Stabilization of locally coupled wave-type systems

Fatiha Alabau-Boussouira*, Matthieu Léautaud†‡

March, 2010

Abstract

In this paper, we consider a system of two wave equations on a bounded domain $\Omega \subset \mathbb{R}^N$, that are coupled by a localized zero order term. Only one of the two equations is supposed to be damped. We show that the energy of smooth solutions of this system decays polynomially at infinity. This result is proved in an abstract setting for coupled second order evolution equations and is then applied to internal and boundary damping for wave and for plate systems. In one space dimension, this yields polynomial stability for any non-empty open coupling and damping regions, in particular if these two regions have empty intersection.

Keywords

Stabilization, indirect damping, hyperbolic systems, wave equation.

Contents

*LMAM, Université de Metz and CNRS (UMR 7122), 57045 Metz, France. e-mail: `alabau@univ-metz.fr`

†Université Pierre et Marie Curie Paris 6, UMR 7598, Laboratoire Jacques-Louis Lions, Paris, F-75005 France; CNRS, UMR 7598 LJLL, Paris, F-75005 France. e-mail: `leautaud@ann.jussieu.fr`

‡Laboratoire POEMS, INRIA Paris-Rocquencourt/ENSTA, CNRS UMR 2706, France.

1 Introduction

1.1 Motivation and general context

The decay properties for the energy of a solution of the damped wave equation are well known since the works [Lio], [Kom] and definitively [BLR92]. More precisely, given a bounded open domain $\Omega \subset \mathbb{R}^N$ with boundary $\Gamma = \partial\Omega$, we consider either the internal damped wave equation

$$\begin{cases} u'' - \Delta u + bu' = 0 & \text{in } (0,\infty) \times \Omega, \\ u = 0 & \text{on } (0,\infty) \times \Gamma, \\ (u, u')(0,\cdot) = (u^0(\cdot), u^1(\cdot)) & \text{in } \Omega, \end{cases} \tag{1}$$

or the boundary damped wave equation

$$\begin{cases} u'' - \Delta u = 0 & \text{in } (0,\infty) \times \Omega, \\ \frac{\partial u}{\partial \nu} + lu + bu' = 0 & \text{on } (0,\infty) \times \Gamma_b, \Gamma_b \subset \Gamma, \\ u = 0 & \text{on } (0,\infty) \times (\Gamma \setminus \Gamma_b), \\ (u, u')(0,\cdot) = (u^0(\cdot), u^1(\cdot)) & \text{in } \Omega. \end{cases} \tag{2}$$

Here, $u = u(t,x)$, u' denotes the time derivative of u and ν stands for the outward unit normal to Γ. In these two cases, the dissipation is due to the damping term bu', where $b = b(x)$ is a non-negative function on Ω in (1), on Γ in (2). The dissipation of the energy

$$E(u(t)) = \frac{1}{2}\int_\Omega \left(|u'|^2 + |\nabla u|^2\right) dx \quad \text{or} \quad E(u(t)) = \frac{1}{2}\int_\Omega \left(|u'|^2 + |\nabla u|^2\right) dx + \frac{1}{2}\int_{\Gamma_b} l(x) u^2 d\gamma$$

is given by

$$E'(u(t)) = -\int_\Omega b(x)|u'|^2 dx \quad \text{or} \quad E'(u(t)) = -\int_{\Gamma_b} b(x)|u'|^2 d\gamma$$

respectively in the internal and the boundary damping case. In both cases, the localization of the damping, supp(b) must satisfy some geometric conditions (see [BLR92], [Lio], [Kom]) in order for the energy of the solutions to decay exponentially, i.e., such that there exist two constants $M, \kappa > 0$ satisfying

$$E(u(t)) \le M e^{-\kappa t} E(u(0)), \qquad t > 0,$$

for all initial data (u^0, u^1) of finite energy.

Besides, when no feedback is applied to the wave equation, i.e., $b = 0$ in Ω in (1) or on Γ in (2), then the energy is conserved, $E(u(t)) = E(u(0))$ for every $t > 0$.

The question we are interested in is what are the stability properties of the following systems, obtained by coupling an exponentially stable wave equation with a conservative one, that is

$$\begin{cases} u_1'' - \Delta u_1 + \delta p u_2 + bu_1' = 0 & \text{in } (0,\infty) \times \Omega, \\ u_2'' - \Delta u_2 + p u_1 = 0 & \text{in } (0,\infty) \times \Omega, \\ u_1 = u_2 = 0 & \text{on } (0,\infty) \times \Gamma, \\ u_j(0,\cdot) = u_j^0(\cdot)\,,\ u_j'(0,\cdot) = u_j^1(\cdot),\ j = 1,2 & \text{in } \Omega, \end{cases} \tag{3}$$

in the case of internal damping, and

$$\begin{cases} u_1'' - \Delta u_1 + \delta p u_2 = 0 & \text{in } (0,\infty) \times \Omega, \\ u_2'' - \Delta u_2 + p u_1 = 0 & \text{in } (0,\infty) \times \Omega, \\ \frac{\partial u_1}{\partial \nu} + l u_1 + b u_1' = 0 & \text{on } (0,\infty) \times \Gamma_b, \Gamma_b \subset \Gamma, \\ u_1 = 0 & \text{on } (0,\infty) \times (\Gamma \setminus \Gamma_b), \\ u_2 = 0 & \text{on } (0,\infty) \times \Gamma, \\ u_j(0,\cdot) = u_j^0(\cdot)\,,\ u_j'(0,\cdot) = u_j^1(\cdot),\ j = 1,2 & \text{in } \Omega, \end{cases} \tag{4}$$

in the case of boundary damping. Here, $\delta > 0$ is a constant and $p \ge 0$ denotes the coupling parameter. The case of a constant coupling p has already been treated in [ACK02] for (3) (internal damping) and in [AB99], [AB02] for (4) (boundary damping). One of the goals of this paper is to generalize these results in cases where the coupling $p = p(x)$ can vanish in some part of Ω. It seems natural that we shall have to suppose some geometric conditions on the localization of the coupling, that is, on the support of the function $p(x)$. We here also point out the work [LZ99] where the exponential stabilization and the polynomial stabilization of a coupled hyperbolic-parabolic system are addressed.

In a more general setting, we are interested in the stability properties of systems of second order evolution equations coupling a conservative equation and an exponentially stable one. The abstract model that we shall refer to is the following

$$\begin{cases} u_1'' + A_1 u_1 + \delta P u_2 + B u_1' = 0 & \text{in } V_1', \\ u_2'' + A_2 u_2 + P^* u_1 = 0 & \text{in } V_2', \\ (u_j, u_j')(0) = (u_j^0, u_j^1) & \in V_j \times H, ;j = 1, 2, \end{cases} \tag{5}$$

where H and $V_j \subset H$, $j = 1, 2$ are separable real Hilbert space, A_j are positive unbounded selfadjoint operators on H. The coupling operator P is assumed to be bounded and P^* is its adjoint. The stabilization operator B will be supposed to be either bounded on H (which corresponds to the case of internal stabilization) or unbounded (which corresponds to the case of boundary stabilization). The total energy of a solution $U = (u_1, u_2)$ of (5) is defined by

$$E(U(t)) = \frac{1}{2}\left(\|u_1'\|_H^2 + \|A_1^{1/2}u_1\|_H^2 + \delta\|u_2'\|_H^2 + \delta\|A_2^{1/2}u_2\|_H^2\right) + \delta(u_1, Pu_2)_H, \tag{6}$$

where $(\cdot,\cdot)_H$ denotes the inner product on H and $\|\cdot\|_H$ the associated norm. Note that we have to consider different operators $A_j, j = 1, 2$, in order to treat the boundary damping case. In the applications to coupled wave equations, A_1 and A_2 will be the same Laplace operator, i.e., with the same speed of propagation, but with different boundary conditions.

The question is now: is the full system (5) stable, and if so, at which rate ?

In the papers [ACK02], [AB02], the authors prove that this system cannot be exponentially stable, since it is a compact perturbation of the decoupled system (obtained by taking $P = 0$ in (5)), that is unstable. However, they prove that the energy decays at least polynomially at infinity, under the assumption that the operator P is coercive on H. Here, since we want to address locally coupled equations (see (3), (4) with $p(x)$ locally supported), we have to remove the coerciveness assumption on P. We shall instead suppose that it is only partially coercive (see Assumption $(A1)$ below). Note that we have to suppose the natural assumption that δ and $p^+ = \|P\|_{\mathcal{L}(H)} = \sup\{\|Pv\|_H, \|v\|_H = 1\}$ are sufficiently small so that the energy is positive.

In this paper, the main result concerning the abstract system (5) is a polynomial stability Theorem under certain assumptions on the operators P and B (see Theorem 2.4 in the case B bounded and Theorem 2.7 in the case B unbounded). This abstract result can then be applied to a large class of second order evolution equations. In Section 4, we treat the case of two locally coupled wave or plate equations, with an internal or a boundary damping. The problem that first motivated this work is the case (3) of partially internally damped wave equations. We now detail the results obtained for this problem, that sum up our study.

1.2 Results for two coupled wave equations

In this section, we consider problem (3) in a domain $\Omega \subset \mathbb{R}^N$ with $\mathcal{C}^2$ boundary. The damping function b and the coupling function p are two bounded real valued functions on Ω, satisfying

$$\begin{cases} 0 \le b \le b^+ \text{ and } 0 \le p \le p^+ \text{ on } \Omega, \\ b \ge b^- > 0 \text{ on } \omega_b, \\ p \ge p^- > 0 \text{ on } \omega_p, \end{cases}$$

for ω_b and ω_p two non-empty open subsets of Ω. As usual for damped wave equations, we have to make some geometric assumptions on the sets ω_b and ω_p so that the energy of a single wave decays sufficiently rapidly at infinity. Here, we shall use the Piecewise Multipliers Geometric Condition (PMGC).

Definition 1.1 (PMGC)**.** We say that $\omega \subset \Omega$ satisfies the PMGC if there exist $\Omega_j \subset \Omega$ having Lipschitz boundary and $x_j \in \mathbb{R}^N$, $j = 1...J$ such that $\Omega_j \cap \Omega_i = \emptyset$ for $j \neq i$ and ω contains a neighborhood in Ω of the set $\bigcup_{j=1}^J \gamma_j(x_j) \cup \left(\Omega \setminus \bigcup_{j=1}^J \Omega_j\right)$, where $\gamma_j(x_j) = \{x \in \partial\Omega_j, (x - x_j) \cdot \nu_j(x) > 0\}$ and ν_j is the outward unit normal to $\partial\Omega_j$.

This geometric assumption was introduced in [Liu97] and further used in [AB04, AB05]. It is a generalization of the usual multiplier geometric condition of [Zua90], saying that ω contains a neighborhood in Ω of the set $\{x \in \partial\Omega, (x - x_0) \cdot \nu(x) > 0\}$, for some $x_0 \in \mathbb{R}^N$. For instance in dimension 2, the PMGC is satisfied in the case where Ω is a disk and ω a neighborhood of one diameter, a situation which is not covered by the condition of [Zua90]. However, the PMGC is of course much more restrictive than the sharp geometric condition proved in [BLR92] (saying for example in the previous case that the neighborhood of a radius is sufficient to obtain the same stability results).

We denote by λ the smallest eigenvalue of the Laplace operator on Ω, with Dirichlet boundary conditions. We also have the identity $\lambda = 1/C_{\mathcal{P}}^2$, where $C_{\mathcal{P}}$ is the Poincaré's constant of Ω. Note that according to (6), the energy of a solution U of (3) is defined by

$$E(U(t)) = \frac{1}{2}\left(\|u_1'\|^2_{L^2(\Omega)} + \|\nabla u_1\|^2_{L^2(\Omega)} + \delta\|u_2'\|^2_{L^2(\Omega)} + \delta\|\nabla u_2\|^2_{L^2(\Omega)}\right) + \delta(pu_1, u_2)_{L^2(\Omega)},$$

and is positive as soon as $p^+ < \lambda$ and $0 < \delta < \frac{\lambda}{p^+}$. With this notations, we can state the stability theorem for system (3).

Theorem 1.2. *(i) Suppose that ω_b and ω_p satisfy the PMGC and that $b, p \in W^{q,\infty}(\Omega)$. Then there exists $p_* \in (0, \lambda]$ such that for all $0 < p^+ < p_*$ there exists $\delta_* = \delta_*(p^+, p^-) \in (0, \frac{\lambda}{p^+}]$, such that for all $\delta \in (0, \delta_*)$, the solution $U(t) = (u_1(t), u_2(t), u_1'(t), u_2'(t))$ of (3) satisfies for $n \in \mathbb{N}, n \leq q$,*

$$E(U(t)) \leq \frac{c}{t^n} \sum_{i=0}^{n} E(U^{(i)}(0)) \quad \forall t > 0,\ U^0 = (u_1^0, u_2^0, u_1^1, u_2^1) \in (H^{n+1} \cap H_0^1)^2 \times (H^n \cap H_0^1)^2,$$

where c is a constant depending on δ, p^+, p^- and n. Besides, if $U^0 \in (H_0^1)^2 \times (L^2)^2$, then $E(U(t))$ converges to zero as t goes to infinity.

(ii) If moreover either $\omega_b \subset \omega_p$ or $\omega_p \subset \omega_b$, then the result holds for $\delta_* = \frac{\lambda}{p^+}$.

This theorem is a consequence of Theorem 2.4. The fact that problem (3) satisfies the assumptions of Theorem 2.4 is postponed in Section 4.1. Note that the constants p_* and δ_* are explicit functions of the parameters of the problem and of the constants coming from the multiplier method. The smoothness assumption on the coefficients p and b comes from Lemma 2.6. If these parameters are not smooth, Theorem 1.2 is still valid for initial data in $\mathcal{D}(\mathcal{A}^n_{P,\delta})$ where the operator $\mathcal{A}_{P,\delta}$ is defined in (9). But in this case, we cannot explicit the space $\mathcal{D}(\mathcal{A}^n_{P,\delta})$ in terms of classical Sobolev spaces.

Some comments can be made about this Theorem. One particularly interesting question for this type of coupled problem is the case $\omega_b \cap \omega_p = \emptyset$. This question first arised in the field of control theory for coupled evolution equations, and, to the authors' knowledge, is still unsolved. More precisely, consider the parabolic system

$$\begin{cases} u_1' - \Delta u_1 + \delta p u_2 = \mathbb{1}_{\omega_b} f & \text{in } (0,T) \times \Omega, \\ u_2' - \Delta u_2 + p u_1 = 0 & \text{in } (0,T) \times \Omega, \\ u_1 = u_2 = 0 & \text{on } (0,T) \times \partial\Omega, \\ u_1(0,\cdot) = u_1^0,\ u_2(0,\cdot) = u_2^0 & \text{in } \Omega, \end{cases}$$

or its hyperbolic counterpart

$$
\begin{cases}
u_1'' - \Delta u_1 + \delta p u_2 = \mathbb{1}_{\omega_b} f & \text{in } (0,T) \times \Omega, \\
u_2'' - \Delta u_2 + p u_1 = 0 & \text{in } (0,T) \times \Omega, \\
u_1 = u_2 = 0 & \text{on } (0,T) \times \partial\Omega, \\
u_1(0,\cdot) = u_1^0 \,,\ u_2(0,\cdot) = u_2^0 & \text{in } \Omega, \\
u_1'(0,\cdot) = u_1^1 \,,\ u_2'(0,\cdot) = u_2^1 & \text{in } \Omega,
\end{cases}
$$

where the function p and the subset ω_b are the same as in the stabilization problem. In these two cases, the null-controllability problem under interest is the following: given a positive time T and initial data, is it possible to find a control function f so that the state has been driven to zero in time T? The Parabolic null-controllability problem is fully solved in the case $\omega_b \cap \omega_p \neq \emptyset$ (see [Ter00], [AKBD06], [GBPG06], [Léa10]). However, this problem is open in the case $\omega_b \cap \omega_p = \emptyset$. Only the approximate controllability has been proved in [KT09] in this case for $\delta = 0$. Concerning the hyperbolic null-controllability problem, only the case of constant coupling p have been considered, to our knowledge, in [AB03] and [You09].

The second reason for which the case $\omega_b \cap \omega_p = \emptyset$ is of particular interest in the stabilization problem (3) is that, in this case, we don't even know if the strong stability property holds, i.e., if the energy goes to zero as t goes to infinity. To our knowledge, the only strong stability result for system (3) is the following.

Proposition 1.3. *Suppose that $\omega_b \cap \omega_p \neq \emptyset$. Then, the energy of every solution of system* (3) *goes to zero as t goes to infinity.*

This is a direct consequence of the unique continuation property for the associated elliptic system proved in [Léa10, Proposition 5.1] and Lasalle's invariance Principle. However this unique continuation property is not known in the case $\omega_b \cap \omega_p = \emptyset$ and the strong stability is open.

Now, concerning the stability Theorem 1.2, it first has to be noted that, in dimension $N \geq 2$ the assumption that both ω_b and ω_p satisfy the PMGC implies that $\omega_b \cap \omega_p \neq \emptyset$ (whereas this is not the case if ω_b and ω_p satisfy the optimal condition of [BLR92]). This Theorem is hence of particular interest in dimension $N = 1$. In this case, $\Omega = (0, L)$ for some $L > 0$, and any non-empty open subinterval ω satisfies the PMGC. As a consequence, we obtain the following corollary of Theorem 1.2 point (i).

Corollary 1.4. *Suppose that $\Omega = (0, L)$. Then, for any non-empty subsets $\omega_b \subset \Omega$ and $\omega_p \subset \Omega$ (i.e., for any non-vanishing non-negative functions p and b), there exists $p_* \in (0, \lambda]$ such that for all $0 < p^+ < p_*$ there exists $\delta_* = \delta_*(p^+, p^-) \in (0, \frac{\lambda}{p^+}]$, such that for all $\delta \in (0, \delta_*)$, the polynomial stability result of Theorem 1.2 holds.*

In particular, this yields in this case a strong stability result with $\omega_b \cap \omega_p = \emptyset$, improving Proposition 1.3. Moreover, this can be a first step to address the hyperbolic and the parabolic control problems in the case $\omega_b \cap \omega_p = \emptyset$.

Remark 1.5. In the sequel, C will denote a generic constant, whose value may change from line to line. Writing $C = C(p, \beta, ...)$ means that this constant depends on the parameters $p, \beta, ...$.

Acknowledgements. The second author wishes to thank O. Glass and J. Le Rousseau for very fruitful discussions and encouragements. Both authors were partially supported by l'Agence Nationale de la Recherche under grant ANR-07-JCJC-0139-01.

2 Abstract formulation and main results

2.1 Abstract setting and well-posedness

Let H and V_j, $j = 1, 2$, be separable real Hilbert spaces such that the injections $V_j \subset H$ are dense and compact. We identify H with its dual space and denote by V_j' the dual space of V_j, so that the injections $V_j \subset H \subset V_j'$ are dense and compact. We denote by $(\cdot,\cdot)_H$ (resp. $(\cdot,\cdot)_{V_j}$) the inner product on H (resp. V_j), $\|\cdot\|_H$ (resp. $\|\cdot\|_{V_j}$) the associated norm and $\|\cdot\|_{V_j'}$ the norm on V_j'. Moreover, we write $\langle\cdot,\cdot\rangle_{V_j',V_j}$ the duality product and A_j the duality mapping from V_j to V_j' defined by

$$\langle A_j v, w\rangle_{V_j',V_j} = (v,w)_H \quad \forall v,w \in V_j.$$

We shall moreover assume that the subspace V_2 is continuously imbedded in V_1, so that we have the following scheme:

$$V_2 \hookrightarrow V_1 \hookrightarrow H \hookrightarrow V_1' \hookrightarrow V_2',$$

where the first and the last inclusions are continuous and the two central ones are dense and compact. We denote $i : V_2 \hookrightarrow V_1$ the natural injection and $\Pi_V : V_1 \mapsto V_2$ the natural projection from V_1 to V_2. We recall that for $u_1 \in V_1$, $\Pi_V u_1$ is characterized by

$$\begin{cases} \langle A_1 i(\Pi_V u_1), i(\phi)\rangle_{V_1',V_1} = \langle A_1 u_1, i(\phi)\rangle_{V_1',V_1} \quad \forall \phi \in V_2\,, u_1 \in V_1, \\ \qquad\qquad \text{and} \quad \Pi_V u_1 \in V_2, \end{cases}$$

and that i, A_1 and A_2 are linked by

$$\langle A_2\phi,\psi\rangle_{V_2',V_2} = \langle A_1 i(\phi), i(\psi)\rangle_{V_1',V_1} \quad \forall \phi\,,\psi \in V_2\ .$$

The coupling operator P is a bounded operator on H and P^* is its adjoint, $\|P\|_{\mathcal{L}(H)} = p^+$. The damping operator B will be supposed to be at least bounded from V_1 to V_1' and symmetric non-negative:

$$\langle Bv, w\rangle_{V_1',V_1} = \langle Bw, v\rangle_{V_1',V_1}\,, \quad \langle Bv, v\rangle_{V_1',V_1} \ge 0 \quad \forall v,w \in V_1. \tag{7}$$

We denote by $\lambda_j, j = 1,2$, the largest constant satisfying

$$\|v\|_{V_j}^2 \ge \lambda_j \|v\|_H^2 \quad \forall v \in V_j,$$

that is, the smallest eigenvalue of the selfadjoint positive operator A_j.

Let us study now the abstract system (5). This linear evolution equation can be rewritten under the form

$$\begin{cases} U' + \mathcal{A}_{P,\delta} U = 0 \\ U(0) = U^0 \in \mathcal{H}, \end{cases} \tag{8}$$

where $\mathcal{H} = V_1 \times V_2 \times H^2$,

$$U = \begin{pmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \end{pmatrix}, U^0 = \begin{pmatrix} u_1^0 \\ u_2^0 \\ u_1^1 \\ u_2^1 \end{pmatrix}, \mathcal{A}_{P,\delta} = \begin{pmatrix} 0 & 0 & -\operatorname{Id} & 0 \\ 0 & 0 & 0 & -\operatorname{Id} \\ A_1 & \delta P & B & 0 \\ P^* & A_2 & 0 & 0 \end{pmatrix}, \mathcal{D}(\mathcal{A}_{P,\delta}) = \{U \in \mathcal{H}, \mathcal{A}_{P,\delta} U \in \mathcal{H}\}. \tag{9}$$

We recall that the energy of this system is given by

$$E(U(t)) = \frac{1}{2}\left(\|u_1'\|_H^2 + \|u_1\|_{V_1}^2 + \delta\|u_2'\|_H^2 + \delta\|u_2\|_{V_2}^2\right) + \delta(u_1, Pu_2)_H, \tag{10}$$

and we will require this energy to be positive for any solution $U(t)$. We have the lower bound for the energy

$$\begin{aligned} E(U(t)) &\ge \frac{1}{2}\left(\|u_1'\|_H^2 + \lambda_1\|u_1\|_H^2 + \delta\|u_2'\|_H^2 + \delta\lambda_2\|u_2\|_H^2\right) - \frac{\delta p^+}{2}(\|u_1\|_H^2 + \|u_2\|_H^2) \\ &\ge \frac{1}{2}(\|u_1'\|_H^2 + \delta\|u_2'\|_H^2) + \frac{1}{2}(\lambda_1 - \delta p^+)\|u_1\|_H^2 + \frac{\delta}{2}(\lambda_2 - p^+)\|u_2\|_H^2. \end{aligned} \tag{11}$$

Therefore, in the sequel, we shall suppose

$$0 < p^+ < \lambda_2, \text{ and } 0 < \delta < \frac{\lambda_1}{p^+} \tag{12}$$

so that (11) holds with positive constants, i.e., E is a positive energy that measures the whole state U.

Remark 2.1. Note that for any $\delta > 0$, the operator $\begin{pmatrix} A_1 & \delta P \\ P^* & A_2 \end{pmatrix}$ is selfadjoint on the space $H \times H$ endowed with the weighted inner product $(u, v)_\delta = \delta(u_1, v_1)_H + (u_2, v_2)_H$ (which is the energy space). This operator is moreover positive under the condition (12). In the case $B = 0$, the operator $\mathcal{A}_{P,\delta}$ is skewadjoint and thus generates a group.

Under the assumptions made above, the system (8) (and thus, (5)) is well-posed in the sense of semigroup theory.

Proposition 2.2. *For all $0 \le p^+ < \lambda_2$ and $0 \le \delta < \frac{\lambda_1}{p^+}$, the operator $\mathcal{A}_{P,\delta}$ is maximal monotone on $\mathcal{H}$. As a consequence, for every $U^0 \in \mathcal{H}$, Problem (8) has a unique solution $U \in \mathcal{C}^0([0, +\infty); \mathcal{H})$. If in addition, $U^0 \in \mathcal{D}(\mathcal{A}^k_{P,\delta})$, for $k \in \mathbb{N}$, then, the solution U is in $\bigcap_{j=0}^k \mathcal{C}^{k-j}([0, +\infty); \mathcal{D}(\mathcal{A}^j_{P,\delta}))$. Moreover, the energy $E(U)$ of the solution defined by (10) is locally absolutely continuous, and for strong solutions, i.e., when $U^0 \in \mathcal{D}(\mathcal{A}_{P,\delta})$, we have*

$$E'(U(t)) = -\langle Bu_1', u_1' \rangle_{V_1', V_1} . \tag{13}$$

2.2 Main results

In all the following, we have to suppose some additional assumptions on the operators P and B, in order to prove the stability results. Let us first precise assumptions $(A1)$ and $(A2)$, related with the operator P. We assume that P is partially coercive, i.e.,

$$(A1) \left\{ \begin{array}{c} \text{there exists an operator } \Pi_P \in \mathcal{L}(H), \|\Pi_P\|_{\mathcal{L}(H)} = 1, \text{ and a number } p^- > 0 \\ \text{such that} \quad (Pv, v)_H \ge p^- \|\Pi_P v\|_H^2 \quad \forall v \in H. \end{array} \right.$$

Note that $p^- \le p^+ = \|P\|_{\mathcal{L}(H)}$ and that $(A1)$ implies that the operators P and P^* are non-negative. We shall moreover make the following assumption $(A2)$ on one decoupled equation, without damping, but with a right hand-side:

$$(A2) \left\{ \begin{array}{l} \exists\, \alpha_2, \beta_2, \gamma_2 > 0 \text{ such that for all } f_2 \in \mathcal{C}^1([0, +\infty); H) \text{ and all } 0 \le S \le T\,, \\ \text{the solution } u_2 \text{ of} \\ \qquad u_2'' + A_2 u_2 = f_2 \text{ in } V_2', \\ \qquad (u_2, u_2')(0) = (u_2^0, u_2^1) \in V_2 \times H, \\ \text{satisfies, with } e_2(t) = 1/2\left(\|u_2'\|_H^2 + \|u_2\|_{V_2}^2\right), \text{ the inequality} \\ \int_S^T e_2(t)\ dt \le \alpha_2(e_2(S) + e_2(T)) + \beta_2 \int_S^T \|f_2(t)\|_H^2\ dt + \gamma_2 \int_S^T \|\Pi_P u_2'(t)\|_H^2 dt. \end{array} \right.$$

This corresponds to the second equation in which the coupling term is viewed as a forcing term. This type of estimate will be proved in the applications below by means of multiplier estimates (for a single equation with a right hand-side). Note that the operator Π_P involved in the estimate of assumption $(A2)$ is the operator given by assumption $(A1)$.

Concerning the operator B, we shall make the following "stability" assumption for a single damped equation:

$$(A3) \left\{ \begin{array}{l} \exists\, \alpha_1, \beta_1, \gamma_1 > 0 \text{ such that for all } f_1 \in \mathcal{C}^1([0, +\infty); H) \text{ and all } 0 \le S \le T\,, \\ \text{the solution } u_1 \text{ of} \\ \qquad u_1'' + A_1 u_1 + Bu_1' = f_1 \text{ in } V_1', \\ \qquad (u_1, u_1')(0) = (u_1^0, u_1^1) \in V_1 \times H, \\ \text{satisfies, with } e_1(t) = 1/2\left(\|u_1'\|_H^2 + \|u_1\|_{V_1}^2\right), \text{ the inequality} \\ \int_S^T e_1(t)\ dt \le \alpha_1(e_1(S) + e_1(T)) + \beta_1 \int_S^T \|f_1(t)\|_H^2\ dt + \gamma_1 \int_S^T \langle Bu_1', u_1' \rangle_{V',V}\, dt. \end{array} \right.$$

Remark 2.3. Assumption $(A3)$ implies in particular that the single damped equation is exponentially stable, since for $f_1 = 0$, we deduce that $e_1(t)$, which is locally absolutely continuous and nonincreasing, satisfies the classical integral inequality (see [Har78], [Kom]).

$$\int_S^T e_1(t)\ dt \le (2\alpha_1 + \gamma_1) e_1(S) \quad \forall 0 \le S \le T.$$

The next assumption we make on the damping operator B depends on its boundedness.

2.2.1 The case B bounded

In the bounded case, we shall moreover suppose

$$(A4b) \qquad \|B\|_{\mathcal{L}(H)} = b^+ \quad \text{and} \quad V_2 = V_1 = V.$$

As a consequence, we have

$$A_1 = A_2 = A, \quad \lambda_1 = \lambda_2 = \lambda, \quad \text{and} \quad i = \Pi_V = \mathrm{Id}_{V_1}.$$

The positivity condition (12) for the energy becomes

$$0 < p^+ < \lambda, \quad \text{and} \quad 0 < \delta < \frac{\lambda}{p^+}. \tag{14}$$

The main result here is

Theorem 2.4. *(i) Suppose* $(A1), (A2), (A3), (A4b)$*. Then there exists* $p_* \in (0, \lambda]$ *such that for all* $0 < p^+ < p_*$ *there exists* $\delta_* = \delta_*(p^+, p^-) \in (0, \frac{\lambda}{p^+}]$*, such that for all* $\delta \in (0, \delta_*)$*, the solution* $U(t) = \exp(-\mathcal{A}_{\delta,P} t)U^0$ *of* (5) *satisfies*

$$E(U(t)) \leq \frac{c}{t^n} \sum_{p=0}^{n} E(U^{(p)}(0)) \quad \forall t > 0,\ U^0 \in \mathcal{D}(\mathcal{A}^n_{\delta,P}),$$

where c *is a constant depending on* δ*,* p^+*,* p^- *and* n*. Besides, if* $U^0 \in \mathcal{H}$*, then* $E(U(t))$ *converges to zero as* t *goes to infinity.*

(ii) Suppose moreover either

$$\|Pv\|^2_H \leq p^+ (Pv, v)_H \ \text{ and } \ (Bv, v)_H \leq \frac{b^+}{p^-}(Pv, v)_H, \quad \forall v \in H, \tag{15}$$

or

$$\text{there exists } b^- > 0 \text{ such that } (Pv, v)_H \leq \frac{p^+}{b^-}(Bv, v)_H, \quad \forall v \in H. \tag{16}$$

Then the result holds for $\delta_* = \frac{\lambda}{p^+}$*.*

Remark 2.5. In Case (ii) of Theorem 2.4, the conclusion is much stronger than in Case (i). As one sees in the proof below, δ_* is very small in Case (i), whereas in Case (ii), the result holds for a large panel of δ, including the interval $(0, 1]$. More precisely, the constants p_* and δ_* are explicit, that is,

- $p_* = \min\left\{\frac{1}{2\gamma_2}, \lambda\right\}$ and $\delta_* = \delta_*(p^+, p^-) = \min\left\{\frac{\lambda}{2\beta_2(p^+)^2 p^-}, \frac{\lambda p^-}{2\beta_1 (p^+)^2}, \frac{\lambda}{p^+}\right\}$ in (i);
- $p_* = \min\left\{\frac{1}{2\beta_1}, \lambda\right\}$ and $\delta_* = \frac{\lambda}{p^+}$ in the first case of (ii);
- $p_* = \min\left\{\frac{1}{2\beta_1}, \frac{1}{2\beta_2}, \lambda\right\}$ and $\delta_* = \frac{\lambda}{p^+}$ in the second case of (ii).

In this case, we are moreover able to give a simple characterization of the space $\mathcal{D}(\mathcal{A}^n_{\delta,P})$, in terms of the spaces $\mathcal{D}(A^n)$, which is useful in the applications. More precisely, setting $\mathcal{H}_n = (\mathcal{D}(A^{\frac{n+1}{2}}))^2 \times (\mathcal{D}(A^{\frac{n}{2}}))^2 \subset \mathcal{H}$, we prove the following lemma, inspired by [ACK02, Lemma 3.1].

Lemma 2.6. *Suppose that for every* $0 < k \leq n - 1$ *(no assumption if* $n = 1$*), we have*

$$P\mathcal{D}(A^{\frac{k+1}{2}}) \subset \mathcal{D}(A^{\frac{k}{2}}) \quad , \quad P^*\mathcal{D}(A^{\frac{k+1}{2}}) \subset \mathcal{D}(A^{\frac{k}{2}}) \quad \text{and} \quad B\mathcal{D}(A^{\frac{k+1}{2}}) \subset \mathcal{D}(A^{\frac{k}{2}}). \tag{17}$$

Then $\mathcal{H}_k = \mathcal{D}(\mathcal{A}^k_{\delta,P})$ *for every* $0 \leq k \leq n$*.*

Proof. We proceed by induction on $n \in \mathbb{N}$. In the case $n = 1$, there is no need of an additional assumption since the operators B and P are bounded on H. Now, we assume that $\mathcal{H}_{n-1} = \mathcal{D}(\mathcal{A}_{\delta,P}^{n-1})$ and prove $\mathcal{H}_n = \mathcal{D}(\mathcal{A}_{\delta,P}^{n})$. We have

$$\begin{aligned}
\mathcal{D}(\mathcal{A}_{\delta,P}^{n}) &= \Big\{U \in \mathcal{D}(\mathcal{A}_{\delta,P}^{n-1}) \ , \ \mathcal{A}_{\delta,P} U \in \mathcal{D}(\mathcal{A}_{\delta,P}^{n})\Big\} \\
&= \Big\{(u_1, u_2, v_1, v_2) \in (\mathcal{D}(A^{\frac{n}{2}}))^2 \times (\mathcal{D}(A^{\frac{n-1}{2}}))^2 \ , \\
&\quad (-v_1, -v_2, Au_1 + \delta P u_2 + B v_1, Au_2 + P^* u_1) \in (\mathcal{D}(A^{\frac{n}{2}}))^2 \times (\mathcal{D}(A^{\frac{n-1}{2}}))^2 \Big\} ,
\end{aligned}$$

when using the induction assumption $\mathcal{D}(\mathcal{A}_{\delta,P}^{n-1}) = \mathcal{H}_{n-1}$. Now using assumption (17) for $k = n-1$, we see that having

$$\begin{cases} v_1 \in \mathcal{D}(A^{\frac{n}{2}}) \quad ; \quad v_2 \in \mathcal{D}(A^{\frac{n}{2}}), \\ Au_1 + \delta P u_2 + B v_1 \in \mathcal{D}(A^{\frac{n-1}{2}}), \\ Au_2 + P^* u_1 \in \mathcal{D}(A^{\frac{n-1}{2}}), \end{cases}$$

is equivalent to having

$$v_1 \in \mathcal{D}(A^{\frac{n}{2}}) \quad ; \quad v_2 \in \mathcal{D}(A^{\frac{n}{2}}) \quad ; \quad Au_1 \in \mathcal{D}(A^{\frac{n-1}{2}}) \quad ; \quad Au_2 \in \mathcal{D}(A^{\frac{n-1}{2}}),$$

that is exactly $(u_1, u_2, v_1, v_2) \in (\mathcal{D}(A^{\frac{n+1}{2}}))^2 \times (\mathcal{D}(A^{\frac{n}{2}}))^2 = \mathcal{H}_n$. This gives $\mathcal{D}(\mathcal{A}_{\delta,P}^{n}) = \mathcal{H}_n$ and concludes the proof of the lemma. $\square$

2.2.2 The case B unbounded

Here, we replace assumption $(A4b)$ by the following

$$(A4u) \begin{cases} \langle Bu_1, i(\phi)\rangle_{V_1', V_1} = 0 \quad \forall \phi \in V_2, \ u_1 \in V_1, \text{ and} \\ \exists \beta > 0, \ \|u_1 - \Pi_V u_1\|_H^2 \leq \beta \ \langle Bu_1, u_1\rangle_{V_1', V_1} \quad \forall u_1 \in V_1 . \end{cases}$$

Assumption $(A4u)$ implies that B satisfies a "weak" coercivity property (since the norm on the left hand side of the second inequality in $(A4u)$ is the weaker H-norm) in the subspace orthogonal to the closed subspace V_2. As will be seen in Section 4, this property is satisfied for most systems (e.g. wave, plate...). We have the analogous of Theorem 2.4.

Theorem 2.7. *Suppose* $(A1), (A2), (A3), (A4u)$. *Then there exists* $p_* \in (0, \lambda_2]$ *such that for all* $0 < p^+ < p_*$ *there exists* $\delta_* = \delta_*(p^+, p^-) \in (0, \frac{\lambda_1}{p^+}]$*, such that for all* $\delta \in (0, \delta_*)$*, the solution* $U(t) = \exp(-\mathcal{A}_{\delta,P} t) U^0$ *of* (5) *satisfies*

$$E(U(t)) \leq \frac{c}{t^n} \sum_{p=0}^{2n} E(U^{(p)}(0)) \quad \forall t > 0, \ U^0 \in \mathcal{D}(\mathcal{A}_{\delta,P}^{2n}),$$

where c is a constant depending on δ, p^+ and n. Besides, if $U^0 \in \mathcal{H}$, then $E(U(t))$ converges to zero as t goes to infinity.

Remark 2.8.
- As in Theorem 2.4, the constants p_* and δ_* here are explicit, that is $p_* = \min\left\{\frac{1}{2\gamma_2}, \lambda_2\right\}$ and $\delta_* = \delta_*(p^+, p^-) = \min\left\{\frac{\lambda_1}{2\beta_2 (p^+)^2 p^-}, \frac{\lambda_2 p^-}{2\beta_1 (p^+)^2}, \frac{\lambda_1}{p^+}\right\}$.
- Note the difference between the conclusions of Theorem 2.4 and 2.7. For $U^0 \in D(\mathcal{A}_{\delta,P}^{2n})$, Theorem 2.4 gives a decay rate of the form C/t^{2n}, whereas Theorem 2.7 only gives a decay rate of the form C/t^n. This comes from the unbounded nature of the operator B (in the applications below, the boundary stabilization).
- Note also that item (ii) of Theorem 2.4 has no counterpart here since P and B are not of the same nature.

3 Proof of the main results, Theorems 2.4 and 2.7

3.1 The stability lemma

In the following, to prove polynomial stability, we shall use the following lemma, which proof can be found in [AB99], [AB02], [ACK02].

Lemma 3.1. *Let $U(t) = \exp(-t\mathcal{A})U(0)$ a strongly continuous semigroup generated by $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$. Suppose that $t \mapsto E(U(t))$ is a nonincreasing, locally absolutely continuous function from $[0, +\infty)$ to $[0, +\infty)$. Assume moreover that there exists $k \in \mathbb{N}^*$ and $c_p > 0$, $p = 0, ..., k$ such that*

$$\int_S^T E(U(t))dt \leq \sum_{p=0}^{k} c_p E(U^{(p)}(t)) \quad \forall 0 \leq S \leq T, \quad \forall U(0) \in \mathcal{D}(\mathcal{A}^k). \tag{18}$$

Then, for every $n \in \mathbb{N}$, there exists $c_n > 0$ such that

$$E(U(t))dt \leq \frac{c_n}{t^n} \sum_{p=0}^{kn} E(U^{(p)}(t)) \quad \forall t > 0, \quad \forall U(0) \in \mathcal{D}(\mathcal{A}^{kn}).$$

Besides, if $U(0) \in \mathcal{H}$, then $E(U(t))$ converges to zero as t goes to infinity.

To prove the stability results Theorem 2.4 and 2.7, we only have to perform energy estimates of the type (18). For this we shall use the dissipation relation (13), that yields, for all $0 \leq S \leq T$,

$$\int_S^T \langle Bu_1', u_1' \rangle_{V',V} \, dt \leq E(U(S)) - E(U(T)) \leq E(U(S)). \tag{19}$$

3.2 Proof of Theorem 2.4, the case B bounded

The link between u_1 and u_2 in the following estimates is given by the following coupling relation.

Lemma 3.2. *Assume (A4b) and (14). Then, for all $U^0 = (u_1^0, u_2^0, u_1^1, u_2^1) \in \mathcal{H}$, the solution $U(t) = \exp(-t\mathcal{A}_{\delta,P})U^0 = (u_1, u_2, v_1, v_2)$ of (5) satisfies for some $C = C(\delta, p^+) > 0$*

$$\delta \int_S^T (Pu_2, u_2)_H dt \leq \int_S^T (Pu_1, u_1)_H dt + \varepsilon \int_S^T (Bu_2, u_2)_H dt + C\left(1 + \frac{1}{\varepsilon}\right) E(U(S)) \tag{20}$$

for all $\varepsilon > 0$ and $0 \leq S \leq T$.

Proof. Assume first that $U^0 \in \mathcal{D}(\mathcal{A}_{\delta,P})$. In this case, the solution $U(t) = \exp(-t\mathcal{A}_{\delta,P})U^0 = (u_1, u_2, v_1, v_2)$ of (5) is in $\mathcal{C}^0([0, +\infty); \mathcal{D}(\mathcal{A}_{\delta,P}) \cap \mathcal{C}^1([0, +\infty); \mathcal{H})$. Hence $U = (u_1, u_2, v_1, v_2)$ satisfies

$$\begin{cases} v_1 = u_1', \quad v_2 = u_2', \\ u_1'' + Au_1 + Bu_1' + \delta P u_2 = 0 & \text{in } H, \\ u_2'' + Au_2 + P^* u_1 = 0 & \text{in } H, \end{cases}$$

As a consequence, we have $\forall 0 \leq S \leq T$,

$$\int_S^T (u_1'' + Au_1 + Bu_1' + \delta P u_2, u_2)_H - (u_2'' + Au_2 + P^* u_1, u_1)_H dt = 0. \tag{21}$$

We first notice that $(Au_1, u_2)_H - (Au_2, u_1)_H = 0$ since A is selfadjoint, and

$$\begin{aligned} \left| \int_S^T (u_1'', u_2)_H - (u_2'', u_1)_H dt \right| &= \left| [(u_1', u_2)_H - (u_2', u_1)_H]_S^T \right| \\ &\leq \frac{1}{2} \sum_{j=1,2} \left(\|u_j'(S)\|_H^2 + \|u_j'(T)\|_H^2 + \|u_j(S)\|_H^2 + \|u_j(T)\|_H^2 \right). \end{aligned}$$

From (11) and (14), each of the terms here is bounded by the energy, i.e., for $j = 1, 2$,

$$\|u_j'(S)\|_H^2 + \|u_j(S)\|_H^2 \le CE(U(S)) \quad \text{and} \quad \|u_j'(T)\|_H^2 + \|u_j(T)\|_H^2 \le CE(U(T)),$$

where $C = C(\delta, p^+)$. Since the energy is decaying and $T \ge S$, we have $E(U(T)) \le E(U(S))$, so that

$$\left| \int_S^T (u_1'', u_2)_H - (u_2'', u_1)_H dt \right| \le CE(U(S)).$$

Now, (21) becomes

$$\begin{aligned} \delta \int_S^T (Pu_2, u_2)_H dt &\le \int_S^T (Pu_1, u_1)_H dt + \left| \int_S^T (Bu_1', u_2)_H dt \right| + CE(U(S)) \\ &\le \int_S^T (Pu_1, u_1)_H dt + \frac{C}{\varepsilon} \int_S^T (Bu_1', u_1')_H dt + \varepsilon \int_S^T (Bu_2, u_2)_H dt + CE(U(S)), \quad (22) \end{aligned}$$

for all $\varepsilon > 0$, since B is selfadjoint on H. From (19), we have $\int_S^T (Bu_1', u_1')_H dt \le CE(U(S))$, and (22) yields

$$\delta \int_S^T (Pu_2, u_2)_H dt \le \int_S^T (Pu_1, u_1)_H dt + \varepsilon \int_S^T (Bu_2, u_2)_H dt + C\left(1 + \frac{1}{\varepsilon}\right) E(U(S)) \qquad (23)$$

for all $\varepsilon > 0$ and $0 \le S \le T$. By a density argument, we deduce that (23) holds for every $U^0 \in \mathcal{H}$. □

We can now prove Theorem 2.4.

Proof of Theorem 2.4. We first prove assertion (i). Assume that $U^0 \in \mathcal{D}(\mathcal{A}_{\delta,P})$, then, the solution U of (5) is in $\mathcal{C}^0([0,+\infty); \mathcal{D}(\mathcal{A}_{P,\delta})) \cap \mathcal{C}^1([0,+\infty); \mathcal{H})$ (see Proposition 2.2). We denote by $e_j(t) = 1/2\left(\|u_j'\|_H^2 + \|u_j\|_V^2\right)$, $j = 1, 2$, the partial energies. The regularity of $U(t)$ gives in particular $Pu_2 \in \mathcal{C}^1([0,+\infty); H)$ and $P^* u_1 \in \mathcal{C}^1([0,+\infty); H)$, so that assumptions $(A2)$ and $(A3)$ yield

$$\int_S^T e_1(t)\ dt \le CE(U(S)) + \beta_1 \int_S^T \|\delta P u_2(t)\|_H^2\ dt + \gamma_1 \int_S^T \langle Bu_1', u_1'\rangle_{V',V}\, dt, \qquad (24)$$

$$\int_S^T e_2(t)\ dt \le CE(U(S)) + \beta_2 \int_S^T \|P^* u_1(t)\|_H^2\ dt + \gamma_2 \int_S^T \|\Pi_P u_2'(t)\|_H^2 dt, \qquad (25)$$

since $e_j(t) \le CE(U(t)) \le CE(U(S))$ for $t \ge S$. From (19), we have $\int_S^T \langle Bu_1', u_1'\rangle_{V',V}\, dt \le E(U(S))$, so that (24) yields

$$\int_S^T e_1(t)\ dt \le CE(U(S)) + \beta_1 \delta^2 (p^+)^2 \int_S^T \|u_2(t)\|_H^2\ dt. \qquad (26)$$

On the other side, assumption $(A1)$ and the coupling relation (20) of Lemma 3.2, applied to $U' \in \mathcal{C}^0([0,+\infty); \mathcal{H})$ give, for all $\varepsilon > 0$,

$$\delta p^- \int_S^T \|\Pi_P u_2'\|_H^2 dt \le \delta \int_S^T (Pu_2', u_2')_H dt \le p^+ \int_S^T \|u_1'\|_H^2 dt + \varepsilon b^+ \int_S^T \|u_2'\|_H^2 dt + C_\varepsilon E(U'(S)). \ (27)$$

Replacing (27) in (25), we obtain

$$\begin{aligned} \delta p^- \int_S^T e_2(t)\ dt \le\ & CE(U(S)) + \beta_2 \delta p^- (p^+)^2 \int_S^T \|u_1\|_H^2\ dt \\ &+ \gamma_2 p^+ \int_S^T \|u_1'\|_H^2 dt + \varepsilon \gamma_2 b^+ \int_S^T \|u_2'\|_H^2 dt + C_\varepsilon E(U'(S)). \quad (28) \end{aligned}$$

Then, recalling that for all $v \in V$, $\|v\|_H^2 \leq 1/\lambda \|v\|_V^2$, and adding (26) and (28), we obtain, for all $\varepsilon > 0$,

$$\left(\frac{1}{2} - \gamma_2 p^+\right) \int_S^T \|u_1'\|_H^2 dt + \left(\frac{1}{2} - \frac{\beta_2 \delta (p^+)^2 p^-}{\lambda}\right) \int_S^T \|u_1\|_V^2 dt + \left(\frac{\delta p^-}{2} - \varepsilon \gamma_2 b^+\right) \int_S^T \|u_2'\|_H^2 dt$$
$$+ \left(\frac{\delta p^-}{2} - \frac{\beta_1 \delta^2 (p^+)^2}{\lambda}\right) \int_S^T \|u_2\|_V^2 dt \leq CE(U(S)) + C_\varepsilon E(U'(S)). \quad (29)$$

We now set $p_* = \min\left\{\frac{1}{2\gamma_2}, \lambda\right\} > 0$ and $\delta_* = \delta_*(p^+, p^-) = \min\left\{\frac{\lambda}{2\beta_2 (p^+)^2 p^-}, \frac{\lambda p^-}{2\beta_1 (p^+)^2}, \frac{\lambda}{p^+}\right\} > 0$. Then for all $p^+ \in (0, p_*)$ and $\delta \in (0, \delta_*)$, one can choose $0 < \varepsilon < \frac{\delta p^-}{2b^+}$, so that the following bound on the energy holds

$$\begin{aligned} \int_S^T E(U(t)) dt &\leq C(\delta, p^+) \int_S^T \left(\|u_1'(t)\|_H^2 + \|u_1(t)\|_V^2 dt + \|u_2'(t)\|_H^2 + \|u_2(t)\|_V^2\right) dt \\ &\leq C(\delta, p^+, p^-) \left(E(U(S)) + E(U'(S))\right), \quad \forall 0 \leq S \leq T, \quad \forall U^0 \in \mathcal{D}(\mathcal{A}_{\delta,P}), \end{aligned}$$

from (29) and the choice of p_* and δ_*. Using now Lemma 3.1, we obtain

$$E(U(t)) \leq \frac{c}{t^n} \sum_{p=0}^n E(U^{(p)}(0)) \quad \forall t > 0, \quad \forall U^0 \in \mathcal{D}(\mathcal{A}_{\delta,P}^n),$$

and (i) is proved.

We now prove the first case of (ii) and suppose (15). Thanks to this assumption, the coupling relation (20) of Lemma 3.2, applied to $U \in \mathcal{C}^1([0, +\infty); \mathcal{H})$ gives for all $\varepsilon > 0$

$$\delta \int_S^T (Pu_2, u_2)_H dt \leq \int_S^T (Pu_1, u_1)_H dt + \frac{\varepsilon b^+}{p^-} \int_S^T (Pu_2, u_2)_H dt + C_\varepsilon E(U(S)).$$

This and assumption (15) yield for all $0 < \varepsilon < \frac{p^- \delta}{b^+}$

$$\int_S^T \|Pu_2\|_H^2 dt \leq p^+ \int_S^T (Pu_2, u_2)_H dt \leq \frac{(p^+)^2}{\delta - \varepsilon b^+ / p^-} \int_S^T \|u_1\|_H^2 dt + C_\varepsilon E(U(S)),$$

so that (24) gives, for all $0 < \varepsilon < \frac{p^- \delta}{b^+}$,

$$\int_S^T e_1(t) dt \leq \frac{\beta_1 \delta^2 (p^+)^2}{\delta - \varepsilon b^+ / p^-} \int_S^T \|u_1\|_H^2 dt + C_\varepsilon E(U(S)).$$

Now we set $\varepsilon = (1 - \eta) \frac{p^- \delta}{b^+}$ and we obtain, for all $\eta \in (0, 1)$,

$$\frac{1}{2} \int_S^T \|u_1'\|_H^2 dt + \left(\frac{1}{2} - \frac{\beta_1 \delta (p^+)^2}{\eta \lambda}\right) \int_S^T \|u_1\|_V^2 dt \leq C_\eta E(U(S)). \quad (30)$$

Choosing $p_* = \min\left\{\frac{1}{2\beta_1}, \lambda\right\}$, we have for every $p^+ < p_*$

$$(p^+)^2 < \frac{\lambda}{2\beta_1 \delta}, \quad (31)$$

since δ is chosen such that $0 < p^+ < \lambda/\delta$. From (31), for all $p^+ \in (0, p_*)$ and $\delta \in (0, \lambda/p^+)$, there exists $0 < \eta < 1$, such that $\frac{2\beta_1 \delta (p^+)^2}{\lambda} < \eta$ and (30) implies the existence of $C = C(\delta, p^+)$, such that

$$\int_S^T e_1(t) dt \leq CE(U(S)). \quad (32)$$

Besides, the coupling relation (20) of Lemma 3.2, applied to $U' \in \mathcal{C}^0([0,+\infty);\mathcal{H})$ implies in this case, for all $\varepsilon > 0$,

$$\delta \int_S^T (Pu_2', u_2')_H dt \le \int_S^T (Pu_1', u_1')_H dt + \frac{\varepsilon b^+}{p^-} \int_S^T (Pu_2', u_2')_H dt + C_\varepsilon E(U'(S)).$$

Proceeding as above, we obtain for all $0 < \varepsilon < p^-\delta/b^+$,

$$p^- \left(\delta - \frac{\varepsilon b^+}{p^-}\right) \int_S^T \|\Pi_P u_2'\|_H^2 dt \le p^+ \int_S^T \|u_1'\|_H^2 dt + C_\varepsilon E(U'(S)). \tag{33}$$

Fixing $\varepsilon \in (0, p^-\delta/b^+)$ and replacing (33) in (25), we obtain for some $C = C(\delta, b^+, p^-, p^+)$,

$$\begin{aligned} \int_S^T e_2(t)\ dt &\le CE(U(S)) + \beta_2 (p^+)^2 \int_S^T \|u_1\|_H^2\ dt + C\int_S^T \|u_1'\|_H^2 dt + CE(U'(S)) \\ &\le C\int_S^T e_1(t)dt + CE(U(S)) + CE(U'(S)). \end{aligned}$$

Estimate (32) on e_1 gives

$$\int_S^T e_2(t) \le CE(U(S)) + CE(U'(S)),$$

so that the following bound on the energy holds

$$\begin{aligned} \int_S^T E(U(t))dt &\le C(\delta, p^+) \int_S^T \left(\|u_1'(t)\|_H^2 + \|u_1(t)\|_V^2 dt + \|u_2'(t)\|_H^2 + \|u_2(t)\|_V^2\right) dt \\ &\le C(\delta, p^+, p^-) \left(E(U(S)) + E(U'(S))\right), \quad \forall 0 \le S \le T, \quad \forall U^0 \in \mathcal{D}(\mathcal{A}_{\delta,P}). \end{aligned}$$

We conclude the proof of the first part of (ii) as in Case (i) with Lemma 3.1.

To conclude the proof of (ii), suppose now assumption (16), i.e., there exists $b^- > 0$ such that $(Pv, v)_H \le p^+/b^-(Bv, v)_H$, for all $v \in H$. In this case, the coupling relation (20) of Lemma 3.2, applied to $U' \in \mathcal{C}^0([0,+\infty);\mathcal{H})$ gives

$$\delta p^- \int_S^T \|\Pi_P u_2'\|_H^2 dt \le \delta \int_S^T (Pu_2', u_2')_H dt \le \frac{p^+}{b^-} \int_S^T (Bu_1', u_1')_H dt + \varepsilon b^+ \int_S^T \|u_2'\|_H^2 dt + C_\varepsilon E(U'(S)). \tag{34}$$

We recall that $\int_S^T (Bu_1', u_1')_H dt \le E(U(S))$. Replacing (34) in (25), we obtain

$$\delta \int_S^T e_2(t)\ dt \le CE(U(S)) + \beta_2 \delta (p^+)^2 \int_S^T \|u_1\|_H^2\ dt + \frac{\varepsilon b^+ \gamma_2}{p^-} \int_S^T \|u_2'\|_H^2 dt + C_\varepsilon E(U'(S)), \tag{35}$$

which, summed with (26), yields

$$\begin{aligned} \frac{1}{2}\int_S^T \|u_1'\|_H^2 dt + \left(\frac{1}{2} - \frac{\beta_2 \delta (p^+)^2}{\lambda}\right) \int_S^T \|u_1\|_V^2 dt + \left(\frac{\delta}{2} - \frac{\varepsilon b^+ \gamma_2}{p^-}\right) \int_S^T \|u_2'\|_H^2 dt \\ + \left(\frac{\delta}{2} - \frac{\beta_1 \delta^2 (p^+)^2}{\lambda}\right) \int_S^T \|u_2\|_V^2 dt \le CE(U(S)) + C_\varepsilon E(U'(S)). \end{aligned} \tag{36}$$

Now, setting $p_* = \min\left\{\frac{1}{2\beta_1}, \frac{1}{2\beta_2}, \lambda\right\}$, we have for every $p^+ < p_*$

$$(p^+)^2 < \min\left\{\frac{\lambda}{2\delta\beta_1}; \frac{\lambda}{2\delta\beta_2}\right\}, \tag{37}$$

since δ is chosen such that $0 < p^+ < \lambda/\delta$. From (37), for all $p^+ \in (0, p_*)$ and $\delta \in (0, \lambda/p^+)$, one can choose $0 < \varepsilon < \frac{\delta p^-}{2b^+\gamma_2}$, so that the following bound on the energy holds

$$\begin{aligned} \int_S^T E(U(t))dt &\le C(\delta, p^+) \int_S^T \left(\|u_1'(t)\|_H^2 + \|u_1(t)\|_V^2 dt + \|u_2'(t)\|_H^2 + \|u_2(t)\|_V^2\right) dt \\ &\le C(\delta, p^+, p^-) \left(E(U(S)) + E(U'(S))\right), \quad \forall 0 \le S \le T, \quad \forall U^0 \in \mathcal{D}(\mathcal{A}_{\delta,P}). \end{aligned}$$

We conclude the proof of the last part of (ii) as before with Lemma 3.1. This ends the proof of Theorem 2.4 $\square$

3.3 Proof of Theorem 2.7, the case B unbounded

We first state the analogous of Lemma 3.2, that provides a coupling relation between u_1 and u_2.

Lemma 3.3. *Assume (A4u) and (12). Then, for all $U^0 = (u_1^0, u_2^0, u_1^1, u_2^1) \in \mathcal{D}(\mathcal{A}_{\delta,P})$, the solution $U(t) = \exp(-t\mathcal{A}_{\delta,P})U^0 = (u_1, u_2, v_1, v_2)$ of (5) satisfies for some $C = C(\delta, p^+) > 0$*

$$\delta \int_S^T (Pu_2, u_2)_H dt \leq \int_S^T (P^* u_1, \Pi_V u_1)_H dt + \varepsilon \int_S^T \|u_2\|_H^2 dt + \frac{C}{\varepsilon} E(U'(S)) + CE(U(S)) \tag{38}$$

for all $\varepsilon > 0$ and $0 \leq S \leq T$.

Remark 3.4. The main difference with the bounded case (Lemma 3.3) is that here, the energy of the derivative of U is needed in the coupling relation.

Proof. Assume first that $U^0 \in \mathcal{D}(\mathcal{A}_{\delta,P}^2)$. In this case, the solution $U(t) = \exp(-t\mathcal{A}_{\delta,P})U^0 = (u_1, u_2, v_1, v_2)$ of (5) is in $\mathcal{C}^0([0,+\infty); \mathcal{D}(\mathcal{A}_{\delta,P}^2)) \cap \mathcal{C}^1([0,+\infty); \mathcal{D}(\mathcal{A}_{\delta,P})) \cap \mathcal{C}^2([0,+\infty); \mathcal{H})$. Hence $U = (u_1, u_2, v_1, v_2)$ satisfies

$$\begin{cases} v_1 = u_1', \quad v_2 = u_2', \\ u_1'' + A_1 u_1 + B u_1' + \delta P u_2 = 0 & \text{in } H, \\ u_2'' + A_2 u_2 + P^* u_1 = 0 & \text{in } H, \end{cases}$$

As a consequence, we have $\forall 0 \leq S \leq T$,

$$\int_S^T (u_1'' + A_1 u_1 + B u_1' + \delta P u_2, u_2)_H - (u_2'' + A_2 u_2 + P^* u_1, \Pi_V u_1)_H dt = 0,$$

i.e., $K_1 + K_2 + K_3 = 0$, with

$$\begin{aligned} K_1 &= \int_S^T (u_1'', u_2)_H - (u_2'', \Pi_V u_1)_H dt\,, \\ K_2 &= \int_S^T (A_1 u_1 + B u_1', u_2)_H - (A_2 u_2, \Pi_V u_1)_H dt\,, \\ K_3 &= \int_S^T \delta(P u_2, u_2)_H - (P^* u_1, \Pi_V u_1)_H dt. \end{aligned}$$

We first consider K_1. Since U^0 is taken in $\mathcal{D}(\mathcal{A}_{\delta,P}^2)$, $u_i \in \mathcal{C}^2([0,+\infty); V_i)$ for $i = 1, 2$. Hence, $(\Pi_V u_1)'' = \Pi_V(u_1'')$ and

$$K_1 = \int_S^T (u_1'' - \Pi_V u_1'', u_2)_H dt + [(\Pi_V u_1', u_2)_H - (\Pi_V u_1, u_2')_H]_S^T\,.$$

As a consequence, for all $\varepsilon > 0$,

$$\begin{aligned} |K_1| \leq \frac{C}{\varepsilon} \int_S^T \|u_1'' - \Pi_V u_1''\|_H^2 + \varepsilon \int_S^T \|u_2\|_H^2 dt \\ + \sum_{j=1,2} \left(\|u_j'(S)\|_H^2 + \|u_j'(T)\|_H^2 + \|u_j(S)\|_H^2 + \|u_j(T)\|_H^2 \right). \end{aligned} \tag{39}$$

From (11) and (12), each of the terms of the sum is bounded by the energy, i.e., for $j = 1, 2$,

$$\|u_j'(S)\|_H^2 + \|u_j(S)\|_H^2 \leq CE(U(S)) \quad \text{and} \quad \|u_j'(T)\|_H^2 + \|u_j(T)\|_H^2 \leq CE(U(T)) \leq CE(U(S)),$$

since the energy is decaying and $T \geq S$. Replacing this in (39), and using assumption $(A4u)$, we obtain, for all $\varepsilon > 0$,

$$|K_1| \leq \frac{C}{\varepsilon} \int_S^T \beta \langle Bu_1'', u_1'' \rangle_{V_1', V_1} \, dt + \varepsilon \int_S^T \|u_2\|_H^2 dt + CE(U(S)).$$

On the other side, Proposition 2.2 gives $E'(U(t)) = -\langle Bu_1', u_1' \rangle_{V_1', V_1}$ for $U^0 \in \mathcal{D}(\mathcal{A}_{\delta,P})$, so that we have $E'(U'(t)) = -\langle Bu_1'', u_1'' \rangle_{V_1', V_1}$ for $U^0 \in \mathcal{D}(\mathcal{A}_{\delta,P}^2)$. Recalling that $E(U(\cdot))$ and $E(U'(\cdot))$ are nonincreasing, we obtain, for all $\varepsilon > 0$,

$$|K_1| \leq \varepsilon \int_S^T \|u_2\|_H^2 dt + \frac{C}{\varepsilon} E(U'(S)) + CE(U(S)).$$

We now consider K_2. From assumption $(A4u)$, we have,

$$(A_1 u_1 + Bu_1', u_2)_H = \langle A_1 u_1 + Bu_1', i(u_2) \rangle_{V_1', V_1} = \langle A_1 u_1, i(u_2) \rangle_{V_1', V_1},$$

since $U^0 \in \mathcal{D}(\mathcal{A}_{\delta,P}^2)$ yields $u_2 \in \mathcal{D}(A_2) \subset V_2$. The definition of Π_V also gives

$$\langle A_1 u_1, i(u_2) \rangle_{V_1', V_1} = \langle A_1 i(\Pi_V u_1), i(u_2) \rangle_{V_1', V_1} = \langle A_2 \Pi_V u_1, u_2 \rangle_{V_2', V_2} = \langle A_2 u_2, \Pi_V u_1 \rangle_{V_2', V_2}.$$

Moreover, since $u_2 \in \mathcal{D}(A_2)$, we have $\langle A_2 u_2, \Pi_V u_1 \rangle_{V_2', V_2} = (A_2 u_2, \Pi_V u_1)_H$, so that

$$(A_1 u_1 + Bu_1', u_2)_H = (A_2 u_2, \Pi_V u_1)_H,$$

and $K_2 = 0$.

Finally, replacing in

$$\int_S^T \delta (Pu_2, u_2)_H dt = \int_S^T (P^* u_1, \Pi_V u_1)_H dt - K_1 - K_2$$

the estimates on $|K_1|$ and K_2, we have for all $\varepsilon > 0$,

$$\int_S^T \delta (Pu_2, u_2)_H dt \leq \int_S^T (P^* u_1, \Pi_V u_1)_H dt + \varepsilon \int_S^T \|u_2\|_H^2 dt + \frac{C}{\varepsilon} E(U'(S)) + CE(U(S)). \tag{40}$$

This concludes the proof of the proposition for an initial datum $U^0 \in \mathcal{D}(\mathcal{A}_{\delta,P}^2)$. By a density argument, we deduce that (40) holds for every $U^0 \in \mathcal{D}(\mathcal{A}_{\delta,P})$. □

We can now prove Theorem 2.7. This proof follows the same steps as in the proof of Theorem 2.4 point (i). We give it for the sake of completeness.

Proof of Theorem 2.7. Assume that $U^0 \in \mathcal{D}(\mathcal{A}_{\delta,P}^2)$, then, the solution U of (5) is in

$$\mathcal{C}^0([0, +\infty); \mathcal{D}(\mathcal{A}_{P,\delta}^2)) \cap \mathcal{C}^1([0, +\infty); \mathcal{D}(\mathcal{A}_{P,\delta})) \cap \mathcal{C}^2([0, +\infty); \mathcal{H})$$

(see Proposition 2.2). We recall the notation $e_j(t) = 1/2 \left(\|u_j'\|_H^2 + \|u_j\|_{V_j}^2 \right)$, $j = 1, 2$. The regularity of $U(t)$ gives in particular $Pu_2 \in \mathcal{C}^1([0, +\infty); H)$ and $P^* u_1 \in \mathcal{C}^1([0, +\infty); H)$, so that assumptions $(A2)$ and $(A3)$ yield

$$\int_S^T e_1(t) \, dt \leq CE(U(S)) + \beta_1 \int_S^T \|\delta P u_2(t)\|_H^2 \, dt + \gamma_1 \int_S^T \langle Bu_1', u_1' \rangle_{V_1', V_1} \, dt, \tag{41}$$

$$\int_S^T e_2(t) \, dt \leq CE(U(S)) + \beta_2 \int_S^T \|P^* u_1(t)\|_H^2 \, dt + \gamma_2 \int_S^T \|\Pi_P u_2'(t)\|_H^2 dt. \tag{42}$$

From (19), we have $\int_S^T \langle Bu_1', u_1'\rangle_{V_1', V_1}\, dt \leq E(U(S))$, so that (41) yields

$$\int_S^T e_1(t)\ dt \leq CE(U(S)) + \beta_1\delta^2(p^+)^2 \int_S^T \|u_2(t)\|_H^2\ dt. \tag{43}$$

On the other side, assumption $(A1)$ and the coupling relation (38) of Lemma 3.3, applied to $U' \in \mathcal{C}^0([0,+\infty); \mathcal{D}(\mathcal{A}_{P,\delta}))$ give, for all $\varepsilon > 0$,

$$\begin{aligned} \delta p^- \int_S^T \|\Pi_P u_2'\|_H^2 dt &\leq \delta \int_S^T (Pu_2', u_2')_H dt \\ &\leq p^+ \int_S^T \|u_1'\|_H^2 dt + \varepsilon \int_S^T \|u_2'\|_H^2 dt + C_\varepsilon E(U''(S)) + CE(U'(S)). \end{aligned} \tag{44}$$

Replacing (44) in (42), we obtain

$$\begin{aligned} \delta p^- \int_S^T e_2(t)\ dt \leq CE(U(S)) + CE(U'(S)) + \beta_2\delta p^-(p^+)^2 \int_S^T \|u_1(t)\|_H^2\ dt \\ + \gamma_2 p^+ \int_S^T \|u_1'\|_H^2 dt + \varepsilon\gamma_2 \int_S^T \|u_2'\|_H dt + C_\varepsilon E(U''(S)). \end{aligned} \tag{45}$$

Then, recalling that for all $v \in V_j$, $\|v\|_H^2 \leq 1/\lambda_j \|v\|_{V_j}^2$, $j = 1, 2$, and adding (43) and (45), we obtain, for all $\varepsilon > 0$,

$$\begin{aligned} \left(\frac{1}{2} - \gamma_2 p^+\right) \int_S^T \|u_1'\|_H^2 dt + \left(\frac{1}{2} - \frac{\beta_2\delta(p^+)^2 p^-}{\lambda_1}\right) \int_S^T \|u_1\|_{V_1}^2 dt + \left(\frac{\delta p^-}{2} - \varepsilon\gamma_2\right) \int_S^T \|u_2'\|_H^2 dt \\ + \left(\frac{\delta p^-}{2} - \frac{\beta_1\delta^2(p^+)^2}{\lambda_2}\right) \int_S^T \|u_2\|_{V_2}^2 dt \leq CE(U(S)) + CE(U'(S)) + C_\varepsilon E(U''(S)). \end{aligned} \tag{46}$$

We now set $p_* = \min\left\{\frac{1}{2\gamma_2}, \lambda_2\right\} > 0$ and $\delta_* = \delta_*(p^+, p^-) = \min\left\{\frac{\lambda_1}{2\beta_2(p^+)^2 p^-}, \frac{\lambda_2 p^-}{2\beta_1(p^+)^2}, \frac{\lambda_1}{p^+}\right\} > 0$. Then for all $p^+ \in (0, p_*)$ and $\delta \in (0, \delta_*)$, one can choose $0 < \varepsilon < \frac{\delta p^-}{2\gamma_2}$, so that the following bound on the energy holds for all $0 \leq S \leq T$ and $U^0 \in \mathcal{D}(\mathcal{A}_{\delta,P}^2)$,

$$\begin{aligned} \int_S^T E(U(t)) dt &\leq C(\delta, p^+) \int_S^T \left(\|u_1'(t)\|_H^2 + \|u_1(t)\|_V^2 dt + \|u_2'(t)\|_H^2 + \|u_2(t)\|_V^2\right) dt \\ &\leq C(\delta, p^+, p^-) \left(E(U(S)) + E(U'(S) + E(U''(S))\right), \end{aligned}$$

from (46) and the choice of p_* and δ_*. Using now Lemma 3.1, we obtain

$$E(U(t)) \leq \frac{c}{t^n} \sum_{p=0}^{2n} E(U^{(p)}(0)) \quad \forall t > 0, \quad \forall U^0 \in \mathcal{D}(\mathcal{A}_{\delta,P}^{2n}),$$

and Theorem 2.7 is proved. □

4 Applications

We now apply the results of Theorem 2.4 and Theorem 2.7 to different second order coupled systems. In each of the following sections, we first explain how the problem can be formulated in the abstract setting of Section 2.1. All these systems are well-posed in the spaces we choose, according to Proposition 2.2. Hence, we only have to check that assumptions $(A1) - (A4)$ in order to apply Theorem 2.4 or Theorem 2.7 and obtain the expected stability results. All this strategy shall be followed in Section 4.1 to address internal stabilization of coupled wave equations, in Section 4.2 to address boundary stabilization of coupled wave equations, and in Section 4.3 to address internal stabilization of coupled plate equations. For the sake of brevity, we do not treat the case of boundary stabilization of coupled plate equations. However, one can prove as well that Theorem 2.7 can be applied in this case.

4.1 Internal stabilization of locally coupled wave equations

Here, we prove Theorem 1.2 in the context presented in the introduction. We recall that $\Omega \subset \mathbb{R}^N$, $\Gamma = \partial\Omega$ is of class $\mathcal{C}^2$, and consider the evolution problem (3). We take $H = L^2(\Omega)$, $V = H_0^1(\Omega)$ with the usual inner products and norms. We moreover take for B and P respectively the multiplication in L^2 by the functions $b, p \in L^\infty(\Omega)$, satisfying

$$\begin{cases} 0 \leq b \leq b^+ \text{ and } 0 \leq p \leq p^+ \text{ on } \Omega, \\ b \geq b^- > 0 \text{ on } \omega_b, \\ p \geq p^- > 0 \text{ on } \omega_p, \end{cases}$$

for ω_b and ω_p two open subsets of Ω, satisfying the PMGC. As a consequence, assumption $(A4b)$ is satisfied and assumption $(A1)$ is fulfilled taking for Π_P the multiplication by $\mathbb{1}_{\omega_p}$. It only remains to check assumptions $(A2)$ and $(A3)$, that are consequences of the following lemma.

Lemma 4.1. *Let ω be a subset of Ω satisfying the PMGC. Then, there exist $\alpha, \beta, \gamma > 0$ such that for all $f \in \mathcal{C}^1(\mathbb{R}^+; L^2(\Omega))$ and all $0 \leq S \leq T$, the solution u of*

$$\begin{cases} u'' - \Delta u = f & \text{in } (0,\infty) \times \Omega, \\ u = 0 & \text{on } (0,\infty) \times \Gamma, \\ (u, u')(0, \cdot) = (u^0(\cdot), u^1(\cdot)) & \in H_0^1(\Omega) \times L^2(\Omega), \end{cases} \tag{47}$$

satisfies, with $e(t) = 1/2 \left(\|u'\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)} \right)$, the inequality

$$\int_S^T e(t)\ dt \leq \alpha(e(S) + e(T)) + \beta \int_S^T \|f(t)\|^2_{L^2(\Omega)}\ dt + \gamma \int_S^T \|\mathbb{1}_\omega u'(t)\|^2_{L^2(\Omega)} dt. \tag{48}$$

This Lemma direcly yields $(A2)$, since ω_p is supposed to satisfy the PMGC. To prove $(A3)$, we note that the solution u of (47) is $\mathcal{C}^1(\mathbb{R}^+; L^2(\Omega))$ and apply the lemma to $u = u_1$, $\omega = \omega_b$ and $f = f_1 - bu_1' \in \mathcal{C}^1(\mathbb{R}^+; L^2(\Omega))$. This yields

$$\begin{aligned} \int_S^T e_1(t)\ dt &\leq \alpha(e_1(S) + e_1(T)) + \beta \int_S^T \|f_1(t) - bu_1'(t)\|^2_{L^2(\Omega)}\ dt + \gamma \int_S^T \|\mathbb{1}_{\omega_b} u_1'(t)\|^2_{L^2(\Omega)} dt \\ &\leq \alpha(e_1(S) + e_1(T)) + 2\beta \int_S^T \|f_1(t)\|^2_{L^2(\Omega)} + \left(2\beta + \frac{\gamma}{(b^-)^2}\right) \int_S^T \|bu_1'(t)\|^2_{L^2(\Omega)}\ dt, \end{aligned}$$

which is $(A3)$ with $\beta_1 = 2\beta$ and $\gamma_1 = \left(2\beta + \frac{\gamma}{(b^-)^2}\right)$.

Now applying Theorem 2.4, we have proved the following theorem.

Theorem 4.2. *(i) Suppose that ω_b and ω_p satisfy the PMGC. Then there exists $p_* \in (0, \lambda]$ such that for all $0 < p^+ < p_*$ there exists $\delta_* = \delta_*(p^+, p^-) \in (0, \frac{\lambda}{p^+}]$, such that for all $\delta \in (0, \delta_*)$, the solution $U(t) = (u_1(t), u_2(t), u_1'(t), u_2'(t))$ of (3) satisfies for $n \in \mathbb{N}$,*

$$E(U(t)) \leq \frac{c}{t^n} \sum_{i=0}^n E(U^{(i)}(0)) \quad \forall t > 0,\ U^0 = (u_1^0, u_2^0, u_1^1, u_2^1) \in \mathcal{D}(\mathcal{A}^n_{P,\delta}),$$

where $\mathcal{D}(\mathcal{A}^n_{P,\delta})$ is defined in (9), and c is a constant depending on δ, p^+, p^- and n. Besides, if $U^0 \in \mathcal{H} = (H_0^1)^2 \times (L^2)^2$, then $E(U(t))$ converges to zero as t goes to infinity.

(ii) If moreover either $\omega_b \subset \omega_p$ or $\omega_p \subset \omega_b$, then the result holds for $\delta_* = \frac{\lambda}{p^+}$.

Theorem 1.2 is now a corollary of Theorem 4.2 in the case of smooth coefficients since Lemma 2.6 allows us to explicit the spaces $\mathcal{D}(\mathcal{A}^n_{P,\delta})$.

Proof of Lemma 4.1. We here prove the energy estimate (48) for the solutions of (47), using the piecewise multiplier method. We proceed as in [Mar99] and [AB05]. The subset ω satisfies the PMGC. Hence, denoting by Ω_j and x_j, $j = 1...J$ the sets and the points given by the PMGC, we have $\omega \supset$

$\mathcal{N}_\varepsilon\left(\bigcup_j \gamma_j(x_j) \cup (\Omega \setminus \bigcup_j \Omega_j)\right) \cap \Omega$. In this expression, $\mathcal{N}_\varepsilon(\mathcal{O}) = \left\{x \in \mathbb{R}^N, d(x, \mathcal{O}) \le \varepsilon\right\}$ with $d(\cdot, \mathcal{O})$ the usual euclidian distance to the subset $\mathcal{O}$ of $\mathbb{R}^N$, and $\gamma_j(x_j) = \{x \in \Gamma_j, (x - x_j) \cdot \nu_j(x) > 0\}$, where ν_j denotes the outward unit normal to $\Gamma_j = \partial\Omega_j$. Let $\varepsilon_0 < \varepsilon_1 < \varepsilon_2 < \varepsilon$ and define

$$Q_i = \mathcal{N}_{\varepsilon_i}\left(\bigcup_j \gamma_j(x_j) \cup (\Omega \setminus \bigcup_j \Omega_j)\right), \quad i = 0, 1, 2.$$

Since $(\overline{\Omega_j} \setminus Q_1) \cap \overline{Q_0} = \emptyset$, we can construct a function $\psi_j \in \mathcal{C}_0^\infty(\mathbb{R}^N)$ which satisfies

$$0 \le \psi_j \le 1, \quad \psi_j = 1 \text{ on } \overline{\Omega_j} \setminus Q_1, \quad \psi_j = 0 \text{ on } Q_0.$$

For $m_j(x) = x - x_j$, we define the $\mathcal{C}^1$ vector field on Ω:

$$h(x) = \begin{cases} \psi_j(x) m_j(x) & \text{if } x \in \Omega_j, \\ 0 & \text{if } x \in \Omega \setminus \cup_j \Omega_j. \end{cases}$$

Multiplying (47) by the multiplier $h \cdot \nabla u$ and integrating on $(S, T) \times \Omega_j$, we obtain

$$\int_S^T \int_{\Omega_j} h(x) \cdot \nabla u (u'' - \Delta u - f)\, dx\, dt = 0\,.$$

For the sake of concision, we will omit the $dx\,dt$ in the following integrals. This gives, after integration by parts

$$\int_S^T \int_{\Gamma_j} \left(\partial_{\nu_j} u h \cdot \nabla u + \frac{1}{2}(h \cdot \nu)(u'^2 - |\nabla u|^2)\right) = \left[\int_{\Omega_j} u' h \cdot \nabla u\right]_S^T$$
$$+ \int_S^T \int_{\Omega_j} \left(\frac{1}{2} \operatorname{div} h \,(u'^2 - |\nabla u|^2) + \sum_{i,k} \frac{\partial h_k}{\partial x_i} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_k} - f h \cdot \nabla u\right). \quad (49)$$

Thanks to the choice of ψ_j, only the boundary term on $(\Gamma_j \setminus \gamma_j(x_j)) \cap \Gamma$ is nonvanishing in the left hand side of (49). But on this part of the boundary $u = 0$, so that $u' = 0$ and $\nabla u = \partial_\nu u\, \nu = \partial_{\nu_j} u\, \nu_j$. Hence, the left hand side of (49) reduces to

$$\frac{1}{2} \int_S^T \int_{(\Gamma_j \setminus \gamma_j(x_j)) \cap \Gamma} (\partial_{\nu_j} u)^2 \psi_j (m_j \cdot \nu_j) \le 0\,.$$

Therefore, since $\psi_j = 0$ on Q_0 and thanks to the above inequality used in (49), we deduce that

$$\left[\int_{\Omega_j} u' h \cdot \nabla u\right]_S^T + \int_S^T \int_{\Omega_j \setminus Q_0} \left(\frac{1}{2} \operatorname{div} h \,(u'^2 - |\nabla u|^2) + \sum_{i,k} \frac{\partial h_k}{\partial x_i} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_k} - f h \cdot \nabla u\right) \le 0.$$

Using $\psi_j = 1$ on $\overline{\Omega_j} - Q_1$ and summing the resulting inequalities on j, we obtain

$$\left[\int_\Omega u' h \cdot \nabla u\right]_S^T + \int_S^T \int_{\Omega \setminus Q_1} \left(\frac{1}{2} (N u'^2 + (2 - N)|\nabla u|^2) - \int_S^T \int_\Omega f h \cdot \nabla u\right.$$
$$\le -\sum_j \int_S^T \left[\int_{\Omega_j \cap Q_1} \frac{1}{2} \operatorname{div}(\psi_j m_j)\,(u'^2 - |\nabla u|^2) + \sum_{i,k} \frac{\partial h_k}{\partial x_i} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_k}\right]$$
$$\le C \int_S^T \int_{\Omega \cap Q_1} (u'^2 + |\nabla u|^2), \quad (50)$$

where C is a positive constant which depends only on ψ_j and m_j. We now use the second multiplier $u(N-1)/2$ and therefore evaluate the term

$$\frac{N-1}{2}\int_S^T\int_\Omega u(u''-\Delta u-f)=0.$$

Hence, one has

$$\frac{N-1}{2}\left[\int_\Omega uu'\right]_S^T+\frac{N-1}{2}\int_S^T\int_\Omega|\nabla u|^2-u'^2-uf)=0. \tag{51}$$

We set $M(u)=h\cdot\nabla u+\frac{N-1}{2}u$. Adding (51) to (50), we obtain

$$\int_S^T e(t)\,dt\le C\int_S^T\int_{\Omega\cap Q_1}u'^2+|\nabla u|^2-\left[\int_\Omega M(u)u'\right]_S^T+\int_S^T\int_\Omega M(u)f. \tag{52}$$

We now estimate the terms on the right hand side of (52) as follows. First, we have

$$\left|\left[\int_\Omega M(u)u'\right]_S^T\right|\le C(e(S)+e(T)). \tag{53}$$

Second, we estimate the last term of (52) as follows

$$\left|\int_S^T\int_\Omega M(u)f\right|\le\frac{C}{\mu}\int_S^T\int_\Omega|f|^2+\mu\int_S^T e(t)\,dt\quad\forall\mu>0\,. \tag{54}$$

The difficulty is now to estimate the first term on the right hand side of (52). This is where the piecewise multiplier method takes its place. We just follow techniques developped in [Mar99]. We give the steps for the sake of the completeness. Since $\overline{\mathbb{R}^N\setminus Q_2}\cap\overline{Q_1}=\emptyset$, there exists a function $\xi\in\mathcal{C}_0^\infty(\mathbb{R}^N)$ such that

$$0\le\xi\le1,\quad\xi=1\text{ on }Q_1,\quad\xi=0\text{ on }\mathbb{R}^N\setminus Q_2.$$

Multiplying (47) by ξu and integrating on $[S,T]\times\Omega$, we obtain after integrations by parts

$$\int_S^T\int_\Omega\xi|\nabla u|^2=\int_S^T\int_\Omega\xi|u'|^2+\frac{1}{2}\Delta\xi u^2+-\left[\int_\Omega\xi uu'\right]_S^T+\int_S^T\int_\Omega\xi uf.$$

We thus have

$$\int_S^T\int_{\Omega\cap Q_1}|\nabla u|^2\le C\int_S^T\int_{\Omega\cap Q_2}|u'|^2+u^2+|f|^2+C(e(S)+e(T)).$$

Since $\overline{\mathbb{R}^N\setminus\omega}\cap\overline{Q_2}=\emptyset$, there exists a function $\beta\in\mathcal{C}_0^\infty(\mathbb{R}^N)$ such that

$$0\le\beta\le1,\quad\beta=1\text{ on }Q_2,\quad\beta=0\text{ on }\mathbb{R}^N\setminus\omega.$$

Proceeding as in [CR93], we fix t and consider the solution z of the following elliptic problem

$$\begin{cases}\Delta z=\beta(x)u & \text{in }\Omega,\\ z=0 & \text{on }\Gamma.\end{cases}$$

Hence, z and z' satisfy the following estimates

$$\|z\|_{L^2(\Omega)}\le C\|u\|_{L^2(\Omega)},\qquad\text{and}\qquad\|z'\|^2_{L^2(\Omega)}\le C\int_\Omega\beta|u'|^2\,. \tag{55}$$

Multiplying (47) by z and integrating on $[S,T]\times\Omega$, we obtain after integrations by parts

$$\int_S^T\int_\Omega\beta u^2=\left[\int_\Omega zu'\right]_S^T+\int_S^T\int_\Omega -z'u'-zf.$$

Hence, using the estimates (55) in the above relation, we obtain for all $\eta > 0$

$$\int_S^T \int_{\Omega \cap Q_2} |u|^2 \le \frac{C}{\eta} \int_S^T \int_\omega |u'|^2 + \frac{C}{\eta} \int_S^T \int_\Omega |f|^2 + \eta \int_S^T e + C(e(S) + e(T)). \tag{56}$$

Combined with the estimates (53), (54) and (56) in (52), this gives for all $\mu > 0$

$$\int_S^T e \le C(e(S) + e(T)) + C\mu \int_S^T e + \frac{C}{\mu} \int_S^T \left[\int_\omega |u'|^2 + \int_\Omega |f|^2 \right].$$

Choosing μ sufficiently small, we finally have

$$\int_S^T e \le C(e(S) + e(T)) + C \int_S^T \left[\int_\omega |u'|^2 + \int_\Omega |f|^2 \right],$$

and the lemma is proved. $\square$

4.2 Boundary stabilization of locally coupled wave equations

Here, we are interested in boundary stabilization. The results given generalize the ones of [AB99] and [AB02] where the case of constant coupling has been considered. Given $\Omega \subset \mathbb{R}^N$ and $\Gamma = \partial\Omega$ of class $\mathcal{C}^2$ we shall use the following Boundary Multiplier Geometric Condition (BMGC).

Definition 4.3 (BMGC)**.** Let $\{\Sigma_1, \Sigma_0\}$ be a partition of Γ such that $\overline{\Sigma}_1 \cap \overline{\Sigma}_0 = \emptyset$. We say that $\{\Sigma_1, \Sigma_0\}$ satisfies the BMGC if there exists $x_0 \in \mathbb{R}^N$ such that $m \cdot \nu \le 0$ on Σ_0 and $m \cdot \nu \ge m^- > 0$ on Σ_1, where $m(x) = x - x_0$.

The most simple situation covered by this condition is the case where Ω is star-shaped with respect to x_0. In this case $\Sigma_0 = \emptyset$ and $\Sigma_1 = \Gamma$. Another interesting and somehow more general situation is the case where $\Omega = \Omega_1 \setminus \overline{\Omega_2}$, with Ω_2 and Ω_1 two open subset of $\mathbb{R}^N$, both star-shaped with respect to x_0, and such that $\overline{\Omega_2} \subset \Omega_1$. In this case, $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$ with a disjoint union, $\Sigma_0 = \partial\Omega_2$ and $\Sigma_1 = \partial\Omega_1$ satisfy the BMGC.

For $\Gamma_b \subset \Gamma$, and $\Gamma_0 = \Gamma \setminus \Gamma_b$, we consider the following coupled stabilization problem

$$\begin{cases} u_1'' - \Delta u_1 + \delta p u_2 = 0 & \text{in } (0,\infty) \times \Omega, \\ u_2'' - \Delta u_2 + p u_1 = 0 & \text{in } (0,\infty) \times \Omega, \\ \frac{\partial u_1}{\partial \nu} + m \cdot \nu (l u_1 + b u_1') = 0 & \text{on } (0,\infty) \times \Gamma_b, \\ u_1 = 0 & \text{on } (0,\infty) \times \Gamma_0, \\ u_2 = 0 & \text{on } (0,\infty) \times \Gamma, \\ u_j(0,\cdot) = u_j^0(\cdot)\,,\ u_j'(0,\cdot) = u_j^1(\cdot),\ j = 1,2 & \text{in } \Omega, \end{cases} \tag{57}$$

where l is a non-negative function on Γ_b. Note that we added $m \cdot \nu$, where $m(x) = x - x_0$ in the stabilization term to avoid some technical estimates. This term can be removed, provided that we do some more assumptions on the functions b and l. Here we make the following assumptions on the coefficients b and p

$$\begin{cases} 0 \le b \le b^+ \text{ on } \Gamma, \text{ and } b \ge b^- > 0 \text{ on } \Gamma_b \\ 0 \le p \le p^+ \text{ on } \Omega, \text{ and } p \ge p^- > 0 \text{ on } \omega_p, \end{cases}$$

Moreover we set $H^1_{\Gamma_0}(\Omega) = \{u \in H^1(\Omega), u = 0 \text{ on } \Gamma_0\}$, and we shall assume for the sake of clarity that $l \ne 0$ or $\text{meas}(\Gamma_0) \ne 0$. We take $H = L^2(\Omega)$ and $V_1 = H^1_{\Gamma_0}(\Omega)$ equipped respectively with the L^2 inner product and the inner product $(u,z)_{V_1} = \int_\Omega \nabla u \cdot \nabla z + \int_{\Gamma_b} m \cdot \nu l u z$ and the corresponding norms. Moreover we take $V_2 = H^1_0(\Omega)$ equipped with the inner product $(u,z)_{V_2} = \int_\Omega \nabla u \cdot \nabla z$ and the associated norm. We define the duality mappings A_1 and A_2 as in Section 2.1. We also define the continuous linear operator B from V_1 to V_1' by

$$\langle Bu, z \rangle_{V_1', V_1} = \int_{\Gamma_b} m \cdot \nu \, b \, u \, z \, d\gamma,$$

that satisfies (7). As in Section 4.2, we take for P the multiplication in L^2 by the function $p \in L^\infty$. With these notations, system (57) can be rewritten under the form (5).

Theorem 4.4. *Suppose that ω_p satisfies the PMGC and $\{\Gamma_b, \Gamma_0\}$ satisfies the BMGC. Then there exists $p_* \in (0, \lambda]$ such that for all $0 < p^+ < p_*$ there exists $\delta_* = \delta_*(p^+, p^-) \in (0, \frac{\lambda}{p^+}]$, such that for all $\delta \in (0, \delta_*)$, the solution $U(t) = (u_1(t), u_2(t), u_1'(t), u_2'(t))$ of (57) satisfies, for $n \geq 1$,*

$$E(U(t)) \leq \frac{c}{t^n} \sum_{i=0}^{2n} E(U^{(i)}(0)) \quad \forall t > 0, \quad \forall U^0 = (u_1^0, u_2^0, u_1^1, u_2^1) \in \mathcal{D}(\mathcal{A}_{\delta,P}^{2n}),$$

where c is a constant depending on δ, p^+, p^- and n. Besides, if $U^0 \in \mathcal{H} = H^1_{\Gamma_0} \times H^1_0 \times (L^2)^2$, then $E(U(t))$ converges to zero as t goes to infinity.

We recall that the operator $\mathcal{A}_{\delta,P}$ is defined in (9). As opposed to the results for internal damping, we do not have here a simple expression of $\mathcal{D}(\mathcal{A}_{\delta,P}^{2n})$ in terms of Sobolev spaces.

To prove this Theorem, we just need to check that the assumptions $(A1) - (A4u)$ are satisfied and then apply Theorem 2.7 in a convenient setting. First, assumption $(A1)$ is satisfied with Π_P the multiplication by $\mathbb{1}_{\omega_p}$, and assumption $(A2)$ is a consequence of Lemma 4.5 as in Section 4.1, since the internal coupling is here the same.

We now check assumption $(A4u)$ and follow the lines of [AB02]. For the sake of clarity, we identify $i(\phi)$ with ϕ for $\phi \in V_2$ (where i is the canonical injection from V_2 in V_1). We first remark that the first equality in assumption $(A4u)$ is satisfied thanks to the definition of B and V_2. We define Π_V and A_2 as in Section 2.1. Then, $\Pi_V u_1$ is the weak solution of

$$\begin{cases} -\Delta \Pi_V u_1 = -\Delta u_1 & \text{in } \Omega, \\ \Pi_V u_1 \in V_2, \end{cases}$$

and A_2 is defined by

$$\langle A_2 \phi, \psi \rangle_{V_2', V_2} = \int_\Omega \nabla \phi \cdot \nabla \psi \, dx \ , \quad \forall \psi, \phi \in V_2.$$

We now check the second relation in $(A4u)$. For this, we set $z = u_1 - \Pi_V u_1$, so that z is the weak solution of

$$\begin{cases} -\Delta z = 0 & \text{in } \Omega, \\ z = u_1, & \text{on } \Gamma. \end{cases}$$

By elliptic regularity, we deduce that there exists a constant $c > 0$ such that $\|z\|_H \leq c \|u_{1|\Gamma_b}\|_{L^2(\Gamma_b)}$. Since we assume the BMGC, $m \cdot \nu \, b \geq m^- b^- > 0$ on Γ_b, there exists then $\beta > 0$ such that

$$\|z\|_H^2 \leq \beta \, \langle B u_1, u_1 \rangle_{V_1', V_1} \quad \forall u_1 \in V_1,$$

and $(A4u)$ is satisfied.

The last assumption $(A3)$ is a direct consequence of the following lemma. Theorem 4.4 follows then from Theorem 2.7.

Lemma 4.5. *Suppose that $\{\Gamma_b, \Gamma_0\}$ satisfies the BMGC. Then, there exist $\alpha, \beta, \gamma > 0$ such that for all $f \in \mathcal{C}^1(\mathbb{R}^+; L^2(\Omega))$ and all $0 \leq S \leq T$, the solution u of*

$$\begin{cases} u'' - \Delta u = f & \text{in } (0, \infty) \times \Omega, \\ \frac{\partial u}{\partial \nu} + m \cdot \nu (lu + bu') = 0 & \text{on } (0, \infty) \times \Gamma_b, \\ u = 0 & \text{on } (0, \infty) \times \Gamma_0, \\ (u, u')(0, \cdot) = (u^0(\cdot), u^1(\cdot)) & \in H^1_0(\Omega) \times L^2(\Omega), \end{cases} \tag{58}$$

satisfies, with $e(t) = 1/2 \left(\|u'\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)} \right)$, the inequality

$$\int_S^T e(t) \, dt \leq \alpha (e(S) + e(T)) + \beta \int_S^T \|f(t)\|^2_{L^2(\Omega)} \, dt + \gamma \int_S^T \int_{\Gamma_b} m \cdot \nu b u'^2 dx dt. \tag{59}$$

Proof. Proceeding as in [Kom, Theorem 8.6], we first use the multiplier $Mu = m(x)\cdot\nabla u + \frac{(N-1)}{2}u$ and set $R = \sup\{|m(x)|, x\in\Omega\}$. Multiplying (58) by Mu, we obtain after integrations by parts

$$\int_S^T e(t)dt = \int_S^T\int_\Omega f\,Mu + \left[\int_\Omega u'\,Mu\right]_S^T + \frac{1}{2}\int_S^T\int_{\Gamma_0} m\cdot\nu\left|\frac{\partial u}{\partial\nu}\right|^2 d\gamma dt$$
$$+\int_S^T\int_{\Gamma_b} m\cdot\nu\left(\frac{1}{2}u'^2 - \frac{1}{2}|\nabla u|^2 - (lu+bu')(m\cdot\nabla u + \frac{N-1}{2}u) + l\frac{u^2}{2}\right).$$

Setting $h = \dfrac{u'^2-|\nabla u|^2}{2} + R|lu+bu'||\nabla u| + |2-N|l\dfrac{u^2}{2} - buu'\dfrac{(N-1)}{2}$, this yields, for all $\eta>0$,

$$\int_S^T e(t)dt \le C(e(S)+e(T)) + \frac{C}{\eta}\int_S^T\int_\Omega f^2 + \eta\int_S^T e(t)dt + \int_S^T\int_{\Gamma_b} m\cdot\nu h. \tag{60}$$

We now estimate the term h. Since $b\ge b^->0$ on Γ_b, we have for all $\varepsilon>0$,

$$h \le C\left(1+\frac{1}{\varepsilon}\right)bu' + Clu^2 + \varepsilon u^2. \tag{61}$$

Concerning the last term, in this equation, we have the trace inequality

$$\varepsilon\int_S^T\int_{\Gamma_b} m\cdot\nu u^2 \le C\varepsilon\int_S^T\left(\int_\Omega\frac{|\nabla u|^2}{2} + \int_{\Gamma_b} m\cdot\nu l u^2\right). \tag{62}$$

Hence, for ε and η sufficiently small, using (62) and (61) in (60), we obtain

$$\int_S^T e(t)dt \le C(e(T)+e(S)) + C\int_S^T\int_\Omega f^2 + \int_S^T\int_{\Gamma_b} m\cdot\nu bu'^2 + \int_S^T\int_{\Gamma_b} m\cdot\nu lu^2 \tag{63}$$

It only remains to treat the last term in this inequality. For this, we use the method introduced in [CR93]. Let z be the solution of the following elliptic problem:

$$\begin{cases} \Delta z = 0 & \text{in } \Omega,\\ z = u & \text{on } \Gamma_b,\\ z=0 & \text{on } \Gamma_0.\end{cases}$$

Note that this definition yields

$$\int_\Omega |\nabla z|^2 + z^2 \le C\int_{\Gamma_b} u^2 \quad\text{and}\quad \int_\Omega \nabla z\cdot\nabla u = \int_\Omega |\nabla z|^2. \tag{64}$$

We multiply (58) by z and integrate on $(S,T)\times\Omega$. Integrating by parts and using (64) and the boundary conditions for u, we obtain

$$\begin{aligned}\int_S^T\int_{\Gamma_b} m\cdot\nu lu^2 &= \int_S^T\int_\Omega (u'z' - |\nabla z|^2 + fz) - \left[\int_\Omega u'z\right]_S^T - \int_S^T m\cdot bu'u\\ &\le \varepsilon\int_S^T\int_\Omega u'^2 + \frac{C}{\varepsilon}\int_S^T\int_\Omega z'^2 + \frac{C}{\varepsilon}\int_S^T\int_\Omega f^2 + C\varepsilon\int_S^T\int_\Omega z^2\\ &\quad + C(e(T)+e(S)) + \frac{C}{\varepsilon}\int_S^T\int_{\Gamma_b} m\cdot\nu bu'^2 + \varepsilon\int_S^T\int_{\Gamma_b} u^2,\end{aligned} \tag{65}$$

for all $\varepsilon>0$. Now noting that

$$\int_S^T\int_\Omega z'^2 \le \int_S^T\int_{\Gamma_b} m^-b^-z'^2 \le \int_S^T\int_{\Gamma_b} m\cdot\nu bz'^2,$$

and

$$\int_S^T \int_\Omega u'^2 + \int_S^T \int_\Omega z^2 + \int_S^T \int_{\Gamma_b} u^2 \leq \int_S^T e(t)dt,$$

the estimate (65) yields, for all $\varepsilon > 0$,

$$\int_S^T \int_{\Gamma_b} m\cdot\nu l u^2 \leq C\varepsilon \int_S^T e(t)dt + \frac{C}{\varepsilon}\int_S^T \int_\Omega f^2 + C(e(T)+e(S)) + \frac{C}{\varepsilon}\int_S^T \int_{\Gamma_b} m\cdot\nu b u'^2.$$

Finally, replacing this in (63) and taking ε sufficiently small, we obtain

$$\int_S^T e(t)dt \leq C(e(T)+e(S)) + C\int_S^T \int_\Omega f^2 + C\int_S^T \int_{\Gamma_b} m\cdot\nu b u'^2,$$

and the lemma is proved. □

4.3 Internal stabilization of locally coupled plate equations

In this last application, we are concerned with a system of two weakly coupled plate equations. This generalize the case of constant coupling investigated in [ACK02]. Here, we assume that the boundary $\Gamma = \partial\Omega$ is at least of class $\mathcal{C}^4$ and we consider the following system:

$$\begin{cases} u_1'' + \Delta^2 u_1 + \delta p u_2 + b u_1' = 0 & \text{in } (0,\infty)\times\Omega, \\ u_2'' + \Delta^2 u_2 + p u_1 = 0 & \text{in } (0,\infty)\times\Omega, \\ u_1 = 0 \text{ and } u_2 = 0 & \text{on } (0,\infty)\times\Gamma, \\ \dfrac{\partial u_1}{\partial\nu} = 0 \text{ and } \dfrac{\partial u_2}{\partial\nu} = 0 & \text{on } (0,\infty)\times\Gamma, \\ u_j(0,\cdot) = u_j^0(\cdot),\ u_j'(0,\cdot) = u_j^1(\cdot),\ j=1,2 & \text{in } \Omega \end{cases} \tag{66}$$

We take $H = L^2(\Omega)$, and $V_1 = V_2 = H_0^2(\Omega)$ endowed with the inner product $(y,z)_{H_0^2(\Omega)} = \int_\Omega \Delta y \Delta z dx$. Hence, $A = \Delta^2$ with Neumann and Dirichlet boundary conditions, and λ denotes its lowest eigenvalue. We moreover take for B and P respectively the multiplication in L^2 by the functions $b, p \in L^\infty(\Omega)$ satisfying, as in Section 4.1,

$$\begin{cases} 0 \leq b \leq b^+ \text{ and } 0 \leq p \leq p^+ \text{ on } \Omega, \\ b \geq b^- > 0 \text{ on } \omega_b, \\ p \geq p^- > 0 \text{ on } \omega_p, \end{cases} \tag{67}$$

for ω_b and ω_p two open subsets of Ω. As for coupled waves, we have the following stability result.

Theorem 4.6. *(i) Suppose that ω_b and ω_p satisfy the PMGC. Then there exists $p_* \in (0,\lambda]$ such that for all $0 < p^+ < p_*$ there exists $\delta_* = \delta_*(p^+,p^-) \in (0, \frac{\lambda}{p^+}]$, such that for all $\delta \in (0,\delta_*)$, the solution $U(t)) = (u_1(t), u_2(t), u_1'(t), u_2'(t))$ of (66) satisfies for $n \in \mathbb{N}$,*

$$E(U(t)) \leq \frac{c}{t^n}\sum_{i=0}^n E(U^{(i)}(0)) \quad \forall t>0,\ U^0 = (u_1^0, u_2^0, u_1^1, u_2^1) \in \mathcal{D}(\mathcal{A}_{P,\delta}^n),$$

where c is a constant depending on δ, p^+, p^- and n. Besides, if $U^0 \in (H_0^2)^2 \times (L^2)^2$, then $E(U(t))$ converges to zero as t goes to infinity.

(ii) If moreover either $\omega_b \subset \omega_p$ or $\omega_p \subset \omega_b$, then the result holds for $\delta_* = \frac{\lambda}{p^+}$.

We recall that the operator $\mathcal{A}_{P,\delta}$ is defined in (9). Under some smoothness assumptions on the coefficients p and b, we can explicit the space $\mathcal{D}(\mathcal{A}_{P,\delta}^n)$ in terms of classical Sobolev spaces thanks to Lemma 2.6. This gives the following corollary.

Corollary 4.7. *(i) Suppose that ω_b and ω_p satisfy the PMGC and that $b, p \in W^{2q,\infty}(\Omega)$. Then there exists $p_* \in (0, \lambda]$ such that for all $0 < p^+ < p_*$ there exists $\delta_* = \delta_*(p^+, p^-) \in (0, \frac{\lambda}{p^+}]$, such that for all $\delta \in (0, \delta_*)$, the solution $U(t)) = (u_1(t), u_2(t), u_1'(t), u_2'(t))$ of (66) satisfies for $n \in \mathbb{N}, n \leq q$,*

$$E(U(t)) \leq \frac{c}{t^n} \sum_{i=0}^{n} E(U^{(i)}(0)) \quad \forall t > 0,\ U^0 = (u_1^0, u_2^0, u_1^1, u_2^1) \in (H^{2n+2} \cap H_0^2)^2 \times (H^{2n} \cap H_0^2)^2,$$

where c is a constant depending on δ, p^+, p^- and n. Besides, if $U^0 \in (H_0^2)^2 \times (L^2)^2$, then $E(U(t))$ converges to zero as t goes to infinity.

(ii) If moreover either $\omega_b \subset \omega_p$ or $\omega_p \subset \omega_b$, then the result holds for $\delta_* = \frac{\lambda}{p^+}$.

To prove Theorem 4.6, we only have to check that assumptions $(A1) - (A4b)$ hold and use Theorem 2.4. From (67), assumption $(A4b)$ is satisfied and assumption $(A1)$ is fulfilled, taking for Π_P the multiplication in L^2 by $\mathbb{1}_{\omega_p}$. It only remains to check assumptions $(A2)$ and $(A3)$, that are consequences of the following lemma.

Lemma 4.8. *Let ω be a subset of Ω satisfying the PMGC. Then, there exist $\alpha, \beta, \gamma > 0$ such that for all $f \in \mathcal{C}^1(\mathbb{R}^+; L^2(\Omega))$ and all $0 \leq S \leq T$, the solution u of*

$$\begin{cases} u'' + \Delta^2 u = f & \text{in } (0,\infty) \times \Omega, \\ u = 0 \text{ and } \dfrac{\partial u}{\partial \nu} = 0 & \text{on } (0,\infty) \times \Gamma, \\ (u, u')(0, \cdot) = (u^0(\cdot), u^1(\cdot)) & \in H_0^2(\Omega) \times L^2(\Omega), \end{cases} \tag{68}$$

satisfies, with $e(t) = 1/2 \left(\|u'\|^2_{L^2(\Omega)} + \|\Delta u\|^2_{L^2(\Omega)} \right)$, the inequality

$$\int_S^T e(t)\ dt \leq \alpha(e(S) + e(T)) + \beta \int_S^T \|f(t)\|^2_{L^2(\Omega)}\ dt + \gamma \int_S^T \|\mathbb{1}_\omega u'(t)\|^2_{L^2(\Omega)} dt. \tag{69}$$

This Lemma direcly yields $(A2)$, since ω_p is supposed to satisfy the PMGC. Proving $(A3)$ is done exactly as in Section 4.1, taking $u = u_1$, $\omega = \omega_b$ and $f = f_1 - bu_1'$ in Lemma 4.8. Theorem 4.6 is then a consequence of Theorem 2.4.

Proof. We give here the details of the piecewise multiplier method for a plate equation, following [AB06], so that the proof is selfcontained. We denote by $\mathcal{N}_\varepsilon\left(\cup_j \gamma_j(x_j) \cup (\Omega \setminus \cup_j \Omega_j)\right)$ the neighborhood given by the PMGC (see Definition 1.1 in the introduction). Let $0 < \varepsilon_0 < \varepsilon_1 < \varepsilon_2 < \varepsilon$ and define for $i = 0, 1, 2$

$$Q_i = \mathcal{N}_{\varepsilon_i}\left(\cup_j \gamma_j(x_j) \cup (\Omega \setminus \cup_j \Omega_j)\right),$$

where Ω_j, x_j and $\gamma_j(x_j)$ are given by the PMGC. Recall that $\Gamma_j = \partial\Omega_j$ and $m_j(x) = x - x_j$. Since $(\overline{\Omega_j} \setminus Q_1) \cap \overline{Q_0} = \emptyset$, we can construct a function $\psi_j \in \mathcal{C}_0^\infty(\mathbb{R}^N)$ which satisfies

$$0 \leq \psi_j \leq 1, \quad \psi_j = 1 \text{ on } \overline{\Omega_j} \setminus Q_1, \quad \psi_j = 0 \text{ on } Q_0.$$

We define the $\mathcal{C}^1$ vector field on Ω:

$$h(x) = \begin{cases} \psi_j(x) m_j(x) & \text{if } x \in \Omega_j, \\ 0 & \text{if } x \in \Omega \setminus \cup_j \Omega_j. \end{cases}$$

Proceeding as in [AB06], we multiply (68) by $h \cdot \nabla u$ and integrate on each $(S, T) \times \Omega_j$

$$\int_S^T \int_{\Omega_j} h(x) \cdot \nabla u (u'' + \Delta^2 u - f)\, dx\, dt = 0\,.$$

For the sake of concision, we will omit the $dx\,dt$ in the following integrals. This gives, after integrations by parts

$$\int_S^T \int_{\Gamma_j} \left[\frac{1}{2} h \cdot \nu_j (|u'|^2 - |\Delta u|^2) + \Delta u \frac{\partial (h \cdot \nabla u)}{\partial \nu} - h \cdot \nabla u \frac{\partial \Delta u}{\partial \nu}\right]$$

$$= \left[\int_{\Omega_j} u' h \cdot \nabla u\right]_S^T + \int_S^T \int_{\Omega_j} \Big(\frac{1}{2} \operatorname{div} h\, (u'^2 - |\Delta u|^2) + \Delta h_k \frac{\partial u}{\partial x_k} \Delta u + 2\nabla h_k \cdot \nabla (\frac{\partial u}{\partial x_k}) \Delta u - f h \cdot \nabla u\Big). \tag{70}$$

Thanks to the choice of ψ_j, only the boundary term on $(\Gamma_j \setminus \gamma_j(x_j)) \cap \Gamma$ is nonvanishing in the left hand side of (70). But on this part of the boundary, we claim that $\partial_\nu (h \cdot \nabla u) = h \cdot \nu \Delta u$ (see also [Lag89] and [Kom]). For this, we first remark that $u = 0 = \partial_\nu u$ there. Hence, $\partial_i u = 0$ for $1 \leq i \leq N$ on $(\Gamma_j \setminus \gamma_j(x_j)) \cap \Gamma$, and we have

$$\partial_\nu (h \cdot \nabla u) = \sum_{j=1}^N \partial_j (\sum_i^N h_i \partial_i u) \nu_j = \sum_{i,j} \partial_j (h_i \partial_i u) \nu_j = \sum_{i,j} h_i \partial_{ij} u \nu_j \,. \tag{71}$$

Setting $v = \partial_j u$, and recalling that $\nabla u = 0$ on $(\Gamma_j \setminus \gamma_j(x_j)) \cap \Gamma$, we have $\nabla v = \partial_\nu v \nu$. Hence, $\partial_i v = \sum_{k=1}^N \partial_k v \nu_k \nu_i$ for all $1 \leq i \leq N$. Coming back to $\partial_j u$, we obtain

$$\partial_{ij} u = \sum_k \partial_{kj} u \nu_k \nu_i \,, \tag{72}$$

and in particular

$$\partial_{jj} u = \sum_k \partial_{kj} u \nu_k \nu_j \,. \tag{73}$$

Using (72) in (71), we deduce that

$$\partial_\nu (h \cdot \nabla u) = \sum_{i,j,k} h_i \partial_{kj} u \nu_k \nu_i \nu_j = \sum_i h_i \nu_i \sum_j \left[\sum_k (\partial_{kj} u \nu_k) \nu_j\right] .$$

Using (73) in this last identity, we obtain

$$\partial_\nu (h \cdot \nabla u) = \sum_i h_i \nu_i \sum_j \partial_{jj} u = h \cdot \nu \Delta u \,,$$

which proves our claim. Since in addition, $u' = 0$ on $(\Gamma_j \setminus \gamma_j(x_j)) \cap \Gamma$, we deduce that the left hand side of (70) reduces to

$$\frac{1}{2} \int_S^T \int_{(\Gamma_j \setminus \gamma_j(x_j)) \cap \Gamma} h \cdot \nu |\Delta u|^2 \leq 0 \,. \tag{74}$$

Therefore, since $\psi_j = 0$ on Q_0 and thanks to the above inequality used in (70), we deduce that

$$\left[\int_{\Omega_j} u' h \cdot \nabla u\right]_S^T + \int_S^T \int_{\Omega_j \setminus Q_0} \Big(\frac{1}{2} \operatorname{div} h\, (u'^2 - |\Delta u|^2)$$

$$+ \Delta h_k \frac{\partial u}{\partial x_k} \Delta u + 2 \nabla h_k \cdot \nabla (\frac{\partial u}{\partial x_k}) \Delta u - f h \cdot \nabla u\Big) \leq 0. \tag{75}$$

Using $\psi_j = 1$ on $\overline{\Omega_j} - Q_1$ and summing the resulting inequalities on j, we obtain

$$\begin{aligned}\Big[\int_\Omega u' h\cdot\nabla u\Big]_S^T &+ \int_S^T\int_{\Omega\setminus Q_1}\Big(\frac{1}{2}\,(Nu'^2+(2-N)|\nabla u|^2) - \int_S^T\int_\Omega f h\cdot\nabla u \\ &\le -\sum_j\int_S^T\Big[\int_{\Omega_j\cap Q_1}\frac{1}{2}\ \mathrm{div}\, h\,(u'^2-|\Delta u|^2)+\Delta h_k\frac{\partial u}{\partial x_k}\Delta u + 2\nabla h_k\cdot\nabla(\frac{\partial u}{\partial x_k})\Delta u\Big] \\ &\le C\int_S^T\int_{\Omega\cap Q_1}(u'^2+|\Delta u|^2+|\nabla u|^2+\frac{\partial^2 u}{\partial x_i\partial x_k}\frac{\partial^2 u}{\partial x_i\partial x_k}), \quad (76)\end{aligned}$$

where C is a positive constant which depends only on ψ_j and m_j. We now use the second multiplier $u(N-2)/2$ and evaluate the term

$$\frac{N-2}{2}\int_S^T\int_\Omega u(u''+\Delta u^2-f)=0\,.$$

Hence, one has

$$\frac{N-2}{2}\Big[\int_\Omega uu'\Big]_S^T+\frac{N-2}{2}\int_S^T\int_\Omega|\Delta u|^2-u'^2-uf=0\,. \quad (77)$$

We set $M(u)=h\cdot\nabla u+\frac{N-2}{2}\,u$. Adding (77) to (76), we obtain

$$2\int_S^T e\,dt\le C\int_S^T\int_{\Omega\cap Q_1}\Big[u'^2+|\Delta u|^2+|\nabla u|^2+\frac{\partial^2 u}{\partial x_i\partial x_k}\frac{\partial^2 u}{\partial x_i\partial x_k}\Big]-\Big[\int_\Omega M(u)u'\Big]_S^T+\int_S^T\int_\Omega M(u)f. \quad (78)$$

We estimate the terms on the right hand side of (78) as follows. First, we have

$$\Big|\Big[\int_\Omega M(u)u'\Big]_S^T\Big|\le C(e(S)+e(T)). \quad (79)$$

Second, we estimate the last term of (78) as follows

$$\Big|\int_S^T\int_\Omega M(u)f\Big|\le\frac{C}{\mu}\int_S^T\int_\Omega|f|^2+\mu\int_S^T e\,dt\qquad\forall\mu>0\,. \quad (80)$$

Using (79) and (80) in (78), we obtain for all $\mu>0$:

$$\begin{aligned}(2-\mu)\int_S^T e\,dt\le C\int_S^T\int_{\Omega\cap Q_1}&\Big[u'^2+|\nabla u|^2+|\Delta u|^2+\frac{\partial^2 u}{\partial x_i\partial x_k}\frac{\partial^2 u}{\partial x_i\partial x_k}\Big]\\ &+C(e(S)+e(T))+\frac{C}{\mu}\int_S^T\int_\Omega|f|^2\,. \quad (81)\end{aligned}$$

We now have to estimate the first term on the right hand side of (81), that we will denote by X:

$$X=\int_S^T\int_{\Omega\cap Q_1}\Big[u'^2+|\nabla u|^2+|\Delta u|^2+\frac{\partial^2 u}{\partial x_i\partial x_k}\frac{\partial^2 u}{\partial x_i\partial x_k}\Big]\,.$$

This is where the piecewise multiplier method takes its place.

Step 1: Estimate of the terms corresponding to second derivatives in space in X:
Since $\overline{\mathbb{R}^N\setminus Q_2}\cap\overline{Q_1}=\emptyset$, there exists a function $\xi\in\mathcal{C}_0^\infty(\mathbb{R}^N)$ such that

$$0\le\xi\le 1,\quad \xi=1\text{ on }Q_1,\quad \xi=0\text{ on }\mathbb{R}^N\setminus Q_2.$$

We need the following result, that is proved in [AB06, Proposition 4.1].

Lemma 4.9. *Let ξ be defined as above. Then for all $v \in H_0^2(\Omega)$, we have*

$$\int_\Omega \nabla\xi \cdot \nabla v \, \Delta v = -\int_\Omega \frac{\partial^2 \xi}{\partial x_i \partial x_k} \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_k} + \frac{1}{2} \int_\Omega \Delta\xi |\nabla v|^2 \,. \tag{82}$$

and

$$\int_\Omega \xi \frac{\partial^2 v}{\partial x_i \partial x_k} \frac{\partial^2 v}{\partial x_i \partial x_k} = \int_\Omega \left[-\frac{\partial^2 \xi}{\partial x_i \partial x_k} \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_k} + \Delta\xi |\nabla v|^2 + \xi |\Delta v|^2 \right] . \tag{83}$$

From (83), we deduce that

$$\int_{\Omega \cap Q_1} \frac{\partial^2 u}{\partial x_i \partial x_k} \frac{\partial^2 u}{\partial x_i \partial x_k} \le \int_\Omega \xi |\Delta u|^2 + C \int_{\Omega \cap Q_2} |\nabla u|^2 \,. \tag{84}$$

We use now the multiplier ξu in the first equation of (68), and consider the expression:

$$\int_S^T \int_\Omega (u'' + \Delta^2 u - f) \xi \, u \, dx \, dt = 0 \,.$$

After integrations by parts, this gives

$$\int_S^T \int_\Omega \xi |\Delta u|^2 = \int_S^T \int_\Omega \xi |u'|^2 - 2\nabla\xi \cdot \nabla u \Delta u - u \Delta u \Delta\xi - \left[\int_\Omega \xi u u' \right]_S^T + \int_S^T \int_\Omega \xi u f \,.$$

Using (82) in this last identity, we obtain

$$\int_S^T \int_\Omega \xi |\Delta u|^2 = \int_S^T \int_\Omega \Bigg[\xi |u'|^2 + 2 \frac{\partial^2 \xi}{\partial x_i \partial x_k} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_k} \\ + \Delta\xi |\nabla u|^2 + u \Delta u \Delta\xi \Bigg] - \left[\int_\Omega \xi u u' \right]_S^T + \int_S^T \int_\Omega \xi u f. \tag{85}$$

On the other hand, one has $\displaystyle\int_\Omega u \Delta u \Delta\xi = \int_\Omega \Big(\frac{u^2}{2} \Delta^2 \xi - |\nabla u|^2 \Delta\xi \Big)$. Using this identity in (85), we obtain

$$\int_S^T \int_\Omega \xi |\Delta u|^2 = \int_S^T \int_\Omega \left[\xi |u'|^2 + 2 \frac{\partial^2 \xi}{\partial x_i \partial x_k} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_k} - \frac{u^2}{2} \Delta^2 \xi \right] - \left[\int_\Omega \xi u u' \right]_S^T + \int_S^T \int_\Omega \xi u f.$$

We estimate the second term on the right hand side of the above inequality as previously (see (79)). Moreover, since $\xi = 1$ on Q_1, whereas $\xi = 0$ on $\mathbb{R}^N \setminus Q_2$, we deduce that

$$\int_S^T \int_\Omega \xi |\Delta u|^2 \le C(e(S) + e(T)) + C \int_S^T \int_{\Omega \cap Q_2} |u'|^2 + |\nabla u|^2 + u^2 + |f|^2. \tag{86}$$

Using (86) in (84), we obtain that

$$\int_{\Omega \cap Q_1} \frac{\partial^2 u}{\partial x_i \partial x_k} \frac{\partial^2 u}{\partial x_i \partial x_k} \le C(e(S) + e(T)) + C \int_S^T \int_{\Omega \cap Q_2} |u'|^2 + |\nabla u|^2 + u^2 + |f|^2, \tag{87}$$

and

$$\int_S^T \int_{\Omega \cap Q_1} |\Delta u|^2 \le C(e(S) + e(T)) + C \int_S^T \int_{\Omega \cap Q_2} |u'|^2 + |\nabla u|^2 + u^2 + |f|^2. \tag{88}$$

Using both (87) and (88) in (81), we obtain for all $\mu > 0$

$$(2 - \mu) \int_S^T e \, dt \le C \int_S^T \int_{\Omega \cap Q_2} \left[u'^2 + |\nabla u|^2 + |u|^2 \right] + C(e(S) + e(T)) + C \left(1 + \frac{1}{\mu} \right) \int_S^T \int_\Omega |f|^2 \,.$$

Step 2: Estimate of the terms corresponding to first derivatives in space in X:

Since $\overline{\mathbb{R}^N \setminus Q_3} \cap \overline{Q_2} = \emptyset$, there exists a function $\beta \in \mathcal{C}_0^\infty(\mathbb{R}^N)$ such that

$$0 \le \beta \le 1, \quad \beta = 1 \text{ on } Q_2, \quad \beta = 0 \text{ on } \mathbb{R}^N \setminus Q_3.$$

We fix t and consider the solution θ of the following elliptic problem:

$$\begin{cases} \Delta^2 \theta = \beta \Delta u & \text{in } \Omega, \\ \theta = 0 = \dfrac{\partial \theta}{\partial \nu} & \text{on } \Gamma. \end{cases}$$

Then, we have

$$\int_\Omega |\Delta\theta|^2 = \int_\Omega \theta\, \Delta^2 \theta = \int_\Omega u \Delta(\beta\theta) = \int_{\Omega\cap Q_3} u\Big(\beta\Delta\theta + 2\nabla\beta\cdot\nabla\theta + \theta\Delta\beta\Big) \le C \int_{\Omega\cap Q_3} |u|^2, \tag{89}$$

and similarly,

$$\int_\Omega |\Delta\theta'|^2 \le C \int_{\Omega\cap Q_3} |u'|^2. \tag{90}$$

We now consider the multiplier θ for (68) and evaluate the expression

$$\int_S^T \int_\Omega \theta\Big(u'' + \Delta u^2 - f\Big) = 0.$$

This yields, after integrations by parts,

$$\int_S^T \int_\Omega \beta u \Delta u - \int_S^T \int_\Omega \theta'\, u' + \Big[\int_\Omega \theta\, u'\Big]_S^T - \int_S^T \int_\Omega \theta\, f = 0\,.$$

Integrating by parts the first term of this expression, we obtain

$$\int_S^T \int_\Omega \beta |\nabla u|^2 = -\int_S^T \int_\Omega \theta'\, u' + \int_S^T \int_\Omega \frac{\Delta\beta}{2}|u|^2 + \Big[\int_\Omega \theta\, u'\Big]_S^T + \int_S^T \int_\Omega \theta\, f.$$

Since $\beta = 1$ on $\Omega \cap Q_2$, whereas $\beta = 0$ on $\Omega \setminus Q_3$, and thanks to (89), we obtain for all $\mu > 0$:

$$\int_S^T \int_{\Omega\cap Q_2} \beta |\nabla u|^2 \le \frac{\mu}{2} \int_S^T \int_\Omega |u'|^2 + \frac{1}{2\mu} \int_S^T \int_\Omega |\theta'|^2$$
$$+ C \int_S^T \int_{\Omega\cap Q_3} |u|^2 + C \int_S^T \int_\Omega |f|^2 + C(e(S) + e(T)).$$

Using, (90) in this last inequality, we deduce that, for all $\mu > 0$

$$\int_S^T \int_{\Omega\cap Q_2} |\nabla u|^2 \le \mu \int_S^T e + \frac{C}{\mu} \int_S^T \int_{\Omega\cap Q_3} |u'|^2$$
$$+ C \int_S^T \int_{\Omega\cap Q_3} |u|^2 + C \int_S^T \int_\Omega |f|^2 + C(e(S) + e(T)).$$

Finally, putting this into (89), we obtain for all $\mu > 0$

$$(2 - C\mu) \int_S^T e\, dt \le C \int_S^T \int_{\Omega\cap Q_3} |u|^2 + C\left(1 + \frac{1}{\mu}\right) \int_S^T \int_{\Omega\cap Q_3} u'^2$$
$$+ C\left(1 + \frac{1}{\mu}\right) \int_S^T \int_\Omega |f|^2 + C(e(S) + e(T)). \tag{91}$$

Step 3: Estimate of the zero order terms in X:

Since $\overline{\mathbb{R}^N \setminus \omega} \cap \overline{Q_3} = \emptyset$, there exists a function $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^N)$ such that

$$0 \leq \psi \leq 1, \quad \psi = 1 \text{ on } Q_3, \quad \psi = 0 \text{ on } \mathbb{R}^N \setminus \omega.$$

We fix t and consider the solution z of the following elliptic problem

$$\begin{cases} \Delta^2 z = \psi u & \text{in } \Omega, \\ z = 0 = \dfrac{\partial z}{\partial \nu} & \text{on } \Gamma. \end{cases}$$

Then, we have

$$\int_\Omega |\Delta z|^2 \leq C \int_\omega \psi |u|^2, \quad \text{and} \quad \int_\Omega |\Delta z'|^2 \leq C \int_\omega |u'|^2. \tag{92}$$

We now consider the multiplier z for (68) and evaluate the expression

$$\int_S^T \int_\Omega z\Big(u'' + \Delta u^2 - f\Big) = 0.$$

This yields

$$\int_S^T \int_\Omega \psi |u|^2 - \int_S^T \int_\Omega z'\, u' + \left[\int_\Omega z\, u'\right]_S^T - \int_S^T \int_\Omega z\, f = 0\,.$$

Integrating by parts the first term of this expression, we obtain

$$\int_S^T \int_\Omega \psi |u|^2 = \int_S^T \int_\Omega z'\, u' - \left[\int_\Omega z\, u'\right]_S^T + \int_S^T \int_\Omega z\, f.$$

Hence, using (92) to estimate the third and fourth terms on the right hand side of the above equality, we obtain for all $\eta > 0\,,\ \mu > 0$:

$$\int_S^T \int_\Omega \psi |u|^2 \leq \frac{\mu}{2} \int_S^T \int_\Omega |u'|^2 + \eta \int_S^T \int_\Omega |z|^2 + \frac{c}{\mu} \int_S^T \int_\Omega |z'|^2 + \frac{c}{\eta} \int_S^T \int_\Omega |f|^2 + C(e(S) + e(T)).$$

Using now the definition of the energy, together with (92), we deduce that

$$(1 - c\eta) \int_S^T \int_\Omega \psi |u|^2 \leq \mu \int_S^T e + \frac{c}{\mu} \int_S^T \int_\omega |u'|^2 + \frac{c}{\eta} \int_S^T \int_\Omega |f|^2 + C(e(S) + e(T)).$$

As a consequence, since $\psi = 1$ on Q_3 and choosing η sufficiently small, we have for all $\mu > 0$:

$$\int_S^T \int_{\Omega \cap Q_3} |u|^2 \leq c\mu \int_S^T e + \frac{c}{\mu} \int_S^T \int_\omega |u'|^2 + c \int_S^T \int_\Omega |f|^2 + C(e(S) + e(T)).$$

Using this last estimate in (91), we obtain for all $\mu > 0$

$$(2 - c\mu) \int_S^T e\, dt \leq C\left(1 + \frac{1}{\mu}\right) \int_S^T \int_\omega |u'|^2 + C\left(1 + \frac{1}{\mu}\right) \int_S^T \int_\Omega |f|^2 + C(e(S) + e(T)).$$

Finally, choosing now μ sufficiently small, we have

$$\int_S^T e\, dt \leq C(e(S) + e(T)) + C \int_S^T \int_\Omega |f|^2 + C \int_S^T \int_\omega |u'|^2,$$

and the lemma is proved. □

References

[AB99] F. Alabau-Boussouira. Stabilisation frontière indirecte de systèmes faiblement couplés. *C. R. Acad. Sci. Paris Sér I Math*, 328:1015–1020, 1999.

[AB02] F. Alabau-Boussouira. Indirect boundary stabilization of weakly coupled systems. *SIAM J. Control Optimization*, 41:511–541, 2002.

[AB03] F. Alabau-Boussouira. A two-level energy method for indirect boundary observability and controllability of weakly coupled hyperbolic systems. *SIAM J. Control Optimization*, 42:871–906, 2003.

[AB04] F. Alabau-Boussouira. Une formule générale pour le taux de décroissance des systèmes dissipatifs non linéaires. *C. R. Acad. Sci. Paris Sér I Math*, 338:35–40, 2004.

[AB05] F. Alabau-Boussouira. Convexity and weighted integral inequalities for energy decay rates of nonlinear dissipative hyperbolic systems. *Appl. Math. and Optimization*, 51:61–105, 2005.

[AB06] F. Alabau-Boussouira. Piecewise multiplier method and nonlinear integral inequalities for petrowsky equation with nonlinear dissipation. *J. Evolution Equations*, 6:95–112, 2006.

[ACK02] F. Alabau, P. Cannarsa, and V. Komornik. Indirect internal damping of coupled systems. *J. Evolution Equations*, 2:127–150, 2002.

[AKBD06] F. Ammar-Khodja, A. Benabdallah, and C. Dupaix. Null controllability of some reaction-diffusion systems with one control force. *J. Math. Anal. Appl.*, 320:928–943, 2006.

[BLR92] C. Bardos, G. Lebeau, and J. Rauch. Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary. *SIAM J. Control Optimization*, 30:1024–1065, 1992.

[CR93] F. Conrad and B. Rao. Decay of solutions of the wave equation in a star-shaped domain with nonlinear boundary feedback. *Asymptotic Anal.*, 7:159–177, 1993.

[GBPG06] M. González-Burgos and R. Pérez-García. Controllability results for some nonlinear coupled parabolic systems by one control force. *Asymptotic Analysis*, 46:123–162, 2006.

[Har78] A. Haraux. Semi-groupes linéaires et équations d'évolution linéaires périodiques. *Publication du Laboratoire d'Analyse Numérique, Université Pierre et Marie Curie*, 78011:123–162, 1978.

[Kom] V. Komornik. *Exact Controllability and Stabilization. The Multiplier Method*, volume 36 of *Collection RMA*. Masson-John Wiley, Paris-Chicester.

[KT09] O. Kavian and L. de Teresa. Unique continuation principle for systems of parabolic equations. *ESAIM Control Optim. Calc. Var., doi:10.1051/cocv/2008077*, 2009.

[Lag89] J. E. Lagnese. Boundary stabilization of thin plates. *SIAM Studies in Applied Mathematics*, 10, 1989.

[Léa10] M. Léautaud. Spectral inequalities for non-selfadjoint elliptic operators and application to the null-controllability of parabolic systems. *J. Funct. Anal.*, 258:2739–2778, 2010.

[Lio] J. L. Lions. *Contrôlabilité exacte et stabilisation de systèmes distribués*, volume 1-2 of *Collection RMA*. Masson, Paris.

[Liu97] K. Liu. Locally distributed control and damping for the conservative systems. *SIAM, J. Control Optimization*, 35:1574–1590, 1997.

[LZ99] G. Lebeau and E. Zuazua. Decay rates for the three-dimensional linear system of thermoelasticity. *Arch. Rational Mech. Anal.*, 148:179–231, 1999.

[Mar99] P Martinez. A new method to obtain decay rate estimates for dissipative systems with localized damping. *Rev. Mat. Complut.*, 12:251–283, 1999.

[Ter00] L. de Teresa. Insensitizing controls for a semilinear heat equation. *Comm. Partial Differential Equations*, 25:39–72, 2000.

[You09] W. Youssef. Contrôle et stabilisation de systèmes élastiques couplés. *Thèse de doctorat de l'université de Metz*, 2009.

[Zua90] E. Zuazua. Exponential decay for the semilinear wave equation with locally distributed damping. *Comm. Partial Differential Equations*, 15:205–235, 1990.