Spectral discretization of the Stokes problem with mixed boundary conditions

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Abstract: The variational formulation of the Stokes problem with three independent unknowns, the vorticity, the velocity and the pressure, was born to handle non standard boundary conditions which involve the normal component of the velocity and the tangential components of the vorticity. We propose an extension of this formulation to the case of mixed boundary conditions in a three-dimensional domain. Next we consider a spectral discretization of this problem. A detailed numerical analysis leads to error estimates for the three unknowns and numerical experiments confirm the interest of the discretization.

Résumé: La formulation variationnelle du problème de Stokes avec trois inconnues indépendantes, le tourbillon, la vitesse et la pression, est née pour traiter des conditions aux limites non usuelles portant sur la composante normale de la vitesse et les composantes tangentielles du tourbillon. Nous proposons une extension de cette formulation au cas de conditions aux limites mixtes dans un domaine tri-dimensionnel. Nous considérons ensuite une discrétisation obtenue par méthode spectrale. Une analyse numérique détaillée permet d'établir des majorations d'erreur pour les trois inconnues et des expériences numériques confirment l'intérêt de la discrétisation.

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1. Introduction.

Let Ω be a bounded connected domain in $\mathbb{R}^3$, with a Lipschitz–continuous boundary $\partial\Omega$. We consider a partition without overlap of $\partial\Omega$ into two connected parts Γ_m and Γ and we introduce the unit outward normal vector $\boldsymbol{n}$ to Ω on $\partial\Omega$. We are interested in the discretization of the Stokes problem, for a positive constant viscosity ν,

$$\begin{cases} -\nu\,\Delta\boldsymbol{u} + \mathbf{grad}\,p = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div}\boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{g} & \text{on } \Gamma, \\ \boldsymbol{u}\cdot\boldsymbol{n} = h & \text{on } \Gamma_m, \\ (\mathbf{curl}\,\boldsymbol{u})\times\boldsymbol{n} = \boldsymbol{k}\times\boldsymbol{n} & \text{on } \Gamma_m, \end{cases} \tag{1.1}$$

where the unknowns are the velocity $\boldsymbol{u}$ and the pressure p of the fluid. Indeed, this type of mixed boundary conditions appears in a large number of physical situations, the simplest one being a tank closed by a membrane on a part of its boundary (the index "m" in Γ_m means membrane). They are also needed for the coupling with other equations, for instance with Darcy's equations when the fluid domain is a crack in a porous medium, see [9] and [10].

In order to handle the boundary conditions on the normal component of the velocity and the tangential components of the vorticity $\mathbf{curl}\,\boldsymbol{u}$ and even if the standard formulation can be used (see [11] for instance), a new formulation was proposed in [16] and [24] (see also [17]). It involves three unknowns, the vorticity, the velocity and the pressure, and seems the more appropriate for the type of boundary conditions enforced on Γ_m. However, in the case of mixed boundary conditions, the lack of regularity of the solution, especially of the vorticity, leads to work with a different variational formulation, first proposed in [1] and [23] in the two-dimensional case and extended to the three-dimensional case in [2]. Relying on these results, we propose a variational problem and prove its well-posedness in the case of the boundary conditions in (1.1).

In the specific case where Ω is a cube and Γ_m one of its faces, we propose a discretization of this problem by the spectral method: The discrete problem relies on high degree polynomial approximation and is constructed by the Galerkin method with numerical integration. Note that the spectral and spectral element discretizations of this formulation have been studied in [5] and [3] in the case of conditions on the normal component of the velocity and the tangential components of the vorticity on the whole boundary. However different arguments are needed here since the variational formulation is different. In particular, we have chosen to work with exactly divergence-free discrete velocities, which seems necessary for the discrete problem to be well-posed. We perform the numerical analysis of this discretization. We thus prove error estimates for the three unknowns. These estimates are optimal for the vorticity and the velocity. But, as standard in spectral methods, the optimality for the pressure depends on the choice of the discrete spaces, even for simpler boundary conditions (see [12, §24–26] and [13]) and does not seem possible here.

We conclude with some numerical experiments both in dimensions 2 and 3. Indeed, we have chosen to present the numerical analysis of the problem only in dimension 3 to keep the uniqueness of the notation (which is different in dimension 2) and also because the analysis in dimension 3 is more complex, see [5] where the two cases are studied simultaneously. However, since three-dimensional computations are much more expensive than two-dimensional ones, we prefer to present both of them to illustrate the previous study in a more complete way. All these experiments confirm the good properties of the discretization.

An outline of the paper is as follows.

- In Section 2, we write the variational formulation of the problem in the case of homogeneous boundary conditions.
- Section 3 is devoted to the description of the spectral discrete problem. We also prove its well-posedness.
- A priori error estimates are derived in Section 4.
- The extension of the previous results to nonhomogeneous boundary conditions is described in Section 5.
- In Section 6, we present some numerical experiments which turn out to be in good agreement with the analysis.
- Some technical results are proved in an Appendix.

2. The velocity, vorticity and pressure formulation.

From now on, we assume for simplicity that Ω is simply-connected and has a connected boundary (we refer to [8, §2.5] for the extension to more general domains in the case where Γ_m is the whole boundary of Ω, but we have no application for that in the case of mixed boundary conditions). We also assume that $\partial\Gamma_m$ is a Lipschitz–continuous submanifold of $\partial\Omega$. In a first step we only consider the case where the data $\boldsymbol{g}$, h and $\boldsymbol{k}$ are zero, indeed we do not want to handle all difficulties together.

According to the approach proposed in [16] and [24], we introduce the vorticity $\boldsymbol{\omega} = \mathbf{curl}\,\boldsymbol{u}$ as a new unknown and observe that system (1.1) is fully equivalent to

$$\begin{cases} \nu\,\mathbf{curl}\,\boldsymbol{\omega} + \mathbf{grad}\,p = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{\omega} = \mathbf{curl}\,\boldsymbol{u} & \text{in } \Omega, \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0 & \text{on } \partial\Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0} & \text{on } \Gamma, \\ \boldsymbol{\omega} \times \boldsymbol{n} = \boldsymbol{0} & \text{on } \Gamma_m. \end{cases} \tag{2.1}$$

We now propose a variational formulation for system (2.1).

In what follows, we need the full scale of Sobolev spaces $H^s(\Omega)$, $s \geq 0$, provided with the usual norm $\|\cdot\|_{H^s(\Omega)}$ and semi-norm $|\cdot|_{H^s(\Omega)}$, together with their subspaces $H^s_0(\Omega)$. We introduce the domain $H(\operatorname{div},\Omega)$ of the divergence operator, namely

$$H(\operatorname{div},\Omega) = \big\{\boldsymbol{v} \in L^2(\Omega)^3;\ \operatorname{div} \boldsymbol{v} \in L^2(\Omega)\big\}. \tag{2.2}$$

Since the normal trace operator: $\boldsymbol{v} \mapsto \boldsymbol{v} \cdot \boldsymbol{n}$ can be defined from $H(\operatorname{div},\Omega)$ onto $H^{-\frac{1}{2}}(\partial\Omega)$, see [18, Chap. I, Thm 2.5], we also consider its kernel

$$H_0(\operatorname{div},\Omega) = \big\{\boldsymbol{v} \in H(\operatorname{div},\Omega);\ \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \partial\Omega\big\}. \tag{2.3}$$

Similarly, we introduce the domain of the **curl** operator

$$H(\mathbf{curl},\Omega) = \big\{\boldsymbol{v} \in L^2(\Omega)^3;\ \mathbf{curl}\,\boldsymbol{v} \in L^2(\Omega)^3\big\}. \tag{2.4}$$

The tangential trace operator: $\boldsymbol{v} \mapsto \boldsymbol{v} \times \boldsymbol{n}$ is defined on $H(\mathbf{curl},\Omega)$ with values in $H^{-\frac{1}{2}}(\partial\Omega)^3$, see [18, Chap. I, Thm 2.11]. It can also be checked that its restriction to Γ maps $H(\mathbf{curl},\Omega)$ into the dual space $H^{\frac{1}{2}}_{00}(\Gamma)'$ of $H^{\frac{1}{2}}_{00}(\Gamma)$ (see [20, Chap. 1, Th. 11.7] for the definition of this last space). So, in view of the fifth line in (2.1), i.e., the last boundary condition on the velocity, we define the space

$$H_*(\mathbf{curl},\Omega) = \big\{\boldsymbol{v} \in H(\mathbf{curl},\Omega);\ \boldsymbol{v} \times \boldsymbol{n} = \boldsymbol{0} \text{ on } \Gamma\big\}. \tag{2.5}$$

We set:

$$\mathbb{X}(\Omega) = H_0(\operatorname{div},\Omega) \cap H_*(\mathbf{curl},\Omega). \tag{2.6}$$

From now on, this space is equipped with the semi-norm

$$\|\boldsymbol{v}\|_{\mathbb{X}(\Omega)} = \big(\|\operatorname{div} \boldsymbol{v}\|^2_{L^2(\Omega)} + \|\mathbf{curl}\, \boldsymbol{v}\|^2_{L^2(\Omega)^3}\big)^{\frac{1}{2}}. \tag{2.7}$$

Indeed, we recall from [4, Cor. 3.16] that, since Ω is simply-connected, this quantity is a norm, equivalent to the graph norm of $H(\operatorname{div}, \Omega) \cap H(\mathbf{curl}, \Omega)$, more precisely that there exists a constant c only depending on Ω such that

$$\forall \boldsymbol{v} \in \mathbb{X}(\Omega), \quad \|\boldsymbol{v}\|_{L^2(\Omega)^3} \le c\, \|\boldsymbol{v}\|_{\mathbb{X}(\Omega)}. \tag{2.8}$$

Note moreover [4, Thm 2.17] that, when Ω is convex, $\mathbb{X}(\Omega)$ is contained in $H^1(\Omega)^3$. Finally, $L^2_\circ(\Omega)$ stands for the space of functions in $L^2(\Omega)$ with a null integral on Ω.

Let us also introduce the bilinear forms

$$\begin{aligned} a(\boldsymbol{\omega}, \boldsymbol{u}; \boldsymbol{v}) &= \nu \int_\Omega \boldsymbol{\omega}(\boldsymbol{x}) \cdot (\mathbf{curl}\, \boldsymbol{v})(\boldsymbol{x})\, d\boldsymbol{x}, \\ b(\boldsymbol{v}, q) &= -\int_\Omega (\operatorname{div} \boldsymbol{v})(\boldsymbol{x}) q(\boldsymbol{x})\, d\boldsymbol{x}, \\ c(\boldsymbol{\omega}, \boldsymbol{u}; \boldsymbol{\varphi}) &= \int_\Omega \boldsymbol{\omega}(\boldsymbol{x}) \cdot \boldsymbol{\varphi}(\boldsymbol{x})\, d\boldsymbol{x} - \int_\Omega (\mathbf{curl}\, \boldsymbol{u})(\boldsymbol{x}) \cdot \boldsymbol{\varphi}(\boldsymbol{x})\, d\boldsymbol{x}. \end{aligned} \tag{2.9}$$

For any data $\boldsymbol{f}$ in $L^2(\Omega)^3$, we now consider the variational problem

Find $(\boldsymbol{\omega}, \boldsymbol{u}, p)$ *in* $L^2(\Omega)^3 \times \mathbb{X}(\Omega) \times L^2_\circ(\Omega)$ *such that*

$$\begin{aligned} &\forall \boldsymbol{v} \in \mathbb{X}(\Omega), \quad a(\boldsymbol{\omega}, \boldsymbol{u}; \boldsymbol{v}) + b(\boldsymbol{v}, p) = \int_\Omega \boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x})\, d\boldsymbol{x}, \\ &\forall q \in L^2_\circ(\Omega), \quad b(\boldsymbol{u}, q) = 0, \\ &\forall \boldsymbol{\varphi} \in L^2(\Omega)^3, \quad c(\boldsymbol{\omega}, \boldsymbol{u}; \boldsymbol{\varphi}) = 0. \end{aligned} \tag{2.10}$$

As standard for mixed boundary conditions, see [8, §2.3], proving the equivalence between the initial system of partial differential equations (2.1) and the variational problem (2.10) is rather complex. For instance, in our case, the density of spaces of smooth vector fields in $\mathbb{X}(\Omega)$ seems unknown. So, we only state a partial result.

Proposition 2.1. *For any data* $\boldsymbol{f}$ *in* $L^2(\Omega)^3$, *any solution* $(\boldsymbol{\omega}, \boldsymbol{u}, p)$ *in* $L^2(\Omega)^3 \times \mathbb{X}(\Omega) \times L^2(\Omega)$ *of problem* (2.10) *is a solution of problem* (2.1) *in the sense of distributions.*

Proof: Let $(\boldsymbol{\omega}, \boldsymbol{u}, \boldsymbol{p})$ be a solution of (2.10). This solution satisfies the fourth and fifth lines of (2.1) owing to the definition of $\mathbb{X}(\Omega)$. The second and third equations in (2.10) imply the second and third lines in (2.1), respectively. To go further, we let $\boldsymbol{v}$ in the first equation of (2.10) run through the space $\mathscr{D}(\Omega)^3$, where $\mathscr{D}(\Omega)$ stands for the space of infinitely differentiable functions with a compact support in Ω. This yields the first line of (2.1). Finally, for any smooth function $\boldsymbol{\mu}$ on $\partial\Omega$ with compact support in Γ_m such that $\boldsymbol{\mu} \cdot \boldsymbol{n}$ is zero on $\partial\Omega$, we introduce a lifting of $\boldsymbol{\mu}$ in $H^1(\Omega)^3$ and note that it belongs to $\mathbb{X}(\Omega)$. Taking $\boldsymbol{v}$ equal to this lifting and using the previous result yield the sixth line in (2.1).

Remark 2.2. It is readily checked from the previous arguments that any solution $(\boldsymbol{\omega}, \boldsymbol{u}, p)$ of (2.1) in the sense of distributions which belongs to $L^2(\Omega)^3 \times \big(H(\operatorname{div}, \Omega) \cap H(\mathbf{curl}, \Omega)\big) \times$

$L^2(\Omega)$ satisfies the second and third equations in (2.10). So only the first equation requires an unknown density result.

We now intend to prove the well-posedness of problem (2.10). The forms $a(\cdot,\cdot;\cdot)$ and $c(\cdot,\cdot;\cdot)$ are obviously continuous on $L^2(\Omega)^3 \times \mathbb{X}(\Omega) \times \mathbb{X}(\Omega)$ and $L^2(\Omega)^3 \times \mathbb{X}(\Omega) \times L^2(\Omega)^3$, respectively. Moreover the form $b(\cdot,\cdot)$ is continuous on $\mathbb{X}(\Omega) \times L^2_\circ(\Omega)$. As a consequence, the kernel

$$V = \big\{\boldsymbol{v} \in \mathbb{X}(\Omega);\ \forall q \in L^2_\circ(\Omega),\ b(\boldsymbol{v}, q) = 0\big\}, \tag{2.11}$$

is a closed subspace of $\mathbb{X}(\Omega)$ and obviously coincides with

$$V = \big\{\boldsymbol{v} \in \mathbb{X}(\Omega);\ \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega\big\}. \tag{2.12}$$

Similarly, the kernel

$$\mathcal{W} = \big\{(\boldsymbol{\vartheta}, \boldsymbol{w}) \in L^2(\Omega)^3 \times V;\ \forall \boldsymbol{\varphi} \in L^2(\Omega)^3,\ c(\boldsymbol{\vartheta}, \boldsymbol{w}; \boldsymbol{\varphi}) = 0\big\}, \tag{2.13}$$

coincides with the space of pairs $(\boldsymbol{\vartheta}, \boldsymbol{w})$ in $L^2(\Omega)^3 \times V$ such that $\boldsymbol{\vartheta} = \mathbf{curl}\, \boldsymbol{w}$ in Ω.

We then observe that, for any solution $(\boldsymbol{\omega}, \boldsymbol{u}, p)$ of problem (2.10), the pair $(\boldsymbol{\omega}, \boldsymbol{u})$ is a solution of the following reduced problem

Find $(\boldsymbol{\omega}, \boldsymbol{u})$ in $\mathcal{W}$ such that

$$\forall \boldsymbol{v} \in V, \quad a(\boldsymbol{\omega}, \boldsymbol{u}; \boldsymbol{v}) = \int_\Omega \boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x})\, d\boldsymbol{x}. \tag{2.14}$$

So we now prove some properties of the form $a(\cdot,\cdot;\cdot)$.

Lemma 2.3. *The following positivity property holds*

$$\forall \boldsymbol{v} \in V \setminus \{0\}, \quad \sup_{(\boldsymbol{\omega}, \boldsymbol{u}) \in \mathcal{W}} a(\boldsymbol{\omega}, \boldsymbol{u}; \boldsymbol{v}) > 0. \tag{2.15}$$

Proof: Let $\boldsymbol{v}$ be an element of V such that $a(\boldsymbol{\omega}, \boldsymbol{u}; \boldsymbol{v})$ is zero for all $(\boldsymbol{\omega}, \boldsymbol{u})$ in $\mathcal{W}$. Taking $(\boldsymbol{\omega}, \boldsymbol{u})$ equal to $(\mathbf{curl}\, \boldsymbol{v}, \boldsymbol{v})$ (which obviously belongs to $\mathcal{W}$) yields that $\mathbf{curl}\, \boldsymbol{v}$ is zero. Thus, it follows from (2.8) and (2.12) that $\boldsymbol{v}$ is zero. This concludes the proof.

Lemma 2.4. *There exists a positive constant α such that the following inf-sup condition holds*

$$\forall (\boldsymbol{\omega}, \boldsymbol{u}) \in \mathcal{W}, \quad \sup_{v \in V} \frac{a(\boldsymbol{\omega}, \boldsymbol{u}; \boldsymbol{v})}{\|\boldsymbol{v}\|_{\mathbb{X}(\Omega)}} \geq \alpha\, \big(\|\boldsymbol{\omega}\|_{L^2(\Omega)^3} + \|\boldsymbol{u}\|_{\mathbb{X}(\Omega)}\big). \tag{2.16}$$

Proof: For any $(\boldsymbol{\omega}, \boldsymbol{u})$ in $\mathcal{W}$, taking $\boldsymbol{v}$ equal to $\boldsymbol{u}$ gives

$$a(\boldsymbol{\omega}, \boldsymbol{u}; \boldsymbol{v}) = \nu \int_\Omega \boldsymbol{\omega}(\boldsymbol{x}) \cdot (\mathbf{curl}\, \boldsymbol{u})(\boldsymbol{x})\, d\boldsymbol{x},$$

whence, owing to the definition of $\mathcal{W}$,

$$a(\boldsymbol{\omega}, \boldsymbol{u}; \boldsymbol{v}) = \nu\, \|\boldsymbol{\omega}\|^2_{L^2(\Omega)^3} = \frac{\nu}{2}\, \|\boldsymbol{\omega}\|^2_{L^2(\Omega)^3} + \frac{\nu}{2}\, \|\mathbf{curl}\, \boldsymbol{u}\|^2_{L^2(\Omega)^3}.$$

Since $\boldsymbol{u}$ is divergence-free, this leads to

$$a(\boldsymbol{\omega}, \boldsymbol{u}; \boldsymbol{v}) = \frac{\nu}{2} \|\boldsymbol{\omega}\|^2_{L^2(\Omega)^3} + \frac{\nu}{2} \|\boldsymbol{u}\|^2_{\mathbb{X}(\Omega)}.$$

This yields the desired inf-sup condition with $\alpha = \frac{\nu}{2}$.

When combining these properties with [8, Thm 1.3.7], we derive that problem (2.14) has a unique solution $(\boldsymbol{\omega}, \boldsymbol{u})$ in $\mathcal{W}$. We also recall the standard inf-sup condition on the form $b(\cdot, \cdot)$.

Lemma 2.5. *There exists a positive constant β such that*

$$\forall q \in L^2_\circ(\Omega), \qquad \sup_{\boldsymbol{v} \in \mathbb{X}(\Omega)} \frac{b(\boldsymbol{v}, q)}{\|\boldsymbol{v}\|_{\mathbb{X}(\Omega)}} \geq \beta \, \|q\|_{L^2(\Omega)}. \tag{2.17}$$

Proof: For each q in $L^2_\circ(\Omega)$, there exists (see [18, Chap. I, Cor. 2.4] for instance) a function $\boldsymbol{v}$ in $H^1_0(\Omega)^3$ such that

$$\operatorname{div} \boldsymbol{v} = -q \quad \text{in } \Omega \qquad \text{and} \qquad |\boldsymbol{v}|_{H^1(\Omega)^3} \leq c \, \|q\|_{L^2(\Omega)}.$$

Thus, the desired inf-sup condition follows thanks to the imbedding of $H^1_0(\Omega)^3$ in $\mathbb{X}(\Omega)$ and the equality (which is derived by integration by parts for smooth functions and extended to $H^1_0(\Omega)^3$ by density)

$$\forall \boldsymbol{v} \in H^1_0(\Omega)^3, \qquad \|\boldsymbol{v}\|_{\mathbb{X}(\Omega)} = |\boldsymbol{v}|_{H^1(\Omega)^3}.$$

The well-posedness of problem (2.10) is a direct consequence of Lemmas 2.3 to 2.5.

Theorem 2.6. *For any data $\boldsymbol{f}$ in $L^2(\Omega)^3$, problem* (2.10) *has a unique solution $(\boldsymbol{\omega}, \boldsymbol{u}, p)$ in $L^2(\Omega)^3 \times \mathbb{X}(\Omega) \times L^2_\circ(\Omega)$. Moreover this solution satisfies*

$$\|\boldsymbol{\omega}\|_{L^2(\Omega)^3} + \|\boldsymbol{u}\|_{\mathbb{X}(\Omega)} + \|p\|_{L^2(\Omega)} \leq c \, \|\boldsymbol{f}\|_{L^2(\Omega)^3}. \tag{2.18}$$

Proof: As already hinted, the existence of a solution $(\boldsymbol{\omega}, \boldsymbol{u})$ of problem (2.14) follows from Lemmas 2.3 and 2.4. Moreover, when combining (2.14) and (2.16), we obtain

$$\|\boldsymbol{\omega}\|_{L^2(\Omega)^3} + \|\boldsymbol{u}\|_{\mathbb{X}(\Omega)} \leq c \, \alpha^{-1} \, \|\boldsymbol{f}\|_{L^2(\Omega)^3},$$

where c is the constant in (2.8). We also derive from (2.14) that the linear form

$$\boldsymbol{v} \quad \mapsto \quad \int_\Omega \boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x}) \, d\boldsymbol{x} - a(\boldsymbol{\omega}, \boldsymbol{u}; \boldsymbol{v}),$$

vanishes on V. So the inf-sup condition (2.17) yields (see [18, Chap. I, Lemma 4.1] for instance) that there exists a p in $L^2_\circ(\Omega)$ such that the first line in (2.10) is satisfied. Thus, the triple $(\boldsymbol{\omega}, \boldsymbol{u}, p)$ is obviously a solution of (2.10) owing to the definitions of V and $\mathcal{W}$. Moreover, the pressure p also satisfies

$$\|p\|_{L^2(\Omega)} \leq \beta^{-1} \big(\|\boldsymbol{f}\|_{L^2(\Omega)^3} + \nu \, \|\boldsymbol{\omega}\|_{L^2(\Omega)^3} \big).$$

By combining all this, we derive (2.18). Finally, since problem (2.10) is linear, the uniqueness of its solution is derived from the fact that any solution of this problem with right-hand side equal to zero is zero, which is a direct consequence of (2.18).

Remark 2.7. It follows from the previous arguments that problem (2.10) is still well-posed for data $\boldsymbol{f}$ in the dual space of $\mathbb{X}(\Omega)$. However we have no applications for this extension.

3. The discrete problem.

We are now interested in the discretization of problem (2.10) in the case where

$$\Omega =]-1,1[^3, \qquad \Gamma_m =]-1,1[^2 \times \{1\}. \tag{3.1}$$

For reasons which appear later on, the discrete spaces are contructed from the finite elements proposed by Nédélec on cubic three-dimensional meshes, see [21, §2]. In order to describe them and for any triple (ℓ, m, n) of nonnegative integers, we introduce the space $\mathbb{P}_{\ell,m,n}(\Omega)$ of restrictions to Ω of polynomials with degree $\leq \ell$ with respect to x, $\leq m$ with respect to y and $\leq n$ with respect to z. When ℓ and m are equal to n, this space is simply denoted by $\mathbb{P}_n(\Omega)$.

Let N be an integer ≥ 2. The space $\mathbb{X}_N$ which approximates $\mathbb{X}(\Omega)$ is defined by

$$\mathbb{X}_N = \mathbb{X}(\Omega) \cap \big(\mathbb{P}_{N,N-1,N-1}(\Omega) \times \mathbb{P}_{N-1,N,N-1}(\Omega) \times \mathbb{P}_{N-1,N-1,N}(\Omega)\big). \tag{3.2}$$

The discrete space of vorticities is the space

$$\mathbb{Y}_N = \mathbb{P}_{N-1,N,N}(\Omega) \times \mathbb{P}_{N,N-1,N}(\Omega) \times \mathbb{P}_{N,N,N-1}(\Omega). \tag{3.3}$$

Finally, for the approximation of $L^2_\circ(\Omega)$, we choose a subspace $\mathbb{M}_N$ of $L^2_\circ(\Omega) \cap \mathbb{P}_{N-1}(\Omega)$ which is made precise later on.

Setting $\xi_0 = -1$ and $\xi_N = 1$, we introduce the $N-1$ nodes ξ_j, $1 \leq j \leq N-1$, and the $N+1$ weights ρ_j, $0 \leq j \leq N$, of the Gauss–Lobatto quadrature formula. Denoting by $\mathbb{P}_n(-1,1)$ the space of restrictions to $[-1,1]$ of polynomials with degree $\leq n$, we recall that the following equality holds

$$\forall \Phi \in \mathbb{P}_{2N-1}(-1,1), \qquad \int_{-1}^{1} \Phi(\zeta)\, d\zeta = \sum_{j=0}^{N} \Phi(\xi_j)\, \rho_j. \tag{3.4}$$

We also recall [12, form. (13.20)] the following property, which is useful in what follows

$$\forall \varphi_N \in \mathbb{P}_N(-1,1), \quad \|\varphi_N\|^2_{L^2(-1,1)} \leq \sum_{j=0}^{N} \varphi_N^2(\xi_j)\, \rho_j \leq 3\, \|\varphi_N\|^2_{L^2(-1,1)}. \tag{3.5}$$

Relying on this formula, we introduce the discrete product, defined on continuous functions u and v by

$$(u,v)_N = \sum_{i=0}^{N} \sum_{j=0}^{N} \sum_{k=0}^{N} u(\xi_i,\xi_j,\xi_k) v(\xi_i,\xi_j,\xi_k)\, \rho_i \rho_j \rho_k. \tag{3.6}$$

It follows from (3.5) that it is a scalar product on $\mathbb{P}_N(\Omega)$. Let finally $\mathcal{I}_N$ denote the Lagrange interpolation operator at the nodes (ξ_i, ξ_j, ξ_k), $0 \leq i,j,k \leq N$, with values in $\mathbb{P}_N(\Omega)$.

We now assume that the function $\boldsymbol{f}$ is continuous on $\overline{\Omega}$. Thus the discrete problem is constructed from (2.10) by using the Galerkin method combined with numerical integration. It reads

Find $(\boldsymbol{\omega}_N, \boldsymbol{u}_N, p_N)$ *in* $\mathbb{Y}_{\mathbb{N}} \times \mathbb{X}_N \times \mathbb{M}_N$ *such that*

$$\begin{aligned}
&\forall \boldsymbol{v}_N \in \mathbb{X}_N, \quad a_N(\boldsymbol{\omega}_N, \boldsymbol{u}_N; \boldsymbol{v}_N) + b_N(\boldsymbol{v}_N, p_N) = (\boldsymbol{f}, \boldsymbol{v}_N)_N, \\
&\forall q_N \in \mathbb{M}_N, \quad b_N(\boldsymbol{u}_N, q_N) = 0, \\
&\forall \boldsymbol{\vartheta}_N \in \mathbb{Y}_N, \quad c_N(\boldsymbol{\omega}_N, \boldsymbol{u}_N; \boldsymbol{\vartheta}_N) = 0,
\end{aligned} \tag{3.7}$$

where the bilinear forms $a_N(\cdot,\cdot;\cdot)$, $b_N(\cdot,\cdot)$ and $c_N(\cdot,\cdot;\cdot)$ are defined by

$$\begin{aligned}
a_N(\boldsymbol{\omega}_N, \boldsymbol{u}_N; \boldsymbol{v}_N) = \nu\,(\boldsymbol{\omega}_N, \mathbf{curl}\, \boldsymbol{v}_N)_N, \qquad b_N(\boldsymbol{v}_N, q_N) = -(\operatorname{div} \boldsymbol{v}_N, q_N)_N, \\
c_N(\boldsymbol{\omega}_N, \boldsymbol{u}_N; \boldsymbol{\varphi}_N) = (\boldsymbol{\omega}_N, \boldsymbol{\varphi}_N)_N - (\mathbf{curl}\, \boldsymbol{u}_N, \boldsymbol{\varphi}_N)_N.
\end{aligned} \tag{3.8}$$

It follows from (3.5) combined with Cauchy–Schwarz inequalities that the forms $a_N(\cdot,\cdot;\cdot)$, $b_N(\cdot,\cdot)$ and $c_N(\cdot,\cdot;\cdot)$ are continuous on $\mathbb{Y}_N \times \mathbb{X}_N \times \mathbb{X}_N$, $\mathbb{X}_N \times \mathbb{M}_N$ and $\mathbb{Y}_N \times \mathbb{X}_N \times \mathbb{Y}_N$, respectively, with norms bounded independently of N. Moreover, as a consequence of (3.4), the forms $b(\cdot,\cdot)$ and $b_N(\cdot,\cdot)$ coincide on $\mathbb{X}_N \times \mathbb{M}_N$.

In order to investigate the well-posedness of problem (3.7), we first identify the spurious modes on the pressure, namely the set

$$Z_N = \big\{ q_N \in L^2_\circ(\Omega) \cap \mathbb{P}_{N-1}(\Omega);\ \forall \boldsymbol{v}_N \in \mathbb{X}_N,\ b_N(\boldsymbol{v}_N, q_N) = 0 \big\}. \tag{3.9}$$

The rather technical proof of the next lemma is given in an Appendix.

Lemma 3.1. *The space* Z_N *has dimension* $8(N-1)$ *and is spanned by the orthogonal projections onto* $L^2_\circ(\Omega)$ *of the polynomials*

$$\begin{aligned}
(L'_N \pm L'_{N-1})(x)\,(L'_N \pm L'_{N-1})(y)\,\varphi_N(z), \qquad (L'_N \pm L'_{N-1})(x)\,\varphi_N(y)\,\chi^-_N(z) \\
\text{and} \qquad \varphi_N(x)(L'_N \pm L'_{N-1})(y)\,\chi^-_N(z),
\end{aligned} \tag{3.10}$$

where φ_N *runs through* $\mathbb{P}_{N-1}(-1,1)$ *and the polynomial* χ^-_N *is defined by*

$$\chi^-_N(z) = \frac{L'_N(z)}{N+1} - \frac{L'_{N-1}(z)}{N-1}. \tag{3.11}$$

In view of this result, from now on, we choose $\mathbb{M}_N$ such that

$$L^2_\circ(\Omega) \cap \mathbb{P}_{N-1}(\Omega) = \mathbb{M}_N \oplus Z_N. \tag{3.12}$$

Proving that problem (3.7) is well-posed now relies on exactly the same arguments as for the continuous problem. We first introduce the kernel

$$V_N = \big\{ \boldsymbol{v}_N \in \mathbb{X}_N;\ \forall q_N \in \mathbb{M}_N,\ b_N(\boldsymbol{v}_N, q_N) = 0 \big\}. \tag{3.13}$$

We have the following property.

Lemma 3.2. *The kernel* V_N *is the space of divergence-free polynomials in* $\mathbb{X}_N$*, i.e., coincides with* $\mathbb{X}_N \cap V$.

Proof: Let $\boldsymbol{v}_N$ be any element of V_N. Its divergence belongs to $L^2_\circ(\Omega) \cap \mathbb{P}_{N-1}(\Omega)$, so that, owing to (3.12), we have

$$\operatorname{div} \boldsymbol{v}_N = q_N^1 + q_N^2,$$

with q_N^1 in $\mathbb{M}_N$ and q_N^2 in Z_N. We have $b_N(\boldsymbol{v}_N, q_N^1) = 0$ thanks to the definition of V_N and $b_N(\boldsymbol{v}_N, q_N^2) = 0$ thanks to the definition of Z_N, so that

$$0 = b_N(\boldsymbol{v}_N, \operatorname{div} \boldsymbol{v}_N) = -(\operatorname{div} \boldsymbol{v}_N, \operatorname{div} \boldsymbol{v}_N)_N = -\int_\Omega (\operatorname{div} \boldsymbol{v}_N)^2(\boldsymbol{x})\, d\boldsymbol{x}.$$

Then, $\operatorname{div} \boldsymbol{v}_N$ is zero, which concludes the proof.

Similarly, we introduce the kernel $\mathcal{W}_N$ of the form $c_N(\cdot, \cdot; \cdot)$:

$$\mathcal{W}_N = \big\{(\boldsymbol{\vartheta}_N, \boldsymbol{v}_N) \in \mathbb{Y}_N \times V_N;\ \forall \boldsymbol{\varphi}_N \in \mathbb{Y}_N,\ c_N(\boldsymbol{\vartheta}_N, \boldsymbol{v}_N; \boldsymbol{\varphi}_N) = 0\big\}. \tag{3.14}$$

It is readily checked that, for any solution of problem (3.7), the pair $(\boldsymbol{\omega}_N, \boldsymbol{u}_N)$ is a solution of the reduced problem

Find $(\boldsymbol{\omega}_N, \boldsymbol{u}_N)$ in $\mathcal{W}_N$ such that

$$\forall \boldsymbol{v}_N \in V_N, \quad a_N(\boldsymbol{\omega}_N, \boldsymbol{u}_N; \boldsymbol{v}_N) = (\boldsymbol{f}, \boldsymbol{v}_N)_N. \tag{3.15}$$

In analogy with Section 2, we now prove some properties of the form $a_N(\cdot, \cdot; \cdot)$ on $\mathcal{W}_N \times V_N$.

Lemma 3.3. *The following positivity property holds*

$$\forall \boldsymbol{v}_N \in V_N \setminus \{0\}, \quad \sup_{(\boldsymbol{\omega}_N, \boldsymbol{u}_N) \in \mathcal{W}_N} a_N(\boldsymbol{\omega}_N, \boldsymbol{u}_N; \boldsymbol{v}_N) > 0. \tag{3.16}$$

Proof: For any $\boldsymbol{v}_N$ in V_N, $\mathbf{curl}\, \boldsymbol{v}_N$ belongs to $\mathbb{Y}_N$, so that $(\mathbf{curl}\, \boldsymbol{v}_N, \boldsymbol{v}_N)$ belongs to $\mathcal{W}_N$. Then, taking $(\boldsymbol{\omega}_N, \boldsymbol{u}_N)$ equal to $(\mathbf{curl}\, \boldsymbol{v}_N, \boldsymbol{v}_N)$ gives

$$a_N(\boldsymbol{\omega}_N, \boldsymbol{u}_N; \boldsymbol{v}_N) = \nu(\mathbf{curl}\, \boldsymbol{v}_N, \mathbf{curl}\, \boldsymbol{v}_N)_N.$$

It thus follows from (3.5) that

$$a_N(\boldsymbol{\omega}_N, \boldsymbol{u}_N; \boldsymbol{v}_N) \geq \nu\, \|\mathbf{curl}\, \boldsymbol{v}_N\|^2_{L^2(\Omega)^3}.$$

Then, if $a_N(\boldsymbol{\omega}_N, \boldsymbol{u}_N; \boldsymbol{v}_N)$ is zero, so is $\mathbf{curl}\, \boldsymbol{v}_N$. Since Lemma 3.2 implies that $\boldsymbol{v}_N$ is also divergence-free and $\mathbb{X}_N$ is contained in $\mathbb{X}(\Omega)$, applying (2.8) yields that $\boldsymbol{v}_N$ is zero. This concludes the proof.

Lemma 3.4. *There exists a positive constant α_* independent of N such that the following inf-sup condition holds*

$$\forall (\boldsymbol{\omega}_N, \boldsymbol{u}_N) \in \mathcal{W}_N, \quad \sup_{\boldsymbol{v}_N \in V_N} \frac{a_N(\boldsymbol{\omega}_N, \boldsymbol{u}_N; \boldsymbol{v}_N)}{\|\boldsymbol{v}_N\|_{\mathbb{X}(\Omega)}} \geq \alpha_* \big(\|\boldsymbol{\omega}_N\|_{L^2(\Omega)^3} + \|\boldsymbol{u}_N\|_{\mathbb{X}(\Omega)}\big). \tag{3.17}$$

Proof: Taking $\boldsymbol{v}_N$ equal to $\boldsymbol{u}_N$ yields

$$a_N(\boldsymbol{\omega}_N, \boldsymbol{u}_N; \boldsymbol{v}_N) = \nu\, (\boldsymbol{\omega}_N, \mathbf{curl}\, \boldsymbol{u}_N)_N,$$

whence, thanks to the definition (3.14) of $\mathcal{W}_N$ and (3.5),

$$a_N(\boldsymbol{\omega}_N, \boldsymbol{u}_N; \boldsymbol{v}_N) = \nu\,(\boldsymbol{\omega}_N, \boldsymbol{\omega}_N)_N \geq \nu\,\|\boldsymbol{\omega}_N\|^2_{L^2(\Omega)^3}.$$

On the other hand, taking $\boldsymbol{\varphi}_N$ equal to $\mathbf{curl}\,\boldsymbol{u}_N$ in the definition of $\mathcal{W}_N$ gives

$$(\mathbf{curl}\,\boldsymbol{u}_N, \mathbf{curl}\,\boldsymbol{u}_N)_N = (\boldsymbol{\omega}_N, \mathbf{curl}\,\boldsymbol{u}_N)_N,$$

whence, by combining (3.5) with the Cauchy–Schwarz inequality,

$$\|\mathbf{curl}\,\boldsymbol{u}_N\|_{L^2(\Omega)^3} \leq 3\,\|\boldsymbol{\omega}_N\|_{L^2(\Omega)^3}.$$

Since $\boldsymbol{u}_N$ is divergence-free, see Lemma 3.2, this yields

$$a_N(\boldsymbol{\omega}_N, \boldsymbol{u}_N; \boldsymbol{v}_N) \geq \frac{\nu}{2}\|\boldsymbol{\omega}_N\|^2_{L^2(\Omega)^3} + \frac{\nu}{18}\,\|\boldsymbol{u}_N\|^2_{\mathbb{X}(\Omega)},$$

which leads to the desired inf-sup condition.

We are now in a position to prove the main result of this Section.

Theorem 3.5. *For any data $\boldsymbol{f}$ continuous on $\overline{\Omega}$, problem (3.7) has a unique solution $(\boldsymbol{\omega}_N, \boldsymbol{u}_N, p_N)$ in $\mathbb{Y}_N \times \mathbb{X}_N \times \mathbb{M}_N$. Moreover this solution satisfies*

$$\|\boldsymbol{\omega}_N\|_{L^2(\Omega)^3} + \|\boldsymbol{u}_N\|_{\mathbb{X}(\Omega)} \leq c\,\|\mathcal{I}_N\boldsymbol{f}\|_{L^2(\Omega)^3}. \tag{3.18}$$

Proof: We prove successively the existence and uniqueness of the solution.
1) It follows from Lemmas 3.3 and 3.4, see [18, Chap. I, Lemma 4.1], that problem (3.15) has a unique solution $(\boldsymbol{\omega}_N, \boldsymbol{u}_N)$. Moreover, applying the inf-sup condition (3.17) yields that

$$\|\boldsymbol{\omega}_N\|_{L^2(\Omega)^3} + \|\boldsymbol{u}_N\|_{\mathbb{X}(\Omega)} \leq \alpha_*^{-1} \sup_{\boldsymbol{v}_N \in V_N} \frac{(\boldsymbol{f}, \boldsymbol{v}_N)_N}{\|\boldsymbol{v}_N\|_{\mathbb{X}(\Omega)}}.$$

We also derive from (3.5) and (2.8) that

$$(\boldsymbol{f}, \boldsymbol{v}_N)_N = (\mathcal{I}_N\boldsymbol{f}, \boldsymbol{v}_N)_N \leq 3\,\|\mathcal{I}_N\boldsymbol{f}\|_{L^2(\Omega)^3}\|\boldsymbol{v}_N\|_{L^2(\Omega)^3} \leq c\,\|\mathcal{I}_N\boldsymbol{f}\|_{L^2(\Omega)^3}\|\boldsymbol{v}_N\|_{\mathbb{X}(\Omega)},$$

which yields (3.18). Next, a direct consequence of the choice (3.12) of $\mathbb{M}_N$ is that, for each $N > 0$, the quantity

$$\beta_N = \inf_{q_N \in \mathbb{M}_N, q_N \neq 0} \sup_{\boldsymbol{v}_N \in \mathbb{X}_N} \frac{b_N(\boldsymbol{v}_N, \boldsymbol{q}_N)}{\|\boldsymbol{v}_N\|_{\mathbb{X}(\Omega)}\|q_N\|_{L^2(\Omega)}}, \tag{3.19}$$

is positive. Thus, since the linear form:

$$\boldsymbol{v}_N \;\mapsto\; (\boldsymbol{f}, \boldsymbol{v}_N)_N - a_N(\boldsymbol{\omega}_N, \boldsymbol{u}_N; \boldsymbol{v}_N),$$

vanishes on V_N, see (3.15), applying [18, Chap. I, Lemma 4.1] yields that there exists a p_N in $\mathbb{M}_N$ such that

$$\forall \boldsymbol{v}_N \in \mathbb{X}_N, \quad b_N(\boldsymbol{v}_N, p_N) = (\boldsymbol{f}, \boldsymbol{v}_N)_N - a_N(\boldsymbol{\omega}_N, \boldsymbol{u}_N; \boldsymbol{v}_N).$$

Thus, the triple $(\boldsymbol{\omega}_N, \boldsymbol{u}_N, p_N)$ is a solution of problem (3.7).
2) Since problem (3.7) is linear, the uniqueness of its solution follows from the fact that any solution of this problem with data $\boldsymbol{f}$ equal to $\mathbf{0}$ is zero. So, let $(\boldsymbol{\omega}_N, \boldsymbol{u}_N, p_N)$ be a solution of (3.7) with $\boldsymbol{f} = \mathbf{0}$. Thus, $(\boldsymbol{\omega}_N, \boldsymbol{u}_N)$ be a solution of (3.15) with $\boldsymbol{f} = \mathbf{0}$ and it follows from the inf-sup condition (3.17) that it is zero. Finally, p_N satisfies

$$\forall \boldsymbol{v}_N \in \mathbb{X}_N, \quad b_N(\boldsymbol{v}_N, p_N) = 0,$$

and, since the constant β_N introduced in (3.19) is positive, it is zero. This concludes the proof.

Remark 3.6. A stability property analogous to (3.18) can also be established for the discrete pressure p_N. However, it involves the constant β_N introduced in (3.19) and, since this constant is likely not bounded independently of N (see the Appendix), we have rather skip this estimate.

4. Error estimates.

In a first step, we derive an a priori error estimate between the solutions $(\boldsymbol{\omega}, \boldsymbol{u})$ of problem (2.14) and $(\boldsymbol{\omega}_N, \boldsymbol{u}_N)$ of problem (3.15). We use the obvious notation for $\mathbb{X}_{N-1}$ defined as in (3.2) with N replaced by $N-1$ and $V_{N-1} = \mathbb{X}_{N-1} \cap V$. Thus, we have the following version of the second Strang's lemma.

Lemma 4.1. *The following error estimate holds between the solutions* $(\boldsymbol{\omega}, \boldsymbol{u})$ *of problem* (2.14) *and* $(\boldsymbol{\omega}_N, \boldsymbol{u}_N)$ *of problem* (3.15):

$$\|\boldsymbol{\omega} - \boldsymbol{\omega}_N\|_{L^2(\Omega)^3} + \|\boldsymbol{u} - \boldsymbol{u}_N\|_{\mathbb{X}(\Omega)} \leq c\Big(\inf_{\boldsymbol{w}_N \in V_{N-1}} \|\boldsymbol{u} - \boldsymbol{w}_N\|_{\mathbb{X}(\Omega)} + \sup_{\boldsymbol{v}_N \in \mathbb{X}_N} \frac{\int_\Omega \boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{v}_N(\boldsymbol{x})\, d\boldsymbol{x} - (\boldsymbol{f}, \boldsymbol{v}_N)_N}{\|\boldsymbol{v}_N\|_{\mathbb{X}(\Omega)}}\Big). \tag{4.1}$$

Proof: For any $\boldsymbol{w}_N$ in V_{N-1}, when setting $\boldsymbol{\zeta}_N = \mathbf{curl}\, \boldsymbol{w}_N$, it is readily checked that the pair $(\boldsymbol{\zeta}_N, \boldsymbol{w}_N)$ belongs to $\mathcal{W}_N$. Thus, we obtain, for all $\boldsymbol{v}_N$ in V_N, first by using (3.15) and the exactness property (3.4), second by using (2.14),

$$\begin{aligned} a_N(\boldsymbol{\omega}_N - \boldsymbol{\zeta}_N, \boldsymbol{u}_N - \boldsymbol{w}_N; \boldsymbol{v}_N) &= (\boldsymbol{f}, \boldsymbol{v}_N)_N - a(\boldsymbol{\zeta}_N, \boldsymbol{w}_N; \boldsymbol{v}_N) \\ &= (\boldsymbol{f}, \boldsymbol{v}_N)_N - \int_\Omega \boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{v}_N(\boldsymbol{x})\, d\boldsymbol{x} + a(\boldsymbol{\omega} - \boldsymbol{\zeta}_N, \boldsymbol{u} - \boldsymbol{w}_N; \boldsymbol{v}_N). \end{aligned}$$

Thus, we derive from the inf-sup condition (3.17)

$$\|\boldsymbol{\omega}_N - \boldsymbol{\zeta}_N\|_{L^2(\Omega)^3} + \|\boldsymbol{u}_N - \boldsymbol{w}_N\|_{\mathbb{X}(\Omega)} \leq c\Big(\|\boldsymbol{\omega} - \boldsymbol{\zeta}_N\|_{L^2(\Omega)^3} + \sup_{\boldsymbol{v}_N \in \mathbb{X}_N} \frac{\int_\Omega \boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{v}_N(\boldsymbol{x})\, d\boldsymbol{x} - (\boldsymbol{f}, \boldsymbol{v}_N)_N}{\|\boldsymbol{v}_N\|_{\mathbb{X}(\Omega)}}\Big).$$

We conclude by using the triangle inequalities

$$\begin{aligned} \|\boldsymbol{\omega} - \boldsymbol{\omega}_N\|_{L^2(\Omega)^3} &\leq \|\boldsymbol{\omega} - \boldsymbol{\zeta}_N\|_{L^2(\Omega)^3} + \|\boldsymbol{\omega}_N - \boldsymbol{\zeta}_N\|_{L^2(\Omega)^3}, \\ \|\boldsymbol{u} - \boldsymbol{u}_N\|_{\mathbb{X}(\Omega)} &\leq \|\boldsymbol{u} - \boldsymbol{w}_N\|_{\mathbb{X}(\Omega)} + \|\boldsymbol{u}_N - \boldsymbol{w}_N\|_{\mathbb{X}(\Omega)}, \end{aligned}$$

and finally

$$\|\boldsymbol{\omega} - \boldsymbol{\zeta}_N\|_{L^2(\Omega)^3} = \|\mathbf{curl}\,(\boldsymbol{u} - \boldsymbol{w}_N)\|_{L^2(\Omega)^3} = \|\boldsymbol{u} - \boldsymbol{w}_N\|_{\mathbb{X}(\Omega)}.$$

Estimating the last term in the right-hand side of (4.1) relies on standard arguments. Indeed, it follows from the exactness property (3.4) that, for all $\boldsymbol{f}_{N-1}$ in $\mathbb{P}_{N-1}(\Omega)^3$,

$$\forall \boldsymbol{v}_N \in \mathbb{X}_N, \quad \int_\Omega \boldsymbol{f}_{N-1}(\boldsymbol{x}) \cdot \boldsymbol{v}_N(\boldsymbol{x})\, d\boldsymbol{x} = (\boldsymbol{f}_{N-1}, \boldsymbol{v}_N)_N.$$

Using this equality and the definition of the interpolation operator $\mathcal{I}_N$ together with (3.5) and (2.8) gives

$$\sup_{\boldsymbol{v}_N \in \mathbb{X}_N} \frac{\int_\Omega \boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{v}_N(\boldsymbol{x})\, d\boldsymbol{x} - (\boldsymbol{f}, \boldsymbol{v}_N)_N}{\|\boldsymbol{v}_N\|_{\mathbb{X}(\Omega)}} \leq c\big(\|\boldsymbol{f} - \boldsymbol{f}_{N-1}\|_{L^2(\Omega)^3} + \|\boldsymbol{f} - \mathcal{I}_N \boldsymbol{f}\|_{L^2(\Omega)^3}\big).$$

Thus, standard arguments [12, Thms 7.1 & 14.2] yield that, if the function $\boldsymbol{f}$ belongs to $H^\sigma(\Omega)^3$, $\sigma > \frac{3}{2}$,

$$\sup_{\boldsymbol{v}_N \in \mathbb{X}_N} \frac{\int_\Omega \boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{v}_N(\boldsymbol{x})\, d\boldsymbol{x} - (\boldsymbol{f}, \boldsymbol{v}_N)_N}{\|\boldsymbol{v}_N\|_{\mathbb{X}(\Omega)}} \le c\, N^{-\sigma}\, \|\boldsymbol{f}\|_{H^\sigma(\Omega)^3}. \tag{4.2}$$

A further lemma is needed to estimate the approximation error $\inf_{\boldsymbol{w}_N \in V_{N-1}} \|\boldsymbol{u} - \boldsymbol{w}_N\|_{\mathbb{X}(\Omega)}$.

Lemma 4.2. *There exists an operator $\mathcal{D}$*
(i) such that, for any function g in $L^2_\circ(\Omega)$, $\mathrm{div}(\mathcal{D}g)$ is equal to g on Ω,
(ii) which is continuous from $H^{s-1}(\Omega) \cap L^2_\circ(\Omega)$ into $\mathbb{X}(\Omega) \cap H^s(\Omega)^3$ for each real number s, $1 \le s < 3.5$.

Proof: We only give an abridged proof of this result since the main arguments can be found in [6, Thms 3.5 and 4.11].
1) Let $\mathcal{O}$ be an open ball such that $\overline{\Omega}$ is contained in $\mathcal{O}$. For any function g in $H^{s-1}(\Omega) \cap L^2_\circ(\Omega)$, we denote by $\overline{g}$ the extension of g into a function of $H^{s-1}(\mathcal{O}) \cap L^2_\circ(\mathcal{O})$ (by a fixed extension operator, see [19, §1.4.3]). Thus, the Stokes problem

$$\begin{cases} -\Delta \boldsymbol{u}_1 + \mathbf{grad}\, p_1 = \mathbf{0} & \text{in } \mathcal{O}, \\ \mathrm{div}\, \boldsymbol{u}_1 = g & \text{in } \mathcal{O}, \\ \boldsymbol{u}_1 = \mathbf{0} & \text{on } \partial\mathcal{O}, \end{cases}$$

has a unique solution $(\boldsymbol{u}_1, p_1)$ in $H^1_0(\mathcal{O})^3 \times L^2_\circ(\mathcal{O})$ and the part $\boldsymbol{u}_1$ of this solution belongs to $H^s(\mathcal{O})^3$. In what follows, we consider the restriction of $\boldsymbol{u}_1$ to Ω and we observe that

$$\|\boldsymbol{u}_1\|_{H^s(\Omega)^3} \le \|\boldsymbol{u}_1\|_{H^s(\mathcal{O})^3} \le c\, \|\overline{g}\|_{H^{s-1}(\mathcal{O})} \le c'\, \|g\|_{H^{s-1}(\Omega)}.$$

2) A divergence-free lifting $\boldsymbol{u}_2$ of the normal trace of $\boldsymbol{u}_1$ on $\partial\Omega$ is constructed in the first part of the proof of [6, Thm 4.11]. Moreover it belongs to $H^s(\Omega)^3$ when $s < 3.5$, and its norm in this space is bounded by a constant times that of $\boldsymbol{u}_1$.
3) Similarly, a divergence-free vector field $\boldsymbol{u}_3$ on $H^s(\Omega)^3$ which has zero normal trace on $\partial\Omega$ and lifts the tangential traces of $\boldsymbol{u}_1 - \boldsymbol{u}_2$ on the five faces of Ω contained in Γ can be constructed as in the second part of the proof of [6, Thm 4.11].
The function $\mathcal{D}g = \boldsymbol{u}_1 - \boldsymbol{u}_2 - \boldsymbol{u}_3$ satisfies all the properties of the lemma.

Lemma 4.3. *The following approximation result holds for any function $\boldsymbol{u}$ in $V \cap H^s(\omega)^3$, $s \ge 1$,*

$$\inf_{\boldsymbol{w}_N \in V_{N-1}} \|\boldsymbol{u} - \boldsymbol{w}_N\|_{\mathbb{X}(\Omega)} \le c\, N^{1-s}\, \|\boldsymbol{u}\|_{H^s(\Omega)^3}. \tag{4.3}$$

Proof: Let Π_N denote the orthogonal projection operator from V onto V_{N-1} for the scalar product associated with the norm of $\mathbb{X}(\Omega)$. The next result is proved in [22, Thm 2.3] for functions in $H^1_0(\Omega)^3$ but can easily be extended to functions satisfying the boundary conditions in $\mathbb{X}(\Omega)$: For any real number $s \ge 3$ and any $\boldsymbol{u}$ in $V \cap H^s(\omega)^3$,

$$\|\boldsymbol{u} - \Pi_N \boldsymbol{u}\|_{\mathbb{X}(\Omega)} \le c\, N^{1-s}\, \|\boldsymbol{u}\|_{H^s(\Omega)^3}.$$

The same result obvious holds for $s = 1$ thanks to the definition of Π_N. So, the desired estimate is derived owing to the principal theorem of interpolation [20, Chap. 1, Th. 5.1]: Indeed, by combining [20, Chap. 1, Th. 14.3] with Lemma 4.2, we observe that the spaces $V \cap H^s(\Omega)^3$ for $1 \le s \le 3$ satisfy the standard interpolation properties.

Inserting (4.2) and (4.3) into (4.1) leads to the error estimate on the vorticity and the velocity.

Theorem 4.4. *Assume that the data* $\boldsymbol{f}$ *belong to* $H^\sigma(\Omega)^3$, $\sigma > \frac{3}{2}$, *and that the solution* $(\boldsymbol{\omega}, \boldsymbol{u})$ *of problem* (2.14) *is such that the velocity* $\boldsymbol{u}$ *belongs to* $H^s(\Omega)^3$, $s \geq 1$. *Then, the following error estimate holds between this solution and the solution* $(\boldsymbol{\omega}_N, \boldsymbol{u}_N)$ *of problem* (3.15)*:*

$$\|\boldsymbol{\omega} - \boldsymbol{\omega}_N\|_{L^2(\Omega)^3} + \|\boldsymbol{u} - \boldsymbol{u}_N\|_{\mathbb{X}(\Omega)} \leq c\left(N^{1-s}\,\|\boldsymbol{u}\|_{H^s(\Omega)^3} + N^{-\sigma}\,\|\boldsymbol{f}\|_{H^\sigma(\Omega)^3}\right). \tag{4.4}$$

Estimate (4.4) is fully optimal and, moreover, it does not require any further regularity of the solution (we recall that, since Ω is convex, $\mathbb{X}(\Omega)$ is imbedded in $H^1(\Omega)^3$). However, as standard for the spectral discretization of the Stokes problem, the estimate on the pressure is no longer optimal.

Before stating this estimate, we must make precise the space $\mathbb{M}_N$ we work with. In view of (3.12), it seems natural to take $\mathbb{M}_N$ equal to the orthogonal of Z_N in $\mathbb{P}_{N-1}(\Omega) \cap L^2_\circ(\Omega)$. However, it is checked in [12, §24] that this space has poor approximation properties. To remedy this difficulty, we use the idea presented for instance in [12, Thm 24.9]: For a fixed real number λ, $0 < \lambda < 1$, denoting by $\pi_{[\lambda N]}$ the orthogonal projection operator from $L^2(-1,1)$ onto $\mathbb{P}_{[\lambda N]}(-1,1)$, where $[\lambda N]$ stands for the integer part of λN, we set

$$\Lambda_{N-1} = L'_N - \pi_{[\lambda N]}\,L'_N, \quad \Lambda_{N-2} = L'_{N-1} - \pi_{[\lambda N]}\,L'_{N-1}, \quad \Xi^-_N = \chi^-_N - \pi_{[\lambda N]}\chi^-_N. \tag{4.5}$$

Definition 4.5. The space $\mathbb{M}_N$ is the orthogonal in $\mathbb{P}_{N-1}(\Omega) \cap L^2_\circ(\Omega)$ of the space $\widetilde{Z}_N$ spanned by the polynomials

$$\begin{gathered}(\Lambda_{N-1} \pm \Lambda_{N-2})(x)\,(\Lambda_{N-1} \pm \Lambda_{N-2})(y)\,\varphi_N(z), \qquad (\Lambda_{N-1} \pm \Lambda_{N-2})(x)\,\varphi_N(y)\,\Xi^-_N(z)\\ \textit{and} \qquad \varphi_N(x)(\Lambda_{N-1} \pm \Lambda_{N-2})(y)\,\Xi^-_N(z),\end{gathered} \tag{4.6}$$

where φ_N runs through $\mathbb{P}_{N-1}(-1,1)$.

It is readily checked that (3.12) holds for this space $\mathbb{M}_N$. Moreover, the next inf-sup condition for this choice is proved in the Appendix.

Lemma 4.6. *For the space* $\mathbb{M}_N$ *introduced in Definition* 4.5, *there exists a constant* $\beta_* > 0$ *such that the following inf-sup condition holds*

$$\forall q_N \in \mathbb{M}_N, \qquad \sup_{\boldsymbol{v}_N \in \mathbb{X}_N} \frac{b_N(\boldsymbol{v}_N, q_N)}{\|\boldsymbol{v}_N\|_{\mathbb{X}(\Omega)}} \geq \beta_*\,N^{-1}\,\|q_N\|_{L^2(\Omega)}. \tag{4.7}$$

We are now in a position to establish the final error estimate. Note that the lack of optimality is of the same order as for standard boundary conditions.

Theorem 4.7. *Assume that the data* $\boldsymbol{f}$ *belong to* $H^\sigma(\Omega)^3$, $\sigma > \frac{3}{2}$, *and that the solution* $(\boldsymbol{\omega}, \boldsymbol{u}, p)$ *of problem* (2.10) *is such that* $(\boldsymbol{u}, p)$ *belongs to* $H^s(\Omega)^3 \times H^{s-1}(\Omega)$, $s \geq 1$. *Then, for the space* $\mathbb{M}_N$ *introduced in Definition* 4.5, *the following error estimate holds between this solution and the solution* $(\boldsymbol{\omega}_N, \boldsymbol{u}_N, p_N)$ *of problem* (3.7)*:*

$$\|p - p_N\|_{L^2(\Omega)} \leq c\left(N^{2-s}\left(\|\boldsymbol{u}\|_{H^s(\Omega)^3} + \|p\|_{H^{s-1}(\Omega)^3}\right) + N^{1-\sigma}\,\|\boldsymbol{f}\|_{H^\sigma(\Omega)^3}\right). \tag{4.8}$$

Proof: By combining Lemma 4.6 with problems (2.10) and (3.7), we easily derive that, for any q_N in $\mathbb{M}_N$,

$$\|p_N - q_N\|_{L^2(\Omega)} \le c\,N \Big(\|\boldsymbol{\omega} - \boldsymbol{\omega}_N\|_{L^2(\Omega)^3} + \|p - q_N\|_{L^2(\Omega)} + \sup_{\boldsymbol{v}_N \in \mathbb{X}_N} \frac{\int_\Omega \boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{v}_N(\boldsymbol{x})\, d\boldsymbol{x} - (\boldsymbol{f}, \boldsymbol{v}_N)_N}{\|\boldsymbol{v}_N\|_{\mathbb{X}(\Omega)}} \Big).$$

We conclude thanks to a triangle inequality, by using (4.2) and (4.4) and noting that the space $\mathbb{M}_N$ contains $\mathbb{P}_{[\lambda N]}(\Omega)$, hence has optimal approximation properties.

5. Extension to nonhomogeneous boundary conditions.

We now intend to handle the case of non-homogeneous boundary conditions which appear in problem (1.1). However, the new formulation (2.1) which is used in all this work leads to write them in a slightly different way:

$$\begin{cases} \nu\,\mathbf{curl}\,\boldsymbol{\omega} + \mathbf{grad}\,p = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{\omega} = \mathbf{curl}\,\boldsymbol{u} & \text{in } \Omega, \\ \boldsymbol{u} \cdot \boldsymbol{n} = g_n & \text{on } \partial\Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = \mathbf{g}_t & \text{on } \Gamma, \\ \boldsymbol{\omega} \times \boldsymbol{n} = \boldsymbol{k} \times \boldsymbol{n} & \text{on } \Gamma_m, \end{cases} \tag{5.1}$$

where the indices "n" and "t" stand for normal and tangential, respectively. With this new notation, the datum h in (1.1) is simply the restriction of g_n to Γ_m while the datum $\boldsymbol{g}$ on Γ is equal to $g_n\boldsymbol{n} - \boldsymbol{g}_t \times \boldsymbol{n}$.

We now set:

$$\overline{\mathbb{X}}(\Omega) = H(\operatorname{div}, \Omega) \cap H(\mathbf{curl}, \Omega), \tag{5.2}$$

and we consider the variational problem

Find $(\boldsymbol{\omega}, \boldsymbol{u}, p)$ in $L^2(\Omega)^3 \times \overline{\mathbb{X}}(\Omega) \times L^2_\circ(\Omega)$ such that

$$\boldsymbol{u} \cdot \boldsymbol{n} = g_n \quad \text{on } \partial\Omega \qquad \text{and} \qquad \boldsymbol{u} \times \boldsymbol{n} = \mathbf{g}_t \quad \text{on } \Gamma, \tag{5.3}$$

and that

$$\begin{aligned} &\forall \boldsymbol{v} \in \mathbb{X}(\Omega), \quad a(\boldsymbol{\omega}, \boldsymbol{u}; \boldsymbol{v}) + b(\boldsymbol{v}, p) = \int_\Omega \boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x})\, d\boldsymbol{x} - \nu\,\langle \boldsymbol{k}, \boldsymbol{v} \times \boldsymbol{n} \rangle_{\Gamma_m}, \\ &\forall q \in L^2_\circ(\Omega), \quad b(\boldsymbol{u}, q) = 0, \\ &\forall \boldsymbol{\varphi} \in L^2(\Omega)^3, \quad c(\boldsymbol{\omega}, \boldsymbol{u}; \boldsymbol{\varphi}) = 0, \end{aligned} \tag{5.4}$$

where $\langle \cdot, \cdot \rangle_{\Gamma_m}$ denotes the duality pairing between $H^{\frac{1}{2}}_{00}(\Gamma_m)^3$ and its dual space. The same arguments as for Proposition 2.1 yield that, for any data $(\boldsymbol{f}, g_n, \boldsymbol{g}_t, \boldsymbol{k})$ in $L^2(\Omega)^3 \times H^{-\frac{1}{2}}(\partial\Omega) \times H^{-\frac{1}{2}}(\Gamma)^3 \times H^{\frac{1}{2}}_{00}(\Gamma_m)^3$, any solution $(\boldsymbol{\omega}, \boldsymbol{u}, p)$ of problem (5.3) – (5.4) is a solution of problem (5.1).

The range of $\overline{\mathbb{X}}(\Omega)$ by the tangential operator: $\boldsymbol{v} \mapsto (\boldsymbol{v} \times \boldsymbol{n})_{|\Gamma}$ is rather difficult to characterize, see [15] for instance. For these reasons, we now state a lifting result which only holds for more regular data g_n and $\boldsymbol{g}_t$; this is not at all restrictive for the applications that we have in mind. On the other hand, we observe that these two functions together give rise to a vector field with components g_x and g_y defined on Γ and g_z defined on $\partial\Omega$.

Lemma 5.1. *For any data $(g_n, \boldsymbol{g}_t)$ in $L^2(\partial\Omega) \times L^2(\Gamma)^3$ satisfying*

$$\int_{\partial\Omega} g_n(\boldsymbol{\tau})\, d\boldsymbol{\tau} = 0, \tag{5.5}$$

and such that the components g_x, g_y and g_z of the function $\boldsymbol{g} = g_n\boldsymbol{n} - \boldsymbol{g}_t \times \boldsymbol{n}$ satisfy

$$g_x \in H^{\frac{1}{2}}(\Gamma), \qquad g_y \in H^{\frac{1}{2}}(\Gamma), \qquad g_z \in H^{\frac{1}{2}}(\partial\Omega), \tag{5.6}$$

there exists a divergence-free function $\boldsymbol{u}_b$ in $\overline{\mathbb{X}}(\Omega)$ such that

$$\boldsymbol{u}_b \cdot \boldsymbol{n} = g_n \quad \text{on } \partial\Omega \qquad \text{and} \qquad \boldsymbol{u}_b \times \boldsymbol{n} = \mathbf{g}_t \quad \text{on } \Gamma, \tag{5.7}$$

and which satisfies

$$\|\boldsymbol{u}_b\|_{\mathbb{X}(\Omega)} \leq c\left(\|g_x\|_{H^{\frac{1}{2}}(\Gamma)} + \|g_y\|_{H^{\frac{1}{2}}(\Gamma)} + \|g_z\|_{H^{\frac{1}{2}}(\partial\Omega)}\right). \tag{5.8}$$

If moreover the data are such that

$$g_x \in H^{\rho+\frac{1}{2}}(\Gamma), \qquad g_y \in H^{\rho+\frac{1}{2}}(\Gamma), \qquad g_z \in H^{\rho+\frac{1}{2}}(\partial\Omega), \tag{5.9}$$

for some real number ρ, $0 < \rho \leq 1$, the function $\boldsymbol{u}_b$ belongs to $H^{\rho+1}(\Omega)^3$ and satisfies

$$\|\boldsymbol{u}_b\|_{H^{\rho+1}(\Omega)^3} \leq c\left(\|g_x\|_{H^{\rho+\frac{1}{2}}(\Gamma)} + \|g_y\|_{H^{\rho+\frac{1}{2}}(\Gamma)} + \|g_z\|_{H^{\rho+\frac{1}{2}}(\partial\Omega)}\right). \tag{5.10}$$

Proof: Let $\overline{\boldsymbol{g}}_t$ stand for an extension of $\boldsymbol{g}_t$ to $\partial\Omega$ such that the function $\boldsymbol{g}^* = g_n\boldsymbol{n} - \overline{\boldsymbol{g}}_t \times \boldsymbol{n}$ belong to $H^{\frac{1}{2}}(\partial\Omega)^3$. Thus, we observe that the part $\boldsymbol{u}_b$ of the solution $(\boldsymbol{u}_b, p_b)$ of the Stokes problem with Dirichlet boundary conditions

$$\begin{cases} -\nu\,\Delta\boldsymbol{u}_b + \mathbf{grad}\,p_b = \mathbf{0} & \text{in } \Omega, \\ \operatorname{div}\boldsymbol{u}_b = 0 & \text{in } \Omega, \\ \boldsymbol{u}_b = \boldsymbol{g}^* & \text{on } \partial\Omega, \end{cases}$$

satisfies all properties stated in the first part of the lemma. The second part of the lemma follows from the regularity properties of the previous Stokes problem in a convex domain, see [19, §7.3].

Remark 5.2. Assumption (5.9) with $\rho > 0$ means that the functions g_x, g_y and g_z belong to $H^{\rho+\frac{1}{2}}(\gamma)$, for each face γ contained either in Γ or in $\partial\Omega$ and moreover that the restrictions of these functions to γ and γ' have the same trace on $\overline{\gamma} \cap \overline{\gamma}'$ for two such faces γ and γ' sharing an edge. However the case $\rho = 0$ is a limit case and these continuity conditions are only enforced in a weaker sense, see [7, Chap. I, Cor. 6.11].

By writing the problem satisfied by $(\boldsymbol{\omega}_0, \boldsymbol{u}_0, p)$, with $\boldsymbol{\omega}_0 = \boldsymbol{\omega} - \mathbf{curl}\,\boldsymbol{u}_b$ and $\boldsymbol{u}_0 = \boldsymbol{u} - \boldsymbol{u}_b$, we easily derive the analogue of Theorem 2.6.

Theorem 5.3. *For any data $(\boldsymbol{f}, g_n, \boldsymbol{g}_t, \boldsymbol{k})$ in $L^2(\Omega)^3 \times L^2(\partial\Omega) \times L^2(\Gamma)^3 \times H^{\frac{1}{2}}_{00}(\Gamma_m)^3$ satisfying (5.5) and (5.6), problem (5.3) – (5.4) has a unique solution $(\boldsymbol{\omega}, \boldsymbol{u}, p)$ in $L^2(\Omega)^3 \times \overline{\mathbb{X}}(\Omega) \times L^2_\circ(\Omega)$.*

We now define the new discrete space

$$\overline{\mathbb{X}}_N = \mathbb{P}_{N,N-1,N-1}(\Omega) \times \mathbb{P}_{N-1,N,N-1}(\Omega) \times \mathbb{P}_{N-1,N-1,N}(\Omega). \tag{5.11}$$

We assume that the data g_n and $\boldsymbol{g}_t$ are continuous on $\partial\Omega$ and $\overline{\Gamma}$, respectively. Next, we fix an interger j^*, $1 \leq j^* \leq N-1$, and set: $\mathcal{J}^* = \{0, 1, \ldots, N\} \setminus \{j^*\}$. We denote by

$i_{N-1}^{\partial\Omega}$ the interpolation operator defined as follows: For any continuous function φ on $\partial\Omega$ and for each face γ contained in $\partial\Omega$, $(i_{N-1}^{\partial\Omega}\varphi)_{|\gamma}$ belongs to $\mathbb{P}_{N-1}(\gamma)$ (with obvious notation for this space) and is equal to φ at all Gauss-Lobatto nodes $(\pm1, \xi_i, \xi_j)$ or $(\xi_i, \pm1, \xi_j)$ or $(\xi_i, \xi_j, \pm1)$, $(i, j) \in \mathcal{J}^* \times \mathcal{J}^*$, which are contained in $\overline{\gamma}$. Thus, the operator i_{N-1}^{Γ} is defined in an obvious way and the discrete product on Γ_m is defined on continuous functions u and v on $\overline{\Gamma}_m$ by

$$(u, v)_N^{\Gamma_m} = \sum_{i=0}^{N}\sum_{j=0}^{N} u(\xi_i, \xi_j)v(\xi_i, \xi_j)\,\rho_i\rho_j. \tag{5.12}$$

Finally, let ψ be the polynomial

$$\psi(x, y, z) = \frac{9}{16}(1 - x^2)(1 - y^2)\frac{1 + z}{2}. \tag{5.13}$$

It is readily checked that the trace of ψ on Γ_m belongs to $\mathbb{P}_2(\Gamma_m)$, vanishes on $\partial\Gamma_m$ and satisfies $\int_{\Gamma_m} \psi(\boldsymbol{\tau})\, d\boldsymbol{\tau} = 1$.

We are thus in a position to write the discrete problem

Find $(\boldsymbol{\omega}_N, \boldsymbol{u}_N, p_N)$ *in* $\mathbb{Y}_{\mathbb{N}} \times \overline{\mathbb{X}}_N \times \mathbb{M}_N$ *such that*

$$\begin{aligned} \boldsymbol{u}_N \cdot \boldsymbol{n} = i_{N-1}^{\partial\Omega} g_n - \Big(\int_{\partial\Omega} i_{N-1}^{\partial\Omega} g_n(\boldsymbol{\tau})\, d\boldsymbol{\tau}\Big)\, \psi \quad \text{on } \partial\Omega \\ \text{and} \qquad \boldsymbol{u}_N \times \boldsymbol{n} = i_{N-1}^{\Gamma} \mathbf{g}_t \quad \text{on } \Gamma, \end{aligned} \tag{5.14}$$

and that

$$\begin{aligned} &\forall \boldsymbol{v}_N \in \mathbb{X}_N, \quad a_N(\boldsymbol{\omega}_N, \boldsymbol{u}_N; \boldsymbol{v}_N) + b_N(\boldsymbol{v}_N, p_N) = (\boldsymbol{f}, \boldsymbol{v}_N)_N - \nu\, (\boldsymbol{k}, \boldsymbol{v}_N \times \boldsymbol{n})_N^{\Gamma_m}, \\ &\forall q_N \in \mathbb{M}_N, \quad b_N(\boldsymbol{u}_N, q_N) = 0, \\ &\forall \boldsymbol{\vartheta}_N \in \mathbb{Y}_N, \quad c_N(\boldsymbol{\omega}_N, \boldsymbol{u}_N; \boldsymbol{\vartheta}_N) = 0. \end{aligned} \tag{5.15}$$

Let $\mathcal{I}_{N-1}^*$ denote the Lagrange interpolation operator at all nodes (ξ_i, ξ_j, ξ_k), $(i, j, k) \in \mathcal{J}^* \times \mathcal{J}^* \times \mathcal{J}^*$, with values in $\mathbb{P}_{N-1}(\Omega)$. When the data g_n and $\boldsymbol{g}_t$ satisfy (5.9) with $\rho > \frac{1}{2}$, the function $\boldsymbol{u}_b$ exhibited in Lemma 5.1 is continuous on $\overline{\Omega}$. It is thus readily checked that the function

$$\boldsymbol{u}_{N0} = \Big(u_{Nx} - \mathcal{I}_{N-1}^* u_{bx}, u_{Ny} - \mathcal{I}_{N-1}^* u_{by}, u_{Nz} - \mathcal{I}_{N-1}^* u_{bz} + \Big(\int_{\partial\Omega} i_{N-1}^{\partial\Omega} g_n(\boldsymbol{\tau})\, d\boldsymbol{\tau}\Big)\, \psi\Big),$$

belongs to $\mathbb{X}_N(\Omega)$ but is no longer divergence-free. However, by combining the inf-sup condition (4.7) with the same arguments as for Theorem 5.3, we easily derive the next result.

Theorem 5.4. *For any data* $\boldsymbol{f}$ *continuous on* $\overline{\Omega}$, g_n *and* $\boldsymbol{g}_t$ *satisfying* (5.9) *with* $\rho > \frac{1}{2}$, *and* $\boldsymbol{k}$ *continuous on* $\overline{\Gamma}_m$, *problem* (5.14) – (5.15) *has a unique solution* $(\boldsymbol{\omega}_N, \boldsymbol{u}_N, p_N)$ *in* $\mathbb{Y}_N \times \overline{\mathbb{X}}_N \times \mathbb{M}_N$.

The same arguments as previously lead to error estimates. However we prefer to treat separately the error related to $\boldsymbol{u}_b$ and that related to $\boldsymbol{u}_0$ since the boundary conditions are most often more regular than the solution (other liftings than $\boldsymbol{u}_b$ can be considered for this, in order to avoid the restriction $\rho \leq 1$).

Theorem 5.5. *Assume that*
(i) the data $\boldsymbol{f}$ *belong to* $H^\sigma(\Omega)^3$, $\sigma > \frac{3}{2}$,
(ii) the data g_n *and* $\boldsymbol{g}_t$ *satisfy* (5.5) *and* (5.9) *with* $\rho > \frac{1}{2}$,
(iii) the data $\boldsymbol{k}$ *belong to* $H^\tau(\Gamma_m)^3$, $\tau > 1$,
(iv) the solution $(\boldsymbol{\omega}, \boldsymbol{u}, p)$ *of problem* (5.3) – (5.4) *is such that* $(\boldsymbol{u}, p)$ *belongs to* $H^s(\omega)^3 \times H^{s-1}(\Omega)$, $1 \leq s \leq \rho + 1$.
Then, for the space $\mathbb{M}_N$ *introduced in Definition* 4.5, *the following error estimate holds between this solution and the solution* $(\boldsymbol{\omega}_N, \boldsymbol{u}_N)$ *of problem* (5.14) – (5.15)*:*

$$\begin{aligned} \|\boldsymbol{\omega} - \boldsymbol{\omega}_N\|_{L^2(\Omega)^3} &+ \|\boldsymbol{u} - \boldsymbol{u}_N\|_{\mathbb{X}(\Omega)} + N^{-1}\,\|p - p_N\|_{L^2(\Omega)} \\ &\leq c\,\Big(N^{1-s}\,\big(\|\boldsymbol{u}\|_{H^s(\Omega)^3} + \|p\|_{H^{s-1}(\Omega)}\big) \\ &\qquad + N^{-\sigma}\,\|\boldsymbol{f}\|_{H^\sigma(\Omega))^3} + N^{\frac{1}{2}-\tau}\,\|\boldsymbol{k}\|_{H^\tau(\Gamma)^3} \\ &\qquad\qquad + N^{-\rho}\,\big(\|g_x\|_{H^{\rho+\frac{1}{2}}(\Gamma)} + \|g_y\|_{H^{\rho+\frac{1}{2}}(\Gamma)} + \|g_z\|_{H^{\rho+\frac{1}{2}}(\partial\Omega)}\big)\Big). \end{aligned} \tag{5.16}$$

As in Section 4, this estimate is fully optimal for both the vorticity and the velocity, but is not for the pressure. However the lack of optimality is the same in the simpler case of homogeneous Dirichlet conditions on the velocity.

6. Some numerical experiments.

We present successively the two-dimensional and the three-dimensional experiments.

In all cases, problem (3.7) results into a square linear system. Its size is the sum of the dimensions of $\mathbb{X}_N$, $\mathbb{Y}_N$ and $\mathbb{M}_N$, i.e., approximatively $4N^2$ in dimension $d = 2$ and $7N^3$ in dimension $d = 3$. For this reason, the numerical experiments are performed with $N = 50$ in dimension $d = 2$ but only with $N = 30$ in dimension $d = 3$. The global system is not symmetric and is solved by the GMRES method.

Two-dimensional computations

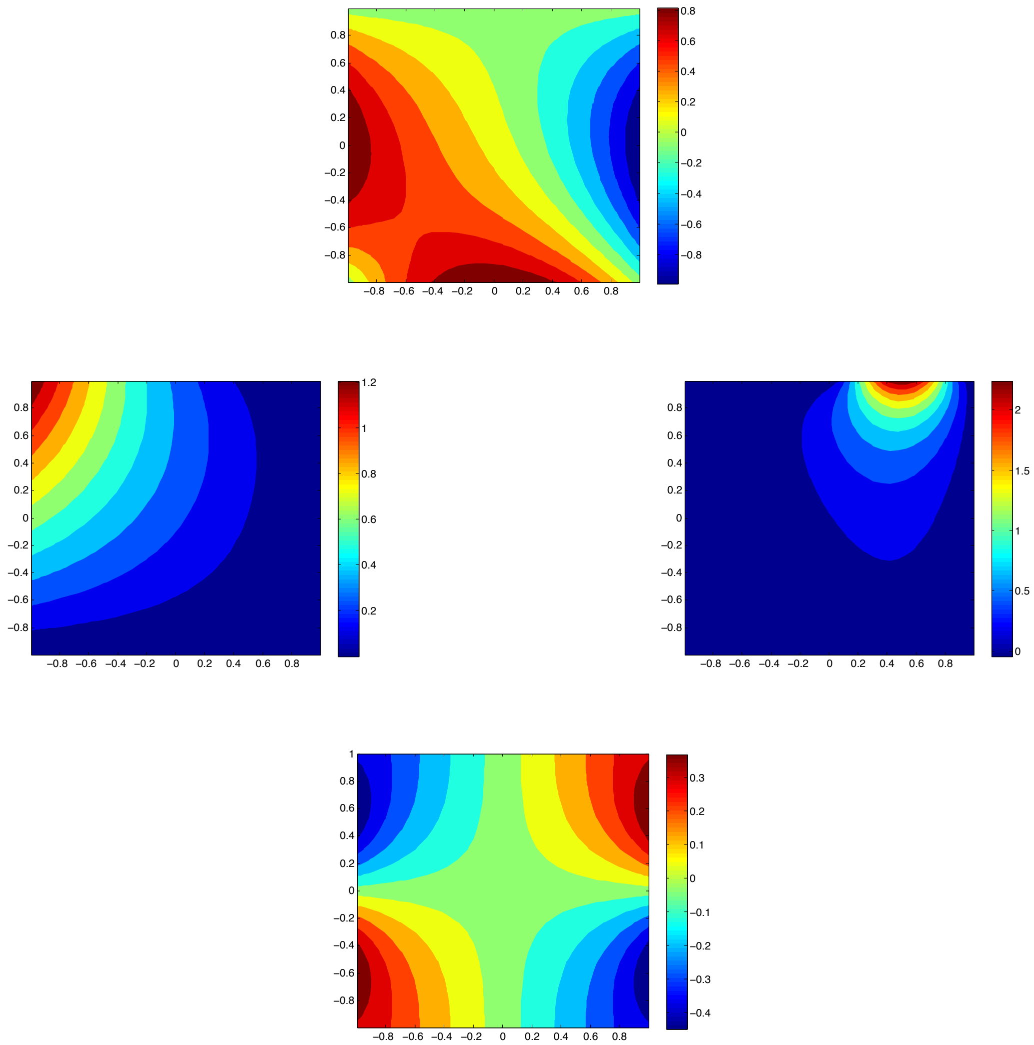

Figure 1: Isovalues of the vorticity, velocity and pressure for the data in (6.5) and $k = 0$

Just for this section, we work with

$$\Omega =]-1,1[^2, \qquad \Gamma_m =]-1,1[\times\{1\}. \tag{6.1}$$

We recall the main modifications with respect to the three-dimensional case:
1) The vorticity ω is now a scalar function in $L^2(\Omega)$, and the sixth line of (1.1) reads

$$\omega = k \quad \text{on } \Gamma_m. \tag{6.2}$$

This yields that the last term in the first line of problems (5.4) and (5.15) must be replaced by $\nu\,\langle k, \boldsymbol{v}\cdot\boldsymbol{\tau}\rangle_{\Gamma_m}$ and $\nu\,(k, \boldsymbol{v}\cdot\boldsymbol{\tau})_N^{\Gamma_m}$, respectively, where $\boldsymbol{\tau}$ denotes the tangential vector to Γ_m which is directly normal to $\boldsymbol{n}$.

2) According to the approach in [5], the discrete space of vorticities is the space $\mathbb{Y}_N = \mathbb{P}_N(\Omega)$. The space Z_N is now of dimension 2, spanned by the orthogonal projections onto $L^2_\circ(\Omega)$ of the polynomials

$$(L'_N \pm L'_{N-1})(x)\,\chi_N^-(y). \tag{6.3}$$

The space $\mathbb{M}_N$ is then defined thanks to an obvious analogue of Definition 4.5, and the inf-sup condition (4.7) remains valid in this case.

With these choices, the results of Theorems 4.4, 4.7 and 5.5 still hold.

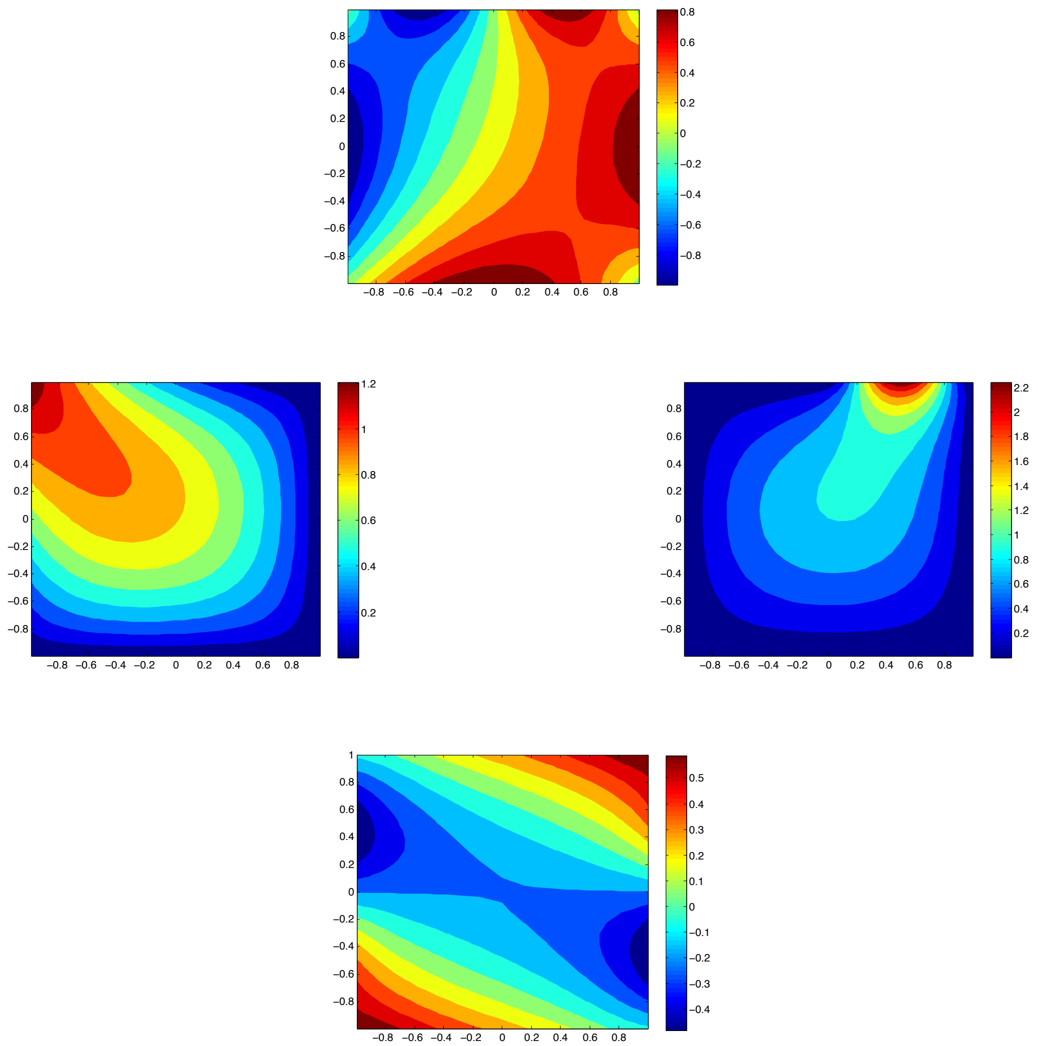

Figure 2: Isovalues of the vorticity, velocity and pressure for the data in (6.5) – (6.6)

We take the coefficient ν given by

$$\nu = 10^{-2}, \tag{6.4}$$

which corresponds to the viscosity of the water. The data $\boldsymbol{f}$ are fixed equal to zero, which means that the effects of gravity are neglected.

We first work with $k = 0$ and the boundary velocity $g = (g_x, g_y)$ given by

$$\begin{aligned} g_x(-1,y) &= \frac{2}{3}(1+y), \qquad g_y(-1,y) = 0, \\ g_x(x,-1) &= g_y(x,-1) = 0, \\ g_x(1,y) &= g_y(1,y) = 0, \\ \boldsymbol{h}skip6truecm g_y(x,1) &= \begin{cases} 0 & \text{if } -1 \le x \le 0, \\ 40x^2(1-x)^2 & \text{if } 0 \le x \le 1. \end{cases} \end{aligned} \tag{6.5}$$

Note that these data satisfy the compatibility condition (5.5). Figure 1 presents from top to bottom and from left to right the isovalues of the vorticity, the isovalues of the two components of the velocity and the isovalues of the pressure, computed with $N = 50$.

In a second step we still work with the data $\boldsymbol{g}$ given in (6.5) but now with the boundary vorticity k given by

$$k(x,1) = \sin(\pi x). \tag{6.6}$$

Figure 2 presents the same quantities as previously but now for these new data.

Three-dimensional computations

The two numerical experiments that we present in this Section are performed on the domain Ω and for the partition of its boundary given in (3.1). In both cases, we still take the coefficient ν defined by (6.4). The data $\boldsymbol{f}$ and $\boldsymbol{k}$ are fixed equal to zero, which means in particular that the effects of gravity are neglected.

In view of Section 5, we must now define the data g_n and $\boldsymbol{g}_t$.

(i) In the first experiment, we take g_y equal to 0 on Γ and g_z equal to 0 on $\partial\Omega$. The function g_x is given by

$$\begin{aligned} g_x(-1,y,z) &= \begin{cases} (1-y^2)z^{\frac{3}{2}}(1-z^{\frac{3}{2}}) & \text{if } 0 \le z \le 1, \\ 0 & \text{if } -1 \le z < 0, \end{cases} \qquad -1 \le y \le 1, \\ g_x(1,y,z) &= \frac{3}{40}(1-y^2)(1-z), \qquad -1 \le y,z \le 1, \\ g_x(x,-1,z) &= g_x(x,1,z) = 0, \qquad -1 \le x,z \le 1, \\ g_x(x,y,-1) &= \begin{cases} \frac{3}{20}x^2(1-y^2) & \text{if } 0 \le x \le 1, \\ 0 & \text{if } -1 \le x < 0, \end{cases} \qquad -1 \le y \le 1. \end{aligned} \tag{6.7}$$

It can be noted that g_n is not zero on the faces contained in the planes $x = \pm 1$ while $\boldsymbol{g}_t$ is not zero on the face contained in the plane $z = -1$. Moreover the function g_n satisfies condition (5.5).

Figure 3 presents from top to bottom and from left to right the isovalue curves of the three components of the vorticity, the three components of the velocity and the pressure in the plane $y = \frac{1}{2}$, for the boundary conditions given in (6.7).

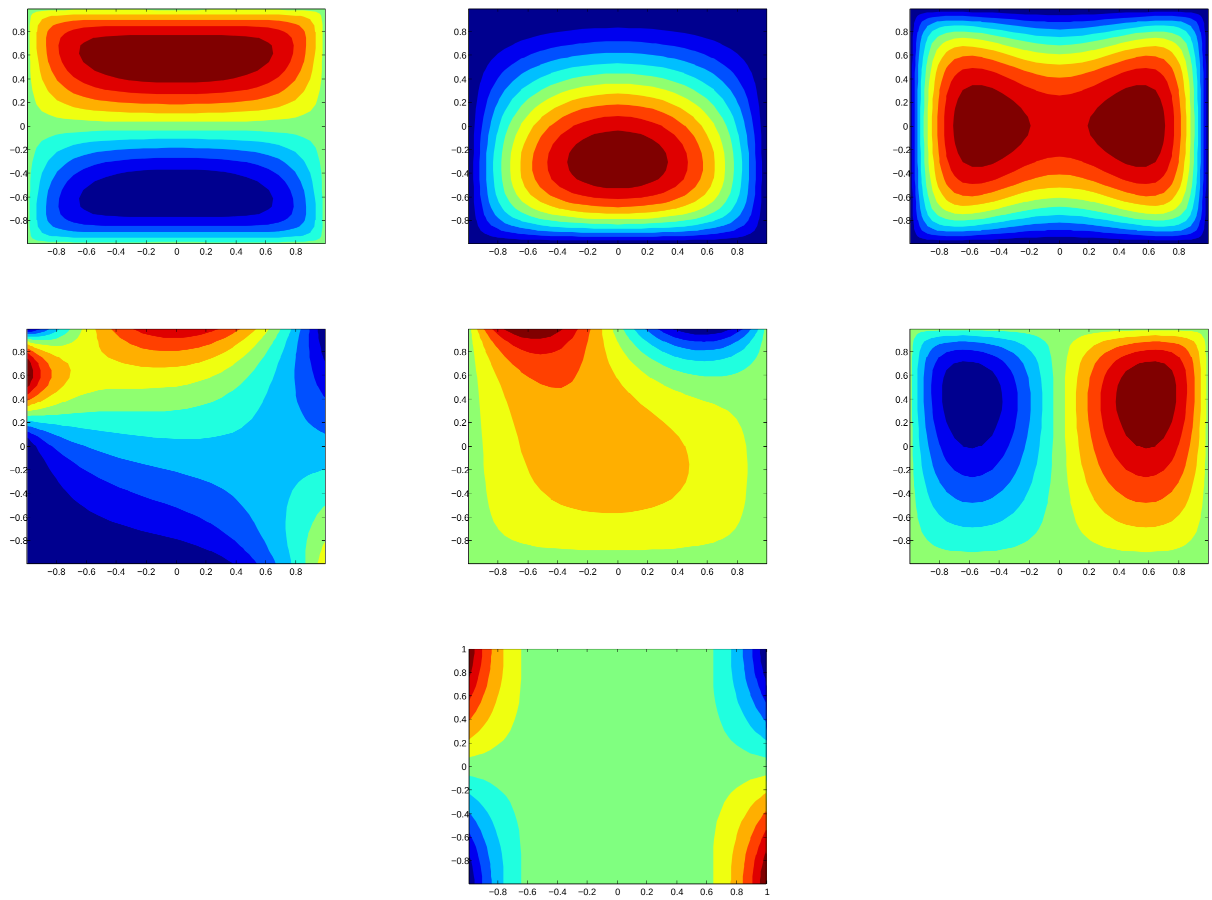

Figure 3: Isovalues of the vorticity, velocity and pressure for the data in (6.7)

(ii) In the second experiment, we work in the case where $\boldsymbol{g}_t$ is equal to zero on Γ and g_n is equal to zero on the faces contained in the planes $x = -1$, $y = \pm 1$ and $z = -1$. Otherwise, the function g_n is given by

$$g_x(1, y, z) = \begin{cases} \frac{256}{5}(1-y^2)^2(\frac{1}{2}+z)(1+z) & \text{if } -1 \leq z \leq -\frac{1}{2}, \\ 0 & \text{if } -\frac{1}{2} < z \leq 1, \end{cases} \quad -1 \leq y \leq 1, \tag{6.8}$$

$$g_z(x, y, 1) = (1-x^2)^2(1-y^2)^2, \qquad -1 \leq x, y \leq 1.$$

Thus, it still satisfies condition (5.5).

Figure 4 presents the same quantities as in Figure 3 but now for the boundary conditions given in (6.8).

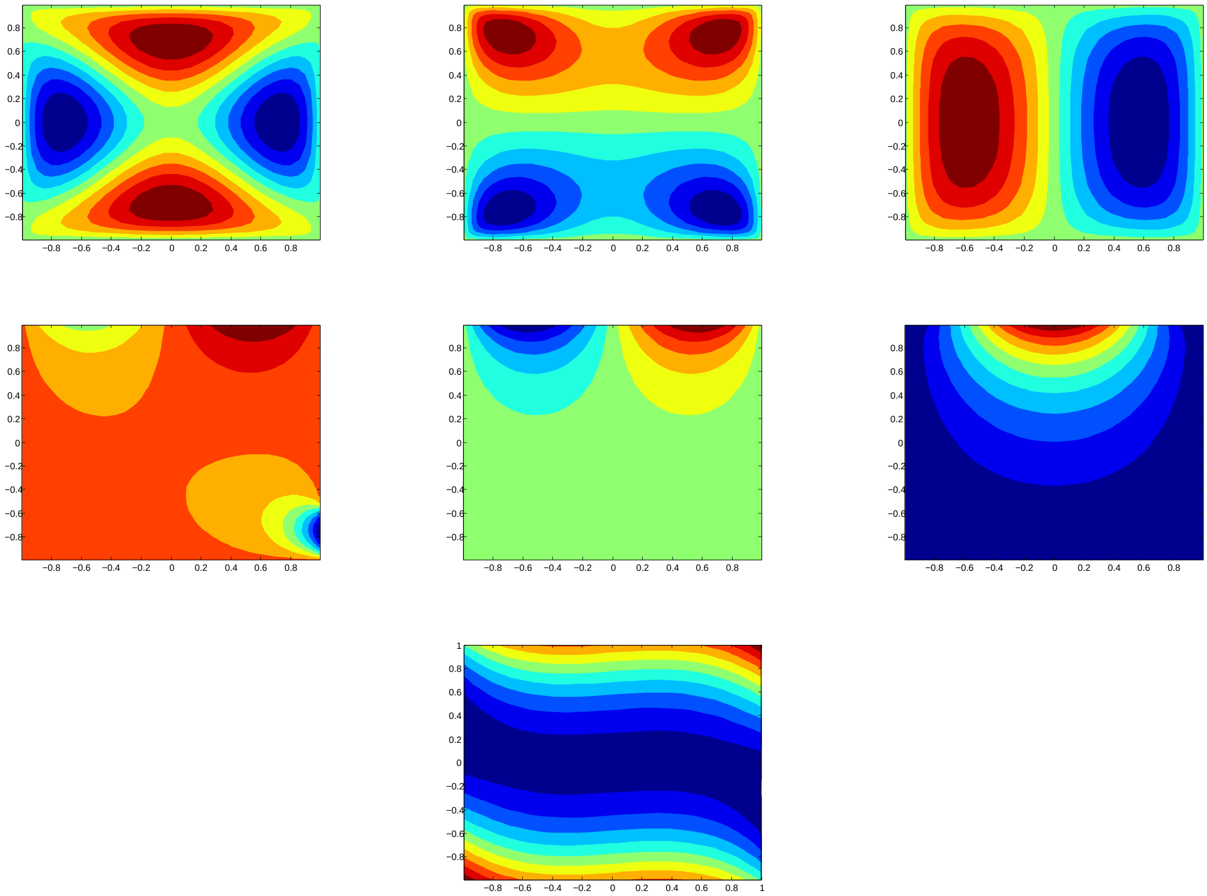

Figure 4: Isovalues of the vorticity, velocity and pressure for the data in (6.8)

Appendix

The aim of this appendix is to present the rather technical proofs of Lemmas 3.1 and 4.6. So, in a first step, we identify the space

$$Z_N = \{q_N \in L^2_\circ(\Omega) \cap \mathbb{P}_{N-1}(\Omega);\ \forall \boldsymbol{v}_N \in \mathbb{X}_N,\ b_N(\boldsymbol{v}_N, q_N) = 0\}. \qquad (a.1)$$

We introduce the polynomials

$$\chi_N^\pm(z) = \frac{L'_N(z)}{N+1} \pm \frac{L'_{N-1}(z)}{N-1}. \qquad (a.2)$$

Lemma a.1. *The space Z_N is spanned by the orthogonal projections onto $L^2_\circ(\Omega)$ of the polynomials*

$$(L'_N \pm L'_{N-1})(x)\,(L'_N \pm L'_{N-1})(y)\,\varphi_N(z), \qquad (L'_N \pm L'_{N-1})(x)\,\varphi_N(y)\,\chi_N^-(z) \quad \text{and} \quad \varphi_N(x)(L'_N \pm L'_{N-1})(y)\,\chi_N^-(z), \qquad (a.3)$$

where φ_N runs through $\mathbb{P}_{N-1}(-1,1)$.

Proof: We first observe from the boundary conditions in the definition of $\mathbb{X}_N$ that, for all $\boldsymbol{v}_N$ in $\mathbb{X}_N$ and q_N in $\mathbb{P}_{N-1}(\Omega)$, $b_N(\boldsymbol{v}_N, q_N)$ cancels if and only if $b_N(\boldsymbol{v}_N, q_N^\circ)$ cancels, where q_N° denotes the orthogonal projection of q_N onto $L^2_\circ(\Omega)$; so, only for this proof, we omit to enforce that q_N belongs to $L^2_\circ(\Omega)$. Owing to the definition of Z_N, any polynomial q_N in Z_N satisfies

$$\forall \boldsymbol{v}_N \in \big(H^1_0(\Omega) \cap \mathbb{P}_{N-1}(\Omega)\big)^3, \quad b(\boldsymbol{v}_N, q_N) = 0.$$

So it follows from [12, Remark 24.1] for instance that Z_N is contained in the sum of the space Z_N^1 spanned by the polynomials in $(a.3)$ and of the space Z_N^2 spanned by

$$\begin{aligned} &L_{N-1}(x), \quad L_{N-1}(y), \quad L_{N-1}(z),\\ &L_{N-1}(x)L_{N-1}(y), \quad L_{N-1}(x)L_{N-1}(z), \quad L_{N-1}(y)L_{N-1}(z),\\ &L_{N-1}(x)L_{N-1}(y)L_{N-1}(z)\\ &(L'_N \pm L'_{N-1})(x)\,\varphi_N(y)\,\chi_N^+(z) \quad \text{and} \quad \varphi_N(x)(L'_N \pm L'_{N-1})(y)\,\chi_N^+(z), \end{aligned} \qquad (a.4)$$

where φ_N runs through the subspace $\mathbb{P}^0_{N-1}(-1,1)$ of $\mathbb{P}_{N-1}(-1,1)$ made of polynomials vanishing in ± 1. So, it remains to check that Z_N^1 is contained in Z_N and that the intersection of Z_N and Z_N^2 is reduced to $\{0\}$.

1) To prove that Z_N^1 is contained in Z_N, we observe that any $\boldsymbol{v}_N = (v_{Nx}, v_{Ny}, v_{Nz})$ in $\mathbb{X}_N$ admits the expansion

$$v_{Nx}(x,y,z) = \sum_{m=1}^{N-2} \alpha_m(x,z)\,(1-y^2)L'_m(y),$$

$$v_{Ny}(x,y,z) = \sum_{m=1}^{N-2} \beta_m(y,z)\,(1-x^2)L'_m(x),$$

$$v_{Nz}(x,y,z) = \sum_{m=1}^{N-2} \gamma_m(y,z)\,(1-x^2)L'_m(x),$$

for polynomials α_m, β_m and γ_m with appropriate degree and satisfying appropriate nullity conditions on the edges of the square $]-1,1[^2$. As a consequence, we have

$$(\operatorname{div} \boldsymbol{v}_N)(x,y,z) = \sum_{m=1}^{N-2} (\partial_x \alpha_m)(x,z)\,(1-y^2) L'_m(y) + \sum_{m=1}^{N-2} (\partial_y \beta_m + \partial_z \gamma_m)(y,z)\,(1-x^2) L'_m(x).$$

We recall from [12, Remark 3.2] the orthogonality properties

$$\int_{-1}^{1} L'_m(\zeta) L'_N(\zeta)\,(1-\zeta^2)\,d\zeta = \int_{-1}^{1} L'_m(\zeta) L'_{N-1}(\zeta)\,(1-\zeta^2)\,d\zeta = 0, \qquad 1 \le m \le N-2, \tag{a.5}$$

so that $b(\boldsymbol{v}_N, q_N)$ is equal to zero for all $\boldsymbol{v}_N$ in $\mathbb{X}_N$ and q_N defined in $(a.3)$. Thus, all polynomials in Z_N^1 belong to Z_N.
2) Let now q_1^*, ... and q_7^* denote the first seven polynomials in $(a.4)$. Setting $\boldsymbol{v}_1^* = (v_{1x}^*, 0, 0)$, with

$$v_{1x}^* = (L_N - L_{N-2})(x)(L_2 - L_0)(y)(L_2 - L_0)(z),$$

we observe that $\boldsymbol{v}_1^*$ belongs to $\mathbb{X}_N$ and that, see [12, Thm 3.3],

$$(\operatorname{div} \boldsymbol{v}_1^*)(x,y,z) = (2N-1) L_{N-1}(x)(L_2 - L_0)(y)(L_2 - L_0)(z).$$

Thus, $b(\boldsymbol{v}_1^*, q_1^*)$ is negative and q_1^* does not belong to Z_N. The same result for q_2^* and q_3^* is obtained by exchanging the variables x, y and z and the components of the function $\boldsymbol{v}_1^*$. Similarly, by taking $\boldsymbol{v}_4^* = (v_{4x}^*, 0, 0)$, with

$$v_{4x}^* = (L_N - L_{N-2})(x)(L_{N-1} - L_{N-3})(y)(L_2 - L_0)(z),$$

we obtain that q_4^* does not belong to Z_N and the same result holds for q_5^* and q_6^*. Finally, taking $\boldsymbol{v}_7^* = (v_{7x}^*, 0, 0)$, with

$$v_{7x}^* = (L_N - L_{N-2})(x)(L_{N-1} - L_{N-3})(y)(L_{N-1} - L_{N-3})(z),$$

yields that q_7^* does not belong to Z_N. On the other hand, taking $\boldsymbol{v}^\sharp = (v_x^\sharp, 0, 0)$, with (note that χ_N^+ vanishes in $z = -1$)

$$v_x^\sharp(x,y) = (L_N - L_{N-2})(x)\varphi_N(y)\,\chi_N^+$$

gives

$$(\operatorname{div} \boldsymbol{v}^\sharp)(x,y,z) = (2N-1) L_{N-1}(x)\varphi_N(y)\chi_N^+.$$

Owing to the formula $L'_N = (2N-1)L_{N-1} + L'_{N-2}$, we derive that none of the polynomials in the first family in the last line of $(a.4)$ belongs to Z_N. The same property for the second family is derived by exchanging the variables x and y.
This concludes the proof.

It can be checked [12, Thm 24.1] that, up to the polynomials

$$(L'_N \pm L'_{N-1})(x)\,(L'_N \pm L'_{N-1})(y)\,\chi_N^-(z),$$

that appear thrice in $(a.3)$, all the polynomials in $(a.3)$ are linearly independent. This leads to the following statement.

Corollary a.2. *The space* Z_N *has dimension* $8(N-1)$.

From now on, for any function φ in $L^2(\Omega)$, we agree to denote by $[\varphi]^\circ$ its orthogonal projection on $L^2_\circ(\Omega)$, namely $[\varphi]^\circ = \varphi - \frac{1}{8}\int_\Omega \varphi(\boldsymbol{x})\,d\boldsymbol{x}$. We now intend to prove an inf-sup condition on the form $b(\cdot,\cdot)$ between $\mathbb{X}_N$ and the space $\mathbb{M}_N$ introduced in Definition 4.5. In order to do this, we observe that each q_N in $\mathbb{M}_N$ admits the expansion

$$q_N = q_N^1 + q_N^2 + q_N^3, \tag{a.6}$$

where
(i) q_N^1 belongs to the orthogonal space $\mathbb{M}_N^1$ of the space $\widetilde{Z}_N$ introduced in Definition 4.5 and of the polynomials introduced in $(a.4)$ in $L^2_\circ(\Omega) \cap \mathbb{P}_{N-1}(\Omega)$;
(ii) q_N^2 belongs to the space $\mathbb{M}_N^2$ spanned by the seven polynomials

$$\begin{aligned} L_{N-1}(x),\quad L_{N-1}(y),\quad L_{N-1}(z),\quad L_{N-1}(x)L_{N-1}(y),\quad L_{N-1}(x)L_{N-1}(z),\\ L_{N-1}(y)L_{N-1}(z),\quad \text{and}\quad L_{N-1}(x)L_{N-1}(y)L_{N-1}(z); \end{aligned} \tag{a.7}$$

(iii) q_N^3 belongs to the space $\mathbb{M}_N^3$ spanned by the polynomials

$$[(L'_N \pm L'_{N-1})(x)\,\varphi_N(y)\,\chi_N^+(z)]^\circ \quad \text{and} \quad [\varphi_N(x)(L'_N \pm L'_{N-1})(y)\,\chi_N^+(z)]^\circ, \tag{a.8}$$

where φ_N runs through $\mathbb{P}^0_{N-1}(-1,1)$.
Indeed, the main idea for proving the inf-sup condition relies on the Boland and Nicolaides argument [14].

We first recall from [12, Thms 24.7 & 24.9] the following result.

Lemma a.3. *For any* q_N^1 *in* $\mathbb{M}_N^1$*, there exists a function* $\boldsymbol{v}_N^1$ *in* $\mathbb{P}_{N-1}(\Omega)^3 \cap H^1_0(\Omega)^3$ *such that* $\operatorname{div}\boldsymbol{v}_N^1 = q_N^1$ *and*

$$\|\boldsymbol{v}_N^1\|_{H^1(\Omega)^3} \le c\,N\,\|q_N^1\|_{L^2(\Omega)}. \tag{a.9}$$

A further property, see [12, Remark 24.1], is that

$$\forall q_N^2 \in \mathbb{M}_N^2,\quad b(\boldsymbol{v}_N^1, q_N^2) = 0 \quad \text{and} \quad \forall q_N^3 \in \mathbb{M}_N^3,\quad b(\boldsymbol{v}_N^1, q_N^3) = 0. \tag{a.10}$$

Lemma a.4. *For any* q_N^2 *in* $\mathbb{M}_N^2$*, there exists a function* $\boldsymbol{v}_N^2$ *in* $\mathbb{X}_N \cap H^1_0(\Omega)^3$ *such that* $\operatorname{div}\boldsymbol{v}_N^2 = q_N^2$ *and*

$$\|\boldsymbol{v}_N^2\|_{H^1(\Omega)^3} \le c\,N\,\|q_N^2\|_{L^2(\Omega)}. \tag{a.11}$$

Proof: With the same notation as in the proof of Lemma a.1, we observe that each q_N^2 in $\mathbb{M}_N^2$ can be written as

$$q_N^2 = \sum_{i=1}^{7} \lambda_i\, q_i^*,$$

and that the q_i^* are orthogonal in $L^2(\Omega)$. Moreover, the same arguments as in the proof of Lemma a.1 yield that there exist functions $\boldsymbol{v}_i^*$ in $\mathbb{X}_N$ such that

$$b(\boldsymbol{v}_i^*, q_i^*) = \|q_i^*\|^2_{L^2(\Omega)} \qquad \text{and} \qquad \|\boldsymbol{v}_i^*\|_{H^1(\Omega)^3} \le c\,N\,\|q_i^*\|_{L^2(\Omega)}.$$

Thus, taking $\boldsymbol{v}_N^2$ equal to $\sum_{i=1}^7 \lambda_i\,\boldsymbol{v}_i^*$ yields the desired result.

There also, since all functions $\boldsymbol{v}_i^*$ belong to $H_0^1(\Omega)^3$, we have

$$\forall q_N^3 \in M_N^3 \quad b(\boldsymbol{v}_N^2, q_N^3) = 0. \qquad (a.12)$$

Lemma a.5. *For any q_N^3 in $\mathbb{M}_N^3$, there exists a function $\boldsymbol{v}_N^3$ in $\mathbb{X}_N$ such that* $\operatorname{div}\boldsymbol{v}_N^3 = q_N^3$ *and*

$$\|\boldsymbol{v}_N^3\|_{H^1(\Omega)^3} \le c\,N\,\|q_N^3\|_{L^2(\Omega)}. \qquad (a.13)$$

Proof: It is performed in two steps.
1) Each function q_N^3 in $\mathbb{M}_N^3$ can be written as

$$q_N^3 = [L_N'(x)\,\varphi_\sharp(y)\,\chi_N^+(z)]^\circ + [L_{N-1}'(x)\,\varphi_\flat(y)\,\chi_N^+(z)]^\circ \\ + [\psi_\sharp(x)L_N'(y)\,\chi_N^+(z)]^\circ + [\psi_\flat(x)L_{N-1}'(y)\,\chi_N^+(z)]^\circ,$$

where the polynomials $\varphi_\sharp$, $\varphi_\flat$, $\psi_\sharp$ and $\psi_\flat$ belong to $\mathbb{P}_{N-1}^0(-1,1)$. Moreover, it follows from $(a.5)$ and the fact that the polynomial $L_N'L_{N-1}'$ is odd that the four terms in the previous expansion are mutually orthogonal. So it suffices to prove the lemma with q_N^3 replaced by each of these terms.
2) Let us set

$$q_\sharp = [L_N'(x)\,\varphi_\sharp(y)\,\chi_N^+(z)]^\circ,$$

and take $\boldsymbol{v}^\sharp = (v_x^\sharp, 0, 0)$, with

$$v_x^\sharp(x,y) = -(L_N - L_*)(x)\varphi_\sharp(y)\,\chi_N^+,$$

and L_* equal to L_0 if N is even, to L_1 if N is odd. The same arguments as in the proof of Lemma a.1 lead to

$$\|q_\sharp\|_{L^2(\Omega)} \le \sqrt{N(N+1)}\,\|\varphi_\sharp\|_{L^2(-1,1)}\|\chi_N^+\|_{L^2(-1,1)},$$

and

$$b(\boldsymbol{v}^\sharp, q_\sharp) \ge N^2\|\varphi_\sharp\|^2_{L^2(-1,1)}\|\chi_N^+\|^2_{L^2(-1,1)}.$$

Finally, applying a standard inverse inequality [12, Thm 5.1] to $\varphi_\sharp$ and χ_N^+ gives

$$\|\boldsymbol{v}^\sharp\|_{H^1(\Omega)^3} \le c\,N^2\|\varphi_\sharp\|_{L^2(-1,1)}\|\chi_N^+\|_{L^2(-1,1)}.$$

Combining all this yields

$$b(\boldsymbol{v}^\sharp, q_\sharp) \ge c\,N^{-1}\|q_N^\sharp\|_{L^2(\Omega)}\|\boldsymbol{v}^\sharp\|_{H^1(\Omega)^3},$$

which is the desired result for $q_N^3 = q_\sharp$. Similar arguments applied to the three other terms in q_N^3 yield the inf-sup condition

$$\forall q_N \in \mathbb{M}_N^3, \qquad \sup_{\boldsymbol{v}_N \in \mathbb{X}_N} \frac{b_N(\boldsymbol{v}_N, q_N)}{\|\boldsymbol{v}_N\|_{\mathbb{X}(\Omega)}} \geq c\, N^{-1}\, \|q_N\|_{L^2(\Omega)}.$$

The desired property is then a direct consequence of this condition, see [18, Chap. I, Lemma 4.1].

We can now conclude thanks to the Boland and Nicolaides argument [14].

Lemma a.6. *For the space $\mathbb{M}_N$ introduced in Definition 4.5, there exists a constant $\beta_* > 0$ such that the following inf-sup condition holds*

$$\forall q_N \in \mathbb{M}_N, \qquad \sup_{\boldsymbol{v}_N \in \mathbb{X}_N} \frac{b_N(\boldsymbol{v}_N, q_N)}{\|\boldsymbol{v}_N\|_{\mathbb{X}(\Omega)}} \geq \beta_*\, N^{-1}\, \|q_N\|_{L^2(\Omega)}. \qquad (a.14)$$

Proof: Relying on the expansion $(a.6)$, we take for some positive constant λ

$$\boldsymbol{v}_N = -\boldsymbol{v}_N^1 - \lambda\, \boldsymbol{v}_N^2 - \lambda\, \boldsymbol{v}_N^3,$$

where the functions $\boldsymbol{v}_N^1$, $\boldsymbol{v}_N^2$ and $\boldsymbol{v}_N^3$ are exhibited in Lemmas a.3 to a.5. Indeed, it follows from these lemmas, $(a.10)$ and $(a.12)$ that

$$b_N(\boldsymbol{v}_N, q_N) = \|q_N^1\|^2_{L^2(\Omega)} + \lambda\, \|q_N^2\|^2_{L^2(\Omega)} + \lambda\, \|q_N^3\|^2_{L^2(\Omega)} \\ - \lambda\, b(\boldsymbol{v}_N^2, q_N^1) - \lambda\, b(\boldsymbol{v}_N^3, q_N^1) - \lambda\, b(\boldsymbol{v}_N^3, q_N^2).$$

Since each $\operatorname{div} \boldsymbol{v}_N^i$ is equal to q_N^i, it follows from the definition of $b(\cdot, \cdot)$ that

$$b_N(\boldsymbol{v}_N, q_N) \geq \|q_N^1\|^2_{L^2(\Omega)} + \lambda\, \|q_N^2\|^2_{L^2(\Omega)} + \lambda\, \|q_N^3\|^3_{L^2(\Omega)} \\ - \lambda\, \|q_N^1\|_{L^2(\Omega)} \|q_N^2\|_{L^2(\Omega)} - \lambda\, \|q_N^1\|_{L^2(\Omega)} \|q_N^3\|_{L^2(\Omega)} - \lambda\, \|q_N^2\|_{L^2(\Omega)} \|q_N^3\|_{L^2(\Omega)},$$

whence

$$b_N(\boldsymbol{v}_N, q_N) \geq \frac{1}{2} \|q_N^1\|^2_{L^2(\Omega)} + \lambda(\frac{1}{2} - \lambda)\, \|q_N^2\|^2_{L^2(\Omega)} + \lambda(\frac{1}{2} - \lambda)\, \|q_N^3\|^2_{L^2(\Omega)}.$$

When taking $\lambda = \frac{1}{4}$, we derive

$$b_N(\boldsymbol{v}_N, q_N) \geq \frac{1}{16} \big(\|q_N^1\|^2_{L^2(\Omega)} + \|q_N^2\|^2_{L^2(\Omega)} + \|q_N^3\|^2_{L^2(\Omega)}\big).$$

On the other hand, we derive from Lemmas a.3 to a.5 that

$$\|\boldsymbol{v}_N\|_{\mathbb{X}(\Omega)} \leq c\, N \big(\|q_N^1\|^2_{L^2(\Omega)} + \|q_N^2\|^2_{L^2(\Omega)} + \|q_N^3\|^2_{L^2(\Omega)}\big)^{\frac{1}{2}}.$$

We also have

$$\|q_N\|_{L^2(\Omega)} \leq \sqrt{3} \big(\|q_N^1\|^2_{L^2(\Omega)} + \|q_N^2\|^2_{L^2(\Omega)} + \|q_N^3\|^2_{L^2(\Omega)}\big)^{\frac{1}{2}}.$$

Combining the last three lines leads to the desired condition.

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