A PRIORI AND A POSTERIORI ESTIMATES FOR THE STOKES PROBLEM
WITH SOME DIFFERENT BOUNDARY CONDITIONS

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Abstract. In this paper we study the Stokes problem with some different boundary conditions. We establish a decoupled variational formulation into a system of velocity and a Poisson equation for the pressure. The continuous and corresponding discrete system do not need an inf-sup condition. Hence, the velocity is approximated with \texttt{curl} conforming finite elements and the pressure with standard continuous elements. Next, we establish optimal a priori and a posteriori estimates and we finish this paper with numerical tests.

Keywords Stokes equations, a priori and a posteriori errors.

1. Introduction.

This paper is devoted to the numerical solution of the the Stokes equations for an incompressible fluid

\[-\nu \Delta \mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega,\]

with the incompressibility condition

\[\text{div } \mathbf{u} = 0 \quad \text{in } \Omega,\]

with boundary conditions

\[\mathbf{u} \times \mathbf{n} = 0, \quad p = 0 \quad \text{on } \partial \Omega,\]

or

\[\mathbf{u} \cdot \mathbf{n} = 0, \quad \text{curl } \mathbf{u} \times \mathbf{n} = 0 \quad \text{on } \partial \Omega\]

where \(\Omega\) is a bounded, simply connected domain of \(\mathbb{R}^3\) with a polyhedral connected boundary \(\Gamma = \partial \Omega\) and \(\mathbf{u}\) the velocity and \(p\) the pressure.

These sets of boundary conditions lend themselves readily to a variational formulation where the Laplacian operator is expressed by a \((\text{curl, curl})\) term and the incompressibility condition by an equation of the form \((\nabla q, \mathbf{v})\). This suggests to use a partially non-confirming finite element method, where just the \texttt{curl} of the velocity is continuous at interface boundaries whereas the pressure is globally continuous.

The convexity assumption on \(\Omega\) is a well-known theoretical consequence of the fact that \(\Gamma\) is not smooth. There is no practical evidence that it is necessary and his assumption is disregarded in practice: instead, we can assume that \(\Omega\) is simply-connected and \(\Gamma\) is connected. A domain with "holes" or a multiply-connected domain can be handled with the techniques of Bendali, Dominguez and Gallic [5]. We refer to Dubois [9] for a good treatment of the potential problem on a domain with a curved and multiply-connected boundary. We also refer to Verfürth [23] for a different approximation of the same potential problem on a curved domain. As far as the theory is concerned, the reader will find in Bègue, Conca, Murat and Pironneau [4] a very comprehensive study of the Navier-Stokes equations with non-standard (and non-homogeneous) boundary conditions on a variety of domains. These authors include a conforming approximation of the Taylor-Hood type for the velocity (the corresponding theoretical analysis is done by Franca and Hughes [11]). We refer also to Girault’s work [14] for a vector potential-vorticity approximation of similar Navier-Stokes type problems and to [13] for the steady-state incompressible Navier-Stokes

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equations with non standard boundary conditions. We also refer to [22] where Repin establishes a posteriori estimates for the velocity, stress and pressure fields for the stationary Stokes problem and where his approach is based on duality theory of the calculus of variations. A posteriori estimates for the Stokes problem and for some viscous flow problems were studied by a number of authors, [3], [24], [19] and [20]. Typically, they have been obtained in the frame of the so-called "residual method" originally proposed in [1] and [2] for the finite element approximations. This type estimates are crucially based on the Galerkin orthogonality condition. Therefore, they are only valid for exact solutions of the corresponding finite dimensional problem which form a very special subset in the natural set of admissible functions.

2. Description and analysis of the model

We denote by (Problem1) the system of equations (1.1), (1.2) and (1.3), and by (Problem2) the system of equations (1.1), (1.2) and (1.4). In all the paper, we suppose that \( f \in L^2(\Omega)^3 \) and we denote by \( C \) a generic positive constant.

In order to write the variational formulation of the previous problems, we introduce some spaces:

\[
W^{m,p}(\Omega) = \{ v \in L^p(\Omega), \partial^\alpha v \in L^p(\Omega), \quad \forall \ |\alpha| \leq m \}, \\
H^m(\Omega) = W^{m,2}(\Omega),
\]

equipped with the following semi-norm and norm:

\[
\| v \|_{m,p,\Omega} = \left\{ \sum_{|\alpha| \leq m} \int_\Omega \partial^\alpha v(x)^p \, dx \right\}^{1/p} \quad \text{and} \quad \| v \|_{m,p,\Omega} = \left\{ \sum_{k \leq m} \| v \|_{k,p,\Omega}^p \right\}^{1/p}
\]

As usual, we shall omit \( p \) when \( p = 2 \) and denote by \((\cdot,\cdot)\) the scalar product of \( L^2(\Omega) \). Also, recall the familiar notation:

\[
H^1_0(\Omega) = \{ v \in H^1(\Omega); v = 0 \text{ on } \Gamma \},
\]

with the Poincaré inequality

\[
\forall v \in H^1_0(\Omega); \quad \|v\|_{0,\Omega} \leq C|v|_{1,\Omega}.
\] (2.1)

Finally, we introduce the spaces:

\[
H(\text{div}, \Omega) = \{ v \in L^2(\Omega)^3, \text{div } v \in L^2(\Omega) \}; \quad H_0(\text{div}, \Omega) = \{ v \in H(\text{div}, \Omega), \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \Gamma \}
\]

\[
H(\text{curl}, \Omega) = \{ v \in L^2(\Omega)^3, \text{curl } v \in L^2(\Omega)^3 \}; \quad H_0(\text{curl}, \Omega) = \{ v \in H(\text{curl}, \Omega), \mathbf{v} \times \mathbf{n} = 0 \text{ on } \Gamma \}
\]

normed respectively by:

\[
\| \mathbf{v} \|_{H(\text{div}, \Omega)} = \left\{ \| \mathbf{v} \|_{0,\Omega}^2 + \| \text{div } \mathbf{v} \|_{0,\Omega}^2 \right\}^{1/2},
\]

and

\[
\| \mathbf{v} \|_{H(\text{curl}, \Omega)} = \left\{ \| \mathbf{v} \|_{0,\Omega}^2 + \| \text{curl } \mathbf{v} \|_{0,\Omega}^2 \right\}^{1/2}.
\]

For the theoretical fondations of these spaces, we can refer to Duvaut & Lions [10] and for the following regularity theorems, we refer to Bernardi [7], Dauge [??], Girault & Raviart [12], Grisvard [15] and Nedelec [18].

**Lemma 2.1.** Let \( \Omega \) be a convex domain. There exists a unique solution \( \mathbf{w} \) in \( H^1(\Omega)/\mathbb{R} \) (resp. \( H^1_0(\Omega) \)) such that:

\[
(\nabla \mathbf{w}, \nabla \mathbf{v}) = (\mathbf{f}, \nabla \mathbf{v}) \quad \forall \mathbf{v} \in H^1(\Omega)/\mathbb{R} \quad \text{(resp } H^1_0(\Omega))
\]

and there exists a positive constant \( C \) such that:

\[
\| \mathbf{w} \|_{1,\Omega} \leq C\| \mathbf{f} \|_{0,\Omega}.
\]
Theorem 2.2. Let $\Omega$ be convex. All functions $v \in L^2(\Omega)^3$ satisfying:
\[ \text{div } v = 0, \quad \text{curl } v \in L^2(\Omega)^3, \quad v \cdot n = 0 \text{ or } v \times n = 0 \text{ on } \Gamma \]
belong to $H^1(\Omega)$ and we have
\[ \|v\|_{1,\Omega} \leq C\|\text{curl } v\|_{0,\Omega}. \]

In view of the relation:
\[ -\Delta u = \text{curl curl } u \quad \text{(as we have } \text{div } u = 0) \]
we can easily prove the following theorems.

Theorem 2.3. Let $\Omega$ be convex.

(Problem 1) has the following week variational formulation:
\[ \text{Find } u \in H_00(\text{curl}, \Omega) \text{ and } p \in H_01(\Omega) \text{ such that} \]
\[ \nu (\text{curl } u, \text{curl } v) + (\nabla p, v) = (f, v) \quad \forall v \in H_00(\text{curl}, \Omega) \quad (2.2) \]
\[ (\nabla q, u) = 0 \quad \forall q \in H_01(\Omega) \quad (2.3) \]

and (Problem 2) has the following week variational formulation:
\[ \text{Find } u \in H(\text{curl}, \Omega) \text{ and } p \in H^1(\Omega)/\mathbb{R} \text{ such that} \]
\[ \nu (\text{curl } u, \text{curl } v) + (\nabla p, v) = (f, v) \quad \forall v \in H(\text{curl}, \Omega) \quad (2.4) \]
\[ (\nabla q, u) = 0 \quad \forall q \in H^1(\Omega) \quad (2.5) \]

Each variational formulation is splitted into a system for the velocity and a Poisson equation for the pressure.

Let us introduce the spaces:
\[ V_0 = \{ v \in H_00(\text{curl}, \Omega); \ (\nabla q, v) = 0 \ \forall q \in H_01(\Omega) \} \]

and
\[ U = \{ v \in H(\text{curl}, \Omega); \ (\nabla q, v) = 0 \ \forall q \in H^1(\Omega) \}. \]

The lemma (2.1) and the theorems (2.2) and (2.3) allow us to establish the following theorem:

Theorem 2.4. The problem (2.2)-(2.3) is equivalent to the problem:
\[ \text{Find } u \in V_0 \text{ such that} \]
\[ \nu (\text{curl } u, \text{curl } v) = (f, v) \quad \forall v \in V_0 \quad (2.6) \]
\[ \text{Find } p \in H_01(\Omega) \text{ such that} \]
\[ (\nabla p, \nabla q) = (f, \nabla q) \quad \forall q \in H_01(\Omega). \quad (2.7) \]

The problem (2.4)-(2.5) is equivalent to the problem:
\[ \text{Find } u \in U \text{ such that} \]
\[ \nu (\text{curl } u, \text{curl } v) = (f, v) \quad \forall v \in U \quad (2.8) \]
\[ \text{Find } p \in H^1(\Omega)/\mathbb{R} \text{ such that} \]
\[ (\nabla p, \nabla q) = (f, \nabla q) \quad \forall q \in H^1(\Omega). \quad (2.9) \]

In both cases, there exists a unique solution and we have the following bounds:
\[ |p|_{1,\Omega} \leq |f|_{0,\Omega}; \quad \|\text{curl } u\|_{0,\Omega} \leq \frac{C_1}{\nu} |f|_{0,\Omega} \]
\[ |u|_{1,\Omega} \leq \frac{C_2}{\nu} |f|_{0,\Omega}; \quad \|\text{curl } u\|_{1,\Omega} \leq \frac{C_3}{\nu} |f|_{0,\Omega}. \]
We introduce a regular family of tetrahedra \((\tau_h)_h\) in the sense that:

- for each \(h\), \(\Omega\) in the union of all elements of \(\tau_h\);
- for each \(h\), the intersection of two different elements of \(\tau_h\), if not empty, is a corner, a whole edge or a whole face of both of them;
- the ratio of the diameter \(h\) of an element \(\kappa\) in \(\tau_h\) to the diameter of its inscribed sphere is bounded by a constant independent of \(\kappa\) and \(h\);

As usual, \(h\) denotes the maximum of the diameters of the elements of \(\tau_h\).

Next, for each \(\kappa\) in \(\tau_h\), we introduce the spaces \(P_0(\kappa)\) of the restrictions to \(\kappa\) of a constant function on \(\mathbb{R}^3\), \(P_1(\kappa)\) of the restrictions to \(\kappa\) of affine function on \(\mathbb{R}\) and the space \(P_k(\kappa)\) of the restrictions to \(\kappa\) of polynomials \(v\) of the form:

\[
v(x) = a + b \times x, \quad a \in \mathbb{R}^3, b \in \mathbb{R}^3.
\]

The space \(P_k(\kappa)\) and the corresponding finite elements are studied in [17].

Their degrees of freedom are the average flux along the edges \(\int_e (v.t)de\), for the six edges \(e\) of \(\kappa\), \(t\) is the direction vector of \(e\).

Hence, we associate the operator \(r_\kappa\) where \(r_\kappa(u)\) is the unique polynomial of \(P_k\) that has the same flux along the edges as \(u\). We define also the operator \(I_\kappa\) where \(I_\kappa(q)\) is the unique polynomial of \(P_1(\kappa)\) that has the same values on the vertex of \(\kappa\) as \(q\).

Next, let us introduce the discrete spaces:

\[
M_h = \{u_h \in H(\text{curl}, \Omega); u_h |_{\kappa} \in P_k(\kappa), \forall \kappa \in \tau_h\}, \quad (3.1)
\]

\[
M_{0h} = M_h \cap H_0(\text{curl}, \Omega), \quad (3.2)
\]

\[
Q_h = \{q_h \in C^0(\Omega); q_h |_{\kappa} \in P_1(\kappa), \forall \kappa \in \tau_h\}, \quad (3.3)
\]

\[
Q_{0h} = Q_h \cap H_0^1(\Omega). \quad (3.4)
\]

With these spaces, the finite dimensional analogues of \(V_0\) and \(U\) are:

\[
V_{0h} = \{v_h \in M_{0h}; (\nabla q_h, v_h) = 0, \quad \forall q_h \in Q_h\}
\]

and

\[
U_h = \{v_h \in M_h; (\nabla q_h, v_h) = 0 \quad \forall q_h \in Q_h\}.
\]

We define the interpolation operators \(r_h\) from \(H^1(\Omega)^3\) onto \(M_h\), \(I_h\) from \(H^2(\Omega)\) onto \(Q_h\) by

\[
\begin{align*}
r_h u &= r_\kappa(u) \text{ on } \kappa, \quad \forall \kappa \in \tau_h \quad \text{(similarly for } I_h \text{).}
\end{align*}
\]

**Theorem 3.1.** Assume that the triangulation \(\tau_h\) is regular. For all \(k \geq 1\) we have:

\[
\| u - r_h u \|_{0, \Omega} + h \| \text{curl}(u - r_h u) \|_{0, \Omega} \leq C h \| u \|_{1, t, \Omega}, \quad \forall u \in W^{1, t}(\Omega)^3, \quad \text{for some } t > 2.
\]

Moreover, when \(u \in (H^k(\Omega))^3\) and \(\forall k \geq 2\) we have:

\[
\| u - r_h u \|_{0, \Omega} \leq C h^k \| u \|_{k, \Omega}
\]

and, when \(u \in (H^{k+1}(\Omega))^3\) and \(\forall k \geq 1\) we have:

\[
\| \text{curl}(u - r_h u) \|_{0, \Omega} \leq C h^k \| u \|_{k+1, \Omega}
\]

There is also an important result given by V. Girault [13] which shows an impact imbedding between the spaces \(V_{0h}\) or \(U_h\) and \(L^2(\Omega)^3\) (or \(L^2(\Omega)\)):
Theorem 3.2. Let $\Omega$ be a convex polyhedron and $\tau_h$ a uniformly regular triangulation of $\Omega$. For each space $V_0h$ and $U_h$, there exists a constant $C$, independent of $h$, such that
\[ \|u_h\|_{2,\Omega} \leq C \|u_h\|_{0,4,\Omega} \leq C' \|\text{curl } u_h\|_{0,\Omega} \quad \forall u_h \in V_0h \text{ or } U_h \] (3.5)

We discretize (Problem 1) by:

Find $u_h \in V_0h$ and $p_h \in Q_0h$ such that
\[ \nu(\text{curl } u_h, \text{curl } v_h) + (\nabla p_h, v_h) = (f, v_h) \quad \forall v_h \in M_0h \] (3.6)

Similarly, we discretize (Problem 2) by:

Find $u_h \in U_h$ and $p_h \in Q_h/\mathbb{R}$ such that
\[ \nu(\text{curl } u_h, \text{curl } v_h) + (\nabla p_h, v_h) = (f, v_h) \quad \forall v_h \in M_h. \] (3.7)

As in the continuous way, the problem (3.6) can be split to

Find $u_h \in V_0h$ such that
\[ \nu(\text{curl } u_h, \text{curl } v_h) = (f, v_h) \quad \forall v_h \in V_0h, \] (3.8)

Find $p_h \in Q_0h$ such that
\[ (\nabla p_h, \nabla q_h) = (f, \nabla q_h), \quad \forall q_h \in Q_0h. \] (3.9)

And the problem (3.7) can be split to

Find $u_h \in U_h$ such that
\[ \nu(\text{curl } u_h, \text{curl } v_h) = (f, v_h), \quad \forall v_h \in U_h, \] (3.10)

Find $p_h \in Q_h/\mathbb{R}$ such that
\[ (\nabla p_h, \nabla q_h) = (f, \nabla q_h), \quad \forall q_h \in Q_h. \] (3.11)

It is easy to show, using theorem (3.2), that these two last discrete problems have a unique solution. The pressure is entirely dissociated from the velocity, i.e. can be computed without knowing the velocity. We have also for both discrete problems:

\[ \|\text{curl } u_h\|_{0,\Omega} \leq \frac{C}{\nu} \|f\|_{0,\Omega} \]

and
\[ \|p_h\|_{1,\Omega} \leq \|f\|_{0,\Omega}. \]

4. A priori error analysis

In this section, we will establish the error estimates for the pressure and the velocity.

Theorem 4.1. The theoretical solution $(u, p)$ of the problem (2.6)-(2.7) (resp. (2.8)-(2.9)) and the numerical solution $(u_h, p_h)$ of the problem (3.8)-(3.9) (resp. (3.10)-(3.11)) verify the error estimates:
\[ \|p - p_h\|_{1,\Omega} = \inf_{q_h \in Q_h} \|p - q_h\|_{1,\Omega} \quad (\text{resp. } Q_h) \] (4.1)

\[ \|\text{curl}(u - u_h)\|_{0,\Omega} \leq C \left( \inf_{v_h \in M_0h} \|\text{curl}(u - v_h)\|_{0,\Omega} + \inf_{q_h \in Q_0h} \|p - p_h\|_{1,\Omega} \right) \quad (\text{resp. } M_h \text{ and } Q_h) \] (4.2)

Proof:
For the pressure, let us choose \( q = q_h \). The difference between (2.7) and (3.9) (resp. (2.9) and (3.11)) gives:

\[
(\nabla (p - p_h), \nabla q_h) = 0, \quad \forall q_h \in Q_{0h} \quad (\text{resp. } Q_h)
\]

then

\[
|p - p_h|_{1, \Omega}^2 = (\nabla (p - p_h), \nabla p) = (\nabla (p - p_h), \nabla (p - q_h)) \leq |p - p_h|_{1, \Omega}|p - q_h|_{1, \Omega}
\]

and we obtain (4.1).

For the velocity, by taking \( \mathbf{v} = \mathbf{v}_h \), the difference between (2.6) and (3.8) (resp. (2.8) and (3.10)) gives:

\[
\nu (\nabla (u - u_h), \nabla \mathbf{v}_h) + (\nabla (p - p_h), \mathbf{v}_h) = 0 \quad \forall \mathbf{v}_h \in M_{0h} \quad (\text{resp. } M_h)
\]

Then for all \( \mathbf{w}_h \) in \( V_{0h} \) (resp. \( U_h \)) we have

\[
\nu (\nabla (u - u_h), \nabla \mathbf{w}_h) + \nu (\nabla (u - u_h), \nabla \mathbf{v}_h) + (\nabla (p - p_h), \mathbf{v}_h) = 0,
\]

By choosing \( \mathbf{v}_h = \mathbf{u}_h - \mathbf{w}_h \in V_{0h} \) and using the relation (3.5), we obtain

\[
|\nabla (u - w_h)||_{0, \Omega} \leq |\nabla (u - w_h)||_{0, \Omega} + C|p - p_h|_{1, \Omega}.
\]

Now we extend this last inequality to all functions \( \mathbf{v}_h \) of \( M_{0h} \) (resp \( M_h \)): Define \( q_h \) in \( Q_{0h} \) (resp. \( Q_h \)) by

\[
(\nabla q_h, \nabla \mu_h) = (\mathbf{v}_h, \nabla \mu_h) \quad \forall \mu_h \in Q_{0h} \quad (\text{resp. } Q_h)
\]

and set \( \mathbf{w}_h = \mathbf{v}_h - \nabla q_h \). Then \( \mathbf{w}_h \) belongs to \( V_{0h} \) (resp. \( U_h \)) and \( \nabla \mathbf{w}_h = \nabla \mathbf{v}_h \) and we obtain

\[
|\nabla (u - \mathbf{v}_h)||_{0, \Omega} = |\nabla (u - \mathbf{w}_h)||_{0, \Omega}
\]

Moreover

\[
|\nabla (u - u_h)||_{0, \Omega} \leq |\nabla (u - \mathbf{v}_h)||_{0, \Omega} + |\nabla (u - \mathbf{w}_h)||_{0, \Omega}
\]

\[
\leq 2|\nabla (u - \mathbf{v}_h)||_{0, \Omega} + C|p - p_h|_{1, \Omega}
\]

and we obtain (4.2).

\[
\square
\]

**Corollary 4.2.** Under the assumption of Theorem 4.1 and when the solution is sufficiently smooth we have

\[
|p - p_h|_{1, \Omega} \leq Ch|p|_{2, \Omega}
\]

and

\[
|\nabla (u - u_h)||_{0, \Omega} \leq Ch(|p|_{2, \Omega} + |u|_{2, \Omega})
\]

5. **A posteriori error analysis**

We now intend to prove a posteriori error estimates between the exact solution \((u, p)\) of the problem (2.6)-(2.7) and the numerical solution \((u_h, p_h)\) of the problem (3.8)-(3.9). By the same way, we can prove a posteriori error estimates between the solution \((u, p)\) of the exact problem (2.8)-(2.9) and \((u_h, p_h)\) of the numerical problem (3.10)-(3.11). In all the rest of the paper, we suppose that \( f \in H(\text{div}, \Omega) \).

We first introduce the space

\[
Z_h = \{ \mathbf{g}_h \in L^2(\Omega)^3; \forall \kappa \in \tau_h, \mathbf{g}_h|_\kappa \in P_0(\kappa) \}
\]

and we fix an approximation \( f_h \) of the data \( f \) in \( Z_h \).

Next, we denote by \( \mathcal{E}_h \) the set of all faces of the elements of \( \tau_h \) that are not contained in \( \partial \Omega \). For every element \( \kappa \) in \( \tau_h \), we denote by \( \mathcal{E}_\kappa \) the set of faces of \( \kappa \) that are not contained in \( \Gamma, \Delta_\kappa \) the set of union of elements of \( \tau_h \) that intersect \( \kappa, \Delta_\kappa \) the union of elements of \( \tau_h \) that intersect the face \( e, h_\kappa \) the diameter of \( \kappa \) and \( h_e \) the diameter of the face \( e \). Also, \( n_\kappa \) stands for the unit outward normal vector to \( \kappa \) on \( \partial \kappa \).
and \([\cdot]_{e}\) the jump through the face \(e\) of \(\kappa\).

For the demonstration of the next theorems, we introduce for an element \(\kappa\) of \(\mathcal{T}_h\), the bull function \(\psi_{\kappa}\) (resp. \(\psi_{e}\) of the face \(e\)) which is equal to the product of the \(d+1\) barycentric coordinates associated with the vertices of \(\kappa\) (resp. of \(e\)) and \(L_e\) the lifting operator from polynomials defined on \(e\) into polynomials defined on the elements \(\kappa\) and \(\kappa'\) contained \(e\), which is constructed by affine transformations from a fixed operator on the reference element.

**Property 5.1.** Denoting by \(P_r(e)\) the polynomial of degrees \(r\) on \(e\), we have

\[
\forall \ v \ poly\ of \ P_r(e) \quad c \parallel v \parallel L^2(e) \leq \parallel \psi_{e}^{1/2} \parallel L^2(e) \leq c' \parallel v \parallel L^2(e)
\]

and \(\forall \ v \ poly\ of \ P_r(e)\) which vanishes on \(\partial e\), we have

\[
\parallel \mathcal{L}_e v \parallel L^2(e) + h e \parallel \mathcal{L}_e v \parallel H^1(\kappa) \leq c h_e^{1/2} \parallel v \parallel L^2(e).
\]

We denote by \(R_h\) the Clément operator [8]. We have for all function \(q \in H_0^1(\Omega)\), \(R_h q \in Q_{0h}\) verifies

\[
\parallel q - R_h q \parallel L^2(e) \leq ch_{\kappa} \parallel q \parallel H^1(\kappa)
\]

\[
\parallel q - R_h q \parallel L^2(e) \leq c h_e^{1/2} \parallel q \parallel H^1(\Delta_e)
\]

and \(\mathcal{R}_h\) the Raviart-Thomas operator: for any smooth enough vectorial function \(v\), \(\mathcal{R}_h v\) belongs to \(M_{0h}\) and satisfies

\[
\forall e \in \mathcal{E}_h, \quad \int_{e} (v - \mathcal{R}_h v).n d\tau = 0.
\]

Moreover, this operator satisfies, see [21]: \(\forall v \in H_0^1(\Omega)^3\) and \(\forall \kappa \in \mathcal{T}_h\),

\[
\parallel v - \mathcal{R}_h v \parallel L^2(\kappa)^3 \leq c h_{\kappa} \parallel v \parallel H^1(\kappa)^3
\]

\[
\parallel v - \mathcal{R}_h v \parallel L^2(e)^3 \leq c h_e^{1/2} \parallel v \parallel H^1(\Delta_e)^3
\]

Let us begin with a posteriori error for the pressure. The error function \(p - p_h\) belongs to \(H_0^1(\Omega)\) and satisfies:

\[
(\nabla (p - p_h), \nabla q) = \langle F, q \rangle, \quad \forall q \in H_0^1(\Omega),
\]

where the "residual" \(F\) belongs to the dual space \(H^{-1}(\Omega)\) and is defined by:

\[
\forall v \in H_0^1(\Omega), \quad \langle F, q \rangle = \int_{\Omega} f \nabla q - \int_{\Omega} \nabla p_h \nabla q.
\]

We deduce that

\[
|p - p_h|_{1,\Omega} \leq \|F\|_{H^{-1}(\Omega)}.
\]

We define the error indicator by

\[
\eta_{\kappa} = \sum_{e \in \mathcal{E}_\kappa} h_{e}^{1/2} \parallel [(f_h - \nabla p_h).n] \parallel L^2(e)
\]

**Lemma 5.2.** The following estimate holds

\[
\|F\|_{H^{-1}(\Omega)} \leq C \left\{ \sum_{\kappa \in \mathcal{T}_h} \left( \eta_{\kappa}^2 + h_{e}^2 \parallel \text{div } f \parallel L^2(\kappa) + \left( \sum_{e \in \mathcal{E}_\kappa} h_{e}^{1/2} \parallel [(f - f_h).n] \parallel L^2(e) \right)^2 \right) \right\}^{1/2}
\]

**Proof :** For any \(q_h \in M_{0h}\), we have

\[
\langle F, q \rangle = \int_{\Omega} f \nabla q - \int_{\Omega} \nabla p_h \nabla (q - q_h)
\]

\[
= \sum_{\kappa \in \mathcal{T}_h} \left( \int_{\kappa} (f - f_h) \nabla (q - q_h) + \int_{\kappa} (f_h - \nabla p_h) \nabla (q - q_h) \right)
\]

By integrating by part, we obtain

\[
\langle F, q \rangle = \sum_{\kappa \in \mathcal{T}_h} \left\{ - \int_{\kappa} \text{div } f \ (q - q_h) + \frac{1}{2} \sum_{e \in \mathcal{E}_\kappa} \int_{e} \left( [(f - f_h).n](q - q_h) + [(f_h - \nabla p_h).n](q - q_h) \right) \right\}
\]
Corollary 5.3. The following a posteriori estimate holds between the solution \( p \) of (2.7) and the solution \( p_h \) of (3.9):
\[
|p - p_h|_{1, \Omega} \leq C \left( \sum_{\kappa \in \tau_h} (\eta^2_h + h^{2}_e) \| \div f \|_{L^2(\kappa)}^2 + \left( \sum_{e \in e_\kappa} h^{1/2}_e \| (f - f_h).n \|_{L^2(e)}^2 \right) \right)^{1/2}.
\]

Proposition 5.4. The error indicators verify the following optimality conditions
\[
\eta_k \leq C \left( \| p - p_h \|_{H^2(\kappa)} + h_e \| \div f \|_{L^2(\kappa)} + \sum_{e \in e_\kappa} h^{1/2}_e \| (f - f_h).n \|_{L^2(e)} \right), \tag{5.6}
\]

Proof: We consider the equation (5.5) with \( q_h = 0 \) and we take \( q = q_e = \mathcal{L}_e((f_h - \nabla q_h).n)\psi_e):\)
\[
\int_{\kappa \cup \kappa'} \nabla (p - p_h) \nabla q_e = - \int_{\kappa \cup \kappa'} \div f q_e + \frac{1}{2} \int_{e_\kappa} \left( (f - f_h).n \right) q_e + q_e^2
\]
then by using the property (5.1)
\[
\| (f_h - \nabla q_h ).n \|_{0, e} \leq C \left( h^{-1/2}_e \| p - p_h \|_{1, \kappa \cup \kappa'} + h^{1/2}_e \| \div f \|_{L^2(\kappa)} + \| (f - f_h).n \|_{L^2(e)} \right),
\]
 multiplying by \( h^{1/2}_e \) and summing over \( e_\kappa \), we obtain the result. \( \square \)

Now, let us establish a posteriori error for the velocity. The error function \( u - u_h \) belongs to \( H_0(\curl, \Omega) \), there exists a function \( \lambda \in H_0^1(\Omega) \) solution of the problem :
\[
\forall \mu \in H_0^1(\Omega), \quad \int_{\Omega} \nabla \lambda \nabla \mu = \int_{\Omega} (u - u_h) \nabla \mu.
\]
Then the function \( w = (u - u_h) - \nabla \lambda \) belongs to \( V_0 \) and we have \( \curl w = \curl (u - u_h) \). We obtain
\[
\| u - u_h \|_{H(\curl, \Omega)}^2 = \| \nabla \lambda \|_{0, \Omega}^2 + \| w \|_{H(\curl, \Omega)}^2 \tag{5.7}
\]
In order to find the upper and lower bounds of \( \| u - u_h \|_{H(\curl, \Omega)}^2 \), we start by finding the upper and lower bounds of the two terms of the left hand side of the last equation.

For the first term of the left hand side of (5.7), we have \( \forall \mu \in H_0^1(\Omega) \),
\[
\int_{\Omega} \nabla \lambda \nabla \mu = \int_{\Omega} \nabla \lambda \nabla \mu = - \int_{\Omega} u_h \nabla (\mu - \mu_h) \quad \forall \mu_h \in Q_0 \Omega.
\]
Let us introduce the problem
\[
\forall \mu \in H_0^1(\Omega), \quad \int_{\Omega} \nabla \lambda \nabla \mu = - \int_{\Omega} u_h \nabla \mu.
\]
The associate "residual" \( G \) belongs to \( H^{-1}(\Omega) \) and satisfies
\[
\int_{\Omega} \nabla \lambda \nabla \mu = \langle G, \mu \rangle \quad \forall \mu \in H_0^1(\Omega),
\]
then \( G \) satisfies
\[
\langle G, \mu \rangle = - \int_{\Omega} u_h \nabla \mu = - \int_{\Omega} u_h \nabla (\mu - \mu_h)
\]
\[
= \sum_{\kappa \in \tau_h} \left( \int_{\kappa} \div u_h (\mu - \mu_h) - \frac{1}{2} \sum_{e \in e_\kappa} \int_{[e]} [u_h.n] (\mu - \mu_h) \right) \tag{5.8}
\]
We introduce the indicators
\[
\xi_h = h_k \| \div u_h \|_{0, \kappa} + \sum_{e \in e_\kappa} h^{1/2}_e \| [u_h.n] \|_{0, e} \tag{5.9}
\]
Theorem 5.5. The following bounds hold

\[ |\lambda|_{1, \Omega} \leq C \left( \sum_{\kappa \in \mathcal{T}_h} \xi_\kappa \right)^{1/2} \]

and

\[ \xi_\kappa \leq C |\lambda|_{1, \Delta_\kappa} \]  \hspace{1cm} (5.10)

**Proof:** We treat this problem exactly as we did to the problem (2.7) and we obtain

\[ |\lambda|_{1, \Omega} \leq \| G \|_{-1, \Omega} \leq C \left( \sum_{\kappa \in \mathcal{T}_h} \xi_\kappa^2 \right)^{1/2} \]  \hspace{1cm} (5.11)

In order to find the lower bound, we take in the equation

\[ \int \nabla \lambda \nabla \mu = \sum_{\kappa \in \mathcal{T}_h} \left( \int \text{div} u_h \mu - \frac{1}{2} \sum_{e \in \mathcal{E}_\kappa} \int_e [u_h \cdot n] \mu \right), \]

\[ \mu = \text{div} u_h \psi_\kappa \] and use the inverse inequality \( |\mu|_{1, \kappa} \leq \chi_\kappa^{-1} \| \mu \|_{0, \kappa} \) to obtain

\[ \| \text{div} u_h \|_{0, \kappa} \leq C \lambda_{1, \kappa}. \]

Then we take \( \mu = L_e([u_h \cdot n][\psi_e]) \) and we obtain

\[ \| [u_h \cdot n] \|_{0, e} \leq C \left( \lambda_{1, \kappa}^{-1/2} + \lambda_{e}^{1/2} \right) \| \text{div} u_h \|_{0, \kappa \cup \mathcal{E}_e}. \]

which leads to

\[ \sum_{e \in \mathcal{E}_\kappa} \lambda_{e}^{1/2} \| [u_h \cdot n] \|_{0, e} \leq 4C \left( \lambda_{1, \kappa} + \lambda_{e} \right) \| \text{div} u_h \|_{0, \kappa \cup \mathcal{E}_e}. \]

Finally, using the fact that \( \lambda_{e} \leq \lambda_{\kappa} \) we have the bound

\[ \xi_\kappa \leq C |\lambda|_{1, \kappa \cup \mathcal{E}_\kappa}. \] \hspace{1cm} (5.12)

Now, we take the second term of the left hand side of (5.7). We begin by \( \forall v \in H_0(\text{curl}, \Omega) \)

\[ \nu \int_\Omega \text{curl}(u - u_h) \text{curl} v + \int_\Omega \nabla(p - p_h)v = \int_\Omega fv - \nu \int_\Omega \text{curl} u_h \text{curl} v - \int_\Omega \nabla p_h v. \]

By replacing \( u - u_h = w + \nabla \xi \) and taking \( v \in V_0 \) we obtain

\[ \nu \int_\Omega \text{curl} w \text{ curl} v = \int_\Omega fv - \nu \int_\Omega \text{curl} u_h \text{curl} v. \]

The associate ”residual” \( L \) belongs to \( V'_0 \) and satisfies

\[ \nu \int_\Omega \text{curl}(u - u_h) \text{curl} v = \langle L, v \rangle \quad \forall v \in V_0, \]

where, \( \forall v \in V_0, \) \( L \) verifies

\[ \langle L, v \rangle = \nu \int_\Omega \text{curl}(u - u_h) \text{curl} v + \int_\Omega \nabla(p - p_h)v \]

\[ = \nu \int_\Omega \text{curl}(u - u_h) \text{curl}(v - v_h) + \int_\Omega \nabla(p - p_h)(v - v_h) \quad \forall v_h \in M_{0h} \]

\[ = \int_\Omega f(v - v_h) - \nu \int_\Omega \text{curl} u_h \text{curl}(v - v_h) - \int_\Omega \nabla p_h (v - v_h) \quad \forall v_h \in M_{0h} \]

\[ = \sum_{\kappa \in \mathcal{T}_h} \left( \int_\kappa (f - f_h)(v - v_h) + \int_\kappa (f_h - \nabla p_h)(v - v_h) - \frac{1}{2} \sum_{e \in \mathcal{E}_e} \int_e ([\text{curl} u_h \times n])(v - v_h) \right) \]

\[ + \frac{1}{2} \sum_{e \in \mathcal{E}_e} \int_e [\text{curl} u_h \times n] \|0, e\| \]

We introduce the indicators

\[ \gamma_\kappa = h_\kappa \| f_h - \nabla p_h \|_{0, \kappa} + \frac{1}{2} \sum_{e \in \mathcal{E}_e} \lambda_{e}^{1/2} \| [\text{curl} u_h \times n] \|_{0, e} \]
Theorem 5.6. The following bounds hold
\[ \| u \|_{H_c(\mathbf{curl}, \Omega)} \leq C \left( \sum_{h \in T_h} \left( h^2 \| \mathbf{f} - \mathbf{f}_h \|_{0,T}^2 + \gamma^2 \right) \right)^{1/2} \] (5.14)
and
\[ \gamma \leq G \left( \| \mathbf{curl} w \|_{0,\Delta_x} + (h^2_\kappa + h_r) \left( \| \mathbf{f} - \mathbf{f}_h \|_{0,\Delta_x} + |p - p_h|_{0,\Delta_x} \right) \right) \] (5.15)

Proof: In the equation (5.13), we take \( \mathbf{v}_h = \mathcal{R}_h \mathbf{v} \) and use the properties of \( \mathcal{R}_h \), we obtain
\[ \| \mathbf{L} \|_{\mathcal{V}_0} \leq C \left( \sum_{h \in T_h} \left( h^2 \| \mathbf{f} - \mathbf{f}_h \|_{0,T}^2 + \gamma^2 \right) \right)^{1/2} \]
which leads to (5.14).

In the other hand, we consider the equation: \( \forall \mathbf{v} \in H_0(\mathbf{curl}, \Omega) \)
\[ \nu \int_{\Omega} \mathbf{curl}(\mathbf{u} - \mathbf{u}_h) \mathbf{curl} \mathbf{v} + \int_{\Omega} \nabla(p - p_h)\mathbf{v} = 0 \]
\[ \sum_{h \in T_h} \left( \int_{\kappa} (f - f_h)\mathbf{v} + \int_{\kappa} (f_h - \nabla p_h)\mathbf{v} - \frac{1}{2} \sum_{e \in e, \kappa} \int_{\kappa} (|\mathbf{curl} \mathbf{u}_h \times \mathbf{n}|)\mathbf{v} \right) \]
First, we take \( \mathbf{v} = (f_h - \nabla p_h)\psi_\kappa \) to obtain the relation:
\[ \| f_h - \nabla p_h \|_{0,\kappa} \leq C \left( h^{-1}_\kappa \| \mathbf{curl} w \|_{0,\kappa} + |p - p_h|_{1,\kappa} + \| f - f_h \|_{0,\kappa} \right) \]
Second, we take \( \mathbf{v} = \mathcal{L}_e ((|\mathbf{curl} \mathbf{u}_h \times \mathbf{n}|)\psi_\kappa) \) to obtain
\[ \| |\mathbf{curl} \mathbf{u}_h \times \mathbf{n}| \|_{0,e} \leq C \left\{ h^{-1/2}_\kappa \| \mathbf{curl} w \|_{0,\kappa} \right\} \]
Using the definition of \( \gamma \) we obtain the relation (5.15).

Corollary 5.7. The optimal a posteriori estimates hold
\[ \| u - u_h \|_{H_0(\mathbf{curl}, \Omega)} + |p - p_h|_{1,\Omega} \leq \left\{ \sum_{h \in T_h} \left( \gamma \kappa^2 + \xi^2 \kappa^2 + \eta^2 \kappa^2 \| \mathbf{f} - \mathbf{f}_h \|_{0,\kappa}^2 \right) + \| \mathbf{f} \|_{L^2(\kappa)}^2 \right\} \]
\[ + \left\{ \sum_{e \in e, \kappa} h^{1/2}_e \| (f - f_h)\mathbf{n} \|_{L^2(\kappa)} \right\}^{1/2} \] (5.16)
where \( \gamma \kappa \), \( \xi \kappa \) and \( \eta \kappa \) are given by the formulas (5.6), (5.10) and (5.15).

6. Numerical results

In order to confirm these results numerically, we did several experiments by using the FreeFem ++ software (see [16]). On the cubic domain \( [0,1] \times [0,1] \times [0,1] \), the numerical velocity and the pressure are taken as \( (u, p) = (\mathbf{curl} \psi, p) \), where:
\[ \psi = (\phi, \phi, \phi) \quad \text{with} \quad \phi(x, y, z) = x^2y^2z^2(x - 1)^2(y - 1)^2(z - 1)^2 \]
and \( p(x, y, z) = x(x - 1)y(y - 1)z(z - 1) \).

We take \( \nu = 1 \) and we denote by \( Nc \) the number of the points on edge of the geometry. We take a mesh with 6000 elements and we obtain the following color comparison between the exact and numerical solutions of the velocity and the pressure:
Next, the graphs related to the velocity’s and pressure’s error estimations have been studied. In logarithmic scale, we represent the errors $\| \text{curl}(u - u_h) \|_{0,\Omega}$ and $\| p - p_h \|_{1,\Omega}$ related to the mesh step $h$ in the following figures:

We can see that the pressure slope is 1.0454 and velocity slope is 1.965, results that are similar to the theoretical ones.
The next figure shows a comparison between the errors $|p - p_h|_{L^2} + \|u - u_h\|_{\text{curl,0}}$ and the sum of the indicators of the second member of the inequality (5.16). As we see, we are compatible with the inequality (5.16).

References

[16] F. Hecht & O. Pironneau FreeFem++, see: http://www.freefem.org