Automatic coupling and finite element discretization of the Navier–Stokes and heat equations

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Abstract: We consider the finite element discretization of the Navier–Stokes equations coupled with the heat equation where the viscosity depends on the temperature. We prove a posteriori error estimates which allow us to automatically determine the zone where the temperature-dependent viscosity must be inserted into the Navier–Stokes equations and also to perform mesh adaptivity in order to optimize the discretization of these equations.

Résumé: Nous considérons une discrétisation par éléments finis des équations de Navier–Stokes couplées avec l'équation de la chaleur là où la viscosité dépend de la température. Nous prouvons des estimations d'erreur a posteriori qui permettent de déterminer automatiquement la zône où cette viscosité doit être insérée dans les équations de Navier–Stokes et simultanément d'adapter le maillage pour optimiser la discrétisation de ces équations.

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1. Introduction.

Let Ω be a connected bounded open set in $\mathbb{R}^d$, $d = 2$ or 3, with a Lipschitz–continuous boundary $\partial\Omega$. The following system models the stationary flow of a viscous incompressible fluid, in the case where the viscosity of the fluid depends on the temperature

$$\begin{cases} -\mathrm{div}\,\big(\nu(T)\,\nabla\boldsymbol{u}\big) + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \mathbf{grad}\,p = \boldsymbol{f} & \text{in } \Omega, \\ \mathrm{div}\,\boldsymbol{u} = 0 & \text{in } \Omega, \\ -\alpha\,\Delta T + (\boldsymbol{u}\cdot\nabla)T = g & \text{in } \Omega. \end{cases} \tag{1.1}$$

The unknowns are the velocity $\boldsymbol{u}$, the pressure p, and the temperature T of the fluid, while the data are the distributions $\boldsymbol{f}$ and g. The function ν is positive and bounded, while the coefficient α is a positive constant. A similar but slightly more complex model has been recently derived and analyzed in [8], see also the references therein.

Indeed, we do think that the equations in (1.1) are a very realistic model for a number of incompressible fluids when the temperature presents high variations. However, simulating such flows is very expensive, in dimension $d = 3$ in particular, and it seems likely that the variations of the temperature are limited to a part of the domain. So our aim is to replace $\nu(T)$ by a very small positive parameter ν_0 where these variations are negligible. There, the solution $(\boldsymbol{u}, p)$ of the first two lines of the system behaves like the viscosity solution of Euler's equations. We refer to [5] for the first study of such a simplification. We then propose a finite element discretization and, relying on the arguments of [7], we establish a priori error estimates for this discretization.

We are mainly interested in the a posteriori analysis of the discretization. Indeed, as first explained in [3] in a general framework, the a posteriori estimates allow us to numerically determine the zone where $\nu(T)$ can be replaced by the constant ν_0 without increasing the global error (see also [1] for a very similar coupling of the Navier–Stokes equations with a turbulence model). On the other hand, we exhibit a second family of error indicators which leads to an optimal adaptation of the mesh, as now standard in finite elements. By combining all these results, we can define a simple strategy which leads us to a very efficient discretization of the initial system.

An outline of the paper is as follows:

- In Section 2, we prove the existence of a solution for problem (1.1) when provided with appropriate boundary conditions on the velocity and the temperature.
- Similar results are derived in Section 3 for the simplified problem, and we prove a first a posteriori estimate for evaluating the distance between the solution of the exact and simplified problems.
- The discrete problem is described in Section 4, and we prove optimal a priori error estimates for the error.
- Section 5 is devoted to the a posteriori analysis of both the simplification and the discretization.
- In Section 6, we describe the final adaptivity strategy which can be deduced from the results in the previous section.

2. The full problem.

We intend to write a variational formulation of system (1.1). We first make precise the assumptions on the function ν: It belongs to $L^\infty(\mathbb{R})$ and satisfies, for two positive constants ν_1 and ν_2,

$$\text{for a.e. } \tau \in \mathbb{R}, \quad \nu_1 \le \nu(\tau) \le \nu_2. \tag{2.1}$$

Note that these assumptions are not at all restrictive.

We also make precise the boundary conditions that must be enforced on the velocity $\boldsymbol{u}$ and the temperature T: For a given function T_0, they read

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{and} \quad T = T_0 \qquad \text{on } \partial\Omega. \tag{2.2}$$

We have chosen to work with a homogeneous condition on the velocity in order to avoid the technical arguments linked to the Hopf lemma, see [14, Chap. IV, Lemma 2.3].

To go further, for any subset $\mathcal{O}$ of Ω with a Lipschitz-continuous boundary $\partial\mathcal{O}$, we consider the full scale of Sobolev spaces $H^s(\mathcal{O})$, $s \in \mathbb{R}$, and also the analogous spaces $H^s(\partial\mathcal{O})$ on its boundary. We need the spaces $W^{m,p}(\mathcal{O})$, for any nonnegative integer m and $1 < p < +\infty$, equipped with the norm $\|\cdot\|_{W^{m,p}(\mathcal{O})}$ and seminorm $|\cdot|_{W^{m,p}(\mathcal{O})}$. We denote by $W_0^{m,p}(\mathcal{O})$ the closure in $W^{m,p}(\mathcal{O})$ of the space $\mathscr{D}(\mathcal{O})$ of infinitely differentiable functions with a compact support in $\mathcal{O}$, by $W^{-m,p'}(\mathcal{O})$ its dual space (with $\frac{1}{p} + \frac{1}{p'} = 1$), and by $W^{m-\frac{1}{p},p}(\partial\mathcal{O})$ the space of traces of functions in $W^{m,p}(\mathcal{O})$ on $\partial\mathcal{O}$.

We also introduce the space

$$L^2_\circ(\mathcal{O}) = \big\{q \in L^2(\mathcal{O});\ \int_{\mathcal{O}} q(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = 0\big\}. \tag{2.3}$$

We thus consider the variational problem

Find $(\boldsymbol{u}, p, T)$ in $H_0^1(\Omega)^d \times L^2_\circ(\Omega) \times H^1(\Omega)$ such that

$$T = T_0 \qquad \text{on } \partial\Omega, \tag{2.4}$$

and that

$$\begin{aligned}
\forall \boldsymbol{v} \in H_0^1(\Omega)^d, \quad & \int_\Omega \nu(T)(\boldsymbol{x})\,(\mathbf{grad}\,\boldsymbol{u})(\boldsymbol{x}) : (\mathbf{grad}\,\boldsymbol{v})(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \\
& + \int_\Omega \big((\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\big)(\boldsymbol{x})\cdot\boldsymbol{v}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} - \int_\Omega (\mathrm{div}\,\boldsymbol{v})(\boldsymbol{x})\,p(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \langle \boldsymbol{f}, \boldsymbol{v}\rangle_\Omega, \\
\forall q \in L^2_\circ(\Omega), \quad & -\int_\Omega (\mathrm{div}\,\boldsymbol{u})(\boldsymbol{x})\,q(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = 0, \\
\forall S \in H_0^1(\Omega), \quad & \alpha \int_\Omega (\mathbf{grad}\,T)(\boldsymbol{x})\cdot(\mathbf{grad}\,S)(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \\
& + \int_\Omega \big((\boldsymbol{u}\cdot\nabla)T\big)(\boldsymbol{x})\,S(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \langle g, S\rangle_\Omega,
\end{aligned} \tag{2.5}$$

where $\langle \cdot, \cdot \rangle_\Omega$ denotes the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$ and also between $H^{-1}(\Omega)^d$ and $H_0^1(\Omega)^d$.

Standard arguments relying on the density of $\mathscr{D}(\Omega)$ in $H_0^1(\Omega)$ lead to the following result.

Proposition 2.1. *Problems* (1.1) – (2.2) *and* (2.4) – (2.5) *are equivalent: Any triple* $(\boldsymbol{u}, p, T)$ *in* $H^1(\Omega)^d \times L^2_\circ(\Omega) \times H^1(\Omega)$ *is a solution of* (1.1) *(in the distribution sense) and* (2.2) *if and only if it is a solution of* (2.4) – (2.5).

The existence of a solution can be established owing to a fixed-point theorem. Its proof requires the kernel

$$\mathbb{V} = \big\{ \boldsymbol{v} \in H_0^1(\Omega)^d;\ \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega \big\}. \tag{2.6}$$

Theorem 2.2. *For any data* $(\boldsymbol{f}, g)$ *in* $H^{-1}(\Omega)^d \times H^{-1}(\Omega)$ *and* T_0 *in* $H^{\frac{1}{2}}(\partial\Omega)$*, problem* (2.4) – (2.5) *admits at least a solution* $(\boldsymbol{u}, p, T)$ *in* $H_0^1(\Omega)^d \times L^2_\circ(\Omega) \times H^1(\Omega)$*. Moreover, this solution satisfies, for a constant c only depending on* ν_1 *and* α*,*

$$\|\boldsymbol{u}\|_{H^1(\Omega)^d} + \|T\|_{H^1(\Omega)} \le c \left(\|\boldsymbol{f}\|_{H^{-1}(\Omega)^d} + \|g\|_{H^{-1}(\Omega)} + \|T_0\|_{H^{\frac{1}{2}}(\partial\Omega)} \right). \tag{2.7}$$

Proof: It is performed in several steps.
1) We refer *e.g.* to [14, Chap. IV, Lemma 2.3] for the following result: For any $\varepsilon > 0$, there exists a lifting $\overline{T}_0$ of T_0 which satisfies

$$\|\overline{T}_0\|_{L^4(\Omega)} \le \varepsilon \, \|T_0\|_{H^{\frac{1}{2}}(\partial\Omega)}, \qquad \|\overline{T}_0\|_{H^1(\Omega)} \le c \, \|T_0\|_{H^{\frac{1}{2}}(\partial\Omega)}, \tag{2.8}$$

where the constant c is independent of ε.
2) Setting $U = (\boldsymbol{u}, T)$ and $V = (\boldsymbol{v}, S)$, we define the mapping Φ from $\mathbb{V} \times H_0^1(\Omega)$ into its dual space by

$$\begin{aligned} \langle \Phi(U), V \rangle = \int_\Omega \nu(T + \overline{T}_0) \, (\mathbf{grad}\, \boldsymbol{u})(\boldsymbol{x}) : (\mathbf{grad}\, \boldsymbol{v})(\boldsymbol{x}) \, d\boldsymbol{x} + \int_\Omega \big((\boldsymbol{u} \cdot \nabla) \boldsymbol{u} \big)(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x}) \, d\boldsymbol{x} \\ + \alpha \int_\Omega (\mathbf{grad}\, (T + \overline{T}_0)(\boldsymbol{x}) \cdot (\mathbf{grad}\, S)(\boldsymbol{x}) \, d\boldsymbol{x} \\ + \int_\Omega \big((\boldsymbol{u} \cdot \nabla)(T + \overline{T}_0) \big)(\boldsymbol{x}) \, S(\boldsymbol{x}) \, d\boldsymbol{x} - \langle \boldsymbol{f}, \boldsymbol{v} \rangle_\Omega - \langle g, S \rangle_\Omega. \end{aligned}$$

It follows from (2.1) and the imbedding of $H^1(\Omega)$ into $L^6(\Omega)$ that Φ is continuous on $\mathbb{V} \times H_0^1(\Omega)$. Moreover, it follows from (2.1), (2.6) and (2.8) and the antisymmetry property

$$\int_\Omega \big((\boldsymbol{u} \cdot \nabla) \overline{T}_0 \big)(\boldsymbol{x}) \, S(\boldsymbol{x}) \, d\boldsymbol{x} = -\int_\Omega \big((\boldsymbol{u} \cdot \nabla) S \big)(\boldsymbol{x}) \, \overline{T}_0(\boldsymbol{x}) \, d\boldsymbol{x},$$

that

$$\begin{aligned} \langle \Phi(U), U \rangle \ge \nu_1 |\boldsymbol{u}|^2_{H^1(\Omega)^d} + \alpha \, |T|^2_{H^1(\Omega)} - \alpha |T|_{H^1(\Omega)} |\overline{T}_0|_{H^1(\Omega)} \\ - \frac{c\varepsilon}{2} \, \|T_0\|_{H^{\frac{1}{2}}(\partial\Omega)} \big(|\boldsymbol{u}|^2_{H^1(\Omega)^d} + |T|^2_{H^1(\Omega)} \big) \\ - \|\boldsymbol{f}\|_{H^{-1}(\Omega)^d} |\boldsymbol{u}|_{H^1(\Omega)^d} - \|g\|_{H^{-1}(\Omega)} |T|_{H^1(\Omega)}. \end{aligned}$$

We now take ε such that

$$c\varepsilon \, \|T_0\|_{H^{\frac{1}{2}}(\partial\Omega)} \le \min\{\nu_1, \alpha\}.$$

Thus, we deduce from the previous inequality that

$$\langle \Phi(U), U\rangle \ge \frac{\min\{\nu_1,\alpha\}}{2}\big(|\boldsymbol{u}|^2_{H^1(\Omega)^d} + |T|^2_{H^1(\Omega)}\big) \\ - \Big(c\alpha\|T_0\|_{H^{\frac{1}{2}}(\Omega)} + \big(\|\boldsymbol{f}\|^2_{H^{-1}(\Omega)^d} + \|g\|^2_{H^{-1}(\Omega)}\big)^{\frac{1}{2}}\Big)\big(|\boldsymbol{u}|^2_{H^1(\Omega)^d} + |T|^2_{H^1(\Omega)}\big)^{\frac{1}{2}}.$$

All this yields that $\langle \Phi(U), U\rangle$ is nonnegative on the sphere of $\mathbb{V} \times H^1_0(\Omega)$ with radius

$$\mu = \frac{2}{\min\{\nu_1,\alpha\}}\Big(c\alpha\|T_0\|_{H^{\frac{1}{2}}(\Omega)} + \big(\|\boldsymbol{f}\|^2_{H^{-1}(\Omega)^d} + \|g\|^2_{H^{-1}(\Omega)}\big)^{\frac{1}{2}}\Big). \tag{2.9}$$

3) We recall from [14, Chap. I, Cor. 2.5] that $\mathscr{D}(\Omega)^d \cap \mathbb{V}$ is dense in $\mathbb{V}$. Thus, there exist an increasing sequence $(\mathbb{V}_n)_n$ of finite-dimensional subspaces of $\mathbb{V}$ and an increasing sequence $(\mathbb{W}_n)_n$ of finite-dimensional subspaces of $H^1_0(\Omega)$ such that $\cup_{n\in\mathbb{N}}\mathbb{V}_n \times \mathbb{W}_n$ is dense in $\mathbb{V} \times H^1_0(\Omega)$. Moreover, the properties of the function Φ established above still hold with $\mathbb{V} \times H^1_0(\Omega)$ replaced by $\mathbb{V}_n \times \mathbb{W}_n$. Thus, applying Brouwer's fixed-point theorem (see [14, Chap. IV, Cor. 1.1] for instance) yields that, for each n, there exists a $U_n = (\boldsymbol{u}_n, T_n)$ satisfying

$$\forall V_n \in \mathbb{V}_n \times \mathbb{W}_n, \quad \langle \Phi(U_n), V_n\rangle = 0 \qquad \text{and} \qquad \big(|\boldsymbol{u}_n|^2_{H^1(\Omega)^d} + |T_n|^2_{H^1(\Omega)}\big)^{\frac{1}{2}} \le \mu. \tag{2.10}$$

4) Since the norms of $\boldsymbol{u}_n$ in $H^1(\Omega)^d$ and of T_n in $H^1(\Omega)$ are bounded by a constant c (due to the Poincaré–Friedrichs inequality on Ω) and owing to the compactness of the imbedding of $H^1(\Omega)$ into $L^4(\Omega)$, there exists a subsequence, still denoted by $(\boldsymbol{u}_n, T_n)_n$ for simplicity, which converges to a pair $(\boldsymbol{u}, \tilde{T})$ of $H^1_0(\Omega)^d \times H^1_0(\Omega)$ weakly in $H^1(\Omega)^d \times H^1(\Omega)$ and strongly in $L^4(\Omega)^d \times L^4(\Omega)$. Next, we observe that, for $m \le n$, these $(\boldsymbol{u}_n, T_n)$ satisfy

$$\forall V_m \in \mathbb{V}_m \times \mathbb{W}_m, \quad \langle \Phi(U_n), V_m\rangle = 0.$$

Passing to the limit on n is obvious for the linear term and follows from the strong convergence in $L^4(\Omega)^d \times L^4(\Omega)$ for the terms $(\boldsymbol{u}_n \cdot \nabla)\boldsymbol{u}_n$ and $(\boldsymbol{u}_n \cdot \nabla)T_n$. On the other hand, due to this strong convergence, the sequence $(\nu(T_n + \overline{T}_0)\,\mathbf{grad}\,\boldsymbol{v}_m)_n$ converges to $\nu(\tilde{T} + \overline{T}_0)\,\mathbf{grad}\,\boldsymbol{v}_m$ a.e. in Ω and its norm is bounded by $\nu_2\,\|\mathbf{grad}\,\boldsymbol{v}_m\|_{L^2(\Omega)^{d\times d}}$, so that using the Lebesgue dominated convergence theorem yields the convergence of $(\nu(T_n + \overline{T}_0)\,\mathbf{grad}\,\boldsymbol{v}_m)_n$ to $\nu(\tilde{T} + \overline{T}_0)\,\mathbf{grad}\,\boldsymbol{v}_m$ in $L^2(\Omega)^{d\times d}$. All this leads to

$$\forall V_m \in \mathbb{V}_m \times \mathbb{W}_m, \quad \langle \Phi(\boldsymbol{u}, \tilde{T}), V_m\rangle = 0,$$

and passing to the limit on m is now easy. Thus, we derive that the pair $(\boldsymbol{u}, T = \tilde{T} + \overline{T}_0)$ satisfies the second and third equations in (2.5) and also

$$\forall \boldsymbol{v} \in \mathbb{V}, \quad \int_\Omega \nu(T)\,(\mathbf{grad}\,\boldsymbol{u})(\boldsymbol{x}) : (\mathbf{grad}\,\boldsymbol{v})(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \\ + \int_\Omega \big((\boldsymbol{u} \cdot \nabla)\boldsymbol{u}\big)(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \langle \boldsymbol{f}, \boldsymbol{v}\rangle_\Omega. \tag{2.11}$$

5) We recall from [14, Chap. I, Cor. 2.4] the following inf-sup condition for a positive constant β

$$\forall q \in L^2_\circ(\Omega), \qquad \sup_{\boldsymbol{v} \in H^1_0(\Omega)^d} \frac{\int_\Omega (\operatorname{div} \boldsymbol{v})(\boldsymbol{x}) q(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}}{\|\boldsymbol{v}\|_{H^1(\Omega)^d}} \geq \beta \, \|q\|_{L^2(\Omega)}. \tag{2.12}$$

Thus, owing to equation (2.11), there exists a p in $L^2_\circ(\Omega)$ such that

$$\forall \boldsymbol{v} \in H^1_0(\Omega)^d, \quad \int_\Omega \nu(T)\, (\mathbf{grad}\, \boldsymbol{u})(\boldsymbol{x}) : (\mathbf{grad}\, \boldsymbol{v})(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} \\ + \int_\Omega \big((\boldsymbol{u} \cdot \nabla)\boldsymbol{u}\big)(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} - \langle \boldsymbol{f}, \boldsymbol{v} \rangle_\Omega = \int_\Omega (\operatorname{div} \boldsymbol{v})(\boldsymbol{x}) p(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}.$$

Then the triple $(\boldsymbol{u}, p, T)$ is a solution of problem (2.4) – (2.5), and estimate (2.7) is easily derived from (2.8) and (2.10).

Proposition 2.3. *Assume that the function ν is Lipschitz-continuous, with Lipschitz constant ν^*. There exist two positive constants $c_\sharp$ and $c_\flat$ such that*
(i) if the data $(\boldsymbol{f}, g)$ in $H^{-1}(\Omega)^d \times H^{-1}(\Omega)$ and T_0 in $H^{\frac{1}{2}}(\partial\Omega)$ satisfy

$$c_\sharp \big(\|\boldsymbol{f}\|_{H^{-1}(\Omega)^d} + \|g\|_{H^{-1}(\Omega)} + \|T_0\|_{H^{\frac{1}{2}}(\partial\Omega)} \big) < 1, \tag{2.13}$$

(ii) if problem (2.4) – (2.5) admits a solution $(\boldsymbol{u}, p, T)$ such that $\boldsymbol{u}$ belongs to $W^{1,q}(\Omega)^d$ with $q > 2$ in dimension $d = 2$ and $q \geq 3$ in dimension $d = 3$, and satisfies

$$c_\flat \nu^* \, |\boldsymbol{u}|_{W^{1,q}(\Omega)^d} < 1, \tag{2.14}$$

then this solution is unique.

Proof: For brevity, we set:

$$c_1 = c \big(\|\boldsymbol{f}\|_{H^{-1}(\Omega)^d} + \|g\|_{H^{-1}(\Omega)} + \|T_0\|_{H^{\frac{1}{2}}(\partial\Omega)} \big),$$

where c is the constant in (2.7). Let $(\boldsymbol{u}_1, p_1, T_1)$ and $(\boldsymbol{u}_2, p_2, T_2)$ be two solutions of problem (2.4) – (2.5), with $\boldsymbol{u}_1$ in $W^{1,q}(\Omega)^d$ satisfying (2.14). Setting for a while $\boldsymbol{u} = \boldsymbol{u}_1 - \boldsymbol{u}_2$, $p = p_1 - p_2$ and $T = T_1 - T_2$, we proceed in three steps.
1) It follows from the third equation in (2.5) that, since T belongs to $H^1_0(\Omega)$,

$$\alpha \, |T|^2_{H^1(\Omega)} = - \int_\Omega \big((\boldsymbol{u}_1 \cdot \nabla) T_1 - (\boldsymbol{u}_2 \cdot \nabla) T_2 \big)(\boldsymbol{x})\, T(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} = - \int_\Omega \big((\boldsymbol{u} \cdot \nabla) T_1(\boldsymbol{x})\, T(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x},$$

whence

$$\alpha \, |T|_{H^1(\Omega)} \leq c_1 c_2 \, |\boldsymbol{u}|_{H^1(\Omega)^d}, \tag{2.15}$$

where c_2 is the square of the norm of the imbedding of $H^1_0(\Omega)$ into $L^4(\Omega)$.
2) Similarly, we derive from the first equation in (2.5) that

$$\int_\Omega \nu(T_2)\, |\mathbf{grad}\, \boldsymbol{u}|^2(\boldsymbol{x})\, d\boldsymbol{x} = - \int_\Omega \big((\boldsymbol{u} \cdot \nabla) \boldsymbol{u}_1(\boldsymbol{x}) \cdot \boldsymbol{u}(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} \\ - \int_\Omega \big(\nu(T_1) - \nu(T_2) \big)\, (\mathbf{grad}\, \boldsymbol{u}_1)(\boldsymbol{x}) : (\mathbf{grad}\, \boldsymbol{u})(\boldsymbol{x})\, d\boldsymbol{x}.$$

Using appropriate Hölder's inequalities thus yields

$$\nu_1\,|\boldsymbol{u}|^2_{H^1(\Omega)^d} \le c_1 c_2\,|\boldsymbol{u}|^2_{H^1(\Omega)^d} + \nu^*\,|\boldsymbol{u}_1|_{W^{1,q}(\Omega)^d}\,c_3\,|T|_{H^1(\Omega)}|\boldsymbol{u}|_{H^1(\Omega)^d},$$

where c_3 stands for the norm of the imbedding of $H^1_0(\Omega)$ into $L^{q^*}(\Omega)$, with $\frac{1}{q}+\frac{1}{q^*}=\frac{1}{2}$. By combining this with (2.15) and choosing $c_\sharp$ and $c_\flat$ such that

$$c_1 c_2\,\nu_1^{-1}\left(1+\alpha^{-1}\nu^*\,|\boldsymbol{u}_1|_{W^{1,q}(\Omega)^d}\,c_3\right)<1,$$

we obtain that $\boldsymbol{u}$ is zero, so that $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are equal.
3) It then follows from (2.15) that T_1 and T_2 are equal. Finally, the function p satisfies

$$\forall \boldsymbol{v}\in H^1_0(\Omega)^d, \qquad -\int_\Omega (\operatorname{div}\boldsymbol{v})(\boldsymbol{x})\,p(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}=0,$$

so that it is zero (see [14, Chap. I, §2] for instance). Thus, p_1 and p_2 coincide.
This concludes the proof.

Assumptions (2.13) and (2.14) are clearly very restrictive and will not be used in what follows. We conclude with a regularity result.

Proposition 2.4. *There exist a real number $q_0>2$ depending on the geometry of Ω and on the ratio ν_2/ν_1 and a real number $q_1>1$ only depending on the geometry of Ω such that, for any q, $2\le q\le q_0$, and q', $1\le q'\le q_1$, and for any data $(\boldsymbol{f},g)$ in the space $W^{-1,q}(\Omega)^d\times L^{q'}(\Omega)$ and T_0 in $W^{2-\frac{1}{q'},q'}(\partial\Omega)$, any solution $(\boldsymbol{u},p,T)$ of problem (2.4)–(2.5) belongs to $W^{1,q}(\Omega)^d\times L^q(\Omega)\times W^{2,q'}(\Omega)$. Moreover, q_1 is $\ge\frac{4}{3}$ for a general domain Ω and ≥ 2 when Ω is convex.*

Proof: Proving the regularity of the velocity follows the approach in [17] and the arguments are exactly the same as for [1, Prop. 3.3]. The regularity of the pressure is a direct consequence of this. Finally, the regularity of the temperature is deduced from the standard properties of the Laplace operator (see [15, Thm 4.3.2.4], [10, Th. 2] or [11, Cor. 3.10]), combined with a boot-strap argument.

Remark 2.5. It can be checked that most of the results proved in this section still hold for more general boundary conditions than in (2.2), namely
(i) when the homogeneous boundary conditions on $\boldsymbol{u}$ are replaced by inhomogeneous ones,
(ii) when the Dirichlet boundary conditions on T are replaced by mixed ones

$$T=T_0 \quad \text{on } \Gamma_D \qquad \text{and} \qquad \partial_n T=T_1 \quad \text{on } \Gamma_N,$$

for any partition $\{\Gamma_D,\Gamma_N\}$ of $\partial\Omega$ without overlap such that both Γ_D and Γ_N have Lipschitz-continuous boundaries.

3. The simplified problem.

For the reasons explained in the introduction, we now assume that the domain Ω admits a partition into Ω_f and Ω_s satisfying

$$\overline{\Omega} = \overline{\Omega}_f \cup \overline{\Omega}_s \qquad \text{and} \qquad \Omega_f \cap \Omega_s = \emptyset, \tag{3.1}$$

where both Ω_f and Ω_s have Lipschitz–continuous boundaries (the indices f and s stand for "full" and "simplified", respectively).

We introduce a modified viscosity function ν^* defined by

$$\forall \tau \in \mathbb{R}, \quad \nu^*(\boldsymbol{x}, \tau) = \begin{cases} \nu(\tau) & \text{for a.e. } \boldsymbol{x} \text{ in } \Omega_f, \\ \nu_0 & \text{for a.e. } \boldsymbol{x} \text{ in } \Omega_s, \end{cases} \tag{3.2}$$

where ν_0 is a positive constant. Next, we consider the reduced problem

$$\begin{cases} -\operatorname{div}\big(\nu^*(\cdot, T^*)\, \nabla \boldsymbol{u}^*\big) + (\boldsymbol{u}^* \cdot \nabla)\boldsymbol{u}^* + \mathbf{grad}\, p^* = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u}^* = 0 & \text{in } \Omega, \\ -\alpha\, \Delta T^* + (\boldsymbol{u}^* \cdot \nabla) T^* = g & \text{in } \Omega, \end{cases} \tag{3.3}$$

still provided with the boundary conditions

$$\boldsymbol{u}^* = \boldsymbol{0} \quad \text{and} \quad T^* = T_0 \qquad \text{on } \partial\Omega. \tag{3.4}$$

There also, we are led to write its variational formulation

Find $(\boldsymbol{u}^*, p^*, T^*)$ in $H_0^1(\Omega)^d \times L_\circ^2(\Omega) \times H^1(\Omega)$ such that

$$T^* = T_0 \qquad \text{on } \partial\Omega, \tag{3.5}$$

and that

$$\begin{aligned} &\forall \boldsymbol{v} \in H_0^1(\Omega)^d, \quad \int_\Omega \nu^*(\boldsymbol{x}, T^*(\boldsymbol{x}))\, (\mathbf{grad}\, \boldsymbol{u}^*)(\boldsymbol{x}) : (\mathbf{grad}\, \boldsymbol{v})(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} \\ &\qquad + \int_\Omega \big((\boldsymbol{u}^* \cdot \nabla)\boldsymbol{u}^*\big)(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} - \int_\Omega (\operatorname{div} \boldsymbol{v})(\boldsymbol{x})\, p^*(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} = \langle \boldsymbol{f}, \boldsymbol{v} \rangle_\Omega, \\ &\forall q \in L_\circ^2(\Omega), \quad -\int_\Omega (\operatorname{div} \boldsymbol{u}^*)(\boldsymbol{x})\, q(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} = 0, \\ &\forall S \in H_0^1(\Omega), \quad \alpha \int_\Omega (\mathbf{grad}\, T^*)(\boldsymbol{x}) \cdot (\mathbf{grad}\, S)(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} \\ &\qquad + \int_\Omega \big((\boldsymbol{u}^* \cdot \nabla) T^*\big)(\boldsymbol{x})\, S(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} = \langle g, S \rangle_\Omega. \end{aligned} \tag{3.6}$$

We skip the proofs of the next two statements, since they are exactly the same as for Proposition 2.1 and Theorem 2.2 (with only ν_1 replaced by $\min\{\nu_0, \nu_1\}$).

Proposition 3.1. *Problems* (3.3) – (3.4) *and* (3.5) – (3.6) *are equivalent, in the sense made precise in Proposition* 2.1.

Theorem 3.2. *For any data* $(\boldsymbol{f}, g)$ *in* $H^{-1}(\Omega)^d \times H^{-1}(\Omega)$ *and* T_0 *in* $H^{\frac{1}{2}}(\partial\Omega)$, *problem* (3.5) – (3.6) *admits at least a solution* $(\boldsymbol{u}^*, p^*, T^*)$ *in* $H^1_0(\Omega)^d \times L^2_\circ(\Omega) \times H^1(\Omega)$.

We also give the regularity result. Its proof is similar to that of Proposition 2.4, however it must be noted that the q_0^* which appears in the next statement highly depends on the ratio $\nu_2/\min\{\nu_0, \nu_1\}$ and tends to 2 when ν_0 tends to zero.

Proposition 3.3. *Let* q_1 *be the real number introduced in Proposition* 2.4. *There exists a real number* $q_0^* > 2$ *depending on the geometry of* Ω *and on the ratio* $\nu_2/\min\{\nu_0, \nu_1\}$ *such that, for any* q, $2 \le q \le q_0^*$, *and* q', $1 \le q' \le q_1$, *and for any data* $(\boldsymbol{f}, g)$ *in the space* $W^{-1,q}(\Omega)^d \times L^{q'}(\Omega)$ *and* T_0 *in* $W^{2-\frac{1}{q'},q'}(\partial\Omega)$, *any solution* $(\boldsymbol{u}^*, p^*, T^*)$ *of problem* (3.5) – (3.6) *belongs to* $W^{1,q}(\Omega)^d \times L^q(\Omega) \times W^{2,q'}(\Omega)$.

To conclude this section, we give a first estimate of the distance between appropriate solutions $(\boldsymbol{u}, p, T)$ of problem (2.4) – (2.5) and $(\boldsymbol{u}^*, p^*, T^*)$ of problem (3.5) – (3.6). The proof relies on the arguments of [18] (see also [20, §2.1]) and requires some further notation. We first introduce the Stokes operator $\mathcal{S}$ which associates with any datum $\boldsymbol{f}$ in $H^{-1}(\Omega)^d$ the part $\boldsymbol{u}$ of the unique solution $(\boldsymbol{u}, p)$ of the problem

$$\begin{cases} -\nu_1\,\Delta\boldsymbol{u} + \mathbf{grad}\,p = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div}\boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{0} & \text{on } \partial\Omega. \end{cases} \tag{3.7}$$

Similarly, we introduce the inverse $\mathcal{L}$ of the Laplace operator which associates with any data (g, T_0) in $H^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ the solution T of the problem

$$\begin{cases} -\alpha\,\Delta T = g & \text{in } \Omega, \\ T = T_0 & \text{on } \partial\Omega. \end{cases} \tag{3.8}$$

Thus, it is readily checked that, when setting $U = (\boldsymbol{u}, T)$, problem (2.4) – (2.5) can be written equivalently as

$$\mathcal{F}(U) = U + \begin{pmatrix} \mathcal{S} & 0 \\ 0 & \mathcal{L} \end{pmatrix} \mathcal{G}(U) = 0, \qquad \text{with} \quad \mathcal{G}(U) = \begin{pmatrix} \operatorname{div}\big(\nu_1 - \nu(T)\big)\nabla\boldsymbol{u}\big) + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \boldsymbol{f} \\ \big((\boldsymbol{u}\cdot\nabla)T - g, T_0\big) \end{pmatrix} \tag{3.9}$$

(it must be noted that the equivalence property requires the inf-sup condition (2.12) already used in the proof of Theorem 2.2). Similarly, when setting $U^* = (\boldsymbol{u}^*, T^*)$, we observe that problem (3.5) – (3.6) can be written equivalently as

$$\mathcal{F}(U^*) = \mathcal{R}(U^*), \quad \text{with} \quad \mathcal{R}(U^*) = \begin{pmatrix} \mathcal{S} & 0 \\ 0 & \mathcal{L} \end{pmatrix} \begin{pmatrix} \operatorname{div}\big((\nu^*(\cdot, T^*) - \nu(T^*))\,\nabla\boldsymbol{u}^*\big) \\ (0,0) \end{pmatrix}. \tag{3.10}$$

Assuming that ν is of class $\mathscr{C}^1$ on $\mathbb{R}$, we now investigate the properties of the operator $D\mathcal{F}(U)$. First, we observe that, for any $(\boldsymbol{F}, G, R_0)$ in $H^{-1}(\Omega)^d \times H^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$, the pair $W = (\boldsymbol{w}, R)$ satisfies

$$D\mathcal{F}(U)W = W + \begin{pmatrix} \mathcal{S} & 0 \\ 0 & \mathcal{L} \end{pmatrix} \begin{pmatrix} \boldsymbol{F} \\ (G, R_0) \end{pmatrix},$$

if and only if there exists a r in $L^2_\circ(\Omega)$ such that the triple $(\boldsymbol{w}, r, R)$ is a solution of the following variational problem: *Find* $(\boldsymbol{w}, r, R)$ *in* $H^1_0(\Omega)^d \times L^2_\circ(\Omega) \times H^1(\Omega)$ *such that*

$$R = R_0 \qquad \text{on } \partial\Omega, \tag{3.11}$$

and that

$$\begin{aligned}
\forall \boldsymbol{v} \in H^1_0(\Omega)^d, \qquad & \int_\Omega \nu(T)\,(\mathbf{grad}\,\boldsymbol{w})(\boldsymbol{x}) : (\mathbf{grad}\,\boldsymbol{v})(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \\
& \quad + \int_\Omega \nu'(T)(\boldsymbol{x}) R(\boldsymbol{x})\,(\mathbf{grad}\,\boldsymbol{u})(\boldsymbol{x}) : (\mathbf{grad}\,\boldsymbol{v})(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \\
& \quad + \int_\Omega \big((\boldsymbol{u}\cdot\nabla)\boldsymbol{w} + (\boldsymbol{w}\cdot\nabla)\boldsymbol{u}\big)(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \\
& \quad - \int_\Omega (\mathrm{div}\,\boldsymbol{v})(\boldsymbol{x})\, r(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \langle \boldsymbol{F}, \boldsymbol{v}\rangle_\Omega, \qquad (3.12)\\
\forall q \in L^2_\circ((\Omega), \qquad & -\int_\Omega (\mathrm{div}\,\boldsymbol{w})(\boldsymbol{x})\, q(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = 0, \\
\forall S \in H^1_0(\Omega), \qquad & \alpha \int_\Omega (\mathbf{grad}\,R)(\boldsymbol{x}) \cdot (\mathbf{grad}\,S)(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \\
& \quad + \int_\Omega \big((\boldsymbol{u}\cdot\nabla)R + (\boldsymbol{w}\cdot\nabla)T\big)(\boldsymbol{x})\, S(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \langle \boldsymbol{G}, S\rangle_\Omega.
\end{aligned}$$

Even if this problem is rather complex, the only difficulty is to give a sense to the term

$$\int_\Omega \nu'(T)(\boldsymbol{x}) R(\boldsymbol{x})\,(\mathbf{grad}\,\boldsymbol{u})(\boldsymbol{x}) : (\mathbf{grad}\,\boldsymbol{v})(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}.$$

Indeed, we observe that, even if ν' is bounded on $\mathbb{R}$, the quantity R only belongs to any $L^q(\Omega)$, $q < +\infty$, in dimension $d = 2$ and to $L^6(\Omega)$ in dimension $d = 3$.

Lemma 3.4. *Assume that ν is of class $\mathscr{C}^1$ on $\mathbb{R}$, with Lipschitz-continuous derivative. For all $\rho > d$, the mapping: $U \mapsto D\mathcal{F}(U)$ is continuous from $W^{1,\rho}_0(\Omega)^d \times W^{1,\rho}(\Omega)$ into the space of endomorphisms of $H^1_0(\Omega)^d \times H^1_0(\Omega)$. Moreover, there exists a $\lambda > 0$ such that it is Lipschitz-continuous on the ball*

$$B(U,\lambda) = \big\{(\boldsymbol{v}, S) \in W^{1,\rho}_0(\Omega)^d \times W^{1,\rho}(\Omega);\, \|\boldsymbol{v} - \boldsymbol{u}\|_{W^{1,\rho}(\Omega)^d} + \|S - T\|_{H^1(\Omega)} \le \lambda\big\}. \tag{3.13}$$

We are now in a position to estimate the distance between a solution $(\boldsymbol{u}, p, T)$ of problem (2.4) – (2.5) and all smooth enough solutions $(\boldsymbol{u}^*, p^*, T^*)$ of problem (3.5) – (3.6) in a neighbourhood of $(\boldsymbol{u}, p, T)$.

Theorem 3.5. *Assume that ν is of class $\mathscr{C}^1$ on $\mathbb{R}$, with Lipschitz-continuous derivative. Let $(\boldsymbol{u}, p, T)$ be a solution of problem (2.4) – (2.5) such that $U = (\boldsymbol{u}, T)$ belongs to $W_0^{1,\rho}(\Omega)^d \times W^{1,\rho}(\Omega)$, $\rho > d$, and $D\mathcal{F}(U)$ is an isomorphism of $H_0^1(\Omega)^d \times H_0^1(\Omega)$. Thus, there exists a neighbourhood of U in $W_0^{1,\rho}(\Omega)^d \times W^{1,\rho}(\Omega)$ and a constant c only depending on U such that the following estimate holds for any solution $(\boldsymbol{u}^*, p^*, T^*)$ of problem (3.5) – (3.6) such that $U^* = (\boldsymbol{u}^*, T^*)$ belongs to this neighbourhood*

$$\begin{aligned} &\|\boldsymbol{u} - \boldsymbol{u}^*\|_{H^1(\Omega)^d} + \|p - p^*\|_{L^2(\Omega)} + \|T - T^*\|_{H^1(\Omega)} \\ &\qquad\qquad \leq c\,\big\|\big(\nu^*(\cdot, T^*) - \nu(T^*)\big)\,\nabla \boldsymbol{u}^*\big\|_{L^2(\Omega)^{d\times d}}. \end{aligned} \tag{3.14}$$

Proof: Owing to Lemma 3.4, applying a slight extension of [18, Thm 1] (see also [20, Prop. 2.1]), we obtain

$$\|\boldsymbol{u} - \boldsymbol{u}^*\|_{H^1(\Omega)^d} + \|T - T^*\|_{H^1(\Omega)} \leq c\,\|\mathcal{R}(U^*)\|_{H^1(\Omega)^d \times H^1(\Omega)},$$

where the constant c only depends on the norm of $D\mathcal{F}(U)^{-1}$. By noting that $\mathcal{S}$ is continuous from $H^{-1}(\Omega)^d$ into $H^1(\Omega)^d$ and also that the divergence operator is continuous from $L^2(\Omega)^{d\times d}$ into $H^{-1}(\Omega)^d$, we obtain the desired estimate for $\|\boldsymbol{u}-\boldsymbol{u}^*\|_{H^1(\Omega)^d}+\|T-T^*\|_{H^1(\Omega)}$. Finally, writing the equation satisfied by $p - p^*$ and using the inf-sup condition (2.12) give the estimate for $\|p - p^*\|_{L^2(\Omega)}$.

The regularity assumptions on $(\boldsymbol{u}, p, T)$ and $(\boldsymbol{u}^*, p^*, T^*)$ are not at all restrictive in dimension $d = 2$, see Propositions 2.4 and 3.3, but they are in dimension $d = 3$. On the other hand, the assumption of non-singularity which is made on $(\boldsymbol{u}, p, T)$, i.e., the fact $D\mathcal{F}(U)$ is an isomorphism, only means that this solution is locally unique and is much less restrictive than the global uniqueness condition, see Proposition 2.3.

4. The discrete problem and its a priori analysis.

From now on, we assume that Ω is a polygon or a polyhedron. In order to describe the discrete problem, we introduce a regular family $(\mathcal{T}_h)_h$ of triangulations of Ω by closed triangles ($d = 2$) or tetrahedra ($d = 3$), in the usual sense that
- for each h, $\overline{\Omega}$ is the union of all elements of $\mathcal{T}_h$,
- for each h, the intersection of two different elements of $\mathcal{T}_h$, if not empty, is a corner, a whole edge or a whole face of both elements,
- the ratio of the diameter h_K of an element K in $\mathcal{T}_h$ to the diameter of its inscribed circle or sphere is bounded by a constant independent of K and h.

As standard, h denotes the maximum of the diameters of the elements of $\mathcal{T}_h$. We make the further assumption that each element K of $\mathcal{T}_h$ is contained either in $\overline{\Omega}_f$ or in $\overline{\Omega}_s$ (this condition is not at all restrictive since our adaptivity strategy, first proposed in [1, §2], consists mainly in moving elements of the triangulation from Ω_s into Ω_f). From now on, c, c', ... stand for generic constants which may vary from line to line but are always independent of h.

For each nonnegative integer m and any K in $\mathcal{T}_h$, let $\mathcal{P}_m(K)$ denote the space of restrictions to K of polynomials with d variables and total degree $\leq m$. As standard for the Stokes problem, we have decided to work with the Taylor–Hood finite elements, see [16] (or also [6, §VI.3] or [14, Chapter II, §4]). Consequently, the discrete spaces of velocities and pressures are defined by

$$\begin{aligned}\mathbb{X}_h &= \big\{\boldsymbol{v}_h \in H_0^1(\Omega)^d;\ \forall K \in \mathcal{T}_h,\ \boldsymbol{v}_h|_K \in \mathcal{P}_2(K)^d\big\},\\ \mathbb{M}_h &= \big\{q_h \in H^1(\Omega) \cap L^2_\circ(\Omega);\ \forall K \in \mathcal{T}_h,\ q_h|_K \in \mathcal{P}_1(K)\big\}.\end{aligned} \tag{4.1}$$

We also use piecewise quadratic functions for approximating the temperature T in order to preserve the convergence order equal to 2 for the previous elements. So we introduce the discrete space

$$\mathbb{Y}_h = \big\{S_h \in H^1(\Omega);\ \forall K \in \mathcal{T}_h,\ S_h|_K \in \mathcal{P}_2(K)\big\}, \qquad \mathbb{Y}_h^0 = \mathbb{Y}_h \cap H_0^1(\Omega). \tag{4.2}$$

To define an approximate boundary condition, we make use of the interpolation operator $i_h^{\partial\Omega}$: For any function φ continuous on $\partial\Omega$,
(i) the restriction of $i_h^{\partial\Omega}\varphi$ to any edge or face $K \cap \partial\Omega$, $K \in \mathcal{T}_h$, belongs to $\mathcal{P}_2(K \cap \partial\Omega)$,
(ii) $i_h^{\partial\Omega}\varphi$ is equal to φ at the vertices and midpoints ($d = 2$), at the vertices and midpoints of the edges ($d = 3$) of $K \cap \partial\Omega$.
We set: $T_{0h} = i_h^{\partial\Omega} T_0$.

The discrete problem is then built from (3.5) – (3.6) by the Galerkin method. It reads

Find $(\boldsymbol{u}_h, p_h, T_h)$ *in* $\mathbb{X}_h \times \mathbb{M}_h \times \mathbb{Y}_h$ *such that*

$$T_h = T_{0h} \qquad \text{on } \partial\Omega, \tag{4.3}$$

and that

$$\begin{aligned}
\forall \boldsymbol{v}_h \in \mathbb{X}_h, \qquad & \int_\Omega \nu^*(\boldsymbol{x}, T_h(\boldsymbol{x}))\,(\mathbf{grad}\,\boldsymbol{u}_h)(\boldsymbol{x}) \,:\, (\mathbf{grad}\,\boldsymbol{v}_h)(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \\
& + \int_\Omega \big((\boldsymbol{u}_h\,\cdot\,\nabla)\boldsymbol{u}_h\big)(\boldsymbol{x})\,\cdot\,\boldsymbol{v}_h(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} - \int_\Omega (\mathrm{div}\,\boldsymbol{v}_h)(\boldsymbol{x})\,p_h(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \langle \boldsymbol{f}, \boldsymbol{v}_h\rangle_\Omega, \\
\forall q_h \in \mathbb{M}_h, \qquad & -\int_\Omega (\mathrm{div}\,\boldsymbol{u}_h)(\boldsymbol{x})\,q_h(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = 0, \\
\forall S_h \in \mathbb{Y}_h^0, \qquad & \alpha \int_\Omega (\mathbf{grad}\,T_h)(\boldsymbol{x})\,\cdot\,(\mathbf{grad}\,S_h)(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \\
& + \int_\Omega \big((\boldsymbol{u}_h\,\cdot\,\nabla)T_h\big)(\boldsymbol{x})\,S_h(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \langle g, S_h\rangle_\Omega.
\end{aligned} \tag{4.4}$$

Remark 4.1. In the implementation of this problem, $\nu^*(\cdot, T_h)$ is most often replaced by the function $\nu_h^*(\cdot, T_h)$ constructed by Lagrange interpolation: For any continuous function τ on $\overline{\Omega}$, we denote by $\nu_h(\tau)$ the function such that its restriction to any K in $\mathcal{T}_h$ belongs to $\mathcal{P}_1(K)$ and which is equal to $\nu(\tau)$ at all vertices of K. Thus, we set:

$$\nu_h^*(\boldsymbol{x}, \tau) = \begin{cases} \nu_h(\tau) & \text{for all } \boldsymbol{x} \text{ in } \overline{\Omega_f}, \\ \nu_0 & \text{for all } \boldsymbol{x} \text{ in } \overline{\Omega}_s. \end{cases} \tag{4.5}$$

We do not take into account this modification in the a priori analysis for simplicity.

We recall the existence of a discrete inf-sup condition between the spaces $\mathbb{X}_h$ and $\mathbb{M}_h$, see [6, §VI.6] and [14, Chap. II, Cor. 4.1]: There exists a constant $\beta^* > 0$ independent of h such that

$$\forall q_h \in \mathbb{M}_h, \qquad \sup_{\boldsymbol{v}_h \in \mathbb{X}_h} \frac{\int_\Omega (\mathrm{div}\,\boldsymbol{v}_h)(\boldsymbol{x}) q_h(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}}{\|\boldsymbol{v}_h\|_{H^1(\Omega)^d}} \ge \beta^*\, \|q_h\|_{L^2(\Omega)}. \tag{4.6}$$

Thus, the existence of a solution for problem (4.3) – (4.4) could be obtained by the same arguments as for Theorem 2.2. However we prefer to derive a more precise result by following the approach in [7]. This requires some further notation.

For any real-valued measurable function τ on Ω, we introduce the modified Stokes operator $\mathcal{S}(\tau)$ which associates with any datum $\boldsymbol{F}$ in $H^{-1}(\Omega)^d$ the part $\boldsymbol{u}$ of the solution $(\boldsymbol{u}, p)$ of the generalized Stokes problem

$$\begin{cases} -\mathrm{div}\,\big(\nu^*(\cdot, \tau)\,\nabla \boldsymbol{u}\big) + \mathbf{grad}\,p = \boldsymbol{F} & \text{in } \Omega, \\ \mathrm{div}\,\boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{0} & \text{on } \partial\Omega. \end{cases} \tag{4.7}$$

The operator $\mathcal{L}$ being defined in (3.8), problem (3.5) – (3.6) can be written as

$$\begin{aligned}
&\mathcal{F}^*(U^*) = U^* + \begin{pmatrix} \mathcal{S}(T^*) & 0 \\ 0 & \mathcal{L} \end{pmatrix} \mathcal{G}^*(U^*) = 0, \\
&\qquad\qquad \text{with} \qquad \mathcal{G}^*(U^*) = \begin{pmatrix} (\boldsymbol{u}^*\,\cdot\,\nabla)\boldsymbol{u}^* - \boldsymbol{f} \\ \big((\boldsymbol{u}^*\,\cdot\,\nabla)T^* - g, T_0\big) \end{pmatrix}.
\end{aligned} \tag{4.8}$$

Similarly, let $\mathcal{S}_h(\tau)$ denote the discrete Stokes operator, *i.e.*, the operator which associates with any data $\boldsymbol{F}$ in $H^{-1}(\Omega)^d$, the part $\boldsymbol{u}_h$ of the solution $(\boldsymbol{u}_h, p_h)$ in $\mathbb{X}_h \times \mathbb{M}_h$ of the Stokes problem

$$\begin{aligned}
&\forall \boldsymbol{v}_h \in \mathbb{X}_h, \quad \int_\Omega \nu^*(\boldsymbol{x},\tau)\, \nabla \boldsymbol{u}_h : \nabla \boldsymbol{v}_h \, \mathrm{d}\boldsymbol{x} - \int_\Omega (\operatorname{div} \boldsymbol{v}_h)(\boldsymbol{x}) p_h(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} = \langle \boldsymbol{F}, \boldsymbol{v}_h \rangle_\Omega, \\
&\forall q_h \in \mathbb{M}_h, \quad -\int_\Omega (\operatorname{div} \boldsymbol{u}_h)(\boldsymbol{x}) q_h(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} = 0.
\end{aligned} \tag{4.9}$$

Let finally $\mathcal{L}_h$ denote the operator which associates with any datum G in $H^{-1}(\Omega)$ and any continuous function R_0 in $H^{\frac{1}{2}}(\partial\Omega)$, the function R_h in $\mathbb{Y}_h$, equal to $i_h^{\partial\Omega} R_0$ on $\partial\Omega$ and which satisfies

$$\forall S_h \in \mathbb{Y}_h^0, \quad \alpha \int_\Omega (\mathbf{grad}\, R_h)(\boldsymbol{x}) \cdot (\mathbf{grad}\, S_h)(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} = \langle G, S_h \rangle_\Omega. \tag{4.10}$$

Due to the inf-sup condition (4.6) and with the notation $U_h = (\boldsymbol{u}_h, T_h)$, problem (4.3) – (4.4) can equivalently be written as

$$\mathcal{F}_h(U_h) = U_h + \begin{pmatrix} \mathcal{S}_h(T_h) & 0 \\ 0 & \mathcal{L}_h \end{pmatrix} \mathcal{G}^*(U_h) = 0. \tag{4.11}$$

We recall the basic properties of the discrete operators $\mathcal{S}_h(\tau)$ and $\mathcal{L}_h$. A simple extension of [14, Chap. II, Thm 4.3] yields that the operator $\mathcal{S}_h(\tau)$ satisfies the following properties: For any $\boldsymbol{F}$ in $H^{-1}(\Omega)^d$,

$$\|\mathcal{S}_h(\tau)\boldsymbol{F}\|_{H^1(\Omega)^d} \le c\, \|\boldsymbol{F}\|_{H^{-1}(\Omega)^d}, \tag{4.12}$$

and, if moreover $\boldsymbol{F}$ belongs to $H^{s-1}(\Omega)^d$ and $\mathcal{S}(\tau)\boldsymbol{F}$ to $H^{s+1}(\Omega)^d$ for a real number s, $0 \le s \le 2$,

$$\|\big(\mathcal{S}(\tau) - \mathcal{S}_h(\tau)\big)\boldsymbol{F}\|_{H^1(\Omega)^d} \le c\, h^s \left(\|\mathcal{S}(\tau)\boldsymbol{F}\|_{H^{s+1}(\Omega)^d} + \|\boldsymbol{F}\|_{H^{s-1}(\Omega)^d}\right). \tag{4.13}$$

The analogous properties concerning the operator $\mathcal{L}_h$ are also standard [2, Chap. X, Th. 1.1 & 1.2]: For any G in $H^{-1}(\Omega)$,

$$\|\mathcal{L}_h(G, 0)\|_{H^1(\Omega)} \le c\, \|G\|_{H^{-1}(\Omega)}, \tag{4.14}$$

and, if moreover $\mathcal{L}G$ belongs to $H^{s+1}(\Omega)$ for a real number s, $0 \le s \le 2$, and R_0 belongs to $H^{\sigma+\frac{1}{2}}(\partial\Omega)$, $\frac{d}{2} - 1 < \sigma \le \frac{5}{2}$,

$$\|(\mathcal{L} - \mathcal{L}_h)(G, R_0)\|_{H^1(\Omega)} \le c\left(h^s\, \|\mathcal{L}G\|_{H^{s+1}(\Omega)} + h^\sigma\, \|R_0\|_{H^{\sigma+\frac{1}{2}}(\partial\Omega)}\right). \tag{4.15}$$

Note that these properties yield the following convergence result, for any $\boldsymbol{F}$ in $H^{-1}(\Omega)^d$ and any G in $H^{-1}(\Omega)$,

$$\lim_{h\to 0} \left(\|\big(\mathcal{S}(\tau) - \mathcal{S}_h(\tau)\big)\boldsymbol{F}\|_{H^1(\Omega)^d} + \|(\mathcal{L} - \mathcal{L}_h)(G, 0)\|_{H^1(\Omega)}\right) = 0. \tag{4.16}$$

We now work with a fixed solution $(\boldsymbol{u}^*, p^*, T^*)$ of problem (3.5) – (3.6). In view of the next lemmas, we are led to make some assumptions on it, which are very similar to the assumptions required for Theorem 3.5. From now on, we denote by $\mathcal{X}(\Omega)$ the product $H_0^1(\Omega)^d \times H_0^1(\Omega)$ and by $\mathcal{E}$ the space of endomorphims of $\mathcal{X}(\Omega)$.

Assumption 4.2. The solution $(\boldsymbol{u}^*, p^*, T^*)$ of problem (3.5) – (3.6) satisfies:
(i) the velocity $\boldsymbol{u}^*$ belongs to $W^{1,\rho}(\Omega)^d$ and the temperature T^* belongs to $W^{1,\rho}(\Omega)$, for some $\rho > d$,
(ii) the pair $U^* = (\boldsymbol{u}^*, T^*)$ is such that $D\mathcal{F}^*(U^*)$ is an isomorphism of $\mathcal{X}(\Omega)$.

The next lemmas require the parameter λ_h defined by

$$\lambda_h = \begin{cases} |\log h_{\min}|^{\frac{1}{2}} & \text{if } d = 2, \\ h_{\min}^{-\frac{1}{2}} & \text{if } d = 3, \end{cases} \qquad \text{with} \quad h_{\min} = \min_{K \in \mathcal{T}_h} h_K. \tag{4.17}$$

Lemma 4.3. *Assume that ν is of class $\mathscr{C}^2$ on $\mathbb{R}$, with bounded derivatives. Let $(\boldsymbol{u}^*, p^*, T^*)$ be a solution of problem (3.5) – (3.6) such that Assumption 4.2 holds and which belongs to $H^{s+1}(\Omega)^d \times H^s(\Omega) \times H^{s+1}(\Omega)$ for a real number s, $0 \le s \le 2$. If the following condition is satisfied*

$$\lim_{h \to 0} \lambda_h \, h^s = 0, \tag{4.18}$$

there exists an $h_0 > 0$ such that, for all $h \le h_0$, $D\mathcal{F}_h(U^*)$ is an isomorphism of $\mathcal{X}(\Omega)$ and the norm of its inverse is bounded independently of h.

Proof: We write the expansion

$$D\mathcal{F}_h(U^*) = D\mathcal{F}^*(U^*) - \begin{pmatrix} (\mathcal{S} - \mathcal{S}_h)(T^*) & 0 \\ 0 & \mathcal{L} - \mathcal{L}_h \end{pmatrix} D\mathcal{G}^*(U^*) - \begin{pmatrix} D(\mathcal{S} - \mathcal{S}_h)(T^*) & 0 \\ 0 & 0 \end{pmatrix} \mathcal{G}^*(U^*).$$

So, owing to Assumption 4.2, we obtain the desired result if the last two terms in the right-hand side tend to zero in the norm of the space $\mathcal{E}$.
1) We observe that

$$D\mathcal{G}^*(U^*).W = \begin{pmatrix} (\boldsymbol{u}^* \cdot \nabla)\boldsymbol{w} + (\boldsymbol{w} \cdot \nabla)\boldsymbol{u}^* \\ ((\boldsymbol{u}^* \cdot \nabla)R + (\boldsymbol{w} \cdot \nabla)T^*, 0) \end{pmatrix}.$$

Thus, when W runs through the unit sphere of $\mathcal{X}(\Omega)$, the compactness of the imbedding of $H^1(\Omega)$ into $L^q(\Omega)$, with $q < \infty$ in dimension $d = 2$ and $q < 6$ in dimension $d = 3$, combined with the regularity of $\boldsymbol{u}^*$, yields that both terms $(\boldsymbol{u}^* \cdot \nabla)\boldsymbol{w} + (\boldsymbol{w} \cdot \nabla)\boldsymbol{u}^*$ and $(\boldsymbol{u}^* \cdot \nabla)R + (\boldsymbol{w} \cdot \nabla)T^*$ belong to a compact subset of $H^{-1}(\Omega)^d$ and $H^{-1}(\Omega)$, respectively. Combining all this with (4.16) leads to

$$\lim_{h \to 0} \left\| \begin{pmatrix} (\mathcal{S} - \mathcal{S}_h)(T^*) & 0 \\ 0 & \mathcal{L} - \mathcal{L}_h \end{pmatrix} D\mathcal{G}^*(U^*) \right\|_{\mathcal{E}} = 0.$$

2) On the other hand, we note that, for any $\boldsymbol{F}$ in $H^{-1}(\Omega)^d$,

$$\begin{aligned}\big(D\mathcal{S}(T^*)R\big)\boldsymbol{F} = \mathcal{S}(T^*)\big(-\mathrm{div}\,(\partial_\tau\nu^*(\cdot,T^*)R\,\nabla\mathcal{S}(T^*)\boldsymbol{F})\big),\\ \big(D\mathcal{S}_h(T^*)R\big)\boldsymbol{F} = \mathcal{S}_h(T^*)\big(-\mathrm{div}\,(\partial_\tau\nu^*(\cdot,T^*)R\,\nabla\mathcal{S}_h(T^*)\boldsymbol{F})\big).\end{aligned} \tag{4.19}$$

By subtracting the second line from the first one, we derive

$$\begin{aligned}\big(D(\mathcal{S}-\mathcal{S}_h)(T^*)R\big)\boldsymbol{F} = (\mathcal{S}-\mathcal{S}_h)(T^*)\big(-\mathrm{div}\,(\partial_\tau\nu^*(\cdot,T^*)R\,\nabla\mathcal{S}(T^*)\boldsymbol{F})\big)\\ + \mathcal{S}_h(T^*)\big(-\mathrm{div}\,(\partial_\tau\nu^*(\cdot,T^*)R\,\nabla(\mathcal{S}-\mathcal{S}_h)(T^*)\boldsymbol{F})\big).\end{aligned}$$

Denoting by $\boldsymbol{F}$ the first component of $\mathcal{G}^*(U^*)$, we see that $\mathcal{S}(T^*)\boldsymbol{F}$ is equal to $-\boldsymbol{u}^*$, see (4.8). First, using the compactness of the imbedding of $H^1(\Omega)$ into $L^r(\Omega)$ for any $r<\infty$ in dimension $d=2$ and $r<6$ in dimension $d=3$, we deduce from the regularity assumption on $\boldsymbol{u}^*$ that, when W runs through the unit sphere of $\mathcal{X}(\Omega)$, the quantity $-\mathrm{div}\,(\partial_\tau\nu^*(\cdot,T^*)R\,\nabla\mathcal{S}(T^*)\boldsymbol{F})$ belongs to a compact subset of $H^{-1}(\Omega)^d$. Thus, the convergence of the first term to zero follows from (4.16). To handle the second term, we observe from (4.12) that it suffices to prove the convergence of $\|\nabla(\mathcal{S}-\mathcal{S}_h)(T^*)\boldsymbol{F})\|_{L^{q^*}(\Omega)^{d\times d}}$, with $\frac{1}{q}+\frac{1}{q^*}=\frac{1}{2}$ for the q introduced in the beginning of the proof. Since $\mathcal{S}(T^*)\boldsymbol{F}$ coincides with $-\boldsymbol{u}^*$, by using a standard inverse inequality [2, Chap. VIII, Prop. 5.1], we obtain for any function $\boldsymbol{w}_h$ in $\mathbb{X}_h$

$$\begin{aligned}\|\nabla(\mathcal{S}-\mathcal{S}_h)(T^*)\boldsymbol{F})\|_{L^{q^*}(\Omega)^{d\times d}} \le \|\boldsymbol{u}^*-\boldsymbol{w}_h\|_{W^{1,q^*}(\Omega)^d}\\ + c\,h_{\min}^{\frac{d}{q^*}-\frac{d}{2}}\big(\|\boldsymbol{u}^*-\boldsymbol{w}_h\|_{H^1(\Omega)^d} + \|(\mathcal{S}-\mathcal{S}_h)(T^*)\boldsymbol{F}\|_{H^1(\Omega)^d}\big).\end{aligned}$$

Thus, taking $\boldsymbol{w}_h$ equal to the Lagrange interpolate of $\boldsymbol{u}^*$ in $\mathbb{X}_h$ and using its standard approximation properties [2, Chap. IX, Th. 1.6] combined with (4.13) (note also that the regularity of $\boldsymbol{F}$ is easily derived from that of the solution $(\boldsymbol{u}^*,p^*,T^*)$) lead to

$$\|\nabla(\mathcal{S}-\mathcal{S}_h)(T^*)\boldsymbol{F})\|_{L^{q^*}(\Omega)^{d\times d}} \le c\,h^{s-\frac{d}{q}}\,\|\boldsymbol{u}^*\|_{H^{s+1}(\Omega)^d} + c(\boldsymbol{u}^*,p^*,T^*)\,h_{\min}^{-\frac{d}{q}}\,h^s,$$

where $c(\boldsymbol{u}^*,p^*,T^*)$ only depends on the norm of the triple $(\boldsymbol{u}^*,p^*,T^*)$ in $H^{s+1}(\Omega)^d\times H^s(\Omega)\times H^{s+1}(\Omega)$. Moreover, in dimension $d=2$, we recall from [19] that the norm of the imbedding of $H^1(\Omega)$ into $L^q(\Omega)$ behaves like $\sqrt{q}$ and we take q equal to $\log h_{\min}$, so that in all cases, the norm of this imbedding times $h_{\min}^{-\frac{d}{q}}$ is $\le c\,\lambda_h$. Thus, owing to assumption (4.18) we derive

$$\lim_{h\to 0}\Big\|\begin{pmatrix} D(\mathcal{S}-\mathcal{S}_h)(T^*) & 0\\ 0 & 0\end{pmatrix}\mathcal{G}^*(U^*)\Big\|_{\mathcal{E}} = 0.$$

This concludes the proof.

Lemma 4.4. *Assume that ν is of class $\mathscr{C}^1$ on $\mathbb{R}$, with Lipschitz-continuous derivative. Then, the mapping $D\mathcal{F}_h$ satisfies the following Lipschitz property, for all V_1 and V_2 in a bounded subset of $\mathcal{X}(\Omega)$,*

$$\|D\mathcal{F}_h(V_1)-D\mathcal{F}_h(V_2)\|_{\mathcal{E}} \le c\,\lambda_h^2\,\|V_1-V_2\|_{\mathcal{X}(\Omega)}, \tag{4.20}$$

where λ_h is defined in (4.17).

Proof: It is readily checked that the stability estimates (4.12) and (4.14) can be replaced by

$$\|\mathcal{S}_h \boldsymbol{F}\|_{H^1(\Omega)^d} \le c \sup_{\boldsymbol{v}_h \in \mathbb{X}_h} \frac{\langle \boldsymbol{F}, \boldsymbol{v}_h \rangle_\Omega}{\|\boldsymbol{v}_h\|_{H^1(\Omega)^d}}, \qquad \|\mathcal{L}_h(G,0)\|_{H^1(\Omega)} \le c \sup_{S_h \in \mathbb{Y}_h^0} \frac{\langle G, S_h \rangle_\Omega}{\|S_h\|_{H^1(\Omega)}}. \quad (4.21)$$

So, by writing the expansion of $D\mathcal{F}_h(V_1) - D\mathcal{F}_h(V_2)$ and using formula (4.19), we have to bound the integrals, for W running through the unit sphere of $\mathcal{X}(\Omega)$ and any $(\boldsymbol{v}_h, S_h)$ in $\mathbb{X}_h \times \mathbb{Y}_h^0$,

$$\int_\Omega \big(\nu^*(\boldsymbol{x}, S_1) - \nu^*(\boldsymbol{x}, S_2)\big)(\boldsymbol{x})(\mathbf{grad}\, \boldsymbol{w})(\boldsymbol{x}) \,:\, (\mathbf{grad}\, \boldsymbol{v}_h)(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x},$$

$$\int_\Omega \big(\partial_\tau \nu^*(\boldsymbol{x}, S_1)\mathbf{grad}\, \boldsymbol{v}_1 - \partial_\tau \nu^*(\boldsymbol{x}, S_2)\mathbf{grad}\, \boldsymbol{v}_2\big)(\boldsymbol{x}) R(\boldsymbol{x}) \,:\, (\mathbf{grad}\, \boldsymbol{v}_h)(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x},$$

$$\int_\Omega \Big(\big((\boldsymbol{v}_1 - \boldsymbol{v}_2)\cdot\nabla\big)\boldsymbol{w} + (\boldsymbol{w}\cdot\nabla)(\boldsymbol{v}_1 - \boldsymbol{v}_2)\Big)(\boldsymbol{x}) \,\cdot\, \boldsymbol{v}_h(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x},$$

$$\int_\Omega \Big(\big((\boldsymbol{v}_1 - \boldsymbol{v}_2)\cdot\nabla\big)R + (\boldsymbol{w}\cdot\nabla)(T_1 - T_2)\Big)(\boldsymbol{x}) S_h(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}.$$

For brevity, we only bound the first two integrals since evaluating the third and fourth ones is easier. We denote them by A_1 and A_2. We now take $q < \infty$ in dimension $d = 2$ and $q = 6$ in dimension $d = 3$.
1) There exists a constant c only depending on the Lipschitz property of ν such that, for q^* defined by $\frac{1}{q} + \frac{1}{q^*} = \frac{1}{2}$,

$$\big|A_1\big| \le c\, \|S_1 - S_2\|_{L^q(\Omega)} \|\boldsymbol{v}_h\|_{W^{1,q^*}(\Omega)^d}.$$

By using a standard inverse inequality [2, Chap. VIII, Prop. 5.1], we obtain

$$\big|A_1\big| \le c\, h_{\min}^{\frac{d}{q^*} - \frac{d}{2}} \|S_1 - S_2\|_{L^q(\Omega)} \|\boldsymbol{v}_h\|_{H^1(\Omega)^d} = c\, h_{\min}^{-\frac{d}{q}} \|S_1 - S_2\|_{L^q(\Omega)} \|\boldsymbol{v}_h\|_{H^1(\Omega)^d}.$$

This gives the right estimate in dimension $d = 3$. In dimension $d = 2$, we use the same argument as in the previous proof: Since the norm of the imbedding of $H^1(\Omega)$ into $L^q(\Omega)$ behaves like $\sqrt{q}$ (see [19]), we take q equal to $\log h_{\min}$. We thus derive

$$\big|A_1\big| \le c\, \lambda_h\, \|S_1 - S_2\|_{H^1(\Omega)} \|\boldsymbol{v}_h\|_{H^1(\Omega)^d}. \quad (4.22)$$

2) By using the boundedness and Lipschitz property of $\partial_\tau \nu$ together with a triangle inequality, we derive (recall that $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ belong to a bounded set of $H^1(\Omega)^d$)

$$\big|A_2\big| \le c\, \big(\|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_{H^1(\Omega)^d} \|R\|_{L^q(\Omega)} \|\boldsymbol{v}_h\|_{W^{1,q^*}(\Omega)} + \|S_1 - S_2\|_{L^q(\Omega)} \|R\|_{L^q(\Omega)} \|\boldsymbol{v}_h\|_{W^{1,q^{**}}(\Omega)}\big),$$

where q^* is defined as previously and q^{**} satisfies $\frac{2}{q} + \frac{1}{q^{**}} = \frac{1}{2}$. Then, the same arguments as in part 1) yield

$$\big|A_2\big| \le c\, \big(\lambda_h\, \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_{H^1(\Omega)^d} + \lambda_h^2\, \|S_1 - S_2\|_{H^1(\Omega)}\big) \|\boldsymbol{v}_h\|_{H^1(\Omega)^d}. \quad (4.23)$$

Combining (4.22), (4.23) and similar estimates for the two last integrals leads to the desired result.

Lemma 4.5. *Assume that ν is of class $\mathscr{C}^2$ on $\mathbb{R}$ and that the solution $(\boldsymbol{u}^*, p^*, T^*)$ of problem* (3.5) – (3.6) *belongs to $H^{s+1}(\Omega)^d \times H^s(\Omega) \times H^{s+1}(\Omega)$ for a real number s, $\frac{d}{2} - 1 < s \leq 2$. Then, the following bound holds*

$$\|\mathcal{F}_h(U^*)\|_{\mathcal{X}(\Omega)} \leq c\, h^s \left(\|\boldsymbol{u}^*\|_{H^{s+1}(\Omega)^d} + \|p^*\|_{H^s(\Omega)} + \|T^*\|_{H^{s+1}(\Omega)}\right). \tag{4.24}$$

Proof: By using equation (4.8), we observe that

$$\|\mathcal{F}_h(U^*)\|_{\mathcal{X}(\Omega)} = \|\mathcal{F}^*(U^*) - \mathcal{F}_h(U^*)\|_{\mathcal{X}(\Omega)} = \left\| \begin{pmatrix} (\mathcal{S} - \mathcal{S}_h)(T^*) & 0 \\ 0 & \mathcal{L} - \mathcal{L}_h \end{pmatrix} \mathcal{G}^*(U^*) \right\|_{\mathcal{X}(\Omega)}.$$

So, the desired estimate follows from (4.13) and (4.15) by noting that:
1) If $\boldsymbol{F}$ denotes the first component of $\mathcal{G}^*(U^*)$, $\mathcal{S}\boldsymbol{F}$ is equal to $-\boldsymbol{u}^*$ and $\boldsymbol{F}$ is equal to $\operatorname{div}\big(\nu^*(\cdot, T^*)\, \nabla \boldsymbol{u}^*\big) - \mathbf{grad}\, p^*$, hence belongs to $H^{s-1}(\Omega)^d$ (note that the complete proof requires [4, Thm 1']);
2) Owing to the trace theorem, the σ in (4.15) is larger than s.

Thanks to Lemmas 4.3 to 4.5, we are now in a position to prove the main result of this section.

Theorem 4.6. *Let $(\boldsymbol{u}^*, p^*, T^*)$ be a solution of problem* (3.5) – (3.6) *which satisfies Assumption* 4.2 *and belongs to $H^{s+1}(\Omega)^d \times H^s(\Omega) \times H^{s+1}(\Omega)$, $\frac{d}{2} - 1 < s \leq 2$. We moreover assume that the function ν is of class $\mathscr{C}^2$ on $\mathbb{R}$ with bounded derivatives and that*

$$\lim_{h \to 0} \lambda_h^2\, h^s = 0, \tag{4.25}$$

where λ_h is defined in (4.17). *Then, there exist positive numbers κ and h_0 such that, for any $h \leq h_0$, problem* (4.3) – (4.4) *has a unique solution $(\boldsymbol{u}_h, p_h, T_h)$ such that $(\boldsymbol{u}_h, T_h)$ belongs to the ball of $\mathcal{X}(\Omega)$ with centre $(\boldsymbol{u}^*, T^*)$ and radius $\kappa\, \lambda_h^{-2}$. Moreover this solution satisfies*

$$\|\boldsymbol{u}^* - \boldsymbol{u}_h\|_{H^1(\Omega)^d} + \|p^* - p_h\|_{L^2(\Omega)} + \|T^* - T_h\|_{H^1(\Omega)} \leq c(\boldsymbol{u}^*, p^*, T^*)\, h^s, \tag{4.26}$$

where the constant $c(\boldsymbol{u}^*, p^*, T^*)$ only depends on the solution $(\boldsymbol{u}^*, p^*, T^*)$.

Proof: By applying the Brezzi–Rappaz–Raviart theorem [7] (this requires Lemmas 4.3 to 4.5), we obtain the existence of a solution, its local uniqueness and the desired estimate for $\|\boldsymbol{u}^* - \boldsymbol{u}_h\|_{H^1(\Omega)^d}$ and $\|T^* - T_h\|_{H^1(\Omega)}$. To go further, we observe that the discrete pressure p_h satisfies

$$\int_\Omega (\operatorname{div} \boldsymbol{v}_h)(\boldsymbol{x})\, p_h(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} = \int_\Omega \nu^*(\boldsymbol{x}, T_h(\boldsymbol{x}))\, (\mathbf{grad}\, \boldsymbol{u}_h)(\boldsymbol{x}) : (\mathbf{grad}\, \boldsymbol{v}_h)(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} + \int_\Omega \big((\boldsymbol{u}_h \cdot \nabla)\boldsymbol{u}_h\big)(\boldsymbol{x}) \cdot \boldsymbol{v}_h(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} - \langle \boldsymbol{f}, \boldsymbol{v}_h \rangle_\Omega,$$

whence, for any q_h in $\mathbb{M}_h$,

$$\begin{aligned} &\int_\Omega (\operatorname{div} \boldsymbol{v}_h)(\boldsymbol{x})\, (p_h - q_h)(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} \\ &\quad = \int_\Omega \big(\nu^*(\boldsymbol{x}, T_h(\boldsymbol{x}))\, (\mathbf{grad}\, \boldsymbol{u}_h)(\boldsymbol{x}) - \nu^*(\boldsymbol{x}, T^*(\boldsymbol{x}))\, (\mathbf{grad}\, \boldsymbol{u}^*)(\boldsymbol{x})\big) : (\mathbf{grad}\, \boldsymbol{v}_h)(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} \\ &\qquad + \int_\Omega \big((\boldsymbol{u}_h \cdot \nabla)\boldsymbol{u}_h - (\boldsymbol{u}^* \cdot \nabla)\boldsymbol{u}^*\big)(\boldsymbol{x}) \cdot \boldsymbol{v}_h(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} + \int_\Omega (\operatorname{div} \boldsymbol{v}_h)(\boldsymbol{x})\, (p - q_h)(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}. \end{aligned}$$

So, by using successively the inf-sup condition (4.6), triangle inequalities and the error estimates on $\boldsymbol{u}^*$ and T^*, we derive the estimate for $\|p^* - p_h\|_{L^2(\Omega)}$.

Estimate (4.26) is fully optimal. Moreover the regularity assumptions on the solution $(\boldsymbol{u}^*, p^*, T^*)$ and condition (4.25) are not all restrictive in dimension $d = 2$ (see Proposition 3.3). But, even if they can be weakened by handling separately the boundary condition T_0, they are not at all likely in dimension $d = 3$. However, condition (4.25) with $s = 2$ seems sufficient to prove the convergence of the discretization.

Remark 4.7. It is readily checked that, in dimension $d = 2$, all the previous results hold for any triple $(\mathbb{X}_h, \mathbb{M}_h, \mathbb{Y}_h)$ of finite element spaces satisfying the inf-sup condition (4.6). In dimension $d = 3$, this requires the further assumption that these elements are of order 2 (in order that (4.25) can hold).

5. A posteriori analysis.

We first recall some notation and define the error indicators which are needed for the a posteriori analysis. Next, we prove successively upper and lower bounds for the error as a function of these indicators. In conclusion, we sum up these rather technical results.

5.1. Some notation and the error indicators.

We denote by $\mathcal{T}_h^{(f)}$ and $\mathcal{T}_h^{(s)}$ the sets of elements of $\mathcal{T}_h$ which are contained in $\overline{\Omega}_f$ and $\overline{\Omega}_s$, respectively. With each K in $\mathcal{T}_h$, we associate:
- the set $\mathcal{E}_K$ of edges ($d = 2$) or faces ($d = 3$) of K which are not contained in $\partial\Omega$,
- the union ω_K of all elements of $\mathcal{T}_h$ that share at least an edge ($d = 2$) or a face ($d = 3$) with K.

We denote by h_K the diameter of any K in $\mathcal{T}_h$ and by h_e the length or diameter of any e in $\mathcal{E}_K$. For each K in $\mathcal{T}_h$ and each e in $\mathcal{E}_K$, being given a unit normal vector $\boldsymbol{n}$ to e, we agree to denote by $[\cdot]_e$ the jump through e in the direction of $\boldsymbol{n}$.

We introduce the approximations $\boldsymbol{f}_h$ of $\boldsymbol{f}$ and g_h of g which are constant on each element K of $\mathcal{T}_h$, equal to the mean values of $\boldsymbol{f}$ and g on K, respectively. As already hinted in Remark 4.1, we also consider the Lagrange interpolate of $\nu(\cdot)$: For any continuous function τ on $\overline{\Omega}$, we denote by $\nu_h(\tau)$ the function such that its restriction to any K in $\mathcal{T}_h$ belongs to $\mathcal{P}_1(K)$ and which is equal to $\nu(\tau)$ at all vertices of K. The function $\nu_h^*(\cdot, \tau)$ is then defined by (4.5).

We are now in a position to define the two kinds of error indicators.

(i) Indicators linked to the simplication of the model
In view of Theorem 3.5, we set, for all K in $\mathcal{T}_h^{(s)}$,

$$\eta_K^{(s)} = \|\big(\nu_0 - \nu_h(T_h)\big)\,\nabla \boldsymbol{u}_h\|_{L^2(K)^{d\times d}}. \tag{5.1}$$

(ii) Indicators linked to the finite element discretization
These indicators are of residual type and are fully standard, see [20, §1.2] for instance. We set, for all K in $\mathcal{T}_h$,

$$\eta_K^{(d)} = \eta_K^{(d)1} + \eta_K^{(d)2}, \tag{5.2}$$

with

$$\begin{aligned}\eta_K^{(d)1} = h_K\,\|\boldsymbol{f}_h + \operatorname{div}\big(\nu_h^*(\cdot, T_h)\,\nabla \boldsymbol{u}_h\big) - (\boldsymbol{u}_h\,\cdot\,\nabla)\boldsymbol{u}_h - \mathbf{grad}\,p_h\|_{L^2(K)^d} \\ + \sum_{e\in\mathcal{E}_K} h_e^{\frac{1}{2}}\,\|\nu_h^*(\cdot, T_h)\,[\partial_n \boldsymbol{u}_h]_e\|_{L^2(e)^d} + \|\operatorname{div}\boldsymbol{u}_h\|_{L^2(K)},\end{aligned} \tag{5.3}$$

and

$$\eta_K^{(d)2} = h_K\,\|g_h + \alpha\,\Delta T_h - (\boldsymbol{u}_h\,\cdot\,\nabla)T_h\|_{L^2(K)} + \sum_{e\in\mathcal{E}_K} h_e^{\frac{1}{2}}\,\|\alpha\,[\partial_n T_h]_e\|_{L^2(e)}. \tag{5.4}$$

It must be noted that all these indicators are easy to compute once the discrete solution $(\boldsymbol{u}_h, p_h, T_h)$ is known.

5.2. Upper bounds for the error.

As standard for multi-step discretizations and since we wish to uncouple the two parts of the error, we use the triangle inequalities

$$\begin{aligned}\|\boldsymbol{u}-\boldsymbol{u}_h\|_{H^1(\Omega)^d} &\le \|\boldsymbol{u}-\boldsymbol{u}^*\|_{H^1(\Omega)^d} + \|\boldsymbol{u}^*-\boldsymbol{u}_h\|_{H^1(\Omega)^d},\\ \|p-p_h\|_{L^2(\Omega)} &\le \|p-p^*\|_{L^2(\Omega)} + \|p^*-p_h\|_{L^2(\Omega)},\\ \|T-T_h\|_{H^1(\Omega)} &\le \|T-T^*\|_{H^1(\Omega)} + \|T^*-T_h\|_{H^1(\Omega)}.\end{aligned} \tag{5.5}$$

Bounding the error between the solutions of the full and simplified problems is a consequence of Theorem 3.5. We first evaluate the quantity

$$\varepsilon_K = \|\big(\nu(T_h)-\nu_h(T_h)\big)\,\nabla\boldsymbol{u}_h\|_{L^2(K)^{d\times d}}. \tag{5.6}$$

Lemma 5.1. *Assume that the function ν is of class $\mathscr{C}^1$ on $\mathbb{R}$, with bounded derivative. Thus, the following estimate holds*

$$\varepsilon_K \le c\,h_K\,\|\mathbf{grad}\,T_h\|_{L^\infty(K)^d}\|\nabla\boldsymbol{u}_h\|_{L^2(K)^{d\times d}}. \tag{5.7}$$

Proof: We have
$$\varepsilon_K \le \|\nu(T_h)-\nu_h(T_h)\|_{L^\infty(K)}\|\nabla\boldsymbol{u}_h\|_{L^2(K)^{d\times d}}.$$
Then, the desired result follows from the fact that $\nu_h(\tau)$ is a Lagrange interpolate of $\nu(\tau)$, see [2, Chap. IX, Lemme 1.1] for instance.

Proposition 5.2. *Assume that ν is of class $\mathscr{C}^1$ on $\mathbb{R}$, with Lipschitz-continuous derivative. Let $(\boldsymbol{u},p,T)$ be a solution of problem (2.4) – (2.5) and $(\boldsymbol{u}^*,p^*,T^*)$ be the corresponding solution of problem (3.5) – (3.6) exhibited in Theorem 3.5. If both $U=(\boldsymbol{u},T)$ and $U^*=(\boldsymbol{u}^*,T^*)$ belong to $W_0^{1,\rho}(\Omega)^d\times W^{1,\rho}(\Omega)$, $\rho>d$, and $D\mathcal{F}(U)$ is an isomorphism of $H_0^1(\Omega)^d\times H_0^1(\Omega)$, the following a posteriori estimate holds for the error between these solutions*

$$\begin{aligned}&\|\boldsymbol{u}-\boldsymbol{u}^*\|_{H^1(\Omega)^d} + \|p-p^*\|_{L^2(\Omega)} + \|T-T^*\|_{H^1(\Omega)}\\ &\qquad\le c\Big(\sum_{K\in\mathcal{T}_h^{(s)}}(\eta_K^{(s)})^2+\varepsilon_K^2\Big)^{\frac12} + c'\,\big(\|\boldsymbol{u}^*-\boldsymbol{u}_h\|_{H^1(\Omega_s)^d}+\|T^*-T_h\|_{H^1(\Omega_s)}\big),\end{aligned} \tag{5.8}$$

where the constants c and c' only depend on the norms of $\boldsymbol{u}^*$ and T^*.

Proof: It follows from Theorem 3.5 that

$$\|\boldsymbol{u}-\boldsymbol{u}^*\|_{H^1(\Omega)^d} + \|p-p^*\|_{L^2(\Omega)} + \|T-T^*\|_{H^1(\Omega)} \le c\,\Big(\sum_{K\in\mathcal{T}_h^{(s)}}(R_K^{(s)})^2\Big)^{\frac12},$$

where each $R_K^{(s)}$ is defined by

$$R_K^{(s)} = \|\big(\nu_0 - \nu(T^*)\big)\,\nabla \boldsymbol{u}^*\|_{L^2(K)^{d\times d}}.$$

We then use triangle inequalities:

$$\begin{aligned} R_K^{(s)} \le \eta_K^{(s)} &+ \|\big(\nu_h(T_h) - \nu(T_h)\big)\,\nabla \boldsymbol{u}_h\|_{L^2(K)^{d\times d}} \\ &+ \|\big(\nu_0 - \nu(T_h)\big)\,\nabla(\boldsymbol{u}^* - \boldsymbol{u}_h)\|_{L^2(K)^{d\times d}} + \|\big(\nu(T_h) - \nu(T^*)\big)\,\nabla \boldsymbol{u}^*\|_{L^2(K)^{d\times d}}. \end{aligned} \tag{5.9}$$

The second term in the right-hand side coincides with ε_K and the third term in the right-hand side is easily bounded:

$$\|\big(\nu_0 - \nu(T_h)\big)\,\nabla(\boldsymbol{u}^* - \boldsymbol{u}_h)\|_{L^2(K)^{d\times d}} \le |\nu_2 - \nu_0|\,\|\boldsymbol{u}^* - \boldsymbol{u}_h\|_{H^1(K)^d}.$$

Bounding the last term follows from a Hölder's inequality, the Lipschitz property of ν and the imbedding of $H^1(\Omega)$ into $L^{\rho'}(\Omega)$, with $\frac{1}{\rho} + \frac{1}{\rho'} = \frac{1}{2}$.

Note that the quantity ε_K comes from the replacement of ν by ν_h (as standard for non constant coefficients, see [20, §3.3]) and is most often negligible with respect to the other terms.

On the other hand, bounding the discretization error again relies on the approach in [18]. Indeed, with the same notation as in Section 3, we observe that the part $U^* = (\boldsymbol{u}^*, T^*)$ of the solution of problem (3.5) – (3.6) satisfies

$$\begin{aligned} \mathcal{F}^{**}(U^*) = U^* + \begin{pmatrix} \mathcal{S} & 0 \\ 0 & \mathcal{L} \end{pmatrix} \mathcal{G}^{**}(U^*) = 0, \\ \text{with} \qquad \mathcal{G}^{**}(U) = \begin{pmatrix} \operatorname{div}\big((\nu_1 - \nu^*(\cdot,T))\,\nabla \boldsymbol{u}\big) + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \boldsymbol{f} \\ \big((\boldsymbol{u}\cdot\nabla)T - g, T_0\big) \end{pmatrix}. \end{aligned} \tag{5.10}$$

Moreover, exactly the same arguments as for the proof of Lemma 3.4 yield that, when ν has a Lipschitz-continuous derivative, this lemma still holds with $\mathcal{F}$ replaced by $\mathcal{F}^{**}$.

Proposition 5.3. *Assume that ν is of class $\mathscr{C}^1$ on $\mathbb{R}$, with Lipschitz-continuous derivative. Let $(\boldsymbol{u}^*, p^*, T^*)$ be a solution of problem (3.5) – (3.6) such that $U^* = (\boldsymbol{u}^*, T^*)$ belongs to $W_0^{1,\rho}(\Omega)^d \times W^{1,\rho}(\Omega)$, $\rho > d$, and $D\mathcal{F}^*(U^*)$ is an isomorphism of $H_0^1(\Omega)^d \times H^1(\Omega)$. Thus, the following a posteriori estimate holds for the error between this solution and the solution $(\boldsymbol{u}_h, p_h, T_h)$ of problem (4.3) – (4.4) exhibited in Theorem 4.6*

$$\begin{aligned} \|\boldsymbol{u}^* - \boldsymbol{u}_h\|_{H^1(\Omega)^d} + \|p^* - p_h\|_{L^2(\Omega)} + \|T^* - T_h\|_{H^1(\Omega)} \\ \le c\Big(\sum_{K\in\mathcal{T}_h} (\eta_K^{(d)})^2\Big)^{\frac{1}{2}} + c'\Big(\sum_{K\in\mathcal{T}_h^{(f)}} \varepsilon_K^2\Big)^{\frac{1}{2}} \\ + c''\Big(\sum_{K\in\mathcal{T}_h} h_K^2\,\big(\|\boldsymbol{f} - \boldsymbol{f}_h\|_{L^2(K)^d}^2 + \|g - g_h\|_{L^2(K)}^2\big)\Big)^{\frac{1}{2}} + c'''\,\|T_0 - T_{0h}\|_{H^{\frac{1}{2}}(\partial\Omega)}, \end{aligned} \tag{5.11}$$

where the constants c, c', c'' and c''' only depend on the norms of $\boldsymbol{u}^*$ and T^*.

Proof: We first derive from the inf-sup condition (2.12) that there exists a function $\boldsymbol{w}$ in $H_0^1(\Omega)^d$ such that

$$\operatorname{div} \boldsymbol{w} = \operatorname{div} \boldsymbol{u}_h \qquad \text{and} \qquad \|\boldsymbol{w}\|_{H^1(\Omega)^d} \leq c\, \|\operatorname{div} \boldsymbol{u}_h\|_{L^2(\Omega)}. \tag{5.12}$$

The pair $\tilde{U}_h = (\tilde{\boldsymbol{u}}_h, T_h)$, with $\tilde{\boldsymbol{u}}_h = \boldsymbol{u}_h - \boldsymbol{w}$, thus satisfies the following residual equation, for all $\boldsymbol{v}$ in $H_0^1(\Omega)^d$ and $\boldsymbol{v}_h$ in $\mathbb{X}_h$,

$$\begin{aligned}
&\int_\Omega \nu^*(\boldsymbol{x}, T_h(\boldsymbol{x}))\, (\mathbf{grad}\, \tilde{\boldsymbol{u}}_h)(\boldsymbol{x}) \,:\, (\mathbf{grad}\, \boldsymbol{v})(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} \\
&\qquad + \int_\Omega \big((\tilde{\boldsymbol{u}}_h \cdot \nabla)\tilde{\boldsymbol{u}}_h\big)(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} - \int_\Omega (\operatorname{div} \boldsymbol{v})(\boldsymbol{x})\, p_h(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} - \langle \boldsymbol{f}, \boldsymbol{v}\rangle_\Omega \\
&= -r(\boldsymbol{w}) - \langle \boldsymbol{f}, \boldsymbol{v} - \boldsymbol{v}_h\rangle_\Omega + \int_\Omega \nu^*(\boldsymbol{x}, T_h(\boldsymbol{x}))\, (\mathbf{grad}\, \boldsymbol{u}_h)(\boldsymbol{x}) \,:\, (\mathbf{grad}\, (\boldsymbol{v} - \boldsymbol{v}_h))(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} \\
&\quad + \int_\Omega \big((\boldsymbol{u}_h \cdot \nabla)\boldsymbol{u}_h\big)(\boldsymbol{x}) \cdot (\boldsymbol{v} - \boldsymbol{v}_h) v(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} - \int_\Omega (\operatorname{div} (\boldsymbol{v} - \boldsymbol{v}_h)(\boldsymbol{x})\, p_h(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x},
\end{aligned} \tag{5.13}$$

and, for any S in $H_0^1(\Omega)$ and S_h in $\mathbb{Y}_h^0$,

$$\begin{aligned}
\alpha &\int_\Omega (\mathbf{grad}\, T_h)(\boldsymbol{x}) \cdot (\mathbf{grad}\, S)(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} + \int_\Omega \big((\boldsymbol{u}_h \cdot \nabla) T_h\big)(\boldsymbol{x})\, S(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} - \langle g, S\rangle_\Omega \\
&= -\langle g, S - S_h\rangle_\Omega + \alpha \int_\Omega (\mathbf{grad}\, T_h)(\boldsymbol{x}) \cdot (\mathbf{grad}\, (S - S_h))(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} \\
&\quad + \int_\Omega \big((\boldsymbol{u}_h \cdot \nabla) T_h\big)(\boldsymbol{x})\, (S - S_h)(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x},
\end{aligned} \tag{5.14}$$

where the quantity $r(\boldsymbol{w})$ is defined by

$$\begin{aligned}
r(\boldsymbol{w}) = &\int_\Omega \nu^*(\boldsymbol{x}, T_h(\boldsymbol{x}))\, (\mathbf{grad}\, \boldsymbol{w})(\boldsymbol{x}) \,:\, (\mathbf{grad}\, \boldsymbol{v})(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} + \int_\Omega \big((\boldsymbol{w} \cdot \nabla)\boldsymbol{w}\big)(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} \\
&+ \int_\Omega \big((\tilde{\boldsymbol{u}}_h \cdot \nabla)\boldsymbol{w}\big)(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} + \int_\Omega \big((\boldsymbol{w} \cdot \nabla)\tilde{\boldsymbol{u}}_h\big)(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}.
\end{aligned}$$

It is clear that the right-hand sides of equations (5.13) and (5.14), together with the final term $T_{0h} - T_0$, represent the residual $\mathcal{F}^*(\tilde{U}_h)$. On the other hand, it follows from [20, Prop. 2.1] for instance that

$$\|\boldsymbol{u}^* - \tilde{\boldsymbol{u}}_h\|_{H^1(\Omega)^d} + \|T^* - T_h\|_{H^1(\Omega)} \leq c\, \|\mathcal{F}^*(\tilde{U}_h)\|_{H_0^1(\Omega)^d \times H_0^1(\Omega)}.$$

To go further, we use the stability property of the operators $\mathcal{S}$ and $\mathcal{L}$, together with the estimate

$$|r(\boldsymbol{w})| \leq c\, \|\boldsymbol{w}\|_{H^1(\Omega)^d} \|\boldsymbol{v}\|_{H^1(\Omega)^d},$$

where owing to estimate (4.26) the constant c only depends on the data; we insert ν_h, $\boldsymbol{f}_h$ and g_h in the previous right-hand sides, we integrate by parts on each K in $\mathcal{T}_h$ and use Cauchy–Schwarz inequalities; finally we take $\boldsymbol{v}_h$ and S_h equal to the images of $\boldsymbol{v}$ and S by a Clément type operator, see [2, §IX.3] for instance. All this proves the desired estimate for $\|\boldsymbol{u}^* - \tilde{\boldsymbol{u}}_h\|_{H^1(\Omega)^d}$ and $\|T^* - T_h\|_{H^1(\Omega)}$. To obtain the bound for $\|\boldsymbol{u}^* - \boldsymbol{u}_h\|_{H^1(\Omega)^d}$, we

use a triangle inequality and (5.12).
Finally, to derive the estimate on the pressure, denoting by $\mathcal{R}_h$ the right-hand side of equation (5.13), we observe that this equation can be written equivalently

$$\begin{aligned}\int_\Omega (\operatorname{div} \boldsymbol{v})(\boldsymbol{x})\,(p-p_h)(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} &= \int_\Omega \nu^*(\boldsymbol{x},T^*(\boldsymbol{x}))\,(\mathbf{grad}\,\boldsymbol{u}^*)(\boldsymbol{x}) : (\mathbf{grad}\,\boldsymbol{v})(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\\ &\quad - \int_\Omega \nu^*(\boldsymbol{x},T_h(\boldsymbol{x}))\,(\mathbf{grad}\,\boldsymbol{u}_h)(\boldsymbol{x}) : (\mathbf{grad}\,\boldsymbol{v})(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\\ &\quad + \int_\Omega \big((\boldsymbol{u}^*\cdot\nabla)\boldsymbol{u}^*\big)(\boldsymbol{x})\cdot\boldsymbol{v}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} - \int_\Omega \big((\boldsymbol{u}_h\cdot\nabla)\boldsymbol{u}_h\big)(\boldsymbol{x})\cdot\boldsymbol{v}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} + \mathcal{R}_h.\end{aligned}$$

We conclude by applying the inf-sup condition (2.12) thanks to exactly the same arguments as previously and the already established estimates.

5.3. Upper bounds for the indicators.

We successively prove upper bounds for the indicators $\eta_K^{(s)}$ on one hand, $\eta_K^{(d)1}$ and $\eta_K^{(d)2}$ on the other hand.

Proposition 5.4. *Assume that ν is Lipschitz-continuous on $\mathbb{R}$ and that $\nu_0 \le \nu_1$. Let $(\boldsymbol{u}^*, p^*, T^*)$ be a solution of problem (3.5) – (3.6) such that $U^* = (\boldsymbol{u}^*, T^*)$ belongs to $W_0^{1,\rho}(\Omega)^d \times W^{1,\rho}(\Omega)$, $\rho > d$. Thus, the following estimate holds for the correponding error indicators $\eta_K^{(s)}$ defined in (5.1),*

$$\begin{aligned}\Big(\sum_{K\in\mathcal{T}_h^{(s)}} (\eta_K^{(s)})^2\Big)^{\frac12} &\le c\Big(\|\boldsymbol{u}-\boldsymbol{u}^*\|_{H^1(\Omega)^d} + \|p-p^*\|_{L^2(\Omega)} + \|T-T^*\|_{H^1(\Omega)}\\ &\quad + \|\boldsymbol{u}^*-\boldsymbol{u}_h\|_{H^1(\Omega_s)^d} + \|T^*-T_h\|_{H^1(\Omega_s)} + \big(\sum_{K\in\mathcal{T}_h^{(s)}} \varepsilon_K^2\big)^{\frac12}\Big).\end{aligned} \tag{5.15}$$

Proof: When subtracting the first equation in (3.6) from the first equation in (2.5), we obtain for all $\boldsymbol{v}$ in $H_0^1(\Omega)^d$,

$$\begin{aligned}&\int_\Omega \nu(T)(\boldsymbol{x})\,\big(\mathbf{grad}\,(\boldsymbol{u}-\boldsymbol{u}^*)\big)(\boldsymbol{x}) : (\mathbf{grad}\,\boldsymbol{v})(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\\ &\quad + \int_\Omega \big(\nu(T)-\nu(T^*)\big)(\boldsymbol{x})\,(\mathbf{grad}\,\boldsymbol{u}^*)(\boldsymbol{x}) : (\mathbf{grad}\,\boldsymbol{v})(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\\ &\quad + \int_\Omega \big((\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - (\boldsymbol{u}^*\cdot\nabla)\boldsymbol{u}^*\big)(\boldsymbol{x})\cdot\boldsymbol{v}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} - \int_\Omega (\operatorname{div}\boldsymbol{v})(\boldsymbol{x})\,(p-p^*)(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\\ &\quad = \int_{\Omega_s} \big(\nu_0-\nu(T^*)\big)(\boldsymbol{x})\,(\mathbf{grad}\,\boldsymbol{u}^*)(\boldsymbol{x}) : (\mathbf{grad}\,\boldsymbol{v})(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}.\end{aligned}$$

We observe that the right-hand member can be written as the sum

$$\int_{\Omega_s} \big(\nu_0 - \nu_h(T_h)\big)(\boldsymbol{x})\,(\mathbf{grad}\,\boldsymbol{u}_h)(\boldsymbol{x}) \,:\, (\mathbf{grad}\,\boldsymbol{v})(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$$

$$+ \int_{\Omega_s} \big(\nu_h(T_h) - \nu(T_h)\big)(\boldsymbol{x})\,(\mathbf{grad}\,\boldsymbol{u}_h)(\boldsymbol{x}) \,:\, (\mathbf{grad}\,\boldsymbol{v})(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$$

$$+ \int_{\Omega_s} \big(\nu_0 - \nu(T_h)\big)(\boldsymbol{x})\,\big(\mathbf{grad}\,(\boldsymbol{u}^* - \boldsymbol{u}_h)\big)(\boldsymbol{x}) \,:\, (\mathbf{grad}\,\boldsymbol{v})(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$$

$$+ \int_{\Omega_s} \big(\nu(T_h) - \nu(T^*)\big)(\boldsymbol{x})\,(\mathbf{grad}\,\boldsymbol{u}^*)(\boldsymbol{x}) \,:\, (\mathbf{grad}\,\boldsymbol{v})(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}.$$

Thus, by taking $\boldsymbol{v}$ equal to $\boldsymbol{u}_h$, noting that, since $\nu_0 - \nu_h(T_h)$ is nonpositive,

$$\sum_{K\in\mathcal{T}_h^{(s)}} (\eta_K^{(s)})^2 \leq -|\nu_2 - \nu_0| \int_{\Omega_s} \big(\nu_0 - \nu_h(T_h)\big)(\boldsymbol{x})\,(\mathbf{grad}\,\boldsymbol{u}_h)(\boldsymbol{x}) \,:\, (\mathbf{grad}\,\boldsymbol{v})(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x},$$

and using a large number of Cauchy-Schwarz inequalities in the previous equations, we obtain the desired result.

Bounding the indicators $\eta_K^{(d)1}$ and $\eta_K^{(d)2}$ relies on fully standard arguments, see [20, Prop. 1.5] for instance. So we only give an abridged proof of the estimate for $\eta_K^{(d)2}$ where the notation is simpler.

Proposition 5.5. *Assume that ν is Lipschitz-continuous on $\mathbb{R}$. Let $(\boldsymbol{u}^*, p^*, T^*)$ be a solution of problem* (3.5) – (3.6) *such that $U^* = (\boldsymbol{u}^*, T^*)$ belongs to $W_0^{1,\rho}(\Omega)^d \times W^{1,\rho}(\Omega)$, $\rho > d$. Thus, the following estimate holds for each error indicator $\eta_K^{(d)1}$ defined in* (5.3)*, $K \in \mathcal{T}_h$,*

$$\eta_K^{(d)1} \leq c\Big(\|\boldsymbol{u}^* - \boldsymbol{u}_h\|_{H^1(\omega_K)^d} + \|p^* - p_h\|_{L^2(\omega_K)} + \|T^* - T_h\|_{H^1(\omega_K)} + \sum_{\kappa\subset\omega_K} \big(h_\kappa\,\|\boldsymbol{f} - \boldsymbol{f}_h\|_{L^2(\kappa)^d} + \varepsilon_\kappa\big)\Big). \tag{5.16}$$

Proposition 5.6. *The following estimate holds for each error indicator $\eta_K^{(d)2}$ defined in* (5.4)*, $K \in \mathcal{T}_h$,*

$$\eta_K^{(d)2} \leq c\Big(\|\boldsymbol{u}^* - \boldsymbol{u}_h\|_{H^1(\omega_K)^d} + \|T^* - T_h\|_{H^1(\omega_K)} + \sum_{\kappa\subset\omega_K} h_\kappa\,\|g - g_h\|_{L^2(\kappa)}\Big). \tag{5.17}$$

Proof: From problems (3.5) – (3.6) and (4.3) – (4.4), we derive the residual equation, for any S in $H_0^1(\Omega)$,

$$\alpha \int_\Omega \big(\mathbf{grad}\,(T^* - T_h)\big)(\boldsymbol{x}) \cdot (\mathbf{grad}\,S)(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} + \int_\Omega \big((\boldsymbol{u}^* \cdot \nabla)T^* - (\boldsymbol{u}_h \cdot \nabla)T_h\big)(\boldsymbol{x})\,S(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$$

$$= \sum_{K\in\mathcal{T}_h} \Big(\int_K (g - g_h)(\boldsymbol{x})S(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} + \int_K \big(g_h + \alpha\,\Delta T_h - (\boldsymbol{u}_h \cdot \nabla)T_h\big)(\boldsymbol{x})S(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$$

$$+ \frac{1}{2}\sum_{e\in\mathcal{E}_K} \int_e \alpha\,[\partial_n T_h]_e(\boldsymbol{\tau})S(\boldsymbol{\tau})\,\mathrm{d}\boldsymbol{\tau}\Big).$$

We then bound separately each term in $\eta_K^{(d)2}$.
1) We first take S equal to S_K, with

$$S_K = \begin{cases} \big(g_h + \alpha\,\Delta T_h - (\boldsymbol{u}_h\,\cdot\,\nabla)T_h\big)\psi_K & \text{in } K, \\ 0, & \text{in } \Omega\setminus K, \end{cases}$$

where ψ_K stands for the bubble function on K, equal to the product of the barycentric coordinates associated with the vertices of K. Thus standard inverse inequalities, see [20, Lemma 3.3], lead to the bound for the first term.
2) For each edge or face e in $\mathcal{E}_K$, denoting by K' the other element of $\mathcal{T}_h$ that contains e, we take S equal to S_e, with

$$S_e = \begin{cases} \mathcal{L}_{e,\kappa}\big(\alpha\,[\partial_n T_h]_e\psi_e\big) & \text{in } \kappa\in\{K,K'\}, \\ 0, & \text{in } \Omega\setminus(K\cup K'), \end{cases}$$

where ψ_e is now the bubble function on e and $\mathcal{L}_{e,\kappa}$ denotes a lifting operator from functions vanishing on ∂e into functions vanishing on $\partial\kappa\setminus e$, constructed from a fixed lifting operator on the reference element. The same inverse inequalities as previously and other ones give the bound for the second term.

5.4. Conclusions.

Up to the terms

$$\Big(\sum_{K\in\mathcal{T}_h} h_K^2\left(\|\boldsymbol{f}-\boldsymbol{f}_h\|^2_{L^2(K)^d} + \|g-g_h\|^2_{L^2(K)}\right)\Big)^{\frac{1}{2}} \qquad \text{and} \qquad \|T_0 - T_{0h}\|_{H^{\frac{1}{2}}(\partial\Omega)}, \tag{5.18}$$

which only depend on the data, the full error

$$\begin{aligned} E = \|\boldsymbol{u}-\boldsymbol{u}^*\|_{H^1(\Omega)^d} + \|p-p^*\|_{L^2(\Omega)} + \|T-T^*\|_{H^1(\Omega)} \\ + \|\boldsymbol{u}^*-\boldsymbol{u}_h\|_{H^1(\Omega)^d} + \|p^*-p_h\|_{L^2(\Omega)} + \|T^*-T_h\|_{H^1(\Omega)} \end{aligned} \tag{5.19}$$

satisfies the following equivalence property

$$\begin{aligned} c\Big(\sum_{K\in\mathcal{T}_h^{(s)}}(\eta_K^{(s)})^2 + \sum_{K\in\mathcal{T}_h}(\eta_K^{(d)})^2 - \sum_{K\in\mathcal{T}_h}\varepsilon_K^2\Big)^{\frac{1}{2}} \le E \\ \le c'\Big(\sum_{K\in\mathcal{T}_h^{(s)}}(\eta_K^{(s)})^2 + \sum_{K\in\mathcal{T}_h}(\eta_K^{(d)})^2 + \sum_{K\in\mathcal{T}_h}\varepsilon_K^2\Big)^{\frac{1}{2}}. \end{aligned} \tag{5.20}$$

This estimate is fully optimal at least when the ε_K are negligible (which can easily be checked thanks to Lemma 5.1).

Moreover, estimates (5.16) and (5.17) are fully local, which means that the indicators $\eta_K^{(d)}$ are a very efficient tool for mesh adaptation. Estimate (5.15) is not local, however the definition of the $\eta_K^{(s)}$ leads us to think that they constitute a good representation of the local modeling error.

6. The adaptivity strategy.

The aim of this section is to propose an iterative algorithm which provides both a partition of Ω satisfying (3.1) and a triangulation $\mathcal{T}_h$ such that the modeling error and the discretization error are of the same order. Let η^* be a fixed tolerance.

Initialization step: We first chose a triangulation $\mathcal{T}_h^0$ of the domain Ω such that the error terms which appear in (5.18) are smaller than the tolerance η^* (the importance of such a choice is brought to light in [12]). We take $\Omega_f^0 = \emptyset$ and $\Omega_s^0 = \Omega$. We then solve problem (4.3) – (4.4). Note that the first and second line in (4.4) on one hand, and the third line on the other hand are fully uncoupled here. Thus we are in a position to compute the error indicators $\eta_K^{(s)}$ (on the whole domain) and $\eta_K^{(d)}$.

Adaptation step: We assume that a partition of Ω into two subdomains Ω_f^n and Ω_s^n satisfying (3.1) is known, together with a triangulation $\mathcal{T}_h^n$. We compute the solution of the associated simplified discrete problem (an algorithm for similar problems is proposed in [9] for instance). Next, we compute the corresponding error indicators $\eta_K^{(s)}$ (only on Ω_s^n) and $\eta_K^{(d)}$, together with the mean values $\overline{\eta}_h^{(s)}$ of the $\eta_K^{(s)}$ and the mean value $\overline{\eta}_h^{(d)}$ of the $\eta_K^{(d)}$. Next, we perform adaptivity.

1. Adaptivity due to modeling error.
All K in $\mathcal{T}_h^{n(s)}$ such that

$$\eta_K^{(s)} \geq \min \{\overline{\eta}_h^{(s)}, \overline{\eta}_h^{(d)}\}, \tag{6.1}$$

are inserted in a new domain $\widetilde{\Omega}_f^{n+1}$. More precisely, this new domain $\widetilde{\Omega}_f^{n+1}$ is the union of Ω_f^n and of these new K.

2. Decomposition regularization.
We perform the following regularization: Any element K which is not imbedded in $\widetilde{\Omega}_f^{n+1}$ but is surrounded by elements which are imbedded in $\widetilde{\Omega}_f^{n+1}$, now belongs to the new domain Ω_f^{n+1}. We skip the details for the construction of Ω_f^{n+1}. Next, we choose Ω_s^{n+1} such that (3.1) holds.

3. Adaptivity due to discretization error.
We perform a standard finite element adaptivity strategy: For each K in $\mathcal{T}_h^n$, the diameter of a new element contained in K or containing K is proportional to h_K times the ratio $\overline{\eta}_h^{(d)}/\eta_K^{(d)}$, with the further condition that this new element is contained either in Ω_f^{n+1} or in Ω_s^{n+1}. We refer to [13] among others for more details on this procedure, especially in dimension $d = 3$. This gives rise to the triangulation $\mathcal{T}_h^{n+1}$.

The adaptation step is of course iterated either a finite number of times or until both quantities

$$\max_{K \in \mathcal{T}_h^n, K \subset \Omega_s^n} \eta_K^{(s)} \qquad \text{and} \qquad \max_{K \in \mathcal{T}_h^n} \eta_K^{(d)},$$

become smaller than the tolerance η^*.

Remark 6.1. At some iterations of the adaptation process, evaluating the ε_K by computing the right-hand member of (5.7) and refining the mesh where they are too large can be necessary.

Remark 6.2. It is also possible to compute the $\eta_K^{(s)}$ for all K in $\mathcal{T}_h^n$ and to move the K for which they are very small from Ω_f^n to Ω_s^{n+1}. This seems useless for the problem under consideration but could be of great interest for its time-dependent analogue.

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