \frac{1}{2}$. Now(2.6) holds true on any bounded subset of V. A simple application of Theorem 2.1 will conclude the proof as soon as boundedness of $u(t)$ in V is established. But is is clear that f satisfies (5.4) with $A = 0$. Then Lemma 5.5 and Lemma 5.3 applied with $\lambda_0 = \lambda_1$ in the Dirichlet case, $\lambda_0 = 1$ in the Neumann case conclude the proof, since the non-increasing character of the energy provides the required $V-$ bound. $\square$

5.3 Multiple equilibria under Neumann boundary conditions

We consider the system

$$\begin{cases} u_{tt} + g(u_t) - \Delta u + f(u) = 0, & \text{in } \mathbb{R}_+ \times \Omega, \\ \frac{\partial u}{\partial n} = 0 \quad \text{on } \mathbb{R}_+ \times \partial\Omega, \end{cases} \tag{5.8}$$

The main result of this paragraph is the following :

Theorem 5.6 *Define V, H as in Section 4.2. Assume that in addition to the conditions of Proposition 4.6 , $f \in C^1(\mathbb{R})$ satisfies* (4.7)*and*

$$\forall s \in \mathbb{R}, \quad |f'(s)| \leq C|\rho(s)|^{p-1}$$

with $(N-2)p < N+2$. Assume that g satisfies (2.3)-(2.4)*. Then any solution $u \in W^{1,1}_{loc}(\mathbb{R}^+, V) \bigcap W^{2,1}_{loc}(\mathbb{R}^+, H)$ of one of these systems converges in V as $t \to \infty$ to a constant $c \in [a,b]$ and we have*

$$\|u(t,.) - c\|_V \leq Mt^{-\frac{1-\alpha p}{(\alpha+1)p-1}}; \quad \|u_t(t,.)\|_H + \|\nabla u(t,.)\|_H \leq M't^{-\frac{p+1}{2[(\alpha+1)p-1]}}$$

Proof. The main assumption to be checked are again (2.6) and boundedness of $u(t)$ in V. Boundedness in V is similar to the previous case since f satisfies (5.4) with $A = 2\max\{|a|,|b|\}+1$ for instance. The relevant Lojasiewicz inequality is uniform on any bounded subset of V and has been proved in Proposition4.9. We now check that the left-handside of (2.6) holds true on any bounded subset of V with $\gamma = \frac{1}{2}$. This is in fact a simple consequence of Lemma 5.2. Indeed under the hypothesis of Theorem 5.5, we have

$$\forall u \in V, \quad \|u\|_V \leq R \Longrightarrow \|\{(u-b)^+\}^p\|_{V'} \leq P(R)\|(u-b)^+\|_{p+1}^{\frac{p+1}{2}} \leq KP(R)(\mathcal{F}(u))^{\frac{1}{2}}$$

Similarly we find

$$\forall u \in V, \quad \|u\|_V \leq R \Longrightarrow \|\{(u-a)^-\}^q\|_{V'} \leq K'Q(R)(\mathcal{F}(u))^{\frac{1}{2}}$$

Then we find that f satisfies (5.1). By applying Proposition 5.1 we conclude that the left-handside of (2.6) holds true on any bounded subset of V with $\gamma = \frac{1}{2}$. Since(2.6) holds true on any bounded subset of V, a simple application of Theorem 2.1 concludes the proof. □

5.4 An example with a fourth order operator in space

Again, in what follows, Ω is a bounded connected open subset of $\mathbb{R}^N$. We consider the following problem

$$\begin{cases} u_{tt} - c\int_\Omega |\nabla u_t|^2 dx^{\frac{\alpha}{2}} \Delta u_t + \Delta^2 u - \lambda_1 u + c_1(u^+)^p - c_2(u^-)^q = 0, & \text{in } \mathbb{R}_+ \times \Omega, \\ u(t,x) = |\nabla u| = 0, \quad \text{on } \mathbb{R}_+ \times \partial\Omega, \end{cases} \tag{5.9}$$

where λ_1 denotes here the first eigenvalue of Δ^2 on $H_0^2(\Omega)$. We define $V = H_0^2(\Omega)$ and $H = L^2(\Omega)$. We assume $c_1 > 0, c_2 > 0$ and $p, q > 1$ with

$$(N-4)\sup\{p,q\} < N+4$$

The main result of this paragraph is the following :

Theorem 5.7 *Assuming that p, q, c_1, c_2 fulfill the conditions above and $0 < \alpha < \frac{1}{r}$, with $r := \sup\{p,q\}$, any solution $u \in W^{1,1}_{loc}(\mathbb{R}^+, V) \bigcap W^{2,1}_{loc}(\mathbb{R}^+, H)$ of (5.9) converges to 0 in V as $t \to \infty$ and we have*

$$\|u(t,.)\|_V \le M t^{-\frac{1}{(\alpha+1)r-1}}; \quad \|u_t(t,.)\|_H \le M' t^{-\frac{r+1}{2[(\alpha+1)r-1]}}$$

where M, M' are some positive constants depending on the solution.

Proof. The proof of the $V-$ bound of $u(t)$ and (2.6) are quite similar to the analogous steps in the proof of Theorem 5.5. The relevant Lojasiewicz inequality is uniform on any bounded subset of V and has been proved in Proposition 4.6 . The proof that the left-handside of (2.6) holds true on any bounded subset of V with $\gamma = \frac{1}{2}$ is already done in proof of Theorem 5.5 . Hence (2.6) holds true on any bounded subset of V. However here Theorem 2.1 cannot be applied as it stands because g is not defined on H, but from H_0^1 to H^{-1}. A thorough inspection of the proof of Theorem 2.1 allows, mutatis mutandis, to obtain the necessary extension, and this concludes the proof. □

6 Convergence and rate of convergence in the non autonomous case

In this section we assume that the hypothesis of Theorem 2.1 are satisfied and we consider the following abstract system

$$\begin{cases} \ddot{u}(t) + g(\dot{u}(t)) + \mathcal{M}(u(t)) = h(t), & t \ge 0, \\ u(0) = u_0, & u_0 \in V, \\ \dot{u}(0) = u_1, & u_1 \in H, \end{cases} \tag{6.1}$$

where $h : \mathbb{R}^+ \to H$ is such that

$$\exists c \ge 0,\ \exists \delta > 0,\ \|h(t)\|_H \le \frac{c}{(1+t)^{1+\delta+\alpha}},\ \text{for all } t \in \mathbb{R}_+. \tag{6.2}$$

The main result of this section is the following :

Theorem 6.1 *Let $u \in W^{1,1}_{loc}(\mathbb{R}^+, V) \bigcap W^{2,1}_{loc}(\mathbb{R}^+, H)$ be a solution of* (6.1) *such that $u(t) \in B$ for t large where B denotes a closed subset of V. Assume that hypothesis* (2.3)*,* (2.4) (2.6)*,* (2.8) *and* (6.2) *are satisfied. Let θ be as in* (2.5) *and* (2.6)*. Let us introduce*

$$\mu = \inf\left\{\frac{1-(\alpha+1)(1-\theta)}{(\alpha+2)(1-\theta)-1}, \frac{\delta}{\alpha+1}\right\}; \quad \nu = \inf\{\frac{1}{(\alpha+2)(1-\theta)-1}, \alpha+1+\delta(\frac{\alpha+2}{\alpha+1})\}.$$

Then there exist $T > 0$, $a \in B$ and some constants $C, M > 0$ such that

$$\forall\, t \geq T, \quad \|u(t) - a\|_H \leq Ct^{-\mu}. \tag{6.3}$$

$$\forall\, t \geq T, \quad \mathcal{E}(u(t)) \leq Mt^{-\nu}. \tag{6.4}$$

6.1 Proof of Theorem 6.1

Let u be a solution of equation (6.1) such that $u \in W^{1,1}_{loc}(\mathbb{R}^+, V) \bigcap W^{2,1}_{loc}(\mathbb{R}^+, H)$ and $u(t) \in B$ for t large. Let us define the nonnegative function

$$E(t) = \frac{1}{2}\|\dot{u}(t)\|_H^2 + \mathcal{E}(u).$$

Now let $0 < \varepsilon \leq 1$ be a real constant. We define the function

$$H(t) = E(t) + \varepsilon E(t)^\beta \langle \mathcal{M}(u(t)), \dot{u}(t)\rangle_{V'} + \int_t^\infty \langle h(s), \dot{u}(s)\rangle_H\, ds + \int_t^\infty E(s)^\beta \|h(s)\|_H^2\, ds,$$

where $\beta = \alpha(1-\theta)$, θ is the Lojasiewicz exponent defined in (2.6).
We have,

$$H(t) \leq c_8 E(t) + \frac{c_9}{(1+t)^\lambda}, \quad \text{for all } t \geq T, \tag{6.5}$$

where $\lambda = \alpha + 1 + \delta(\frac{\alpha+2}{\alpha+1})$. In fact, we have

$$H(t) \leq E(t) + \varepsilon E(t)^\beta \|\mathcal{M}(u)\|_{V'}\|\dot{u}\|_{V'} + \int_t^\infty \|h\|_H\|\dot{u}\|_H\, ds + \int_t^\infty E(s)^\beta\|h(s)\|_H^2\, ds.$$

We have by (2.3)

$$E'(t) = -\langle g(\dot{u}), \dot{u}\rangle_{V',V} + \langle h, \dot{u}\rangle_H \leq -\rho_1\|\dot{u}\|_H^{\alpha+2} + \|h\|_H\|\dot{u}\|_H.$$

Therefore we have by Young's inequality

$$\int_t^\infty \|\dot{u}\|_H^{\alpha+2}\, ds \leq c_2 E(t) + c_3\int_t^\infty \|h\|_H^{\frac{\alpha+2}{\alpha+1}}\, ds.$$

Now, since E is bounded then we have

$$\int_t^\infty E(s)^\beta\|h(s)\|_H^2\, ds \leq c_5\int_t^\infty \|h(s)\|_H^2\, ds.$$

On the other hand, by the above inequalities together with (2.6) we get

$$H(t) \leq E(t) + \varepsilon c_6 E(t)^{\alpha(1-\theta)+\gamma+\frac{1}{2}} + c_3 \int_t^\infty \|h\|_H^{\frac{\alpha+2}{\alpha+1}} + c_2 E(t) + c_5 \int_t^\infty \|h(s)\|_H^2 \, ds. \qquad (6.6)$$

Since $E(t)^{\alpha(1-\theta)+\gamma+\frac{1}{2}} \leq c_7 E(t)$, then by observing that $2 > \frac{\alpha+2}{\alpha+1}$ and using the assumption (6.2) we get

$$H(t) \leq c_8 E(t) + \frac{c_9}{(1+t)^\lambda},$$

where $\lambda = \alpha + 1 + \delta(\frac{\alpha+2}{\alpha+1})$.
We have

$$H'(t) = E'(t) - \langle h, \dot{u}\rangle_H + \varepsilon\beta\langle \dot{u}(t), h(t) - g(\dot{u}(t))\rangle_{V,V'} E(t)^{\beta-1}\langle \mathcal{M}(u(t)), \dot{u}(t)\rangle_{V'} + \varepsilon E(t)^\beta \langle \mathcal{M}'(u(t))\dot{u}(t), \dot{u}(t)\rangle$$

$$+\varepsilon E(t)^\beta \langle \mathcal{M}(u(t)), h(t) - g(\dot{u}(t))\rangle_{V'} - \varepsilon E(t)^\beta \|\mathcal{M}(u(t))\|_{V'}^2 - E(t)^\beta \|h(t)\|_H^2.$$

We have by Cauchy-schwarz inequality

$$\varepsilon\beta\langle \dot{u}(t), h(t)\rangle_{V,V'} E(t)^{\beta-1}\langle \mathcal{M}(u(t)), \dot{u}(t)\rangle_{V'} \leq \varepsilon\beta c_{10}\|\dot{u}\|_H^2 \|\mathcal{M}(u(t))\|_{V'}\|h\|_H E(t)^{\beta-1}.$$

Then we obtain since $\|\dot{u}\|_H^2 \leq 2E(t)$

$$\varepsilon\beta\langle \dot{u}(t), h(t)\rangle_{V,V'} E(t)^{\beta-1}\langle \mathcal{M}(u(t)), \dot{u}(t)\rangle_{V'} \leq \varepsilon\beta c_{11} E(t)^\beta \|h\|_H \|\mathcal{M}(u(t))\|_{V'}.$$

By Cauchy-Schwarz inequality together with the last inequality we get

$$\varepsilon E(t)^\beta \langle \mathcal{M}(u(t)), h(t)\rangle_{V,V'} + \varepsilon\beta\langle \dot{u}(t), h(t)\rangle_{V'} E(t)^{\beta-1}\langle \mathcal{M}(u(t)), \dot{u}(t)\rangle_{V'} \leq \varepsilon E(t)^\beta \|h\|_H \|\mathcal{M}(u(t))\|_{V'}$$

$$+\varepsilon\beta c_{11} E(t)^\beta \|h\|_H \|\mathcal{M}(u(t))\|_{V'}.$$

Once again by applying Cauchy-Schwarz inequality we get

$$\varepsilon E(t)^\beta \langle \mathcal{M}(u(t)), h(t)\rangle_{V,V'} + \varepsilon\beta\langle \dot{u}(t), h(t)\rangle_{V'} E(t)^{\beta-1}\langle \mathcal{M}(u(t)), \dot{u}(t)\rangle_{V'} \leq \frac{\varepsilon}{4} E(t)^\beta \|\mathcal{M}(u(t))\|_{V'}^2$$

$$+\varepsilon c_{13} E(t)^\beta \|h(t)\|_H^2.$$

On the other hand we have by the calculations of section 1

$$-\langle \dot{u}(t), g(\dot{u}(t))\rangle_{V,V'} E(t)^{\beta-1}\langle \mathcal{M}(u(t)), \dot{u}(t)\rangle_{V'} \leq c_{14}\langle \dot{u}(t), g(\dot{u}(t))\rangle_{V,V'}. \qquad (6.7)$$

$$\varepsilon E(t)^\beta \langle \mathcal{M}'(u(t))\dot{u}(t), \dot{u}(t)\rangle_{V'} \leq \varepsilon c_{15}\|\dot{u}(t)\|_H^{\alpha+2} + \frac{\varepsilon}{4} E(t)^\beta \|\mathcal{M}(u(t))\|_{V'}^2. \qquad (6.8)$$

$$\varepsilon E(t)^\beta \langle \mathcal{M}(u(t)), g(\dot{u}(t))\rangle_{V'} \leq \frac{\varepsilon}{4} E(t)^\beta \|\mathcal{M}(u(t))\|_{V'}^2 + \varepsilon c_{16}\|\dot{u}(t)\|_H^{\alpha+2}. \qquad (6.9)$$

Then by combining the last 4 inequalities we obtain for a fixed ε small enough

$$H'(t) \leq -c_{17}\left(\|\dot{u}(t)\|_H^{\alpha+2} + E(t)^\beta \|\mathcal{M}(u(t))\|_{V'}^2\right). \qquad (6.10)$$

Then by following the steps of the proof of Theorem 2.1 and by using (6.5) we get the following differential inequality

$$c_{18}H(t)^{(\alpha+2)(1-\theta)} + c_{19}H'(t) \le \frac{c_{20}}{(1+t)^{\lambda(1-\theta)(\alpha+2)}}.$$

Therefore by applying Lemma 2.1 from [2] we obtain

$$H(t) \le c_{21}(1+t)^{-\nu}, \tag{6.11}$$

where $\nu = \inf\{\frac{1}{(\alpha+2)(1-\theta)-1}, \lambda\}$.

Now, by using (6.10) together with the last inequality we have for all $t \ge T$

$$\int_t^{2t} \|\dot{u}\|_H^{\alpha+2}\, ds \le c_{22}(1+t)^{-\nu}.$$

Since we have

$$\int_t^{2t} \|\dot{u}\|_H\, ds \le t^{\frac{\alpha+1}{\alpha+2}}(\int_t^{2t} \|\dot{u}\|_H^{\alpha+2}\, ds)^{\frac{1}{\alpha+2}},$$

we obtain

$$\int_t^{2t} \|\dot{u}\|_H\, ds \le \frac{c_{23}}{(1+t)^{-\mu}},$$

where $\mu = \inf\left\{\frac{1-(\alpha+1)(1-\theta)}{(\alpha+2)(1-\theta)-1}, \frac{\delta}{\alpha+1}\right\}$.

Then the conclusion follows as in the proof of Theorem 2.1 and we have $\dot{u} \in L^1(T,\infty,H)$ for $T > 0$ large enough. Hence, $u(t)$ has a limit a in H as $t \to \infty$ and

$$\|u(t) - a\|_H \le Ct^{-\mu}.$$

On the other hand we have by the calculation leading to (6.6)

$$H(t) \ge E(t) - \varepsilon c_6 E(t)^{\alpha(1-\theta)+\gamma-\frac{1}{2}} - c_3\int_t^{\infty} \|h\|_H^{\frac{\alpha+2}{\alpha+1}} - c_2E(t) - c_5\int_t^{\infty} \|h(s)\|_H^2\, ds.$$

Then by choosing ε small enough we get

$$H(t) \ge \frac{E(t)}{2} - \frac{c_{24}}{(1+t)^{\lambda}}.$$

Therefore we have

$$E(t) \le 2H(t) + \frac{c_{24}}{(1+t)^{\lambda}},$$

and then thanks to (6.11) we obtain, since $\lambda \ge \nu$

$$E(t) \le \frac{c_{25}}{(1+t)^{\nu}}.$$

It follows that there exists a constant M such that

$$\mathcal{E}(u(t)) \le M(1+t)^{-\nu}.$$

6.2 Applications

Theorem 6.1 is applicable to perturbations of any of the particular systems considered in Sections 3 and 5 by a sufficiently fast decaying forcing term. To avoid heavy repetitions, the details of application to those examples are left to the reader.

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