Rate of decay to 0 of the solutions to a nonlinear parabolic equation

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Résumé: On étudie l'ordre de convergence vers 0, quand $t \longrightarrow +\infty$ de la solution de l'équation $\psi_t - \Delta\psi + |\psi|^{p-1}\psi = 0$ avec les conditions aux limites de Neumann dans un ouvert connexe borné de $\mathbb{R}^n$ où $p > 1$. On montre que soit $\psi(t,\cdot)$ converge vers 0 exponentiellement, soit $\psi(t,\cdot)$ décroît comme $t^{-\frac{1}{(p-1)}}$.

Abstract: We study the decay rate to 0, as $t \longrightarrow +\infty$ of the solution of equation $\psi_t - \Delta\psi + |\psi|^{p-1}\psi = 0$ with Neumann boundary conditions in a bounded smooth open connected domain of $\mathbb{R}^n$ where $p > 1$. We show that either $\psi(t,\cdot)$ converges to 0 exponentially fast or $\psi(t,\cdot)$ decreases like $t^{-\frac{1}{(p-1)}}$.

Keywords: rate of decay, parabolic equation

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1 Introduction and main results.

In this paper we consider the following nonlinear parabolic equation

$$\begin{cases} \psi_t - \Delta\psi + g(\psi) = 0 \quad in \quad \mathbb{R}^+ \times \Omega, \\ \dfrac{\partial\psi}{\partial n} = 0 \quad on \quad \mathbb{R}^+ \times \partial\Omega. \end{cases} \tag{1.1}$$

where Ω is a bounded smooth open connected domain of $\mathbb{R}^n$ and $g \in C^1(\mathbb{R})$ satisfies

$$g(0) = 0 \tag{1.2}$$

and for some $p > 1$

$$\exists c > 0, \forall s \in \mathbb{R}, \qquad 0 \leq g'(s) \leq c|s|^{p-1}. \tag{1.3}$$

From (1.2)-(1.3)we deduce that $g(s)$ has the sign of s and

$$\forall s \in \mathbb{R}, \quad |g(s)| \leq \frac{c}{p}|s|^p. \tag{1.4}$$

We define the operator A by

$$D(A) = \{\psi \in H^2(\Omega), \frac{\partial\psi}{\partial n} = 0 \quad on \quad \partial\Omega\}$$

and

$$\forall \psi \in D(A), \quad A\psi = -\Delta\psi$$

On the other hand the operator B defined by

$$D(B) = \{\psi \in L^2(\Omega), -\Delta\psi + g(\psi) \in L^2(\Omega) \text{ and } \frac{\partial\psi}{\partial n} = 0 \quad on \quad \partial\Omega\}$$

and

$$\forall \psi \in D(B), \quad B\psi = -\Delta\psi + g(\psi)$$

is well-known to be maximal monotone in $L^2(\Omega)$. As a consequence of [1, 2] for any $\psi_0 \in L^2(\Omega)$ there exists a unique weak solution of the equation

$$\psi' + B\psi = 0 \quad on \quad \mathbb{R}^+; \quad \psi(0, x) = \psi_0. \tag{1.5}$$

In addition it is well known that if $\psi_0 \in L^\infty(\Omega)$, $\psi(t, \cdot)$ remains in $L^\infty(\Omega)$ for all $t > 0$. Finally [8] contains an estimate of the solution in $C(\bar{\Omega})$ and $C^1(\bar{\Omega})$ for $t > 0$, which is valid for any sufficiently regular domain.

Concerning the behaviour for t large, in [5], A.Haraux established in the case of a pure power nonlinearity the exponential convergence to 0 of the projection on the range of A of the solution of equation (1.1). Moreover in [4], the study of the equation $u'' + u' - \Delta u + g(u) = 0$ with Neumann boundary conditions and where g satisfies $\exists C, c > 0, \forall s \in \mathbb{R},\ c|s|^{p-1} \leq g'(s) \leq C|s|^{p-1}$ for some $p > 1$, showed that either $u(t)$ converges to 0 exponentially fast, or $\|u(t)\|_{H_0^1(\Omega)} \geq \gamma t^{-1/(p-1)}$ with $\gamma > 1$ for $t \geq 1$.

Several authors have treated some variants of equation (1.1), for example in [6] with $g(u) = c|u|^{p-1}u - \lambda_1 u$ and with Dirichlet boundary conditions, was studied the decay rate at the infinity of solutions to (1.1), where $\lambda_1 > 0$ is the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$. The result obtained there is optimal for positive solutions.

According to La Salle's invariance principle, cf. [3, 7], any solution ψ of (1.1), having a precompact range on $\mathbb{R}^+$ with values to $L^\infty(\Omega)$, converges to a continuum of stationary solutions of equation (1.1), which reduces here to the constants of some sub-interval of $g^{-1}(0)$. By monotonicity, it is in fact known that ψ converges to some constant $a \in g^{-1}(0)$.

Our first result is valid without any additional hypothesis on g

Theorem 1.1. *Let g satisfy* (1.2) *and* (1.3). *Then any solution ψ of (1.1) satisfies the following alternative as $t \to \infty$: either*

$$\|\psi(t,\cdot)\|_\infty \leq Ce^{-\lambda_2 t}, \tag{1.6}$$

or

$$\exists c' > 0, \forall t \geq 1, \quad \Big|\int_\Omega \psi(t,x)dx\Big| \geq c' t^{-\frac{1}{p-1}}, \tag{1.7}$$

where $\lambda_2 > 0$ is the second eigenvalue of A in $D(A)$.

Our second result provides a more accurate estimate when $g(\psi) = |\psi|^{p-1}\psi$

Theorem 1.2. *Let us consider the nonlinear parabolic problem* (1.1) *with $g(\psi) = |\psi|^{p-1}\psi$, then any solution ψ of* (1.1) *satisfies the following alternative as $t \to \infty$: either*

$$\|\psi(t,\cdot)\|_\infty \leq Ce^{-\lambda_2 t}, \tag{1.8}$$

or

$$\forall t \geq 1, \qquad \||\psi(t,\cdot)| - ((p-1)t)^{-\frac{1}{p-1}}\|_\infty \leq K t^{-\frac{1}{(p-1)}-1}, \tag{1.9}$$

where $K, C > 0$, $p > 1$ and $\lambda_2 > 0$ is the second eigenvalue of A in $D(A)$.

In the following Proposition, we consider two special cases showing that both possibilities in the second result in the Theorem 1.1 can actually happen.

Proposition 1.3. *Let g satisfy* (1.2) *and* (1.3)*. Then we have*

- (i) *If Ω is symmetric around 0, g is odd and $\psi(0,\cdot)$ is an odd function in Ω, then any solution of* (1.1) *satisfies* (1.6)*.*
- (ii) *If $\psi(0,\cdot) \geq 0$ and ψ does not vanish a.e in Ω, then any solution of* (1.1) *satisfies* (1.7)*.*

Finally, our last result shows that the second possibility is sharp for a class of functions g more general than the pure power

Proposition 1.4. *Under the additional hypothesis*

$$\exists k_1 > 0, \forall s \in \mathbb{R}, |g(s)| \geq k_1 |s|^p \tag{1.10}$$

for any solution ψ of (1.1)*, we have*

$$\forall t \geq 1, \quad \|\psi(t,\cdot)\|_\infty \leq \left\{ \frac{1}{k_1(p-1)} \right\}^{\frac{1}{p-1}} t^{-\frac{1}{p-1}} \tag{1.11}$$

2 Proof of Proposition 1.4.

Proof. If $\psi(0,\cdot) = 0$, we have $\psi(t,\cdot) \equiv 0$ and the result is obvious. Otherwise let z be defined by

$$z(t) = \left\{ \frac{1}{\|\psi(0,\cdot)\|_\infty^{1-p} + k_1(p-1)t} \right\}^{\frac{1}{p-1}}$$

Then z is a solution of the following nonlinear ODE problem

$$\left\{ \begin{array}{l} z' + k_1 z^p = 0, \\ z(0) = \|\psi(0,\cdot)\|_\infty. \end{array} \right. \tag{2.1}$$

Under the additional condition(1.10), we will show that z is a super solution of (1.1). Indeed, we have

$$\begin{aligned} z_t - \Delta z + g(z) &= -k_1(z(0)^{1-p} + k_1(p-1)t)^{-\frac{p}{p-1}} + g(z) \\ &\geq -k_1(z(0)^{1-p} + k_1(p-1)t)^{-\frac{p}{p-1}} + k_1|z|^p \\ &\geq 0. \end{aligned}$$

Since $\psi(0,\cdot) \leq z(0)$ we deduce, by standard comparison principle, that $\psi(t,\cdot) \leq z(t)\ \forall t \geq 1$.
A similar calculation shows that $\psi(t,\cdot) \geq -z(t), \quad \forall t \geq 1$ which concludes the proof. ■

3 A general result on the range component.

Defining the orthogonal projection $P : H \longrightarrow N$, where

$$H = L^2(\Omega), N = \ker(A) \text{ and } P\psi(t,\cdot) = \frac{1}{|\Omega|}\int_\Omega \psi(t,x)dx,$$

as already mentioned in the introduction, [5] showed that for $g(\psi) = |\psi|^{p-1}\psi$, the following estimate holds

$$\|\psi(t) - P\psi(t)\|_{L^2(\Omega)} \leq Ke^{-\lambda_2 t},$$

for some constant $K > 0$. In this section, we will show that we have the same result for any function g satisfying (1.3).

Proposition 3.1. *Let $\psi \in C(\mathbb{R}^+, L^\infty)$ be any solution of (1.1). Assume that g is a locally Lipschitz non-decreasing function. Then we have ,*

$$\|\psi(t) - P\psi(t)\|_2 \leq \|\psi(0) - P\psi(0)\|_2\ e^{-\lambda_2 t}, \tag{3.1}$$

where $\|.\|_2$ denotes the norm in $L^2(\Omega)$ and $\lambda_2 > 0$ is the second eigenvalue of $-\Delta$ in $L^2(\Omega)$ with Neumann boundary conditions .

Proof. We denote by (u, v) the inner product of two functions u, v of $L^2(\Omega)$. Since g is a nondecreasing function, for all $\psi \in L^\infty(\Omega)$, we have a.e. in $x \in \Omega$

$$(g(\psi) - g(P\psi))(\psi - P\psi) \geq 0$$

and then by integrating over Ω

$$(g(\psi), \psi - P\psi) - (g(P\psi), \psi - P\psi) \geq 0. \tag{3.2}$$

Since $g(P\psi)$ is a constant and $(\psi - P\psi) \in N^\perp$, we deduce that $(g(P\psi), \psi - P\psi) = 0$. Hence from (3.2),

$$(g(\psi), \psi - P\psi) \geq 0.$$

Setting

$$w = \psi - P\psi,$$

we have since $\Delta P\psi = P\Delta\psi = 0$

$$w' - \Delta w = \psi' - \Delta\psi - P\psi' + \Delta P\psi = (I-P)(\psi' - \Delta\psi) = -(I-P)g(\psi).$$

Thus , since $(w,(I-P)g(\psi)) = ((I-P)w, g(\psi)) = (w, g(\psi)) = (g(\psi), \psi - P\psi)$ we find

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|_2^2 = (w, \Delta w) - (g(\psi), \psi - P\psi) \leq -\lambda_2 |w|^2.$$

By integrating we obtain (3.1). ∎

4 Proof of Theorem 1.1.

We set $\psi = u + w$, where $u = P\psi$ and $w = (I-P)\psi$. By projecting (1.1) on N we obtain

$$u' + P(g(\psi)) = 0, \tag{4.1}$$

where we have used that $P(A\psi) = 0$, since $R(A) \subset N^{\perp}$. Noticing that

$$u' + P(g(u)) + P(g(\psi) - g(u)) = u' + g(u) + P(g(\psi) - g(u)),$$

we can rewrite the equation (4.1) as

$$u' + g(u) = -P(g(\psi) - g(u)).$$

By the assumption (1.3), we deduce that

$$|P(g(\psi) - g(u))| \leq \frac{1}{|\Omega|}\|g(\psi) - g(u)\|_1 \leq \frac{c}{|\Omega|}(\|\psi\|_{2p-2}^{p-1} + \|u\|_{2p-2}^{p-1})\|w\|_2.$$

But ψ and u are uniformly bounded and from Proposition 3.1 we have the estimate $\|w(t)\|_2 \leq Ke^{-\lambda_2 t}$. Therefore

$$|P(g(\psi) - g(u))| \leq K' e^{-\lambda_2 t},$$

with $K' > 0$. Which leads us to study the equation:

$$u' + g(u) = f(t) \quad \text{in} \quad \mathbb{R}^+, \tag{4.2}$$

where

$$f(t) = P(g(\psi) - g(u))$$

and

$$|f(t)| \leq K' e^{-\lambda_2 t}.$$

Using the same method as in [5], we show the following result:

Lemma 4.1. *Let $c > 0$, $\gamma > 0$, $p > 1$ and g satisfying* (1.2) *and* (1.3) *Let $M > 0$ such that*

$$M \leq \left(\frac{\gamma}{2c}\right)^{\frac{1}{p-1}}, \tag{4.3}$$

$c_1 > 0$ with

$$c_1 \leq \frac{\gamma}{2}M,$$

then for every function f satisfying

$$|f(t)| \leq c_1 e^{-\gamma t},$$

there exists a unique function $v \in C^1(\mathbb{R}^+)$ satisfying

$$\forall t \geq 0, \quad v' + g(v) = f(t) \tag{4.4}$$

and

$$\sup_{t\in(0,+\infty)} \left\{e^{\gamma t}|v(t)|\right\} \leq M. \tag{4.5}$$

Proof. Since any solution of (4.4).(4.5) satisfies the integral equation

$$v(t) = -\int_t^{+\infty} (f(s) - g(v(s))ds. \tag{4.6}$$

We look for a solution of (4.6). It is then natural to introduce the following function space :

$$X = \{v \in C(0, +\infty); \sup_{t\in(0,+\infty)} e^{\gamma t}|v(t)| \leq M\},$$

equipped with the distance associated to the norm

$$\|v\|_\gamma = \sup_{t\in(0,+\infty)} e^{\gamma t}|v(t)|. \tag{4.7}$$

We consider the operator $\mathcal{T} : X \to C(0, +\infty)$ defined by

$$\mathcal{T}v(t) = -\int_t^{+\infty} (f(s) - g(v(s))ds.$$

From (1.4), we have the estimate

$$\forall s \in \mathbb{R}^+; \quad |g(v(s))| \leq \frac{c}{p}|v(s)|^p.$$

First we will show that $\mathcal{T}(X) \subset X$. Let $v \in X$, then for all $t \geq 0$,

$$\begin{aligned}
|\mathcal{T}v(t)| &\leq \int_t^{+\infty} |f(s)|ds + \int_t^{+\infty} |g(v(s))|ds \\
&\leq \int_t^{+\infty} |f(s)|ds + \frac{c}{p}\int_t^{+\infty} |v(s)|^p ds \\
&\leq \frac{c_1}{\gamma}e^{-\gamma t} + \frac{c}{p}M^p \int_t^{+\infty} e^{-p\gamma s}ds \\
&\leq \left(\frac{c_1}{\gamma} + \frac{cM^p}{p^2\gamma}\right) e^{-\gamma t} \\
&\leq \left(\frac{M}{2} + Mc\frac{M^{p-1}}{p^2\gamma}\right) e^{-\gamma t} \\
&\leq \left(\frac{M}{2} + \frac{M}{2p^2}\right) e^{-\gamma t}.
\end{aligned}$$

Since $p > 1$, it follows that

$$|\mathcal{T}v(t)| \leq Me^{-\gamma t}.$$

Hence by (4.5), we obtain that $\mathcal{T}v \in X$, with

$$\|\mathcal{T}v(t)\|_\gamma \leq M.$$

Secondly, we will prove that $\mathcal{T}$ is a contraction on X. In fact, for $x, \bar{x} \in X$ and for all $t \geq 0$

$$\begin{aligned}
|\mathcal{T}x(t) - \mathcal{T}\bar{x}(t)| &\leq \int_t^{+\infty} |g(x(s)) - g(\bar{x}(s))|ds \\
&\leq cM^{p-1}\int_t^{+\infty} e^{-p\gamma s}e^{\gamma s}|x(s) - \bar{x}(s)|ds \\
&\leq \frac{cM^{p-1}}{p\gamma}\|x - \bar{x}\|_\gamma e^{-\gamma t}.
\end{aligned}$$

Then we have

$$|\mathcal{T}x(t) - \mathcal{T}\bar{x}(t)|e^{\gamma t} \leq \frac{cM^{p-1}}{p\gamma}\|x - \bar{x}\|_\gamma.$$

Therefore, since M^{p-1} satisfies (4.3), we conclude that $\forall x, \bar{x} \in X$

$$\|\mathcal{T}x - \mathcal{T}\bar{x}\|_\gamma \leq \frac{1}{2}\|x - \bar{x}\|_\gamma.$$

Thus $\mathcal{T}$ is a $\frac{1}{2}$-Lipschitz functional on the complete metric space X and the result follows from the Banach fixed point theorem. From (4.6) it follows easily that v satisfies (4.4). Then the uniqueness of v follows from the uniqueness of the solution of (4.6) ($\mathcal{T}$ is a contraction) and the fact that any solution of (4.4) satisfies (4.6). The existence comes from the fact that conversely any solution of (4.6) satisfies (4.4). ■

Proof of Theorem 1.1 continued. Consequently, we have a solution v that satisfies the equation (4.4) and for all $t \geq T_0$

$$|v(t)| \leq Me^{-\lambda_2 t}, \tag{4.8}$$

where $M = M'e^{\gamma T_0}$ and $M' > 0$. If we subtract (4.4) from (4.2) we obtain

$$(u-v)' + g(u) - g(v) = 0.$$

Setting $z = u - v$, we complete the proof analyzing two cases.

1^{st} case: If $z(T_0) = 0$, then for all $t \geq T_0$, $z(t) = 0$. Hence $u \equiv v$ and from (4.8) it follows that

$$|u(t)| \leq Me^{-\lambda_2 t}.$$

Then, using (3.1), we obtain

$$\|\psi(t)\|_2 \leq M'e^{-\lambda_2 t}.$$

Finally by reasoning as in [5], [6] we obtain (1.6).

2^{nd} case: If $z(T_0) \neq 0$ then $\forall t \geq T_0, z(t) \neq 0$ and we have

$$z'(t) + \frac{g(u(t)) - g(v(t))}{u(t) - v(t)} z(t) = 0.$$

Since g is a monotonic function,

$$\alpha(t) := \frac{g(u(t)) - g(v(t))}{u(t) - v(t)}$$

is a strictly positive function. Moreover, $\exists \theta \in]0, 1[$,

$$\alpha(t) = g'(\theta u(t) + (1-\theta)v(t)) \leq c|\theta u(t) + (1-\theta)v(t)|^{p-1} \tag{4.9}$$

We distinguish 2 cases:
– $p > 2$ then by convexity of the pth power we have

$$|\theta u(t) + (1-\theta)v(t)|^{p-1} \leq \theta|u|^{p-1} + (1-\theta)|v|^{p-1} \leq |u|^{p-1} + |v|^{p-1}.$$

– $1 < p < 2$ we study the function $(x+y)^a - x^a$ for $0 < a < 1$ and $x, y > 0$ we prove that $X \longrightarrow (1+X)^a - X^a$ is a decreasing function on $(0, +\infty)$, we deduce $0 < (1+X)^a - X^a < 1$ and it follows by homogeneity that $(x+y)^a < x^a + y^a$ by letting $X = \frac{x}{y}$. Then we conclude that $|\theta u(t) + (1-\theta)v(t)|^{p-1} \leq |u|^{p-1} + |v|^{p-1}$. Consequently we obtain

$$\forall p > 1, \quad \alpha(t) \leq c(|u|^{p-1} + |v|^{p-1}). \tag{4.10}$$

Setting $y = |z|$, we have that

$$y' + \alpha(t)y \geq 0.$$

Then the estimate (4.10) implies that

$$y' \geq -c(|u|^{p-1} + |v|^{p-1})y \geq -c(|z+v|^{p-1} + |v|^{p-1})y.$$

Hence there exists some constants $c_2, c_3 > 0$ such that

$$y' \geq -c_2(|z|^{p-1})y - c_3|v|^{p-1}y.$$

Since $y = |z|$,

$$y' \geq -c_2 y^p - c_3|v|^{p-1}y.$$

Putting $a(t) = c_3|v|^{p-1}$, we deduce that

$$y' + a(t)y \geq -c_2 y^p. \tag{4.11}$$

We set

$$A(t) = -c_3 \int_t^{+\infty} |v|^{p-1} ds, \qquad \omega(t) = e^{A(t)}y$$

and by replacing ω in (4.11), we obtain

$$\omega(t) \geq \left\{ \frac{1}{\omega(0)^{1-p} + (p-1)c_4 t} \right\}^{\frac{1}{p-1}}$$

with $c_4 = \left(\int_0^{+\infty} a(s)ds \right)^{-(p-1)}$. Then for t large enough we have

$$\omega(t) \geq Kt^{-\frac{1}{p-1}}.$$

Since $t \longrightarrow e^{A(t)}$ is a bounded function, we conclude (1.7) by observing that $u = z + v$ and v tends to 0 exponentially at infinity.

■

5 Proof of Theorem 1.2.

Considering $g(\psi) = |\psi|^{p-1}\psi$, g satisfies (1.2) and (1.3) with $c = p$. Hence Lemma 4.1 is applicable with $c = p$, therefore we assume

$$M \leq \left(\frac{\lambda_2}{2p}\right)^{\frac{1}{p-1}}.$$

In (4.2), we replace $g(\psi) = |\psi|^{p-1}\psi$, we can subtract (4.4) from (4.2), we deduce:

$$(u - v)' + |u|^{p-1}u - |v|^{p-1}v = 0. \tag{5.1}$$

We will study two cases.

<u>1^{st} case</u>: If $z(T_0) = 0$ then for all $t \geq T_0, z(t) = 0$. Hence $u \equiv v$ and from (4.8) it follows that

$$|u(t)| \leq Me^{-\lambda_2 t}.$$

Moreover, using (3.1), we obtain (1.8).

<u>2^{nd} case</u>: if $z(T_0) \neq 0$ then for all $t \geq T_0$ $z(t) \neq 0$ and we have

$$z'(t) + |u(t)|^{p-1}u(t) - |v(t)|^{p-1}v(t) = z'(t) + \alpha(t)z(t) = 0.$$

with

$$\begin{aligned}
\alpha(t) &= \frac{|u(t)|^{p-1}u(t) - |v(t)|^{p-1}v(t)}{u(t) - v(t)} \\
&= \frac{|z(t) + v(t)|^{p-1}(z(t) + v(t)) - |v(t)|^{p-1}v(t)}{z(t)} \\
&= \frac{|z(t)|^{p-1}|1 + \frac{v(t)}{z(t)}|^{p-1}(z(t) + v(t)) - |v(t)|^{p-1}v(t)}{z(t)}.
\end{aligned}$$

In that case $\alpha(t) > 0$, indeed $t \mapsto |u(t)|^{p-1}u(t)$ is non decreasing function. By applying Taylor's formula, we obtain

$$\alpha(t) = |z(t)|^{p-1} + \beta(t), \tag{5.2}$$

with $|\beta(t)| \leq Be^{-\eta t}$, where $\eta > 0$ is any positive number smaller than $(p-1)\lambda_2$ and $B > 0$. Replacing α by its expression in (5.2), equation (5.1) becomes

$$z' + |z|^{p-1}z + \beta(t)z = 0. \tag{5.3}$$

Let $y = |z|$, we obtain

$$y' + y^p + \beta(t)y = 0. \tag{5.4}$$

Setting $\xi(t) = e^{A(t)}y(t)$, with $A(t) = -\int_t^{+\infty} \beta(s)ds$ we find

$$|A(t)| \leq \int_t^{+\infty} |\beta(s)|ds \leq \frac{B}{\eta} e^{-\eta t}. \tag{5.5}$$

By Taylor's formula we have

$$\forall h \in [-1, 1], \quad |e^h - 1| \leq 2|h|, \tag{5.6}$$

we can give the following estimate

$$|\xi(t) - y(t)| \leq y(t)|e^{A(t)} - 1| \leq 2|A(t)|y(t) \leq \frac{2B}{\eta}\|y\|_\infty e^{-\eta t},$$

we conclude

$$|\xi(t) - y(t)| \leq ke^{-\delta t}, \tag{5.7}$$

where $\delta = \eta$ and $k = \frac{2B}{\eta}\|y\|_\infty$.
Replacing $\xi(t)$ in (5.3), we have

$$\begin{aligned}
&- e^{-A(t)}\xi'(t) = e^{-pA(t)}\xi^p(t) \\
&\Leftrightarrow -\frac{\xi'(t)}{\xi^p(t)} = e^{(1-p)A(t)} \\
&\Leftrightarrow (\frac{1}{\xi^{p-1}(t)})' = (p-1)e^{(1-p)A(t)}.
\end{aligned}$$

$t \mapsto e^{-(p-1)A(t)}$ is bounded and tends to 1 at infinity, ξ is given by

$$\xi(t)^{p-1} = \frac{1}{\xi(t_0)^{1-p} + (p-1)\int_{t_0}^t e^{-(p-1)A(s)}ds}.$$

we set

$$D(t) = \xi(t_0)^{1-p} + (p-1)\int_{t_0}^t e^{-(p-1)A(s)}ds \quad \text{and} \quad h(t) = -(p-1)A(t),$$

then we show that $D(t) - (p-1)t$ is bounded.

From (5.5), we know that $t \mapsto h(t)$ is an integrable function, then by (5.6), $(e^{-(p-1)A(t)} - 1)$ is also integrable. In order to show (1.9), we proceed as follows

$$\begin{aligned}|D(t)-(p-1)t| &= |\xi(t_0)^{1-p}+(p-1)\int_{t_0}^{t} e^{-(p-1)A(s)}ds-(p-1)t| \\ &= |\xi(t_0)^{1-p}+(p-1)\int_{t_0}^{t} (e^{-(p-1)A(s)}-1)ds-(p-1)t_0|\end{aligned}$$

Using (5.6) we obtain

$$\begin{aligned}|D(t)-(p-1)t| &\leq |K|+2(p-1)^2\int_{t_0}^{t}|A(s)|ds \\ &\leq M_1 = |K|+2(p-1)^2\frac{B}{\eta^2}\end{aligned}$$

with $K=\xi(t_0)^{1-p}-(p-1)t_0$. Setting $d(t)=D(t)-(p-1)t$, we obtain

$$\begin{aligned}\left|\,\xi(t)-\left(\frac{1}{(p-1)t}\right)^{\frac{1}{(p-1)}}\right| &= \left|\left(\frac{1}{D(t)}\right)^{\frac{1}{(p-1)}}-\left(\frac{1}{(p-1)t}\right)^{\frac{1}{(p-1)}}\right| \\ &= \left|\left(\frac{1}{D(t)-(p-1)t+(p-1)t}\right)^{\frac{1}{(p-1)}}-\left(\frac{1}{(p-1)t}\right)^{\frac{1}{(p-1)}}\right| \\ &= \left|\left(\frac{1}{d(t)+(p-1)t}\right)^{\frac{1}{(p-1)}}-\left(\frac{1}{(p-1)t}\right)^{\frac{1}{(p-1)}}\right| \\ &= \left|\left(\frac{1}{(p-1)t}\right)^{\frac{1}{(p-1)}}\left(\frac{1}{\frac{d(t)}{(p-1)t}+1}\right)^{\frac{1}{(p-1)}}-\left(\frac{1}{(p-1)t}\right)^{\frac{1}{(p-1)}}\right| \\ &= \left(\frac{1}{(p-1)t}\right)^{\frac{1}{(p-1)}}\left|\left(\frac{d(t)}{(p-1)t}+1\right)^{-\frac{1}{(p-1)}}-1\right| \\ &= \left(\frac{1}{(p-1)t}\right)^{\frac{1}{(p-1)}}\left(1-\left(\frac{d(t)}{(p-1)t}+1\right)^{-\frac{1}{(p-1)}}\right).\end{aligned}$$

Since $\mid \frac{d(t)}{(p-1)t} \mid< 1$ for t large enough. Let $\eta(t)=\left(1+\frac{d(t)}{(p-1)t}\right)^{-\frac{1}{(p-1)}}$, by the mean value Theorem and if we suppose that $|\frac{d(t)}{(p-1)t}|\leq\frac{1}{2}$, we obtain

$$|\eta'(t)|\leq\frac{1}{p-1}\times 2^{1+\frac{1}{p-1}}.$$

Therefore

$$|1-\left(1+\frac{d(t)}{(p-1)t}\right)^{-\frac{1}{(p-1)}}|\leq\frac{1}{p-1}\times 2^{1+\frac{1}{p-1}}|\frac{d(t)}{(p-1)t}|.$$

As we have seen above, $d(t)$ is bounded by M_1 then we conclude

$$|\,\xi(t)-\left(\frac{1}{(p-1)t}\right)^{\frac{1}{(p-1)}}\,|\;\leq Ct^{-1-\frac{1}{p-1}}.$$

With $C=(\frac{1}{p-1})^{\frac{p}{p-1}}\times 2^{1+\frac{1}{p-1}}M_1$.

We recall that z has a constant sign on $[T_0,+\infty[$ and z and u have the same sign. As in Section 4, we set $u=v+z$ and $\psi=u+w$, we distinguish two cases

- If $z>0$, this implies that for t large enough $u>0$ and $|\psi|=\psi$.
Then

$$||\psi|-\left(\tfrac{1}{(p-1)t}\right)^{\frac{1}{(p-1)}}\,|\;\leq|u-\left(\tfrac{1}{(p-1)t}\right)^{\frac{1}{(p-1)}}\,|\;+|w|\leq Kt^{-1-\frac{1}{p-1}}.$$

Indeed,

$$\begin{aligned}|u-\left(\frac{1}{(p-1)t}\right)^{\frac{1}{(p-1)}}\,|\;&\leq|u-z|+|z-\xi|+|\xi-\left(\frac{1}{(p-1)t}\right)^{\frac{1}{(p-1)}}\,|\\&\leq Me^{-\lambda_2 t}+ke^{-\delta t}+Ct^{-1-\frac{1}{p-1}}.\end{aligned}$$

Since we have (5.7), we obtain (1.9).

- If we suppose that $z<0$, then implies that $u<0$. By similar calculations we obtain the same result.

Indeed, $|\psi|=-\psi$, and we have

$$
\begin{aligned}
\left||\psi| - \left(\frac{1}{(p-1)t}\right)^{\frac{1}{(p-1)}}\ \right| &\leq \left| -u - \left(\frac{1}{(p-1)t}\right)^{\frac{1}{(p-1)}}\ \right| + |w| \\
&\leq K t^{-1-\frac{1}{p-1}}
\end{aligned}
$$

since

$$
\begin{aligned}
\left| -u - \left(\frac{1}{(p-1)t}\right)^{\frac{1}{(p-1)}}\ \right| &\leq |u - z| + |-z - \xi| + \left|\xi - \left(\frac{1}{(p-1)t}\right)^{\frac{1}{(p-1)}}\ \right| \\
&\leq Me^{-\lambda_2 t} + ke^{-\delta t} + Ct^{-1-\frac{1}{p-1}}.
\end{aligned}
$$

Also since we have (5.7), finally we obtain (1.9).

6 Proof of Proposition 1.3

(*i*) If ψ is an odd function, then $\psi(0,-x) = -\psi(0,x)$. It implies for all $t > 0, \quad \psi(t,-x) = -\psi(t,x)$.

In that case

$$
u(t) = P\psi(t,x) = \frac{1}{|\Omega|}\int_\Omega \psi(t,x)dx = 0.
$$

Moreover, we know that $\psi = u + w$, where $w(t) \in N^\perp$ for all $t \geq 0$ and (cf. Proposition 3.1) we have

$$
\|w(t)\|_{L^\infty(\Omega)} \leq Ke^{-\lambda_2 t},
$$

hence the solution ψ satisfies (1.8).

(*ii*) If $\psi(0,x) \geq 0$ and ψ does not vanish a.e in Ω, then for all $t \geq 0$ $\psi(t,x) > 0$, it implies that

$$
\int_\Omega \psi(t,x)dx > 0. \tag{6.1}
$$

We suppose that we have $\|\psi(t,\cdot)\|_\infty \leq Ce^{-\lambda_2 t}$ and we consider the problem (1.1) then we integrate on Ω, we obtain

$$\int_\Omega \psi_t(t,x)dx = -\int_\Omega g(\psi(t,x))dx. \tag{6.2}$$

An elementary calculation shows that we have

$$\begin{aligned}\int_\Omega g(\psi(t,x))dx &\leq \frac{c}{p}\int_\Omega |\psi(t,x)|^p dx \\ &\leq \frac{c}{p}\int_\Omega |\psi(t,x)|^{p-1}\psi(t,x)dx \\ &\leq \frac{c}{p}\int_\Omega \|\psi(t,x)\|_\infty^{p-1}\psi(t,x)dx \\ &\leq \frac{c}{p}C^{p-1}e^{-(p-1)\lambda_2 t}\int_\Omega \psi(t,x)dx\end{aligned}$$

Now we set $y(t) = \int_\Omega \psi(t,x)dx$. From (6.2), we deduce

$$y'(t) \geq -Me^{-\delta t}y(t) \tag{6.3}$$

with $M = \frac{c}{p}C^{p-1}$ and $\delta = (p-1)\lambda_2$.
Since $y(t) > 0$ by (6.1), we can integrate in the interval $[0,t]$, we obtain

$$y(t) \geq y(0)\exp\left\{-M\int_0^t e^{-\delta s}ds\right\} \geq y(0)\exp\left\{-\frac{M}{\delta}\right\} > 0. \tag{6.4}$$

Hence y does not tend to 0 for t large, this contradicts our hypothesis and we conclude that y satisfies (1.7).

References

[1] P. Bénilan, H. Brézis, *Solutions faibles d'équations d'évolution dans les espaces de Hilbert*, Annales de l'institut Fourier, 22 no. 2 (1972), p. 311-329.

[2] H.Brézis, *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert.*

[3] C. M.Dafermos, *Asymptotic behavior of solutions of evolution equations*, Nonlinear Evolution Equations, M. G. Crandall Ed, Academic Press, New-York 1978, 103-123.

[4] A. Haraux, *Slow and fast decay of solutions to some second order evolution equations*, J. Analyse Mathématiques. 95 (2005), 297-321.

[5] A. Haraux, *Decay rate of the range component of solutions to some semilinear evolution equations*, J. Analyse Mathématiques.13 (2006), 435-445.

[6] A. Haraux, M. A. Jendoubi and O. Kavian, *Rate of decay to equilibrium in some semilinear parabolic equations*, Journal of Evolution equations 3 (2003), 463-484.

[7] A. Haraux, *Systèmes dynamiques dissipatifs et applications*, Collection R. M. A. 17, Collection dirigé par P. G. Ciarlet et J. L. Lions, Masson, Paris 1991.

[8] A. Haraux, M. Kirane, *Estimations C^1 pour des problèmes paraboliques semi-linéaires*, Annales de la faculté des sciences de Toulouse Sér. 5, 5 no. 3-4 (1983), p. 265-280.