NONCONFORMING VECTOR FINITE ELEMENTS FOR $\boldsymbol{H}(\mathbf{curl};\boldsymbol{\Omega}) \cap \boldsymbol{H}(\mathbf{div};\boldsymbol{\Omega})$

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Abstract. We present a family of nonconforming vector finite elements of arbitrary order for problems posed on the space $H(\mathrm{curl};\Omega) \cap H(\mathrm{div};\Omega)$, where $\Omega \subset \mathbb{R}^2$. This result was first stated as a conjecture by Brenner and Sung in [1]. In contrast an extension of the same conjecture to domains of $\mathbb{R}^3$ is disproved.

Let Ω be a domain of $\mathbb{R}^d$ where $d \in \{2,3\}$. As explained in [1] several problems involving the space $H(\mathrm{curl};\Omega) \cap H(\mathrm{div};\Omega)$, such as the cavity resonance problem and the acoustic fluid-structure interaction problem, can be solved using nonconforming finite element methods. In contrast conforming finite element methods cannot capture the solution of these problems under certain conditions.

The accuracy of the approximate numerical solution of these problems can be improved if one uses finite elements which are not piecewise linear, but piecewise quadratic or of higher degree. For that purpose a quadratic nonconforming vector finite element for $H(\mathrm{curl};\Omega) \cap H(\mathrm{div};\Omega)$ was introduced in [1], in the case of a bidimensional domain $\Omega \subset \mathbb{R}^2$. The paper [1] also contains a conjecture which suggests a way of constructing nonconforming vector finite elements of arbitrary degree k for $H(\mathrm{curl};\Omega) \cap H(\mathrm{div};\Omega)$, for domains of $\mathbb{R}^2$ and of $\mathbb{R}^3$.

In order to state this conjecture and to formulate our results, we need to introduce some notations. We use boldfaced letters to represent vectors. The space of polynomials of total degree $\leq k$ in d variables is denoted by $P_k(\mathbb{R}^d)$, and the space of homogeneous harmonic polynomials of degree k in d variables is denoted by $\mathcal{H}_k(\mathbb{R}^d)$. For each $k \geq 1$ and $d \in \{2,3\}$ we define a space $\mathcal{P}_{k,d}$ of vector fields on $\mathbb{R}^d$ as follows

$$\mathcal{P}_{k,d} := [P_k(\mathbb{R}^d)]^d \oplus \left(\nabla \mathcal{H}_{k+2}(\mathbb{R}^d) \oplus \cdots \oplus \nabla \mathcal{H}_{2k}(\mathbb{R}^d)\right) \tag{1}$$

For any triangle T if $d = 2$ (resp. tetrahedron T if $d = 3$) we consider the set $\mathcal{N}_{k,d}(T)$ of linear functionals on $\mathcal{P}_{k,d}$ defining the moments on the $d+1$ edges (resp. faces) of T up to order $k-1$ and the moments on T up to order $k-2$, for the d components of the vector fields.

Brenner and Sung formulated in [1] a series of conjectures, which depend on two parameters $d \in \{2,3\}$ and $k \geq 1$.

$$\begin{array}{ll} \mathrm{Conj}(k,d): & \text{For any } T \text{ the elements of } \mathcal{P}_{k,d} \text{ are uniquely determined} \\ & \text{by the linear functionals in } \mathcal{N}_{k,d}(T). \end{array} \tag{2}$$

The conjectures $\mathrm{Conj}(1,2)$ and $\mathrm{Conj}(1,3)$ are true and correspond to the nonconforming Crouzeix-Raviart P_1 vector finite element. It was established in [1] that

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Conj$(2,2)$ is true, thus defining piecewise quadratic nonconforming vector finite elements in two space dimensions.

The purpose of this paper is to establish the following result :

Theorem. *For any $k \geq 3$ the conjecture* Conj$(k,2)$ *holds. In contrast the conjecture* Conj$(2,3)$ *does not hold.*

Our result therefore validates the construction of bi-dimensional vector finite elements of arbitrary degree proposed in [1]. On the contrary the three-dimensional quadratic vector finite element is invalid. Our result does not completely close the conjecture as the cases of three-dimensional vector finite elements of cubic or higher degree remain unsolved.

It was established in [1] that for all $d \in \{2,3\}$, all $k \geq 1$ and all T, one has

$$\dim \mathcal{P}_{k,d} = \#\mathcal{N}_{k,d}(T).$$

Hence the conjecture Conj(k,d) is equivalent to the following property :

$$\begin{gathered}\text{For all } T \text{ and all } \boldsymbol{v} \in \mathcal{P}_{k,d},\\ \text{if } \ (l(\boldsymbol{v}) = 0 \text{ for all } l \in \mathcal{N}_{k,d}(T)) \ \text{ then } \boldsymbol{v} = 0.\end{gathered} \tag{3}$$

In the first section of this paper we establish this property in the bi-dimensional case $d = 2$ and for an arbitrary $k \geq 1$, while the second section gives a counter example in the three-dimensional case $d = 3$ and $k = 2$.

1. Proof of the bi-dimensional result

In this section the integer $k \geq 1$ is arbitrary but fixed. If $\boldsymbol{v} = (v_1, v_2) \in \mathcal{P}_{k,2}$ we remark that

$$\nabla \times \boldsymbol{v} = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \in P_{k-1}(\mathbb{R}^2) \quad \text{and} \quad \nabla \cdot \boldsymbol{v} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} \in P_{k-1}(\mathbb{R}^2).$$

Our first lemma extends to degree k an argument used in the initial paper [1].

Lemma. *Let $\boldsymbol{v} \in \mathcal{P}_{k,2}$. Let T be a triangle and let us assume that $l(\boldsymbol{v}) = 0$ for all $l \in \mathcal{N}_{k,2}(T)$. Then*

$$\nabla\cdot \boldsymbol{v} = \nabla\times \boldsymbol{v} = 0. \tag{4}$$

Proof. We first notice that $\nabla\cdot \boldsymbol{v}$ and $\nabla\times \boldsymbol{v}$ are polynomials of degree $k-1$, and that the components of $\nabla \times (\nabla \times \boldsymbol{v})$ and of $\nabla(\nabla \cdot \boldsymbol{v})$ are polynomials of degree $k-2$. In view of Green's theorem and the vanishing moments of $\boldsymbol{v}$, we have

$$\int_T (\nabla \times \boldsymbol{v})(\nabla \times \boldsymbol{v})\, dx = \int_{\partial T} (\boldsymbol{n} \times \boldsymbol{v})(\nabla \times \boldsymbol{v})\, ds + \int_T \boldsymbol{v} \cdot \nabla \times (\nabla \times \boldsymbol{v})\, dx = 0$$

where $\boldsymbol{n}$ is the outer unit normal along ∂T. Similarly, we have

$$\int_T (\nabla \cdot \boldsymbol{v})(\nabla \cdot \boldsymbol{v})\, dx = \int_{\partial T} (\boldsymbol{n} \cdot \boldsymbol{v})(\nabla \cdot \boldsymbol{v})\, ds - \int_T \boldsymbol{v} \cdot \nabla(\nabla \cdot \boldsymbol{v})\, dx = 0.$$

The results follow. $\diamond$

We now rephrase the conjecture (2) in terms of complex functions, and for that purpose we introduce some definitions.

Definition. *For any pair $\boldsymbol{v} = (v_1, v_2)$ of real valued functions we define a complex valued function $P_{\boldsymbol{v}}$ as follows*

$$P_{\boldsymbol{v}}(x+iy) := v_1(x,y) - i\, v_2(x,y) \ \text{for all } (x,y) \in \mathbb{R}^2.$$

We now notice that the equations $\nabla\cdot\boldsymbol{v}=\nabla\times\boldsymbol{v}=0$ are equivalent to the Cauchy-Riemann equations of $P_{\boldsymbol{v}}$, namely

$$\frac{\partial\,\Re(P_{\boldsymbol{v}})}{\partial x}=\frac{\partial\,\Im(P_{\boldsymbol{v}})}{\partial y}\ \text{and}\ \frac{\partial\,\Re(P_{\boldsymbol{v}})}{\partial y}=-\frac{\partial\,\Im(P_{\boldsymbol{v}})}{\partial x}$$

where $\Re:\mathbb{C}\to\mathbb{R}$ and $\Im:\mathbb{C}\to\mathbb{R}$ respectively refer to the real and imaginary part. These equations characterize holomorphic functions. Let us introduce for all $m\geq 1$ the space $\mathbb{C}_m$ of polynomials in the complex variable $z=x+iy$ and of degree less or equal to m

$$\mathbb{C}_m:=\left\{P=\sum_{r=0}^{m}a_r z^r\ ;\ (a_0,\cdots,a_r)\in\mathbb{C}^m\right\}.$$

If $\boldsymbol{v}\in\mathcal{P}_{k,2}$ satisfies $l(\boldsymbol{v})=0$ for all $l\in\mathcal{N}_{k,2}$, then $P_{\boldsymbol{v}}$ satisfies the Cauchy-Riemann equations according to (4), and therefore $P_{\boldsymbol{v}}\in\mathbb{C}_{2k-1}$.

Definition. *For any continuous function $P:\mathbb{C}\to\mathbb{C}$ and any $z_1,z_2\in\mathbb{C}$ we define*

$$I_{z_1,z_2}(P)=\int_{t=0}^{1}P(z_1+t(z_2-z_1))(z_2-z_1)dt=\int_S P(z)dz \tag{5}$$

where $S\subset\mathbb{C}$ is the oriented segment from z_1 to z_2.

Let S be an edge of a triangle T with endpoints (x_1,y_1) and (x_2,y_2) and let $z_1=x_1+iy_1$ and $z_2=x_2+iy_2$ be their complex coordinates. Let $\boldsymbol{v}=(v_1,v_2)\in\mathcal{P}_{k,2}$ be such that $l(\boldsymbol{v})=0$ for all $l\in\mathcal{N}_{k,2}(T)$, and let $Q(x+iy):=R_1(x,y)+iR_2(x,y)$ where $R_1,R_2\in P_{k-1}(\mathbb{R}^2)$ are arbitrary. Since $\boldsymbol{v}$ has vanishing moments up to order $k-1$ on the edges of T we have

$$I_{z_1,z_2}(P_{\boldsymbol{v}}Q)=(z_2-z_1)\int_{t=0}^{1}\big((v_1R_1+v_2R_2)+i(v_1R_2-v_2R_1)\big)(x(t),y(t))dt=0, \tag{6}$$

Where we used the notations $x(t):=x_1+t(x_2-x_1)$ and $y(t):=y_1+t(y_2-y_1)$.

We now define a bilinear function which is related to our conjecture.

Definition. *For all $Z=(z_1,z_2,z_3)\in\mathbb{C}^3$ we define a bilinear form $q_Z:\mathbb{C}_{2k-1}\times(\mathbb{C}_{k-1}\times\mathbb{C}_{k-1})\to\mathbb{C}$ as follows*

$$q_Z(P,(Q_1,Q_2)):=I_{z_1,z_2}(PQ_1)+I_{z_1,z_3}(PQ_2).$$

Let T be a triangle and let $z_1=x_1+iy_1$, $z_2=x_2+iy_2$ and $z_3=x_3+iy_3$ be the complex coordinates of the vertices of T. If $\boldsymbol{v}\in\mathcal{P}_{k,2}$ is such that $l(\boldsymbol{v})=0$ for all $l\in\mathcal{N}_{k,2}(T)$ then $P_{\boldsymbol{v}}\in\mathbb{C}_{2k-1}$ as previously noticed. Furthermore, specializing (6) to polynomials $Q\in\mathbb{C}_{k-1}$ we obtain

$$q_Z(P_{\boldsymbol{v}},(Q_1,Q_2))=0\text{ for all }(Q_1,Q_2)\in\mathbb{C}_{k-1}\times\mathbb{C}_{k-1}. \tag{7}$$

The purpose of the rest of this section is to show that the bilinear form q_Z is nondegenerate. It then follows from (7) that $P_{\boldsymbol{v}}=0$ and therefore that $\boldsymbol{v}=0$ which concludes the proof of the conjecture Conj$(k,2)$.

We denote by $B:=(1,z,\cdots,z^{2k-1})$ the canonical basis of $\mathbb{C}_{2k-1}$, and by $B^*:=((1,0),(z,0)\cdots,(z^{k-1},0),\ (0,1),\cdots,(0,z^{k-1}))$ the canonical basis of $\mathbb{C}_{k-1}\times C_{k-1}$. We denote by $M(Z)$, or $M(z_1,z_2,z_3)$, the matrix of q_Z in the basis B and B^*. Hence for all $1\leq i\leq 2k$ and all $1\leq j\leq k$ we have

$$M(Z)_{i,j}=I_{z_1,z_2}(z^{i-1}z^{j-1})\ \text{ and }\ M(Z)_{i,j+k}=I_{z_1,z_3}(z^{i-1}z^{j-1})$$

It follows that

$$M(Z)_{i,j} = \frac{z_2^{i+j-1} - z_1^{i+j-1}}{i+j-1} \text{ and } M(Z)_{i,j+k} = \frac{z_3^{i+j-1} - z_1^{i+j-1}}{i+j-1}. \tag{8}$$

For example if $k = 2$ we have

$$M(Z) = \begin{pmatrix} z_2 - z_1 & \frac{z_2^2 - z_1^2}{2} & z_3 - z_1 & \frac{z_3^2 - z_1^2}{2} \\ \frac{z_2^2 - z_1^2}{2} & \frac{z_2^3 - z_1^3}{3} & \frac{z_3^2 - z_1^2}{2} & \frac{z_3^3 - z_1^3}{3} \\ \frac{z_2^3 - z_1^3}{3} & \frac{z_2^4 - z_1^4}{4} & \frac{z_3^3 - z_1^3}{3} & \frac{z_3^4 - z_1^4}{4} \\ \frac{z_2^4 - z_1^4}{4} & \frac{z_2^5 - z_1^5}{5} & \frac{z_3^4 - z_1^4}{4} & \frac{z_3^5 - z_1^5}{5} \end{pmatrix} \tag{9}$$

Our next proposition gives an explicit expression of $\det M(Z)$, therefore showing that q_Z is non-degenerate. In the following Z always refers to the triplet of complex variables $Z = (z_1, z_2, z_3)$.

Proposition. *One has*

$$\det M(Z) = \alpha (z_1 - z_2)^{k^2} (z_2 - z_3)^{k^2} (z_3 - z_1)^{k^2}$$

where $\alpha = \frac{\left(\prod_{0 \le i \le k-1} i!\right)^5}{\prod_{0 \le i \le k-1} (2k+i)!} > 0$. *Therefore* q_Z *is non-degenrate whenever* z_1, z_2 *and* z_3 *are pairwise distinct.*

Proof. We denote by $\mathbb{S}$ the collection of all permutations σ of the set $\{1, \cdots, 2k\}$, and by $\varepsilon(\sigma)$ be the algebraic signature of such a permutation. We recall that

$$\det M(Z) := \sum_{\sigma \in \mathbb{S}} \varepsilon(\sigma) \prod_{j=1}^{2k} M(Z)_{\sigma(j), j}. \tag{10}$$

For any permutation $\sigma \in \mathbb{S}$ one has

$$\sum_{j=1}^{k} (j + \sigma(j) - 1) + \sum_{j=1}^{k} (j + \sigma(k+j) - 1) = 3k^2.$$

It follows from (8) that $\det M(Z)$ is a homogeneous polynomial in the variables z_1, z_2, z_3 and of degree $3k^2$. We also note for future use that

$$\sum_{j=1}^{k} (j + \sigma(j) - 1) \ge k^2 \tag{11}$$

with equality if and only if σ leaves invariant the sets $\{1, \cdots, k\}$ and $\{k+1, \cdots, 2k\}$. For any $c \in \mathbb{C}$ we define two $2k \times 2k$ triangular matrices $P(c)$ and $P^*(c)$ associated with the following changes of basis on $\mathbb{C}_{2k-1}$ and $\mathbb{C}_{k-1}$ respectively

$$\begin{aligned} P(c)B &= (\ 1,\ z + c, \cdots,\ (z+c)^{2k-1}) \\ P^*(c)B^* &= (\ (1,0), \cdots,\ ((z+c)^{k-1}, 0),\ (0,1), \cdots,\ (0, (z+c)^{k-1})\ ) \end{aligned}$$

One easily sees that the matrices $P(c)$ and $P^*(c)$ are lower-triangular and have ones on the diagonal, hence $\det P(c) = \det P^*(c) = 1$.

Since

$$I_{z_1+c, z_2+c}(z^i z^j) = I_{z_1, z_2}((z+c)^i (z+c)^j)$$

we obtain

$$M(z_1 + c,\ z_2 + c,\ z_3 + c) = P(c)^{\mathrm{T}} M(Z) P^*(c).$$

Recalling that $\det P(c) = \det P^*(c) = 1$ and choosing $c = -z_1$ we obtain

$$\det M(0,\ z_2 - z_1,\ z_3 - z_1) = \det M(Z)$$

For example if $k = 2$,

$$M(0,\ z_2 - z_1,\ z_3 - z_1) = \begin{pmatrix} z_2 - z_1 & \frac{(z_2-z_1)^2}{2} & z_3 - z_1 & \frac{(z_3-z_1)^2}{2} \\ \frac{(z_2-z_1)^2}{2} & \frac{(z_2-z_1)^3}{3} & \frac{(z_3-z_1)^2}{2} & \frac{(z_3-z_1)^3}{3} \\ \frac{(z_2-z_1)^3}{3} & \frac{(z_2-z_1)^4}{4} & \frac{(z_3-z_1)^3}{3} & \frac{(z_3-z_1)^4}{4} \\ \frac{(z_2-z_1)^4}{4} & \frac{(z_2-z_1)^5}{5} & \frac{(z_3-z_1)^4}{4} & \frac{(z_3-z_1)^5}{5} \end{pmatrix}$$

It follows from (10) and (11) that the polynomial $\det M(Z)$ is a multiple of $(z_2 - z_1)^{k^2}$. Similarly, $\det M(Z)$ is a multiple of $(z_3 - z_1)^{k^2}$.

Substracting column $k + i$ from column i, for all $1 \le i \le k$, we find that

$$\det M(z_3, z_2, z_1) = (-1)^k \det M(Z)$$

and therefore $\det M(Z)$ is also a multiple of $(z_3 - z_2)^{k^2}$. Since $\det M(Z)$ is a polynomial of degree $3k^2$ in the complex variables z_1, z_2, z_3, and since $(z_1 - z_2)^{k^2}$, $(z_2 - z_3)^{k^2}$ and $(z_3 - z_1)^{k^2}$ have no common factors, there exists a constant $\alpha \in \mathbb{C}$ such that

$$\det M(Z) = \alpha\ (z_1 - z_2)^{k^2} (z_2 - z_3)^{k^2} (z_3 - z_1)^{k^2}.$$

In order to compute the constant α, and to show that $\alpha \neq 0$, we remark that it is the coefficient of z^{k^2} in the polynomial $\det M(0, z, 1) = \alpha(-z)^{k^2}(z - 1)^{k^2}$. If $k = 2$ this matrix has the following form

$$M(0, z, 1) = \begin{pmatrix} z & \frac{z^2}{2} & 1 & \frac{1}{2} \\ \frac{z^2}{2} & \frac{z^3}{3} & \frac{1}{2} & \frac{1}{3} \\ \frac{z^3}{3} & \frac{z^4}{4} & \frac{1}{3} & \frac{1}{4} \\ \frac{z^4}{4} & \frac{z^5}{5} & \frac{1}{4} & \frac{1}{5} \end{pmatrix}$$

The contribution of a permutation $\sigma \in \mathbb{S}$ to $\det M(0, z, 1)$ is a monomial which has degree k^2 if and only if (11) is an equality. Denoting by $\mathbb{S}^*$ the collection of permutations of the set $\{1, \cdots, k\}$, we obtain that $\det M(0, z, 1)$ equals

$$\left(\sum_{\sigma_1 \in \mathbb{S}^*} \varepsilon(\sigma_1) \prod_{j=1}^{k} M(0, z, 1)_{j,\sigma_1(j)}\right) \left(\sum_{\sigma_2 \in \mathbb{S}^*} \varepsilon(\sigma_2) \prod_{j=1}^{k} M(0, z, 1)_{j+k,\, \sigma_2(j)+k}\right) + \mathcal{O}(z^{k^2+1}).$$

Hence using (8)

$$\det M(0, z, 1) = z^{k^2} \det\left(\frac{1}{i + j - 1}\right)_{1 \le i,j \le k} \det\left(\frac{1}{i + j + k - 1}\right)_{1 \le i,j \le k} + \mathcal{O}(z^{k^2+1})$$

This expression gives the value of α as the product of two Cauchy determinants, which can be computed using the formula, established in [2] §I.1.3,

$$\det\left(\frac{1}{a_i + b_j}\right)_{1 \le i,j \le k} = \frac{\prod_{1 \le i < j \le k} (a_i - a_j) \prod_{1 \le i < j \le k} (b_i - b_j)}{\prod_{1 \le i,j \le k} (a_i + b_j)}.$$

This concludes the computation of $\det M(Z)$. $\diamond$

2. A counter example in three space dimensions

Let T_0 be the simplex of vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ and let P_0 be the harmonic polynomial of degree 4

$$\begin{aligned}P_0 \;:=\; & 3x+10x^3-15x^4+3y-18xy-15x^2y+30x^3y-15xy^2+45x^2y^2+10y^3\\ & +30xy^3-15y^4+3z-18xz-15x^2z+30x^3z-18yz+240xyz-180x^2yz\\ & -15y^2z-180xy^2z+30y^3z-15xz^2+45x^2z^2-15yz^2-180xyz^2+45y^2z^2\\ & +10z^3+30xz^3+30yz^3-15z^4.\end{aligned}$$

We define

$$u_0 := \nabla P_0 \in \mathcal{P}_{2,3}.$$

One can easily check using a formal computing program that all the linear functionals in $\mathcal{N}_{3,3}(T_0)$ vanish on u_0, which shows that the conjecture Conj(2,3), on quadratic vector fields in three dimensions, is not valid. The interested reader can download on the website *www.ann.jussieu.fr/˜ mirebeau/* a Mathematica® file that contains these verifications.

It thus remains an open question to find a quadratic vector finite element for $H(\mathrm{curl};\Omega) \cap H(\mathrm{div};\Omega)$ in dimension 3. Let us finally mention that, up to a multiplicative constant, u_0 is the only element of $\mathcal{P}_{2,3}$ on which all the linear functionals $\mathcal{N}_{3,3}(T_0)$ vanish.

References

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