A REDUCED BASIS FOR OPTION PRICING *

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Abstract.
We introduce a one dimensional Galerkin basis to numerically solve parabolic partial (integro-)differential equations which arise in option pricing theory. Basis functions are designed on Black-Scholes solutions and this choice is driven by the two main constraints: the numerical efficiency in the computation of the basis and of the Galerkin matrices and the suitable global shape and correct asymptotic behaviour for boundary condition. A convergence proof is given and numerical tests are performed. The basis is tried also for calibration of local volatilities.

Key words. Option pricing, jump diffusion, Galerkin basis, reduced basis

AMS subject classifications. 37M25, 65N99

1. Introduction.

Any option pricing problem is equivalent to solving a partial (integro-) differential equation (PIDE). When the random evolution of the underlying asset is driven by a Lévy process or more generally a time inhomogeneous jump-diffusion process, the Feynmann-Kac theorem relates the conditional expectation of the value of a contract payoff function under the risk-neutral measure to the solution of a PIDE. Due to the nonlocal integral term, the discretization of this kind of problem leads to dense matrices with important resolution cost. Efficient numerical methods are required to rapidly price complex contracts and calibrate various financial models. Various technics have then been introduced to speed this computation up as the one presented in [3] by the use of wavelets or in [5] for an implicit-explicit scheme. *Reduced-Order Models* (ROM) can also be used to efficiently approximate the behaviour of an costly problem with a smaller one: for example Proper Orthogonal Decomposition (POD) have been applied in [6] and [7].

Galerkin methods use variational formulation to numerically approximate solution of this kind of equation on a finite basis of functions.

The choice of this basis is important and has to be driven by at least the two following main constraints: the numerical efficiency of the basis computation and the suitability of its "global" shape and asymptotic behaviour. With this purpose in mind, our idea is to introduce a one dimensional Galerkin reduced basis designed on the Black-Scholes solution under the same initial problem and to prove that this is a good choice to efficiently solve PIDE problems. This methodology can theoretically be applied to any payoff that has a (semi-)closed form.

In a first part, we introduce the problem of option pricing and introduce some notation. We then recall the Galerkin method and introduce our choice of basis. In section 4,we study the convergence and we highlight the efficiency of our basis in section 5 and 6 with option pricing and calibration example.

2. Problem.

In the Partial Integro-Differential Equation (PIDE) framework, we are looking for an efficient ways of solving an time dependent partial integro-differential equation of the kind:

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$$\partial_t w - \mathcal{L}w = f, \quad w(T) = \phi \tag{2.1}$$

For the pricing of options modeled with a Lévy process and adapted probability measures, the operator $\mathcal{L}$ is the sum of the local volatility Black-Scholes operator $\mathcal{L}^\sigma$ and the integral jump operator $\mathcal{L}^J$:

$$\begin{aligned}
&\mathcal{L} = \mathcal{L}^\sigma + \mathcal{L}^J, \\
&\mathcal{L}^\sigma w(S,t) = -\frac{1}{2}\sigma^2(S,t)S^2\partial_{SS}w(S,t) - (r-d)S\partial_S w(S,t) + rw(S,t) \\
&\mathcal{L}^J w(S,t) = -\int_{\mathbb{R}} \left[w(Se^z,t) - w(S,t) - S(e^z-1)\partial_S w(S,t)\right] J(z)dz
\end{aligned} \tag{2.2}$$

Among the popular choices for the Lévy density $J(z)$ are the variance Gamma, CGMY or Merton.

2.1. Change of variable. We introduce the following change of variable with

$$\begin{aligned}
&\tau = T - t, \text{ the time to maturity.} \\
&y = e^{(r-d)(T-t)}\frac{S}{K}, \text{ the forward moneyness price,} \\
&x = \ln y \ \ (i.e.\ S = Ke^{x-(r-d)\tau}), \text{ the log forward moneyness (LFM) price,}
\end{aligned} \tag{2.3}$$

PROPOSITION 2.1. *Let $C(s,t)$ be an option verifying (2.1) with $f = 0$. Let $v(y,\tau) = \frac{e^{(r-d)(T-t)}}{K}C(S,t)$ and $u(x,\tau) = v(y,\tau)$; then*

$$\begin{aligned}
&\partial_\tau v - \frac{1}{2}\sigma^2 y^2 \partial_{yy} v + \mathcal{L}^J v = 0 \quad \text{in } \mathbb{R}^+ \times (0,T), \ \ v(y,0) = v_0(y) := \phi(Ky) \\
&\partial_\tau u + \mathcal{L}^\sigma u + \mathcal{L}^J u = 0 \quad \text{in } \mathbb{R} \times (0,T), \ \ u(x,0) = u_0(x) := \phi(Ke^x) \\
&\text{where} \\
&\mathcal{L}^\sigma u = \frac{1}{2}\sigma^2\partial_x u - \frac{1}{2}\sigma^2\partial_{xx}u \\
&\mathcal{L}^J v = \int_{\mathbb{R}} v(ye^z)J(z)dz - v(y)\int_{\mathbb{R}} J(z)dz - y\partial_y v(y)\int_{\mathbb{R}}(e^z-1)J(z)dz \\
&\mathcal{L}^J u = \int_{\mathbb{R}} u(x+z)J(z)dz - u(x)\int_{\mathbb{R}} J(z)dz - \partial_x u(x)\int_{\mathbb{R}}(e^z-1)J(z)dz
\end{aligned} \tag{2.4}$$

Proof. By elementary algebra of composite functions:

$$\begin{aligned}
\partial_t \left[\tfrac{e^{(r-d)(T-t)}}{K}C(S,t)\right] - \mathcal{L}^\sigma\left[\tfrac{e^{(r-d)(T-t)}}{K}C(S,t)\right] &= \partial_\tau v - \tfrac{\sigma^2(yKe^{(r-d)\tau},T-\tau)}{2}y^2\partial_{yy}v \\
&= \partial_\tau u + \tfrac{\sigma^2(Ke^{x-(r-d)\tau},T-\tau)}{2}\partial_x u - \tfrac{\sigma^2(Ke^{x-(r-d)\tau},T-\tau)}{2}\partial_{xx}u = \partial_\tau u + \mathcal{L}^\sigma u
\end{aligned}$$

and similarly for the other operator.
□

2.2. Reduction to homogeneous initial conditions . In order to obtain a good asymptotic behaviour at infinity, we choose a constant volatility Σ and introduce

$$\pi(x,\tau) = [u - u_\Sigma](x,\tau), \tag{2.5}$$

where u_Σ is the solution of

$$\partial_\tau u(x,\tau) + \mathcal{L}^\Sigma u(x,\tau) = 0, \ \ u(x,0) = u_0(x) \tag{2.6}$$

The equation for π has a source term:

$$\begin{cases} \partial_\tau \pi(x,\tau) + \mathcal{L}^\sigma \pi(x,\tau) + \mathcal{L}^J \pi(x,\tau) = f(x,\tau) \\ \pi(x,0) = 0, \end{cases} \tag{2.7}$$

with $f(x,\tau) := -\partial_\tau u_\Sigma(x,\tau) - \mathcal{L}^\sigma u_\Sigma(x,\tau) - \mathcal{L}^J u_\Sigma(x,\tau)$.

3. Galerkin method.

3.1. Introduction. The Galerkin method is a general and robust methodology to approximate a partial differential equation via its variational formulation. It has already been applied to the PIDE (2.7): the reader is sent to [3] and [8] for more details and existence results. If (u,w) denotes the integral on $\mathbb{R}$ of $u(\cdot)w(\cdot)$, one seeks $\pi(\cdot,t) \in H^1(\mathbb{R})$, the Sobolev space of order 1, solution of

$$\begin{aligned} &\partial_\tau(\pi,\omega) + a^\sigma(\pi,\omega) + \left(\mathcal{L}^J\pi,\omega\right) = (f,\omega)\,,\ \ \forall\omega \in H^1(\mathbb{R}) \\ &\text{with } a^\sigma(\pi,\omega) = \int_{-\infty}^{+\infty} \left[\tfrac{\sigma^2}{2}\partial_x\pi\partial_x\omega + \left(\sigma\partial_x\sigma + \tfrac{\sigma^2}{2}\right)\partial_x\pi\,\omega\right] \end{aligned} \tag{3.1}$$

The problem is localized in the interval $\Omega = (-A,+A)$ and integrals on $\mathbb{R}$ are replaced by integrals on Ω. A finite number of independent functions $\omega_i \in H^1(\Omega)$ are chosen to generate a subset $H_N \subset H^1(\Omega)$:

$$H_N = \mathrm{Sp}\{\omega_i\}_{i=1,..,N}.$$

and we seek for an approximation of $\pi_N \in H_N$ of π :

$$\pi_N(x,\tau) = \sum_{j=1}^N \alpha_j(\tau)\omega_j(x)$$

by solving

$$\partial_\tau(\pi_N,\omega_i) + a^\sigma(\pi_N,\omega_i) + \left(\mathcal{L}^J\pi_N,\omega_i\right) = (f,\omega_i)\,,\ \ \forall i = 1,..,N \tag{3.2}$$

In effect this is a linear system of differential equations of the form

$$\begin{aligned} &M\dot\alpha(\tau) + A\alpha(\tau) = F(\tau), \qquad \text{where } \alpha(\tau) = \{\alpha_J(\tau)\}_{j=1,\dots,N} \\ &\text{and } A_{i,j} = a^\sigma(\omega_j,\omega_i) + \left(\mathcal{L}^J\omega_j,\omega_i\right) \text{ and with } M_{i,j} = (\omega_j,\omega_i)\,,\ \ F_i = (f,\omega_i)\,. \end{aligned}$$

The Euler implicit scheme is used to discretize the time derivative: ∂_τ:

$$(M + \delta\tau A)\alpha^{n+1} = M\alpha^n + \delta\tau F^n \tag{3.3}$$

3.2. Choice of a basis. The key point in this work is the following choice for the Galerkin basis, with $N = 2n$:

$$\begin{aligned} \omega_i(x) &= \omega_i^\sigma(x) := \mathcal{L}^\sigma[u_{\sigma_i}](x,T) \qquad \forall i = 1,\dots,n \\ \omega_{i+n}(x) &= \omega_i^J(x) := \mathcal{L}^J[u_{\sigma_i}](x,T) \qquad \forall i = 1,\dots,n \end{aligned} \tag{3.4}$$

where u_{σ_i} is the Black-Scholes solution with constant volatility σ_i, i.e. the solution of (2.6) with $\Sigma = \sigma_i$.

Notice that the basis does not depend on time.

$spot$	K	T	r	d	σ_0
40	42	5	0.1	0.02	0.15

TABLE 3.1
Option pricing data

3.2.1. Numerical computation. With the notation $v_i = \sigma_i\sqrt{T}, \forall i = 1,\dots,n$ a straightforward computation sketched in section 4) below leads to:

$$\omega_i^\sigma(x) = -\frac{\sigma^2}{2v_i}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x}{v_i}-\frac{v_i}{2}\right)^2}$$

On the other hand a closed formula for $\omega_i^J(x)$ is more difficult to obtain in general except with Merton's kernel as shown below. Nevertheless since $\int_{\mathbb{R}} u(x+z)J(z)dz$ is a convolution, efficient algorithm using Fourier transform and FFT are available when the characteristic function of the jump density function is known (see [4])

Remarks: $\int_{\mathbb{R}} u(x+z)J(z)dz$ needs to be computed for a range of x. The method of Carr and Madan [1] compute the option pricing for a range of strikes and cannot then be applied here.

PROPOSITION 3.1. *For Merton's model with*

$$J(z) = \frac{\lambda}{\delta\sqrt{2\pi}}e^{-\frac{(z-\mu)^2}{2\delta^2}} \tag{3.5}$$

as jump density probability, we have the following expression for $\omega_i^J(x)$ basis function

$$\omega_i^J(x) = \lambda\left(e^{x+\mu+\frac{\delta^2}{2}}\mathcal{N}(d_1) - \mathcal{N}(d_2) - u_{\sigma_i}(x,T) - \left[e^{\mu+\frac{\delta^2}{2}}-1\right]\partial_x u_{\sigma_i}(x,T)\right)$$

where $d_1 = \frac{x+\mu+\delta^2+\frac{v_i^2}{2}}{\sqrt{\delta^2+v_i^2}}$, $d_2 = \frac{x+\mu-\frac{v_i^2}{2}}{\sqrt{\delta^2+v_i^2}}$,

$$\text{with as usual } \mathcal{N}(y) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-\frac{t^2}{2}}dt = \frac{1}{2}\left[1+\mathit{erf}\left(\frac{y}{\sqrt{2}}\right)\right]$$
$$\text{and } \mathit{erf}(y) = \frac{2}{\sqrt{\pi}}\int_0^y e^{-t^2}dt. \tag{3.6}$$

The proof is in appendix A.

3.2.2. Example. In this section we plot the basis function associated to various model: Black-Scholes (B-S), Constant Elasticity of Variance (CEV introduced in [10]) and Merton for a range of volatility :

$$\{\sigma_i\}_{i=1..5} = 0.070, 0.124, 0.221, 0.393, 0.7 \tag{3.7}$$

Option pricing common data for every model are defined in the following table:

The B-S basis function are plotted in Fig. 3.1.

The CEV model defines the volatility σ in (2.6) as:

$$\sigma(x,\tau) = \alpha\left(Ke^{x-(r-d)\tau}\right)^\beta, \qquad \alpha\in\mathbb{R}^+, \beta\le 0$$

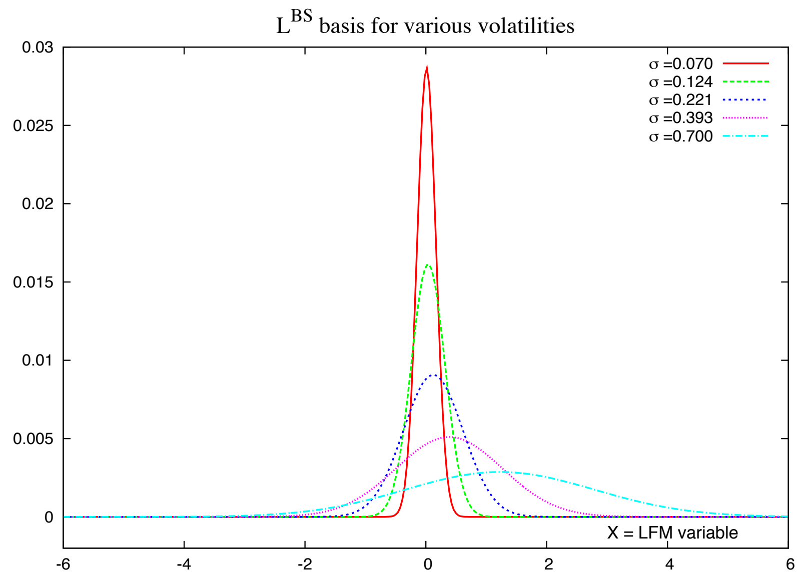

FIG. 3.1. *Plots of* ω_i^{BS} *for the 5* σ_i *of (3.7) versus x, the Log Forward Moneyness (LFM).*

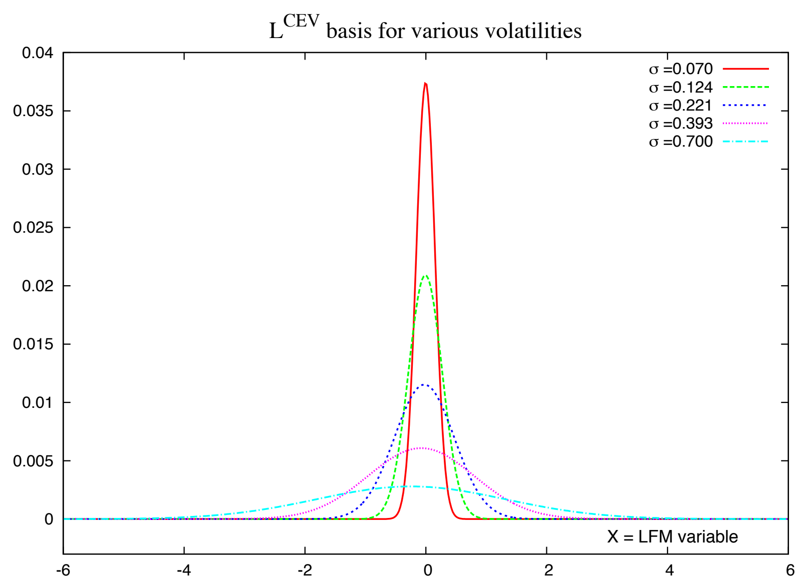

FIG. 3.2. *Plots of* ω_i^{CEV} *for the 5* σ_i *of (3.7) versus x, the Log Forward Moneyness (LFM).*

and the associated basis functions are plotted in Fig. 3.2, with $\beta = -0.3$ and $\alpha = \sigma_0 K^{-\beta}$.

Finally, we plot in Fig. 3.3 the basis function associated to the Merton's model as jump density probability with arbitrary parameter $\lambda = 0.4$, $\mu = 0.5$ and $\delta = 0.6$.

As you can see the chosen basis functions decay exponentially fast to zero at infinity and we can correctly deal with the boundary conditions without any specific treatments.

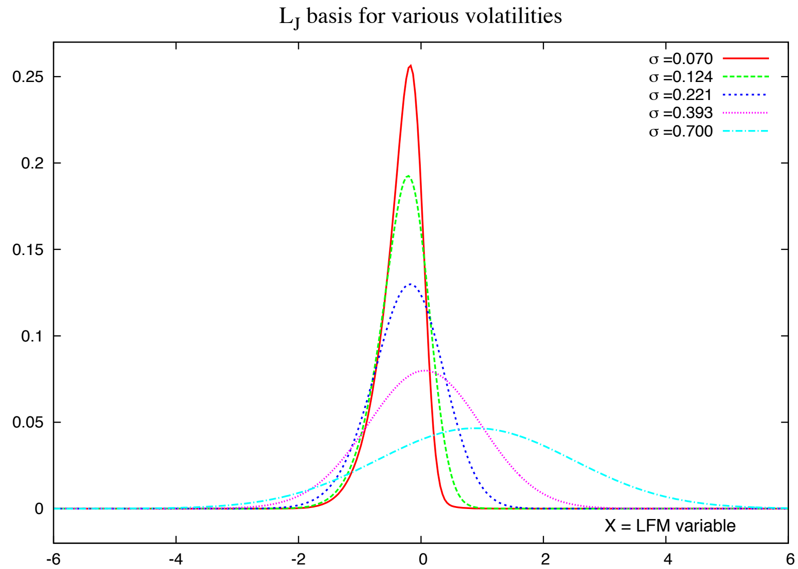

FIG. 3.3. *Plots of ω_i^J for the 5 σ_i of (3.7) versus x, the Log Forward Moneyness (LFM).*

3.3. Preliminary numerical validation of the basis. For CEV model as diffusion (respectively Merton's model for the jumps), the problem for a call option has an "exact" semi-analytical solution see [11] (respectively [9]). We can then observe the efficiency of the basis defined above, by projecting this "exact" solution $\pi(x, \tau)$ on our basis $\{\omega_i\}_{i=1,..,N}$.

To numerically compute the projection, we build the matrix M and the vector F as defined in section 3 where in this case the function f is $\pi(x, \tau)$, the "exact" solution associated to the model and X is the coordinates in our basis of the "exact" solution as the solution of the linear system $MX = F$ (numerically solved with GMRES).

We define Ω the computational domain for numerical integration and Ω_ϵ is a small domain around spot price where errors are computed. Ω_ϵ is defined as $[spot.e^{-\sigma_m\sqrt{T}}, spot.e^{\sigma_m\sqrt{T}}]$ for an arbitrary σ_m ($\sigma_m = 0.15$ for CEV and $\sigma_m = \sqrt{\sigma^2 + \lambda(\mu^2 + \delta^2)} = 0.5162$ according to data defined in Table 3.1). To study the accuracy of the projection we define the following error metric:

$$\epsilon_p(\tau) = \frac{\int_{\Omega_\epsilon} [\pi_N(x,\tau) - \pi(x,\tau)]^2 dx}{\int_{\Omega_\epsilon} [\pi(x,\tau)]^2 dx} \tag{3.8}$$

and we plot for both models the above error metric $\epsilon_p(T)$ expressed in percentage (%) as function of n. The σ_i are distributed in the segment $[\Sigma_{min}, \Sigma]$ according to the inverse of the square root, as advised by the theory (see Proposition 4.1).

For CEV model, the reduced basis of size $N = 2n$ is defined as

$$\begin{aligned} \omega_i(x) &= \omega_i^{BS}(x) := \mathcal{L}^{\sigma_0}[u_{\sigma_i}](x, T) \qquad \forall i = 1, \dots, n \\ \omega_{i+n}(x) &= \omega_i^{CEV}(x) := \mathcal{L}^{\sigma(x,T)}[u_{\sigma_i}](x, T) \qquad \forall i = 1, \dots, n \end{aligned} \tag{3.9}$$

with $\Sigma_{min} = 0.03$ and $\Sigma = 0.3$ arbitrary chosen. The results obtained are shown in Fig. 3.4.

For Merton model, $2n$ basis functions are defined as in (3.4) with $\sigma = \sigma_0$, $\Sigma_{min} = 0.07$ and $\Sigma = 0.7$ are arbitrary chosen. In Fig. 3.5, we plot the results obtained.

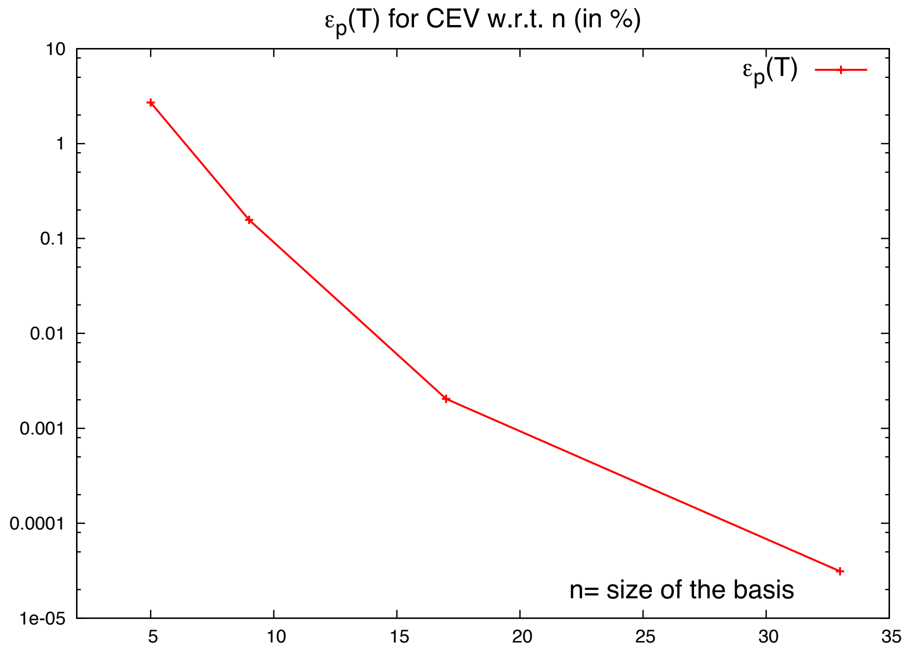

FIG. 3.4. $\epsilon_p(T)$ *in log scale for CEV volatility model w.r.t.* n *on* $\Omega_\epsilon = [28.6, 56]$

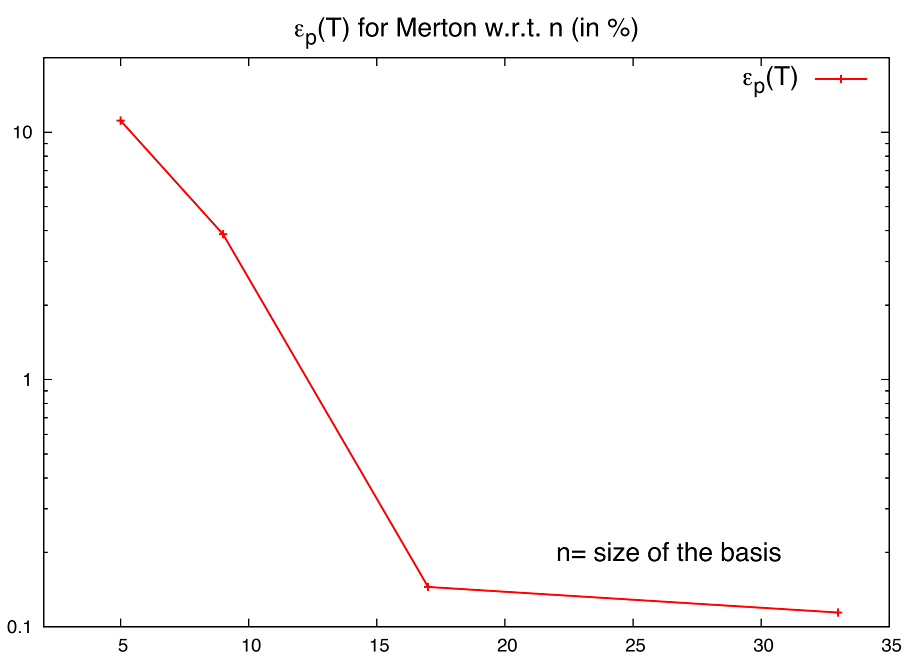

FIG. 3.5. $\epsilon_p(T)$ *in log scale for Merton model w.r.t.* n *(%) on* $\Omega_\epsilon = [12.6, 127]$

We observe in both case the good behaviour of the basis as the accuracy grows with the size of the basis n.

4. Convergence. . In this section we proceed to show that the basis chosen is indeed sufficient to represent any solution of Black-Scholes' equation (2.1) after translation with or without jumps.

4.1. The basis for diffusion operator.

4.1.1. Convergence. We choose to work with calls and with the formulation using the moneyness y and time-to-maturity τ. The basis functions are solution at time $\tau = T$ of

$$\partial_\tau v - \frac{\sigma^2 y^2}{2}\partial_{yy} v = 0, \text{ in } R\times[0,T], \ \ v(y,0) = (y-1)^+$$

for some σ. Using the well known analytical solution of the Black-Scholes equation with constant volatility σ_i:

$$\begin{aligned} v &:= \tfrac{e^{r\tau}}{K} Call(S,K,\sigma_i,t,T) \\ &= \tfrac{e^{r\tau}}{K}\left[S\mathcal{N}\left(\tfrac{\ln y}{\sigma_i\sqrt{\tau}} + \tfrac{\sigma_i\sqrt{\tau}}{2}\right) - Ke^{-r(T-t)}\mathcal{N}\left(\tfrac{\ln y}{\sigma_i\sqrt{\tau}} - \tfrac{\sigma_i\sqrt{\tau}}{2}\right)\right] \\ &= \tfrac{1}{2}\left[y(1 - \text{erf}\left(-\tfrac{\ln y}{\sigma_i\sqrt{2\tau}} - \tfrac{\sigma_i\sqrt{\tau}}{2\sqrt{2}}\right) - 1 + \text{erf}\left(\tfrac{\ln y}{\sigma_i\sqrt{2\tau}} - \tfrac{\sigma_i\sqrt{\tau}}{2\sqrt{2}}\right)\right] \end{aligned}$$

Therefore

$$\begin{aligned} \mathcal{L}^{\sigma(y,T)} v &:= -\frac{\sigma^2(y,T)y^2}{2}\partial_{yy} v \\ &= -\tfrac{\sigma^2(y,T)}{\sqrt{2\pi}2\sigma_i\sqrt{\tau}} e^{-\frac{1}{2}\left[\frac{\ln^2 y}{\sigma_i^2\tau} - \ln y + \frac{\sigma_i^2\tau}{4}\right]} = C_0\,\sigma^2(y,T)\sqrt{y}e^{-\frac{\ln^2 y}{2\sigma_i^2\tau}} \end{aligned} \tag{4.1}$$

where $C_0 = -\frac{e^{-\frac{1}{8}\sigma_i^2\tau}}{\sqrt{2\pi}2\sigma_i\sqrt{\tau}}$ is not a function of y.

REMARK 1. *Notice that $\sigma^{-2}(y,T)y^{-1/2}\mathcal{L}^{\sigma(y,T)}v$ is an even function of* $\ln y$.

Accordingly, functions which do not have the above symmetry cannot be written as a sum of such $\mathcal{L}^{\sigma(y,T)}v$.

PROPOSITION 4.1. *Consider the set of constant volatilities $\sigma_i = i^{-1/2}c, \ i = 1,2,3...$ for some real c. Let v_i be the Black-Scholes Call at time T with constant volatility σ_i and let $\omega^i = \mathcal{L}^{\sigma(y,T)}v_i$; Then $\{\omega_i\}$ is a basis for the set of continuous functions $f : \mathbb{R}^+ \to \mathbb{R}$ which decay exponentially fast at $+\infty$ and are such that*

$$f\left(\frac{1}{y}\right) = f(y)\frac{\sigma^2(y,T)}{y\sigma^2\left(\frac{1}{y},T\right)} \ \ \forall y > 0.$$

Proof. : Let $x = \ln y$. According to (4.1), ω_i is proportional to $\sqrt{y}\sigma^2(y,T)e^{-i\frac{x^2}{2c^2T}}$. Consider the algebra generated by $\{\exp(-n\alpha x^2)\}_{n\in N}$ for a given real $\alpha \neq 0$; by the Stone-Weiestrass theorem it is a basis for the continuous *even* functions on $\mathbb{R}^+$ which decay exponentially fast at $\pm\infty$ because $\exp(-\alpha x^2)$ is a separating function on $\mathbb{R}^+$ (i.e. $f(x) \neq f(x')$ for all $x \neq x' \geq 0, \ x \geq 0$).

Given a function $y \to f(y)$ we can decompose $g(\ln y) := f(y)\sigma^{-2}(y,T)/\sqrt{y}$ on the basis w^i only if $g(x)$ is even in $\ln y$, i.e. $g(-\ln y) = g(\ln y)$; this corresponds to the following restriction on f:

$$\frac{f(e^{-x})}{\sqrt{e^{-x}}} = \frac{f(e^x)}{\sqrt{e^x}} \qquad \text{i.e. } yf\left(\frac{1}{y}\right) = f(y) \tag{4.2}$$

□

PROPOSITION 4.2. *If $\sigma(y,\tau)=\sigma\left(\frac{1}{y},\tau\right)$ and $f(y,\tau)=yf\left(\frac{1}{y},\tau\right)$ for all $y>0$ then the solution of*

$$\partial_\tau v-\frac{\sigma^2(y,\tau)y^2}{2}\partial_{yy}v=f,\ \ v(y,0)=0,\ in\ \mathbb{R}^+\times[0,T] \tag{4.3}$$

is invariant under the transformation $v(y)\to y.v\left(\frac{1}{y}\right)$.

Proof. : Let us prove that $yv\left(\frac{1}{y},\tau\right)$ satisfies the PDE when v does. Let $w(y,\tau):=yv\left(\frac{1}{y},\tau\right)$. Notice that

$$\partial_{yy}w(y,\tau)=\frac{1}{y^3}\partial_{zz}v(z,\tau)|_{z=\frac{1}{y}}.$$

Therefore

$$\frac{1}{y}\partial_\tau w=\partial_\tau v\left(\frac{1}{y},\tau\right)=\frac{\sigma^2\left(\frac{1}{y},\tau\right)}{2y^2}\partial_{zz}v(z,\tau)|_{z=\frac{1}{y}}=\frac{\sigma^2(y,\tau)y^2}{2}\partial_{yy}w(y,\tau)$$

which means that w verifies also the PDE. Equation (4.2) differentiated at $y=1$ gives $f(1)=2f'(1)$, therefore if v^+ is the unique solution of (4.3) on $[1,\infty)$ and v^- is constructed from v^+ on $[0,1]$ by (4.2), then ${v^+}'(1)={v^-}'(1)$ and so $v^\pm$ is the unique solution of (4.3)on $\mathbb{R}^+$.
□

Consequently we have the following result:

THEOREM 4.3. *Let Σ,c be real positive numbers and $\sigma_i=c/\sqrt{i}$. Let u_Σ,u_{σ_i} be the solutions of the Black-Scholes equations with the corresponding volatilities. Then the solution of the Black-Scholes equation u_σ for a non-constant volatility σ can be written as*

$$u_{\sigma(S,t)}(S,t)=u_\Sigma(S,t)+\sum_{i=1}^{\infty}\alpha_i(t)\mathcal{L}^{\sigma(S,0)}u_{\sigma_i}(S,0)$$

for some time dependent but S-independent α_i provided that $\sigma(S,t)=\sigma(e^{-2(r-d)(T-t)}\frac{K^2}{S},t)$ for all S and t.

Proof. : The data f and σ have the properties required by Proposition 4.2, so $u_\sigma-u_\Sigma$ can be decomposed on the basis $\{w^i\}$. □

PROPOSITION 4.4. *Let assume that $x_m\le x\le x_M$, $0<z_m\le z\le z_M<1$, $\forall i\in\mathbb{N}$, $0<\sigma_m\le\partial_x^i\sigma\le\sigma_M$ and $\forall i\in\mathbb{N}$, $\pi_m\le\partial_x^i\pi\le\pi_M$. We also assume that σ checks the above symmetry properties. Then $u_\Sigma(S,t)+\sum_{i=1}^N\alpha_i(t)\mathcal{L}^\sigma u_{\sigma_i}(S,0)$ tends exponentially fast in N to $u_\sigma(S,t)$*

Proof. : Let $z=e^{-\alpha x^2}$ with α a real positive constant and $f(z)=\frac{\pi(x)}{\sqrt{y}\sigma^2(y,T)}$. For a given z_0 such that $0<z_m\le z_0\le z_M<1$ and $\xi\in[z,z_0]$, we can write

$$f(z)=\sum_{i=0}^{N}\frac{(z-z_0)^i}{i!}f^{(i)}(z_0)+f^{(N+1)}(\xi)\frac{(z-z_0)^{N+1}}{(N+1)!} \tag{4.4}$$

Let assume the following recurrence relation for $i\in\mathbb{N}$ with $\beta_i(x)\le\beta_M$ and $\partial_x\beta_i\le\beta_M$:

$$f^{(i)}(z)=\left[\frac{(-1)^ie^{\alpha i\ x^2}}{(2\alpha x)^i}\right]\frac{1}{\sigma^2(y,T)\sqrt{y}}\left[\sum_{j=0}^{i}\beta_j(x)u^{(j)}\right]$$

This relation is true for $i = 1$. For $i + 1$ we have:

$$\begin{aligned}
f^{(i+1)}(z) &= \partial_z x \partial_x f^{(i)}(z) \\
&= \left[\frac{-e^{\alpha x^2}}{2\alpha x}\right] \partial_x \left[(-1)^i \frac{e^{\alpha i\ x^2}}{(2\alpha x)^i} \frac{1}{\sigma^2(y,T)\sqrt{y}} \sum_{j=0}^{i} \beta_j(x) u^{(j)}\right] \\
&= (-1)^{i+1} \frac{e^{\alpha(i+1)x^2}}{(2\alpha x)^{i+1}} \frac{1}{\sigma^2(y,T)\sqrt{y}} \left(\left[2ix - \frac{i}{x} - \frac{1}{2} - 2\frac{\partial_x \sigma}{\sigma}\right] \sum_{j=0}^{i} \beta_j(x) u^{(j)} + \sum_{j=0}^{i} \left[\partial_x \beta_j(x) u^{(j)} + \beta_j(x) u^{(j+1)}\right]\right) \\
&= (-1)^{i+1} \frac{e^{\alpha(i+1)x^2}}{(2\alpha x)^{i+1}} \frac{1}{\sigma^2(y,T)\sqrt{y}} \sum_{j=0}^{i+1} \beta'_j(x) u^{(j)}
\end{aligned}$$

The recurrence relationship is then true and for C and C' real constants:

$$\begin{aligned}
|f^{(i)}(z)| &= \left| \frac{(-1)^i e^{\alpha i\ x^2}}{(2\alpha x)^i} \frac{1}{\sigma^2(y,T)\sqrt{y}} \sum_{j=0}^{i} \beta_j(x) u^{(j)} \right| \\
&\leq \frac{e^{\alpha i\ x_M^2 - \frac{1}{2} x_m}}{(2\alpha x_m)^i \sigma_m^2} i\, C \\
&\leq \frac{e^{\alpha i\ x^2}}{(2\alpha x_m)^i} i\, C'
\end{aligned}$$

We then introduce $h = z - z_0$ and let

$$\begin{aligned}
\epsilon_{N-1} &:= f(z) - \sum_{i=0}^{N-1} \frac{h^i}{i!} f^{(i)}(z_0) \\
&= f^{(N)}(\xi) \frac{h^N}{N!}
\end{aligned}$$

$$\begin{aligned}
\epsilon_{N-1} &\leq |f^{(N)}(\xi)| \frac{|z - z_0|^N}{N!} \\
&\leq \frac{e^{\alpha N x_M^2}}{(2x_m)^N} N C' \frac{h^N}{N!} \\
&\leq \frac{e^{\alpha N x_M^2}}{(2x_m)^N} N C' \frac{h^N}{\sqrt{2\pi N} \left(\frac{N}{e}\right)^N} \text{ via Stirling's formula} \\
&\leq C'' \frac{e^{N(x_M^2+1)}}{(2x_m)^N} \frac{h^N}{N^{N-\frac{1}{2}}} \\
&\leq C'' \sqrt{N} \left(\frac{e^{(x_M^2+1)}}{2x_m} \frac{h}{N}\right)^N \to 0 \text{ as } N \to +\infty \text{ as fast as } N^{-N}
\end{aligned}$$

□

4.1.2. Numerical Validation. In this section, we numerically check the importance of symmetry condition in the volatility parameter. In this purpose we introduce a "Gaussian volatility" (GV) model with $\sigma(x,\tau) = e^{-0.1x^2}$. We chose to compare $\epsilon_p(T)$ relative error introduced before for 3 various cases: the Black-Scholes model and Gaussian volatility model which both check the symmetry condition, and the CEV model which doesn't. We numerically solve a call option problem with Finite Elements method according to these three volatility models on fine mesh of $2^{13}-1$ vertices and 501 time step. We then project the approximated solution on a reduced basis of size $N = n$ defined as

$$\text{For BS model by } \omega_i(x) = \omega_i^{BS} := \mathcal{L}^{\sigma_0}[u_{\sigma_i}](x,T) \qquad \forall i = 1,\dots,n$$

$$\text{For GV model by } \omega_i(x) = \omega_i^{GV} := \mathcal{L}^{\sigma_{GV}(x,T)}[u_{\sigma_i}](x,T) \qquad \forall i = 1,\dots,n$$

$$\text{For CEV model by } \omega_i(x) = \omega_i^{CEV} := \mathcal{L}^{\sigma_{CEV}(x,T)}[u_{\sigma_i}](x,T) \qquad \forall i = 1,\dots,n$$

We plot in Fig. 4.1 the results for a growing number of basis function. We observe a good

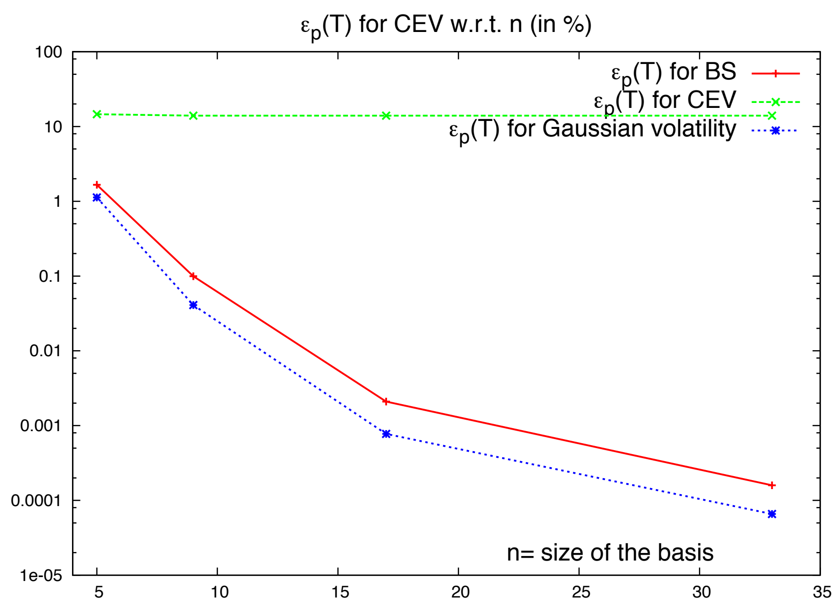

FIG. 4.1. *Relative errors $\epsilon_p(T)$ on $\mathcal{L}^\sigma$ basis only on $\Omega_\epsilon = [28.6, 56]$. The curve for CEV is to be compared with Fig .3.4; it shows that when the symmetry condition is not satisfied $\omega_i(x) = \omega_i^{CEV}$ are not sufficient.*

convergence behaviour for the two symmetric cases and a threshold for CEV's model where the chosen $\{\omega_i\}$ basis functions don't achieve to capture the solution.

4.2. The basis in a special case of CEV's model. As the CEV volatility doesn't check the symmetry property, we use the basis presented in (3.9) and prove the following property:

PROPOSITION 4.5. *For CEV model, i.e. $\sigma(y,T) = \alpha y^\beta$, then $\{\mathcal{L}^{\sigma_0} u_{\sigma_i}\}_i \cup \{\mathcal{L}^{\sigma(y,T)} u_{\sigma_i}\}_i$ is a basis of the space of C^∞ functions which decay exponentially at infinity.*

Proof. : Let $x = \ln y$ and $h_i = \exp(-\frac{x2}{2\sigma_i 2\tau})$ with the constants σ_i chosen so as to make a basis for the even functions of x which decays exponentially at infinity.

We wish to show that any fast decaying function at infinity f can be written as

$$f(x) = e^{\frac{x}{2}} \sum_i (a_i + b_i e^{2\beta x}) h_i(x)$$

Let $g(x) = f(x)e^{-\frac{x}{2}}/(e^{2\beta x} - e^{-2\beta x})$. As $g(x) + g(-x)$ is even there exist a_i such that

$$g(x) + g(-x) = \sum_i a_i h_i(x)$$

Similarly there exist b_i such that

$$e^{-2\beta x} g(x) + e^{2\beta x} g(-x) = \sum_i b_i h_i(x)$$

By elimination of $g(-x)$ we find

$$(e^{2\beta x} - e^{-2\beta x})g(x) = e^{2\beta x} \sum_i a_i h_i(x) - \sum_i b_i h_i(x)$$

In terms of f it gives

$$(e^{2\beta x} - e^{-2\beta x})f(x)e^{-\frac{x}{2}}/(e^{2\beta x} - e^{-2\beta x}) = \sum_i (e^{2\beta x} a_i - b_i) h_i(x)$$

which proves the result. □

4.3. The basis in a special case of Merton's model. To prove that $\{\mathcal{L}^J u_{\sigma_i}\}_i$ forms a basis of a subspace U of the square integrable functions which decay exponentially fast at infinity we shall use the property that if $\mathcal{L}$ is continuous from U to $\mathcal{L}U$ and u_i is a basis of U then $\mathcal{L}u_i$ is a basis of $\mathcal{L}U$.

Recall that $\mathcal{L}^\sigma$ is continuous and injective from $H^2(\mathbb{R})$ to $L^2(\mathbb{R})$ and $\mathcal{L}^J$ is continuous from $H^1(\mathbb{R})$ to $L^2(\mathbb{R})$.

We know from the previous theorem that $\mathcal{L}^\sigma(u_{\sigma_i} - u_\Sigma)$ is a basis for the functions of $H^2(\mathbb{R})$, u, which decay exponentially fast at infinity and are such that $u(-x) = e^{-x}u(x)$; therefore, since $(\mathcal{L}^\sigma)^{-1}$ is continuous, $\{u_{\sigma_i} - u_\Sigma\}_{i\in N}$ is a basis for the functions of $H^2(\mathbb{R})$ with exponential decay at infinity and which satisfy the same symmetry condition, because $\mathcal{L}^\sigma$ preserves the condition.

Consequently $\{\mathcal{L}^J(u_{\sigma_i} - u_\Sigma)\}_{i\in N}$ is a basis for the space of functions which satisfy the transported-by-$\mathcal{L}^J$ symmetry condition, which we proceed to establish now.

By definition of $\mathcal{L}^J$ and by (3.5) we have

$$\mathcal{L}^J u(-x) = \int_{\mathbb{R}} u(z-x)J(z)\mathrm{d}z - \lambda u(-x) - c\partial_x u(-x)$$

with $c = \lambda(e^{\frac{\delta^2}{2}+\mu} - 1)$. Now $u(-x) = e^{-x}u(x)$ implies that $\partial_x u(-x) = -e^{-x}(u(x) - \partial_x u(x))$, so with $z' = -z$

$$\mathcal{L}^J u(-x) = -e^{-x}[\int_{\mathbb{R}} e^{-z'} u(x+z')J(-z')\mathrm{d}z' + (\lambda - c)u(x) + c\partial_x u(x)]$$

The condition on $\mathcal{L}^J u(x)$ seems difficult to find except if

$$J(z) = -e^{-z}J(-z) \text{ and } c = 0. \quad (4.5)$$

in which case one has

$$\mathcal{L}^J u(-x) = e^{-x}\mathcal{L}^J u(x)$$

Therefore any smooth function v which decays exponentially fast at infinity and which satisfies $u(-x) = e^{-x}u(x)$ can be written on the basis $\{\mathcal{L}^J u_{\sigma_i}\}_{i \in N}$. Because this symmetry is the opposite of the previous one, all this leads to the following proposition:

THEOREM 4.6. *If $\sigma(S,t) = \sigma(e^{-2r(T-t)}\frac{K^2}{S}, t)$ for all S and t and if (4.5) holds then any solution of the Black-Scholes-Merton model (2.7) is a linear combination of $\{\mathcal{L}^\sigma u_{\sigma_i}\}_i \cup \{\mathcal{L}^J u_{\sigma_i}\}_i$.*

Proof. Let v be any smooth function decaying exponentially fast at infinity. The following identity holds

$$v(x) = \frac{1}{2}(v(x) + e^x v(-x)) + \frac{1}{2}(v(x) - e^x v(-x)).$$

It shows that v has been written as a sum of a function verifying the first symmetry condition plus a function verifying the second symmetry condition. □

REMARK 2. *When neither σ nor J satisfy the conditions of the last two theorems, still we suspect that we have a basis because each operator has a restriction and it doesn't look like both restrictions are the same. However we cannot prove it.*

For Numerical computations the following is useful

PROPOSITION 4.7. *When σ is constant and J is given by (3.5) then*

$$\mathcal{L}^J u_\sigma(x,\tau) = \mathcal{N}(d_2) - e^{x+\mu+\frac{\delta^2}{2}}\mathcal{N}(d_1) - \lambda u_\sigma - \lambda e^x \mathcal{N}(\frac{x}{\tau} + \frac{\tau}{2})(e^{\frac{\delta^2}{2}+\mu} - 1)$$

with $d_1 = (x + \mu + \delta^2 + \frac{\sigma^2\tau}{2})/\sqrt{\delta^2 + \sigma^2\tau}$ and $d_2 = (x + \mu - \frac{\sigma^2\tau}{2})/\sqrt{\delta^2 + \sigma^2\tau}$.

The proof is in appendix B.

5. Numerical results. We now solve the Galerkin method describe in §3 for a call option under different models: CEV and Merton. In both cases, the basis is the one chosen in section 3.3.

To study the efficiency of the method, we work on two distinct metrics of error expressed in percentage. The first one is the classic relative error reminded in (5.1). Another appropriate error metric for financial application is the relative error expressed in term of Black-Scholes (B-S) implied volatility. We then define

$$\epsilon(S) = \frac{|u_N(S,0) - u_{Exact}(S,0)|}{|u_{Exact}(S,0)|} \quad \text{and} \quad \epsilon_\Sigma(S) = \frac{|\Sigma_N(S,0) - \Sigma_{Exact}(S,0)|}{|\Sigma_{Exact}(S,0)|} \tag{5.1}$$

where Σ is the implied volatility computed by the inversion of B-S formula with respect to the volatilty σ. All the parameters for models are those presented in §3.2.2 . The linear system associated to the Galerkin discretization is solved with a `GMRES` solver.

5.1. CEV model. We first present the numerical results obtained for CEV model. In Fig. 5.1, we first plot the relative error $\epsilon(S)$ in option price and then in term of implied volatility (Fig. 5.2 and 5.3). The convergence results are shown in the two tables 5.1 and 5.2. We recall that $\Omega_\epsilon = [28.6, 56]$.

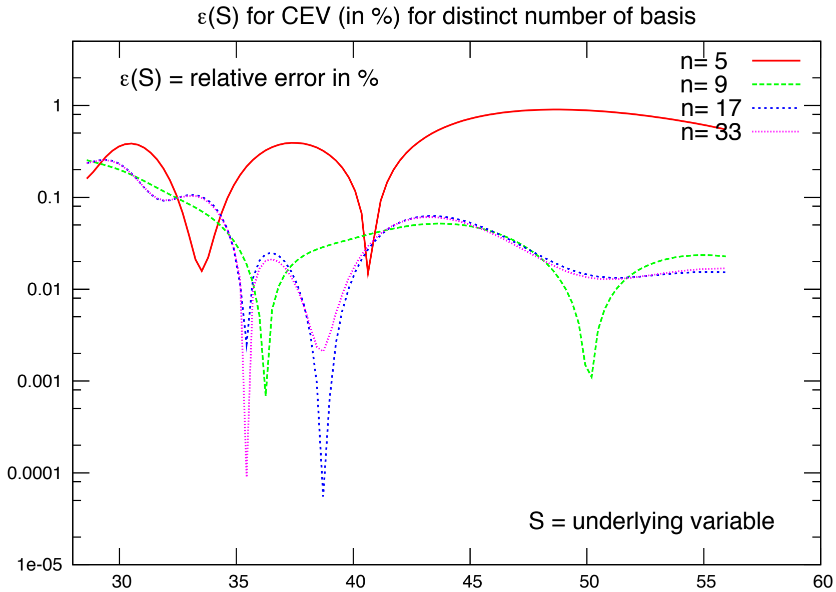

FIG. 5.1. *Relative errors $\epsilon(S)$ for CEV's solution on Ω_ϵ*

n	# Basis	$C_N(Spot, 0)$	$\epsilon(Spot)$	$L^2[\epsilon(S)]$	$L^\infty[\epsilon(S)]$
5	10	11.492	0.132 %	0.428 %	0.902 %
9	18	11.511	0.036 %	0.078 %	0.254 %
17	34	11.509	0.014 %	0.075 %	0.256 %
33	66	11.509	0.017 %	0.074 %	0.253 %
Exact		11.507			

TABLE 5.1
Relative errors $\epsilon(S)$ for CEV's model w.r.t. n ($S \in \Omega_\epsilon$)

The results show that 18 basis (i.e. $n = 9$) is sufficient to reach an acceptable accuracy in both error metrics. The gain obtained by adding new basis is small. This may be due to at least two facts: the CEV's local volatility model doesn't check the symmetry assumption presented in §4 and the associated Galerkin matrix is ill-conditioned.

5.2. Merton model. We then present the numerical results obtained under Merton's model. We plot in Fig. 5.4, 5.5 and 5.6 and in tables 5.3 and 5.4 the same indicators as before. We recall $\Omega_\epsilon = [12.6, 127]$

The results obtained for Merton's model are pretty good even if as CEV's case, symmetry assumptions and ill-conditioned difficulties. Our basis is efficient and as accurate as a standard finite element method on a refined mesh but for a lesser computational cost as the linear system to solve at every time step is much smaller for the given accuracy.

n	# Basis	$\Sigma_N(Spot, 0)$	$\epsilon_\Sigma(Spot)$	$L^2[\epsilon_\Sigma(S)]$	$L^\infty[\epsilon_\Sigma(S)]$
5	10	0.1503	0.646 %	7.281 %	20.03 %
9	18	0.1515	0.174 %	0.322 %	1.274 %
17	34	0.1514	0.067 %	0.277 %	0.815 %
33	66	0.1514	0.081 %	0.277 %	0.897 %
Exact		0.1513			

TABLE 5.2
Relative error for CEV's model in term of B-S implied volatility w.r.t. n ($S \in \Omega_\epsilon$)

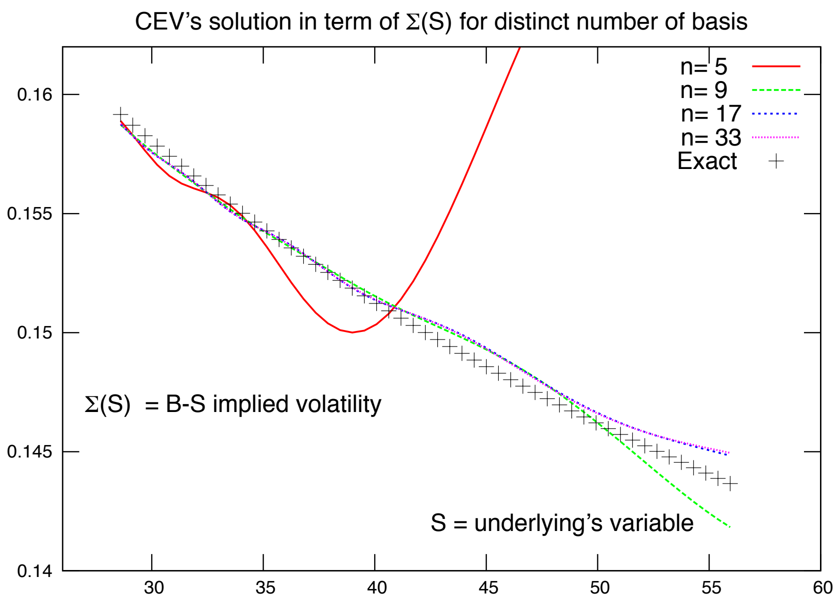

FIG. 5.2. *CEV solution in term of B-S implied volatility on Ω_ϵ*

5.2.1. Study of the spectrum. Here we are interested in the spectrum of the matrix $C = M + \delta\tau A$ of the Galerkin scheme obtained for 66 basis (see (3.3)).

Figure 5.7 illustrates the exponential decay of the C-matrix eigenvalues (normalized by the trace of the matrix), a key ingredient in the search for a reduced basis as previously explained. Note that the reduction of the decay rate that occurs after the fourtieth eigenvalue is surely due to numerical noise.

In Fig. 5.8 we show the relative error of the Galerkin scheme on the eigenvector basis. We observe that 20 basis are sufficient to achieve formal convergence. Moreover the convergence rate is similar to the one observed with the basis w_i, which reinforces our confidence in the fact that this basis is a good one and further reduction is unnecessary.

6. Calibration. The fact that one can write any call C_σ , solution of the Black-Scholes equation for a general volatility $\sigma(S, t)$, as

$$C_\sigma(S, t) = C_\Sigma(S, t) + \sum_{i=1}^{I} \alpha_i(t)(C_{\sigma_i}(S, 0) - C_\Sigma(S, 0)) \qquad (6.1)$$

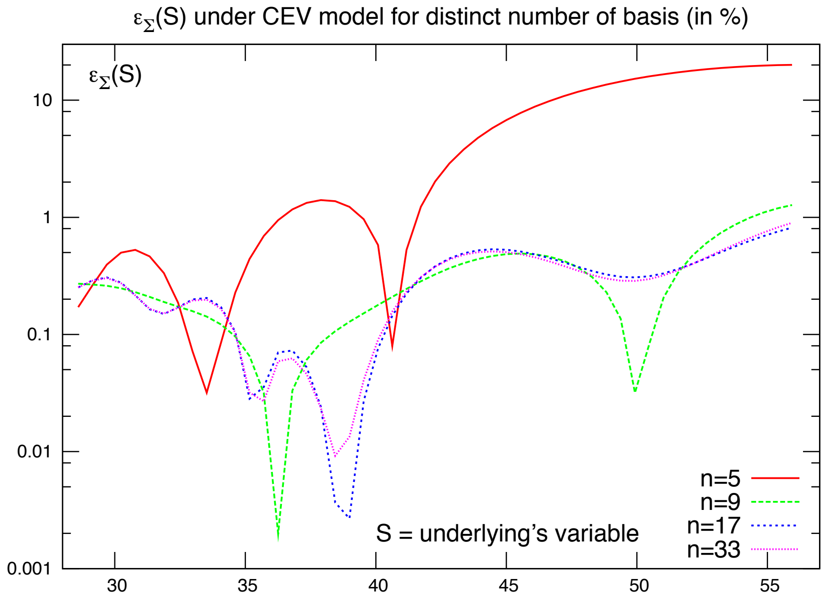

FIG. 5.3. *Relative Error of B-S implied volatility: $\epsilon_\Sigma(S)$ on Ω_ϵ*

n	# Basis	$C_h(Spot)$	$\epsilon(Spot)$	$L^\infty[\epsilon(S)]$	$L^2[\epsilon(S)]$
5	10	22.588	0.160 %	2.323 %	8.549 %
9	18	22.634	0.045 %	0.089 %	0.269 %
17	34	22.626	0.006 %	0.046 %	0.134 %
33	66	22.627	0.012 %	0.046 %	0.104 %
Exact		22.624			

TABLE 5.3
Relative errors for Merton's model in bp wrt n

has interesting consequences for calibration.

In general one observes at $t = 0$ some calls $\{u_j\}_{j=1}^J$ all based on the same asset S; these have strikes K_j and maturity T_j.

Suppose one wishes to calibrate σ so as to reproduce these calls; according to Dupire's equation

$$\partial_T u_\sigma - \frac{\sigma^2}{2}\partial_{KK} u_\sigma + r\partial_K u_\sigma = 0, \;\; u_\sigma(K,0) = (S-K)^+ \tag{6.2}$$

one may solve

$$\sigma = \text{arg min}_\sigma \sum_{j=1}^J |u_\sigma(K_j,T_j) - u_j|^2 \tag{6.3}$$

When a decompostion similr to (6.1) but with K, T as variables, then (6.3) is a sum of independent problems at each time T_j: for each T^* one solves

$$\alpha(T^*) = \text{arg}\min_\alpha \sum_{j:T_j=T^*} |u(K_j,T^*;\alpha) - u_j|^2 \;\; :$$

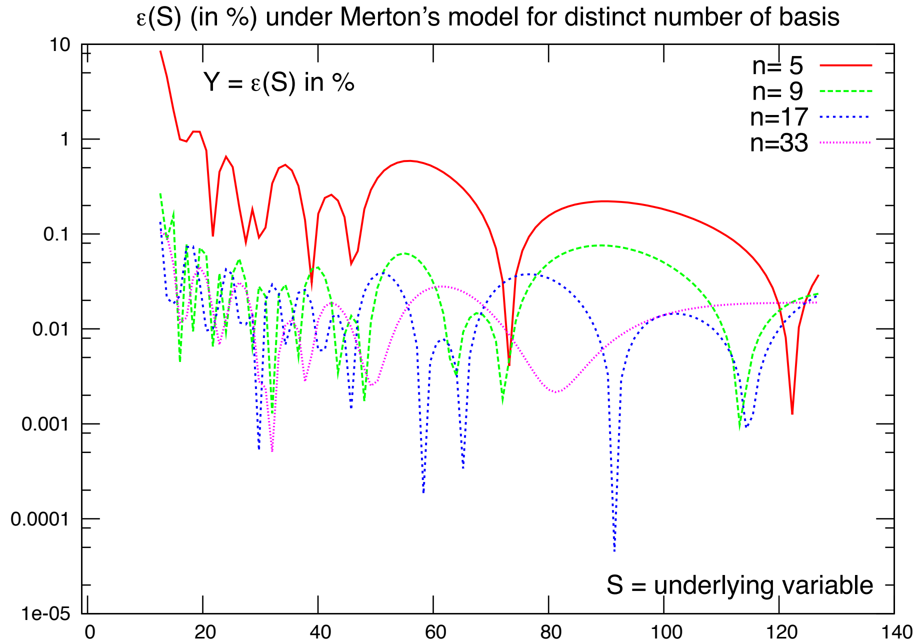

FIG. 5.4. *Relative error $\epsilon(T, S)$ for Merton solution*

n	# Basis	$\Sigma_h(Spot)$	$\epsilon_\Sigma(Spot)$	$L^2[\epsilon_\Sigma(S)]$	$L^\infty[\epsilon_\Sigma(S)]$
5	10	0.6706	0.271 %	1.913 %	5.721 %
9	18	0.6729	0.076 %	0.189 %	0.340 %
17	34	0.6725	0.011 %	0.084 %	0.182 %
33	66	0.6725	0.021 %	0.076 %	0.151 %
Exact		0.6723			

TABLE 5.4
Relative error in term of B-S' implied volatility ($S \in \Omega_\epsilon$)

$$u(K_j, T^*; \alpha) = u_\Sigma(K_j, T^*) + \sum_{i=1}^{I} \alpha_i \left[u_{\sigma_i}(K_j, T_M) - u_\Sigma(K_j, T_M)\right] \quad (6.4)$$

where $T_M = \max T_j$ is the reference time chosen to build the basis. The volatility surface is recovered from Dupire's equation and $u_\sigma(K,T) = u(K,T;\alpha)$; the derivatives with respect to K are computed analytically.

The method is tested on the data shown in Table 6.1 . The rate r is constant: $r = 0.03$. The spot price is 1418.3. In this example there are 6 times T^*. The basis is made of Black-scholes calls with volatility $0.3/\sqrt{i}$, i=2..9. The Black-Scholes solution used for the translation corresponds to $\Sigma = 0.3$. At each T^* a set of 8 α_i are computed by solving (6.4) by a conjugate gradient method with a maximum of 300 iterations. We found the optimization more efficient if α_i is replaced by $10 \sin \alpha_i$.

The method is very fast, even faster than fitting with an implied volatility, but it gives the local volatility only at the times corresponding to the maturity of an observation. The

Strike	1 Month	2 Months	6 Months	12 Months	24 Months	36 Months
700						733
800						650.6
900						569.8
1000					467.8	
1100					385.3	
1150					345.4	
1175			265.2			
1200			242	266.1	306.6	
1215				253.4		
1225			219	245		
1250			196.6	224.2	269.2	
1275			174.5	203.9	251	
1300			152.9	184.1	233.2	
1325			131.9	164.9	215.8	
1350			111.7	146.3	198.9	
1365			100			
1375	50.6	60	92.5	139	182.6	
1380	46.1	55.8				
1385	41.8	51.8				
1390	37.5	47.9				
1395	33.4	44				
1400	29.4	40.3	74.5	128.4	166.7	215.9
1405	25.6	36.7				
1410	21.9	33.2				
1415	18.7	29.8				
1420	15.4	26.6				
1425	12.7	23.8	58	111.4	151.5	
1430	10	20.7				
1435	8	18.2				
1440	6.3	15.7				
1445	4.4	13.4				
1450	3.1	11.3	43.3	95.2	136.9	187.5
1455	2.05	9.6				
1460	1.45	7.9				
1475			30.6	80.2		
1500			20.3	54	109.6	160.8
1525			12.6	42.7		
1550			7.5	33		
1575				24.7		
1600			1.95	18.2	64.5	113,9
1700					32.7	75,7
1800					15.5	
1900					5.2	

TABLE 6.1
The prices of a family of calls on the same asset (Eurostoxx50)

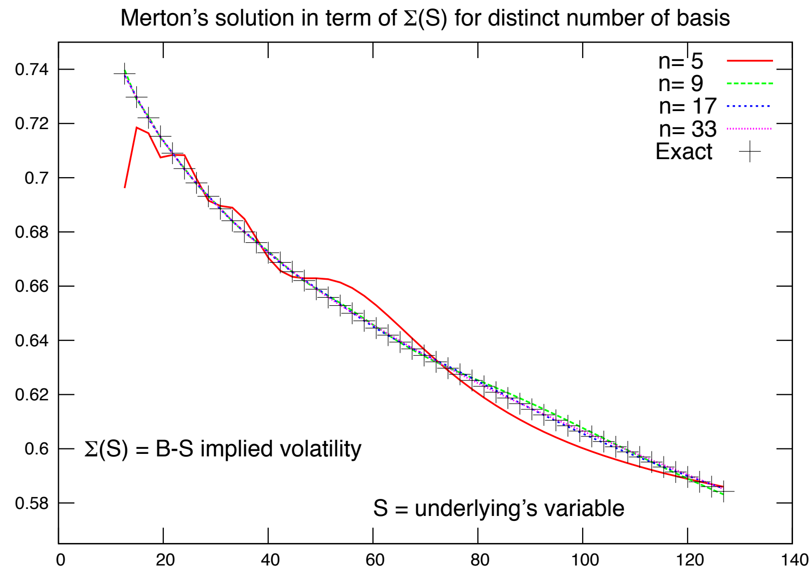

FIG. 5.5. *Merton's solutions in term of B-S implied volatility for various n and "exact" solution*

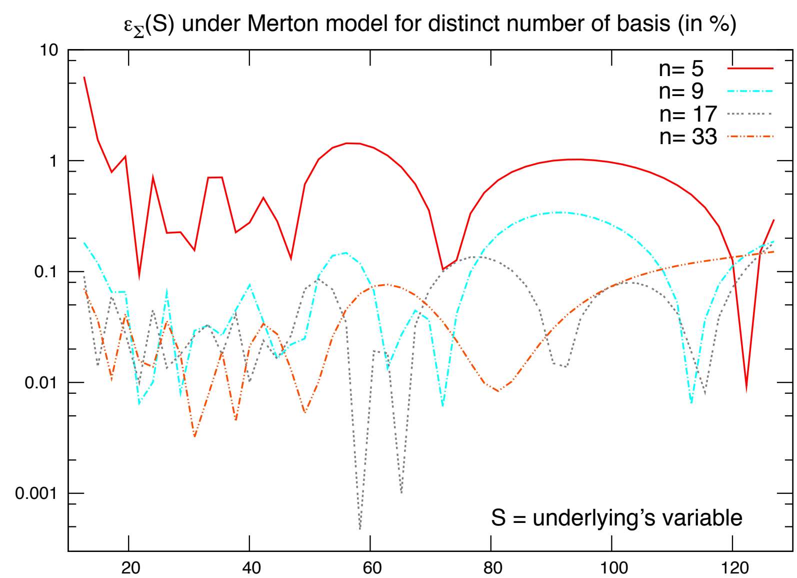

FIG. 5.6. *Relative error of B-S' implied volatility: $\epsilon_\Sigma(x, T)$*

method seems stable and accurate. Restrictions on α such as $\alpha \in (0, 1)$ can be applied for more stability but it may deteriorate the accuracy.

7. Conclusion. We design a one dimensional reduced basis to solve parabolic partial integro-differential equation. Our basis computed on Black-scholes closed form solution allows efficient numerical computation and a good asymptotic behaviour. We prove the convergence and the accuracy of the numerical results shows the efficiency of our basis choice.

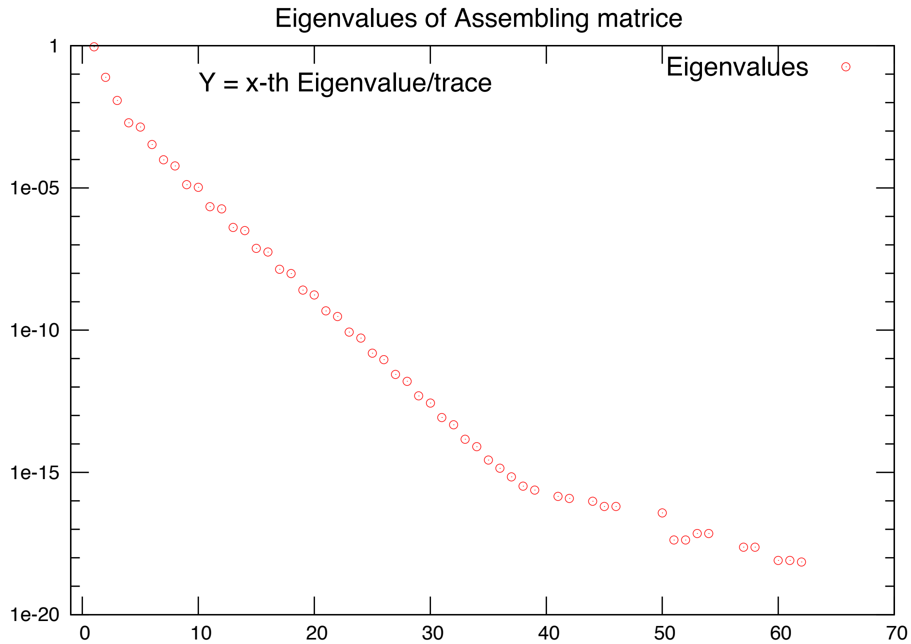

FIG. 5.7. *Eigenvalues of the matrix C normalized by its trace, in log-scale*

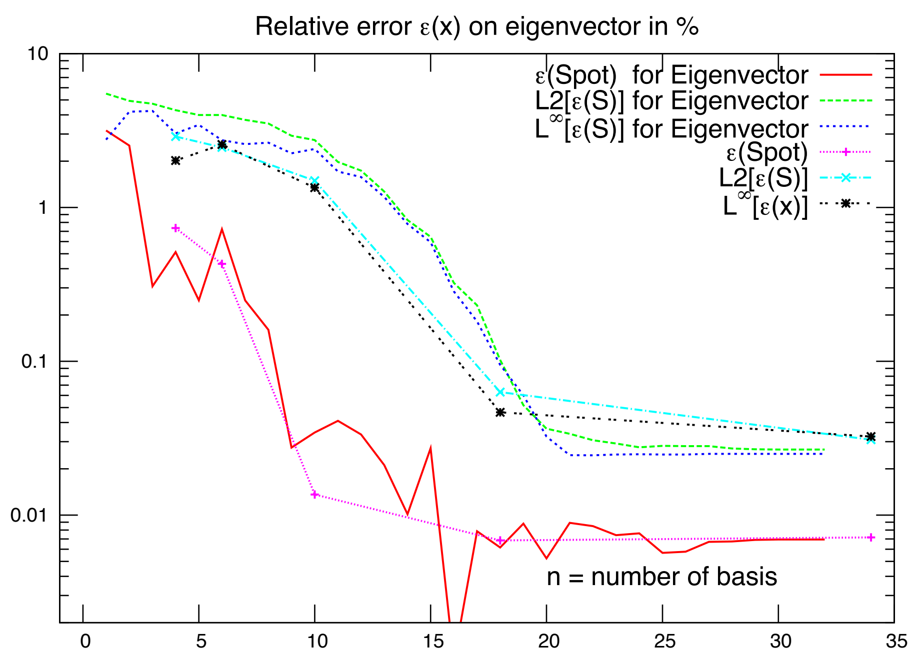

FIG. 5.8. *Relative error in several norms of $\epsilon(S,T)$ in logscale : evolution with the number of eigenvectors in the basis and comparison with the same numbers of w_i vectors in place of eigenvectors.*

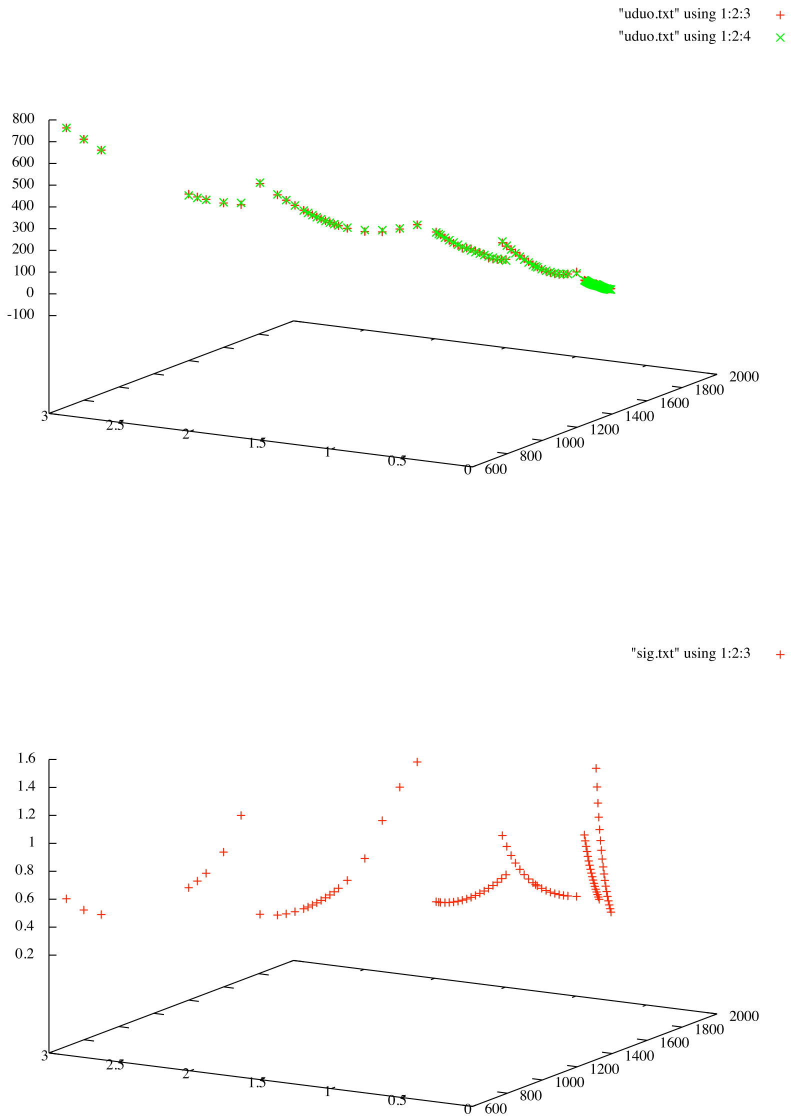

FIG. 6.1. *Difference between observations and model predictions (top). The average error is between 1 and 9 and average 4 at each point. Bottom: the local volatility recovered from the Dupire equation at the observed points.*

Less than twenty basis are needed to efficiently solve our problem. This basis is also tried for calibration of a local volatility.

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Appendix A: proof of Proposition of Proposition 3.1. With the log forward moneyness variable x, the probability density function under the Black-Scholes model is: $\rho_\tau(x) = \frac{1}{\sqrt{2\pi}\sigma\sqrt{\tau}} e^{-\frac{1}{2\sigma^2\tau}\left(x+\frac{\sigma^2\tau}{2}\right)^2}$
and we introduce $v = \sigma\sqrt{\tau}$

$$\begin{aligned}\omega^J(x) := \mathcal{L}^J[u_\sigma](x,v) &= \int_{\mathbb{R}} [u_\sigma(x+z,v) - u_\sigma(x,v) - (e^z-1)\partial_x u_\sigma(x,v)]J(z)dz \\ &= \int_{\mathbb{R}} u_\sigma(x+z,v)J(z)dz - \lambda u_\sigma(x,v) - \lambda(e^{\frac{\delta^2}{2}+\mu} - 1)\partial_x u_\sigma(x,v)\end{aligned}$$

We note $\gamma = x+z+s$ and $X = x - \frac{\sigma^2\tau}{2}$

$$\begin{aligned}\int_{\mathbb{R}} u_\sigma(x+z,v)J(z)dz &= \int_{\mathbb{R}}\int_{\mathbb{R}} \left(e^{x+z+s}-1\right)^+ \rho_\tau(s)dsk(z)dz \\ &= \int_{\mathbb{R}} (e^\gamma - 1)^+ \left[\int_{\mathbb{R}} \rho_\tau(\gamma-(x+z))k(z)dz\right] d\gamma \\ &= \frac{\lambda}{\sqrt{2(\delta^2+v^2)\pi}} \int_0^\infty (e^\gamma-1)\, e^{-\frac{1}{2[\delta^2+v^2]}[\gamma-X-\mu]^2} d\gamma \\ &= \lambda\left[e^{x+\mu+\frac{\delta^2}{2}} \mathcal{N}\left(\frac{x+\mu+\delta^2+\frac{v^2}{2}}{\sqrt{\delta^2+v^2}}\right) - \mathcal{N}\left(\frac{x+\mu-\frac{v^2}{2}}{\sqrt{\delta^2+v^2}}\right)\right]\end{aligned}$$

Appendix B: proof of Proposition 4.7.

Appendix C: Study of the spectrum of $\mathcal{L}^\sigma$. By definition, w^i, scaled so as to have a L^2 norm equal to 1, is:

$$w^i = \frac{\exp(-\frac{1}{4}(\frac{x}{v_i} - v_i)^2)}{\sqrt{2\sqrt{2\pi}v_i}},\ \text{with}\ v_i = \sqrt{\frac{\sigma_i^2 T}{2}}\ \text{because}\ \int_{\mathbb{R}} e^{-\frac{y^2}{2}}\mathrm{d}y = 2\sqrt{2\pi}$$

When $v_i = i^{-\frac{1}{2}}$ with some algebra one finds that

$$(w^i, w^j) = \sqrt{\frac{v_i v_j}{2(v_i^2+v_j^2)}}\exp(-\frac{1}{4}\frac{(v_i^2-v_j^2)^2}{v_i^2+v_j^2}) = \sqrt{\frac{\sqrt{ij}}{\sqrt{2}(i+j)}}\exp(-\frac{1}{4}\frac{(i-j)^2}{ij(i+j)})$$

We need to establish similar expressions for $(\partial_x w_i, w_j)$ and $(\partial_x w_i, \partial_x w_j)$.

$$(\partial_x w_i, w_j) = \sqrt{\frac{v_i v_j}{2(v_i^2+v_j^2)}}\exp(-\frac{1}{4}\frac{(v_i^2-v_j^2)^2}{v_i^2+v_j^2})\frac{v_j^2-v_i^2}{2(v_i^2+v_j^2)} = (w_i,w_j)\frac{i-j}{2(i+j)}$$

$$\begin{aligned}(\partial_x w_i, \partial_x w_j) = &\sqrt{\frac{v_i v_j}{16\pi(v_i^2+v_j^2)}}\exp(-\frac{1}{4}\frac{(v_i^2-v_j^2)^2}{v_i^2+v_j^2}) \\ &\frac{a^2}{v_i v_j}[\int_{\mathbb{R}} y^2 e^{-\frac{y^2}{4}}\mathrm{d}y + (2a - \frac{v_i^2}{a})(2a - \frac{v_j^2}{a})\int_{\mathbb{R}} e^{-\frac{y^2}{4}}\mathrm{d}y] \\ &= (w_i,w_j)\frac{v_i v_j}{v_i^2+v_j^2}(2 - \frac{(v_i^2-v_j^2)^2}{v_i^2+v_j^2}) \\ &= (w_i,w_j)\frac{\sqrt{ij}}{i+j}(2 - \frac{(i-j)^2}{ij(i+j)})\end{aligned}$$

The variational form of the problem leads to the linear system for the vector $\alpha(t)$:

$$B\dot{\alpha} + \frac{\sigma^2}{2}A\alpha = B\tilde{f}$$

where $B_{ij} = (w_i, w_j),\ A_{ij} = (\partial_x w_i, \partial_x w_j) + (w_i, \partial_x w_j)$ and $\tilde{f}_i$ is the component on w_i of the right hand side, i.e. $f(x,t) = \sum \tilde{f}_i(t) w_i(x)$.

The following tables display the first eigenvalues of A,B and A with respect to B for different values of n, the number of σ-basis functions.

Notice the very rapid decay of the eigenvalues of A and the very rapid growth of those of B.

$$\begin{bmatrix} 2.8874 \\ 0.5872 \\ 0.05843 \\ 0.002407 \\ 0.00003339 \end{bmatrix} \begin{bmatrix} 5.6688 \\ 1.1587 \\ 0.2167 \\ 0.02506 \\ 0.001685 \\ 0.00007929 \\ 0.000002448 \\ 0.00000005106 \\ 0.0000000008225 \\ -0.0000000002322 \end{bmatrix} \begin{bmatrix} 8.4913 \\ 1.6717 \\ 0.3736 \\ 0.06279 \\ 0.006536 \\ 0.00052435 \\ 0.00003150 \\ 0.0000015210 \\ 0.00000005600 \\ 0.000000001366 \\ -0.0000000009870 \\ -0.0000000001615 \end{bmatrix} \begin{bmatrix} 11.3288 \\ 2.1722 \\ 0.5175 \\ 0.1076 \\ 0.01421 \\ 0.001498 \\ 0.0001236 \\ 0.000008636 \\ 0.0000004925 \\ 0.00000002433 \\ -0.000000001522 \\ 0.000000001300 \end{bmatrix}$$

First 12 eigenvalues of the matrix $A = ((\mathcal{L}^\sigma w^i, w^j))$ for $n = 5, 10, 15, 20$ when $\sigma^2 T = 2$

$$\begin{bmatrix} 0.000003812 \\ 0.0003502 \\ 0.01271 \\ 0.2203 \\ 3.3020 \end{bmatrix} \begin{bmatrix} -0.0000000007761 \\ -6.630 \times 10^{-11} \\ 0.000000003418 \\ 0.0000001979 \\ 0.000007586 \\ 0.0002038 \\ 0.003919 \\ 0.05291 \\ 0.4838 \\ 6.530 \end{bmatrix} \begin{bmatrix} 0.0000000001159 \\ 0.0000000005094 \\ 0.0000000009339 \\ 0.000000003368 \\ 0.0000001086 \\ 0.000002650 \\ 0.00005220 \\ 0.0008265 \\ 0.01037 \\ 0.09952 \\ 0.7257 \\ 9.7701 \end{bmatrix}$$

Last 12 eigenvalues of the matrix $B = ((w^i, w^j))$ for $n = 5, 10, 15$

$$\begin{bmatrix} 9.1544 + 0.9648\,i \\ 9.1544 - 0.9648\,i \\ 0.8117 \\ 4.1003 \\ 2.2851 \\ \\ \\ \\ \\ \\ \\ \end{bmatrix} \begin{bmatrix} 16.4818 + 3.1441\,i \\ 16.4818 - 3.1441\,i \\ 9.7760 + 1.665\,i \\ 9.7760 - 1.6652\,i \\ 5.2870 + 0.7487\,i \\ 5.2870 - 0.7487\,i \\ 3.53077 \\ 0.7664 + 0.1090\,i \\ 0.7664 - 0.1090\,i \\ 1.7905 \\ \\ \end{bmatrix} \begin{bmatrix} 20.6981 + 4.2722\,i \\ 20.6981 - 4.2722\,i \\ 12.8389 + 3.2087\,i \\ 12.8389 - 3.2087\,i \\ 0.7675 \\ 0.5482 \\ 1.5402 \\ 7.9943 + 1.9323\,i \\ 7.9943 - 1.9323\,i \\ 1.7395 \\ 2.6983 \\ 4.1079 + .3209\,i \end{bmatrix}$$

First 12 eigenvalues of $B^{-1}A = ((w^i, w^j))^{-1}((\mathcal{L}^\sigma w^i, w^j))$ for $n = 5, 10, 15,\ \sigma^2 T = 2$

These computations where done with the following `Maple9` program:

```
n:=5;
M:=matrix(n,n,(i,j)->sqrt(sqrt(i*j)/(i+j))*exp(-(i-j)^2/(4*i*j*(i+j))));
N:=matrix(n,n,(i,j)->sqrt(sqrt(i*j)/(i+j))*exp(-(i-j)^2/(4*i*j*(i+j)))*
          ((i-j)/(i+j)/2 + sqrt(i*j)*(2-(i-j)^2/((i+j)*i*j))/(i+j)));
evalf(Eigenvals(N,M,vects));print(vects);vects:='vects';
evalf(Eigenvals(N,vects));print(vects);vects:='vects';
evalf(Eigenvals(M,vects));print(vects);vects:='vects';
```

7.1. Consequence. In the eigenbasis v^i such that $Av^i = \lambda_i B v^i$, the solution is

$$u = \sum \beta_i v^i \ \text{ with } \ \beta_i(t) = e^{-\lambda_i t}(\tilde{f}_i(0) + \int_0^t e^{-\lambda_i \tau} \tilde{f}_i(\tau) \mathrm{d}\tau$$

so the size of β_i is driven by the size of $\tilde{f}_i$, the component of f on the basis vector w^i.

The reduced basis idea is to compute the eigenvectors q^i of B and to keep those associated with the biggest eigenvalues. However what we have done in this study is to keep a small number of w^i vectors.

$$\begin{bmatrix} .6644688842 & 0.6546577055 & 0.6508321972 & 0.6508321972 \end{bmatrix}$$

Last eigenvalue of B divided by n

$$0.257210^{-9}, 0.270010^{-9}, 0.187410^{-5}, 0.779210^{-4}, 0.002064, 0.03545, 0.3819, 5.237$$

$$\begin{bmatrix}
0.0002582 & -0.002256 & -0.01432 & -0.0661 & 0.2267 & 0.5494 & -0.7411 & 0.3047 \\
-0.006571 & 0.03993 & 0.1584 & 0.4091 & -0.6387 & -0.3730 & -0.3668 & 0.3522 \\
0.05858 & -0.2323 & -0.5186 & -0.5691 & 0.02227 & -0.4506 & -0.1159 & 0.3642 \\
-0.2460 & 0.5560 & 0.4710 & -0.2041 & 0.3732 & -0.3012 & 0.04883 & 0.3662 \\
0.5486 & -0.4674 & 0.2889 & 0.2809 & 0.3848 & -0.1030 & 0.1622 & 0.3646 \\
-0.6684 & -0.2196 & -0.3079 & 0.4186 & 0.1911 & 0.09106 & 0.2436 & 0.3614 \\
0.42029 & 0.5599 & -0.4275 & 0.1563 & -0.1079 & 0.2659 & 0.3039 & 0.3574 \\
-0.1066 & -0.2342 & 0.3501 & -0.4289 & -0.4521 & 0.4189 & 0.3497 & 0.3533
\end{bmatrix}$$