Optimally adaped meshes for finite elements of arbitrary order and $W^{1,p}$ norms

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January 21, 2010

Abstract

Given a function f defined on a bounded domain $\Omega \subset \mathbb{R}^2$ and a number $N > 0$, we study the properties of the triangulation $\mathcal{T}_N$ that minimizes the distance between f and its interpolation on the associated finite element space, over all triangulations of at most N elements. The error is studied in the $W^{1,p}$ norm for $1 \leq p < \infty$ and we consider Lagrange finite elements of arbitrary polynomial order $m - 1$. We establish sharp asymptotic error estimates as $N \to +\infty$ when the optimal anisotropic triangulation is used. A similar problem has been studied in [1, 2, 7, 6, 10, 15], but with the error measured in the L^p norm. The extension of this analysis to the $W^{1,p}$ norm is crucial in order to match more closely the needs of numerical PDE analysis, and it is not straightforward. In particular, the meshes which satisfy the optimal error estimate are characterized by a metric describing the local aspect ratio of each triangle and by a geometric constraint on their maximal angle, a second feature that does not appear for the L^p error norm. Our analysis also provides with practical strategies for designing meshes such that the interpolation error satisfies the optimal estimate up to a fixed multiplicative constant. We discuss the extension of our results to finite elements on simplicial partitions of a domain $\Omega \subset \mathbb{R}^d$, and we provide with some numerical illustration in 2-d.

Key words : anisotropic finite elements, $W^{1,p}$ norm, adaptive meshes, interpolation, nonlinear approximation.
AMS subject classifications : 65D05, 65N15, 65N50

1 Introduction

In finite element approximation, a usual distinction is between *uniform* and *adaptive* methods. In the latter, the elements defining the mesh may vary strongly in size and shape for a better adaptation to the local features of the approximated function f. This naturally raises the objective of characterizing and constructing an *optimal mesh* for a given function f. Depending on the context, the function f may be fully known to us, either through an explicit formula or a discrete sampling, or observed through noisy measurements, or implicitly defined as the solution of a given partial differential equation.

In this paper, we assume that f is a function defined on a polygonal bounded domain $\Omega \subset \mathbb{R}^2$. For a given conforming triangulation $\mathcal{T}$ of Ω, and an arbitrary but fixed integer $m \geq 1$, we denote by $\mathrm{I}_{\mathcal{T}}^m$ the standard interpolation operator on the Lagrange finite elements of degree m space associated to $\mathcal{T}$. Given a norm X of interest and a number $N > 0$, the objective of finding the optimal mesh for f can be formulated as solving the optimization problem

$$\min_{\#(\mathcal{T}) \leq N} \|f - \mathrm{I}_{\mathcal{T}}^m f\|_X,$$

where the minimum is taken over all conforming triangulations of cardinality N. A general objective is to establish sharp asymptotic error estimates that precisely describe the behavior of the above quantity as $N \to +\infty$. Estimates of that type were obtained in [7, 1] in the particular case of linear finite elements and with the error measured in $X = L^p$. They have the form

$$\limsup_{N \to +\infty} \Big(N \min_{\#(\mathcal{T}) \leq N} \|f - \mathrm{I}_{\mathcal{T}}^1 f\|_{L^p} \Big) \leq C \|\sqrt{|\det(d^2 f)|}\|_{L^\tau}, \quad \frac{1}{\tau} = \frac{1}{p} + 1, \tag{1.1}$$

which reveals that the convergence rate is governed by the quantity $\sqrt{|\det(d^2 f)|}$, which depends nonlinearly on the Hessian $d^2 f$. This is heavily tied to the fact that we allow triangles with possibly highly anisotropic shape. The convergence estimate (1.1) has been extended to arbitrary approximation order in [15], where the quantity governing the convergence rate for finite elements of arbitrary degree $m-1$ was identified. This quantity depends nonlinearly on the m-th order derivative $d^m f$. See also the book chapter [10] for an introduction to the subject of adaptive and anisotropic piecewise polynomial approximation.

The purpose of the present article is investigate this problem when the L^p-norm is replaced by the $W^{1,p}$ semi-norm which plays a critical role in PDE analysis. This semi-norm is defined as

$$|f|_{W^{1,p}(\Omega)} := \|\nabla f\|_{L^p(\Omega)} = \left(\int_\Omega |\nabla f|^p\right)^{1/p}.$$

Our second objective is to propose simple and practical ways of designing meshes which behave similar to the optimal one, in the sense that they satisfy the sharp error estimate up to a fixed multiplicative constant.

1.1 Main results and layout

We denote by $\mathbb{P}_m$ the space of polynomials of total degree less or equal to m and by $\mathbb{H}_m$ the space of homogeneous polynomials of total degree m,

$$\mathbb{P}_m := \mathrm{Span}\{x^k y^l\ ;\ k+l \leq m\} \text{ and } \mathbb{H}_m := \mathrm{Span}\{x^k y^l\ ;\ k+l = m\}.$$

For any triangle T, we denote by I_T^m the local interpolation operator acting from $C^0(T)$ onto $\mathbb{P}_m$. For any continuous fonction $\nu \in C^0(T)$, the interpolating polynomial $\mathrm{I}_T^m \nu \in \mathbb{P}_m$ is defined by the conditions

$$\mathrm{I}_T^m \nu(\gamma) = \nu(\gamma),$$

for all points $\gamma \in T$ with barycentric coordinates in the set $\{0, \frac{1}{m}, \frac{2}{m}, \cdots, 1\}$. This interpolation operator is invariant by translation, in particular for any polynomial $\pi \in \mathbb{H}_m$, triangle T and offset z we have

$$|\pi - \mathrm{I}_T^{m-1} \pi|_{W^{1,p}(T)} = |\pi - \mathrm{I}_T^{m-1} \pi|_{W^{1,p}(z+T)}. \tag{1.2}$$

If $\mathcal{T}$ is a triangulation of a domain Ω, then $\mathrm{I}_{\mathcal{T}}^m$ refers to the interpolation operator which coincides with I_T^m on each triangle $T \in \mathcal{T}$.

A key ingredient in this paper is the *shape function* $L_{m,p}$, which is defined by a *shape optimization problem*: for any fixed $1 \leq p \leq \infty$ and for any $\pi \in \mathbb{H}_m$, we define

$$L_{m,p}(\pi) := \inf_{|T|=1} |\pi - \mathrm{I}_T^{m-1} \pi|_{W^{1,p}(T)}. \tag{1.3}$$

Here, the infimum is taken over all triangles of area $|T| = 1$. From the homogeneity of π, it is easily checked that

$$\inf_{|T|=A} |\pi - \mathrm{I}_T^{m-1} \pi|_{W^{1,p}(T)} = L_{m,p}(\pi) A^{\frac{m-1}{2}+\frac{1}{p}}. \tag{1.4}$$

The solution to this optimization problem thus describes the shape of the triangles of a given area which are best adapted to the polynomial π in the sense of minimizing the interpolation error measured in $W^{1,p}$.

The function $L_{m,p}$ is the natural generalisation of the function $K_{m,p}$ introduced in [15] for the study of optimal anisotropic triangulations in the sense of the L^p interpolation error

$$K_{m,p}(\pi) := \inf_{|T|=1} \|\pi - \mathrm{I}_T^{m-1} \pi\|_{L^p(\Omega)}.$$

Our asymptotic error estimate for the optimal triangulation is given by the following theorem.

Theorem 1.1 *For any polygonal domain $\Omega \subset \mathbb{R}^2$, and any function $f \in C^m(\Omega)$, and any $1 \leq p < \infty$, there exists a sequence of triangulations $(\mathcal{T}_N)_{N \geq N_0}$, with $\#(\mathcal{T}_N) \leq N$ such that*

$$\limsup_{N\to\infty} N^{\frac{m-1}{2}} |f - \mathrm{I}_{\mathcal{T}_N}^{m-1} f|_{W^{1,p}(\Omega)} \leq \left\| L_{m,p}\left(\frac{d^m f}{m!}\right)\right\|_{L^\tau(\Omega)}, \text{ where } \frac{1}{\tau} := \frac{m-1}{2} + \frac{1}{p} \tag{1.5}$$

In the above convergence estimate, the number N_0 is independent of f and refers to the minimal cardinality of a conforming triangulation of Ω. The m-th derivative $d^m f(z)$ at each point z is identified to a homogeneous polynomial $\pi_z \in \mathbb{H}_m$:

$$\frac{d^m f(z)}{m!} \sim \pi_z = \sum_{k+l=m} \frac{\partial^m f}{\partial x^k \partial y^l}(z) \frac{x^k}{k!} \frac{y^l}{l!}. \tag{1.6}$$

An important feature of this estimate is the "lim sup". Recall that the upper limit of a sequence $(u_N)_{N \geq N_0}$ is defined by

$$\limsup_{N \to \infty} u_N := \lim_{N \to \infty} \sup_{n \geq N} u_n,$$

and is in general stricly smaller than the supremum $\sup_{N \geq N_0} u_N$. It is still an open question to find an appropriate upper estimate of $\sup_{N \geq N_0} N^{\frac{m-1}{2}} |f - \mathrm{I}_{\mathcal{T}_N}^{m-1} f|_{W^{1,p}(\Omega)}$ when optimally adapted anisotropic triangulations are used.

In order to illustrate the sharpness of (1.5), we introduce a slight restriction on sequences of triangulations, following an idea in [2]: a sequence $(\mathcal{T}_N)_{N \geq N_0}$ of triangulations is said to be *admissible* if $\#(\mathcal{T}_N) \leq N$ and

$$\sup_{T \in \mathcal{T}_N} \operatorname{diam}(T) \leq C_A N^{-\frac{1}{2}}. \tag{1.7}$$

for some constant $C_A > 0$ independent of N. The following theorem shows that the estimate (1.5) cannot be improved when we restrict our attention to admissible sequences. It also shows that this class is reasonably large in the sense that (1.5) is ensured to hold up to small perturbation.

Theorem 1.2 *Let $\Omega \subset \mathbb{R}^2$ be a compact polygonal domain, let $f \in C^m(\Omega)$ and $1 \leq p < \infty$. We define $\frac{1}{\tau} := \frac{m-1}{2} + \frac{1}{p}$. For all* admissible *sequences of triangulations $(\mathcal{T}_N)_{N \geq N_0}$, one has*

$$\liminf_{N \to \infty} N^{\frac{m-1}{2}} |f - \mathrm{I}_{\mathcal{T}_N}^{m-1} f|_{W^{1,p}(\Omega)} \geq \left\| L_{m,p}\left(\frac{d^m f}{m!}\right) \right\|_{L^\tau(\Omega)}. \tag{1.8}$$

Furthermore, for all $\varepsilon > 0$ there exists an admissible *sequence of triangulations $(\mathcal{T}_N^\varepsilon)_{N \geq N_0}$ such that*

$$\limsup_{N \to \infty} N^{\frac{m-1}{2}} |f - \mathrm{I}_{\mathcal{T}_N^\varepsilon}^{m-1} f|_{W^{1,p}(\Omega)} \leq \left\| L_{m,p}\left(\frac{d^m f}{m!}\right) \right\|_{L^\tau(\Omega)} + \varepsilon. \tag{1.9}$$

Note that the sequences $(\mathcal{T}_N^\varepsilon)_{N \geq N_0}$ satisfy the admissibility condition (1.7) with a constant $C_A(\varepsilon)$ which may grow to $+\infty$ as $\varepsilon \to 0$. The proofs of these two theorems are given in §3. Theorem 1.1 is a direct consequence of Theorem 1.2, by considering a sequence of triangulations of the type $\mathcal{T}_N^{\varepsilon_N}$ with $\varepsilon_N \to 0$ as $N \to +\infty$. The proof of the upper estimate in Theorem 1.2 involves the construction of an optimal mesh based on a patching strategy adapted from the one encountered in [2]. However, inspection of the proof reveals that this construction becomes effective as the number of triangles N becomes very large. Therefore it may not be useful in practical applications.

Remark 1.3 *It can easily be shown that if $(\mathcal{T}_N)_{N \geq N_0}$ is an admissible sequence of triangulations and $f \in \mathcal{C}^m(\Omega)$, then $\|f - \mathrm{I}_{\mathcal{T}_N}^m f\|_{L^p(\Omega)}$ decays with the rate $N^{-m/2}$ which is faster than the decay rate obtained for the $W^{1,p}$ error. Therefore, our convergence estimates are also valid in the $W^{1,p}$ norm*

$$\|f\|_{W^{1,p}(\Omega)} := \left(\|f\|_{L^p(\Omega)}^p + |f|_{W^{1,p}(\Omega)}^p \right)^{1/p}.$$

We show in §2 that in order to satisfy the optimal estimate (1.5) up to a fixed multiplicative constant, it suffices to build a triangulation which obeys four general principles:

(i) The interpolation error should be evenly distributed on all triangles,

(ii) The triangles should adopt locally a specific aspect ratio, dictated by the local value of $d^m f$,

(iii) the largest angle of the triangles should be bounded away from $\boldsymbol{\pi} = 3.14159\ldots$

(iv) the triangulation $\mathcal{T}$ should be sufficiently refined in order to adapt to the local features of f.

The third point (iii) is the main new ingredient of this paper compared to [15], and is necessary for obtaining $W^{1,p}$ error estimates (but not for L^p error estimates). Roughly speaking, two triangles having the same optimized aspect ratio imposed by (ii) may greatly differ in term of their largest angle, and the most acute triangle should be preferred when error is measured in $W^{1,p}$ rather than L^p. The influence of large angles in mesh adaptation has already been studied in [3, 12, 16]. The heuristic guideline is that large angles should be avoided in general, since they lead to oscillations of the gradient of the interpolant. On the contrary, extremely thin triangles and very small angles can be necessary for optimal mesh adaptation.

A practical approach for mesh generation is discussed in §4, and consists in deriving a distorted metric from the exact or approximate data of $d^m f$ at each point $x \in \Omega$. We restrict in this section to the case of linear and quadratic finite elements, and we provide simple mesh generation procedures and numerical results. To any $\pi \in \mathbb{H}_m$, we associate a symmetric positive definite matrix $h_\pi \in S_2^+$. If $z \in \Omega$ and $d^m f(z)$ is close to π, then the triangle T containing z should be isotropic in the metric h_π. The requirements (i) and (ii) above, which are respectively linked to the size and shape of the triangles, can then be summarized through a global metric on Ω given by

$$h(z) = s(\pi_z) h_{\pi_z}, \ \ \pi_z = \frac{d^m f(z)}{m!}, \tag{1.10}$$

where $s(\pi_z)$ is a scalar factor which depends on the desired accuracy of the finite element approximation. Once this metric has been properly identified, fast algorithms such as in [19, 18, 5] can be used in order to design a near-optimal mesh based on it. Recently it has been rigorously proved in [14, 4], that several algorithms terminate and produce good quality meshes, under certain conditions. Although we are not aware that the angle constraint (iii) is guaranteed in such algorithms, it seems to hold in practice. Computing the map

$$\pi \in \mathbb{H}_m \mapsto h_\pi \in S_2^+, \tag{1.11}$$

is therefore of key use in applications (S_2^+ refers to the set of 2×2 symmetric and non-negative matrices). This problem is solved in [14], in the case of linear elements ($m = 2$): the matrix h_π is then defined as the square of the matrix associated to the quadratic form π. We give a simple expression of h_π for piecewise quadratic finite elements ($m = 3$). The optimality of this construction is proved theoretically, and numerical experiments confirm its adequacy. An open source implementation for the mesh generator FreeFem++ [18] is provided at [17].

The shape function $L_{m,p}$ does not always have a simple analytic expression from the coefficients of π. For this reason we introduce in §5 explicit functions $\pi \mapsto \mathbf{L}_m(\pi)$ which are defined as the root of a polynomial in the coefficients of π, and are equivalent to $L_{m,p}$, leading therefore to similar asymptotic error estimates up to multiplicative constants. We finally discuss in §6 the possible extension of our analysis to simplicial elements in dimension $d > 2$.

Notations

Throughout this paper, we define $L_m := L_{m,\infty}$, where $L_{m,p}$ is defined at Equation (1.3). We prove further in Lemma 2.7 that for all $1 \le p \le \infty$

$$cL_m \le L_{m,p} \le L_m \text{ on } \mathbb{H}_m, \tag{1.12}$$

where the constant $c > 0$ depends only on m. For any compact set $E \subset \mathbb{R}^d$, we denote by $\mathrm{bary}(E)$ its barycenter. For any pair of vector $u, v \in \mathbb{R}^d$, we denote by $\langle u, v\rangle$ their inner product, and by

$$|u| := \sqrt{\langle u, u\rangle},$$

the euclidean norm of u. When g is a vector valued function, we denote by $\|g\|_{L^p(E)}$ the L^p norm of $x \mapsto |g(x)|$ on E.

We denote by $\mathrm{M}_d(\mathbb{R})$ the set of all $d \times d$ real matrices, equiped with the norm

$$\|A\| := \max_{|u| \le 1} |Au|.$$

We denote by $\mathrm{GL}_d \subset \mathrm{M}_d(\mathbb{R})$ the linear group of invertible matrices and by $\mathrm{SL}_d \subset \mathrm{GL}_d$ the special linear group of matrices of determinant 1.

$$\mathrm{GL}_d := \{A \in \mathrm{M}_d(\mathbb{R}) \ ; \ \det A \neq 0\} \text{ and } \mathrm{SL}_d := \{A \in \mathrm{M}_d(\mathbb{R}) \ ; \ \det A = 1\}.$$

For $A \in \mathrm{GL}_d$, we denote by

$$\kappa(A) := \|A\| \, \|A^{-1}\|, \tag{1.13}$$

its condition number. We denote by $S_d \subset \mathrm{M}_d(\mathbb{R})$ the subset of symmetric matrices, and by $S_d^+ \subset S_d$ the subset of non-negative symmetric matrices.

$$S_d := \{S \in \mathrm{M}_d(\mathbb{R}) \ ; \ S^{\mathrm{t}} = S\} \text{ and } S_d^+ := \{S \in S_d \ ; \ z^{\mathrm{t}} S z \geq 0 \text{ for all } z \in \mathbb{R}^d\}.$$

For any two symmetric matrices $S, S' \in S_d$, we write $S \leq S'$ if and only if $S' - S \in S_d^+$.

We equip the spaces $\mathbb{P}_m$ and $\mathbb{H}_m$ with the norm

$$\|\pi\| := \max_{|u| \leq 1} |\pi(u)|. \tag{1.14}$$

Note that the greek letter π always refers to an homogeneous polynomial $\pi \in \mathbb{H}_m$, while the large and bold notation $\boldsymbol{\pi}$ refers to the mathematical constant $\boldsymbol{\pi} = 3.14159\ldots$.

Recall that if g is a C^m function, we identify $d^m g(x)$ to a polynomial in $\mathbb{H}_m$. We then denote

$$\|d^m g\|_{L^\infty(E)} := \max_{z \in E} \|d^m g(z)\| \tag{1.15}$$

with $\|\cdot\|$ the previously defined norm on $\mathbb{H}_m$.

2 The shape function $L_{m,p}$ and local error estimates

In this section, we study the function $L_{m,p}$ and obtain local $W^{1,p}$ error estimates for functions of two variables. These estimates naturally give rise to a heuristic method for the design of "near optimal" triangulations adapted to a function f, in other words triangulations satisfying the estimate (1.5) up to a fixed multiplicative constant, and it is put into practice in §4 in the case of piecewise linear and piecewise quadratic finite elements. The results of this section are also useful to the proof, in §3, of the optimal error estimates presented in Theorems 1.1 and 1.2.

We first introduce the measure of sliverness $S(T)$ of a triangle T. Given two triangles T, T', there are precisely 6 affine transformations Ψ such that $\Psi(T) = T'$. Each of these affine transformations Ψ defines a linear transformation ψ and we set

$$\mathrm{d}(T, T') := \ln\left(\inf\{\kappa(\psi) \ ; \ \Psi(T) = T'\}\right), \tag{2.16}$$

where $\kappa(\psi)$ is the condition number defined in (1.13). Clearly $d(T, T') \geq 0$, $d(T, T') = d(T', T)$ and $d(T, T'') \leq d(T, T') + d(T', T'')$. Furthermore $d(T, T') = 0$ if and only if T can be transformed into T' through a translation, a rotation and a dilatation. Therefore $d(\cdot, \cdot)$ defines a distance between shapes of triangles. The heuristic guideline of the papers [12, 3] is that obtuse shapes should be avoided when possible in the design of Finite Element meshes. We therefore introduce the set of all acute triangles

$$\mathbb{A} := \{T \ ; \ \theta_{\max}(T) \leq \boldsymbol{\pi}/2\},$$

and we define the *measure of sliverness*

$$S(T) := \exp \mathrm{d}(T, \mathbb{A}) = \inf\{\kappa(\psi) \ ; \ \Psi(T) \in \mathbb{A}\}. \tag{2.17}$$

This quantity reflects the distance from T to the set of acute triangles: in particular $S(T) = 1$ if and only if $T \in \mathbb{A}$, and $S(T) > 1$ otherwise. It has an analytic expression, which is given in the following proposition.

Proposition 2.1 *For any triangle T with largest angle θ, one has $S(T) = \max\{1, \tan\frac{\theta}{2}\}$.*

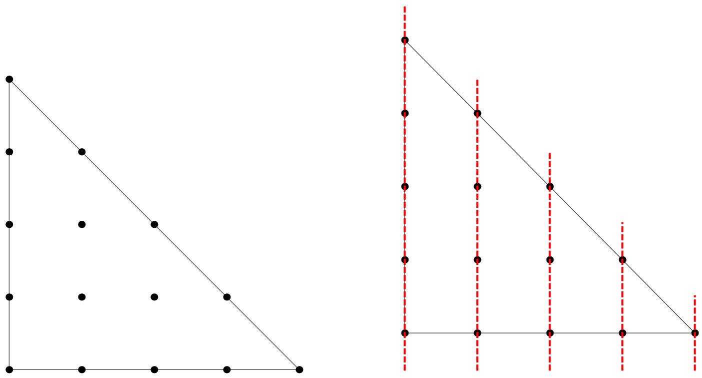

Figure 1: The interpolation points on the triangle T_0 are aligned vertically.

Proof: The result of this proposition is trivial if the triangle T is acute, we therefore assume that T is obtuse. We can assume without loss of generality that the vertices of T are 0, αu and βv, where $\alpha, \beta > 0$, $u, v \in \mathbb{R}^2$, $|u| = |v| = 1$ and $\langle u, v\rangle = \cos\theta$. Note that $|u - v| = 2\sin(\theta/2)$ and $|u + v| = 2\cos(\theta/2)$. Let Ψ be such that $\Psi(T) \in \mathbb{A}$, and let ψ be the associated linear transform. Since $\Psi(T)$ is acute we have $\langle \psi(u), \psi(v)\rangle \geq 0$ and therefore $|\psi(u) - \psi(v)| \leq |\psi(u) + \psi(v)|$. It follows that

$$\kappa(\psi) = \|\psi\| \, \|\psi^{-1}\| \geq \frac{|u - v|}{|u + v|} \times \frac{|\psi(u) + \psi(v)|}{|\psi(u) - \psi(v)|} \geq \frac{2\sin(\theta/2)}{2\cos(\theta/2)} = \tan\frac{\theta}{2}.$$

Therefore $S(T) \geq \tan\frac{\theta}{2}$. Furthermore, let ψ be defined by $\psi(u) = (0, 1)$ and $\psi(v) = (1, 0)$. Obviously $\psi(T)$ has one of its angle equal to $\boldsymbol{\pi}/2$ and is therefore acute. On the other hand, one easily checks that $\kappa(\psi) = \tan(\theta/2)$ and therefore $S(T) \leq \tan\frac{\theta}{2}$. This concludes the proof of this proposition. $\diamond$

This proposition implies in particular that $S(T)$ is equivalent to the quantity $\frac{1}{\sin\theta}$ used in [12, 3]. The following lemma shows that the interpolation process is stable with respect to the L^∞ norm of the gradient if the measure of sliverness $S(T)$ is controlled. Let us mention that a slightly different formulation of this result was already proved in [12], yet not exactly adapted to our purposes.

Lemma 2.2 *There exists a constant $C = C(m)$ such that for any triangle T and $f \in C^1(T)$ one has*

$$\|\nabla \, \mathrm{I}_T^{m-1} f\|_{L^\infty(T)} \leq C S(T) \|\nabla f\|_{L^\infty(T)}, \tag{2.18}$$

Proof: Let T_0 be the triangle of vertices $(0, 0)$, $(1, 0)$ and $(0, 1)$, and let $f \in C^1(T_0)$. We define $g(x, y) := f(x, 0)$ and $h(x, y) := f(x, y) - f(x, 0)$. Since g does not depend on y and since the lagrange interpolation points on T_0 are aligned vertically, as illustrated on Figure 1, the Lagrange interpolant $\mathrm{I}_{T_0}^{m-1} g$ does not depend on y either. Futhermore, for all $(x, y) \in T_0$, we have $|h(x, y)| = |\int_{s=0}^{y} \frac{\partial f}{\partial x}(x, s) ds| \leq \|\frac{\partial f}{\partial y}\|_{L^\infty(T_0)}$. Hence

$$\begin{aligned}
\left\|\frac{\partial \, \mathrm{I}_{T_0}^{m-1} f}{\partial y}\right\|_{L^\infty(T_0)} &= \left\|\frac{\partial \, \mathrm{I}_{T_0}^{m-1} h}{\partial y}\right\|_{L^\infty(T_0)} \\
&\leq C_0 \| \, \mathrm{I}_{T_0}^{m-1} h\|_{L^\infty(T_0)} \\
&\leq C_0 C_1 \|h\|_{L^\infty(T_0)} \\
&\leq C_0 C_1 \left\|\frac{\partial f}{\partial y}\right\|_{L^\infty(T_0)},
\end{aligned}$$

where the constants C_0 and C_1 are the $L^\infty(T_0)$ norms of the operators $g \mapsto \frac{\partial g}{\partial y}$ restricted to $\mathbb{P}_{m-1}$ and $g \mapsto \mathrm{I}_{T_0}^{m-1} g$ respectively.

Let e be an edge vector of T. There exists an affine change of coordinates Ψ_e, with linear part ψ_e, such that $T = \Psi_e(T_0)$ and $\psi_e(e) = e_0$ where $e_0 = (0, 1)$ is the vertical edge vector of T_0. Noticing that

$$\langle e, \nabla \mathrm{I}_T^{m-1}(g \circ \Psi_e)\rangle = \langle e, \nabla((\mathrm{I}_{T_0}^{m-1} g) \circ \Psi_e)\rangle = \langle e_0, (\nabla \mathrm{I}_{T_0}^{m-1} g) \circ \Psi_e\rangle = \frac{\partial \mathrm{I}_{T_0}^{m-1} g}{\partial y} \circ \Psi_e,$$

we obtain

$$\|\langle e, \nabla \mathrm{I}_T^{m-1}(g \circ \Psi_e)\rangle\|_{L^\infty(T)} = \left\|\frac{\partial \mathrm{I}_{T_0}^{m-1} g}{\partial y}\right\|_{L^\infty(T_0)} \le C_0 C_1 \left\|\frac{\partial g}{\partial y}\right\|_{L^\infty(T_0)} = C_0 C_1 \|\langle e, \nabla(g \circ \Psi_e)\rangle\|_{L^\infty(T)}. \tag{2.19}$$

Applying this inequality to $g = f \circ \Psi_e^{-1}$ we obtain that

$$\|\langle e, \nabla \mathrm{I}_T^{m-1} f\rangle\|_{L^\infty(T)} \le C_0 C_1 \|\langle e, \nabla f\rangle\|_{L^\infty(T)}, \tag{2.20}$$

for all edge vectors $e \in \{a, b, c\}$ of T. We next define a norm on $\mathbb{R}^2$ as follows

$$|v|_T := |a|^{-1}|\langle a, v\rangle| + |b|^{-1}|\langle b, v\rangle| + |c|^{-1}|\langle c, v\rangle|.$$

It follows from inequality (2.20), that

$$\|\ |\nabla \mathrm{I}_T^{m-1} f|_T\ \|_{L^\infty(T)} \le 3C_0C_1 \||\nabla f|_T\|_{L^\infty(T)}. \tag{2.21}$$

We next observe that if θ denotes the maximal angle of T,

$$\cos(\theta/2)|v| \le |v|_T \le 3|v|,$$

where $|\cdot|$ is the euclidean norm: the upper inequality is trivial and the lower one is implied by the fact that at least one of the edge vector makes an angle less than $\theta/2$ with v. Combining this with (2.21), we obtain

$$\|\nabla \mathrm{I}_T^{m-1} f\ \|_{L^\infty(T)} \le \frac{9C_0C_1}{\cos(\theta/2)} \|\nabla f\|_{L^\infty(T)}.$$

Since $\theta > \pi/3$ we have $\frac{1}{\cos(\theta/2)} \le 2\tan(\theta/2) \le 2S(T)$ according to Proposition 2.1, which concludes the proof with $C = 18C_0C_1$. ◇

Remark 2.3 *The following example in the simple case of piecewise linear approximation illustrates the sharpness of inequality (2.18). Let T be a triangle having an obtuse angle θ at a vertex v, and edges neighbouring v of length l and l'. Let $f(z) := |z - v|^2$. A simple computation shows that*

$$\|\nabla \mathrm{I}_T^1 f\|_{L^\infty(T)} = \frac{\operatorname{diam} T}{\sin\theta} \quad \textit{and} \quad \|\nabla f\|_{L^\infty(T)} = 2\max(l, l').$$

It follows that

$$\|\nabla \mathrm{I}_T^1 f\|_{L^\infty(T)} = \lambda(T)S(T)\|\nabla f\|_{L^\infty(T)},$$

with

$$\lambda(T) := \frac{\operatorname{diam}(T)}{2\sin(\theta)S(T)\max\{l, l'\}} = \frac{\operatorname{diam}(T)}{4\sin(\theta/2)^2\max\{l, l'\}} \in [\frac{1}{4}, 1],$$

which shows the sharpness of Lemma 2.2 in this context.

We now introduce for each polynomial $\pi \in \mathbb{H}_m$, a special set $\mathcal{A}_\pi \subset \mathrm{M}_2(\mathbb{R})$ of linear maps.

$$\mathcal{A}_\pi := \{A \in \mathrm{M}_2(\mathbb{R})\ ;\ |\nabla\pi(z)| \le |Az|^{m-1} \text{ for all } z \in \mathbb{R}^2\}. \tag{2.22}$$

This set has a geometrical interpretation : since $\nabla\pi$ is homogeneous of degree $m-1$, we find that $A \in \mathcal{A}_\pi$ if and only if the ellipse $\{z \in \mathbb{R}^2\ ;\ |Az| \le 1\}$ is included in the set $\{z \in \mathbb{R}^2\ ;\ |\nabla\pi(z)| \le 1\}$ which is limited by the algebraic curve $\{|\nabla\pi(z)| = 1\}$. If T is a triangle that contains the origin and if $A \in \mathcal{A}_\pi$, we observe that

$$\|\nabla\pi\|_{L^\infty(T)} \le \operatorname{diam}(A(T))^{m-1}. \tag{2.23}$$

We define

$$\gamma_m(\pi) := \inf\{|\det A| \; ; \; A \in \mathcal{A}_\pi\},$$

so that $\frac{\boldsymbol{\pi}}{\gamma_m(\pi)}$ is the maximal area of an ellipse contained in $\{z \in \mathbb{R}^2 \; ; \; |\nabla\pi(z)| \le 1\}$.

Remark 2.4 *Similar concepts have been introduced in [6] for the purpose of studying the L^p interpolation error of anisotropic finite elements, therefore with $|\pi(z)|$ in place of $|\nabla\pi(z)|$.*

The following result shows that a certain power of γ_m is equivalent to the shape function L_m.

Lemma 2.5 *There exists a constant $C = C(m)$ such that for all $\pi \in \mathbb{H}_m$*

$$C^{-1} L_m(\pi) \le \gamma_m(\pi)^{\frac{m-1}{2}} \le C L_m(\pi). \tag{2.24}$$

Proof: We first prove the left part of (2.24). Let $\pi \in \mathbb{H}_m$ and $A \in \mathcal{A}_\pi$ such that A is invertible. The matrix A admits a singular value decomposition

$$A = UDV,$$

where U, V are unitary and $D = \operatorname{diag}(\lambda_1, \lambda_2)$ with $\lambda_i > 0$ such that λ_i^2 are the eigenvalues of $A^{\mathrm{t}}A$. Let T_0 be the triangle of vertices $(0,0)$, $(0,\sqrt{2})$ and $(\sqrt{2},0)$. We define the triangle

$$T := \sqrt{|\det A|}V^{\mathrm{t}} D^{-1}(T_0),$$

which satisfies $|T| = |T_0| = 1$ and has an angle of $\boldsymbol{\pi}/2$ at the origin so that $S(T) = 1$. Denoting by C the constant in Lemma 2.2 and using (2.23), we obtain

$$(1+C)^{-1}\|\nabla\pi - \nabla \mathrm{I}_T^{m-1}\pi\|_{L^\infty(T)} \le \|\nabla\pi\|_{L^\infty(T)} \le \operatorname{diam}(A(T))^{m-1} = |\det A|^{\frac{m-1}{2}} \operatorname{diam}(T_0)^{m-1}.$$

Taking the infimum over all invertible $A \in \mathcal{A}_\pi$ and remarking that this set is dense in $\mathcal{A}_\pi$, we conclude the proof of the left part of (2.24). For the right part, we define for all $q_1, q_2 \in \mathbb{H}_{m-1}$, and any triangle T,

$$\|(q_1,q_2)\|_T := \inf_{r_1,r_2 \in \mathbb{P}_{m-2}} \|(q_1,q_2) - (r_1,r_2)\|_{L^\infty(T)}. \tag{2.25}$$

We denote by T_{eq} an equilateral triangle centered at the origin and of area 1. Since the functions $\|\cdot\|_{T_{\mathrm{eq}}}$ and $\|\cdot\|_{L^\infty(T_{\mathrm{eq}})}$ are norms on $\mathbb{H}_{m-1} \times \mathbb{H}_{m-1}$ there exists a constant C_* such that $\|\cdot\|_{L^\infty(T_{\mathrm{eq}})} \le C_*\|\cdot\|_{T_{\mathrm{eq}}}$. Let T be a triangle satisfying $|T| = 1$ and $\operatorname{bary}(T) = 0$. Then there exists a linear change of coordinates $\phi \in \mathrm{SL}_2$ such that $T = \phi(T_{\mathrm{eq}})$. We then obtain

$$\|(q_1,q_2)\|_{L^\infty(T)} = \|(q_1\circ\phi, q_2\circ\phi)\|_{L^\infty(T_{\mathrm{eq}})} \le C_*\|(q_1\circ\phi, q_2\circ\phi)\|_{T_{\mathrm{eq}}} = C_*\|(q_1,q_2)\|_T$$

We now choose a polynomial $\pi \in \mathbb{H}_m$ and set $(q_1,q_2) := \nabla\pi$. It follows from the previous equation and (2.25) that

$$\|\nabla\pi\|_{L^\infty(T)} \le C_*\|\nabla\pi\|_T \le C_*\|\nabla\pi - \nabla \mathrm{I}_T^{m-1}\pi\|_{L^\infty(T)}$$

We define a linear map $A \in \mathrm{GL}_2$ associated to π and T as follows

$$A := \|\nabla\pi\|_{L^\infty(T)}^{\frac{1}{m-1}} \lambda^{-1}\phi^{-1},$$

where $\lambda = 3^{-3/4}$ is the minimal distance from 0 to ∂T_{eq}. Then for all $z \in \partial T$ we have $\phi^{-1}(z) \in \partial T_{\mathrm{eq}}$ and hence $|\phi^{-1}(z)| \ge \lambda$. Therefore,

$$|A(z)|^{m-1} = \|\nabla\pi\|_{L^\infty(T)} (\lambda^{-1}|\phi^{-1}(z)|)^{m-1} \ge \|\nabla\pi\|_{L^\infty(T)} \ge |\nabla\pi(z)|.$$

By homogeneity, we thus find that

$$|\nabla\pi(z)| \le |A(z)|^{m-1},$$

for all $z \in \mathbb{R}^2$ which means that $A \in \mathcal{A}_\pi$. Furthermore, since $\det\phi = 1$, we have

$$|\det A|^{\frac{m-1}{2}} = \lambda^{-(m-1)}\|\nabla\pi\|_{L^\infty(T)} \le C_*\lambda^{-(m-1)}\|\nabla\pi - \nabla\,\mathrm{I}_T^{m-1}\,\pi\|_{L^\infty(T)}.$$

Hence taking the infimum over all triangles T satisfying $|T| = 1$ and $\mathrm{bary}(T) = 0$ we obtain

$$\gamma_m(\pi)^{\frac{m-1}{2}} \le C \inf_{\substack{|T|=1\\ \mathrm{bary}(T)=0}} \|\nabla\pi - \nabla\,\mathrm{I}_T^{m-1}\,\pi\|_{L^\infty(T)} \tag{2.26}$$

with $C = C_*\lambda^{-(m-1)}$. Using the invariance of the interpolation error under translation, as expressed by (1.2), we find that the right hand side of (2.26) is $CL_m(\pi)$, which concludes the proof. ◇

We next introduce a measure of the isotropy of a triangle T with respect to the euclidean metric:

$$\rho(T) := \frac{\mathrm{diam}(T)^2}{|T|}. \tag{2.27}$$

If T is an obtuse triangle, an elementary computation shows that $4S(T) \le \rho(T)$. Indeed, if the largest angle of T is $\theta \ge \boldsymbol{\pi}/2$, and if the edges neighbouring the angle θ have length l_1, l_2, we obtain using $l_1^2 + l_2^2 \ge 2l_1l_2$ that

$$\rho(T) = \frac{l_1^2 + l_2^2 - 2l_1l_2\cos\theta}{\frac{1}{2}l_1l_2\sin\theta} \ge 4\frac{1-\cos\theta}{\sin\theta} = 4\tan\frac{\theta}{2} = 4S(T)$$

Since the minimal value of ρ is $4/\sqrt{3}$ (for the equilateral triangle), and since $S(T) = 1$ for acute triangles, we obtain that for any triangle T

$$\rho(T) \ge \frac{4}{\sqrt{3}}S(T). \tag{2.28}$$

The functions S and ρ have a different behavior : $\rho(T)$ increases as T becomes thinner, while $S(T)$ increases only if an angle of T approaches $\boldsymbol{\pi}$.

In the follow up of this paper, we frequently distort the measure of isotropy ρ by a linear transform. If $A \in \mathrm{GL}_2$, then $\rho(A(T))$ reflects the isotropy of T *measured in the metric* $A^{\mathrm{t}}A$. In particular $\rho(A(T))$ is minimal, i.e. equal to $4/\sqrt{3}$, if and only if the ellipse $\mathcal{E}$ containing T and of minimal area is of the form

$$\mathcal{E} = \{z \in \mathbb{R}^2\ ;\ |A(z - \mathrm{bary}(T))| \le r\}$$

for some $r > 0$.

We may now state the main theorem of this section.

Theorem 2.6 *There exists a constant $C = C(m)$ such that for all $\pi \in \mathbb{H}_m$, all $A \in \mathcal{A}_\pi$ and any triangle T, we have*

$$|\pi - \mathrm{I}_T^{m-1}\,\pi|_{W^{1,p}(T)} \le C|T|^{\frac{1}{\tau}}\ S(T)\ \rho(A(T))^{\frac{m-1}{2}}|\det A|^{\frac{m-1}{2}} \tag{2.29}$$

where $\frac{1}{\tau} := \frac{m-1}{2} + \frac{1}{p}$. Furthermore for any triangle T and any $g \in C^m(T)$, we have

$$|g - \mathrm{I}_T^{m-1}\,g|_{W^{1,p}(T)} \le C|T|^{\frac{1}{\tau}}\ S(T)\ \rho(T)^{\frac{m-1}{2}}\|d^m g\|_{L^\infty(T)}, \tag{2.30}$$

where $\|d^m g\|_{L^\infty(T)}$ is defined by (1.15).

Before proving this result, we make some observations on its consequences. Combining the two estimates contained in this theorem, we obtain a mixed anisotropic-isotropic estimate, that can be used as a guideline for producing triangulations adapted to a function $f \in C^m(\Omega)$. Let T be a triangle, let $f \in C^m(T)$, $\pi \in \mathbb{H}_m$ and $A \in \mathcal{A}_\pi$. Then

$$\begin{aligned} |f - \mathrm{I}_T^{m-1}\,f|_{W^{1,p}(T)} &\le |\pi - \mathrm{I}_T^{m-1}\,\pi|_{W^{1,p}(T)} + |(f-\pi) - \mathrm{I}_T^{m-1}(f-\pi)|_{W^{1,p}(T)}, \\ &\le C|T|^{\frac{1}{\tau}}S(T)\left(\rho(A(T))^{\frac{m-1}{2}}|\det A|^{\frac{m-1}{2}} + \rho(T)^{\frac{m-1}{2}}\|d^m f - d^m\pi\|_{L^\infty(T)}\right), \end{aligned} \tag{2.31}$$

where $C = C(m)$. Note that the left term in the parenthesis is an "anisotropic" contribution to the error, while the right term is an "isotropic" contribution.

Let $\varepsilon > 0$ and $1 \leq p < \infty$. We now explain how the requirements (i), (ii), (iii) and (iv) heuristically exposed in the introduction can be mathematically stated, and show that the estimate

$$\#(\mathcal{T})^{\frac{m-1}{2}} |f - \mathrm{I}_{\mathcal{T}}^{m-1} f|_{W^{1,p}(\Omega)} \leq C \left\| L_m(\pi_z) + \varepsilon \right\|_{L^\tau(\Omega)},$$

is met when the triangulation satisfies these requirements. Consider a polygonal and bounded domain Ω, a function $f \in C^m(\Omega)$ and a triangulation $\mathcal{T}$. For each $z \in \Omega$, we denote by $T_z \in \mathcal{T}$ the triangle containing z and define $\pi_z = \frac{d^m f(z)}{m!} \in \mathbb{H}_m$. The adaptation of $\mathcal{T}$ with respect to f for the $W^{1,p}$ semi-norm, can be measured by the smallest constant $C_{\mathcal{T}} \geq 1$ such that the following criterions are met:

(i) (Equilibrated errors) There exists a constant $\delta > 0$ such that for all $z \in \Omega$,

$$C_{\mathcal{T}}^{-1} \delta \leq |T_z|^{\frac{1}{\tau}} (L_m(\pi_z) + \varepsilon) \leq C_{\mathcal{T}} \delta. \tag{2.32}$$

(ii) (Optimized shapes) For all $z \in \Omega$, there exists $A_z \in \mathcal{A}_{\pi_z}$, such that

$$\rho(A_z(T_z)) \leq C_{\mathcal{T}} \text{ and } |\det A_z|^{\frac{m-1}{2}} \leq C_L (L_m(\pi_z) + \varepsilon), \tag{2.33}$$

where C_L is the constant that appears in Lemma (2.5). According to this lemma, such an A_z always exists for any $\varepsilon > 0$.

(iii) (Bounded sliverness in average) The averaged $l^p(\mathcal{T})$ norm of S is bounded as follows

$$\left(\frac{1}{\#(\mathcal{T})} \sum_{T \in \mathcal{T}} S(T)^p \right)^{\frac{1}{p}} \leq C_{\mathcal{T}}. \tag{2.34}$$

This condition is less stringent than asking that $S(T) \leq C_{\mathcal{T}}$ for *all* $T \in \mathcal{T}$, and turns out to be sufficient for proving the optimal error estimate.

(iv) (Sufficient refinement) The mesh $\mathcal{T}$ is sufficiently fine in such way that the local interpolation error estimate(2.31) is controlled by the "anisotropic" component. More precisely, for all $z \in \Omega$,

$$\rho(T_z)^{\frac{m-1}{2}} \| d^m f - d^m f(z) \|_{L^\infty(T_z)} \leq C_{\mathcal{T}} (L_m(\pi_z) + \varepsilon). \tag{2.35}$$

This condition is ensured by sufficient refinement of the triangulation due to the following observation: If T_z' is the image of T_z by a homothetic size reduction around z, then $\rho(T_z') = \rho(T_z)$ while $\|d^m f - d^m f(z)\|_{L^\infty(T_z')}$ tends to zero due to the continuity of $d^m f$.

We now produce a global error estimate from these four assumptions. For a given $z \in \Omega$ we inject successively $\pi = \pi_z$, (2.33), (2.35) and (2.32) into the estimate (2.31) and obtain

$$\begin{aligned}
|f - \mathrm{I}_{T_z}^{m-1} f|_{W^{1,p}(T_z)} &\leq C |T_z|^{\frac{1}{\tau}} S(T_z) \left(\rho(A_z(T_z))^{\frac{m-1}{2}} |\det A_z|^{\frac{m-1}{2}} + \rho(T_z)^{\frac{m-1}{2}} \| d^m f - d^m \pi_z \|_{L^\infty(T)} \right) \\
&\leq C |T_z|^{\frac{1}{\tau}} S(T_z) (C_{\mathcal{T}}^{\frac{m-1}{2}} C_L (L_m(\pi_z) + \varepsilon) + \rho(T_z)^{\frac{m-1}{2}} \| d^m f - d^m f(z) \|_{L^\infty(T)}) \\
&\leq C |T_z|^{\frac{1}{\tau}} S(T_z) (C_{\mathcal{T}}^{\frac{m-1}{2}} C_L + C_{\mathcal{T}}) (L_m(\pi_z) + \varepsilon) \\
&\leq C_0 \delta S(T_z)
\end{aligned}$$

where $C_0 = C_0(m, C_{\mathcal{T}}, C_L)$. Using (2.34) we obtain

$$|f - \mathrm{I}_{\mathcal{T}}^{m-1} f|_{W^{1,p}(\Omega)}^p = \sum_{T \in \mathcal{T}} |f - \mathrm{I}_T^{m-1} f|_{W^{1,p}(T)}^p \leq C_0^p \delta^p \sum_{T \in \mathcal{T}} S(T)^p \leq (C_0 C_{\mathcal{T}})^p \delta^p \#(\mathcal{T}). \tag{2.36}$$

On the other hand, the left side of inequality (2.32) provides an upper estimate of δ as follows.

$$C_{\mathcal{T}}^{-\tau} \delta^\tau \#(\mathcal{T}) = C_{\mathcal{T}}^{-\tau} \delta^\tau \int_\Omega \frac{dz}{|T_z|} \leq \int_\Omega (L_m(\pi_z) + \varepsilon)^\tau dz = \| L_m(\pi_z) + \varepsilon \|_{L^\tau(\Omega)}^\tau. \tag{2.37}$$

Combining (2.36) with (2.37) we eliminate the variable δ and obtain

$$\#(\mathcal{T})^{\frac{m-1}{2}} |f - \mathrm{I}_{\mathcal{T}}^{m-1} f|_{W^{1,p}(\Omega)} \leq C \, \|L_m(\pi_z) + \varepsilon\|_{L^\tau(\Omega)}, \tag{2.38}$$

where $C = C(m, C_{\mathcal{T}})$. Hence the optimal asymptotic estimate (1.5) is satisfied up to a multiplicative constant depending only on the degree $m-1$ of interpolation and the quality of the mesh reflected by $C_{\mathcal{T}}$. Note however that the properties (2.32), (2.33), (2.34) and (2.35) required for $C_{\mathcal{T}}$ may lead to a very pessimistic constant $C = C(m, C_{\mathcal{T}}, C_L)$ in inequality (2.38). Finer estimates and weaker conditions on the mesh $\mathcal{T}$ can be obtained from (2.31).

In the context of the $H^1 = W^{1,2}$ semi-norm and of piecewise linear and quadratic elements we present numerical results in §4.2 and discuss the quality of a numerical mesh $\mathcal{T}$ using three quantities $\sigma(\mathcal{T})$, $\rho(\mathcal{T})$ and $S(\mathcal{T})$ that are related to the conditions (i), (ii) and (iii) respectively. We also discuss in §4 a reformulation of the requirements of size (2.32) and shape (2.33) for the triangles T of the mesh $\mathcal{T}$ in terms of Riemannian metrics, a more convenient form for mesh generation.

Proof of Theorem 2.6 : Let T be a triangle and let $h \in C^1(T)$. Using lemma 2.2, we obtain

$$\begin{aligned} |h - \mathrm{I}_T^{m-1} h|_{W^{1,p}(T)} &= \|\nabla h - \nabla \mathrm{I}_T^{m-1} h\|_{L^p(T)} \\ &\leq |T|^{\frac{1}{p}} \|\nabla h - \nabla \mathrm{I}_T^{m-1} h\|_{L^\infty(T)} \\ &\leq C|T|^{\frac{1}{p}} (1 + S(T)) \|\nabla h\|_{L^\infty(T)}. \end{aligned} \tag{2.39}$$

Replacing h with π in inequality (2.39) and combining it with (2.23), we obtain that if T contains the origin, then for all $A \in \mathcal{A}_\pi$

$$|\pi - \mathrm{I}_T^{m-1} \pi|_{W^{1,p}(T)} \leq C|T|^{1/p} (1 + S(T)) \operatorname{diam}(A(T))^{m-1}.$$

The left and right quantities in the above inequality are invariant by translation of T and therefore this inequality remains valid for any T. Combining it with the identity

$$\operatorname{diam}(A(T))^2 = |T| \, |\det A| \, \rho(A(T)),$$

this leads to the first inequality (2.29) of Theorem 2.6. For the second inequality, we take $g \in C^m(T)$ and $z_0 = (x_0, y_0) \in T$. We now take for h the remainder of the Taylor development of g at z_0,

$$h(x,y) := g(x,y) - \sum_{k+l \leq m-1} \frac{\partial^{k+l} g}{\partial x^k \partial y^l}(z_0) \frac{(x-x_0)^k}{k!} \frac{(y-y_0)^l}{l!}.$$

Therefore $h \in C^m(T)$ and

$$h(z_0) = dh(z_0) = \cdots = d^{m-1} h(z_0) = 0. \tag{2.40}$$

It follows that

$$\|\nabla h\|_{L^\infty(T)} \leq C_1 (\operatorname{diam} T)^{m-1} \|d^m h\|_{L^\infty(T)} = C_1 |T|^{\frac{m-1}{2}} \rho(T)^{\frac{m-1}{2}} \|d^m h\|_{L^\infty(T)}, \tag{2.41}$$

where $C_1 = C_1(m)$. Combining (2.39) and (2.41), we obtain

$$|h - \mathrm{I}_T^{m-1} h|_{W^{1,p}(T)} \leq C|T|^{\frac{1}{\tau}} S(T) \rho(T)^{\frac{m-1}{2}} \|d^m h\|_{L^\infty(T)}, \tag{2.42}$$

where $C = C(m)$. We now observe that $g - h \in \mathbb{P}_{m-1}$, hence $d^m g = d^m h$ and $h - \mathrm{I}_T^{m-1} h = g - \mathrm{I}_T^{m-1} g$. Injecting this into the last equation we conclude the proof of inequality (2.30) and of Theorem 2.6 $\diamond$

As a conclusion to this section we prove inequality (1.12), which links the functions $L_{m,p}$ and $L_m = L_{m,\infty}$.

Lemma 2.7 *There exists a constant $c = c(m) > 0$ such that for all $1 \leq p_1 \leq p_2 \leq \infty$,*

$$cL_m \leq L_{m,p_1} \leq L_{m,p_2} \leq L_m \text{ on } \mathbb{H}_m, \tag{2.43}$$

Proof: Let $T_{\rm eq}$ be an equilateral triangle of area 1. Since all norms are equivalent on the finite dimensional space $\mathbb{P}_{m-1}^2$, there exists a constant $c = c(m) > 0$ such that all $(q_1, q_2) \in \mathbb{P}_{m-1} \times \mathbb{P}_{m-1}$,

$$c\|(q_1, q_2)\|_{L^\infty(T_{\rm eq})} \leq \|(q_1, q_2)\|_{L^1(T_{\rm eq})}, \tag{2.44}$$

Furthermore, since $T_{\rm eq}$ has area 1, we have

$$\|(q_1, q_2)\|_{L^{p_1}(T_{\rm eq})} \leq \|(q_1, q_2)\|_{L^{p_2}(T_{\rm eq})}, \tag{2.45}$$

for all $1 \leq p_1 \leq p_2 \leq \infty$. If T is a triangle satisfying $|T| = 1$, there exists an affine change of coordinates Ψ such that $T = \Psi(T_{\rm eq})$ and we have $\|(q_1, q_2)\|_{L^p(T)} = \|(q_1\circ\Psi, q_2\circ\Psi)\|_{L^p(T_{\rm eq})}$ for all $(q_1, q_2) \in \mathbb{P}_{m-1}\times\mathbb{P}_{m-1}$ and $1 \leq p \leq \infty$. Combining this invariance property with inequalities (2.44) and (2.45) we obtain

$$c\|(q_1, q_2)\|_{L^\infty(T)} \leq \|(q_1, q_2)\|_{L^{p_1}(T)} \leq \|(q_1, q_2)\|_{L^{p_2}(T)} \leq \|(q_1, q_2)\|_{L^\infty(T)}. \tag{2.46}$$

We now choose a polynomial $\pi \in \mathbb{H}_m$, we set $(q_1, q_2) := \nabla\pi - \nabla\, \mathrm{I}_T^{m-1}\pi$, and we take the infimum of (2.46) among all triangles T of area 1. This leads to the announced inequality (2.43) which concludes the proof. $\diamond$

3 Proof of Theorems 1.1 and 1.2

The domain Ω, the integer m, the function $f \in C^m(\Omega)$ and the exponent $1 \leq p < \infty$ are fixed in this section which is devoted to the proof of the lower estimate (1.8) and the upper estimates (1.5) and (1.9) which are stated in Theorems 1.1 and 1.2.

We denote by μ_{z_0} the Taylor polynomial of f and of degree m at the point $z_0 = (x_0, y_0) \in \Omega$

$$\mu_{z_0}(x, y) := \sum_{k+l\leq m} \frac{\partial^{k+l} f(z_0)}{\partial x^k\, \partial y^l} \frac{(x - x_0)^k}{k!} \frac{(y - y_0)^l}{l!}.$$

Note that π_z is the homogeneous component of degree m in μ_z. Therefore $d^m\pi_z = d^m\mu_z = d^m f(z)$ and for any triangle T

$$\pi_z - \mathrm{I}_T^{m-1}\pi_z = \mu_z - \mathrm{I}_T^{m-1}\mu_z. \tag{3.47}$$

3.1 Proof of the lower estimate (1.8)

The following lemma allows to bound by below the interpolation error of f on a triangle T.

Lemma 3.1 *Let $\frac{1}{\tau} := \frac{m-1}{2} + \frac{1}{p}$. For any triangle $T \subset \Omega$ and $z \in T$ we have*

$$|f - \mathrm{I}_T^{m-1} f|_{W^{1,p}(T)} \geq |T|^{\frac{1}{\tau}} \left(L_{m,p}(\pi_z) - \omega(\operatorname{diam} T)\rho(T)^{\frac{m-1}{2}} S(T)\right),$$

where the function ω is positive, depends only on f and m, and satisfies $\omega(\delta) \to 0$ as $\delta \to 0$.

Proof: Let $h := f - \mu_z$. Using Equation (3.47) we obtain

$$\begin{aligned} |f - \mathrm{I}_T^{m-1} f|_{W^{1,p}(T)} &\geq |\pi_z - \mathrm{I}_T^{m-1}\pi_z|_{W^{1,p}(T)} - |h - \mathrm{I}_T^{m-1} h|_{W^{1,p}(T)} \\ &\geq |T|^{\frac{1}{\tau}} L_{m,p}(\pi_z) - |h - \mathrm{I}_T^{m-1} h|_{W^{1,p}(T)}. \end{aligned}$$

and we have seen in Theorem 2.6 that

$$|h - \mathrm{I}_T^{m-1} h|_{W^{1,p}(T)} \leq C_0|T|^{\frac{1}{\tau}} S(T)\rho(T)^{\frac{m-1}{2}} \|d^m h\|_{L^\infty(T)}.$$

for some constant $C_0 > 0$ depending only on m. We then remark that

$$\|d^m h\|_{L^\infty(T)} = \|d^m f - d^m\pi_z\|_{L^\infty(T)} = \|d^m f - d^m f(z)\|_{L^\infty(T)}.$$

Therefore, defining

$$\omega(\delta) := C_0 \sup_{z,z' \in \Omega \,;\, |z-z'| \le \delta} \|d^m f(z) - d^m f(z')\|, \tag{3.48}$$

we conclude the proof of this lemma. ⋄

We now consider an admissible sequence of triangulations $(\mathcal{T}_N)_{N \ge N_0}$. For all $N \ge N_0$, $T \in \mathcal{T}_N$ and $z \in T$, we define $\phi_N(z) := |T|$ and

$$\psi_N(z) := \left(L_{m,p}(\pi_z) - \omega(\mathrm{diam}(T))\rho(T)^{\frac{m-1}{2}} S(T)\right)_+,$$

where $\lambda_+ := \max\{\lambda, 0\}$. Holder's inequality gives, with $\frac{1}{\tau} := \frac{m-1}{2} + \frac{1}{p}$,

$$\int_\Omega \psi_N^\tau \le \left(\int_\Omega \phi_N^{\frac{(m-1)p}{2}} \psi_N^p\right)^{\frac{\tau}{p}} \left(\int_\Omega \phi_N^{-1}\right)^{\frac{(m-1)\tau}{2}} \tag{3.49}$$

Note that $\int_\Omega \phi_N^{-1} = \#\mathcal{T}_N \le N$. Furthermore if $T \in \mathcal{T}_N$ and $z \in T$ then according to Lemma 3.1

$$\phi_N(z)^{\frac{(m-1)p}{2}} \psi_N(z)^p = |T|^{\frac{p}{\tau}-1} \psi_N(z)^p \le \frac{1}{|T|} |f - \mathrm{I}_T^{m-1} f|_{W^{1,p}(T)}^p,$$

hence

$$\int_\Omega \phi_N^{\frac{(m-1)p}{2}} \psi_N^p \le \sum_{T \in \mathcal{T}_N} \frac{1}{|T|} \int_T |f - \mathrm{I}_T^{m-1} f|_{W^{1,p}(T)}^p = |f - \mathrm{I}_T^{m-1} f|_{W^{1,p}(\Omega)}^p$$

Inequality (3.49) therefore leads to

$$\|\psi_N\|_{L^\tau(\Omega)} \le |f - \mathrm{I}_{\mathcal{T}_N}^{m-1} f|_{W^{1,p}(\Omega)} N^{\frac{m-1}{2}}. \tag{3.50}$$

Since the sequence $(\mathcal{T}_N)_{N \ge N_0}$ is admissible, there exists a constant $C_A > 0$ such that for all N and all $T \in \mathcal{T}_N$ we have $\mathrm{diam}(T) \le C_A N^{-\frac{1}{2}}$. We introduce a subset of $\mathcal{T}_N' \subset \mathcal{T}_N$ which gathers the most degenerate triangles

$$\mathcal{T}_N' = \{T \in \mathcal{T}_N \,;\, \rho(T) \ge \omega(C_A N^{-\frac{1}{2}})^{\frac{-1}{m+1}}\},$$

where ω is the function from Lemma 3.1. We denote by Ω_N' the portion of Ω covered by $\mathcal{T}_N'$. For all $z \in \Omega \backslash \Omega_N'$, recalling from (2.28) that $\rho \ge S$, we obtain

$$\psi_N(z) \ge L_{m,p}(\pi_z) - \sqrt{\omega(C_A N^{-\frac{1}{2}})}.$$

Hence

$$\begin{aligned} \|\psi_N\|_{L^\tau(\Omega)}^\tau &\ge \left\| \left(L_{m,p}(\pi_z) - \sqrt{\omega(C_A N^{-\frac{1}{2}})}\right)_+ \right\|_{L^\tau(\Omega \backslash \Omega_N')}^\tau \\ &\ge \left\| \left(L_{m,p}(\pi_z) - \sqrt{\omega(C_A N^{-\frac{1}{2}})}\right)_+ \right\|_{L^\tau(\Omega)}^\tau - C^\tau |\Omega_N'|, \end{aligned}$$

where $C := \max_{z \in \Omega} L_{m,p}(\pi_z)$. We next observe that $|\Omega_N'| \to 0$ as $N \to +\infty$: indeed for all $T \in \mathcal{T}_N'$ we have

$$|T| = \mathrm{diam}(T)^2 \rho(T)^{-1} \le C_A^2 N^{-1} \omega(C_A N^{-\frac{1}{2}})^{\frac{1}{m+1}}.$$

Since $\#\mathcal{T}_N' \le N$, we obtain $|\Omega_N'| \le C_A^2 \omega(C_A N^{-\frac{1}{2}})^{\frac{1}{m+1}}$, which tends to 0 as $N \to \infty$. We thus obtain

$$\liminf_{N \to \infty} \|\psi_N\|_{L^\tau(\Omega)} \ge \lim_{N \to \infty} \left\| \left(L_{m,p}(\pi_z) - \sqrt{\omega(C_A N^{-\frac{1}{2}})}\right)_+ \right\|_{L^\tau(\Omega)} = \|L_{m,p}(\pi_z)\|_{L^\tau(\Omega)}.$$

Combining this result with (3.50) we conclude the proof of the announced estimate (1.8).

3.2 Proof of the upper estimates (1.5) and (1.9)

The proof of the upper estimate is based on an explicit construction of the triangulations $\mathcal{T}_N$, which is adapted from the construction in [2]. Roughly speaking, the idea of this construction is to produce a first mesh $\mathcal{R}$ of the domain Ω, composed of elements sufficiently small so that f can be regarded as a polynomial π_R on each triangle $R \in \mathcal{R}$. Each element $R \in \mathcal{R}$ is then tiled with small triangles optimally adapted to π_R, and some technical manipulations are done in order to preserve the conformity at the interfaces of the elements of $\mathcal{R}$. The main difference with the construction first proposed in [2], and used later in [15], is that the measure of sliverness S of the generated triangles should be kept under control.

Let T be a triangle with vertices (z_0, z_1, z_2). We define the symmetrized triangle $\tilde{T}$ of vertices $(z_1, z_2, z_1 + z_2 - z_0)$ so that $T \cup \tilde{T}$ is a parallelogram. We define a tiling $\mathcal{P}_T$ of the plane $\mathbb{R}^2$ as follows

$$\mathcal{P}_T := \{\alpha(z_1 - z_0) + \beta(z_2 - z_0) + T' \ ; \ \alpha, \beta \in \mathbb{Z}, \mathrm{T}' \in \{\mathrm{T}, \tilde{\mathrm{T}}\}\}. \tag{3.51}$$

If $\pi \in \mathbb{H}_m$, note that $|\pi - \mathrm{I}_{T'}^{m-1}\pi|_{W^{1,p}(T')}$ is constant among all triangles $T' \in \mathcal{P}_T$. We also define

$$\mathcal{P}_{T,n} := \frac{1}{n}\mathcal{P}_T$$

the tiling obtained by rescaling $\mathcal{P}_T$ by a factor $\frac{1}{n}$. We may use this rescaled tiling in order to subdivide an arbitrary triangle R, up to a few additional triangles located near the boundary of R, as expressed by the following lemma.

Lemma 3.2 *Let R and T be two triangles. There exists a family $(\mathcal{P}_{T,n}(R))_{n\geq 0}$, of conforming triangulations of R such that*

1. *Nearly all the elements of $\mathcal{P}_{T,n}(R)$ belong to $\mathcal{P}_{T,n}$, in the sense that*

$$\lim_{n\to\infty} \frac{\#(\mathcal{P}^1_{T,n}(R))}{n^2} = \frac{|R|}{|T|} \quad \textit{and} \quad \lim_{n\to\infty} \frac{\#(\mathcal{P}^2_{T,n}(R))}{n^2} = 0. \tag{3.52}$$

   where

$$\mathcal{P}^1_{T,n}(R) := \mathcal{P}_{T,n}(R) \cap \mathcal{P}_{T,n} \quad \textit{and} \quad \mathcal{P}^2_{T,n}(R) := \mathcal{P}_{T,n}(R) \setminus \mathcal{P}_{T,n} \tag{3.53}$$

2. *The vertices of $\mathcal{P}_{T,n}(R)$ on the boundary of R are exactly those of the form $\frac{k}{n}a + (1-\frac{k}{n})b$, where $0 \leq k \leq n$ and a, b are vertices of R.*

3. *There exists constants $C_1 = C_1(R,T)$ and $C_2 = C_2(R,T)$ such that*

$$\sup_{n\geq 0}\left(n \max_{T\in\mathcal{P}_{T,n}(R)} \mathrm{diam}(T)\right) \leq C_1 \quad \textit{and} \quad \sup_{n\geq 0} \max_{T\in\mathcal{P}_{T,n}(R)} S(T) \leq C_2. \tag{3.54}$$

Proof: See appendix. ◇

For any $M > 0$, we define the compact set of triangles

$$\mathbf{T}_M := \{T \text{ triangle } \ ; \ |T| = 1,\ \mathrm{diam}(T) \leq M \text{ and } \mathrm{bary}(T) = 0\}.$$

Note that

$$\rho(T) \leq M^2,$$

for all $T \in \mathbf{T}_M$. We also define the function

$$L_M(\pi) := \min_{T\in\mathbf{T}_M} |\pi - \mathrm{I}_T^{m-1}\pi|_{W^{1,p}(T)}. \tag{3.55}$$

Since $\mathbf{T}_M$ is compact for the Hausdorff distance between sets and since $T \mapsto |\pi - \mathrm{I}_T^{m-1}\pi|_{W^{1,p}(T)}$ is continuous with respect to this distance on the set of all triangles, we find that this minimum is indeed attained and L_M is continuous. We also observe that $M \mapsto L_M(\pi)$ is a decreasing function of M and that

$$\lim_{M\to\infty} L_M(\pi) = \inf_{|T|=1,\mathrm{bary}(T)=0} |\pi - \mathrm{I}_T^{m-1}\pi|_{W^{1,p}(T)} = \inf_{|T|=1} |\pi - \mathrm{I}_T^{m-1}\pi|_{W^{1,p}(T)} = L_{m,p}(\pi),$$

where we have used the invariance under translation of the interpolation error (1.2) for the second equality.

The constant $M > 0$ is now fixed until the last step of this proof. Let $\pi \in \mathbb{H}_m$ and let $T \subset \Omega$ be homothetic to a triangle achieving the minimum in the definition of $L_M(\pi)$. Then,

$$\begin{aligned} |f - \mathrm{I}_T^{m-1} f|_{W^{1,p}(T)} &\le |\pi - \mathrm{I}_T^{m-1}\pi|_{W^{1,p}(T)} + |(f-\pi) - \mathrm{I}_T^{m-1}(f-\pi)|_{W^{1,p}(T)} \\ &\le |T|^{\frac{1}{\tau}} L_M(\pi) + C|T|^{\frac{1}{\tau}} \rho(T)^{\frac{m}{2}} S(T) \|d^m f - d^m\pi\|_{L^\infty(T)} \\ &\le |T|^{\frac{1}{\tau}} (L_M(\pi) + CM^{m+2}\|d^m f - d^m \pi\|_{L^\infty(T)}), \end{aligned} \tag{3.56}$$

where we have used inequality (2.30) in the second line, and in third line the fact that

$$\rho(T)^{\frac{m}{2}} S(T) \le \rho(T)^{\frac{m}{2}+1} \le M^{m+2},$$

since $S \le \rho$ and T is homothetic to an element of $\mathbf{T}_M$.

Let $\delta > 0$ which value will be specified later. Since $d^m f$ is continuous, we can choose a sufficiently fine mesh $\mathcal{R} = \mathcal{R}(M,\delta)$ of Ω in such way that,

$$CM^{m+2}\|d^m f(x) - d^m f(y)\|_{L^\infty(T)} \le \delta, \text{ for all } R \in \mathcal{R} \text{ and } x, y \in R. \tag{3.57}$$

For any triangle $R \in \mathcal{R}$ we define

$$z_R := \operatorname*{argmin}_{z\in R} L_M(\pi_z) \quad \text{and} \quad \pi_R := \pi_{z_R}. \tag{3.58}$$

We also define

$$T_R := (L_M(\pi_R) + \delta)^{-\frac{\tau}{2}} T_*, \tag{3.59}$$

where $T_* \in \mathbf{T}_M$ achieves the minimum in the definition of $L_M(\pi_R)$. We denote by $\mathcal{P}_n(R) = \mathcal{P}_{T_R,n}(R)$ the triangulation of Lemma 3.2 built from the two triangles R and T_R, and similarly $\mathcal{P}_n^1(R) = \mathcal{P}_{T_R,n}^1(R)$ and $\mathcal{P}_n^2(R) = \mathcal{P}_{T_R,n}^2(R)$. We define for all n the global mesh of Ω

$$\mathcal{T}_n^{M,\delta} = \bigcup_{R\in\mathcal{R}} \mathcal{P}_n(R),$$

which coincides with $\mathcal{P}_n(R)$ on each $R \in \mathcal{R}$. Since all the meshes $\mathcal{P}_n(R)$ are conforming, and since $\mathcal{P}_n(R)$ has by construction $n+1$ equispaced vertices on each edge of R, the mesh $\mathcal{T}_n^{M,\delta}$ is also conforming. According to Equations (3.52) and (3.58), we have

$$\lim_{n\to\infty} \frac{\#\left(\mathcal{T}_n^{M,\delta}\right)}{n^2} = \sum_{R\in\mathcal{R}} \left(\lim_{n\to\infty} \frac{\#(\mathcal{P}_n(R))}{n^2}\right) = \sum_{R\in\mathcal{R}} |R|(L_M(\pi_R)+\delta)^\tau \le \int_\Omega (L_M(\pi_z)+\delta)^\tau dz. \tag{3.60}$$

For $T \in \mathcal{P}_n^1(R)$, we combine (3.56), (3.57) and (3.59) to obtain

$$|f - \mathrm{I}_T^{m-1} f|_{W^{1,p}(T)} \le n^{-\frac{2}{\tau}} \text{ for all } T \in \mathcal{P}_n^1(R). \tag{3.61}$$

For $T \in \mathcal{P}_n^2(R)$, we invoke the isotropic estimate (2.30) to obtain

$$|f - \mathrm{I}_T^{m-1} f|_{W^{1,p}(T)}^p \le C|T|^{\frac{1}{p}} S(T)\,\mathrm{diam}(T)^{m-1}\|d^m f\|_{L^\infty(\Omega)} \le CS(T)\,\mathrm{diam}(T)^{\frac{2}{\tau}}\|d^m f\|_{L^\infty(\Omega)} \tag{3.62}$$

where C is the constant from (2.30). Using the third item in Lemma 3.2, we find that that there exists constants $C_1 = C_1(M,\delta)$ and $C_2 = C_2(M,\delta)$ such that

$$\sup_{n\ge 0}\Big(n \max_{T\in\mathcal{T}_n^{M,\delta}} \mathrm{diam}(T)\Big) \le C_1 \quad \text{and} \quad \sup_{n\ge 0} \max_{T\in\mathcal{T}_n^{M,\delta}} S(T) \le C_2, \tag{3.63}$$

so that, combining with (3.62), we have for all $T \in \mathcal{P}_n^2(R)$

$$|f - \mathrm{I}_T^{m-1} f|^p_{W^{1,p}(T)} \le C_0 n^{-\frac{2}{\tau}}, \tag{3.64}$$

with $C_0 = C_0(M, \delta)$. Combining (3.61) and (3.64), and using the first item in Lemma 3.2, we obtain

$$|f - \mathrm{I}^{m-1}_{\mathcal{T}_n^{M,\delta}} f|^p_{W^{1,p}(\Omega)} = \sum_{T \in \mathcal{T}_n^{M,\delta}} |f - \mathrm{I}_T^{m-1} f|^p_{W^{1,p}(T)} \le \sum_{R \in \mathcal{R}} \left(\#(\mathcal{P}_n^1(R)) n^{-\frac{2p}{\tau}} + \#(\mathcal{P}_n^2(R)) C_1^p n^{-\frac{2p}{\tau}} \right)$$

therefore

$$\begin{aligned} \limsup_{n\to\infty} n^{\frac{2p}{\tau}-2} |f - \mathrm{I}^{m-1}_{\mathcal{T}_n^{M,\delta}} f|^p_{W^{1,p}(\Omega)} &\le \sum_{R\in\mathcal{R}} \lim_{n\to\infty} \frac{\#(\mathcal{P}_n^1(R)) + \#(\mathcal{P}_n^2(R)) C_1^p}{n^2} \\ &= \sum_{R\in\mathcal{R}} |R| (L_M(\pi_R) + \delta)^\tau \\ &\le \int_\Omega (L_M(\pi_z) + \delta)^\tau dz. \end{aligned}$$

Combining this with (3.60) we obtain

$$\limsup_{n\to\infty} \#(\mathcal{T}_n^{M,\delta})^{\frac{m-1}{2}} \left| f - \mathrm{I}^{m-1}_{\mathcal{T}_n^{M,\delta}} f \right|_{W^{1,p}(\Omega)} \le \|L_M(\pi_z) + \delta\|_{L^\tau(\Omega)}. \tag{3.65}$$

Let $\varepsilon > 0$. Since

$$\lim_{M\to\infty} \lim_{\delta\to 0} \|L_M(\pi_z) + \delta\|_{L^\tau(\Omega)} = \lim_{M\to\infty} \|L_M(\pi_z)\|_{L^\tau(\Omega)} = \|L_{m,p}(\pi_z)\|_{L^\tau(\Omega)}$$

we can choose adequately M and δ in such way that $\|L_M(\pi_z) + \delta\|_{L^\tau(\Omega)} \le \|L_{m,p}(\pi_z)\|_{L^\tau(\Omega)} + \varepsilon$.

Let $n = n(N, M, \delta)$ be the largest integer such that $\#(\mathcal{T}_n^{M,\delta}) \le N$, we define

$$\mathcal{T}_N^\varepsilon := \mathcal{T}_n^{M,\delta}.$$

so that $N^{-1}\#(\mathcal{T}_N^\varepsilon) \to 1$ as $N \to \infty$. If follows from (3.60) and (3.63) that the sequence of triangulations $(\mathcal{T}_N^\varepsilon)$ is admissible, and inequality (3.65) gives

$$\limsup_{N\to\infty} N^{\frac{m-1}{2}} |f - \mathrm{I}^{m-1}_{\mathcal{T}_N^\varepsilon} f|_{W^{1,p}(\Omega)} \le \|L_{m,p}(\pi_z)\|_{L^\tau(\Omega)} + \varepsilon.$$

which is the upper estimate (1.9) announced. Last we choose for all N large enough $\varepsilon(N) > 0$ such that

$$N^{\frac{m-1}{2}} |f - \mathrm{I}^{m-1}_{\mathcal{T}_N^{\varepsilon(N)}} f|_{W^{1,p}(\Omega)} \le \|L_{m,p}(\pi_z)\|_{L^\tau(\Omega)} + 2\varepsilon(N).$$

and such that $\varepsilon(N) \to 0$ as $N \to \infty$. The sequence of triangulations $\mathcal{T}_N := \mathcal{T}_N^{\varepsilon(N)}$ fulfills the estimate (1.5) which concludes the proof.

4 Optimal metrics for linear and quadratic elements

The proof of the upper estimate (1.9) exposed in the previous section involves the construction of meshes $\mathcal{T}_N^\varepsilon$ by tiling each element R of the "coarse" triangulation $\mathcal{R}$ using the finer mesh $\mathcal{P}_n(R)$. In practice, such a construction may require a very large number of triangles in order to match the optimal error estimate. More commonly used strategies for mesh generation are based on the prescription of a non-euclidean metric depending on f for which each triangle should be isotropic. In this section, we explain how to design such metric in order to derive near-optimal error estimates and we give analytic expressions in the particular case of $\mathbb{P}_1$ and $\mathbb{P}_2$ finite elements.

4.1 Optimal metrics

As a first step, we express the requirements (i), (ii) and (iv) of mesh adaptation in terms of metrics. We therefore use the following notations : we consider a polygonal domain Ω, an integer $m \geq 2$, an exponent $1 \leq p < \infty$, and a function $f \in C^m(\Omega)$ to be approximated in the $W^{1,p}$ semi-norm by $\mathbb{P}_{m-1}$ finite element interpolation on a triangulation of Ω. We also consider two real numbers $\varepsilon > 0$ and $\delta > 0$.

We define for all $\pi \in \mathbb{H}_m$

$$\mathcal{A}'_\pi := \{M \in S_2^+ \; ; \; |\nabla \pi(z)|^2 \leq (z^{\mathrm{t}} M z)^{m-1} \text{ for all } z \in \mathbb{R}^2\} = \{A^{\mathrm{t}} A \; ; \; A \in \mathcal{A}_\pi\} \tag{4.66}$$

and we consider a continuous field $\mathcal{M}$ of symmetric positive definite matrices satisfying $\mathcal{M}(z) \in \mathcal{A}'_{\pi_z}$ for all $z \in \Omega$ and

$$C_2^{-1}(L_m(\pi_z) + \varepsilon) \leq (\det \mathcal{M}(z))^{\frac{m-1}{4}} \leq C_2(L_m(\pi_z) + \varepsilon), \tag{4.67}$$

where C_2 is larger than the constant appearing in Lemma 2.5. Such a field exists thanks to the continuity of $z \mapsto \pi_z$, and we explain in the sequel of this section a practical construction in the case of piecewise linear and bilinear elements. We then define a field of symmetric positive definite matrices h on Ω by

$$h(z) := \delta^{-\tau} (\det \mathcal{M}(z))^{\frac{-\tau}{2p}} \mathcal{M}(z) \tag{4.68}$$

where $\frac{1}{\tau} := \frac{m-1}{2} + \frac{1}{p}$. Such a field h is called a Riemannian metric. Under some assumptions on the metric h and on the domain Ω, which are discussed in [14, 4], it is possible to produce a triangulation $\mathcal{T}$ of Ω satisfying for all $T \in \mathcal{T}$ and $z \in T$

$$C_1^{-1} \leq |T| \sqrt{\det h(z)} \leq C_1 \quad \text{and} \quad \rho\left(\sqrt{h(z)}(T)\right) \leq C_1 \tag{4.69}$$

where the constant $C_1 \geq 1$ reflects the quality of the adaptation of the mesh to the metric h. (In the second inequality the square root is meant in the sense of symmetric positive matrices). Examples of such mesh generators are [18, 19, 5]. We shall not discuss in this paper the conditions under which such a mesh can be generated. Let us only mention that, if one ignores a few outliers at the corners of Ω, these conditions hold if δ is small enough.

Note that $(\det h(z))^{\frac{1}{2\tau}} = \delta^{-1} (\det \mathcal{M}(z))^{\frac{m-1}{4}}$, therefore if (4.69) holds we find that for all $T \in \mathcal{T}$ and $z \in T$,

$$(C_1 C_2)^{-1} \delta \leq |T|^{\frac{1}{\tau}} (L_m(\pi_z) + \varepsilon) \leq C_1 C_2 \delta,$$

hence condition (i) of equilibrated errors, as stated in (2.32), holds provided $C_{\mathcal{T}} \geq C_1 C_2$. Furthermore for all $z \in \Omega$ let us define $A_z := \sqrt{\mathcal{M}(z)}$ and note that $A_z \in \mathcal{A}_{\pi_z}$ and $\det A_z = \sqrt{\det \mathcal{M}(z)}$. Using (4.67) and (4.69) we find that condition (ii) of optimal shapes, as stated in (2.33), holds provided $C_{\mathcal{T}} \geq C_2$. Condition (iv) holds when the mesh $\mathcal{T}$ is sufficiently refined, which is the case if δ is small enough.

In summary, given a map $\mathcal{M} : \Omega \to S_2^+$ satisfying $\mathcal{M}(z) \in \mathcal{A}'_{\pi_z}$, (4.67) and such that $\mathcal{M}(z)$ is positive definite, state of the art mesh generators allow us to build triangulations $\mathcal{T}$ that match the conditions (i), (ii) and (iv). In order to prove the near-optimal estimate

$$\#(\mathcal{T})^{\frac{m-1}{2}} |f - \mathrm{I}_{\mathcal{T}}^{m-1}|_{W^{1,p}(\Omega)} \leq C \|L_m(\pi_z) + \varepsilon\|_{L^\tau(\Omega)},$$

it is also necessary that the generated meshes satisfy condition (iii) of bounded measure of sliverness, as stated in (2.34). Unfortunately, the author has not heard of theoretical results that would guarantee this condition when a mesh is built by such algorithms. We discuss in §4.2. the observed behaviour of $S(T)$ when using the mesh generation software [18].

For $m \in \{2, 3\}$, which correspond to $\mathbb{P}_1$ and $\mathbb{P}_2$ elements, we give in the sequel a simple expression of a continuous map $\mathcal{M}_m : \mathbb{H}_m \to S_2^+$ satisfying $\mathcal{M}_m(\pi) \in \mathcal{A}'_\pi$ for all $\pi \in \mathbb{H}_m$ and

$$K^{-1} L_m(\pi) \leq (\det \mathcal{M}_m(\pi))^{\frac{m-1}{4}} \leq K L_m(\pi) \tag{4.70}$$

for some absolute constant $K \geq 1$. It is not hard to build from $\mathcal{M}_m(\pi_z)$ a matrix $\mathcal{M}(z)$ satisfying (4.67). For practical uses one usually takes

$$\mathcal{M}(z) := \mathcal{M}_m(\pi_z) + \mu \, \mathrm{Id}\,.$$

where the constant $\mu \geq 0$ is here to avoid degeneracy problems.

Let us mention that there exists radically different approaches to anisotropic mesh generation, which are not based on Riemannian metrics. For example the hierarchical refinement procedure exposed in [8], which was proved in [15] to yield the best possible estimate (1.1) in the case of piecewise linear interpolation of bidimensional convex functions with the error measured in L^p norm. This approach does not seem to adapt well to the $W^{1,p}$ norm: the main problem arises again from condition (iii) of bounded measure of sliverness.

4.2 The case of linear and quadratic elements

We now give analytic expression of matrix fields $\mathcal{M}_2$ and $\mathcal{M}_3$ satisfying (4.70), which correspond to linear and quadratic elements. In the simplest and already well established case of $\mathbb{P}_1$ elements, a more detailed analysis can be found in [16]. In contrast, the results for quadratic elements are new.

Let $\pi \in \mathbb{H}_2$, $\pi = ax^2 + 2bxy + cy^2$. For any such quadratic form, we use the notation

$$[\pi] = \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

We define

$$\mathcal{M}_2(\pi) := 4[\pi]^2 = 4\begin{pmatrix} a & b \\ b & c \end{pmatrix}^2 = 4\begin{pmatrix} a^2+b^2 & b(a+c) \\ b(a+c) & b^2+c^2 \end{pmatrix} \tag{4.71}$$

For all $z \in \mathbb{R}^2$ one has $\nabla\pi(z) = 2[\pi]z$, and therefore $|\nabla\pi(z)|^2 = z^{\mathrm{t}}\mathcal{M}_2(\pi)z$. It follows that $\mathcal{M}_2(\pi) \in \mathcal{A}'_\pi$ and

$$\det \mathcal{M}_2(\pi) = \inf\{\det M \ ; \ M \in \mathcal{A}'_\pi\}$$

which implies (4.70) according to Lemma 2.5.

Let $\pi \in \mathbb{H}_3$, $\pi = ax^3 + 3bx^2y + 3cxy^2 + dy^3$. We define

$$\mathcal{M}_3(\pi) := \sqrt{[\partial_x\pi]^2 + [\partial_y\pi]^2} = 3\sqrt{\begin{pmatrix} a & b \\ b & c \end{pmatrix}^2 + \begin{pmatrix} b & c \\ c & d \end{pmatrix}^2} = 3\sqrt{\begin{pmatrix} a^2+2b^2+c^2 & ab+2bc+cd \\ ab+2bc+cd & b^2+2c^2+d^2 \end{pmatrix}} \tag{4.72}$$

In the sense of symmetric matrices, we have

$$\mathcal{M}_3(\pi) = \sqrt{[\partial_x\pi]^2 + [\partial_y\pi]^2} \geq \sqrt{[\partial_x\pi]^2} = |[\partial_x\pi]|.$$

It follows that

$$|\nabla\pi(z)|^2 = |\partial_x\pi(z)|^2 + |\partial_y\pi(z)|^2 \leq 2(z^{\mathrm{t}}\mathcal{M}_3(\pi)z)^2,$$

hence $\sqrt{2}\mathcal{M}(\pi) \in \mathcal{A}'(\pi)$. Note that

$$\det \mathcal{M}_3(\pi) = 9\sqrt{(a^2+2b^2+c^2)(b^2+2c^2+d^2) - (ab+2bc+cd)^2}. \tag{4.73}$$

It remains to establish (4.70). This point is postponed to §5, right after (5.87), as we develop a general method for obtaning simple equivalents of the functions L_m. Let us finally mention the work [13] in which approximate solutions to the optimization problem $\inf\{\det M \ ; \ M \in \mathcal{A}'_\pi\}$ are obtained through numerical optimization. This approach works for general m but is harder to use than the algebraic expressions of $\mathcal{M}_2(\pi)$ and $\mathcal{M}_3(\pi)$ given here.

4.3 Limiting the anisotropy in mesh adaptation

The measure of non degeneracy of a triangle and of its image by a linear transform can be linked by the following result.

Proposition 4.1 *There exists an absolute constant $c > 0$ such that for any triangle T and any $A \in \mathrm{GL}_2$,*

$$\frac{c}{\rho(A(T))} \le \frac{\rho(T)}{\|A\|\|A^{-1}\|} \le \rho(A(T)). \tag{4.74}$$

Proof: We use in this proof the identity $|\det B| = \|B\|\|B^{-1}\|^{-1}$ which holds for all $B \in \mathrm{GL}_2$. Let T' be a triangle and let $A' \in \mathrm{GL}_2$, then

$$\rho(A'(T')) = \frac{\mathrm{diam}(A'(T'))^2}{|A'(T')|} \le \frac{\|A'\|^2}{|\det A'|}\frac{\mathrm{diam}(T')^2}{|T'|} = \|A'\|\|A'^{-1}\|\rho(T').$$

with the particular choice $A' = A^{-1}$ and $T' = A(T)$ we obtain the right side of (4.74). Let T_{eq} be an equilateral triangle of area 1, and let μ be the diameter of the largest ball included in T_{eq}. Up to a translation on T_{eq} we can assume that there exists $B \in \mathrm{GL}_2$ such that $T = B(T_{\mathrm{eq}})$. We then have

$$\mathrm{diam}(B(T_{\mathrm{eq}}))\,\mathrm{diam}(AB(T_{\mathrm{eq}})) \ge \mu^2\|B\|\|AB\| \ge \mu^2\|B\|\|A\|\|B^{-1}\|^{-1} = \mu^2\|A\||\det B|$$

Hence, since $|T| = |B(T_{\mathrm{eq}})| = |\det B|$,

$$\rho(T)\rho(A(T)) = \frac{\mathrm{diam}(T)^2\,\mathrm{diam}(A(T))^2}{|T||A(T)|} \ge \frac{(\mu^2\|A\||\det B|)^2}{|\det B|^2|\det A|} = \mu^4\|A\|\|A^{-1}\|$$

which establishes the left part of (4.74) with $c = \mu^4 = \frac{2^4}{3^3}$. ◇

A consequence of the above lemma is that if $\mathcal{T}$ is a mesh adapted to a metric h in the sense of (4.69), then for all $T \in \mathcal{T}$ and $z \in T$ we have

$$cC_1^{-1}\sqrt{\|h(z)\|\|h(z)^{-1}\|} \le \rho(T) \le C_1\sqrt{\|h(z)\|\|h^{-1}(z)\|}.$$

The measure of non-degeneracy $\rho(T)$ is thus large when $h(z)$ is ill conditioned. Although this property is desirable in order to adapt to highly anisotropic features of the function f to be approximated, excessive degeneracy can cause mesh generation problems, which are discussed in §4.2. In the following, we explain how to slightly modify the construction of $\mathcal{M}_2$ and $\mathcal{M}_3$ in order to control the value of $\rho(T)$.

According to (4.68) we have $\|h(z)\|\|h^{-1}(z)\| = \|\mathcal{M}(z)\|\|\mathcal{M}(z)^{-1}\|$, and thus

$$\rho(T) \le C_1\sqrt{\|\mathcal{M}(z)\|\|\mathcal{M}(z)^{-1}\|}.$$

This leads us to define for all $\alpha \ge 1$,

$$\mathcal{A}'_{\pi,\alpha} := \{M \in \mathcal{A}'_\pi \ ; \ \|M\|\|M^{-1}\| \le \alpha^2\}.$$

Let $M \in S_+^2$, let R be a rotation and let $\lambda \ge \mu \ge 0$ be the eigenvalues of M in such way that $M = R^{\mathrm{t}}\,\mathrm{diag}(\lambda, \mu)R$. We define for any $\alpha \ge 1$

$$M^{(\alpha)} := R^{\mathrm{t}}\begin{pmatrix} \lambda & 0 \\ 0 & \max(\lambda\alpha^{-2}, \mu)\end{pmatrix}R. \tag{4.75}$$

Clearly $M^{(\alpha)} \ge M$, and if $M \neq 0$ then $\|M^{(\alpha)}\|\|\left(M^{(\alpha)}\right)^{-1}\| \le \alpha^2$. Hence for all $M \in \mathcal{A}'_\pi$ we have $M^{(\alpha)} \in \mathcal{A}'_{\pi,\alpha}$.

In the case of piecewise linear elements we therefore have $\mathcal{M}_2^{(\alpha)}(\pi) \in \mathcal{A}'_{\pi,\alpha}$ for all $\pi \in \mathbb{H}_2$, and one easily shows that $\det \mathcal{M}_2^{(\alpha)}(\pi) = \inf\{\det M \ ; \ M \in \mathcal{A}'_{\pi,\alpha}\}$. This suggests that constructing $\mathcal{M}(z)$ from $\mathcal{M}_2^{(\alpha)}(\pi_z)$ instead of $\mathcal{M}_2(\pi_z)$ leads to a near-optimal mesh adaptation to the function f, under the constraint $\rho(T) \le C_1\alpha$ for all triangles T in the triangulation. The following proposition implies the same in the case of piecewise quadratic finite elements.

Proposition 4.2 *Let $\pi \in \mathbb{H}_3$ and $\alpha \ge 1$. Then $\sqrt{2}\mathcal{M}_3^{(\alpha)}(\pi) \in \mathcal{A}'_{\pi,\alpha}$ and*

$$\det \mathcal{M}_3^{(\alpha)}(\pi) \le K\inf\{\det M \ ; \ M \in \mathcal{A}'_{\pi,\alpha}\} \tag{4.76}$$

where the constant K is independent of π and α.

Proof: We already know that $\sqrt{2}\mathcal{M}_3^{(\alpha)}(\pi) \in \mathcal{A}'_{\pi,\alpha}$. If $\mathcal{M}_3^{(\alpha)}(\pi) = \mathcal{M}_3(\pi)$ then (4.76) holds as a consequence of (4.70) and Lemma 2.5. We therefore assume in the following that $\mathcal{M}_3^{(\alpha)}(\pi) \neq \mathcal{M}_3(\pi)$. Let

$$\lambda_*(\pi) := \|\nabla\pi\|_{L^\infty(D)} = \sup_{|z|\le 1} |\nabla\pi(z)|,$$

where $D = \{z \in \mathbb{R}^2 \; ; \; |z| \le 1\}$ is the unit disc of $\mathbb{R}^2$. The largest ball inscribed in $\{z \in \mathbb{R}^2 \; ; \; |\nabla\pi(z)| \le 1\}$ is $\lambda_*(\pi)^{-\frac{1}{2}}D$. Let $M \in \mathcal{A}'_{\pi,\alpha}$ and let $\lambda_1 \ge \lambda_2 > 0$ be its eigenvalues. The ellipse $\{z \in \mathbb{R}^2 \; ; \; z^{\mathrm{t}}Mz \le 1\}$ contains the ball $\lambda_1^{-\frac{1}{2}}D$, hence $\lambda_1 \ge \lambda_*(\pi)$. Furthermore $\lambda_2 \ge \alpha^{-2}\lambda_1$, hence

$$\det M = \lambda_1\lambda_2 \ge \alpha^{-2}\lambda_1^2 \ge \alpha^{-2}\lambda_*(\pi)^2. \tag{4.77}$$

Let $\lambda(\pi)$ be the largest eigenvalue of $\mathcal{M}_3(\pi)$, and assume that $\pi = ax^2 + 3bx^2y + 3cxy^2 + dy^3$. We obtain from (4.72) that

$$\lambda(\pi) \le \sqrt{\operatorname{Tr}\mathcal{M}_3(\pi)^2} = \sqrt{a^2 + 3b^2 + 3c^2 + d^2}.$$

Since the norms $\|\nabla\pi\|_{L^\infty(D)}$ and $\sqrt{a^2 + 3b^2 + 3c^2 + d^2}$ are equivalent on the vector space $\mathbb{H}_3$, there exists a constant $C_0 > 0$ independent of $\pi \in \mathbb{H}_3$ such that $\lambda(\pi) \le C_0\lambda_*(\pi)$. Since $\mathcal{M}_3^{(\alpha)}(\pi) \neq \mathcal{M}_3(\pi)$, the eigeinvalues of $\mathcal{M}_3^{(\alpha)}(\pi))$ are $\lambda(\pi)$ and $\alpha^{-2}\lambda(\pi)$. Hence

$$\det \mathcal{M}_3^{(\alpha)}(\pi)) = \alpha^{-2}\lambda(\pi)^2 \le C_0^2\alpha^{-2}\lambda_*(\pi)^2.$$

Combining this with (4.77) we conclude the proof, with $K = C_0^2$. ⋄

Let us finally mention that, although they are derived from the coefficients of π, the maps $\pi \mapsto \mathcal{M}_m(\pi)$ and $\pi \mapsto \mathcal{M}_m^{(\alpha)}(\pi)$ for $m \in \{2,3\}$ are invariant under rotation, and therefore not tied to the chosen system of coordinate (x,y), as expressed by the following result.

Proposition 4.3 *For $m \in \{2,3\}$, if $\pi \in \mathbb{H}_m$ and if U is unitary, then*

$$\mathcal{M}_m(\pi \circ U) = U^{\mathrm{t}}\mathcal{M}_m(\pi)U.$$

Furthermore, if $\alpha \ge 1$, then $\mathcal{M}_m^{(\alpha)}(\pi \circ U) = U^{\mathrm{t}}\mathcal{M}_m^{(\alpha)}(\pi)U$.

Proof: We only prove the invariance under unitary transformation of $\mathcal{M}_3$, since the proof for $\mathcal{M}_2$ is elementary, as well at the result for $\mathcal{M}_m^{(\alpha)}$. Let $\pi \in \mathbb{H}_3$, let $D_x = [\partial_x\pi]$ and $D_y = [\partial_y\pi]$. Let $U = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}$ be unitary, then

$$[\partial_x(\pi \circ U)] = u_{11}U^{\mathrm{t}}D_xU + u_{12}U^{\mathrm{t}}D_yU \quad \text{and} \quad [\partial_y(\pi \circ U)] = u_{21}U^{\mathrm{t}}D_xU + u_{22}U^{\mathrm{t}}D_yU$$

Hence

$$[\partial_x(\pi\circ U)]^2 + [\partial_y(\pi\circ U)]^2 = (u_{11}^2 + u_{21}^2)U^{\mathrm{t}}D_x^2U + (u_{11}u_{12} + u_{21}u_{22})U^{\mathrm{t}}(D_xD_y + D_yD_x)U + (u_{12}^2 + u_{22}^2)U^{\mathrm{t}}D_y^2U$$

which equals $U^{\mathrm{t}}D_x^2U + U^{\mathrm{t}}D_y^2U$ since U is unitary. Eventually

$$\mathcal{M}_3(\pi \circ U) = \sqrt{U^{\mathrm{t}}D_x^2U + U^{\mathrm{t}}D_y^2U} = U^{\mathrm{t}}\sqrt{D_x^2 + D_y^2}\,U = U^{\mathrm{t}}\mathcal{M}_3(\pi)U$$

which concludes the proof. ⋄

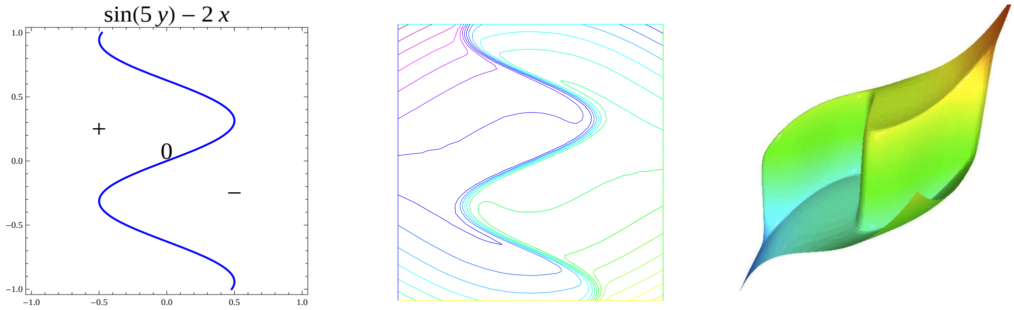

Figure 2: Description of the function f_δ, $\delta = 0.1$.

4.4 Numerical results

The envisionned applications for the theory developped in this paper are mainly in the field of partial differential equations that exhibit "shocks", and strongly anisotropic features, in particular conservation laws and fluid dynamics. We shall therefore test the quality of our meshes on a synthetic function that mimics the typical behavior of functions encountered in these contexts. For all $\delta > 0$, our test function $f_\delta : [-1,1]^2 \to \mathbb{R}$ is defined as follows

$$f_\delta(x,y) := \tanh\left(\frac{2x - \sin(5y)}{\delta}\right) + x^3 + xy^2.$$

In all numerical results, we choose $\delta := 0.1$. This function f_δ, although smooth, exhibits a "smoothed jump" of height 2 along to the curve defined by the equation $2x = \sin(5y)$, on a layer of width δ. On the rest of the domain, f_δ is dominated by the polynomial part $x^3 + xy^2$. The level lines and a 3D plot of f_δ are presented on the two rightmost pictures of Figure 2.

Our purpose is to produce four triangulations $\mathcal{T}_{H^1,\mathbb{P}_1}$, $\mathcal{T}_{H^1,\mathbb{P}_2}$, $\mathcal{T}_{L^2,\mathbb{P}_1}$ and $\mathcal{T}_{L^2,\mathbb{P}_2}$ containing 2000 triangles each and which, for this cardinality, produce respectivelly the smallest possible interpolation errors $\|\nabla f - \nabla \mathrm{I}^1_{\mathcal{T}} f\|_2$, $\|\nabla f - \nabla \mathrm{I}^2_{\mathcal{T}} f\|_2$, $\|f - \mathrm{I}^1_{\mathcal{T}} f\|_2$ and $\|f - \mathrm{I}^2_{\mathcal{T}} f\|_2$. It is clearly out of reach to find the triangulations leading exactly to the smallest error. Following the analysis developed in the beginning of this section we have generated $\mathcal{T}_{H^1,\mathbb{P}_1}$ and $\mathcal{T}_{H^1,\mathbb{P}_2}$ based on the metrics

$$\begin{array}{rclll} h_{H^1,\mathbb{P}_1}(z) & = & \lambda_1(\det \mathcal{M}_2^{(100)}(\pi_z))^{-\frac{1}{4}} \mathcal{M}_2^{(100)}(\pi_z) & \text{where} & \pi_z := \frac{d^2 f_\delta(z)}{2}, \\ h_{H^1,\mathbb{P}_2}(z) & = & \lambda_2(\det \mathcal{M}_3(\pi_z))^{-\frac{1}{6}} \mathcal{M}_3(\pi_z) & \text{where} & \pi_z := \frac{d^3 f_\delta(z)}{6}, \end{array} \tag{4.78}$$

where the positive constants λ_1, λ_2 are adjusted in such way that the meshes generated have 2000 elements. Mesh generation was performed by the open source program FreeFEM++ [18] and results are illustrated on Figure 3. Note that we have used $\mathcal{M}_2^{(100)}$ (defined as in (4.75)) instead of $\mathcal{M}_2$ which would lead to a different triangulation $\mathcal{T}^*_{H^1,\mathbb{P}_1}$, also displayed on Figure 3, and associated to the metric

$$h^*_{H^1,\mathbb{P}_1}(z) := \lambda_1^*(\det \mathcal{M}_2(\pi_z))^{-\frac{1}{4}} \mathcal{M}_2(\pi_z) \text{ where } \pi_z := \frac{d^2 f_\delta(z)}{2},$$

with again λ_1^* adjusted to obtain 2000 elements. The use of $\mathcal{M}_2^{(100)}$ in place of $\mathcal{M}_2$ is justified by mesh generation issues which are discussed in the next subsection.

Similarly, and following the study developed in [15], we have generated $\mathcal{T}_{L^2,\mathbb{P}_1}$ and $\mathcal{T}_{L^2,\mathbb{P}_2}$ from the metrics

$$\begin{array}{rclll} h_{L^2,\mathbb{P}_1}(z) & = & \mu_1(\det \mathcal{N}_2(\pi_z))^{-\frac{1}{6}} \mathcal{N}_2(\pi_z) & \text{where} & \pi_z := \frac{d^2 f_\delta(z)}{2}, \\ h_{L^2,\mathbb{P}_2}(z) & = & \mu_2(\det \mathcal{N}_3(\pi_z))^{-\frac{1}{8}} \mathcal{N}_3(\pi_z) & \text{where} & \pi_z := \frac{d^3 f_\delta(z)}{6}, \end{array}$$

where again μ_1, μ_2 are adjusted in order to generate a mesh with 2000 elements. Here $\mathcal{N}_2(\pi) := \sqrt{\mathcal{M}_2(\pi)}$ and

$$\mathcal{N}_3(\pi) := \operatorname{argmin}\{\det M \; ; \; M \in S_2^+ \text{ and } |\pi(z)| \le (z^{\mathrm{t}} M z)^{\frac{3}{2}} \text{ for all } z \in \mathbb{R}^2\}.$$

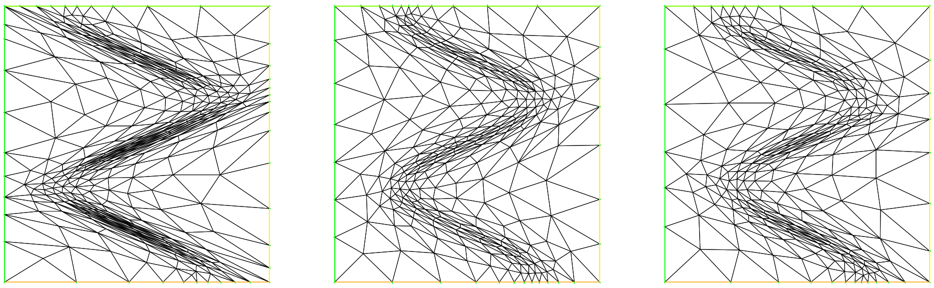

Figure 3: The meshes $\mathcal{T}^*_{H^1,\mathbb{P}_1}$, $\mathcal{T}_{H^1,\mathbb{P}_1}$ and $\mathcal{T}_{H^1,\mathbb{P}_2}$ adapted to f_δ (500 triangles only).

We have obtained the following results, which confirm the use of the metric adapted to a given norm and interpolation degree produces the triangulation that yields the smallest interpolation error in this case (at least among these four triangulations).

$\#\mathcal{T} = 2000$	$\mathcal{T}_{H^1,\mathbb{P}_1}$	$\mathcal{T}_{H^1,\mathbb{P}_2}$	$\mathcal{T}_{L^2,\mathbb{P}_1}$	$\mathcal{T}_{L^2,\mathbb{P}_2}$
$\lvert f_\delta - \mathrm{I}^1_{\mathcal{T}} f_\delta\rvert_{H^1_0}$	1.35	1.47	1.43	1.63
$10\lvert f_\delta - \mathrm{I}^2_{\mathcal{T}} f_\delta\rvert_{H^1_0}$	1.66	1.17	1.89	1.47
$10^2\lVert f_\delta - \mathrm{I}^1_{\mathcal{T}} f_\delta\rVert_{L^2}$	1.54	2.73	0.759	1.18
$10^4\lVert f_\delta - \mathrm{I}^2_{\mathcal{T}} f_\delta\rVert_{L^2}$	6.64	6.61	4.73	3.17

(4.79)

4.5 Quality of a triangulation generated from a metric

Given a metric $h : \Omega \to S_2^+$, there does not always exists a triangulation $\mathcal{T}$ adapted to h, i.e. satisfying (4.69) for some constant $C_1 \geq 1$ not too large. Such a triangulation exists only if h satisfies some constraints which are analyzed in [14]. Instead of analysing the metric h prior to the process of mesh generation, we choose here the simpler option of evaluating a posteriori the quality of a triangulation $\mathcal{T}$.

Since we are interested in the $H^1 = W^{1,2}$ semi norm we define following (2.34)

$$S(\mathcal{T}) := \left(\frac{1}{\#\mathcal{T}} \sum_{T\in\mathcal{T}} S(T)^2\right)^{\frac{1}{2}}.$$

For all $T \in \mathcal{T}$ we define $h_T := h(\mathrm{bary}(T)) \in S_2^+$. We also define the sets

$$E := \left\{\ln\left(|T|\sqrt{\det h_T}\right) \ ;\ T \in \mathcal{T}\right\} \text{ and } F := \left\{\rho\left(\sqrt{h_T}(T)\right) \ ;\ T\in\mathcal{T}\right\}.$$

According to (4.69), the quality of $\mathcal{T}$ is reflected by the quantities

$$\exp(\max E - \min E) \quad \text{and} \quad \max F.$$

However these quantities give a rather pessimistic account of the adaptation of $\mathcal{T}$ to h, and *heuristically* we find it more fruitful to consider averages. We therefore define

$$\rho(\mathcal{T}, h) := \frac{1}{\#(\mathcal{T})} \sum_{T\in\mathcal{T}} \rho\left(\sqrt{h_T}(T)\right).$$

and

$$\sigma(\mathcal{T}, h) := \exp\left(\frac{1}{\#(E)} \sum_{e\in E} \left| e - \frac{1}{\#(E)} \sum_{e\in E} e\right|\right).$$

The following table shows that the quantities $S(\mathcal{T})$, $\rho(\mathcal{T}, h)$ and $\sigma(\mathcal{T}, h)$ are abnormally large for the triangulation $\mathcal{T}^*_{H^1,\mathbb{P}_1}$ generated from the metric $h^*_{H^1,\mathbb{P}_1}$ but reasonable for the triangulations generated

$\mathcal{T}_{H^1,\mathbb{P}_1}$ and $\mathcal{T}_{H^1,\mathbb{P}_2}$ generated from the metrics (4.78).

$\#\mathcal{T} = 2000$	$\mathcal{T}^*_{H^1,\mathbb{P}_1}$	$\mathcal{T}_{H^1,\mathbb{P}_1}$	$\mathcal{T}_{H^1,\mathbb{P}_2}$
$S(\mathcal{T})$	14.2	3.14	4.04
$\rho(\mathcal{T}, h)$	10.6	6.02	4.18
$\sigma(\mathcal{T}, h)$	2.39	2.25	1.70

In practice $\mathcal{T}^*_{H^1,\mathbb{P}_1}$ led to a poor interpolation error, contrary to $\mathcal{T}_{H^1,\mathbb{P}_1}$. We believe that the poor quality of $\mathcal{T}^*_{H^1,\mathbb{P}_1}$ is due to the excessively wild behavior of the metric $h^*_{H^1,\mathbb{P}_1}$ and not to a deficiency of the excellent mesh generator BAMG [18].

5 Polynomial equivalents of the shape function

The optimal error estimates established in Theorem 1.1 involve the quantity $L_{m,p}(\frac{d^m f}{m!})$. The function $\pi \mapsto L_{m,p}(\pi)$ is obtained by solving an optimization problem, and it does not have an explicit analytic expression in terms of the coefficients of $\pi \in \mathbb{H}_m$. In this section, we introduce quantities which are equivalent to $L_m(\pi)$, and therefore to $L_{m,p}(\pi)$ for all p, and which can be written in analytic form in terms of the coefficients of $\pi \in \mathbb{H}_m$.

Given a pair of non negative functions Q and R on $\mathbb{H}_m$ we write $Q \sim R$ if and only if there exists a constant $C > 0$ such that $C^{-1}Q \leq R \leq CQ$ uniformly on $\mathbb{H}_m$. We sometimes slightly abuse notations and write $Q(\pi) \sim R(\pi)$. We say that a function Q is a polynomial on $\mathbb{H}_m$ if there exists a polynomial P of $m+1$ real variables such that for all $a_0, \cdots, a_m \in \mathbb{R}$,

$$Q\left(\sum_{i=0}^{m} a_i x^i y^{m-i}\right) = P(a_0, \cdots, a_m).$$

We define $\deg Q := \deg P$, and we say that Q is homogeneous if P is homogeneous. For all $m \geq 2$, we shall build an homogeneous polynomial Q on $\mathbb{H}_m$ such that

$$L_m \sim \sqrt[r]{|Q|} \ \text{ with } \ r := \deg Q, \tag{5.80}$$

where the constants in the equivalence only depend on m.

We first introduce for all $\pi \in \mathbb{H}_m$ the set

$$\mathcal{B}_\pi := \{B \in \mathrm{M}_2(\mathbb{R}) \ ; \ |\pi(z)| \leq |Bz|^m \text{ for all } z \in \mathbb{R}^2\},$$

and the function

$$K^{\mathcal{E}}_m(\pi) := \inf\{|\det B|^{\frac{m}{2}} \ ; \ B \in \mathcal{B}_\pi\}. \tag{5.81}$$

According to Lemma 2.5 we have for any $m \geq 2$

$$L_m(\pi) \sim \sqrt{K^{\mathcal{E}}_{2m-2}(|\nabla\pi|^2)} \tag{5.82}$$

where $|\nabla\pi|^2 = (\partial_x\pi)^2 + (\partial_y\pi)^2 \in \mathbb{H}_{2m-2}$. The function $K^{\mathcal{E}}_m$ has been extensively studied in [15] (more precisely, due to different conventions, the function studied under this name in [15] is $\boldsymbol{\pi}^{-\frac{m}{2}} K^{\mathcal{E}}_m$).

For $m = 2$ and $m = 3$, it was proved in [15] that

$$K^{\mathcal{E}}_2(\pi) \sim \sqrt{|\det[\pi]|}, \tag{5.83}$$

and

$$K^{\mathcal{E}}_3(\pi) \sim \sqrt[4]{|\operatorname{disc}(\pi)|} \tag{5.84}$$

where $\operatorname{disc}(\pi)$ denotes the discriminant of a polynomial $\pi \in \mathbb{H}_3$, namely

$$\operatorname{disc}(ax^3 + bx^2y + cxy^2 + dy^3) = b^2c^2 - 4ac^3 - 4b^3d + 18abcd - 27a^2d^2.$$

More generally, it was proved in [15] that for all $m \geq 2$, the function $K^{\mathcal{E}}_m$ has an equivalent of the form $\sqrt[r]{|Q|}$, where Q is an homogeneous polynomial of degree r on $\mathbb{H}_m$. Combining this result with (5.82) we obtain the main result of this section.

Proposition 5.1 *Let $m \geq 2$ and let Q be an homogeneous polynomial on $\mathbb{H}_{2m-2}$ such that $K^{\mathcal{E}}_{2m-2} \sim \sqrt[r]{|Q|}$, where $r = \deg Q$. Let Q_* be the polynomial defined for all $\pi \in \mathbb{H}_m$ by*

$$Q_*(\pi) := Q(|\nabla\pi|^2).$$

then $L_m \sim \sqrt[2r]{Q_*}$ on $\mathbb{H}_m$.

Let $\pi \in \mathbb{H}_2$ and let us observe that $|\nabla\pi(z)|^2 = |2[\pi]z|^2 = 4z^{\mathrm{t}}[\pi]^2 z$. Using (5.83) we therefore obtain

$$L_2(\pi) \sim \sqrt{K^{\mathcal{E}}_2(|\nabla\pi|^2)} \sim \sqrt{\sqrt{\det(4[\pi]^2)}} = 2\sqrt{|\det[\pi]|}. \tag{5.85}$$

The construction suggested by Theorem 5.1 uses an equivalent of $K^{\mathcal{E}}_{2m-2}$ to produce an equivalent to L_m. Unfortunately, as m increases, the practical construction of Q such that $\sqrt[r]{|Q|}$ is equivalent to $K^{\mathcal{E}}_m$ becomes more involved and the degree r quickly raises. In the following theorem, we build an equivalent to L_m from an equivalent of $K^{\mathcal{E}}_{m-1}$ instead of $K^{\mathcal{E}}_{2m-2}$ which is therefore simpler.

Theorem 5.2 *Let $m \geq 3$ and let Q be an homogeneous polynomial on $\mathbb{H}_{m-1}$ such that $K^{\mathcal{E}}_{m-1} \sim \sqrt[r]{|Q|}$, where $r = \deg Q$. Let $(Q_k)_{0 \leq k \leq r}$ be the homogeneous polynomials of degree r on $\mathbb{H}_{m-1} \times \mathbb{H}_{m-1}$ such that for all $u, v \in \mathbb{R}$ and all $\pi_1, \pi_2 \in \mathbb{H}_m$ we have*

$$Q(u\pi_1 + v\pi_2) = \sum_{0 \leq k \leq r} \binom{r}{k} u^k v^{r-k} Q_k(\pi_1, \pi_2), \tag{5.86}$$

where $\binom{r}{k} := \frac{r!}{k!(r-k)!}$. Let Q_* be the polynomial defined for all $\pi \in \mathbb{H}_m$ by

$$Q_*(\pi) := \sum_{0 \leq k \leq r} \binom{r}{k} Q_k\left(\partial_x\pi, \partial_y\pi_2\right)^2$$

then $L_m \sim \sqrt[2r]{Q_*}$ on $\mathbb{H}_m$.

Proof: See Appendix. ◇

Using this construction and (5.83) we obtain an equivalent of L_3 as follows. Let $\pi_1 = ax^2 + 2bxy + cy^2$ and $\pi_2 = a'x^2 + 2b'xy + c'y^2$ be two elements of $\mathbb{H}_2$. We obtain

$$\begin{aligned} \det([u\pi_1 + v\pi_2]) &= (ua + va')(uc + vc') - (ub + vb')^2 \\ &= u^2(ac - b^2) + uv(ac' + a'c - 2bb') + v^2(a'c' - b'^2). \end{aligned}$$

Applying the construction of Theorem 5.2 to $\pi = ax^3 + 3bx^2y + 3cxy^2 + dy^3 \in \mathbb{H}_3$ we obtain

$$L_3(\pi) \sim 3\sqrt[4]{(ac - b^2)^2 + (ad - bc)^2/2 + (bd - c^2)^2}. \tag{5.87}$$

Remarking that

$$2[(ac - b^2)^2 + (ad - bc)^2/2 + (bd - c^2)^2] = (a^2 + 2b^2 + c^2)(b^2 + 2c^2 + d^2) - (ab + 2bc + cd)^2,$$

and using equation (4.73) we obtain that $L_3(\pi) \sim \sqrt{\det \mathcal{M}_3(\pi)}$. This point was announced in §4.1 and establishes that the map $\mathcal{M}_3$ defined in (4.72) can be used for optimal mesh adaptation for quadratic finite elements.

Using (5.84) and the construction of Theorem 5.2, we also obtain an equivalent of $L_4(\pi)$

$$\begin{aligned} L_4(\pi)^8 \sim\ & (3b^2c^2 - 4ac^3 - 4b^3d + 6abcd - a^2d^2)^2 \\ +\ & (2bc^3 - 6ac^2d + 4abd^2 - 4b^3e + 6abce - 2a^2de)^2/4 \\ +\ & (3c^4 - 6bc^2d + 8b^2d^2 - 6acd^2 - 6b^2ce + 6ac^2e + 2abde - a^2e^2)^2/6 \\ +\ & (2c^3d - 4ad^3 - 6bc^2e + 4b^2de + 6acde - 2abe^2)^2/4 \\ +\ & (3c^2d^2 - 4bd^3 - 4c^3e + 6bcde - b^2e^2)^2. \end{aligned}$$

The following proposition identifies the polynomials $\pi \in \mathbb{H}_m$ for which $L_m(\pi) = 0$, and therefore the values of $d^m f$ for which anisotropic mesh adaptation may lead to *super-convergence*.

Proposition 5.3 *Let $m \geq 2$ and let $t_m := \left\lfloor \frac{m+3}{2} \right\rfloor$. Then for all $\pi \in \mathbb{H}_m$,*

$$L_m(\pi) = 0 \text{ if and only if } \pi = (\alpha x + \beta y)^{t_m} \tilde{\pi} \text{ for some } \alpha, \beta \in \mathbb{R} \text{ and } \tilde{\pi} \in \mathbb{H}_{m-t_m}. \tag{5.88}$$

Proof: According to (5.82), $L_m(\pi) = 0$ if and only if $K^{\mathcal{E}}_{2m-2}(|\nabla\pi|^2) = 0$. On the other hand, it was proved in [15] that $K^{\mathcal{E}}_{2m-2}(\pi_*) = 0$ if and only if $\pi_* \in \mathbb{H}_{2m-2}$ has a linear factor of multiplicity m. Therefore $L_m(\pi) = 0$ if and only $|\nabla\pi|^2$ is a multiple of l^m, where l is of the form $l = \alpha x + \beta y$.

Let us first assume that $|\nabla\pi|^2$ has such a form. Clearly $(\partial_x\pi)^2$ and $(\partial_y\pi)^2$ are both multiples of l^m. Therefore $\partial_x\pi$ and $\partial_y\pi$ are multiples of l^s where s is an integer such that $2s \geq m$, hence $s \geq t_m - 1$. We therefore have

$$\partial_x\pi = l^s\pi_1 \text{ and } \partial_y\pi = l^s\pi_2 \text{ where } \pi_x, \pi_y \in \mathbb{H}_{m-s}$$

Recalling that $l = \alpha x + \beta y$ we obtain

$$0 = \partial^2_{yx}\pi - \partial^2_{xy}\pi = l^s(\partial_y\pi_1 - \partial_x\pi_2) + sl^{s-1}(\beta\pi_1 - \alpha\pi_2),$$

hence $\beta\pi_1 - \alpha\pi_2$ is a multiple of l. Since π is homogenous of degree m it obeys the Euler identity $m\pi(z) = \langle z, \nabla\pi(z)\rangle$ for all $z \in \mathbb{R}^2$. Assuming without loss of generality that $\alpha \neq 0$, we therefore obtain

$$m\pi(x,y) = l^s(x\pi_1 + y\pi_2) = l^s\left((\alpha x + \beta y)\frac{\pi_1}{\alpha} + \frac{y}{\alpha}(\alpha\pi_2 - \beta\pi_1)\right)$$

which shows that π is a multiple of l^{s+1}, hence of l^{t_m}.

Conversely if π is a multiple of l^{t_m} then $\partial_x\pi$ and $\partial_y\pi$ are both multiples of l^{t_m-1}. Since $2(t_m-1) \geq m$ the polynomial $|\nabla\pi|^2$ is a multiple of l^m which concludes the proof. ⋄

6 Extension to higher dimension

This section partially extends the results exposed in the previous sections to functions of d variables. We give in §6.1 the generalisations of the shape function $L_{m,p}$ and of the measure of sliverness S.

Subsection §6.2 is devoted to interpolation error estimates. We prove a local d-dimensional error estimate in Theorem 6.6 which generalises Theorem 2.6. We then establish an asymptotic lower error estimate in Theorem 6.7 which generalises Theorem 1.2. We give sufficient conditions under which the interpolation on a d-dimensional mesh $\mathcal{T}$ achieves this optimal lower bound up to a multiplicative constant. However due to technical issues linked to the measure of sliverness S we were not able to construct such meshes, and we therefore state the upper bound as a conjecture.

We discuss in subsection §6.3 the construction of optimal metrics for practical mesh generation. We partially extend the results of §4 and raise open questions.

6.1 Generalisation of the shape function and of the measure of sliverness.

We extend in this section the tools used in our analysis of optimally adapted triangulations to arbitrary dimension d. We begin with the spaces of polynomials. Let

$$\mathbb{H}_{m,d} := \mathrm{Span}\{x_1^{\alpha_1} \cdots x_d^{\alpha_d} \ ; \ |\alpha| = m\} \text{ and } \mathbb{P}_{m,d} := \mathrm{Span}\{x_1^{\alpha_1} \cdots x_d^{\alpha_d} \ ; \ |\alpha| \leq m\}$$

For any simplex T the Lagrange interpolation operator $\mathrm{I}_T^m : C^0(T) \to \mathbb{P}_{m,d}$ is defined by imposing $f(\gamma) = \mathrm{I}_T^m f(\gamma)$ for all points γ with barycentric coordinates in the set $\{0, \frac{1}{m}, \frac{2}{m}, \cdots, 1\}$ with respect to the vertices of T. For all $\pi \in \mathbb{H}_{m,d}$ we define

$$L_{m,d,p}(\pi) := \inf_{|T|=1} \|\nabla\pi - \nabla \mathrm{I}_T^{m-1}\pi\|_{L^p(T)},$$

where the infimum is taken on the set of d-dimensional simplices of unit volume. Similarly to (1.12) the functions $L_{m,d,p}$, $1 \leq p \leq \infty$, are uniformly equivalent on $\mathbb{H}_{m,d}$. We define $L_{m,d} := L_{m,d,\infty}$.

The distance defined at (2.16) between triangles extends easily to simplices. Given two d-dimensional simplices T, T' there are precisely $(d+1)!$ affine transformations Ψ such that $\Psi(T) = T'$. For each such Ψ, we denote by ψ its linear part and we define

$$\mathrm{d}(T,T') := \ln\Big(\inf\{\kappa(\psi)\ ;\ \Psi(T) = T'\}\Big).$$

We say that a d-dimensional simplex T is acute if the exterior normals n, n' to any two faces F, F' of T have a negative scalar product $\langle n, n'\rangle$. In other words if all faces of T form acute dihedral angles. We denote the set of acute simplices by $\mathbb{A}$ and generalise the measure of sliverness to arbitrary dimension d as follows

$$S(T) := \exp \mathrm{d}(T, \mathbb{A}) = \inf\{\kappa(\psi)\ ;\ \Psi(T) \in \mathbb{A}\}. \tag{6.89}$$

Similarly to (2.17), the quantity $S(T)$ reflects the distance from a simplex T to the set of acute simplexes $\mathbb{A}$. The definition (6.89) of $S(T)$ raises a legitimate question : how to produce an affine transformation Ψ such that $\Psi(T)$ has acute angles, and $\kappa(\psi)$ is comparable to $S(T)$? This question is answered by the following proposition. In the sequel, we use the notation

$$\alpha_d := \sqrt{\frac{d(d+1)}{2}} \tag{6.90}$$

Proposition 6.1 *Let T be a d-dimensional simplex with vertices $(v_i)_{0\leq i\leq d}$. We define the symmetric matrix*

$$M_T := \sum_{0\leq i<j\leq d} e_{ij} e_{ij}^{\mathrm{t}}, \text{ where } e_{ij} := \frac{v_i - v_j}{|v_i - v_j|}. \tag{6.91}$$

Then $M_T^{-\frac{1}{2}}(T)$ is an acute simplex and

$$S(T) \leq \kappa(\sqrt{M_T}) \leq \alpha_d S(T). \tag{6.92}$$

Proof: We only observe here that $1 \leq \|\sqrt{M_T}\| = \sqrt{\|M_T\|} \leq \alpha_d$ (since this is used further in this section), and we leave the rest of the proof to the appendix. $\diamond$

Remark 6.2 *In the paper [12] an alternative measure of sliverness $S'(T)$ of a simplex T is introduced, and defined as*

$$S'(T) := \left(\inf_{|u|=1} \max_{i<j} |\langle u, e_{i,j}\rangle|\right)^{-1}.$$

We now prove that this quantity is equivalent to $S(T)$. For any $u \in \mathbb{R}^d$ we have

$$\max_{i<j} |\langle u, e_{i,j}\rangle| \leq \sqrt{\sum_{i<j} \langle e_{ij}, u\rangle^2} = \sqrt{u^{\mathrm{t}} M_T u} = \left|M_T^{\frac{1}{2}} u\right| \leq \alpha_d \max_{i<j} |\langle u, e_{i,j}\rangle|$$

its follows easily that $\alpha_d^{-1} S'(T) \leq \|M_T^{-\frac{1}{2}}\| \leq S'(T)$. Hence $\alpha_d^{-1} S(T) \leq S'(T) \leq \alpha_d^2 S(T)$ for all T. Our approach therefore introduces a new geometrical interpretation to the quantity S' introduced in [12], as the distance from a given simplex to the set of acute simplices.

The following lemma generalises Lemma 2.2 and shows that the interpolation process is stable in the L^∞ norm of the gradient if the measure of sliverness is controlled. Let us mention that a slightly different version of this lemma can be found in [12], yet not exactly adapted to our purposes.

Lemma 6.3 *For all $m \geq 2$ and $d \geq 2$ there exists a constant $C = C(m,d)$ such that for any d-dimensional simplex T and any $f \in C^1(T)$, one has*

$$\|\nabla \mathrm{I}_T^m f\|_{L^\infty(T)} \leq C S(T) \|\nabla f\|_{L^\infty(T)}. \tag{6.93}$$

Proof: The proof this lemma is extremely similar to the proof of Lemma (2.2). Let T_0 be the simplex which vertices are the origin and the canonical basis of $\mathbb{R}^d$. For the same reason as in Lemma 2.2, if a function $g(x_1, x_2, \cdots, x_d)$ does not depend on the coordinate x_d, then $\mathrm{I}_{T_0}^{m-1} g$ does not depend on x_d either. Using the same reasonning as in Lemma 2.2 we obtain that there exists a constant $C_0 = C_0(m, d)$ such that for all $f \in C^1(T_0)$

$$\left\| \frac{\partial \mathrm{I}_{T_0}^m f}{\partial x_d} \right\|_{L^\infty(T_0)} \leq C_0 \left\| \frac{\partial f}{\partial x_d} \right\|_{L^\infty(T_0)}.$$

Again similarly to the proof of Lemma 2.2 we obtain using a change of variables that for any simplex T, any $f \in C^1(T)$ and any edge vector u of T

$$\|\langle u, \nabla \mathrm{I}_T^m f \rangle\|_{L^\infty(T)} \leq C_0 \|\langle u, \nabla f \rangle\|_{L^\infty(T)}.$$

We use the notations of Proposition 6.1 and we define a norm $|v|_T$ on $\mathbb{R}^d$ by

$$|v|_T^2 := v^{\mathrm{t}} M_T v = \sum_{0 \leq i < j \leq d} \langle v, e_{ij} \rangle^2.$$

Observe that

$$\|M_T^{-\frac{1}{2}}\|^{-1} |v| \leq |v|_T \leq \|M_T^{\frac{1}{2}}\| |v|. \tag{6.94}$$

Then, since e_{ij} is proportional to an edge vector of T,

$$\begin{aligned} \| \, |\nabla \mathrm{I}_T^m f|_T \|_{L^\infty(T)}^2 &\leq \sum_{0 \leq i < j \leq d} \|\langle e_{ij}, \nabla \mathrm{I}_T^m f \rangle\|_{L^\infty(T)}^2 \\ &\leq C_0 \sum_{0 \leq i < j \leq d} \|\langle e_{ij}, \nabla f \rangle\|_{L^\infty(T)}^2 \leq C_0 \alpha_d^2 \| \, |\nabla f|_T \|_{L^\infty(T)}^2. \end{aligned}$$

Combining this result with (6.94) we obtain $\|M_T^{-\frac{1}{2}}\|^{-1} \| \, |\nabla \mathrm{I}_T^m f|_T \|_{L^\infty(T)} \leq C_0 \alpha_d^2 \|M_T^{\frac{1}{2}}\| \| \, |\nabla f|_T \|_{L^\infty(T)}$ and we conclude the proof using (6.92). ◇

The oscillation of the gradient of the interpolated function is an important problem encountered by numerical methods that try to take advantage of highly anisotropic meshes, see the discussion in [16]. As the previous lemma shows, such oscillations are kept under control if $S(T)$ is bounded on the mesh of interest. For checking this property in pratical situations one needs an equivalent of the sliverness S that can be computed at low numerical cost. The formula (6.89) is clearly not adapted, since it involves a complicated optimisation procedure. Instead we propose to use

$$\hat{S}(T) := \sqrt{\mathrm{Tr}(M_T^{-1})}. \tag{6.95}$$

and we observe that $\|M_T^{-\frac{1}{2}}\| \leq \hat{S}(T) \leq \sqrt{d} \|M_T^{-\frac{1}{2}}\|$. Recalling that $1 \leq \|M_T\| \leq \alpha_d$ we obtain

$$\alpha_d^{-1} \hat{S}(T) \leq S(T) \leq \sqrt{d} \alpha_d \hat{S}(T)$$

Note that $\hat{S}(T)$ has an analytic expression in terms of the coordinates of $T:$ the square root of the ratio of two polynomials in the positions of the vertices of T.

Remark 6.4 *We illustrate the sharpness of inequality (6.93) in a simple example. Let x, y, z be the coordinates on $\mathbb{R}^3$ and let $\pi_0 := x^2 \in \mathbb{H}_{2,3}$. Let T_λ be the tetrahedron of vertices $(-\lambda, 0, 0)$, $(\lambda, 0, 0)$, $(\lambda, 1, 0)$ and $(0, 0, 1)$. Simple computations show that*

$$\|\nabla \mathrm{I}_{T_\lambda}^1 \pi_0\|_{L^\infty(T_\lambda)} = \lambda^2, \ \|\nabla \pi_0\|_{L^\infty(T_\lambda)} = 2\lambda \textit{ and } \lim_{\lambda \to \infty} \frac{\hat{S}(T_\lambda)}{\lambda} = \sqrt{\frac{5}{7}}.$$

Let T'_λ be defined by replacing the vertex $(-\lambda, 0, 0)$ of T_λ with $(0, 0, 0)$. Then

$$\|\nabla \mathrm{I}_{T_\lambda}^1 \pi_0\|_{L^\infty(T'_\lambda)} = \lambda, \ \|\nabla \pi_0\|_{L^\infty(T'_\lambda)} = 2\lambda \textit{ and } \lim_{\lambda \to \infty} \hat{S}(T'_\lambda) = \frac{3}{2}.$$

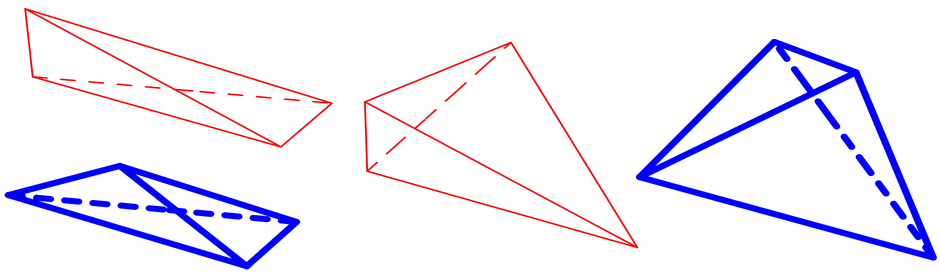

Figure 4: Examples of Good anisotropy (Thin lines, $S(T) \sim 1$), and Bad anisotropy (Thick lines, $S(T) \gg 1$) .

Hence the simplices T_λ and T'_λ have very different interpolation properties for large λ, although they have a similar aspect ratio. They are representatives of "bad" and "good" anisotropy respectively. The tetrahedrons $T_{\frac{3}{2}}$ and $T'_{\frac{3}{2}}$ are illustrated on the left of Figure 4, bottom and top respectively.

For any d-dimensional simplex T, we define its measure of non degeneracy by

$$\rho(T) := \frac{\operatorname{diam}(T)^d}{|T|}.$$

We now generalize inequality (2.28). Let T_* be a fixed d-dimensional acute simplex. For any d-dimensional simplex T let $\psi \in \mathrm{GL}_d$ and $z \in \mathbb{R}^d$ be such that $T = z + \psi(T_*)$. Since T_* is acute, we obtain

$$S(T) \leq \kappa(\psi) \leq \|\psi\|^d |\det \psi|^{-1} \leq \frac{\operatorname{diam}(T)^d}{\mu(T_*)^d} \frac{|T_*|}{|T|} = C(d)\rho(T).$$

where $\mu(T_*)$ is the diameter of the largest ball incribed in T_*, and where we have used the inequality $|\det(\psi^{-1})| \geq \|\psi^{-1}\| \|\psi\|^{-(d-1)}$. This last inequality can be derived by using the singular value decomposition $\psi = U\mathrm{diag}(\lambda_1, \cdots, \lambda_d)V$ with $0 < \lambda_i \leq \lambda_{i+1}$ and noting that $\|\psi\| = \lambda_d$ and $\|\psi^{-1}\| = \lambda_1^{-1}$.

Generalizing (2.22), we define for all $\pi \in \mathbb{H}_{m,d}$,

$$\mathcal{A}_\pi := \{A \in \mathrm{M}_d(\mathbb{R}) \ ; \ |\nabla\pi(z)| \leq |Az|^{m-1} \text{ for all } z \in \mathbb{R}^d\}.$$

Geometrically, one has $A \in \mathcal{A}_\pi$ if and only if the ellipsoid $\{z \in \mathbb{R}^d \ ; \ |Az| \leq 1\}$ is included in the algebraic set $\{z \in \mathbb{R}^d \ ; \ |\nabla\pi(z)| \leq 1\}$. This leads us to the generalisation of Lemma 2.5.

Lemma 6.5 *For all $m \geq 2$ and $d \geq 2$ there exists a constant $C = C(m,d)$ such that for all $\pi \in \mathbb{H}_{m,d}$, we have*

$$C^{-1} L_{m,d}(\pi) \leq \inf\{|\det A|^{\frac{m-1}{d}} \ ; \ A \in \mathcal{A}_\pi\} \leq C L_{m,d}(\pi).$$

Proof: The proof of this lemma is completely similar to the proof of its bidimensional version Lemma 2.5. The only point that needs to be properly generalized is the following : Given a matrix $A \in \mathrm{GL}_d$, how to construct an acute simplex $T = T(A)$ such that $\rho(A(T))$ is bounded independently of A?

The following construction is not the simplest but will be useful in our subsequent analysis. Let $A = UDV$, be the singular value decomposition of A, where U, V are orthogonal matrices and D is a diagonal matrix with positive diagonal entries $(\lambda_i)_{1 \leq i \leq d}$. We define the Kuhn simplex T_0

$$T_0 := \{x \in [0,1]^d \ ; \ x_1 \geq x_2 \geq \cdots \geq x_d\} \quad \text{and} \quad T := V^{\mathrm{t}} D^{-1} T_0. \tag{6.96}$$

Then $\rho(A(T)) = \rho(U(T_0)) = \rho(T_0) = d! d^{d/2}$ which is independent of A. We now show that T is an acute simplex.

Let $(e_1, \cdots, e_d)$ be the canonical basis of $\mathbb{R}^d$, and let by convention $e_0 = e_{d+1} = 0$. For $0 \leq i \leq d$, an easy computation shows that the the exterior normal to the face F_i of T_0, opposite to the vertex $v_i = \sum_{0 \leq k \leq i} e_k$, is $n_i = \frac{e_i - e_{i+1}}{\|e_i - e_{i+1}\|}$. It follows that the exterior normal n'_i to the face $D^{-1}(F_i)$ of the simplex $D^{-1}(T_0)$ is

$$n'_i = \frac{D(n_i)}{|D(n_i)|} = \frac{\lambda_i e_i - \lambda_{i+1} e_{i+1}}{|\lambda_i e_i - \lambda_{i+1} e_{i+1}|}.$$

Hence $\langle n'_i, n'_j \rangle = 0$ if $|i-j| > 1$, and $\langle n'_i, n'_{i+1} \rangle < 0$ for all $0 \le i \le d-1$. It follows that the simplex $D^{-1}(T_0)$ is acute, and therefore $T = V^{\mathrm{t}} D^{-1}(T_0)$ is also acute since V is a rotation. ◇

6.2 Generalisation of the error estimates

We present in this section the generalisation to higher dimension of our anisotropic error estimates. We prove a local error estimate in theorem 6.6 and an asymptotic lower estimate in 6.7. We also point out in conjecture 6.8 a technical point which, if proved, would lead to the optimal asymptotic upper estimates (6.101) and (6.102).

Theorem 6.6 *For all $m \ge 2$ and $d \ge 2$ there exists a constant $C = C(m,d)$ such that for all $\pi \in \mathbb{H}_{m,d}$, all $A \in \mathcal{A}_\pi$ and any simplex T we have*

$$|\pi - \mathrm{I}_T^{m-1} \pi|_{W^{1,p}(T)} \le C|T|^{\frac{1}{\tau}} S(T)\, \rho(A(T))^{\frac{m-1}{d}} |\det A|^{\frac{m-1}{d}}, \tag{6.97}$$

where $\frac{1}{\tau} := \frac{m-1}{d} + \frac{1}{p}$. Furthermore for any $g \in C^m(T)$ we have

$$|g - \mathrm{I}_T^{m-1} g|_{W^{1,p}(T)} \le C|T|^{\frac{1}{\tau}} S(T)\, \rho(T)^{\frac{m-1}{d}} \|d^m g\|_{L^\infty(T)}.$$

Proof: It is a straightforward generalization of the proof of Theorem 2.6. ◇

Combining these two estimates, we can obtain a mixed estimate similar to (2.31), with the new value of τ and the generalised S and ρ. For all $m \ge 2$ and $d \ge 2$ there exists a constant $C = C(m,d)$ such that for any simplex T, any $f \in C^m(T)$, any $\pi \in \mathbb{H}_m$ and any $A \in \mathcal{A}_\pi$

$$|f - \mathrm{I}_T^{m-1} f|_{W^{1,p}(T)} \le C|T|^{\frac{1}{\tau}} S(T) \left(\rho(A(T))^{\frac{m-1}{d}} |\det A|^{\frac{m-1}{d}} + \rho(T)^{\frac{m-1}{d}} \|d^m f - d^m \pi\|_{L^\infty(T)} \right). \tag{6.98}$$

This leads us to a straightforward generalisation of the points (i) to (iv) exposed in (2.32). Similarly to the bidimensional case (2.38) if a triangulation $\mathcal{T}$ meets these requirements, then it satisfies the error estimate

$$\#(\mathcal{T})^{\frac{m-1}{d}} |f - \mathrm{I}_{\mathcal{T}}^{m-1} f|_{W^{1,p}(\Omega)} \le C \|L_m(\pi_z) + \varepsilon\|_{L^\tau(\Omega)}. \tag{6.99}$$

Generalizing (1.7), we say that a sequence $(\mathcal{T}_N)_{N \ge N_0}$ of simplicial meshes of a d-dimensional polygonal domain is admissible if $\#(\mathcal{T}_N) \le N$ and there exists a constant $C_A > 0$ such that

$$\sup_{T \in \mathcal{T}_N} \mathrm{diam}(T) \le C_A N^{-\frac{1}{d}}.$$

Similar to (1.8), it can shown that (6.99) cannot be improved for an admissible sequence of triangulations, in the following asymptotical sense.

Theorem 6.7 *Let $(\mathcal{T}_N)_{N \ge N_0}$ be an admissible sequence of triangulations of a domain Ω, let $f \in C^m(\Omega)$ and $1 \le p < \infty$. Then*

$$\liminf_{N \to \infty} N^{\frac{m-1}{d}} |f - \mathrm{I}_{\mathcal{T}_N}^{m-1} f|_{W^{1,p}(\Omega)} \ge \left\| L_{m,d,p}\left(\frac{d^m f}{m!}\right) \right\|_{L^\tau(\Omega)} \tag{6.100}$$

where $\frac{1}{\tau} := \frac{m-1}{d} + \frac{1}{p}$.

Proof: It is is identical to the proof of the bidimensional estimate (1.8), which is exposed in §3.1. ◇

In contrast, the upper estimates (1.5) and (1.9) do not generalize easily to higher dimension. A first problem is that the bidimensional mesh $\mathcal{P}_T$ defined in Equation (3.51) has no equivalent in higher dimension, in the sense that we cannot exactly tile the space by simplices of optimal shape. We may however build a tiling made of near optimal simplices, based on the following procedure: for any permutation $\sigma \in \Sigma_d$ of $\{1, \cdots, d\}$ we define

$$T_\sigma := \{x \in [0,1]^d \ ; \ x_{\sigma(1)} \geq \cdots \geq x_{\sigma(d)}\}.$$

Let $A \in \mathrm{GL}_d(\mathbb{R})$, and let $A = UDV$ be the singular value decomposition of A, where U and V are unitary and D is diagonal. We define

$$\mathcal{P}_A := \{V^{\mathrm{t}} D^{-1}(T_\sigma + z) \ ; \ \sigma \in \Sigma_d, \ z \in \mathbb{Z}^{\mathrm{d}}\},$$

which is a tiling of $\mathbb{R}^d$ built of *acute* simplices T satisfying $\rho(A(T)) = d! d^{d/2}$ (these properties are established in the proof of Lemma 6.3). Using such tiling, we would like to build partitions $\mathcal{P}_{A,n}(R)$ of any d-dimensional simplex R, with properties similar to those expressed in Lemma 3.2 for the triangulations $\mathcal{P}_{T,n}(R)$. At the present stage we do not know how to properly adapt the construction of $\mathcal{P}_{A,n}(R)$ near the boundary of R in order to respect the condition on the measure of sliverness. The following conjecture, if established, would serve as a generalisation of Lemma 3.2.

Conjecture 6.8 *Let R be a d-dimensional simplex, and let $A \in \mathrm{GL}_d(\mathbb{R})$. There exists a sequence $(\mathcal{P}_{A,n}(R))_{N \geq 0}$, of conformal triangulations of R such that*

- *Nearly all the elements of $\mathcal{R}_N$ belong to $\mathcal{P}_{A,n} := \frac{1}{n}\mathcal{P}_A$, in the sense that*

$$\lim_{n\to\infty} \frac{\#(\mathcal{P}^1_{A,n}(R))}{n^d} = \frac{d!|R|}{|\det A|} \quad \text{and} \quad \lim_{n\to\infty} \frac{\#(\mathcal{P}^2_{A,n}(R))}{n^d} = 0.$$

  where

$$\mathcal{P}^1_{A,n}(R) := \mathcal{P}_{A,n}(R) \cap \mathcal{P}_{A,n} \quad \text{and} \quad \mathcal{P}^2_{A,n}(R) := \mathcal{P}_{A,n}(R) \setminus \mathcal{P}_{A,n}$$

- *The restriction of $\mathcal{P}_{A,n}(R)$ to a face F of R is its standard periodic tiling with n^{d-1} elements.*

- *The sequence $(\mathcal{P}_{A,n}(R))_{n\geq 0}$ satisfies*

$$\sup_{n\geq 0} \left(n \max_{T \in \mathcal{P}_{A,n}(R)} \mathrm{diam}(T) \right) < \infty \quad \text{and} \quad \sup_{n\geq 0} \max_{T\in\mathcal{P}_{A,n}(R)} S(T) < \infty.$$

The validity of this conjecture would imply the following result using the same proof as for the estimates (1.5) and (1.9) established in §3.2.

Conjecture 6.9 *For all $m \geq 2$ there exists a constant $C = C(m,d)$ such that the following holds. Let $\Omega \subset \mathbb{R}^d$ be polygonal domain, let $f \in C^m(\Omega)$ and $1 \leq p < \infty$. Then there exists a sequence $(\mathcal{T}_N)_{N\geq N_0}$ of simplicial meshes of Ω such that $\#(\mathcal{T}_N) \leq N$ and*

$$\limsup_{N\to\infty} N^{\frac{m-1}{d}} |f - \mathrm{I}^{m-1}_{\mathcal{T}_N} f|_{W^{1,p}(\Omega)} \leq C \left\| L_{m,d}\left(\frac{d^m f}{m!}\right) \right\|_{L^\tau(\Omega)} \tag{6.101}$$

where $\frac{1}{\tau} := \frac{m-1}{d} + \frac{1}{p}$. Furthermore, for all $\varepsilon > 0$, there exists an admissible sequence of simplicial meshes $(\mathcal{T}^\varepsilon_N)_{N\geq N_0}$ of Ω such that $\#(\mathcal{T}^\varepsilon_N) \leq N$ and

$$\limsup_{N\to\infty} N^{\frac{m-1}{d}} |f - \mathrm{I}^{m-1}_{\mathcal{T}^\varepsilon_N} f|_{W^{1,p}(\Omega)} \leq C \left\| L_{m,d}\left(\frac{d^m f}{m!}\right) \right\|_{L^\tau(\Omega)} + \varepsilon. \tag{6.102}$$

6.3 Optimal metrics and Algebraic expressions of the shape function.

The theory of anisotropic mesh generation in dimension three or higher is only at its infancy. However efficient software already exists such as [19] for tetrahedral mesh generation in domains of $\mathbb{R}^3$. A description of such an algorithm can be found in [11] as well as some applications to computational mechanics. These software take as input a field $h(z)$ of symmetric positive definite matrices and attempt to create a mesh satisfying (4.69). Defining the set of symmetric matrices $\mathcal{A}'_\pi$ in a similar way as in the two dimensional case (4.66), let us assume that there exists a continuous function $\mathcal{M}_{m,d} : \mathbb{H}_{m,d} \to S_d^+$ satisfying for all $\pi \in \mathbb{H}_{m,d}$,

$$\mathcal{M}_{m,d}(\pi) \in \mathcal{A}'_\pi \text{ and } \det \mathcal{M}_{m,d}(\pi) \leq K \inf\{\det M \; ; \; M \in \mathcal{A}'_\pi\} \tag{6.103}$$

for some fixed constant K. Let $\mathcal{M}(z) := \mathcal{M}_{m,d}(d^m f(z)) + \mu \operatorname{Id}$, where the constant $\mu \geq 0$ is here to avoid degeneracy problems. Let $\delta > 0$ and let

$$h(z) := \delta^{-\tau} (\det \mathcal{M}(z))^{\frac{-\tau}{dp}} \mathcal{M}(z). \tag{6.104}$$

A heuristic analysis similar to the one developed in §4 suggests that mesh generation based on this metric leads to a mesh $\mathcal{T}$ of Ω optimally adapted for approximating f with $\mathbb{P}_{m-1}$ elements in the $W^{1,p}$ semi norm. This justifies the search for functions $\mathcal{M}_{m,d}$ satisfying (6.103).

The form of $\mathcal{M}_{2,d}$, which corresponds to piecewise linear finite elements, is already established, see for instance [16], but we recall it for completeness. The same analysis as in §4.2 shows that

$$\mathcal{M}_{2,d}(\pi) := 4[\pi]^2$$

satisfies $\mathcal{M}_{2,d}(\pi) \in \mathcal{A}'_\pi$ and $\det \mathcal{M}_{2,d}(\pi) = \inf\{\det M \; ; \; M \in \mathcal{A}'_\pi\}$. As a byproduct we obtain from Lemma 6.5 that there exists a constant $C = C(d)$ such that for all $\pi \in \mathbb{H}_{2,d}$

$$C^{-1} \sqrt[d]{|\det[\pi]|} \leq L_{2,d}(\pi) \leq C \sqrt[d]{|\det[\pi]|}.$$

For piecewise quadratic elements, we generalise (4.72) and define

$$\mathcal{M}^*_{3,d}(\pi) := \sqrt{[\partial_{x_1}\pi]^2 + \cdots + [\partial_{x_d}\pi]^2}.$$

Then $\sqrt{d}\mathcal{M}^*_{3,d}(\pi) \in \mathcal{A}'_\pi$, but we have found that the inequality

$$\det \mathcal{M}^*_{3,d}(\pi) \leq K \inf\{\det M \; ; \; M \in \mathcal{A}'_\pi\}$$

does *not* hold uniformly on $\mathbb{H}_{3,d}$ for any constant $K > 0$. The metric $\mathcal{M}^*_{3,d}$ may still be used for mesh adaptation through the formula (6.104) but this metric may not be optimal in the area where $\pi_z = \frac{d^3 f(z)}{6}$ is such that $\det \mathcal{M}^*_{3,d}(\pi_z)$ is not well controlled by $\inf\{\det M \; ; \; M \in \mathcal{A}'_{\pi_z}\}$.

7 Final remarks and conclusion

In this paper, we have introduced asymptotic estimates for the finite element interpolation error measured in the $W^{1,p}$ semi-norm, when the mesh is optimally adapted to a function of two variables and the degree of interpolation $m-1$ is arbitrary. The approach used is an adaptation of the ideas developped in [15] for the L^p interpolation error, and leads to asymptotically sharp error estimates, exposed in Theorems 1.1 and 1.2. These estimates involve a shape function $L_{m,p}$ which generalises the determinant which appears in estimates for piecewise linear interpolation. The shape function has equivalents of polynomial form for all values of m, as established in theorems 5.1 and 5.2. Up to a fixed multiplicative constant, our estimates can therefore be written under analytic form in terms of the derivatives of the function to be approximated.

In the case of piecewise linear and piecewise quadratic finite elements, we have presented in §4 metrics which allow to produce near optimal meshes. This metric is new in the case of quadratic elements. Some numerical experiments presented in §4.2 demonstrate the efficiency of this procedure, and the C++ source code is freely available on the internet [17].

We have partially extended these results to higher dimension, in particular we provide a local error estimate (6.98) which leads to sufficient conditions for building meshes that satisfy the best possible estimate up to a multiplicative constant. A multidimensional asymptotical lower error estimate is proved in Theorem 6.7 and generalises the bidimensional study. The corresponding asymptotical upper estimate is presented in 6.9 but not proved.

One of the main tools used throughout this paper for the construction of an optimal partition is the measure of sliverness $S(T)$ of a simplex, defined in (6.89), which has a geometrical interpretation as the distance from T to the set of acute simplices. This measure accurately distinguishes between good anisotropy, that leads to optimal error estimates, and bad anisotropy that leads to oscillation of the gradient of the interpolated function. Equivalent quantities can be found in [3, 12], but had not been used in the context of optimal mesh adaptation.

APPENDIX

A Proof of Lemma 3.2

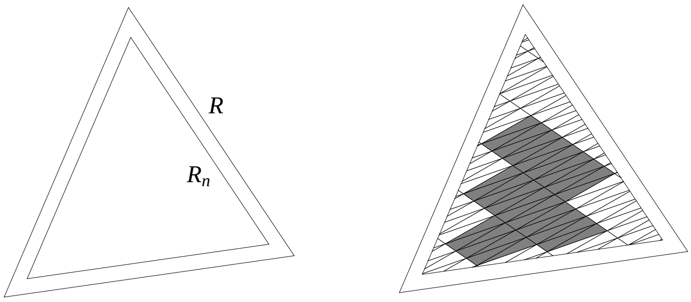

Figure 5: Left : The triangles R and R_n. Right : The partition $\mathcal{P}'_n$ of R_n.

Let R_n be the homothetic contraction of R by the factor $1 - n^{-1}$ and with the same barycenter. We define a partition $\mathcal{P}'_n$ of R_n into convex polygons as follows $\mathcal{P}'_n := \{R_n \cap T' \; ; \; T' \in \mathcal{P}_{T,n}\}$. The triangles R, R_n, and the partition $\mathcal{P}'_n$ are illustrated on Figure 5. Note that the normals to the faces of the polygons building the partition $\mathcal{P}'_n$ belong to a family of only 6 elements $(\mathbf{n}_i)_{1\leq i\leq 6}$: the normals to the faces of of R, and the normals to the faces of T. Hence only 6×5 different angles can appear in $\mathcal{P}'_n$, and we denote the largest of these by $\alpha < \boldsymbol{\pi}$.

We now partition into triangles each convex polygon $C \in \mathcal{P}'_n$ using the Delaunay triangulation of its vertices. Note that the angles of the triangles partitionning a convex polygon C are smaller than the maximal angle of C, hence than α. We denote by $\mathcal{P}''_n$ the resulting triangulation of R_n, as illustrated on the left of Figure 6.

We denote by E_n the collection of n equidistributed points on each edge of R, described in item 2 of Lemma 3.2. We denote by E'_n the set of vertices of the triangles in $\mathcal{P}''_n$ that fall on $\partial R'_n$. For each point $p \in E_n$, we draw an edge between p and the point of E'_n which is the closest to p. This produces a partition of $R \setminus R_n$ into triangles and convex quadrilaterals. Eventually we partition each of these polygons C into triangles using the Delaunay triangulation of the point set $\overline{C} \cap (E_n \cup E'_n)$, which produces a triangulation

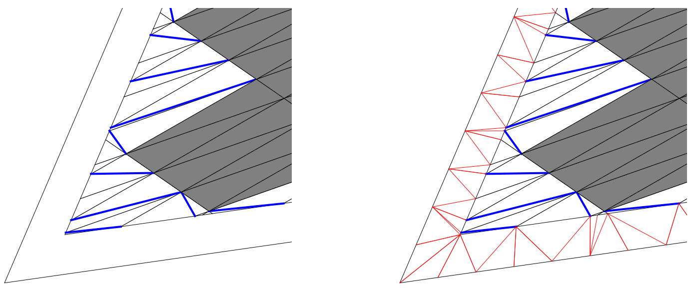

Figure 6: Left : detail of the partition $\mathcal{P}''_n$ of R_n. Right : the partition $\mathcal{P}_n = \mathcal{P}_{T,n}(R) = \mathcal{P}''_n \cup \widetilde{\mathcal{P}}_n$ of R.

$\widetilde{\mathcal{P}}_n$ of $R \setminus R_n$ illustrated on the right of Figure 6. The triangles $T' \in \widetilde{\mathcal{P}}_n$ obey

$$\operatorname{diam}(T') \le (2 \operatorname{diam} R + \operatorname{diam} T) n^{-1} = C n^{-1}.$$

Furthermore let L be the length of the edge of T' included in $\partial R \cup \partial R_n$, and let H be the height of the triangle T' such that $LH = 2|T'|$. Then

$$H \ge \min\{|z - z'|\ ;\ z \in \partial R,\ z' \in \partial R_n\} = c n^{-1}.$$

where $c > 0$ is independent of n. Let L' be another edge of T', and let θ be the angle of T' between the edges L and L'. Then

$$2|T'| = LL' \sin\theta = LH.$$

hence $\sin\theta \ge \frac{H}{\operatorname{diam}(T')} \ge \frac{c}{C}$, and therefore $\arcsin(\frac{c}{C}) \le \theta \le \pi - \arcsin(\frac{c}{C})$. It follows that all the angles of T' are smaller than $\pi - \arcsin(\frac{c}{C})$. We eventually define $\mathcal{P}_{T,n}(R) := \mathcal{P}''_n \cup \widetilde{\mathcal{P}}_n$ and observe that the largest angle of a triangle in $\mathcal{P}_{T,n}(R)$ is bounded by the constant $\beta(R,T) = \max\{\alpha, \pi - \arcsin(\frac{c}{C})\} < \boldsymbol{\pi}$ which is independent of n. Hence

$$\sup_n \sup_{T \in \mathcal{P}_{T,n}(R)} S(T) \le \tan\left(\frac{\beta(R,T)}{2}\right) < \infty$$

The other properties of $\mathcal{P}_{T,n}(R)$ mentionned in 3.2 are easily checked.

B Proof of Theorem 5.2

Let $m \ge 2$ be arbitrary and let $s_m := \lfloor \frac{m}{2} \rfloor + 1$. We have proved in [15], Proposition 2.1, that for all $\pi \in \mathbb{H}_m$ the three following properties are equivalent

$$\left[\begin{array}{l} K^{\mathcal{E}}_m(\pi) = 0, \\ \text{There exists } \alpha, \beta \in \mathbb{R} \text{ and } \tilde\pi \in \mathbb{H}_{m-s_m} \text{ such that } \pi = (\alpha x + \beta y)^{s_m} \tilde\pi, \\ \text{There exists a sequence } (\phi_n)_{n\ge 0}, \phi_n \in \mathrm{SL}_2 \text{ such that } \pi \circ \phi_n \to 0. \end{array}\right. \tag{B.105}$$

We also proved in [15], Appendix B, the following invariance property : Let Q be a polynomial on $\mathbb{H}_m$ such that $K^{\mathcal{E}}_m \sim \sqrt[r]{|Q|}$ where $r = \deg Q$. Then

$$Q(\pi \circ \phi) = (\det \phi)^{\frac{rm}{2}} Q(\pi) \text{ for all } \pi \in \mathbb{H}_m \text{ and } \phi \in \mathrm{M}_2(\mathbb{R}). \tag{B.106}$$

This property clearly transfers to the polynomials $(Q_k)_{0 \le k \le r}$ defined in Equation (5.86), in such way that

$$Q_k(\pi_1 \circ \phi, \pi_2 \circ \phi) = (\det \phi)^{\frac{rm}{2}} Q_k(\pi_1, \pi_2) \text{ for all } \pi_1, \pi_2 \in \mathbb{H}_m \text{ and all } \phi \in \mathrm{M}_2(\mathbb{R}). \tag{B.107}$$

We define two functions on $\mathbb{H}_m \times \mathbb{H}_m$

$$K_*(\pi_1, \pi_2) := \sqrt[2r]{\sum_{0 \le k \le r} Q_k(\pi_1, \pi_2)^2} \text{ and } K(\pi_1, \pi_2) := \sqrt[2\tilde r]{\tilde Q(\pi_1^2 + \pi_2^2)},$$

where $\tilde Q$ is such that $K^{\mathcal{E}}_{2m} \sim \sqrt[\tilde r]{\tilde Q}$, $\tilde r := \deg \tilde Q$. We show below that $K \sim K_*$ on $\mathbb{H}_m \times \mathbb{H}_m$. This result combined with (5.82) concludes the proof of Theorem 5.2. (Note that m is replaced with $m+1$ in the statement of this theorem.) Using (B.107) and remarking the invariance property $\tilde Q(\pi \circ \phi) = (\det \phi)^{\tilde r m} Q(\pi)$, for the same reasons as (B.106), we obtain

$$\text{for all } \pi_1, \pi_2 \in \mathbb{H}_m \text{ and all } \phi \in \mathrm{M}_2(\mathbb{R}), \left\{ \begin{array}{lcl} K(\pi_1 \circ \phi, \pi_2 \circ \phi) & = & |\det \phi|^{\frac{m}{2}} K(\pi_1, \pi_2), \\ K_*(\pi_1 \circ \phi, \pi_2 \circ \phi) & = & |\det \phi|^{\frac{m}{2}} K_*(\pi_1, \pi_2). \end{array}\right. \tag{B.108}$$

If $K(\pi_1, \pi_2) = 0$, then $\pi_1^2 + \pi_2^2 \in \mathbb{H}_{2m}$ has a linear factor of multiplicity $s_{2m} = m + 1$ according to (B.105), and therefore π_1 and π_2 have a common linear factor of multiplicity s_m.

If $K_*(\pi_1,\pi_2)=0$, then for all k, $0\le k\le r$, we have $Q_k(\pi_1,\pi_2)=0$. Using (5.86) we obtain that for all $u,v\in\mathbb{R}$ we have $K_m(u\pi_1+v\pi_2)=0$. It follows from (B.105) that for all $u,v\in\mathbb{R}$ the polynomial $u\pi_1+v\pi_2\in\mathbb{H}_m$ has a linear factor of multiplicity s_m, hence that π_1 and π_2 have a common linear factor of multiplicity s_m.

Hence the following properties are equivalent

$$\left[\begin{array}{l} K(\pi_1,\pi_2)=0,\\ K_*(\pi_1,\pi_2)=0,\\ \text{There exists } \alpha,\beta\in\mathbb{R} \text{ and } \tilde\pi_1,\tilde\pi_2\in\mathbb{H}_{m-s_m} \text{ such that } \pi_1=(\alpha x+\beta y)^{s_m}\tilde\pi_1, \text{ and } \pi_2=(\alpha x+\beta y)^{s_m}\tilde\pi_2. \end{array}\right. \tag{B.109}$$

Using (B.105) we find that these properties are also equivalent to

$$\left[\begin{array}{l} K_{2m}^{\mathcal{E}}(\pi_1^2+\pi_2^2)=0,\\ \text{There exists a sequence } (\phi_n)_{n\ge0}, \phi_n\in\mathrm{SL}_2, \text{ such that } (\pi_1\circ\phi_n)^2+(\pi_2\circ\phi_n)^2\to0,\\ \text{There exists a sequence } (\phi_n)_{n\ge0}, \phi_n\in\mathrm{SL}_2, \text{ such that } \pi_1\circ\phi_n\to0 \text{ and } \pi_2\circ\phi_n\to0. \end{array}\right. \tag{B.110}$$

We now define the norm $\|(\pi_1,\pi_2)\|:=\sup_{|u|\le1}|(\pi_1(u),\pi_2(u))|$ on $\mathbb{H}_m\times\mathbb{H}_m$ and

$$\mathcal{F}:=\{(\pi_1,\pi_2)\in\mathbb{H}_m\times\mathbb{H}_m\ ;\ \|(\pi_1,\pi_2)\|=1 \text{ and } \|(\pi_1\circ\phi,\pi_2\circ\phi)\|\ge1 \text{ for all } \phi\in\mathrm{SL}_2\}.$$

$\mathcal{F}$ is compact subset of $\mathbb{H}_m\times\mathbb{H}_m$ and K as well as K_* do not vanish on $\mathcal{F}$ according to (B.109) and (B.110). Since these functions are continuous, there exists a constant $C_0>0$ such that

$$C_0^{-1}K\le K_*\le C_0K \text{ on } \mathcal{F}. \tag{B.111}$$

Let $\pi_1,\pi_2\in\mathbb{H}_m$. If there exists a sequence $(\phi_n)_{n\ge0}$, $\phi_n\in\mathrm{SL}_2$, such that $\pi_1\circ\phi_n\to0$ and $\pi_2\circ\phi_n\to0$, then $K(\pi_1,\pi_2)=K_*(\pi_1,\pi_2)=0$. Otherwise, consider a sequence $(\phi_n)_{n\ge0}$, $\phi_n\in\mathrm{SL}_2$ such that

$$\lim_{n\to\infty}\|(\pi_1\circ\phi_n,\pi_2\circ\phi_n)\|=\inf_{\phi\in\mathrm{SL}_2}\|(\pi_1\circ\phi,\pi_2\circ\phi)\|.$$

By compactness there exists a pair $(\tilde\pi_1,\tilde\pi_2)\in\mathbb{H}_m\times\mathbb{H}_m$ and a subsequence $(\phi_{n_k})_{k\ge0}$ such that

$$(\pi_1\circ\phi_{n_k},\pi_2\circ\phi_{n_k})\to(\tilde\pi_1,\tilde\pi_2).$$

Note that $\frac{(\tilde\pi_2,\tilde\pi_2)}{\|(\tilde\pi_2,\tilde\pi_2)\|}\in\mathcal{F}$. Using (B.108) we obtain

$$\frac{K(\pi_1,\pi_2)}{K_*(\pi_1,\pi_2)}=\lim_{n\to\infty}\frac{K(\pi_1\circ\phi_n,\pi_2\circ\phi_n)}{K_*(\pi_1\circ\phi_n,\pi_2\circ\phi_n)}=\frac{K(\tilde\pi_1,\tilde\pi_2)}{K_*(\tilde\pi_1,\tilde\pi_2)}$$

Using (B.111) and the homogeneity of K and K_*, we obtain that $C_0^{-1}K\le K_*\le C_0K$ on $\mathbb{H}_m\times\mathbb{H}_m$ which concludes the proof.

C Proof of Proposition 6.1

We denote by T_{eq} a d-dimensional equilateral simplex such that $\mathrm{bary}(T_{\mathrm{eq}})=0$ and such that its vertices q_i, $0\le i\le d$, belong to the unit sphere, i.e. $|q_i|=1$. Since the vertices of T_{eq} play symmetrical roles there exists a constant $\xi\in\mathbb{R}$ such that

$$\text{For all } 0\le i\le d,\ \ 0\le j\le d,\ \ 0\le k\le d, \text{ one has } \langle q_i-q_j,q_k\rangle=\xi(\delta_{ik}-\delta_{jk}), \tag{C.112}$$

where δ is the Kronecker symbol : $\delta_{ij}=1$ if $i=j$, and 0 otherwise. Using the relation $q_0+\cdots+q_d=0$ we obtain $\xi d=\sum_{j=0}^d\langle q_0-q_j,q_0\rangle=d+1$ hence $\xi=1+\frac{1}{d}$. Note also that the unit exterior normal to the face of T_{eq} opposite to the vertex q_i is $-q_i$.

We recall the following property : if $A\in\mathrm{GL}_d$ and if $\mathbf{n}$ is the exterior normal to a face F of a simplex T, then the exterior normal to the face $A(F)$ of $A(T)$ is

$$\mathbf{n}'=\frac{(A^{-1})^{\mathrm{t}}\mathbf{n}}{|(A^{-1})^{\mathrm{t}}\mathbf{n}|}. \tag{C.113}$$

We first prove the fact that for any simplex T, the simplex $M_T^{-\frac{1}{2}}(T)$ is acute. Without loss of generality we can assume that $\mathrm{bary}(T) = 0$, hence there exists $A \in \mathrm{GL}_d$ such that $T = A(T_{\mathrm{eq}})$. Since the vertices of T are $v_i = Aq_i$ for $0 \le i \le d$, we obtain from definition (6.91) that

$$M_T = A \left(\sum_{0 \le i < j \le d} \frac{(q_i - q_j)(q_i - q_j)^{\mathrm{t}}}{|A(q_i - q_j)|^2} \right) A^{\mathrm{t}}.$$

According to Equation (C.113), the exterior normal to the face of the simplex $T_{\mathrm{ac}} := M_T^{-\frac{1}{2}}(T) = M_T^{-\frac{1}{2}} A(T_{\mathrm{eq}})$ opposite to the vertex v_i is

$$\mathbf{n}_i = -\nu_i M_T^{\frac{1}{2}} (A^{-1})^{\mathrm{t}} q_i$$

where $\nu_i > 0$. For all $0 \le a < b \le d$, we therefore obtain using Equation (C.112)

$$\begin{aligned}
\nu_a \nu_b \langle \mathbf{n}_a, \mathbf{n}_b \rangle &= \left\langle M_T^{\frac{1}{2}} (A^{-1})^{\mathrm{t}} q_a,\ M_T^{\frac{1}{2}} (A^{-1})^{\mathrm{t}} q_b \right\rangle \\
&= q_a^{\mathrm{t}} A^{-1} M_T (A^{-1})^{\mathrm{t}} q_b \\
&= q_a^{\mathrm{t}} \left(\sum_{0 \le i < j \le d} \frac{(q_i - q_j)(q_i - q_j)^{\mathrm{t}}}{|A(q_i - q_j)|^2} \right) q_b \\
&= \xi^2 \sum_{0 \le i < j \le d} \frac{(\delta_{ai} - \delta_{aj})(\delta_{bi} - \delta_{bj})}{|A(q_i - q_j)|^2} \\
&= \frac{-\xi^2}{|A(q_a - q_b)|^2} < 0.
\end{aligned}$$

This establishes that the simplex $T_{\mathrm{ac}} := M_T^{-\frac{1}{2}}(T)$ is acute.

It remains to prove the inequality (6.92). For this we need the following lemma.

Lemma C.1 *For any acute simplex T_{ac} one has $M_{T_{\mathrm{ac}}} \ge \mathrm{Id}$.*

Proof: Without loss of generality we can assume that $\mathrm{bary}(T_{\mathrm{ac}}) = 0$, hence there exists $A \in \mathrm{GL}_d$ such that $T_{\mathrm{ac}} = A(T_{\mathrm{eq}})$. The vertices of T_{ac} are $c_i = Aq_i$ for $0 \le i \le d$, and the exterior normal to the face of T_{ac} opposite c_i is

$$\mathbf{m}_i = -\mu_i (A^{-1})^{\mathrm{t}} q_i.$$

where $\mu_i > 0$. We define for all $0 \le i < j \le d$

$$\lambda_{ij} := -\xi^{-2} |c_i - c_j|^2 \mu_i \mu_j \langle \mathbf{m}_i, \mathbf{m}_j \rangle.$$

Since T_{ac} is acute we have $\langle \mathbf{n}_i, \mathbf{n}_j \rangle \le 0$ and therefore $\lambda_{ij} \ge 0$. We now introduce the symmetric matrix

$$M := \sum_{0 \le i < j \le d} \lambda_{ij} f_{ij} f_{ij}^{\mathrm{t}} \text{ where } f_{ij} := \frac{c_i - c_j}{|c_i - c_j|}. \tag{C.114}$$

For all $0 \le a < b \le d$ we obtain using the relation $\mu_i \langle \mathbf{m}_i, c_j \rangle = -\langle q_i, q_j \rangle$ that

$$\begin{aligned}
\mu_a \mu_b\ \mathbf{m}_a^{\mathrm{t}} M \mathbf{m}_b &= \sum_{0 \le i < j \le d} \lambda_{ij} \frac{\langle q_a, q_i - q_j \rangle \langle q_b, q_i - q_j \rangle}{|c_i - c_j|^2} \\
&= \xi^2 \sum_{0 \le i < j \le d} \lambda_{ij} \frac{(\delta_{ai} - \delta_{aj})(\delta_{bi} - \delta_{bj})}{|c_i - c_j|^2} \\
&= \frac{-\xi^2 \lambda_{ab}}{|c_a - c_b|^2} = \mu_a \mu_b \langle \mathbf{m}_a, \mathbf{m}_b \rangle.
\end{aligned}$$

Therefore $\mathbf{m}_a^{\mathrm{t}} M \mathbf{m}_b = \mathbf{m}_a^{\mathrm{t}} \mathbf{m}_b$ for all $0 \leq a < b \leq d$, which implies that $M = \mathrm{Id}$. Furthermore for all $0 \leq a < b \leq d$, we have

$$1 = |f_{ab}|^2 = f_{ab}^{\mathrm{t}} M f_{ab} = \sum_{0 \leq i < j \leq d} \lambda_{ij} \langle f_{ab}, f_{ij} \rangle^2 \geq \lambda_{ab}.$$

It follows that in the sense of symmetric matrices,

$$M_{T_{\mathrm{ac}}} := \sum_{0 \leq i < j \leq d} f_{ij} f_{ij}^{\mathrm{t}} \geq \sum_{0 \leq i < j \leq d} \lambda_{ij} f_{ij} f_{ij}^{\mathrm{t}} = M = \mathrm{Id}\,.$$

$\diamond$

We now conclude the proof of inequality (6.92). Let T be an arbitrary simplex. Since $M_T^{-\frac{1}{2}}(T)$ is acute the definition (6.89) of S implies that $S(T) \leq \kappa(\sqrt{M_T})$ which proves the left part of inequality (6.92). On the other hand let $\psi \in \mathrm{GL}_d$ be such that the simplex $T_{\mathrm{ac}} := \psi(T)$ is acute. Let v_i, $0 \leq i \leq d$ be the vertices of T and $c_i = \psi(v_i)$ the vertices of T_{ac}. We define the vectors e_{ij} and f_{ij} similarly to (6.91) and (C.114)

$$e_{ij} := \frac{v_i - v_j}{|v_i - v_j|}, \quad f_{ij} := \frac{c_i - c_j}{|c_i - c_j|} = \frac{\psi(e_{ij})}{|\psi(e_{ij})|}.$$

for all $0 \leq i < j \leq d$. For any $v \in \mathbb{R}^d$ we therefore have

$$\begin{aligned} v^{\mathrm{t}} M_{T_{\mathrm{ac}}} v &= \sum_{0 \leq i < j \leq d} \frac{\langle \psi(e_{ij}), v \rangle^2}{|\psi(e_{ij})|^2} \\ &\leq \|\psi^{-1}\|^2 \sum_{0 \leq i < j \leq d} \langle \psi(e_{ij}), v \rangle^2 \\ &= \|\psi^{-1}\|^2 \, (\psi^{\mathrm{t}} v)^{\mathrm{t}} M_T (\psi^{\mathrm{t}} v). \end{aligned}$$

Using the previous lemma and defining $u := \psi^{\mathrm{t}} v$ we obtain

$$|u|^2 \leq \|\psi\|^2 |v|^2 \leq \|\psi\|^2 v^{\mathrm{t}} M_{T_{\mathrm{ac}}} v \leq \|\psi\|^2 \|\psi^{-1}\|^2 u^{\mathrm{t}} M_T u = \left(\|\psi\| \|\psi^{-1}\| |M_T^{\frac{1}{2}} u| \right)^2,$$

hence $\|M_T^{\frac{-1}{2}}\| \leq \kappa(\psi)$. Recalling that $\|\sqrt{M_T}\| \leq \alpha_d$ we obtain

$$\kappa(\sqrt{M_T}) = \|M_T^{\frac{1}{2}}\| \|M_T^{\frac{-1}{2}}\| \leq \alpha_d \kappa(\psi).$$

We conclude the proof by taking the infimum among all ψ such that $\psi(T)$ is acute.

Acknowledgement

I am extremely grateful to my Ph.D advisor Albert Cohen for his support in the elaboration of this paper.

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