On a second order dissipative ODE in Hilbert space with an integrable source term

Alain HARAUX
1- UPMC Univ Paris 06, UMR 7598, Laboratoire Jacques-Louis Lions, F-75005, Paris, France.
2- CNRS, UMR 7598, Laboratoire Jacques-Louis Lions, Boîte courrier 187, 75252 Paris Cedex 05, France
haraux@ann.jussieu.fr

Mohamed Ali JENDOUBI
Faculté des sciences de Bizerte
Département de Mathématiques
7021 Jarzouna Bizerte, Tunisie
ma.jendoubi@fsb.rnu.tn

Abstract

Asymptotic behaviour of solutions is studied for some second order equations including the model case $\ddot{x}(t) + \gamma\dot{x}(t) + \nabla\Phi(x(t) = h(t)$ with $\gamma > 0$ and $h \in L^1(0, +\infty; H)$, Φ being continuouly differentiable with locally Lipschitz continuous gradient and bounded from below. In particular when Φ is convex, all solutions tend to minimize the potential F as time tends to infinity and the existence of one bounded trajectory implies the weak convergence of all solutions to equilibrium points.

AMS classification numbers: 34A12, 34Dxx, 49Mxx.

Keywords: dissipative dynamical system, asymptotic behaviour, gradient system, heavy ball with friction.

The authors gratefully acknowledge support by the France-Tunisia cooperation under the auspices of the CNRS/DGRSRT agreement No. 21817: Systèmes dynamiques et équations d'évolution.
Part of this work was done during a sojourn of the second author at Laboratoire Jacques-Louis Lions under the auspices of the Fondation Sciences Mathématiques de Paris.

1 Introduction

Let H be a real Hilbert space. Let us consider a mapping $\Phi : H \longrightarrow \mathbb{R}$ satisfying the following conditions :

$$(\mathcal{H}) \qquad \begin{cases} \Phi \text{ is continuously differentiable on } H, \\ \Phi \ \text{ is bounded from below on } H, \\ \nabla\Phi \ \text{ is Lipschitz continuous on the bounded subsets of } H. \end{cases}$$

The main object of this paper is to study the asymptotic behaviour of solutions to the second order equation

$$\ddot{x}(t) + \gamma\dot{x}(t) + \nabla\Phi(x(t) = h(t). \tag{1}$$

under the conditions $\gamma > 0$ and

$$h \in L^1(0, +\infty; H). \tag{2}$$

The case $h = 0$ has been studied by Alvarez [1] (cf. Also [2]) as a natural extension of the fundamental result of [5]. When Φ is convex and assuming $S := \operatorname{argmin} \Phi \neq \emptyset$ he established the weak convergence of any solution of (1) to some point in S. The consideration of L^1 perturbations was initiated by H. Brezis [4] in the context of contractive systems generated by maximal monotone operators. In that context asymptotic behaviour can be studied for quite general subgradient operators and even periodic forcing terms can be treated (cf. e.g. [7], [3]). Both extensions are impossible for the second order problems : lack of regularity for Φ lead to hard problems even in the definition of solutions (cf. [10]), while periodic forcings even for scalar equations of the form (1) can produce the period doubling phenomenon.

On the other hand, L^1 forcing terms allow much more general extensions (cf [6]). Under the hypothesis $(\mathcal{H})$ it is possible to obtain some rather precise results on the global behavior of solutions x to the more general equation

$$A\ddot{x}(t) + \gamma(\dot{x}(t)) + \nabla\Phi(x(t) = h(t). \tag{3}$$

where A is a linear symmetric positive topological isomorphism $H \to H$ and the map γ satisfies some coerciveness and continuity assumptions. The plan of the paper is the following: in Section 2 we study the global properties of solutions to the general equation (3). In Section 3 more precise results, similar to those of Alvarez [1] are established for solutions of an equation slightly more general than (1) but less general than (3) when Φ is convex.

2 General results

Throughout this section we make the following hypotheses on A and γ

$$A \in L(H); \quad A^* = A \tag{4}$$

$$\exists \alpha > 0, \quad \forall z \in H, \quad \langle Az, z \rangle = \|A^{1/2} z\|^2 \geq \alpha \|z\|^2 \tag{5}$$

$$\gamma \text{ is Lipschitz continuous on the bounded subsets of } H \tag{6}$$

$$\exists p \geq 2, \quad \exists c > 0, \quad \forall y \in H, \quad \langle \gamma(y), y \rangle \geq c\|y\|^p \tag{7}$$

Theorem 2.1. *Let us assume that $\Phi : H \longrightarrow \mathbb{R}$ satisfies assumptions (H) and h satisfies (2). Then the following properties hold*

(i) *for all $(x_0, \dot{x}_0) \in H \times H$, there exists a unique solution $x(t) \in W^{2,1}_{loc}([0, +\infty), H)$ of* (3) *which satisfies the initial conditions $x(0) = x_0$ and $\dot{x}(0) = \dot{x}_0$.*

(ii) *Every solution $x(t)$ of* (3) *satisfies*

$$\dot{x} \in L^\infty(0, +\infty; H) \cap L^p(0, +\infty; H).$$

In addition the energy $E(t)$ defined as in [6] by

$$E(t) = \frac{1}{2}\|A^{1/2}\dot{x}(t)\|^2 + \phi(x(t)) + \int_t^{+\infty} \langle h(s), \dot{x}(s) \rangle \, ds \tag{8}$$

is nonincreasing on $[0, +\infty)$ and bounded from below, and hence converges to some real value E_∞.

(iii) *Assuming moreover that $x \in L^\infty(0, +\infty; H)$, then we have*
- $\lim_{t \to +\infty} \dot{x}(t) = 0$,
- $\lim_{t \to +\infty} \nabla\Phi(x(t)) = 0$ *and* $\lim_{t \to +\infty} \Phi(x(t)) = E_\infty$.

Proof. First note that (3) can be written as a first order system in $H \times H$:

$$\dot{Y}(t) = F(t, Y(t)), \tag{9}$$

with

$$Y(t) = (x(t), \dot{x}(t))^t, \quad F(t, u, v) = (v, A^{-1}(h(t) - \gamma(v) - \nabla\Phi(u))).$$

For any choice of initial conditions $(x_0, \dot{x}_0) \in H \times H$, the existence and uniqueness of a local solution for (3) (or equivalently (9)), follows from the Cauchy-Lipschitz Theorem. Let $(x(t), \dot{x}(t))$ denote the corresponding maximal solution which is defined on some interval $[0, T_{max})$ with $0 < T_{max} \leq +\infty$. By using the hypothesis $(\mathcal{H})$, we obtain that x is $W^{2,1}_{loc}$ on $[0, T_{max})$. In order to prove that $T_{max} = +\infty$, let us show that $\dot{x}(t)$ is bounded.
Since Φ is bounded from below, there exists $K \in \mathbb{R}$ such that $\Phi(x) + K > 0$ for all $x \in H$. Now we define

$$\Psi(t) = \frac{1}{2}\|A^{1/2}\dot{x}(t)\|^2 + \phi(x(t)) + K.$$

Ψ is well defined $[0, T_{max})$ and we have

$$\|\dot{x}(t)\| \leq \sqrt{\frac{2}{\alpha}}\sqrt{\Psi(t)} \tag{10}$$

By differentiating Ψ, using equations (3) and (10), we obtain for all $t \in [0, T_{max})$,

$$\begin{aligned}
\dot{\Psi}(t) &= \langle A\ddot{x}(t), \dot{x}(t)\rangle + \langle \nabla\phi(x(t)), \dot{x}(t)\rangle \\
&= \langle h(t), \dot{x}(t)\rangle - \langle \gamma(\dot{x}(t)), \dot{x}(t)\rangle \qquad (11)\\
&\leq \|h(t)\|\|\dot{x}(t)\| \\
&\leq \sqrt{\frac{2}{\alpha}}\|h(t)\|\sqrt{\Psi(t)}. \qquad (12)
\end{aligned}$$

Now since $\Psi(t) > 0$, we get

$$\frac{d}{dt}[\sqrt{\Psi(t)}] \leq \frac{1}{\sqrt{2\alpha}}\|h(t)\|.$$

By integrating this last inequality on $(0, t)$ ($t \in [0, T_{max})$), we obtain

$$\sqrt{\Psi(t)} \leq \frac{1}{\sqrt{2\alpha}}\int_0^t \|h(s)\|\,ds + \sqrt{\Psi(0)} \leq \frac{1}{\sqrt{2\alpha}}\int_0^{+\infty} \|h(s)\|\,ds + \sqrt{\Psi(0)}.$$

Now by using again (10), we obtain that

$$\sup_{t\in[0,T_{max})} \|\dot{x}(t)\| := C < +\infty.$$

Now we claim that $T_{max} = +\infty$. In fact assume that $T_{max} < +\infty$. We have

$$\|x(t)\| \leq \|x(0)\| + CT_{max}$$

Therefore $(x, \dot{x})$ is bounded on $[0, T_{max})$, a contradiction with $T_{max} < +\infty$. Consequently (i) is proved.

Next by integrating (11), using the fact that $\Psi(t) > 0$ and that $\dot{x} \in L^\infty(0, +\infty; H)$, we get for all $t \in \mathbb{R}_+$

$$\begin{aligned}
c\int_0^T \|\dot{x}\|^p\,ds &\leq -\Psi(T) + \Psi(0) + \int_0^T \langle h(s), \dot{x}(s)\rangle\,ds \\
&\leq \Psi(0) + \int_0^T \|h(s)\|\|\dot{x}(s)\|\,ds \\
&\leq \Psi(0) + \|\dot{x}\|_{L^\infty(0,\infty;H)}\int_0^\infty \|h(s)\|\,ds.
\end{aligned}$$

By letting T go to $+\infty$, we obtain $\dot{x} \in L^p(0, +\infty; H)$. We have already proved that $\dot{x} \in L^\infty(0, +\infty; H)$. Finally we have

$$\frac{d}{dt}\left[\frac{1}{2}\|A^{1/2}\dot{x}(t)\|^2 + \phi(x(t)) + \int_t^{+\infty} \langle h(s), \dot{x}(s)\rangle\,ds\right] = -\langle \gamma(\dot{x}(t)), \dot{x}(t)\rangle \leq 0 \qquad (13)$$

Therefore the proof of assertion (ii) is complete.

Next, we assume that $x \in L^\infty(0, +\infty; H)$. Then the fact that $\nabla\Phi$ is bounded on the bounded subsets of H, together with equation (3) and assumption (2) imply that $\dot{x}$ is uniformly continuous. This together with $\dot{x} \in L^p(0,+\infty;H)$ yields $\lim_{t\to+\infty} \dot{x}(t) = 0$.

Now we claim that $\lim_{t\to+\infty} \nabla\Phi(x(t)) = 0$. In fact, by using the equation (3) we have for all $\delta > 0$,

$$\lim_{t\to+\infty} \int_t^{t+\delta} \nabla\Phi(x(s))\, ds = \lim_{t\to+\infty} A(\dot{x}(t) - \dot{x}(t+\delta)) - \int_t^{t+\delta} \gamma(\dot{x}(s))\, ds + \int_t^{t+\delta} h(s)\, ds = 0.$$

Now since $\dot{x} \in L^\infty(0,+\infty;H)$, x is Lipschitz and bounded and then $\nabla\Phi(x(t))$ is also Lipschitz continuous in t with some Lipschitz constant M. The claim follows easily. Indeed

$$\|\nabla\Phi(x(t))\| \leq \frac{1}{\delta}\|\int_t^{t+\delta} \nabla\Phi(x(s))ds\| + \frac{1}{\delta}\int_t^{t+\delta} M|t-s|ds$$

$$\leq \frac{M\delta}{2} + \frac{1}{\delta}\|\int_t^{t+\delta} \nabla\Phi(x(s))ds\|$$

Letting t tend to infinity results in

$$\limsup_{t\to\infty} \|\nabla\Phi(x(t))\| \leq \frac{M\delta}{2}$$

and then the conclusion follows by letting δ tend to 0. To prove that $\lim_{t\to+\infty} \Phi(x(t))$ exists, we have just to use the convergence of $E(t)$ defined in (8) and remark that $\lim_{t\to+\infty} \|\dot{x}(t)\| = \lim_{t\to+\infty} \int_t^{+\infty} \langle h(s), \dot{x}(s)\rangle\, ds = 0$. □

Corollary 2.2. *Assume that $\Phi : H \longrightarrow \mathbb{R}$ satisfies the assumptions $(\mathcal{H})$. Assume moreover that Φ is coercive :*

$$\lim_{\|x\|\to+\infty} \Phi(x) = +\infty.$$

Then x is in $L^\infty(0,+\infty;H)$ and the conclusions of Theorem 2.1 hold.

Proof. From (13), we deduce that

$$\phi(x(t)) \leq \frac{1}{2}\|A^{1/2}\dot{x}_0\|^2 + \phi(x_0) + 2\int_0^\infty \|h(s)\|\|\dot{x}(s)\|\, ds < +\infty.$$

Clearly the coerciveness of Φ imply that x remains bounded, i.e. $x \in L^\infty(0,+\infty;H)$. □

3 The convex case

This Section is devoted to a generalization of Theorem 2.1 from [1]. We consider the case $\gamma(v) = \gamma Av$ for some $\gamma > 0$, therefore the equation we are dealing with is

$$A\ddot{x}(t) + \gamma A\dot{x}(t) + \nabla\Phi(x(t) = h(t). \tag{14}$$

where h satisfies (2)

The first result is valid for all solutions of (14).

Theorem 3.1. *Assume that Φ is convex and satisfies assumption $(\mathcal{H})$. Then for all $(x_0, \dot{x}_0) \in H \times H$, the unique solution x of (14) satifies*

$$\lim_{t\to+\infty} \Phi(x(t)) = \inf \Phi. \tag{15}$$

$$\lim_{t\to+\infty} \|\dot{x}(t)\| = 0. \tag{16}$$

Proof of Theorem 3.1. We know already that $\dot{x} \in L^\infty(0, +\infty; H) \cap L^2(0, +\infty; H)$. Following [1] for any $z \in H$ we set

$$\varphi(t) := \frac{1}{2}\|A^{1/2}(x(t) - z)\|^2.$$

We have

$$\begin{aligned} \dot{\varphi}(t) &= \langle A\dot{x}(t), x(t) - z\rangle \\ \ddot{\varphi}(t) &= \|A^{1/2}\dot{x}(t)\|^2 + \langle A\ddot{x}(t), x(t) - z\rangle. \end{aligned}$$

It follows by using equation (14) and the convexity of Φ that

$$\begin{aligned} \ddot{\varphi}(t) + \gamma\dot{\varphi}(t) &= \|A^{1/2}\dot{x}(t)\|^2 + \langle h(t) - \nabla\Phi(x(t)), x(t) - z\rangle \\ &\leq \|A^{1/2}\dot{x}(t)\|^2 + \langle h(t), x(t) - z\rangle + \Phi(z) - \Phi(x(t)) \\ &= \frac{3}{2}\|A^{1/2}\dot{x}(t)\|^2 + \langle h(t), x(t) - z\rangle + \Phi(z) - E(t) + \int_t^{+\infty} \langle h(s), \dot{x}(s)\rangle\, ds \end{aligned}$$

Given $\theta > 0$, for any $\tau \in [0, \theta]$ we have since $E(\theta) \leq E(\tau)$

$$\ddot{\varphi}(\tau) + \gamma\dot{\varphi}(\tau) \leq \frac{3}{2}\|A^{1/2}\dot{x}(\tau)\|^2 + \langle h(\tau), x(\tau) - z\rangle + \Phi(z) - E(\theta) + \int_\tau^{+\infty} \langle h(s), \dot{x}(s)\rangle\, ds$$

hence

$$\forall \tau \in [0, \theta], \quad \ddot{\varphi}(\tau) + \gamma\dot{\varphi}(\tau) \leq k(\tau) + \Phi(z) - E(\theta)$$

with

$$k(\tau) := \frac{3}{2}\|A^{1/2}\dot{x}(\tau)\|^2 + \|h(\tau)\|\|x(\tau) - z\| + \int_\tau^{+\infty} \|h(s)\|\|\dot{x}(s)\|\, ds$$

By integrating in τ we find

$$\forall \theta > 0, \quad \dot{\varphi}(\theta) \leq e^{-\gamma\theta}\dot{\varphi}(0) + \frac{1 - e^{-\gamma\theta}}{\gamma}[\Phi(z) - E(\theta)] + \int_0^\theta e^{-\gamma(\theta-\tau)}k(\tau)d\tau$$

Then given $t > 0$, since $E(\theta) \geq E(t)$ whenever $t \geq \theta$, by integrating once more we obtain

$$0 \leq \varphi(t) \leq C + \frac{\gamma t - 1 + e^{-\gamma t}}{\gamma^2}[\Phi(z) - E(t)] + \int_0^t \int_0^\theta e^{-\gamma(\theta-\tau)} k(\tau) d\tau d\theta$$

and therefore

$$\limsup_{t\to\infty}[E(t) - \Phi(z)] \leq \gamma \limsup_{t\to\infty} \frac{1}{t} \int_0^t \int_0^\theta e^{-\gamma(\theta-\tau)} k(\tau) d\tau d\theta \tag{17}$$

To end the proof we estimate the rigth hand-side as follows. First by Fubini's Theorem

$$\int_0^t \int_0^\theta e^{-\gamma(\theta-\tau)} k(\tau) d\tau d\theta = \int_0^t k(\tau) \int_\tau^t e^{-\gamma(\theta-\tau)} d\theta d\tau \leq \frac{1}{\gamma} \int_0^t k(\tau) d\tau$$

and we find

$$\limsup_{t\to\infty}[E(t) - \Phi(z)] \leq \limsup_{t\to\infty} \frac{1}{t} \int_0^t k(\tau) d\tau \tag{18}$$

To conclude we use the following easy Lemma

Lemma 3.2. *For any nonnegative function $g(t) \in L^1([0, +\infty))$, we have*

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \tau g(\tau) d\tau = 0$$

Proof. Let

$$G(t) = \int_t^\infty g(s) ds$$

we have

$$[sG(s)]' = G(s) - sg(s)$$

therefore

$$\int_0^t sg(s) ds = \int_0^t G(s) ds - tG(t) \leq \int_0^t G(s) ds$$

and finally

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \tau g(\tau) d\tau \leq \lim_{t\to\infty} \frac{1}{t} \int_0^t G(s) ds = 0$$

since G tends to 0 at infinity □

We now establish

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t k(\tau) d\tau = 0 \tag{19}$$

Indeed we clearly have

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \|A^{1/2} \dot{x}(\tau)\|^2 d\tau = 0$$

Moreover we observe that since $\|x(t)\|$ grows at most linearly at infinity,

$$g(t) := \frac{\|h(t)\|\|x(t)-z\|}{t+1} \in L^1([0,+\infty))$$

then as a consequence of Lemma 3.2

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t g(\tau)d\tau = 0$$

Finally we consider the function

$$p(\tau) := \int_\tau^{+\infty} \|h(s)\|\|\dot{x}(s)\|\, ds \leq CH(\tau)$$

with

$$H(t) := \int_t^{+\infty} \|h(s)\|\, ds$$

and C a uniform bound for $\|\dot{x}(s)\|$. Fubini's Theorem gives immediately

$$\frac{1}{t}\int_0^t H(\tau)d\tau = \frac{1}{t}\int_0^\infty \|h(s)\|\Big(\int_0^{\inf\{s,t\}} d\tau\Big)ds = \frac{1}{t}\int_0^\infty \|h(s)\|\inf\{s,t\}ds$$

$$= \int_t^\infty \|h(s)\|ds + \frac{1}{t}\int_0^t s\|h(s)\|ds$$

and this tends to 0 at infinity by applying once more Lemma 3.2.

By applying (18) we now find

$$\forall z \in H, \quad \limsup_{t\to\infty}[E(t) - \Phi(z)] \leq 0$$

and therefore

$$\limsup_{t\to\infty} E(t) \leq \inf \Phi$$

which implies both conclusions of Theorem 3.1 due to the expression of the energy $E(t)$. □

Theorem 3.3. *Under the hypotheses of Theorem 3.1 the following properties are equivalent*

i) For some $(x_0, \dot{x}_0) \in H \times H$*, the unique solution* x *of* (14) *is bounded.*
ii) $S := \operatorname{argmin} \Phi \neq \emptyset$.
iii) For all $(x_0, \dot{x}_0) \in H \times H$*, the unique solution* x *of* (14) *is bounded.*

In addition then for all $(x_0, \dot{x}_0) \in H \times H$*, there exists* $\bar{x} \in \operatorname{argmin} \Phi$ *such that* $x(t) \rightharpoonup \bar{x}$ *weakly in* H *as* $t \to +\infty$*. Moreover,*

$$\lim_{t\to+\infty} \Phi(x(t)) = \min \Phi$$

Proof of Theorem 3.3. The last conclusion already follows from the previous theorem. For the other properties we proceed in 3 steps.

Step 1. i) $\Rightarrow$ ii). Let $t_n \to +\infty$ with $x(t_n) \rightharpoonup \bar{x}$ weakly in H. Since Φ is weakly lower semicontinuous (because Φ is convex and continuous), then

$$\liminf_{n\to+\infty} \Phi(x(t_n)) \geq \Phi(\bar{x}).$$

Finally as a consequence of (15) we have $\Phi(\bar{x}) \leq \inf \Phi$, that is $\bar{x} \in \operatorname{argmin} \Phi = S$.

Step 2. Assuming ii), let $a \in \operatorname{argmin} \Phi$ and let us prove that any solution x is bounded. We define

$$Z(t) = \frac{1}{2}\|A^{1/2}\dot{x}(t)\|^2 + \Phi(x(t)) - \Phi(a) + \frac{\gamma^2}{4}\|A^{1/2}(x(t)-a)\|^2 + \frac{\gamma}{2}\langle A(x(t)-a), \dot{x}(t)\rangle.$$

Clearly $Z(t) \geq 0$ for all t. We also have

$$\|\dot{x}(t)\| + \|x(t)-a\| \leq K[Z(t)]^{\frac{1}{2}} \tag{20}$$

where $K > 0$ is a postive constant. Now by differentiating Z, using (14) and the fact that Φ is convex we get

$$\begin{aligned}
\frac{d}{dt}Z(t) &= \langle h(t) - \gamma A\dot{x}(t), \dot{x}(t)\rangle + \frac{\gamma^2}{2}\langle A\dot{x}(t), x(t)-a\rangle + \frac{\gamma}{2}\|A^{1/2}\dot{x}(t)\|^2 + \frac{\gamma}{2}\langle x(t)-a, A\ddot{x}(t)\rangle \\
&\leq -\frac{\gamma}{2}\|A^{1/2}\dot{x}(t)\|^2 + \langle h(t), \dot{x}(t)\rangle + \frac{\gamma}{2}\langle h(t) - \nabla\Phi(x(t)), x(t)-a\rangle \\
&\leq \langle h(t), \dot{x}(t)\rangle + \frac{\gamma}{2}\langle h(t), x(t)-a\rangle \\
&\leq \|h(t)\|[\|\dot{x}(t)\| + \frac{\gamma}{2}\|x(t)-a\|] \\
&\leq K'\|h(t)\|[Z(t)]^{\frac{1}{2}}
\end{aligned}$$

with $K' = K\max\{1, \frac{\gamma}{2}\}$

Now let $\varepsilon > 0$. We have

$$\frac{d}{dt}[Z(t)+\varepsilon]^{\frac{1}{2}} = \frac{1}{2}\frac{d}{dt}Z(t)[Z(t)+\varepsilon]^{-\frac{1}{2}} \leq \frac{K'}{2}\|h(t)\|.$$

By integrating this last inequality, using (20) and hypothesis (2), we prove that x is bounded, hence iii).

Step 3. Proof of the convergence result: let

$$S = \operatorname{argmin} \Phi = \{x \in H \,:\, \nabla\Phi(x) = 0\}.$$

We shall use Lemma 1 p. 192 of [9] in the Hilbert space H endowed the new Hilbertian norm p given by

$$p(u) := \|A^{1/2}u\|$$

in the following modified form (cf. e.g. [8], Lemma 38 p. 236.)

Lemma 3.4. *Let H be a Hilbert space with norm p and $x : [0,+\infty) \longrightarrow H$ be a function such that there exists a non empty subset S of H which verifies*

(1) *$\forall t_n \to +\infty$ with $x(t_n) \rightharpoonup \bar{x}$ weakly in H, we have $\bar{x} \in S$;*

(2) *$\forall z \in S$, $\lim_{t\to+\infty} p(x(t) - z)$ exists.*

Then, $x(t)$ weakly converges as $t \to +\infty$ to some element $\bar{x}$ of S.

Proof of step 3 continued

(1) Let $t_n \to +\infty$ with $x(t_n) \rightharpoonup \bar{x}$ weakly in H. Then by repeating the argument in the proof of step 1 we find $\bar{x} \in \operatorname{argmin} \Phi = S$.

(2) Now given $z \in S$, we prove that $\lim_{t\to+\infty} \|A^{1/2}(x(t) - z)\|$ exists. Define as previously $\varphi(t) := \frac{1}{2}\|A^{1/2}(x(t) - z)\|^2$. It follows by using equation (14) and the convexity of Φ that

$$\begin{aligned} \ddot{\varphi}(t) + \gamma\dot{\varphi}(t) &= \|A^{1/2}\dot{x}(t)\|^2 + \langle h(t) - \nabla\Phi(x(t)), x(t) - z\rangle \\ &\leq K''\|\dot{x}(t)\|^2 + \|h(t)\|\|x(t) - z\|. \end{aligned}$$

Since x is bounded, we obtain that the right term in the last inequality is in $L^1(0,+\infty; H)$. To conclude the proof we use the following lemma which is due to Alvarez [1].

Lemma 3.5. *(Alvarez [1]) Let $\varphi \in C^1(0,+\infty;\mathbb{R}_+)$ satisfy the following differential inequality*

$$\ddot{\varphi}(t) + \gamma\dot{\varphi}(t) \leq \zeta(t)$$

with $\zeta \in L^1(0,+\infty;\mathbb{R}_+)$. Then, $(\dot{\varphi})_+$ the positive part of $\dot{\varphi}$ belongs to $L^1(0,+\infty;\mathbb{R})$ and, as a consequence, $\lim_{t\to+\infty} \varphi(t)$ exists.

□

References

[1] F. Alvarez, On the minimizing property of a second order dissipative system in Hilbert space, SIAM J. Control Optim. **38**, No. 4 (2000), 1102-1119.

[2] H. Attouch, X. Goudou, and P. Redont, *The heavy ball with friction method, I. The continuous dynamical system: global exploration of the local minima of a real-valued function asymptotic by analysis of a dissipative dynamical system*, Commun. Contemp. Math. **2** (2000),1–34.

[3] J.B. Baillon, A . Haraux, Comportement à l'infini pour les équations d'évolution avec forcing périodique, Arch. Rational Mech. Anal. **67** (1977), no. 1, 101–109.

[4] H. Brézis, Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert. North-Holland Mathematics Studies, No. 5. Notas de Matematica (50). North-Holland Publishing Co., Amsterdam-London, New York (1973).

[5] R. Bruck, Asymptotic convergence of nonlinear contraction semigroups in Hilbert space. J. Funct. Anal. **18** (1975), 15-26.

[6] R. Chill , M. A. Jendoubi, Convergence to steady states in asymptotically autonomous semilinear evolution equations, Nonlinear Anal. **53** (2000), no 7-8, 1017–1039.

[7] A. Haraux, Equations d'évolution non linéaires: solutions bornées et périodiques. Ann. Inst. Fourier (Grenoble) **28** (1978), no. 2, vii, 201–220.

[8] A. Haraux, Nonlinear evolution equations-global behavior of solutions. Lecture Notes in Mathematics, 841. Springer-Verlag, Berlin-New York, 1981

[9] Z. Opial, Weak convergence of the sequence of successive approximations for nonexpansive mappings, Bull. Amer. Math. Soc. **73** (1967), 591597.

[10] M. Schatzman, Le système différentiel $(d^2u/dt^2) + \partial\varphi(u) \ni f$ avec conditions initiales. (English summary) C. R. Acad. Sci. Paris. A-B **284** (1977), no. 11, A603–A606.