AN EXAMPLE OF FUNCTIONAL WHICH IS WEAKLY LOWER SEMICONTINUOUS ON $W_0^{1,p}$ FOR EVERY $p > 2$ BUT NOT ON H_0^1

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ABSTRACT. In this note we give an example of functional which is defined and coercive on $H_0^1(\Omega)$, which is sequentially weakly lower semicontinuous on $W_0^{1,p}(\Omega)$ for every $p > 2$, but which is not sequentially lower semicontinuous on $H_0^1(\Omega)$. This functional is non local.

1. Results and comments

Let Ω be a bounded open subset of $\mathbb{R}^N$, with $0 \in \Omega$ if $N \geq 2$, and $\Omega = (0, R_0)$ if $N = 1$. In this note, we give an example of functional which is defined and coercive on $H_0^1(\Omega)$, which has quadratic growth with respect to $\|Dv\|_2 = \|Dv\|_{(L^2(\Omega))^N}$, which is sequentially weakly lower semicontinuous on $W_0^{1,p}(\Omega)$ for every $p > 2$, but which is not sequentially weakly lower semicontinuous on $H_0^1(\Omega)$.

More precisely, when $N \geq 3$, we recall the Hardy-Sobolev inequality (see e.g. Theorems 21.7 and 21.8 in [5], Lemma 17.1 in [6], and also 4.1 in the Appendix below)

$$m_N^2 \int_{\mathbb{R}^N} \frac{|v|^2}{|x|^2} dx \leq \int_{\mathbb{R}^N} |Dv|^2 dx \qquad \forall v \in H_0^1(\mathbb{R}^N), \tag{1.1}$$

where m_N^2 denotes the best possible constant in the inequality, i.e.

$$m_N^2 = \inf_{v \in H_0^1(\mathbb{R}^N)} \frac{\displaystyle\int_{\mathbb{R}^N} |Dv|^2 dx}{\displaystyle\int_{\mathbb{R}^N} \frac{|v|^2}{|x|^2} dx}. \tag{1.2}$$

It is well known that m_N^2 is given by (see the references above)

$$m_N^2 = \frac{(N-2)^2}{4}.$$

We consider a function φ which is defined and continuous on $[0, \infty]$, which is non negative, non increasing and which satisfies

$$\varphi(0) > m_N^2 \text{ and } \varphi(\infty) < \frac{m_N^2}{2}. \tag{1.3}$$

Finally we define the functional J by

$$J(v) = \int_\Omega |Dv|^2 dx - \varphi(\|Dv\|_2^2) \int_\Omega \frac{|v|^2}{|x|^2} dx \qquad \forall v \in H_0^1(\Omega). \tag{1.4}$$

Our main result is the following

Theorem 1.1. *Let $N \geq 3$ and let Ω be a bounded open subset of $\mathbb{R}^N$, with $0 \in \Omega$. Assume that φ is a continuous, non negative and non increasing function on $[0, \infty]$ satisfying* (1.3)*, where m_N^2 is given by* (1.2)*; then the functional J defined by* (1.4) *satisfies*

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(i) *there exists a constant $C > 0$ such that*

$$-C + \frac{1}{2}\int_\Omega |Dv|^2 dx \le J(v) \le \int_\Omega |Dv|^2 dx \qquad \forall v \in H_0^1(\Omega); \tag{1.5}$$

(ii) *the functional J is sequentially weakly lower semicontinuous on $W_0^{1,p}(\Omega)$ for every $p > 2$, i.e.*

$$J(v) \le \liminf_{n\to\infty} J(v_n) \text{ if } v_n \rightharpoonup v \text{ in } W_0^{1,p}(\Omega) \text{ weakly;} \tag{1.6}$$

(iii) *the functional J is not sequentially weakly lower semicontinuous on $H_0^1(\Omega)$; more precisely, there exists a sequence of functions $w_n \in H_0^1(\Omega)$ such that $w_n \rightharpoonup 0$ in $H_0^1(\Omega)$ weakly and*

$$\liminf_{n\to\infty} J(w_n) < J(0). \tag{1.7}$$

Theorem 1.1 is proved in Section 2 below.

On the other hand, when $N = 2$ we consider a bounded open subset Ω of $\mathbb{R}^2$, with $0 \in \Omega$ and some R_0 for which[1] $\overline{\Omega} \subset B_{R_0}$. We recall the Hardy-Sobolev inequality (see e.g. Theorems 4.2 and 5.4 in [1] and Lemma 17.4 in [6])

$$m_2^2 \int_\Omega \frac{|v|^2}{|x|^2 \log^2 \frac{|x|}{R_0}} dx \le \int_\Omega |Dv|^2 dx \qquad \forall v \in H_0^1(\Omega), \tag{1.8}$$

where m_2^2 denotes the best possible constant in the inequality, i.e.

$$m_2^2 = \inf_{v \in H_0^1(\Omega)} \frac{\displaystyle\int_\Omega |Dv|^2 dx}{\displaystyle\int_\Omega \frac{|v|^2}{|x|^2 \log^2 \frac{|x|}{R_0}} dx}. \tag{1.9}$$

It is well known that m_2^2 is given by (see the references above)

$$m_2^2 = \frac{1}{4}.$$

We consider a function φ which is defined and continuous on $[0, \infty]$, which is non negative, non increasing and which satisfies

$$\varphi(0) > m_2^2 \text{ and } \varphi(\infty) < \frac{m_2^2}{2}, \tag{1.10}$$

and we define the functional J by

$$J(v) = \int_\Omega |Dv|^2 dx - \varphi(\|Dv\|_2^2) \int_\Omega \frac{|v|^2}{|x|^2 \log^2 \frac{|x|}{R_0}} dx \qquad \forall v \in H_0^1(\Omega). \tag{1.11}$$

In this case, we prove the following

Theorem 1.2. *Let $N = 2$ and let Ω be a bounded open subset of $\mathbb{R}^2$, with $0 \in \Omega$ and $\overline{\Omega} \subset B_{R_0}$. Assume that φ is a continuous, non negative and non increasing function on $[0, \infty]$ satisfying* (1.10)*, where m_2^2 is given by* (1.9)*; then the functional J defined by* (1.11) *satisfies the conditions* (i), (ii) *and* (iii) *of Theorem 1.1.*

Theorem 1.2 is proved in Section 3 below.

Finally, in the 1-D case, let Ω be the interval $\Omega = (0, R_0)$. We recall the Hardy-Sobolev inequality (see e.g. Theorem 327 in [3] and Lemma 1.3 in [5])

$$m_1^2 \int_0^\infty \frac{|v|^2}{|x|^2} dx \le \int_0^\infty |v'|^2 dx \quad \forall v \in H_0^1(0, \infty), \tag{1.12}$$

[1] In this note we denote by B_R the open ball of $\mathbb{R}^N$ of radius R and center 0.

where m_1^2 denotes the best possible constant in the inequality, i.e.

$$m_1^2 = \inf_{v \in H_0^1(0,\infty)} \frac{\int_0^\infty |v'|^2 dx}{\int_0^\infty \frac{|v|^2}{|x|^2} dx}. \tag{1.13}$$

It is well known that m_1^2 is given by (see the references above)

$$m_1^2 = \frac{1}{4}.$$

We consider a function φ which is defined and continuous on $[0, \infty]$, which is non negative, non increasing and which satisfies

$$\varphi(0) > m_1^2 \text{ and } \varphi(\infty) < \frac{m_1^2}{2}, \tag{1.14}$$

and we define the functional J by

$$J(v) = \int_0^{R_0} |v'|^2 dx - \varphi(\|v'\|_2^2) \int_0^{R_0} \frac{|v|^2}{|x|^2} dx \qquad \forall v \in H_0^1(0, R_0). \tag{1.15}$$

In this case we prove the following

Theorem 1.3. *In the* 1*-D case, let* Ω *be the interval* $\Omega = (0, R_0)$*. Assume that* φ *is a continuous, non negative and non increasing function on* $[0, \infty]$ *satisfying* (1.14)*, where* m_1^2 *is given by* (1.13)*; then the functional* J *defined by* (1.15) *satisfies the conditions* (i), (ii) *and* (iii) *of Theorem 1.1.*

The proof of Theorem 1.3 is the same as the one of Theorem 1.1 and is left to the reader.

Remark 1.1. Observe that, in contrast with the case $N \geq 2$, the functions $v \in H_0^1(0, R_0)$ vanish in 0 in the 1-D case.

Remark 1.2. Consider a functional of the (integral) form

$$J(v) = \int_\Omega F(x, v, Dv) dx \quad \forall v \in W^{1,p}(\Omega), \tag{1.16}$$

where $F : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function satisfying

$$a_0(x) + c_0|\xi|^p \leq F(x, s, \xi) \leq a_1(x) + b_1|s|^p + c_1|\xi|^p \qquad \text{for a.e. } x \in \Omega,\ \forall (s, \xi) \in \mathbb{R} \times \mathbb{R}^N,$$

where $p > 1$, $c_0 > 0$ and $a_0, a_1 \in L^1(\Omega)$. It is well known (see e.g. Theorems 3.1 and 3.4 in [2] and Theorem 2.4 in [4]) that the functional J is sequentially weakly lower semicontinuous on $W^{1,p}(\Omega)$ if and only if $F(x, s, \cdot)$ is a convex function for a.e. $x \in \Omega$ and for every $s \in \mathbb{R}$; moreover, in this case, the functional J is sequentially weakly lower semicontinuous on $W^{1,q}(\Omega)$ for every $q > 1$.

It is therefore impossible to write the functionals defined by (1.4), (1.11) and (1.15) in the integral form (1.16).

Remark 1.3. Using the result 4.3 of the Appendix below, we can prove an assertion which is stronger than assertion (ii), namely: if $N \geq 3$ then

$$\begin{cases} J(v) \leq \liminf\limits_{n\to\infty} J(v_n) \\ \\ \text{if } v_n \rightharpoonup v \text{ in } H_0^1(\Omega) \text{ weakly with } |Dv_n| \text{ equi-integrable in } L^2(\Omega). \end{cases}$$

The same result continues to hold for $N = 1$ and $N = 2$. Assertion (ii) of Theorems 1.1, 1.2 and 1.3 is a special case of this assertion since Ω is assumed to be bounded.

Remark 1.4. Actually in dimension $N \geq 3$, Theorem 1.1 continues to hold (with the same proof) if the Hardy-Sobolev inequality (1.1) is replaced by the Sobolev inequality

$$m^2 \left(\int_{\mathbb{R}^N} |v|^{2^*} dx \right)^{\frac{2}{2^*}} \leq \int_{\mathbb{R}^N} |Dv|^2 dx \quad \forall v \in H_0^1(\mathbb{R}^N), \tag{1.17}$$

where 2^* is the Sobolev's exponent defined by $2^* = 2N/(N-2)$ and where m^2 is the best possible constant in (1.17), and if in the definition (1.4) of the functional J the integral $\displaystyle\int_\Omega \frac{|v|^2}{|x|^2}dx$ is replaced by $\displaystyle\left(\int_\Omega |v|^{2^*}dx\right)^{\frac{2}{2^*}}$. More than that, Theorem 1.1, Theorem 1.2 and Theorem 1.3 still continue to hold (with the same proof) if the inequalities (1.1), (1.8), (1.12) and (1.17) are replaced by an inequality of the type

$$m_{X(\Omega)}\|v\|_{X(\Omega)} \le \|Dv\|_2,$$

where $X(\Omega)$ is a Banach space such that the embedding $H_0^1(\Omega) \hookrightarrow X(\Omega)$ is not compact while the embedding $W_0^{1,p}(\Omega) \hookrightarrow X(\Omega)$ is compact for any $p > 2$. The non compactness of the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega;\omega(x)dx)$ and the compactness of the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^2(\Omega;\omega(x)dx)$ for $p > 2$, where

$$\omega(x) = \begin{cases} \frac{1}{|x|^2} & \text{if } N = 1 \text{ or } N \ge 3, \\ \frac{1}{|x|^2 \log^2 \frac{|x|}{R_0}} & \text{if } N = 2, \end{cases}$$

(see 4.2 and 4.3 in the Appendix below), are indeed at the root of the proofs of (iii) and (ii). This explains why Theorem 1.1 continues to hold by replacing the Hardy-Sobolev inequality by the Sobolev inequality.

In contrast, if the embedding $H_0^1(\Omega) \hookrightarrow X(\Omega)$ is compact (e.g. in the case $X(\Omega) = L^2(\Omega)$ for Ω bounded), it is straightforward to prove that the functional

$$J(v) = \int_\Omega |Dv|^2dx - \varphi(\|Dv\|_2^2)\|v\|_{X(\Omega)}^2 \quad \forall v \in H_0^1(\Omega)$$

is sequentially weakly lower semicontinuous on $H_0^1(\Omega)$ whenever φ is non increasing: just take a sequence v_n such that $v_n \rightharpoonup v$ in $H_0^1(\Omega)$ weakly, and observe that in this framework

$$\int_\Omega |Dv|^2dx \le \liminf_{n\to\infty} \int_\Omega |Dv_n|^2dx,$$

$$-\varphi(\|Dv\|_2^2) \le \liminf_{n\to\infty} -\varphi(\|Dv_n\|_2^2),$$

$$\lim_{n\to\infty} \|v_n\|_{X(\Omega)}^2 = \|v\|_{X(\Omega)}^2.$$

Remark 1.5. Observe finally that in the proof of Theorem 1.1 below (for $N \ge 3$) it is not necessary to know the explicit value of the best constant m_N^2 in the inequality (1.1), whenever the function φ is chosen such that (1.3) holds.

In contrast, the proof of Theorem 1.2 below (where $N = 2$) uses the fact that the constant m_2^2 coincides with m_1^2. If one does not want to use the fact that $m_2^2 = m_1^2$, it would be sufficient to assume in (1.10) that $\varphi(0) > m_1^2$ in place of $\varphi(0) > m_2^2$ (see also the proof of (iii) in Theorem 1.2).

Also it should be observed that the fact that the best constant $m_{X(\Omega)}$ is attainable or not does not play any role in the proofs below, in contrast with the fact that the embedding $H_0^1(\Omega) \hookrightarrow X(\Omega)$ is not compact for Ω bounded, which is crucial.

2. Proof of Theorem 1.1

Proof of (i). By the definition of $J(v)$ we have

$$J(v) \le \int_\Omega |Dv|^2dx,$$

since φ is non negative.

It remains to prove the inequality on the left-hand side of (i). Since φ is continuous and satisfies (1.3), there exists $t_0 > 0$ such that $\varphi(t_0) = m_N^2/2$.

If $\|Dv\|_2^2 \geq t_0$ then $\varphi(\|Dv\|_2^2) \leq m_N^2/2$. Therefore

$$
\begin{aligned}
J(v) &\geq \int_\Omega |Dv|^2 dx - \frac{m_N^2}{2} \int_\Omega \frac{|v|^2}{|x|^2} dx \\
&\geq \int_\Omega |Dv|^2 dx - \frac{1}{2} \int_\Omega |Dv|^2 dx \\
&\geq \frac{1}{2} \int_\Omega |Dv|^2 dx,
\end{aligned}
$$

and the inequality on the left-hand side of (i) holds.

On the other hand, if $\|Dv\|_2^2 \leq t_0$, then

$$
\begin{aligned}
J(v) &\geq \int_\Omega |Dv|^2 dx - \varphi(0) \int_\Omega \frac{|v|^2}{|x|^2} dx \\
&\geq \int_\Omega |Dv|^2 dx - \frac{\varphi(0)}{m_N^2} \int_\Omega |Dv|^2 dx \\
&\geq \left(1 - \frac{\varphi(0)}{m_N^2} \right) \int_\Omega |Dv|^2 dx \\
&\geq \left(1 - \frac{\varphi(0)}{m_N^2} \right) t_0,
\end{aligned}
$$

in view of (1.3). If we choose a constant C such that

$$
\frac{\varphi(0)}{m_N^2} t_0 \leq C,
$$

we have

$$
J(v) \geq t_0 - \frac{\varphi(0)}{m_N^2} t_0 \geq \int_\Omega |Dv|^2 dx - C,
$$

and the inequality on the left-hand side of (i) is again proved.

This proves (i).

Proof of (ii). Let $p > 2$. Assume that $v_n \rightharpoonup v$ in $W_0^{1,p}(\Omega)$ weakly. Since Ω is bounded, $v_n \rightharpoonup v$ in $H_0^1(\Omega)$ weakly and there exists $\alpha \geq 0$ such that

$$
\liminf_{n\to\infty} \|Dv_n\|_2^2 = \|Dv\|_2^2 + \alpha. \tag{2.1}
$$

Since φ is continuous and non increasing, there esists $\beta \geq 0$ such that

$$
\liminf_{n\to\infty} -\varphi(\|Dv_n\|_2^2) = -\varphi(\|Dv\|_2^2) + \beta. \tag{2.2}
$$

Moreover, by the compactness of the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^2\left(\Omega; \frac{1}{|x|^2} dx\right)$ for $p > 2$ (see 4.3 in the Appendix below), we get

$$
\lim_{n\to\infty} \int_\Omega \frac{|v_n|^2}{|x|^2} dx = \int_\Omega \frac{|v|^2}{|x|^2} dx. \tag{2.3}
$$

(Note that (2.3) continues to hold if we assume (4.3) in place of $v_n \rightharpoonup v$ in $W_0^{1,p}(\Omega)$ weakly. This allows one to prove the assertion of Remark 1.3.)

Combining (2.1), (2.2) and (2.3), we obtain

$$
\liminf_{n\to\infty} J(v_n) \geq J(v) + \alpha + \beta \int_\Omega \frac{|v|^2}{|x|^2} dx \geq J(v),
$$

which proves (ii).

Proof of (iii). Let λ be such that $m_N^2 < \lambda < \varphi(0)$ (such a λ exists in view of (1.3)). Recalling the definition (1.2) of m_N^2, there exists a function $\psi \in C_0^\infty(\mathbb{R}^N)$ such that

$$\lambda \int_{\mathbb{R}^N} \frac{|\psi|^2}{|x|^2} dx > \int_{\mathbb{R}^N} |D\psi|^2 dx.$$

Since φ is continuous and satisfies (1.3), there exists $t_1 > 0$ such that $\varphi(t_1) = \lambda$. Take s such that $0 < s^2 \|D\psi\|_2^2 \leq t_1$. The function $w = s\psi$ belongs to $C_0^\infty(\mathbb{R}^N)$ and satisfies

$$\varphi(\|Dw\|_2^2) \geq \lambda, \tag{2.4}$$

as well as

$$\lambda \int_{\mathbb{R}^N} \frac{|w|^2}{|x|^2} dx > \int_{\mathbb{R}^N} |Dw|^2 dx. \tag{2.5}$$

Define the sequence w_n by

$$w_n(x) = n^{\frac{N-2}{2}} w(nx);$$

then

$$Dw_n(x) = n^{\frac{N}{2}} Dw(nx).$$

For n sufficiently large, the function w_n belongs to $H_0^1(\Omega)$ and

$$\int_\Omega |Dw_n|^2 dx = \int_{\mathbb{R}^N} |Dw|^2 dx \quad \text{and} \quad \int_\Omega \frac{|w_n|^2}{|x|^2} dx = \int_{\mathbb{R}^N} \frac{|w|^2}{|x|^2} dx.$$

Therefore, for n sufficiently large, the sequence w_n is bounded in $H_0^1(\Omega)$ with $w_n \rightharpoonup 0$ in $H_0^1(\Omega)$ weakly, and

$$J(w_n) = \int_{\mathbb{R}^N} |Dw|^2 dx - \varphi(\|Dw\|_2^2) \int_{\mathbb{R}^N} \frac{|w|^2}{|x|^2} dx.$$

Therefore $J(w_n) < 0$ in view of (2.4) and (2.5). This proves (iii).

3. Proof of Theorem 1.2

The proof of Theorem 1.2 is similar to the one of Theorem 1.1, but differs by technical aspects.
Proof of (i). Condition (1.5) is proved exactly as in the proof of Theorem 1.1.
Proof of (ii). Let $p > 2$. Assume that $v_n \rightharpoonup v$ in $W_0^{1,p}(\Omega)$ weakly. Then, as in the proof of

Theorem 1.1, we have for some $\alpha \geq 0$ and $\beta \geq 0$

$$\liminf_{n\to\infty} \|Dv_n\|_2^2 = \|Dv\|_2^2 + \alpha, \tag{3.1}$$

$$\liminf_{n\to\infty} \big\{ -\varphi(\|Dv_n\|_2^2) \big\} = -\varphi(\|Dv\|_2^2) + \beta. \tag{3.2}$$

Moreover, since $p > N = 2$, we have that $v_n \to v$ uniformly in Ω, and, since

$$\frac{1}{|x|^2 \log^2 \frac{|x|}{R_0}} \in L^1(\Omega),$$

we have

$$\lim_{n\to\infty} \int_\Omega \frac{|v_n|^2}{|x|^2 \log^2 \frac{|x|}{R_0}} dx = \int_\Omega \frac{|v|^2}{|x|^2 \log^2 \frac{|x|}{R_0}} dx. \tag{3.3}$$

Combining (3.1), (3.2) and (3.3), we obtain

$$\liminf_{n\to\infty} J(v_n) = J(v) + \alpha + \beta \int_\Omega \frac{v^2}{|x|^2 \log^2 \frac{|x|}{R_0}} dx \geq J(v),$$

which proves (ii).
Proof of (iii). Let λ be such that $m_1^2 < \lambda$, where m_1^2 is the best constant (defined by (1.13)) in the 1-D Hardy-Sobolev inequality (1.12). Then there exists $\psi \in C_0^\infty(0,\infty)$ such that

$$\lambda \int_0^\infty \frac{|\psi(t)|^2}{t^2} dt > \int_0^\infty |\psi'(t)|^2 dt.$$

Since φ is a continuous and satisfies (1.10), and since the best constant m_2^2 (defined by (1.9)) in the 2-D Hardy-Sobolev inequality (1.8) coincides with m_1^2, we can choose λ such that $m_2^2 = m_1^2 < \lambda < \varphi(0)$ (if we do not want to use the property $m_2^2 = m_1^2$, it would be sufficient to assume in (1.10) that $\varphi(0) > m_1^2$ in place of $\varphi(0) > m_2^2$). Then, there exists $t_1 > 0$ such that $\varphi(t_1) = \lambda$. Take s such that $0 < 2\pi s^2 \|\psi'\|_2^2 \le t_1$. The function $w = s\psi$ belongs to $C_0^\infty(0,\infty)$ and satisfies

$$\varphi\left(2\pi \int_0^\infty |w'(t)|^2 dt\right) \ge \lambda, \tag{3.4}$$

as well as

$$\lambda \int_0^\infty \frac{|w(t)|^2}{t^2} dt > \int_0^\infty |w'(t)|^2 dt. \tag{3.5}$$

Define the sequence w_n by

$$w_n(x) = \begin{cases} \frac{1}{\sqrt{n}} w\left(-n \log \frac{|x|}{R_0}\right) & \text{if } |x| \le R_0, \\ 0 & \text{if } |x| \ge R_0, \end{cases}$$

then

$$Dw_n(x) = \begin{cases} -\sqrt{n} w'\left(-n \log \frac{|x|}{R_0}\right) \frac{x}{|x|^2} & \text{if } |x| \le R_0, \\ 0 & \text{if } |x| \ge R_0. \end{cases}$$

For n sufficiently large, the function w_n belongs to $H_0^1(\Omega)$ and

$$\int_\Omega |Dw_n|^2 dx = 2\pi \int_0^{R_0} \left| w'\left(-n \log \frac{r}{R_0}\right)\right|^2 \frac{n}{r} dr = 2\pi \int_0^\infty |w'(t)|^2 dt,$$

while

$$\int_\Omega \frac{|w_n|^2}{|x|^2 \log^2 \frac{|x|}{R_0}} dx = 2\pi \int_0^{R_0} \frac{\left| w\left(-n \log \frac{r}{R_0}\right)\right|^2}{nr \log^2 \frac{r}{R_0}} dr = 2\pi \int_0^\infty \frac{|w(t)|^2}{t^2} dt.$$

Therefore, for n sufficiently large, the sequence w_n is bounded in $H_0^1(\Omega)$ with $w_n \rightharpoonup 0$ in $H_0^1(\Omega)$ weakly, and

$$J(w_n) = 2\pi \int_0^\infty |w'(t)|^2 dt - 2\pi \varphi\left(2\pi \int_0^\infty |w'(t)|^2 dt\right) \int_0^\infty \frac{|w(t)|^2}{t^2} dt.$$

Therefore $J(w_n) < 0$ in view of (3.4) and (3.5). This proves (iii).

4. Appendix

In this Appendix we recall some facts about the Hardy-Sobolev inequality in dimension $N \ge 3$, some of whom are well known.

4.1. A classical proof of (1.1) is to write, for every $v \in C_0^\infty(\mathbb{R}^N)$

$$0 \le \int_{\mathbb{R}^N} \left| Dv + cv \frac{x}{|x|^2} \right|^2 dx = \int_{\mathbb{R}^N} \left(|Dv|^2 + 2cv \frac{x \cdot Dv}{|x|^2} + c^2 \frac{|v|^2}{|x|^2} \right) dx.$$

Integrating by parts the second term, one gets

$$\int_{\mathbb{R}^N} 2v \frac{x \cdot Dv}{|x|^2} dx = -(N-2) \int_{\mathbb{R}^N} \frac{|v|^2}{|x|^2} dx,$$

and therefore

$$0 \leq \int_{\mathbb{R}^N} |Dv|^2 dx - \left((N-2)c - c^2\right) \int_{\mathbb{R}^N} \frac{|v|^2}{|x|^2} dx.$$

The choice $c = (N-2)/2$ proves (1.1) with $m_N^2 = (N-2)^2/4$.

4.2. Let us now prove by mean of a counterexample that, when $0 \in \Omega$, the embedding $H_0^1(\Omega) \hookrightarrow L^2\left(\Omega; \frac{1}{|x|^2} dx\right)$ is not compact. For that we consider the functions

$$u_n(x) = \frac{1}{\sqrt{n}} T_n\left(G_{R_0}(x)\right), \tag{4.1}$$

where $G_{R_0} : \mathbb{R}^N \to \mathbb{R}$ is the function defined by

$$G_{R_0}(x) = \begin{cases} \dfrac{1}{|x|^{N-2}} - \dfrac{1}{R_0^{N-2}} & \text{if } |x| \leq R_0, \\ 0 & \text{if } |x| \geq R_0, \end{cases} \tag{4.2}$$

with $R_0 > 0$ such that the ball $B_{R_0} \subset \Omega$, and where $T_n : \mathbb{R} \to \mathbb{R}$ is the truncation at height n, i.e.

$$T_n(t) = \begin{cases} t & \text{if } |t| \leq n, \\ n\dfrac{t}{|t|} & \text{if } |t| \geq n. \end{cases}$$

Then

$$\int_\Omega |Du_n|^2 dx = \int_{B_{R_0}} |Du_n|^2 dx = \frac{(N-2)^2 S_{N-1}}{n} \int_{r_n}^{R_0} \frac{1}{r^{N-1}} dr,$$

where S_{N-1} is the area of the unit sphere of $\mathbb{R}^N$ and where r_n is defined by

$$\frac{1}{r_n^{N-2}} - \frac{1}{R_0^{N-2}} = n.$$

Therefore

$$\int_\Omega |Du_n|^2 dx = (N-2) S_{N-1},$$

and $u_n \rightharpoonup 0$ in $H_0^1(\Omega)$ weakly. On the other hand, one has

$$\int_\Omega \frac{|u_n|^2}{|x|^2} dx \geq \int_{B_{r_n}} \frac{|u_n|^2}{|x|^2} dx = S_{N-1} n \int_0^{r_n} r^{N-3} dr = \frac{S_{N-1}}{N-2} n r_n^{N-2},$$

and then

$$\lim_{n\to\infty} \int_\Omega \frac{|u_n|^2}{|x|^2} dx \geq \frac{S_{N-1}}{N-2}.$$

This proves that the embedding $H_0^1(\Omega) \hookrightarrow L^2\left(\Omega; \frac{1}{|x|^2} dx\right)$ is not compact.

In dimension $N = 2$, this counterexample continues to hold by defining in (4.2) the function G_{R_0} by $G_{R_0}(x) = -\log \frac{|x|}{R_0}$ if $|x| \leq R_0$. In dimension $N = 1$, one uses the continuous piecewise affine functions u_n such that $u_n(0) = 0$, $u_n(R_0/n) = 1/\sqrt{n}$ and $u_n(R_0) = 0$.

4.3. Let us finally prove that, when

$$u_n \rightharpoonup u \text{ in } H_0^1(\Omega) \text{ weakly with } |Du_n| \text{ equi-integrable in } L^2(\Omega), \tag{4.3}$$

then $u_n \to u$ in $L^2\left(\Omega; \frac{1}{|x|^2} dx\right)$. Note that every sequence satisfying $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ weakly, with $p > 2$, satisfies (4.3) since Ω is bounded; therefore this claim implies that the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^2\left(\Omega; \frac{1}{|x|^2} dx\right)$ is compact for $p > 2$.

Let $\delta > 0$ be small. We write

$$\int_\Omega \frac{|u_n - u|^2}{|x|^2} dx = \int_{\Omega \setminus B_\delta} \frac{|u_n - u|^2}{|x|^2} dx + \int_{B_\delta} \frac{|u_n - u|^2}{|x|^2} dx, \tag{4.4}$$

where B_δ is the ball of radius δ. Since $\frac{1}{|x|^2} \in L^\infty(\Omega \setminus B_\delta)$ and since the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact for Ω bounded, the first term of (4.4) tends to zero when $n \to \infty$.

Let ψ_δ be the radial function defined by

$$\psi_\delta(x) = \begin{cases} 1 & \text{if } |x| \le \delta, \\ 2 - \frac{|x|}{\delta} & \text{if } \delta \le |x| \le 2\delta, \\ 0 & \text{if } |x| \ge 2\delta. \end{cases}$$

For δ sufficiently small, ψ_δ has compact support in Ω, and using Hardy-Sobolev inequality (1.1) we have

$$\begin{aligned} m^2 \int_{B_\delta} \frac{|u_n - u|^2}{|x|^2} dx &\le m^2 \int_\Omega \frac{|\psi_\delta (u_n - u)|^2}{|x|^2} dx \\ &\le \int_\Omega |D(\psi_\delta (u_n - u))|^2 dx \\ &\le 2 \int_\Omega |D\psi_\delta|^2 |u_n - u|^2 dx + 2 \int_\Omega |D(u_n - u)|^2 |\psi_\delta|^2 dx \\ &\le 2 \int_\Omega |D\psi_\delta|^2 |u_n - u|^2 dx + 2 \int_{B_{2\delta}} |D(u_n - u)|^2 dx. \end{aligned}$$

For δ fixed, the first term tends to zero when $n \to \infty$ (still because the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact), while the second term is small uniformly in n when δ is small in view of the equi-integrability assumption (4.3). This proves the claim. This also proves the assertion of Remark 1.3.

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