The Łojasiewicz gradient inequality in the infinite dimensional Hilbert space framework.

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Résumé

On donne une réponse raisonnablement optimale à la question de savoir sous quelles conditions supplémentaires une fonction analytique sur un espace de Hilbert de dimension infinie satisfait l'inégalité du gradient de Łojasiewicz.

Abstract

We provide a reasonably optimal answer to the natural question of the conditions under which an analytic function on an infinite dimensional Hilbert space satisfies the Łojasiewicz gradient inequality.

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1 Introduction

Let V be a real Hilbert space. It is a natural question to ask whether an analytic function $F : V \to \mathbb{R}$ satisfies the Łojasiewicz gradient inequality which means that for any $a \in V$ there exists $\theta \in (0, 1/2)$, a neighborhood W of a in V and $c > 0$ for which

$$\forall u \in W, \quad \|DF(u)\|_{V'} \geq c|F(u) - F(a)|^{1-\theta}$$

where V' is the topological dual of V. After the pioneering works of S. Łojasiewicz, many results of this type have been proved in the literature in various contexts with main applications to partial differential equations, the main objectives being convergence results of bounded solutions to stationary ones or decay estimates of the difference between the solution and its limiting equilibrium, cf. for instance [1]-[16], [20]. Usually in the PDE framework one makes use of a compactness hypothesis of the resolvent of the linearization of $DF(u)$ around an arbitrary equilibrium. It is however reasonable to wander what would be a minimal framework to extend the Łojasiewicz theory to analytic functionals in infinite dimensions. This paper gives rather simple answers to this question, first in the linear case where the gradient inequality can already fail without additional assumptions, and secondly in the semilinear case where the situation turns out to be slightly more complicated.

2 Quadratic forms and the linear case

Throughout this section we consider a real Hilbert space H and a linear operator A such that

$$A \in L(H); \quad A^* = A \tag{1}$$

and the associated quadratic form $\Phi : H \longrightarrow \mathbb{R}$ defined by

$$\forall u \in H, \quad \Phi(u) = \frac{1}{2}\langle Au, u\rangle. \tag{2}$$

We denote by $|u|$ the norm of a vector $u \in H$. Our main result is the following

Theorem 2.1. *The following properties are equivalent*
i) 0 is not an accumulation point of sp(A)
ii) For some $\rho > 0$ we have

$$\forall u \in \ker(A)^{\perp}, \quad |Au| \geq \rho|u|$$

iii) Φ satisfies the gradient inequality at the origin for some $\theta > 0$
iv) Φ satisfies the gradient inequality at any point for $\theta = \frac{1}{2}$.

Proof. We establish [i) $\Rightarrow$ ii) $\Rightarrow$ iv)] and the contraposition of [iii) $\Rightarrow$ i)]. Since [vi) $\Rightarrow$ iii)] is obvious, the result follows.

Step 1. Assuming that 0 is an accumulation point of sp(A) we prove that the Łojasiewicz gradient inequality at 0 fails. We state first an easy

Lemma 2.2. *Assume that for some $\eta > 0$ we have*

$$\forall u \in H, \quad |Au - \lambda u| \geq \eta |u|.$$

Then $\lambda \notin sp(A)$.

Proof. Indeed since A is bounded , $A - \lambda I$ has closed graph and consequently $(A - \lambda I)^{-1}$, which is well defined on H has also closed graph and is therefore bounded. $\square$

As a consequence of this Lemma, we can find a sequence λ_n of positive numbers tending to 0 and a sequence of vectors $u_n \in H$ for which

$$\forall n \in \mathbb{N}, \quad |Au_n - \lambda_n u_n| < \frac{\lambda_n}{2}|u_n|.$$

In particular

$$\forall n \in \mathbb{N}, \quad |Au_n| < \frac{3\lambda_n}{2}|u_n|$$

and since

$$\forall n \in \mathbb{N}, \quad |\langle Au_n, u_n\rangle - \lambda_n |u_n|^2| < \frac{\lambda_n}{2}|u_n|^2$$

we find

$$\forall n \in \mathbb{N}, \quad \langle Au_n, u_n\rangle > \frac{\lambda_n}{2}|u_n|^2.$$

By homogeneity we can change u_n in order to achieve $|u_n| = \rho$. Then we find

$$\forall n \in \mathbb{N}, \quad |Au_n| < \frac{3\lambda_n}{2}\rho; \quad \Phi(u_n) > \frac{\lambda_n}{4}\rho^2.$$

And therefore no Łojasiewicz gradient inequality of the form

$$|Au| \geq \delta \Phi(u)^{1-\theta}$$

with $\delta, \theta > 0$ can be satisfied in a neighborhood of 0.

Step 2. i) $\Rightarrow$ ii). As a consequence of Theorem VIII.4 p. 260 from [19] , up to an isometric isomorphism we may assume $H = L^2(\Omega, d\mu)$ where $(\Omega, d\mu)$ is some positively measured space and

$$\forall u \in H, \quad (Au)(x) = a(x)u(x), \quad \mu - a.e. \,\mathrm{in}\, \Omega.$$

We define

$$\Omega_+ = \{x \in \Omega, \quad a(x) > 0\}; \quad \Omega_- = \{x \in \Omega, \quad a(x) < 0\}; \quad \Omega_0 = \{x \in \Omega, \quad a(x) = 0\}.$$

First if $0 \notin sp(A)$, then A is an isomorphism and then the result is obvious. Indeed in that case

$$|\Phi(u)| \leq |Au||u| \leq C|Au|^2.$$

On the other hand if $0 \in sp(A)$ and 0 is isolated in $sp(A)$, it means that for some $\rho > 0$ we have

$$[-\rho, \rho] \cap sp(A) = \{0\}.$$

We claim that

$$\mu(a^{-1}(0,\rho) \cap \Omega_+) = 0. \tag{3}$$

Indeed assuming $\mu(a^{-1}(0,\rho) \cap \Omega_+) > 0$, there is first of all $\eta \in (0,\rho)$ for which

$$\mu(a^{-1}(\eta,\rho) \cap \Omega_+) > 0.$$

Then we have either

$$\mu(a^{-1}(\eta, \frac{\rho+\eta}{2}) \cap \Omega_+) > 0$$

or

$$\mu(a^{-1}(\frac{\rho+\eta}{2}, \rho) \cap \Omega_+) > 0$$

and by inductive dichotomy we find a sequence of integers $k_n \in [0, 2^n - 1]$ for which, setting

$$I_n = [\eta + k_n \frac{\rho-\eta}{2^n}, \eta + (k_n+1)\frac{\rho-\eta}{2^n}]$$

the following properties hold

$$I_n \subset I_{n-1} ... \subset I_1$$

and

$$\forall n \in \mathbb{N}, \quad \mu(a^{-1}(I_n) \cap \Omega_+) > 0.$$

Let

$$\rho_* := \bigcap_{n \geq 1} I_n.$$

It is clear that

$$\forall \varepsilon > 0, \quad \mu(a^{-1}(B(\rho_*, \varepsilon) \cap \Omega_+) > 0.$$

Letting

$$\omega_\varepsilon = a^{-1}(B(\rho_*, \varepsilon) \cap \Omega_+; \quad \phi_\varepsilon = 1_{\omega_\varepsilon}$$

we find

$$\forall \varepsilon > 0, \quad |(A - \rho_*)\phi_\varepsilon|^2 \leq \varepsilon^2 |\phi_\varepsilon|^2.$$

Hence $\rho_* \in sp(A)$, a contradiction. Similarly we have

$$\mu(a^{-1}(-\rho, 0) \cap \Omega_-) = 0. \tag{4}$$

Finally given $u \in H$, we have

$$|Au|^2 = \int_\Omega |a(x)u(x)|^2 d\mu(x) \geq \rho^2 [\int_{\Omega_+} u^2(x) d\mu(x) + \int_{\Omega_-} u^2(x) d\mu(x)]$$

and the result is now obvious since

$$\forall u \in \ker(A)^{\perp}, \quad \int_{\Omega_+} u^2(x)d\mu(x) + \int_{\Omega_-} u^2(x)d\mu(x) = |u|^2.$$

Step 3. Given $u \in H$, let $v = Qu$ be the orthogonal projection of u on $\ker(A)^{\perp}$. We have clearly

$$2|\Phi(u)| = 2|\Phi(v)| \leq \|A\||v|^2 \leq \frac{\|A\|}{\rho^2}|Av|^2 = \frac{\|A\|}{\rho^2}|Au|^2$$

which is precisely the gradient inequality at 0 with $\theta = \frac{1}{2}$. The gradient inequality for any $1 \geq \theta > 0$ is trivially satisfied at any point x with $Ax \neq 0$ and if $Ax = 0$ we have $|\Phi(u) - \Phi(x)| = |\Phi(u)|$, so that at such a point x the gradient inequality reduces to the same inequality at 0.

□

3 What happens in the nonlinear case ?

A natural question is whether the necessary and sufficient condition of Theorem 2.1 gives the right condition for the second derivative at a critical point a in order for an analytic functional $F : H \to \mathbb{R}$ to fulfill a Łojasiewicz inequality near a. Since in the quadratic case the Łojasiewicz inequality is either false, or satisfied with the best possible value $\theta = \frac{1}{2}$, it is clear that some additional difficulties will appear. The next result shows that if the second derivative is "bad", at least the functional cannot satisfy the Łojasiewicz inequality with $\theta = \frac{1}{2}$.

Proposition 3.1. *Let $F : U \to \mathbb{R}$ be an analytic functional where $U \subset H$ is an open neighborhood of* 0 *and assume*

$$F(0) = |DF(0)| = 0.$$

If 0 *is an accumulation point of $sp(D^2F(0))$, then an inequality*

$$\forall u \in U, \quad |\nabla F(u)| \geq c|F(u)|^{1-\theta}$$

for some $c > 0$ implies $\theta \leq \frac{1}{3}$.

Proof. As a consequence of the hypothesis, as in the proof of Theorem 2.1 we can find a sequence λ_n of positive numbers tending to 0 and a sequence of vectors $u_n \in H$ with $|u_n| = \rho$ which can be taken arbitrarily small independently of λ_n , and for which

$$\forall n \in \mathbb{N}, \quad |Au_n| < \frac{3}{2}\lambda_n\rho; \quad \Phi(u_n) > \frac{\lambda_n}{4}\rho^2$$

where $A = D^2F(0)$ and Φ is the quadratic part of F. Then by taking the next (second order) approximation we find for some fixed constants $C, D \geq 0$

$$|\nabla F(u_n)| \leq 2\lambda_n \rho + C\rho^2; \quad F(u_n) \geq \frac{\lambda_n}{4}\rho^2 - D\rho^3.$$

Choosing $\rho = \varepsilon\lambda_n$ with $\varepsilon > 0$ small enough we find for some $M > 0$

$$(\varepsilon^2 \frac{{\lambda_n}^3}{8})^{1-\theta} \leq M{\lambda_n}^2.$$

Therefore ${\lambda_n}^{1-3\theta}$ is bounded and by letting n go to infinity we conclude that $\theta \leq \frac{1}{3}$. $\square$

Remark 3.2. We have been unable to construct an example of the above situation in which $\theta = \frac{1}{3}$. The next example shows that $\theta = \frac{1}{4}$ can actually happen.

Proposition 3.3. *Let $H = L^2(0,1)$ and $F : H \to \mathbb{R}$ be the analytic functional given by*

$$F(u) := \frac{1}{4}(\int_0^1 u^2(x)dx)^2 + \frac{1}{2}\int_0^1 xu^2(x)dx.$$

Then

$$\forall u \in H, \quad |(\nabla F)(u)| \geq |F(u)|^{\frac{3}{4}}.$$

However $[0,1] \subset sp(D^2F(0))$.

Proof. It is easily verified that F is an analytic (actually polynomial) functional with

$$(\nabla F)(u) = xu + (\int_0^1 u^2 dx)u.$$

In particular

$$|(\nabla F)(u)|^2 = \int_0^1 [(x + (\int_0^1 u^2 dx)u]^2 dx \geq \int_0^1 x^2u^2 dx + (\int_0^1 u^2 dx)^3.$$

On the other hand

$$(\int_0^1 u^2 dx)^2 = [(\int_0^1 u^2 dx)^{\frac{3}{2}}]^{\frac{4}{3}} \leq |(\nabla F)(u)|^{\frac{4}{3}}.$$

In addition

$$(\int_0^1 xu^2(x)dx)^{\frac{3}{2}} \leq (\int_0^1 x^2u^2 dx)^{\frac{3}{4}}(\int_0^1 u^2 dx)^{\frac{3}{4}} \leq \int_0^1 x^2u^2 dx + [(\int_0^1 u^2 dx)^{\frac{3}{4}}]^4$$

$$= \int_0^1 x^2u^2 dx + (\int_0^1 u^2 dx)^3$$

therefore

$$(\int_0^1 xu^2(x)dx) \leq [\int_0^1 x^2u^2dx + (\int_0^1 u^2dx)^3]^{\frac{2}{3}} \leq |(\nabla F)(u)|^{2\times\frac{2}{3}} = |(\nabla F)(u)|^{\frac{4}{3}}$$

and we end up with

$$F(u) := \frac{1}{4}(\int_0^1 u^2(x)dx)^2 + \frac{1}{2}\int_0^1 xu^2(x)dx \leq \frac{3}{4}|(\nabla F)(u)|^{\frac{4}{3}}$$

which clearly implies the result. $\square$

Remark 3.4. In the opposite direction it is natural to wander whether the condition on the spectrum of $D^2F(0)$ is enough to ensure the existence of a Łojasiewicz gradient inequality. The following example where $\ker D^2F(0) = H$ shows that it is not the case.

Proposition 3.5. *Let $H = l^2(\mathbb{N})$ and $F : H \to \mathbb{R}$ be the analytic functional given by*

$$F(u_1, u_2, ...u_n, ...) := \sum_{k=2}^{\infty} \frac{|u_k|^{2k+2}}{(2k+2)!}.$$

Then F satisfies no Łojasiewicz gradient inequality.

Proof. First we note that $D^2F(0) = 0$, hence $sp(D^2F(0)) = \{0\}$ and in particular 0 is isolated in $sp(D^2F(0))$. Defining $(e_i)_j = \delta_{ij}$, an immediate calculation shows that

$$\forall t > 0, \quad F(te_k) = \frac{t^{2k+2}}{(2k+2)!}; \quad |\nabla F(te_k)| = \frac{t^{2k+1}}{(2k+1)!}.$$

In particular for each $\theta > 0$ we have

$$\frac{F(te_k)^{1-\theta}}{|\nabla F(te_k)|} = c(\theta, k)t^{1-(2k+2)\theta}.$$

Choosing k large enough gives a contradiction for t small. $\square$

4 A framework adapted to unbounded operators.

In the application, in particular to PDE problems, the basic space will not be identified with its dual since the gradient operators we are dealing with are semilinear perturbations of an unbounded self-adjoint linear operator. Therefore the Hilbert space H will be replaced by a space V that we shall not identify with ist topological dual V' . We shall denote by $J : V \to V'$ the duality map, by $\|v\|$ the norm of a vector $v \in V$, by $\|f\|_*$ the norm of a continuous linear form $f \in V'$ and the duality pairing will be represented by

$$\forall v \in V, \forall f \in V', \quad f(v) =: \langle f, v\rangle.$$

Throughout this section we consider a linear operator $A \in L(V, V')$ which is symmetric :

$$\forall u \in V, \forall v \in V, \quad \langle Au, v\rangle = \langle Av, u\rangle \tag{5}$$

and the associated quadratic form $\Phi : V \longrightarrow \mathbb{R}$ defined by

$$\forall u \in V, \quad \Phi(u) = \frac{1}{2}\langle Au, u\rangle \tag{6}$$

so that A is the derivative of Φ at any point.

Theorem 4.1. *The following properties are equivalent*
i) 0 *is not an accumulation point of* $sp(J^{-1}A)$.
ii) For some $\rho > 0$ *we have*

$$\forall v \in \ker(A)^{\perp}, \quad \|Av\|_* \geq \rho\|v\|.$$

iii) Φ *satisfies the gradient inequality at the origin for some* $\theta > 0$.
iv) Φ *satisfies the gradient inequality at any point for* $\theta = \frac{1}{2}$.

Proof. The result follows from Theorem 2.1 applied to $\tilde{A} = J^{-1}A \in L(V)$. □

5 A basically optimal nonlinear result.

The examples of Section 3 suggest that the following result is essentially optimal. For the statement of this result we consider two real Hilbert spaces V, H where $V \subset H$ with continuous and dense imbedding and H', the topological dual of H is identified with H . therefore

$$V \subset H = H' \subset V'$$

with continuous and dense imbeddings. The duality product of $\phi \in V'$ with $v \in V$ is denoted as $\langle \phi, v\rangle$.

Theorem 5.1. *Let* $F : U \to \mathbb{R}$ *be an analytic functional where* $U \subset V$ *is an open neighborhood of* 0 *and assume*

$$F(0) = 0; \quad DF(0) = 0.$$

We assume the two conditions

- *i)* $N := \ker D^2F(0)$ *is finite dimensional .*
- *ii) There is* $\rho > 0$ *for which*

$$\forall u \in N^{\perp}, \quad \|D^2F(0)u\|_{V'} \geq \rho\|u\|_V.$$

Then there exists $\theta \in (0, 1/2)$, *a neighborhood* W *of* 0 *and* $c > 0$ *for which*

$$\forall u \in W, \quad \|DF(u)\|_{V'} \geq c|F(u)|^{1-\theta}.$$

Proof. We set $A = D^2F(0) \in L(V, V')$ and we introduce the orthogonal projection Π in H on $N = \ker(A)$. First we show that the linear operator $\mathcal{L} := \Pi + A$ restricted to V is one to one and onto. Actually we shall see that for some $\eta > 0$

$$\forall u \in V, \quad \|\mathcal{L}u\|_{V'} \geq \eta \|u\|_V. \tag{7}$$

Thanks to the fact that A is symmetric: $V \to V'$ we have $R(A) \subset N^\perp$ and in particular

$$R(A) \cap N = \{0\}.$$

Then (7) becomes an immediate consequence of the next Lemma

Lemma 5.2. *Let W be a real Hilbert space endowed with the norm $\|.\|_W$ and N, F two closed subspaces with N finite dimensional. Then, assuming*

$$F \cap N = \{0\}$$

there is a constant $\sigma > 0$ such that

$$\forall (n, f) \in N \times F, \quad \|n + f\|_W \geq \sigma(\|n\|_W + \|f\|_W).$$

Proof. First we denote by Q the projection onto $F^\perp$ in the sense of W. We observe that the function

$$n \in N \to p(n) = \|Qn\|_W$$

is a norm on N and since N is finite dimensional we find immediately the existence of $\nu > 0$ for which

$$\forall n \in N, \quad \|Qn\|_W \geq \nu \|n\|_W.$$

Now we have

$$n + f = Qn + (I - Q)n + f$$

and $(I - Q)n + f \in F$, therefore by orthogonality in W we deduce

$$\|n + f\|_W \geq \|Qn\|_W \geq \nu \|n\|_W.$$

Then it suffices to observe that

$$\|f\|_W = \|n + f - n\|_W \leq \|n + f\|_W + \|n\|_W \leq (1 + \nu^{-1})\|n + f\|_W$$

and the result follows with

$$\sigma = \frac{\nu}{\nu + 2}.$$

$\square$

In order to prove (7) it suffices to apply Lemma 5.2 with $W = V'$, $F = R(A), N = \ker A \subset V \subset V'$, $n = \Pi u$, $f = Au$, $n + f = \mathcal{L}u$. Indeed we have

$$\forall u \in V, \|u\|_V \le \|u - \Pi u\|_V + \|\Pi u\|_V \le \rho^{-1}\|A(u - \Pi u)\|_{V'} + K\|\Pi u\|_{V'}$$

by using ii) and the equivalence of the norms in V and V' in the finite dimensional space N. Then

$$\forall u \in V, \|u\|_V \le \max\{\rho^{-1}, K\})(\|Au\|_{V'} + \|\Pi u\|_{V'})$$

and the Lemma gives

$$\forall u \in V, \|u\|_V \le \sigma^{-1} \max\{\rho^{-1}, K\}\|\mathcal{L}u\|_{V'}$$

that is (7) with $\eta = \sigma \min\{\rho, K^{-1}\}$. Now we observe that (7) implies the one to one character of $\mathcal{L}$. In addition since $\mathcal{L}$ is symmetric we have $\overline{R(\mathcal{L})} = [\ker \mathcal{L}]^\perp = V'$. Then from (7) it follows that $\mathcal{L}$ is onto . Indeed given any $\varphi \in V'$, there exists a sequence $\varphi_n = \mathcal{L}u_n$ with $\|\varphi_n - \varphi\|_{V'} \to 0$. In particular φ_n is a Cauchy sequence in V', and by(7) u_n is Cauchy in V. Setting $u = \lim_V u_n$ we clearly conclude that $\varphi = \mathcal{L}u \in R(\mathcal{L})$. Finally, by Banach's theorem we have also

$$\mathcal{L}^{-1} \in L(V', V).$$

Let now

$$\begin{aligned} \mathcal{N} : V &\longrightarrow V' \\ u &\longmapsto \Pi u + DF(u). \end{aligned}$$

By using the hypotheses, we deduce that $\mathcal{N}$ is analytic in the neighborhood of 0 and $D\mathcal{N}(0) = \mathcal{L}$. Applying the local inversion theorem (analytic version cf [21] corollary 4.37 p. 172), we can find a neighborhood of 0, $W_1(0)$ in V, a neighborhood of 0, $W_2(0)$ in V' and an analytic map

$$\Psi : W_2(0) \longrightarrow W_1(0) \quad \text{which satisfies}$$
$$\mathcal{N}(\Psi(f)) = f \qquad \forall f \in W_2(0)$$
$$\Psi(\mathcal{N}(u)) = u \qquad \forall u \in W_1(0)$$
$$\|\Psi(f) - \Psi(g)\|_V \le C_1\|f - g\|_{V'} \quad \forall f, g \in W_2(0) \quad C_1 > 0. \tag{8}$$

Since F is C^1, we also have

$$\|DF(u) - DF(v)\|_{V'} \le C_2\|u - v\|_V \quad \forall (u, v) \in W_1(0). \tag{9}$$

For $\xi \in \mathbb{R}^m$ small enough to achieve $\displaystyle\sum_{j=1}^{m} \xi_j\varphi_j \in W_2(0)$, we define the map Γ by

$$\Gamma(\xi) = F(\Psi(\sum_{j=1}^{m} \xi_j\varphi_j)).$$

By using the chain rule, since $F : U \longrightarrow \mathbb{R}$ is analytic, the function Γ is real analytic in some neighborhood of 0 in $\mathbb{R}^m$.

Let $u \in W_1(0)$ such that $\Pi(u) = \sum_{j=1}^{m} \xi_j \varphi_j \in W_2(0)$. For any $k \in \{1, \cdots m\}$ we have the formula

$$\frac{\partial \Gamma}{\partial \xi_k} = \frac{d}{ds} E(\Psi[\sum_{j \neq k} \xi_j \varphi_j + (\xi_k + s)\varphi_k])|_{s=0} = \langle DF(\Psi(\sum_{j=1}^{m} \xi_j \varphi_j)), \Psi'(\sum_{j=1}^{m} \xi_j \varphi_j)\varphi_k \rangle.$$

By (8), it is clear that for each $k \in \{1, \cdots m\}$, $\Psi'(\sum_{j=1}^{m} \xi_j \varphi_j)\varphi_k$ is bounded in V. Then by using (8) and (9) we obtain

$$\begin{aligned}
\|\nabla \Gamma(\xi)\|_{\mathbb{R}^m} &\leq C_3 \|DF(\Psi(\sum_{j=1}^{m} \xi_j \varphi_j))\|_{V'} \\
&= C_3 \|DF(\Psi(\Pi(u)))\|_{V'} \\
&= C_3 \|DF(\Psi(\Pi(u))) - DF(u) + DF(u)\|_{V'} \\
&\leq C_3 \|DF(u)\|_{V'} + C_4 \|\Psi(\Pi(u)) - u\|_V \\
&= C_3 \|DF(u)\|_{V'} + C_5 \|\Psi(\Pi(u)) - \Psi(\Pi u + DF(u))\|_V \\
&\leq C_3 \|DF(u)\|_{V'} + C_5 \|DF(u)\|_{V'}
\end{aligned}$$

hence

$$\|\nabla \Gamma(\xi)\|_{\mathbb{R}^m} \leq C_6 \|DF(u)\|_{V'}. \tag{10}$$

On the other hand

$$\begin{aligned}
|E(u) - \Gamma(\xi)| &= |E(u) - E(\Psi(\Pi(u)))| \\
&= |\int_0^1 \frac{d}{dt}[E(u + t(\Psi(\Pi(u)) - u)] \, dt \,| \\
&= |\int_0^1 (DF(u + t(\Psi(\Pi(u)) - u)), \Psi(\Pi(u)) - u) \, dt \,| \\
&\leq \|\Psi(\Pi(u)) - u\|_V \int_0^1 \|DF(u + t(\Psi(\Pi(u)) - u)\|_{V'} \, dt \\
&\leq [\int_0^1 (\|DF(u)\|_{V'} + t\, C_7 \|\Psi(\Pi(u)) - u\|_V) \, dt \,] \, \|\Psi(\Pi(u)) - u\|_V \\
&\leq C_8 \|DF(u)\|_{V'} \|\Psi(\Pi(u)) - \Psi(\Pi(u) + DF(u))\|_V
\end{aligned}$$

hence

$$|E(u) - \Gamma(\xi)| \, \leq C_9 \|DF(u)\|^2_{V'}. \tag{11}$$

Applying the classical Łojasiewicz inequality to Γ, we now obtain:

$$|E(u)|^{1-\theta} \leq |\Gamma(\xi)|^{1-\theta} + |\Gamma(\xi) - E(u)|^{1-\theta} \leq \|\nabla \Gamma(\xi)\|_{\mathbb{R}^m} + |\Gamma(\xi) - E(u)|^{1-\theta}. \tag{12}$$

By combining (10), (11), (12) we obtain

$$|E(u)|^{1-\theta} \leq C_6 \|DF(u)\|_{V'} + C_9^{1-\theta} \|DF(u)\|_{V'}^{2(1-\theta)}.$$

Then since $2(1-\theta) \geq 1$, there exist $\sigma > 0$, $c > 0$ such that

$$\|DF(u)\|_{V'} \geq c|E(u)|^{1-\theta} \quad \text{for all } u \in V \text{ such that } \|u\|_V < \sigma.$$

Theorem 5.1 is completely proved. □

6 Remarks and application.

Theorem 5.1 suggests a few observations

Remark 6.1. Theorem 5.1 is of course applicable near any equilibrium point

$$a \in \mathcal{E} = \{v \in V, \quad DF(v) = 0\}$$

and gives, assuming that $A = D^2F(a)$ satisfies the relevant hypothesis, the existence of a neighbourhood W of a in V such that

$$\forall u \in W, \quad \|DF(u)\|_{V'} \geq c|F(u) - F(a)|^{1-\theta}.$$

Remark 6.2. Theorem 4.1 shows that Theorem 5.1 is essentially optimal, since in order for such a general result to be true we at least need it to apply to quadratic forms. The finite dimensionality hypothesis on N is motivated by the example of Proposition 3.5.

Remark 6.3. If the imbedding $V \to H$ is compact and for some $m_0 \in \mathbb{R}$ the operator $D^2F(0) + m_0 I$ is invertible, then Lemma 6.1 from [12] shows that the condition is automatically fulfilled. Actually we have the following more general result

Proposition 6.4. *Let V, W be two reflexive Banach spaces spaces and $L \in \mathcal{L}(V, W)$ be such that for some compact operator $K \in \mathcal{L}(V, W)$, $L + K$ is invertible. Let X be a closed linear subspace of V such that*

$$X \cap \ker L = \{0\}.$$

Then there exists $c > 0$ such that

$$\forall v \in X, \quad \|Lv\|_W \geq c\|v\|_V.$$

Proof. If $X = \{0\}$ there is nothing to prove. Otherwise, assuming that the result is not true, for each integer $n \geq 1$, let $w_n \in X$ be such that

$$w_n \in X, \quad \|w_n\|_V = 1, \quad \|Lw_n\|_W \leq \frac{1}{n}.$$

Then we can replace w_n by a subsequence (still denoted w_n) such that

$$w_n \to w \quad \text{weakly in } V \qquad \text{and} \qquad Lw_n \to 0 \quad \text{strongly in } W.$$

Since L is continuous from (V, weak) to (W, weak) we have $Lw = 0$. Since X is closed, hence weakly sequentially closed in V, we also have $w \in X$, hence $w = 0$. In particular, we have

$$(L+K)w_n = Lw_n + Kw_n \to 0 \quad \text{strongly in } W$$

therefore

$$w_n \to 0 \quad \text{strongly in } V,$$

which contradicts

$$\|w_n\|_V = 1.$$

□

Corollary 6.5. *Let V, H be as in the statement of Theorem 5.1, let $F : U \to \mathbb{R}$ be an analytic functional where $U \subset V$ is an open neighborhood of a and assume $DF(a) = 0$. We assume the two conditions*

i) $N := \ker D^2F(0)$ is finite dimensional .

ii) For some compact operator $K \in \mathcal{L}(V, V')$, $L + K$ is invertible.
Then there exists $\theta \in (0, 1/2)$, a neighborhood W of a and $c > 0$ for which

$$\forall u \in W, \quad \|DF(u)\|_{V'} \geq c|F(u) - F(a)|^{1-\theta}.$$

Proof. This result is an immediate consequence of Theorem 5.1 and Proposition 6.4 □

Corollary 6.6. *Let Ω be a bounded open interval of $\mathbb{R}$, let G be an analytic function and $g = G'$. The functional Φ defined by*

$$\forall u \in H_0^1(\Omega), \quad \Phi(u) = \int_\Omega [\frac{1}{2}u_x^2 + G(u)]dx$$

is such that for any solution φ of

$$\varphi \in H_0^1(\Omega), \quad -\Delta\varphi + g(\varphi) = 0$$

there is $\theta \in (0, \frac{1}{2})$ and $\varepsilon > 0$, $C > 0$ for which

$$\forall u \in H_0^1(\Omega), \quad \|u - \varphi\|_{H_0^1(\Omega)} \leq \varepsilon \Rightarrow |\Phi(u) - \Phi(\varphi)|^{1-\theta} \leq C\| - \Delta u + g(u)\|_{H^{-1}(\Omega)}.$$

Proof. We have for any φ as above

$$\forall u \in H_0^1(\Omega), \quad D^2\Phi(\varphi)(u) = -\Delta u + g'(\varphi)(u).$$

In particular for $m > 0$ large enough, the operator

$$\Lambda = D^2\Phi(\varphi) + mI$$

where I stands for the identity operator is coercive, thus invertible as an operator from $H_0^1(\Omega)$ to $H^{-1}(\Omega)$. In addition the kernel of $D^2\Phi(\varphi)$ is clearly finite dimensional. Since $I : H_0^1(\Omega) \to H^{-1}(\Omega)$ is compact, the result follows from Corollary 6.5 applied with $V = H_0^1(\Omega)$, $H = L^2(\Omega)$. $\square$

Corollary 6.7. *Let Ω be a bounded open interval of $\mathbb{R}^N$ with $N \leq 3$, let G be an analytic function and $g = G'$. The functional Φ defined by*

$$\forall u \in H_0^2(\Omega), \quad \Phi(u) = \int_\Omega [\frac{1}{2}|\Delta u|^2 + G(u)]dx$$

is such that for any solution φ of

$$\varphi \in H_0^2(\Omega), \quad \Delta^2\varphi + g(\varphi) = 0$$

there is $\theta \in (0, \frac{1}{2})$ and $\varepsilon > 0$, $C > 0$ for which

$$\forall u \in H_0^2(\Omega), \quad \|u - \varphi\|_{H_0^2(\Omega)} \leq \varepsilon \to |\Phi(u) - \Phi(\varphi)|^{1-\theta} \leq C\|\Delta^2 u + g(u)\|_{H^{-2}(\Omega)}.$$

Proof. We have for any φ as above

$$\forall u \in H_0^2(\Omega), \quad D^2\Phi(\varphi)(u) = \Delta^2 u + g'(\varphi)(u)$$

In particular for $m > 0$ large enough, the operator

$$\Lambda = D^2\Phi(\varphi) + mI$$

is coercive, thus invertible as an operator from $H_0^2(\Omega)$ to $H^{-2}(\Omega)$. In addition the kernel of $D^2\Phi(\varphi)$ is clearly finite dimensional. Since $I : H_0^2(\Omega) \to H^{-2}(\Omega)$ is compact, the result follows from Corollary 6.5 applied with $V = H_0^2(\Omega)$, $H = L^2(\Omega)$. $\square$

Remark 6.8. In higher dimensions, Theorem 5.1 cannot be applied directly since the non-linear perturbation is no longer analytic in the topology of $V = H_0^1(\Omega)$ for the first example when $N > 1$, and the topology of $V = H_0^2(\Omega)$ when $N > 3$ in the second example. In higher dimensions one makes use of the fact that the equilibria are smoother and the analyticity of the functional is used on a smaller Banach space in order to be able to treat the finite dimensional term at the end, cf. [20, 16, 10, 8] for precise statements.

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