Γ–CONVERGENCE OF 2D GINZBURG-LANDAU FUNCTIONALS WITH VORTEX CONCENTRATION ALONG CURVES

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Abstract. We study the variational convergence of a family of two-dimensional Ginzburg-Landau functionals arising in the study of superfluidity or thin-film superconductivity, as the Ginzburg-Landau parameter ε tends to 0. In this regime and for large enough applied rotations (for superfluids) or magnetic fields (for superconductors), the minimizers acquire quantized point singularities (vortices). We focus on situations in which an unbounded number of vortices accumulate along a prescribed Jordan curve or a simple arc in the domain. This is known to occur in a circular annulus under uniform rotation, or in a simply connected domain with an appropriately chosen rotational vector field. We prove that, suitably normalized, the energy functionals Γ-converge to a classical energy from potential theory. Applied to global minimizers, our results describe the limiting distribution of vortices along the curve in terms of Green equilibrium measures.

Keywords: Calculus of variations, Ginzburg–Landau model, Γ-convergence, Partial differential equations, Equilibrium measures.

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1. Introduction

The Ginzburg-Landau theories have had an enormous influence on both physics and mathematics. Physicists employ Ginzburg-Landau models in modeling superconductivity, superfluidity, and, more recently, for rotating Bose-Einstein condensates (BECs), all systems which present quantized defects commonly known as vortices. In mathematics, starting with the work by Bethuel, Brezis & Hélein [6], many powerful methods have been developed to study the physical London limit, *i.e.*, as the Ginzburg-Landau parameter ε tends to 0. This limit corresponds to the Thomas-Fermi regime in BEC, and to an analogous regime in superfluids where the characteristic length scale ε is very small. In a two-dimensional setting, vortices are essentially characterized as isolated zeroes of the order parameter carrying a winding number, and in the London limit as point defects where energy concentration occurs. The question of whether energy minimizers develop vortices, where they appear in the domain, and how many there should be (for given boundary conditions, constant applied fields or angular velocities) has been analyzed in many contexts and parameter regimes.

In this paper, we will focus on the following Ginzburg-Landau energy, arising for instance in the physical context of a rotating superfluid. Considering a bounded simply connected domain $\mathcal{D} \subset \mathbb{R}^2$, a smooth vector field $V : \mathbb{R}^2 \to \mathbb{R}^2$, $\Omega > 0$ and $\varepsilon > 0$, we define the functional

$$u \in H^1(\mathcal{D};\mathbb{C}) \mapsto \mathcal{F}_\varepsilon(u) := \int_{\mathcal{D}} \left\{ \frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}(1-|u|^2)^2 - \Omega\, V(x)\cdot j(u) \right\} dx\,.$$

Identifying $\mathbb{R}^2$ with the complex plane $\mathbb{C}$, we denote by

$$j(u) := u \wedge \nabla u \in L^1(\mathcal{D};\mathbb{R}^2)\,,$$

the *pre-Jacobian* of u. The L^1-vector field $j(u)$ may also be written as $j(u) = (iu, \nabla u)$, where $(\cdot,\cdot)$ is the standard inner product of two complex numbers, viewed as vectors in $\mathbb{R}^2$.

In the case of uniform rotation, that is $V(x) = x^\perp = (-x_2, x_1)$ and with $\mathcal{D}$ a disk, Serfaty [17] studied minimizers of a closely related functional (see Remark 1.5) to determine the critical value $\Omega_1 = \Omega_1(\varepsilon)$ of the angular speed Ω at which vortices first appear (see also [10,11] for BECs). She finds that minimizers acquire vorticity at $\Omega_1 = k(\mathcal{D})|\ln\varepsilon| + O(\ln|\ln\varepsilon|)$ for an explicitly determined constant $k(\mathcal{D})$. In a series of papers, culminating with the publication of the research monograph [16], Sandier & Serfaty developed powerful tools to study vortices in Ginzburg-Landau models. Although they primarily work with the full Ginzburg-Landau model with magnetic field, the methods apply as well to the functional $\mathcal{F}_\varepsilon$ above. In particular, their results apply to the near-critical regime in *simply connected* domains. In our setting, their results show that for any simply connected domain $\mathcal{D}$, the first order expansion of the critical value Ω_1 for vortex existence in minimizing configurations is also given by $k(\mathcal{D})|\ln\varepsilon|$ for some constant $k(\mathcal{D})$. Moreover the locus of concentration of vortices for $\Omega = \Omega_1 + o(|\ln\varepsilon|)$ is given by the set of maxima of $|\zeta|$, with ζ the solution of the following boundary-value problem:

$$\begin{cases} -\Delta\zeta = \operatorname{curl} V & \text{in } \mathcal{D}\,, \\ \zeta = 0 & \text{on } \partial\mathcal{D}\,. \end{cases} \tag{1.1}$$

The constant $k(\mathcal{D})$ is then determined by

$$k(\mathcal{D}) = \frac{1}{2|\zeta|_{\max}}\,,$$

where $|\zeta|_{\max}$ denotes the maximum value of $|\zeta|$. If, for instance, V is real-analytic and $\operatorname{curl} V$ is nonnegative, then so is the solution ζ, and the maximum is generically attained at a finite number of points in $\mathcal{D}$ (see *e.g.* [7]). In this situation, if $\Omega = \Omega_1 + o(|\ln\varepsilon|)$, minimizers exhibit *vortex concentration at isolated points*, and the number of vortices remains uniformly bounded whenever $\Omega - \Omega_1$ is of order $O(\ln|\ln\varepsilon|)$, see [16,17,10,11].

The case of a multiply connected domain provides a slightly different qualitative picture. In a work on rotating Bose-Einstein condensates, Aftalion, Alama & Bronsard [1] considered a similar functional in a domain given by a circular annulus $\mathcal{A}$ (centered at the origin) and again with uniform rotation $V(x) = x^\perp$ (see Remark 1.5). Unlike the simply connected case, minimizers in the annulus may have vorticity without vortices, as the hole acquires positive winding at bounded rotation Ω. Then point vortices are nucleated inside the interior of $\mathcal{A}$ at a critical value Ω_1, again of leading order $|\ln\varepsilon|$. Solving equation (1.1) in the annulus $\mathcal{A}$, one finds out that the set of maxima of the function ζ is given by a circle inside $\mathcal{A}$ (see Example 5.1). Hence one can expect that, rather than accumulating at isolated points, vortices concentrate along this circle in the limit $\varepsilon \to 0$. The main feature proved in [1] is that if $\Omega \sim \Omega_1 + O(\ln|\ln\varepsilon|)$, then vortices are indeed essentially supported by a circle Σ and that the total degree of these vortices is of order $\ln|\ln\varepsilon|$. In other words, in the limit $\varepsilon \to 0$, infinitely many vortices concentrate on Σ, a phenomenon that we call *vortex concentration along a curve*. However the question of the distribution of the limiting vorticity around the circle was left open. Subsequent results of Alama & Bronsard [2,3] extend the result of [1] to multiply connected domains and to the full Ginzburg-Landau model with magnetic field and pinning potential. In contrast with the previous case, concentration on curves might not be a generic phenomenon for the Ginzburg-Landau model with magnetic field in a general multiply connected domain. Indeed, in this setting the vector field V represents the electromagnetic potential and it is an unknown of the problem. The results in [2,3] show that, for near-critical external applied fields, the concentration set of vortices is also given by the set of maxima of a certain potential related to V. This set may contain finitely many points and/or closed loops. Assuming that it contains closed loops, they prove that *vortex concentration along a curve* occurs, but the determination of the limiting vorticity was again left open.

To effectively separate the question of the nature of the concentration set from the question of localizing vortices, we instead start with a *simply connected* domain $\mathcal{D}$, and we prescribe the function ζ with $\zeta \geq 0$ in $\mathcal{D}$ and $\zeta|_{\partial\mathcal{D}} = 0$, in such a way that ζ is maximized on a *prescribed curve* $\Sigma \subset\subset \mathcal{D}$. Then, we define

$$V(x) := -\nabla^{\perp}\zeta(x) = \left(\frac{\partial\zeta}{\partial x_2}, -\frac{\partial\zeta}{\partial x_1}\right)$$

as our vector field. We will prove that vortices will be forced to accumulate on Σ as ε tends to 0. The curve Σ can be either a smooth Jordan curve or a smooth embedded simple arc, compactly contained in $\mathcal{D}$. In this setting, we shall resolve the problem of distribution of vortices along curves, both for minimizers and in the more general setting of Γ-convergence. In the last section we will show that in a multiply connected domain, and for more general vector fields V, the problem does not differ too much in nature, and that a similar analysis can be performed.

To state our main result we must give more specific hypotheses on ζ and the angular speed Ω. We assume that ζ satisfies the following assumptions:

(H1) $\zeta \in C_0^{0,1}(\mathcal{D})$, $\zeta \geq 0$ in $\overline{\mathcal{D}}$, and $\zeta_{\max} := \max_{x\in\overline{\mathcal{D}}} \zeta(x) > 0$;

(H2) $\Sigma := \{x \in \mathcal{D} : \ \zeta(x) = \zeta_{\max}\} \subset\subset \mathcal{D}$ is a Jordan curve or a simple embedded arc of class C^2.

We further assume that $\Omega = \Omega(\varepsilon)$ is near to the critical value needed for the presence of vortices. More precisely,

$$\Omega_\varepsilon := \frac{|\ln\varepsilon|}{2\zeta_{\max}} + \omega(\varepsilon)\,, \tag{1.2}$$

for some function $\omega : (0,+\infty) \to (0,+\infty)$ satisfiying $\omega(\varepsilon) \to +\infty$ with $|\ln\varepsilon|^{-1}\omega(\varepsilon) \to 0$ as $\varepsilon \to 0$.

For $u \in H^1(\mathcal{D};\mathbb{C})$ we consider the rescaled functional

$$F_\varepsilon(u) := \frac{1}{\omega^2(\varepsilon)} \int_{\mathcal{D}} \left\{\frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}(1-|u|^2)^2 + \Omega_\varepsilon \nabla^{\perp}\zeta \cdot j(u)\right\} dx\,,$$

and for a nonnegative Radon measure μ on $\mathcal{D}$, we define

$$I(\mu) := \frac{1}{2}\iint_{\mathcal{D}\times\mathcal{D}} G(x,y)\,d\mu(x)\,d\mu(y)\,,$$

where the function G denotes the Dirichlet Green's function of the domain $\mathcal{D}$, *i.e.*, for every $y\in\mathcal{D}$, $G(\cdot,y)$ is the solution of

$$\begin{cases} -\Delta G(\cdot,y) = \delta_y & \text{in } \mathscr{D}'(\mathcal{D})\,,\\ G(\cdot,y) = 0 & \text{on } \partial\mathcal{D}\,. \end{cases} \tag{1.3}$$

Our main result deals with the Γ-convergence of the family of functionals $\{F_\varepsilon\}_{\varepsilon>0}$ as $\varepsilon\to 0$, and it is stated (as usual) in terms of the vorticity distribution given by the weak Jacobian, that is half the distributional curl of the pre-Jacobian (see *e.g.* [16]).

Theorem 1.1. *Assume that* (H1), (H2) *and* (1.2) *hold. Let* $\varepsilon_n\to 0^+$ *be an arbitrary sequence. Then,*

(i) for any $\{u_n\}_{n\in\mathbb{N}}\subset H^1(\mathcal{D};\mathbb{C})$ *satisfying* $\sup_n F_{\varepsilon_n}(u_n)<+\infty$*, there exist a subsequence (not relabelled) and a nonnegative Radon measure* μ *in* $H^{-1}(\mathcal{D})$ *supported by* Σ *such that*

$$\frac{1}{\omega(\varepsilon_n)}\,\mathrm{curl}\,j(u_n) \underset{n\to+\infty}{\longrightarrow} \mu \quad \textit{strongly in } (C_0^{0,1}(\mathcal{D}))^*\,; \tag{1.4}$$

(ii) for any $\{u_n\}_{n\in\mathbb{N}}\subset H^1(\mathcal{D};\mathbb{C})$ *such that* (1.4) *holds for some nonnegative Radon measure* μ *in* $H^{-1}(\mathcal{D})$ *supported by* Σ*, we have*

$$\liminf_{n\to+\infty} F_{\varepsilon_n}(u_n) \geq I(\mu) - \zeta_{\max}\,\mu(\mathcal{D})\,;$$

(iii) for any nonnegative Radon measure μ *in* $H^{-1}(\mathcal{D})$ *supported by* Σ*, there exists a sequence* $\{u_n\}_{n\in\mathbb{N}}\subset H^1(\mathcal{D};\mathbb{C})$ *such that* (1.4) *holds and*

$$\lim_{n\to+\infty} F_{\varepsilon_n}(u_n) = I(\mu) - \zeta_{\max}\,\mu(\mathcal{D})\,.$$

As it is well known, the Γ-convergence theory is well suited to study asymptotics in minimization problems (see *e.g.* [9]). In this context, we shall derive from Theorem 1.1 the following convergence result for the vorticity of global minimizers, and hence solving the problem on the limiting distribution of vortices along Σ, see Remark 1.3 below.

Corollary 1.1. *Assume that* (H1), (H2) *and* (1.2) *hold. Let* $\varepsilon_n\to 0^+$ *be an arbitrary sequence. For every integer* $n\in\mathbb{N}$*, let* $u_n\in H^1(\mathcal{D};\mathbb{C})$ *be a minimizer of* F_{ε_n}*. Then,*

$$\frac{1}{\omega(\varepsilon_n)}\,\mathrm{curl}\,j(u_n) \underset{n\to+\infty}{\longrightarrow} \frac{\zeta_{\max}}{2I_*}\,\mu_* \quad \textit{strongly in } (C_0^{0,1}(\mathcal{D}))^*\,,$$

where μ_* *is the unique minimizer of* I *over all probability measures supported on* Σ*, and* $I_* := I(\mu_*)$*.*

Remark 1.1. As a direct application of the results in [16] (see also [12]), we shall see in Section 2 that for configurations $\{u_\varepsilon\}$ with F_ε-energy uniformly bounded from above, the vorticity distribution $\mathrm{curl}\,j(u_\varepsilon)$ can be approximated (with respect to the $(C_0^{0,1}(\mathcal{D}))^*$–topology) by a measure of the form $2\pi\sum_{i\in I_\varepsilon} d_i\delta_{a_i}$ for some finite set of points $\{a_i\}_{i\in I_\varepsilon}\subset\mathcal{D}$ and integers $\{d_i\}_{i\in I_\varepsilon}\subset\mathbb{Z}$. In other words, each point a_i can be viewed as an approximate vortex with winding number d_i. Thus the integer $D_\varepsilon=\sum_{i\in I_\varepsilon}|d_i|$ may be refered to as to *approximate total vorticity* of the configuration u_ε. It is commonly known that approximate vortices carry a kinetic energy essentially greater than or equal to $\pi D_\varepsilon |\ln\varepsilon|$ (see Section 2 for more details). With such an estimate in hand, and using the arguments of Section 3, we actually obtain a more refined lower bound for the energy than the one given by Theorem 1.1, claim *(ii)*. More precisely, one has

$$\liminf_{\varepsilon\to 0} \frac{1}{\omega^2(\varepsilon)} \left(\int_{\mathcal{D}} \frac{1}{2}|\nabla u_\varepsilon|^2 + \frac{1}{4\varepsilon^2}(1-|u_\varepsilon|^2)^2\, dx - \pi D_\varepsilon |\ln\varepsilon| \right) \geq I(\mu)\,,$$

and

$$\liminf_{\varepsilon\to 0} \frac{1}{\omega^2(\varepsilon)} \left(\Omega_\varepsilon \int_{\mathcal{D}} \nabla^\perp \zeta \cdot j(u_\varepsilon)\, dx + \pi D_\varepsilon |\ln\varepsilon| \right) \geq -\zeta_{\max}\, \mu(\mathcal{D})\,.$$

As a consequence, if $\{u_\varepsilon\}$ is any recovery sequence (in the sense of *(iii)* of Theorem 1.1), the $\liminf$'s above become limits, and equality holds in each case. In analogy with [6], we may then say that $I(\mu)$ plays the role of renormalized energy.

Remark 1.2 (Minimizers). From Corollary 1.1 and Remark 1.1, we deduce that if u_ε is energy minimizing, then

$$D_\varepsilon = \frac{\zeta_{\max}}{4\pi I_*}\,\omega(\varepsilon) + o(\omega(\varepsilon)) \quad \text{as } \varepsilon \to 0\,,$$

and from Theorem 1.1, the minimal value of the energy expands as

$$\min_{H^1(\mathcal{D};\mathbb{C})} \omega^2(\varepsilon) F_\varepsilon = -\frac{\zeta_{\max}^2}{4I_*}\,\omega^2(\varepsilon) + o(\omega^2(\varepsilon))\,.$$

Remark 1.3 (Equilibrium measures). The value $I(\mu)$ gives the electrostatic energy of a positive charge distribution μ on the set $\Sigma \subset\subset \mathcal{D}$. The minimizer μ_* of I over all probability measures on Σ is called the *Green equilibrium measure* in $\mathcal{D}$ associated to the set Σ, and gives the equilibrium charge distribution of a charged conductor inside of a neutral conducting shell, represented by $\partial\mathcal{D}$. The value $1/I_*$ is refered to as to the *capacity of the condenser* $(\Sigma, \partial\mathcal{D})$. The interested reader can find in [14] many results on the existence and general (regularity) properties of the equilibrium measures as well as some examples. For instance, if $\mathcal{D}$ is a disc and Σ is a concentric circle, then the equilibrium measure μ_* is the normalized arclength measure on Σ, see [14, Example II.5.13], and thus vortices are asymptotically equidistributed along Σ as $\varepsilon \to 0$. However for an arbitrary curve Σ, the distribution is of course non-uniform in general. In case where Σ is an embedded arc, it is even singular at the endpoints, see [14, Example II.5.14].

Remark 1.4. In the present results the structure and regularity assumptions on the set Σ given in (H2) are mainly motivated by the physical context of [1,2,3]. However it will be clear that (H2) can be relaxed into weaker statments. More precisely, the proof of Theorem 1.1 relies on (H2) only for the $\Gamma - \limsup$ inequality, *i.e.*, claim *(iii)*. The construction of the recovery sequence (see Section 4) could be applied with minor modifications if the set Σ is for instance a finite union of piecewise C^2 arcs/Jordan curves. Actually Σ could even have a more general structure such as a non-empty interior. In this later case we assume that $\partial\Sigma$ is made by finitely many arcs and Jordan curves of class C^2. Then, given a nonnegative Radon measure $\mu \in H^1(\mathcal{D})$ supported by Σ, one can construct a recovery sequence for μ applying the approximation techniques of Section 4 to $\mu \llcorner \partial\Sigma$ and a more standard regularization procedure for $\mu \llcorner \operatorname{int}(\Sigma)$ as in [16].

In Section 5 we will show how to apply the method to a multiply connected domain and a more general vector field V. For simplicity, we shall consider only domains $\mathcal{A}$ which are topological annuli, *i.e.*, $\mathcal{A} = \mathcal{D}\setminus\overline{\mathcal{B}}$, where $\mathcal{D}$, $\mathcal{B}$ are simply connected and $\mathcal{B} \subset\subset \mathcal{D}$. For multiply connected domains and/or a general field V, there is an extra step involved in the analysis. Indeed, the leading order term in the minimal energy (of order $|\ln\varepsilon|^2$) is due to the curl-free part of V which induces a diverging phase in any minimal configuration, and to the vorticity in the hole $\mathcal{B}$ which acts as a sort of giant vortex (by analogy with [1]). In other words, to get information on the internal vorticity, one has to perform a second order Γ-convergence analysis. As in [2] we first describe minimizing vortexless configurations (*i.e.*, energy minimizers over $\mathbb{S}^1$-valued maps) which nonetheless have vorticity around the hole. Then we show that for arbitrary configurations, the energy due to the curl-free part of V and the hole decouples nearly exactly (see Proposition 5.2). After separating out this contribution to the energy, the residual energy functional resembles F_ε above, and the Γ-convergence analysis can be done in much the same way, with some care taken to control the residual vorticity around the hole. The result is stated in Theorem 5.2. The Γ-limit again involves a Green energy for measures supported by a prescribed set.

Remark 1.5 (Related funtionals). As mentioned earlier, the methods here may also be applied to other Ginzburg-Landau functionals which have the same structure as $\mathcal{F}_\varepsilon$. A simple variant is

$$\tilde{\mathcal{F}}_\varepsilon(u) = \int_{\mathcal{D}} \left\{ \frac{1}{2}|(\nabla - i\Omega_\varepsilon V)u|^2 + \frac{1}{4\varepsilon^2}(1-|u|^2)^2 \right\} dx\,.$$

This energy has been studied in more than one context. Serfaty considers in [17] the minimization of this energy for rotating superfluids, with $V = x^\perp$ and under a Dirichlet boundary condition $u|_{\partial\mathcal{D}} = 0$. The minimization of the same energy under natural (Neumann) boundary conditions also arises in a simplified model of thin-film superconductors introduced by Chapman, Du & Gunzburger [8] (see also Alama, Bronsard & Galvão-Sousa [4]). In this setting, $\Omega_\varepsilon V(x) = A_\varepsilon(x)$ represents the magnetic vector potential of the externally applied magnetic field $h_{ex} = \Omega_\varepsilon \mathrm{curl}\, V$. With Ω_ε satisfying (1.2), and a sequence $\{u_\varepsilon\}$ in the energy regime of Theorem 1.1 (which holds for minimizers under a homogeneous Neumann boundary condition), the two energies agree very closely,

$$\tilde{\mathcal{F}}_\varepsilon(u_\varepsilon) = \mathcal{F}_\varepsilon(u_\varepsilon) + \frac{\Omega_\varepsilon^2}{2}\int_{\mathcal{D}} |V|^2\,dx + o(1) \quad \text{as } \varepsilon \to 0\,.$$

In the case of the homogeneous Dirichlet boundary condition some care must be taken since a singular boundary layer arises near $\partial\mathcal{D}$ as $\varepsilon \to 0$ (see [17]), but otherwise the result of Theorem 1.1 should remain essentially the same.

For annular domains, another functional which resembles $\mathcal{F}_\varepsilon$ has been used in the modeling of rotating BECs in certain anharmonic traps (see [1]),

$$\mathcal{G}_\varepsilon(u) = \int_{\mathcal{A}} \left\{ \frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}(a(x)-|u|^2)^2 - \Omega_\varepsilon\, V(x)\cdot j(u) \right\} dx\,.$$

Here $V(x) = x^\perp$ is the velocity field of uniform rotation, and the function $a(x)$, positive in $\mathcal{A}$, gives the trapping potential which contains the condensate. It is shown in [1] that $\mathcal{G}_\varepsilon$-minimizers develop as $\varepsilon \to 0$, infinitely many vortices concentrating along a circle for sufficently high rotation Ω_ε. We believe that similar results to the ones in Section 5 should hold, but the method is not directly applicable here. Indeed, our analysis is based on "global energy estimates" of [16] (see (2.1) in Proposition 2.1) and the presence of the inhomogeneity $a(x)$ requires local estimates. Moreover the analysis of vortices for the energy $\mathcal{G}_\varepsilon$ is complicated by the fact that $a(x)$ vanishes on the boundary, and some delicate estimates are required so as not to lose too much information near the boundary.

One may also consider the full Ginzburg-Landau model of superconductivity for complex order parameter $u : \mathcal{D} \to \mathbb{C}$ and magnetic vector potential $A : \mathcal{D} \to \mathbb{R}^2$. For the Ginzburg-Landau model in a simply connected domain with constant applied magnetic field, Sandier & Serfaty [16, Theorem 9.1] have proven a Γ-convergence theorem of the form of Theorem 1.1 for applied fields of the form $h_{ex} = H_{C1} + \omega(\varepsilon)$, where in that case it is appropriate to take $\ln|\ln\varepsilon| \lesssim \omega(\varepsilon) \ll |\ln\varepsilon|$. This problem exhibits *vortex concentration at points*, and the limiting energy is obtained by rescaling around the points of concentration. The rescaled vorticity measures of minimizers converge to an equilibrium measure associated to a different problem in potential theory, a "Gauss variation" problem whereby the charges are to be optimally placed in $\mathbb{R}^2$ subject to an applied electric field (see [14]). Concentration on curves is possible in multiply connected regions (see [2,3]). The methods described here should apply in this situationl once adapted to the magnetic setting, although the energy obtained as a Γ-limit should be for the Helmholtz (and not the Laplace) Green's function, $-\Delta_x H(x,y) + H(x,y) = \delta_y(x)$. However, for the full Ginzburg-Landau functional with magnetic field in general domains, it is an interesting open problem in PDE to determine which (if any) non-symmetric multiply connected domains exhibit vortex concentration on curves.

We finally mention a recent paper by Kashmar [13] exhibiting concentration on a circle for a magnetic Ginzburg-Landau functional in the disc with an inhomogeneity $a(x)$ as above described by a radial step function (modelling for instance a superconducting body made by two different species). Here again we believe that similar results should hold but the method does not directly apply due to the inhomogeneity of the Ginzburg-Landau energy density.

The plan of the paper is as follows. In Section 2 we prove assertion *(i)* of Theorem 1.1. Section 3 tackles part *(ii)* of the theorem. The upper bound of statement *(iii)* is derived in Section 4, completing the proof of Theorem 1.1. The proof of Corollary 1.1 is presented at the end of Section 4. Section 5 sketches how the preceeding arguments must be modified to treat annular domains.

Notations. For any open set $B \subset \mathcal{D}$ and any admissible map u, we denote by

$$E_\varepsilon(u, B) := \int_B \left\{ \frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}(1-|u|^2)^2 \right\} dx$$

the so-called Ginzburg-Landau energy of u in B. Given $\varepsilon > 0$ we define

$$\mathcal{D}_\varepsilon := \{x \in \mathcal{D} : \ \mathrm{dist}\,(x, \partial\mathcal{D}) > \varepsilon\},$$

and for a sequence $\varepsilon_n \to 0^+$, we shall write $\omega_n := \omega(\varepsilon_n)$ and $\Omega_n := \Omega_{\varepsilon_n}$.

2. Compactness of normalized weak Jacobians

This section is devoted to the proof of claim *(i)* in Theorem 1.1. The key ingredient to prove compactness of normalized weak Jacobians is the so-called *"vortex balls construction"* taken from [16, Theorem 4.1].

Proposition 2.1. *There exists a constant $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ and any $u \in H^1(\mathcal{D};\mathbb{C}) \cap C^1(\mathcal{D})$ satisfying $E_\varepsilon(u, \mathcal{D}) \le \sqrt{\varepsilon}$, the following holds. For any $C_0\varepsilon^{1/4} < r < 1$ there exists a finite collection of disjoint closed balls $\mathcal{B}_r = \{\overline{B}(a_i, \rho_i)\}_{i \in I_r}$ such that, writing $B_i := \overline{B}(a_i, \rho_i)$,* *(i)* $r = \sum_{i \in I_r} \rho_i$;

(ii) $\{|1-|u|| \geq \varepsilon^{1/8}\} \cap \mathcal{D}_\varepsilon \subset \mathcal{D}_\varepsilon \cap \bigcup_{i \in I_r} B_i =: V$;

(iii) *setting* $d_i := \deg(u, \partial B_i)$ *if* $B_i \subset \mathcal{D}_\varepsilon$, *and* $d_i := 0$ *otherwise,*

$$E_\varepsilon(u, V) \geq \pi D_r \left(\ln \left(\frac{r}{\varepsilon D_r} \right) - C_1 \right), \tag{2.1}$$

where $D_r := \sum_{i \in I_r} |d_i|$ *is assumed to be positive;*

(iv) $D_r \leq C_2 |\ln \varepsilon|^{-1} E_\varepsilon(u, \mathcal{D})$;

and C_0, C_1, C_2 *are universal constants. Moreover, if* $C_0 \varepsilon^{1/4} < r_1 < r_2 < 1$ *and* $\mathcal{B}_1$, $\mathcal{B}_2$ *are the corresponding families of balls, then every ball of* $\mathcal{B}_1$ *is included in a ball of* $\mathcal{B}_2$.

In the remainder of this section, we consider an arbitrary sequence $\varepsilon_n \to 0^+$. In the following lemma, we prove an upper bound on the Ginzburg-Landau energy for a sequence having an F_{ε_n}-energy uniformly bounded from above. It will allow us to apply the previous proposition to such a sequence.

Lemma 2.1. *Assume that* (H1) *and* (1.2) *hold. Let* $\{u_n\}_{n \in \mathbb{N}}$ *be a sequence in* $H^1(\mathcal{D}; \mathbb{C})$ *such that* $\sup_n F_{\varepsilon_n}(u_n) < +\infty$. *Then there exists a constant* C *independent of* n *such that* $\|u_n\|_{L^4(\mathcal{D})} \leq C$ *and* $E_{\varepsilon_n}(u_n, \mathcal{D}) \leq C |\ln \varepsilon_n|^2$.

Proof. Observe that

$$\begin{aligned} E_{\varepsilon_n}(u_n, \mathcal{D}) = \omega_n^2 F_{\varepsilon_n}(u_n) - \Omega_n \int_{\mathcal{D}} \nabla^\perp \zeta \cdot j(u_n)\, dx \leq \Omega_n \|\nabla \zeta\|_\infty \int_{\mathcal{D}} |u_n||\nabla u_n| dx + O(\omega_n^2) \\ \leq \frac{1}{4} \int_{\mathcal{D}} |\nabla u_n|^2 dx + \Omega_n^2 \|\nabla \zeta\|_\infty^2 \int_{\mathcal{D}} |u_n|^2 dx + O(\omega_n^2)\,. \end{aligned} \tag{2.2}$$

In particular,

$$\int_{\mathcal{D}} \big(|u_n|^4 - 2(1 + 2\varepsilon_n^2 \Omega_n^2 \|\nabla \zeta\|_\infty^2) |u_n|^2 + 1 \big) dx \leq O(\varepsilon_n^2 \omega_n^2)\,,$$

so that $\|u_n\|_{L^4(\mathcal{D})} \leq C$ for a constant C independent of n. Inserting this estimate in (2.2), the announced result follows easily. □

The first step in proving compactness of the normalized Jacobians is to show that the *approximate total vorticity* is bounded by the excess rotation ω_n. We emphasize that here Σ could be any compact subset of $\mathcal{D}$.

Proposition 2.2. *Assume that* (H1) *and* (1.2) *hold. Let* $\{u_n\}_{n \in \mathbb{N}}$ *be a sequence in* $H^1(\mathcal{D}; \mathbb{C}) \cap C^1(\mathcal{D})$ *such that* $\sup_n F_{\varepsilon_n}(u_n) < +\infty$. *Then there exist* $r_n \to 0^+$ *and a sequence of families of balls* $\mathcal{B}_{r_n} = \{\overline{B}(a_i^n, \rho_{i,n})\}_{i \in I_{r_n}}$ *as in Proposition 2.1 such that* $D_{r_n} \leq C \omega_n$ *for some constant* $C > 0$ *independent of* n.

Proof. Let $r_n := |\ln \varepsilon_n|^{-4}$. In view of Lemma 2.1 we can apply Proposition 2.1 to u_n with $r = r_n$ and n large enough. For each such n we denote by $\mathcal{B}_{r_n} = \{\overline{B}(a_i^n, \rho_{i,n})\}_{i \in I_{r_n}}$ the corresponding family of balls. For convenience we write $I_n := I_{r_n}$, $B_i^n := \overline{B}(a_i^n, \rho_{i,n})$, $d_{i,n} := \deg(u, \partial B_i^n)$ if $B_i^n \subset \mathcal{D}_{\varepsilon_n}$, $d_{i,n} = 0$ otherwise, and $D_n := D_{r_n}$.

Step 1. We first claim that

$$\Omega_n \int_{\mathcal{D}} \nabla^\perp \zeta \cdot j(u_n)\, dx = -2\pi\Omega_n \sum_{i\in I_n} d_{i,n}\, \zeta(a_i^n) + o(1) \quad \text{as } n \to +\infty\,. \tag{2.3}$$

Indeed, applying [16, Theorem 6.1] with the balls $\{B_i^n\}_{i\in I_n}$ we derive the estimate

$$\Big\| \operatorname{curl} j(u_n) - 2\pi \sum_{i\in I_n} d_{i,n}\, \delta_{a_i^n} \Big\|_{(C_0^{0,1}(\mathcal{D}))^*} \leq C r_n \big(1 + E_{\varepsilon_n}(u_n, \mathcal{D})\big) \leq C |\ln \varepsilon_n|^{-2}\,, \tag{2.4}$$

thanks to Lemma 2.1. Then (2.3) follows since $\Omega_n = O(|\ln \varepsilon_n|)$ and $\zeta \in C_0^{0,1}(\mathcal{D})$.

Step 2. Without loss of generality we may assume that $\omega_n = O(D_n)$, otherwise there is nothing to prove. In view of claim *(iv)* in Proposition 2.1 and Lemma 2.1, we have $D_n \leq O(|\ln \varepsilon_n|)$. Next, combining the lower bound (2.1) with (2.3), we infer that

$$O(\omega_n^2) \geq \omega_n^2 F_{\varepsilon_n}(u_n) \geq \pi D_n \big(|\ln \varepsilon_n| - C \ln|\ln \varepsilon_n|\big) - 2\pi\Omega_n \sum_{i\in I_n} d_{i,n} \zeta(a_i^n) + \\ + \int_{\mathcal{D}\setminus \cup_{i\in I_n} B_i^n} |\nabla u_n|^2\, dx + o(1) \tag{2.5}$$

as $n \to +\infty$.

Let us now fix a sequence $\eta_n \to 0^+$ such that

$$\max\{\omega_n, \ln|\ln \varepsilon_n|\} = o(\eta_n |\ln \varepsilon_n|) \quad \text{as } n \to +\infty\,. \tag{2.6}$$

We group the vortex balls into the following classes and we define:

$$D_n^* := \sum_{i\in I_n^*} d_{i,n}\,, \qquad I_n^* := \big\{ i \in I_n \,:\, d_{i,n} \geq 0 \text{ and } \zeta(a_i^n) > \zeta_{\max} - \eta_n \big\}\,;$$

$$D_n^+ = \sum_{i\in I_n^+} d_{i,n}\,, \qquad I_n^+ := \big\{ i \in I_n \,:\, d_{i,n} \geq 0 \text{ and } \zeta(a_i^n) \leq \zeta_{\max} - \eta_n \big\}\,;$$

$$D_n^- := \sum_{i\in I_n^-} |d_{i,n}|\,, \qquad I_n^- := \big\{ i \in I_n \,:\, d_{i,n} < 0 \big\}\,.$$

Observe that $D_n = D_n^* + D_n^+ + D_n^-$. We claim that

$$D_n^+ \leq C\, \frac{\max\{\omega_n, \ln|\ln \varepsilon_n|\} D_n}{\eta_n |\ln \varepsilon_n|}\,, \tag{2.7}$$

$$D_n^- \leq C\, \frac{\max\{\omega_n, \ln|\ln \varepsilon_n|\} D_n}{|\ln \varepsilon_n|}\,, \tag{2.8}$$

$$D_n \leq C \max\{\omega_n, \ln|\ln \varepsilon_n|\}\,, \tag{2.9}$$

for a constant $C > 0$ independent of n. In particular, if $\omega_n^2 = o(|\ln \varepsilon_n|)$, then we can choose η_n satisfying in addition $\max\{\omega_n^2, (\ln|\ln \varepsilon_n|)^2\} = o(\eta_n |\ln \varepsilon_n|)$, and consequently (2.9) yields $D_n^+ = D_n^- = 0$ for n large enough.

We evaluate the lower bound for each class of vortex ball separately. First, we use the explicit form of Ω_n (see (1.2)) and the bound $\zeta(x) \leq \zeta_{\max}$ to obtain,

$$\pi D_n^* |\ln \varepsilon_n| - 2\pi\Omega_n \sum_{i\in I_n^*} d_{i,n} \zeta(a_i^n) \geq -2\pi \omega_n \zeta_{\max} D_n^*\,. \tag{2.10}$$

For negative degrees we have the simple estimate

$$\pi D_n^- |\ln \varepsilon_n| - 2\pi\Omega_n \sum_{i\in I_n^-} d_{i,n} \zeta(a_i^n) \geq \pi D_n^- |\ln \varepsilon_n|\,. \tag{2.11}$$

Then for the vortex balls staying away from Σ, we have

$$\pi D_n^+|\ln\varepsilon_n| - 2\pi\Omega_n \sum_{i\in I_n^+} d_{i,n}\zeta(a_i^n) \geq (\pi|\ln\varepsilon_n| - 2\pi\Omega_n\zeta_{\max})\, D_n^+ + \\ + 2\pi\Omega_n \sum_{i\in I_n^+} d_{i,n}(\zeta_{\max} - \zeta(a_i^n)) \geq D_n^+\,(-2\pi\omega_n\zeta_{\max} + 2\pi\Omega_n\eta_n) \geq \\ \geq C\Omega_n\eta_n D_n^+\,, \quad (2.12)$$

since $\omega_n = o(\eta_n\Omega_n)$. We now insert (2.10), (2.11) and (2.12) into (2.5),

$$O(\omega_n^2) \geq \pi D_n\,(|\ln\varepsilon_n| - C_1\ln|\ln\varepsilon_n|) - 2\pi\Omega_n \sum_{i\in I_n} d_{i,n}\zeta(a_i^n) + \int_{\mathcal{D}\setminus\cup B_i^n} |\nabla u_n|^2 dx + o(1) \\ \geq -\pi C_1 D_n \ln|\ln\varepsilon_n| - 2\pi\omega_n\zeta_{\max}D_n^* + \pi D_n^-|\ln\varepsilon_n| + C\eta_n\Omega_n D_n^+ + \\ + \int_{\mathcal{D}\setminus\cup B_i^n} |\nabla u_n|^2 dx\,. \quad (2.13)$$

Rearranging all terms we derive

$$D_n^-|\ln\varepsilon_n| + D_n^+\eta_n\Omega_n \leq C\left(\omega_n D_n^* + \ln|\ln\varepsilon_n|D_n + \omega_n^2\right) \\ \leq C\max\{\omega_n, \ln|\ln\varepsilon_n|\}D_n\,,$$

which proves (2.7) and (2.8).

To prove (2.9), we argue as in [2, pg. 58–60] to obtain a constant $C' > 0$ (independent of n) such that

$$\int_{\mathcal{D}\setminus\cup B_i^n} |\nabla u_n|^2 dx \geq C' D_n^2\,. \quad (2.14)$$

Accepting (2.14) we return to the lower bound (2.13) to deduce

$$D_n^2 - C\,(\omega_n + \ln|\ln\varepsilon_n|)\,D_n \leq O(\omega_n^2)\,,$$

so that $D_n \leq C\max\{\omega_n, \ln|\ln\varepsilon_n|\}$ and estimate (2.9) is established.

It remains to show (2.14). To this aim we identify an annular band lying outside of Σ and use the fact that the total degree is approximately constant in that band. Since the boundary $\partial\mathcal{D}$ is assumed to be smooth, there exists $0 < \delta_0 < \frac{1}{2}\mathrm{dist}\,(\Sigma, \partial\mathcal{D})$ such that the function $\varrho(x) := \mathrm{dist}\,(x, \partial\mathcal{D})$ is smooth in $\mathcal{D}_{\varepsilon_n}^{\delta_0}$ with

$$\mathcal{D}_{\varepsilon_n}^{\delta_0} := \{x \in \mathcal{D} \,:\, \varepsilon_n < \varrho(x) \leq \delta_0\}\,.$$

The level sets

$$\mathcal{C}_t := \{x \in \mathcal{D} :\ \varrho(x) = t\}\,,$$

are smooth and diffeomorphic to $\partial\mathcal{D}$ for all $t \in [0, \delta_0]$. Define the set $T_n \subset [0, \delta_0]$ by

$$T_n := \{t \in (\varepsilon_n, \delta_0] \,:\, \mathcal{C}_t \cap \cup_{i\in I_n} B_i^n = \emptyset\}\,.$$

By the choice of r_n and claim *(i)* in Proposition 2.1, T_n is a finite union of disjoint intervals and $\mathcal{L}^1((\varepsilon_n, \delta_0]\setminus T_n) \leq O(|\ln\varepsilon_n|^{-4})$. From claim *(ii)* in Proposition 2.1, we can define the degree of u_n on $\mathcal{C}_t$ for every $t \in T_n$, *i.e.*,

$$D_n(t) := \deg\left(\frac{u_n}{|u_n|}, \mathcal{C}_t\right) = \frac{1}{2\pi}\int_{\mathcal{C}_t} \frac{u_n}{|u_n|} \wedge \nabla_\tau\left(\frac{u_n}{|u_n|}\right) d\mathcal{H}^1\,,$$

where ∇_τ denotes the tangential derivative along $\mathcal{C}_t$ oriented counterclockwise. Setting $I_n(t) := \{i \in I_n \,:\, \varrho(a_i^n) \geq \rho_{i,n} + t\}$ ($\rho_{i,n}$ being the radius of the ball B_i^n), we have

$$D_n(t) = \sum_{i \in I_n(t)} d_{i,n}\,.$$

Using (2.7) and (2.8) we infer that for n large enough,

$$D_n(t) \geq D_n - 2D_n^- - D_n^+ \geq \frac{1}{2} D_n \quad \text{for every } t \in T_n\,.$$

Denote $v_n := u_n/|u_n|$. In view of claim *(ii)* in Proposition 2.1, using the Coarea Formula and Jensen Inequality we can estimate for n large enough,

$$\int_{\mathcal{D}\setminus\cup B_i^n} |\nabla u_n|^2 dx \geq \int_{\mathcal{D}_{\varepsilon_n}^{\delta_0}\setminus\cup B_i^n} |u_n|^2 |\nabla v_n|^2 dx \geq \frac{1}{2}\int_{\varepsilon_n}^{\delta_0}\left(\int_{\mathcal{C}_t} |\nabla v_n|^2 d\mathcal{H}^1\right) dt \geq$$
$$\geq \frac{1}{2}\int_{T_n}\left(\int_{\mathcal{C}_t} |v_n \wedge \nabla_\tau v_n|^2 d\mathcal{H}^1\right) dt \geq 2\pi^2 \int_{T_n} \frac{|D_n(t)|^2}{\mathcal{H}^1(\mathcal{C}_t)}\, dt \geq C' D_n^2\,,$$

which completes the proof of (2.14).

Step 3. If $\ln|\ln\varepsilon_n| \leq o(\omega_n)$ the conclusion follows from (2.7), (2.8) and (2.9). If $\omega_n \leq O(\ln|\ln\varepsilon_n|)$ we must refine our lower bound by growing the vortex balls. First observe that in this regime, (2.7) and (2.8) ensures that

$$D_n^- = D_n^+ = 0 \tag{2.15}$$

for n large so that each ball $B_i^n \subset \mathcal{D}_{\varepsilon_n}$ carries a nonnegative degree and $D_n = D_n^*$.

We choose a new radius $s_n := e^{-\sqrt{\omega_n}}$ and thus $s_n > r_n$ for n large enough. We now reapply Proposition 2.1 with $r = s_n$ to obtain a new family of larger balls $\{\tilde{B}_j^n\}_{j\in J_n}$, each new ball $\tilde{B}_j^n$ containing one or more of the smaller balls $\{B_i^n\}$. By claim *(ii)* in Proposition 2.1 and (2.15) we have $D_{s_n} = D_n$. Using the lower bound (2.1) together with (2.3) and (2.15), we can argue as in Step 2 to derive

$$O(\omega_n^2) \geq \omega_n^2 F_{\varepsilon_n}(u_n) \geq \pi D_n \left(\ln\left(\frac{s_n}{\varepsilon_n D_n}\right) - C_1\right) - 2\pi\Omega_n \zeta_{\max} D_n + C' D_n^2$$
$$\geq \pi D_n \left(\ln\left(\frac{s_n}{D_n}\right) - C_1\right) - 2\pi\omega_n \zeta_{\max} D_n + C' D_n^2\,. \tag{2.16}$$

Next we distinguish two cases. First assume that $\ln D_n \leq O(\omega_n)$. In this case, (2.16) yields the inequality $D_n^2 - C\omega_n D_n \leq O(\omega_n^2)$ (with $C > 0$ independent of n) so that $D_n \leq O(\omega_n)$ as claimed. If $\omega_n = o(\ln D_n)$, we obtain the bound $D_n^2 - CD_n \ln D_n \leq O(\omega_n^2)$ which also yields $D_n \leq O(\omega_n)$, and the proof of Proposition 2.2 is complete. □

We are now ready to prove claim *(i)* in Theorem 1.1.

Theorem 2.1. *Assume that* (H1) *and* (1.2) *hold. Let* $\varepsilon_n \to 0^+$ *and let* $\{u_n\}_{n\in\mathbb{N}}$ *be a sequence in* $H^1(\mathcal{D};\mathbb{C})$ *such that* $\sup_n F_{\varepsilon_n}(u_n) < +\infty$*. Then there exist a subsequence (not relabelled) and a nonnegative Radon measure* μ *supported by* Σ *such that*

$$\mu_n := \frac{1}{\omega_n} \operatorname{curl} j(u_n) \underset{n\to+\infty}{\longrightarrow} \mu \quad \textit{strongly in } \left(C_0^{0,1}(\mathcal{D})\right)^*\,.$$

Proof. *Step 1.* We first assume that $\{u_n\}_{n\in\mathbb{N}} \subset H^1(\mathcal{D};\mathbb{C}) \cap C^1(\mathcal{D})$. Using the notations of the previous proof, we apply Proposition 2.2 to obtain the family of vortex balls $\{B_i^n\}_{i\in I_n}$. Define the measure

$$\bar{\mu}_n := \frac{2\pi}{\omega_n} \sum_{i\in I_n} d_{i,n} \delta_{a_i^n} .$$

Since $D_n \le O(\omega_n)$ we have $|\bar{\mu}_n|(\mathcal{D}) \le C$ for a constant C independent of n. Therefore, up to a subsequence, $\bar{\mu}_n \rightharpoonup \mu$ as $n \to +\infty$ weakly* in the sense of measures on $\mathcal{D}$ for some finite Radon measure μ. We claim that μ is nonnegative and supported by Σ. Indeed, decompose $\bar{\mu}_n$ in its Hahn decomposition, *i.e.*, write $\bar{\mu}_n = \bar{\mu}_n^+ - \bar{\mu}_n^-$ where $\bar{\mu}_n^+$ and $-\bar{\mu}_n^-$ are respectively the positive and the negative parts of $\bar{\mu}_n$. Then we have

$$\bar{\mu}_n^-(\mathcal{D}) = \frac{2\pi D_n^-}{\omega_n} \to 0 \quad \text{as } n \to +\infty ,$$

thanks to (2.8), and the nonnegativity of μ follows. Now consider the sequence of sets $\mathcal{V}_n := \{\zeta(x) \le \zeta_{\max} - \eta_n\}$ where η_n is given by (2.6). In view of (2.7), we have

$$\bar{\mu}_n^+(\mathcal{V}_n) = \frac{2\pi D_n^+}{\omega_n} \to 0 ,$$

which clearly implies that $\operatorname{supp}\mu \subset \Sigma$.

By the compact embedding $(C_0^0(\mathcal{D}))^* \hookrightarrow (C_0^{0,1}(\mathcal{D}))^*$, we deduce that $\bar{\mu}_n \to \mu$ strongly in $(C_0^{0,1}(\mathcal{D}))^*$. On the other hand, (2.4) yields

$$\big\|\mu_n - \bar{\mu}_n\big\|_{(C_0^{01}(\mathcal{D}))^*} \underset{n\to+\infty}{\longrightarrow} 0 ,$$

and the conclusion follows.

Step 2. We now consider the general case. In view of the strong continuity of the functional F_{ε_n} under strong H^1-convergence, we can find a sequence $\{\tilde{u}_n\}_{n\in\mathbb{N}} \subset H^1(\mathcal{D};\mathbb{C}) \cap C^1(\mathcal{D})$ such that for every n,

$$\|u_n - \tilde{u}_n\|_{H^1(\mathcal{D})} \le \varepsilon_n , \tag{2.17}$$

and

$$F_{\varepsilon_n}(\tilde{u}_n) \le F_{\varepsilon_n}(u_n) + 1 , \tag{2.18}$$

so that $\sup_n F_{\varepsilon_n}(\tilde{u}_n) < +\infty$.

Given an arbitrary $\varphi \in C_0^{0,1}(\mathcal{D})$ satisfying $|\nabla\varphi| \le 1$, we estimate

$$\left| \int_{\mathcal{D}} \big(j(u_n) - j(\tilde{u}_n)\big) \cdot \nabla^\perp \varphi \, dx \right| \le$$
$$\le \|u_n - \tilde{u}_n\|_{L^2(\mathcal{D})} \|\nabla u_n\|_{L^2(\mathcal{D})} + \|\tilde{u}_n\|_{L^2(\mathcal{D})} \|\nabla(u_n - \tilde{u}_n)\|_{L^2(\mathcal{D})} \le C\varepsilon_n |\ln \varepsilon_n| ,$$

using (2.17) and (2.18) together with Lemma 2.1. As a consequence, setting $\tilde{\mu}_n := \omega_n^{-1} \operatorname{curl} j(\tilde{u}_n)$, we have

$$\big\|\mu_n - \tilde{\mu}_n\big\|_{(C_0^{01}(\mathcal{D}))^*} \underset{n\to+\infty}{\longrightarrow} 0 . \tag{2.19}$$

Applying Step 1 to $\{\tilde{u}_n\}$, up to a subsequence we have $\tilde{\mu}_n \to \mu$ in $(C_0^{0,1}(\mathcal{D}))^*$ for some nonnegative Radon measure μ supported by Σ. Then (2.19) yields $\mu_n \to \mu$ in $(C_0^{0,1}(\mathcal{D}))^*$ and the proof is complete. $\square$

3. The lower bound inequality

This section is devoted to the proof of claim *(ii)* in Theorem 1.1 that we summarize in the following result.

Theorem 3.1. *Assume that* (H1) *and* (1.2) *hold. Let $\varepsilon_n \to 0^+$ be an arbitrary sequence and let $\{u_n\}_{n\in\mathbb{N}} \subset H^1(\mathcal{D};\mathbb{C})$ be such that*

$$\mu_n = \frac{1}{\omega_n} \operatorname{curl} j(u_n) \underset{n\to+\infty}{\longrightarrow} \mu \quad \textit{strongly in } (C_0^{0,1}(\mathcal{D}))^* \,, \tag{3.1}$$

for some nonnegative Radon measure μ supported by Σ. Then,

$$\liminf_{n\to+\infty} F_{\varepsilon_n}(u_n) \geq I(\mu) - \zeta_{\max}\, \mu(\mathcal{D}) \,. \tag{3.2}$$

In particular, if the left hand side in (3.2) is finite, then $\mu \in H^{-1}(\mathcal{D})$.

Proof. We will use in this proof the notations of the previous section. Without loss of generality, we may assume that

$$\liminf_{n\to+\infty} F_{\varepsilon_n}(u_n) = \lim_{n\to+\infty} F_{\varepsilon_n}(u_n) < +\infty \,. \tag{3.3}$$

Moreover, by Step 2 in the proof of Theorem 2.1, we may also assume that $\{u_n\}_{n\in\mathbb{N}} \subset H^1(\mathcal{D};\mathbb{C}) \cap C^1(\mathcal{D})$. We shall distinguish two cases.

Case 1. We first assume that $\ln|\ln \varepsilon_n| \leq o(\omega_n)$. We consider the family of vortex balls $\{B_i^n\}_{i\in I_n}$ constructed in the proof of Proposition 2.2, and we refer to it for the notations. Arguing as in (2.3) we obtain

$$\begin{aligned} \Omega_n \int_{\mathcal{D}} \nabla^\perp \zeta \cdot j(u_n)\, dx &= -\frac{\pi|\ln \varepsilon_n|}{\zeta_{\max}} \sum_{i\in I_n} d_{i,n} \zeta(a_i^n) + \omega_n \int_{\mathcal{D}} \nabla^\perp \zeta \cdot j(u_n)\, dx + o(1) \\ &\geq -\pi D_n |\ln \varepsilon_n| - \omega_n^2 \langle \mu_n, \zeta\rangle + o(1) \end{aligned} \tag{3.4}$$

as $n \to +\infty$, where $\langle \cdot,\cdot\rangle$ denotes the duality pairing $(C_0^{0,1})^*$–$C_0^{0,1}$.

Combining the lower bound (2.1) with (3.4), we infer that

$$\begin{aligned} \omega_n^2 F_{\varepsilon_n}(u_n) &\geq \pi D_n \left(\ln\left(\frac{r_n}{\varepsilon_n D_n}\right) - C_1 \right) + \frac{1}{2} \int_{\mathcal{D}\setminus \cup B_i^n} |\nabla u_n|^2 dx + \Omega_n \int_{\mathcal{D}} \nabla^\perp \zeta \cdot j(u_n)\, dx \\ &\geq \pi D_n \left(\ln\left(\frac{r_n}{D_n}\right) - C_1 \right) + \frac{1}{2} \int_{\mathcal{D}\setminus \cup B_i^n} |\nabla u_n|^2 dx - \omega_n^2 \langle \mu_n, \zeta\rangle + o(1) \,, \end{aligned}$$

Since $r_n = |\ln \varepsilon_n|^{-4}$ and $D_n \leq O(\omega_n)$ by Proposition 2.2, dividing the previous inequality by ω_n^2 yields

$$\begin{aligned} F_{\varepsilon_n}(u_n) &\geq \frac{1}{2\omega_n^2} \int_{\mathcal{D}\setminus \cup B_i^n} |\nabla u_n|^2 dx - \langle \mu_n, \zeta\rangle + o(1) \\ &\geq \frac{1}{2\omega_n^2} \int_{\mathcal{D}\setminus \cup B_i^n} |\nabla u_n|^2 dx - \zeta_{\max}\mu(\mathcal{D}) + o(1) \,. \end{aligned}$$

In the last inequality, we have used (3.1) and the fact that μ is supported by Σ. In view of claim *(ii)* in Proposition 2.1, we estimate

$$\int_{\mathcal{D}\setminus \cup B_i^n} |\nabla u_n|^2 dx \geq \frac{1}{1+\varepsilon_n^{1/4}} \int_{\mathcal{D}_{\varepsilon_n}\setminus \cup B_i^n} |j(u_n)|^2 dx \,.$$

Next we define

$$\tilde{j}_n(x) := \begin{cases} \omega_n^{-1} j(u_n(x)) & \text{if } x \in \mathcal{D}_{\varepsilon_n} \setminus \bigcup_{i\in I_n} B_i^n \,, \\ 0 & \text{otherwise}\,, \end{cases} \tag{3.5}$$

so that by (3.3),

$$O(1) \geq F_{\varepsilon_n}(u_n) \geq \frac{1}{2}\int_{\mathcal{D}} |\tilde{j}_n(x)|^2 dx - \zeta_{\max}\mu(\mathcal{D}) + o(1)\,. \tag{3.6}$$

Hence there exist a subsequence $\varepsilon_n \to 0$ (not relabelled) and $j_* \in L^2(\mathcal{D};\mathbb{R}^2)$ such that $\tilde{j}_n \rightharpoonup j_*$ weakly in $L^2(\mathcal{D};\mathbb{R}^2)$ as $n \to +\infty$. By lower semicontinuity, we have

$$\lim_{n\to+\infty} F_{\varepsilon_n}(u_n) \geq \frac{1}{2}\int_{\mathcal{D}} |j_*|^2 dx - \zeta_{\max}\mu(\mathcal{D})\,. \tag{3.7}$$

It remains to tie the limit j_* to the limit μ of the normalized weak Jacobians. To this aim we fix $\varphi \in \mathscr{D}(\mathcal{D})$. Using Lemma 2.1, claim *(i)* in Proposition 2.1 and Hölder Inequality, we estimate

$$\begin{aligned} \left| \int_{\cup B_i^n} \nabla^\perp\varphi \cdot j(u_n)\, dx \right| &\leq \big(\mathcal{L}^2\big(\cup_{i\in I_n} B_i^n\big)\big)^{1/4} \|\nabla\varphi\|_\infty \|u_n\|_{L^4(\mathcal{D})} \|\nabla u_n\|_{L^2(\mathcal{D})} \\ &\leq C\, r_n^{1/2}\, |\ln\varepsilon_n| = o(1)\,. \end{aligned}$$

Since $\operatorname{supp}\varphi \subset \mathcal{D}_{\varepsilon_n}$ for n large enough, we deduce that

$$\begin{aligned} \int_{\mathcal{D}} \nabla^\perp\varphi \cdot j_*\, dx &= \lim_{n\to+\infty} \int_{\mathcal{D}} \nabla^\perp\varphi \cdot \tilde{j}_n\, dx = \lim_{n\to+\infty} \omega_n^{-1} \int_{\mathcal{D}_{\varepsilon_n}\setminus\cup B_i^n} \nabla^\perp\varphi \cdot j(u_n)\, dx = \\ &= \lim_{n\to+\infty} \omega_n^{-1} \int_{\mathcal{D}} \nabla^\perp\varphi \cdot j(u_n)\, dx = -\lim_{n\to+\infty} \langle \mu_n, \varphi\rangle = -\int_{\mathcal{D}} \varphi\, d\mu\,. \end{aligned}$$

Consequently,

$$\operatorname{curl} j_* = \mu \quad \text{in } \mathscr{D}'(\mathcal{D})\,. \tag{3.8}$$

In particular, $\mu \in H^{-1}(\mathcal{D})$ since $j_* \in L^2(\mathcal{D};\mathbb{R}^2)$.

Next we introduce $h_\mu \in H_0^1(\mathcal{D})$ to be the unique solution of

$$\begin{cases} -\Delta h_\mu = \mu & \text{in } H^{-1}(\mathcal{D})\,, \\ h_\mu = 0 & \text{on } \partial\mathcal{D}\,. \end{cases}$$

In view of (3.8) and the definition of h_μ, we have

$$\operatorname{curl}\big(j_* + \nabla^\perp h_\mu\big) = 0 \quad \text{in } H^{-1}(\mathcal{D})\,,$$

so that we can find $g_\mu \in H^1(\mathcal{D})$ satisfying $\nabla g_\mu = j_* + \nabla^\perp h_\mu$. Therefore,

$$\int_{\mathcal{D}} |j_*|^2 dx = \int_{\mathcal{D}} |\nabla h_\mu|^2 dx + \int_{\mathcal{D}} |\nabla g_\mu|^2 dx\,,$$

since an integration by parts yields $\int_{\mathcal{D}} \nabla^\perp h_\mu \cdot \nabla g_\mu = 0$ (using the fact h_μ is constant on $\partial\mathcal{D}$). Going back to (3.7), we infer that

$$\lim_{n\to+\infty} F_{\varepsilon_n}(u_n) \geq \frac{1}{2}\int_{\mathcal{D}} |\nabla h_\mu|^2 dx - \zeta_{\max}\mu(\mathcal{D})\,.$$

On the other hand, using the Green representation of h_μ, we have

$$\frac{1}{2}\int_{\mathcal{D}} |\nabla h_\mu|^2\, dx = \frac{1}{2}\iint_{\mathcal{D}\times\mathcal{D}} G(x,y)\, d\mu(x)\, d\mu(y) = I(\mu)\,,$$

and the conclusion follows.

Case 2. We now treat the case $\omega_n \leq O(\ln|\ln\varepsilon_n|)$. Consider the family of vortex balls $\{\tilde{B}^n_j\}_{j\in J_n}$ (of size $s_n = e^{-\sqrt{\omega_n}}$) constructed in Step 3 in the proof of Proposition 2.2. Recall that this family satisfies $D_{s_n} = D_n$ for n large. Combining the lower bound (2.1) for the family $\{\tilde{B}^n_j\}_{j\in J_n}$ with (3.4), we derive

$$\omega_n^2 F_{\varepsilon_n}(u_n) \geq \pi D_n \left(\ln\left(\frac{s_n}{D_n}\right) - C_1\right) + \frac{1}{2}\int_{\mathcal{D}\setminus\cup\tilde{B}^n_j} |\nabla u_n|^2 dx - \omega_n^2\langle\mu_n,\zeta\rangle + o(1)\,.$$

Then arguing as in (3.6), we infer that

$$F_{\varepsilon_n}(u_n) \geq \frac{1}{2}\int_{\mathcal{D}} |\hat{j}_n(x)|^2 dx - \zeta_{\max}\mu(\mathcal{D}) + o(1)\,,$$

where

$$\hat{j}_n(x) := \begin{cases} \omega_n^{-1} j(u_n(x)) & \text{if } x \in \mathcal{D}_{\varepsilon_n}\setminus \cup_{j\in J_n}\tilde{B}^n_j\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

As previously, up to a subsequence we have $\hat{j}_n \rightharpoonup j_*$ weakly in $L^2(\mathcal{D};\mathbb{R}^2)$, and

$$\lim_{n\to+\infty} F_{\varepsilon_n}(u_n) \geq \frac{1}{2}\int_{\mathcal{D}} |j_*(x)|^2 dx - \zeta_{\max}\mu(\mathcal{D})\,.$$

Now it remains to show that $\operatorname{curl} j_* = \mu$ in $\mathscr{D}'(\mathcal{D})$, and then the proof can be completed as in Step 1.

We proceed as before, taking an arbitrary $\varphi \in \mathscr{D}(\mathcal{D})$ and using the weak formulation of the Jacobians. The key observation is that the contribution of the vortex balls will be negligible provided we can restrict our choice of test functions φ to functions *constant* in each vortex ball. This can be achieved thanks to [16, Proposition 9.6], *i.e.*, given an arbitrary $\varphi \in \mathscr{D}(\mathcal{D})$, there exists a modified function $\tilde{\varphi}_n$ which is constant on each ball B^n_j and such that

$$\|\varphi - \tilde{\varphi}_n\|_{C^{0,\alpha}(\mathcal{D})} \leq C s_n^{1-\alpha}\,, \qquad \|\nabla\varphi - \nabla\tilde{\varphi}_n\|_{L^1(\mathcal{D})} \leq C s_n \tag{3.9}$$

for each $0 \leq \alpha \leq 1$. Moreover, $\tilde{\varphi}_n$ has compact support in $\mathcal{D}$ for n large enough. From (3.9) and the L^2-boundedness of the normalized currents $\hat{j}_n$, we derive

$$\left|\int_{\mathcal{D}} \left(\nabla^\perp\varphi - \nabla^\perp\tilde{\varphi}_n\right)\cdot \hat{j}_n\, dx\right| \leq \|\hat{j}_n\|_{L^2(\mathcal{D})}\|\nabla\varphi - \nabla\tilde{\varphi}_n\|_{L^2(\mathcal{D})} \leq C s_n^{1/2}\,. \tag{3.10}$$

Then (3.9), (3.10) and the strong convergence of μ_n to μ yield

$$\int_{\mathcal{D}} \nabla^\perp\varphi\cdot j_*\, dx = \lim_{n\to+\infty}\int_{\mathcal{D}} \nabla^\perp\varphi\cdot\hat{j}_n\, dx = \lim_{n\to+\infty}\int_{\mathcal{D}} \nabla^\perp\tilde{\varphi}_n\cdot\hat{j}_n\, dx\ =$$
$$= \lim_{n\to+\infty}\omega_n^{-1}\int_{\mathcal{D}} \nabla^\perp\tilde{\varphi}_n\cdot j(u_n)\, dx = \lim_{n\to+\infty}\left(-\int_{\mathcal{D}}\tilde{\varphi}_n d\mu + \langle\mu-\mu_n,\tilde{\varphi}_n\rangle\right) = -\int_{\mathcal{D}}\varphi\, d\mu\,,$$

and the conclusion follows. □

4. The upper bound inequality

Throughout this section we shall use the following notation. For a nonnegative Radon measure $\mu \in H^{-1}(\mathcal{D})$ compactly supported in $\mathcal{D}$, we denote by $h_\mu \in H^1_0(\mathcal{D})$ the solution of

$$\begin{cases} -\Delta h_\mu = \mu & \text{in } H^{-1}(\mathcal{D})\,, \\ h_\mu = 0 & \text{on } \partial\mathcal{D}\,. \end{cases} \tag{4.1}$$

Using the Green representation $h_\mu(x) = \int_{\mathcal{D}} G(x,y)\,d\mu(y)$, we have

$$I(\mu) = \frac{1}{2}\int_{\mathcal{D}} |\nabla h_\mu|^2\,dx\,,$$

and the following elementary lemma holds.

Lemma 4.1. *Let $\{\mu_n\}_{n\in\mathbb{N}}$ be a sequence of nonnegative Radon measure in $H^{-1}(\mathcal{D})$ with compact support in $\mathcal{D}$. Assume that $\mu_n \rightharpoonup \mu$ weakly* as measures on $\mathcal{D}$ as $n \to +\infty$, for some $\mu \in H^{-1}(\mathcal{D})$ compactly supported in $\mathcal{D}$. Then $I(\mu_n) \to I(\mu)$ if and only if $h_{\mu_n} \to h_\mu$ strongly in $H^1(\mathcal{D})$ as $n \to +\infty$.*

We may now start the proof of claim *(iii)* in Theorem 1.1 in the case where the measure μ is absolutely continuous with respect to $\mathcal{H}^1 \llcorner \Sigma$.

Proposition 4.1. *Assume that* (H1)*,* (H2) *and* (1.2) *hold. Let $\varepsilon_n \to 0^+$ be an arbitrary sequence. For every nonnegative Radon measure μ of the form*

$$\mu = f(x)\,\mathcal{H}^1 \llcorner \Sigma \tag{4.2}$$

with $f \in L^\infty(\Sigma)$, there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subset H^1(\mathcal{D};\mathbb{C})$ such that

$$\frac{1}{\omega_n}\operatorname{curl} j(u_n) \underset{n\to+\infty}{\longrightarrow} \mu \quad \textit{strongly in } (C^{0,1}_0(\mathcal{D}))^*\,,$$

and

$$\lim_{n\to+\infty} F_{\varepsilon_n}(u_n) = I(\mu) - \zeta_{\max}\mu(\mathcal{D})\,.$$

Proof. Without loss of generality we may assume that $f \not\equiv 0$, the case $\mu = 0$ being easily true. It is well known that a measure of the form (4.2) belongs to $H^{-1}(\mathcal{D})$, see *e.g.* [18, Theorem 4.7.5].

We recall that the Dirichlet Green's function G in $\mathcal{D}$ defined by (1.3) satisfies

(i) $G(x,y) \geq 0$ for every $x \in \overline{\mathcal{D}} \setminus \{y\}$ and for every $y \in \mathcal{D}$;
(ii) for any compact set $K \subset\subset \mathcal{D}$ there exists a constant C_K such that

$$\left| G(x,y) + \frac{1}{2\pi}\ln|x-y| \right| \leq C_K \tag{4.3}$$

for all $y \in K$ and $x \in \overline{\mathcal{D}}$.

In the three first steps below, we assume that the density function f does not vanish on Σ, *i.e.*, $f \geq \delta$ $\mathcal{H}^1$-a.e. on Σ for some constant $\delta > 0$. The general case is considered in Step 4.

Step 1. We will construct a trial function using the Green function $G(x,y)$ in the spirit of [15]. Let $\gamma : [0,\ell] \to \Sigma$ be an arclength parametrization of the curve Σ (so that $\ell = \mathcal{H}^1(\Sigma)$), and define for $t \in [0,\ell]$,

$$M(t) := \mu\big(\gamma([0,t])\big)\,.$$

Then $M(\cdot)$ is strictly increasing, $M(0)=0$, and $M(\ell)=\mu(\Sigma)=\mu(\mathcal{D})$. Moreover $M(\cdot)$ is continuous since $\mu\in H^{-1}(\mathcal{D})$ and thus atomless.

Next we introduce for n large enough,

$$D_n := \left[\frac{\omega_n\mu(\Sigma)}{2\pi}\right]\,,$$

where $[\cdot]$ denotes the integer part. Since M is continuous and increasing, we can define for $k=0,\ldots,D_n$,

$$t_{k,n} := M^{-1}\big(2\pi k\omega_n^{-1}\big)\,.$$

Now we set for $k=0,\ldots,D_n$,

$$a_k^n := \gamma(t_{k,n})\,,$$

and we claim that

$$C\omega_n^{-1} \le |a_k^n - a_{k-1}^n| \le 2\pi\delta^{-1}\omega_n^{-1} \tag{4.4}$$

for each $k\in\{1,\ldots,D_n\}$ and for some constant $C>0$ independent of n. Write $\Sigma_k := \gamma([t_{k-1,n},t_{k,n}])$ for $k=1,\ldots,D_n$ so that Σ_k is a smooth curve whose end-points are a_k^n and a_{k-1}^n. By construction, we have

$$2\pi\omega_n^{-1} = M(t_{k,n})-M(t_{k-1,n}) = \mu\big(\Sigma_k\big) \ge \delta\mathcal{H}^1\big(\Sigma_k\big) \ge \delta|a_k^n-a_{k-1}^n|\,.$$

Now the curve Σ being smooth and $|a_k^n-a_{k-1}^n|$ small for n large, we deduce

$$2\pi\omega_n^{-1} = \mu\big(\Sigma_k\big) \le \|f\|_{L^\infty(\Sigma)}\mathcal{H}^1\big(\Sigma_k\big) \le C\|f\|_{L^\infty(\Sigma)}|a_k^n-a_{k-1}^n|\,.$$

Define the family of measures

$$\bar\mu_n := \frac{2\pi}{\omega_n}\sum_{k=1}^{D_n}\delta_{a_k^n}\,.$$

We claim that $\bar\mu_n \rightharpoonup \mu$ weakly* as measures on $\mathcal{D}$ and $\bar\mu_n(\mathcal{D})\to\mu(\mathcal{D})$. To prove the weak* convergence of $\bar\mu_n$, we fix an arbitrary function $\psi\in C_0^0(\mathcal{D})$. Observe that by (4.4) and the smoothness of Σ, we have $\operatorname{diam}(\Sigma_k)\le C_0\omega_n^{-1}$ for a constant C_0 independent of k and n. Therefore, using $\mu(\Sigma_k)=2\pi\omega_n^{-1}$ we derive

$$\left|\int_{\mathcal{D}}\psi\,d\mu-\int_{\mathcal{D}}\psi\,d\bar\mu_n\right| = \left|\sum_{k=1}^{D_n}\int_{\Sigma_k}\big(\psi(x)-\psi(a_k^n)\big)d\mu\right|+o(1) \le C\mu(\mathcal{D})\operatorname{osc}(\psi,C_0\omega_n^{-1})+o(1)\,,$$

where

$$\operatorname{osc}(\psi,C_0\omega_n^{-1}) := \sup_{|x-y|\le C_0\omega_n^{-1}}|\psi(x)-\psi(y)| \underset{n\to+\infty}{\longrightarrow} 0\,,$$

and the claim is proved.

Next we must regularize the measure $\bar\mu_n$. Let us define for $k=1,\ldots,D_n$,

$$f_{k,n}(x) := \frac{1}{\pi\varepsilon_n^2}\,\chi_{B_{\varepsilon_n}(a_k^n)}\,,$$

where $\chi_{B_{\varepsilon_n}(a_k^n)}$ denotes the characteristic function of the ball $B_{\varepsilon_n}(a_k^n)$. Set

$$f_n(x) := \frac{2\pi}{\omega_n}\sum_{k=1}^{D_n}f_{k,n}(x) \quad\text{and}\quad \hat\mu_n := f_n(x)\mathcal{L}^2\llcorner\mathcal{D}\,. \tag{4.5}$$

Since $\varepsilon_n = o(\omega_n^{-1})$, the functions $f_{k,n}$ have disjoint supports for n large by (4.4). As a consequence, $\hat{\mu}_n(\mathcal{D}) = \bar{\mu}_n(\mathcal{D}) \to \mu(\mathcal{D})$. Since $\bar{\mu}_n \rightharpoonup \mu$, one may also easily check that $\hat{\mu}_n \rightharpoonup \mu$ weakly* as measures on $\mathcal{D}$.

Step 2. According to (4.1), we introduce

$$h_n := h_{\omega_n \hat{\mu}_n} = \omega_n h_{\hat{\mu}_n}\,.$$

Then

$$\int_{\mathcal{D}} |\nabla h_n|^2 dx = \omega_n^2 \iint_{\mathcal{D}\times\mathcal{D}} G(x,y)\, f_n(x) f_n(y)\, dx dy \tag{4.6}$$

$$= 4\pi^2 \sum_{i,j=1}^{D_n} \iint_{\mathcal{D}\times\mathcal{D}} G(x,y) f_{i,n}(x) f_{j,n}(y)\, dx dy\,. \tag{4.7}$$

We need to estimate the integral term in the right handside of (4.6). We proceed as in [1] and we provide some details for the reader's convenience. Let $\mathcal{N}_0 \subset\subset \mathcal{D}$ be a small tubular neighborhood of Σ. Let $0 < \alpha < 1$ be given small, and set $\Delta_\alpha := \{(x,y) \in \mathcal{D}\times\mathcal{D} \,:\, |x-y| < \alpha\}$. Since G is continuous on $(\mathcal{N}_0 \times \mathcal{N}_0) \setminus \Delta_\alpha$, and the support of $\hat{\mu}_n$ lies in $\mathcal{N}_0$ for n large, by the weak* convergence of $\hat{\mu}_n$ to μ we have

$$I_\alpha := \lim_{n\to+\infty} \iint_{(\mathcal{D}\times\mathcal{D})\setminus\Delta_\alpha} G(x,y)\, f_n(x) f_n(y) dx dy =$$

$$= \iint_{(\mathcal{N}_0\times\mathcal{N}_0)\setminus\Delta_\alpha} G(x,y) d\mu(x) d\mu(y) \leq \iint_{\mathcal{D}\times\mathcal{D}} G(x,y) d\mu(x) d\mu(y) = 2I(\mu)\,. \tag{4.8}$$

Near the diagonal Δ_α we split the sum in (4.7) in two terms. Using (4.3) we estimate

$$II_\alpha^n := 4\pi^2 \sum_{i=1}^{D_n} \iint_{\Delta_\alpha} G(x,y)\, f_{i,n}(x) f_{i,n}(y) dx dy =$$

$$= 4 \sum_{i=1}^{D_n} \iint_{B_1(0)\times B_1(0)} G\left(a_i^n + \varepsilon_n z_1, a_i^n + \varepsilon_n z_2\right) dz_1 dz_2$$

$$\leq 4 \sum_{i=1}^{D_n} \iint_{B_1(0)\times B_1(0)} \left(\frac{1}{2\pi} \ln\left(\frac{1}{\varepsilon_n |z_1 - z_2|}\right) + C\right) dz_1 dz_2$$

$$\leq 2\pi D_n |\ln \varepsilon_n| + O(\omega_n)\,, \tag{4.9}$$

and

$$III_\alpha^n := 4\pi^2 \sum_{0<|a_i^n - a_j^n|<\alpha} \iint_{\Delta_\alpha} G(x,y)\, f_{i,n}(x)\, f_{j,n}(y) dx dy \leq$$

$$\leq C \sum_{0<|a_i^n - a_j^n|<\alpha} \left|\ln|a_i^n - a_j^n|\right|\,. \tag{4.10}$$

By the smoothness of Σ and (4.4), there exists a constant $c_0 > 0$ independent of n such that for every $i \neq j$ and every $(x,y) \in \Sigma_i \times \Sigma_j$,

$$|a_i^n - a_j^n| \geq c_0 |x-y|\,.$$

Since $\mu \in H^{-1}(\mathcal{D})$ and it is supported by Σ, the map $(x,y) \mapsto \big|\ln(c_0|x-y|)\big|$ belongs to $L^1(\Sigma\times\Sigma, \mu\otimes\mu)$. Therefore, by the Mean Value Theorem, for every $i \neq j$ we can find a pair $(x_i^n, y_j^n) \in \Sigma_i \times \Sigma_j$ such that

$$\frac{\omega_n^2}{4\pi^2} \iint_{\Sigma_i \times \Sigma_j} \big| \ln(c_0|x-y|)\big| d\mu(y)d\mu(x) = \big| \ln(c_0|x_i^n - y_j^n|)\big| \,,$$

noticing that $4\pi^2\omega_n^{-2} = \mu(\Sigma_i)\mu(\Sigma_j)$. Applying the previous inequality to (x_i^n, y_j^n) we deduce from (4.10) that for n large,

$$\begin{aligned} III_\alpha^n &\leq C \sum_{0<|a_i^n - a_j^n|<\alpha} \big| \ln(c_0|x_i^n - y_j^n|)\big| \\ &\leq C\omega_n^2 \sum_{0<|a_i^n - a_j^n|<\alpha} \iint_{\Sigma_i \times \Sigma_j} \big| \ln(c_0|x-y|)\big| d\mu(y)d\mu(x) \\ &\leq C\omega_n^2 \iint_{(\Sigma\times\Sigma)\cap\Delta_{2\alpha}} \big| \ln(c_0|x-y|)\big| d\mu(y)d\mu(x)\,. \end{aligned} \tag{4.11}$$

On the other hand $\mu \in H^{-1}(\mathcal{D})$ so it is atomless, and thus $\mu \otimes \mu$ does not charge $\{x = y\} \cap \mathcal{D} \times \mathcal{D}$. Consequently the integral term in the right handside of (4.11) vanishes as $\alpha \to 0^+$. Hence,

$$\lim_{\alpha\to 0^+} \limsup_{n\to+\infty} \omega_n^{-2} III_\alpha^n = 0\,. \tag{4.12}$$

Gathering (4.8), (4.9) and (4.12) yields

$$\begin{aligned} &\limsup_{n\to+\infty} \left(\iint_{\mathcal{D}\times\mathcal{D}} G(x,y)\, f_n(x) f_n(y)\, dxdy - 2\pi D_n |\ln\varepsilon_n|\,\omega_n^{-2} \right) \leq \\ &\qquad \leq \limsup_{\alpha\to 0^+} \left(I_\alpha + \limsup_{n\to+\infty} \left((II_\alpha^n - 2\pi D_n |\ln\varepsilon_n|)\omega_n^{-2} + \omega_n^{-2} III_\alpha^n \right) \right) \leq 2I(\mu)\,, \end{aligned}$$

and in view of (4.6), we conclude that

$$\frac{1}{2}\int_{\mathcal{D}} |\nabla h_n|^2 dx \leq \pi D_n\,|\ln\varepsilon_n| + \omega_n^2 I(\mu) + o(\omega_n^2)\,. \tag{4.13}$$

Step 3. We shall now define a complex-valued order parameter u_n associated to h_n. We proceed as follows. Since

$$\text{curl}\,\left(-\nabla^\perp h_n\right) = -\Delta h_n = \omega_n f_n \tag{4.14}$$

is supported by $\cup_k \overline{B}_{\varepsilon_n}(a_k^n)$, we may locally define a phase ϕ_n in $\mathcal{D} \setminus \cup_k \overline{B}_{\varepsilon_n}(a_k^n)$ by

$$\nabla\phi_n(x) = -\nabla^\perp h_n(x) \quad \text{for } x \in \mathcal{D} \setminus \cup_k \overline{B}_{\varepsilon_n}(a_k^n)\,.$$

In fact, since the balls $B_{\varepsilon_n}(a_k^n)$ are pairwise disjoint (assuming n large enough) and the mass of $\omega_n f_n$ is quantized in each such ball, it is easy to show that ϕ_n is single-valued modulo 2π, *i.e.*, for any smooth Jordan curve Θ inside $\mathcal{D} \setminus \cup_k \overline{B}_{\varepsilon_n}(a_k^n)$,

$$\frac{1}{2\pi}\int_\Theta \nabla\phi_n \cdot \tau \,\in \mathbb{Z}\,,$$

where $\tau : \Theta \to \mathbb{S}^1$ is any smooth vector field tangent to Θ. Hence $\exp(i\phi_n(x))$ is well defined for every $x \in \mathcal{D} \setminus \cup_k \overline{B}_{\varepsilon_n}(a_k^n)$.

Then consider a smooth cut-off function $\rho : \mathbb{R} \to [0,1]$ such that $\rho(t) \equiv 1$ for $t \geq 2$, and $\rho(t) \equiv 0$ for $t \leq 1$. Define

$$\rho_n(x) := \begin{cases} \rho\left(\dfrac{|x - a_k^n|}{\varepsilon_n}\right) & \text{if } x \in B_{2\varepsilon_n}(a_k^n) \text{ for some } k = 1, \ldots, D_n\,, \\ 1 & \text{otherwise}\,, \end{cases} \tag{4.15}$$

and observe that

$$E_{\varepsilon_n}(\rho_n, \mathcal{D}) = O(\omega_n)\,. \tag{4.16}$$

Then set

$$u_n(x) := \begin{cases} \rho_n(x) e^{i\phi_n(x)} & \text{for } x \in \mathcal{D} \setminus \cup_k \overline{B}_{\varepsilon_n}(a_k^n)\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

One may easily check that $u_n \in H^1(\mathcal{D};\mathbb{C})$. We claim that

$$\mu_n = \omega_n^{-1} \text{curl}\, j(u_n) \underset{n\to+\infty}{\longrightarrow} \mu \quad \text{strongly in } (C_0^{0,1}(\mathcal{D}))^*\,. \tag{4.17}$$

A simple computation gives

$$j(u_n) = \rho_n^2 \nabla \phi_n = -\rho_n^2 \nabla^\perp h_n \quad \text{a.e. in } \mathcal{D}\,.$$

Given $\varphi \in C_0^{0,1}(\mathcal{D})$ satisfying $|\nabla\varphi| \le 1$, we deduce from (4.14) and the previous indentity,

$$\begin{aligned} \langle \mu_n, \varphi \rangle &= \frac{1}{\omega_n} \int_{\mathcal{D}} \nabla\varphi \cdot \nabla h_n dx + \frac{1}{\omega_n} \int_{\mathcal{D}} (\rho_n^2 - 1) \nabla\varphi \cdot \nabla h_n dx \\ &= \langle \hat{\mu}_n, \varphi \rangle + \frac{1}{\omega_n} \int_{\cup_k B_{2\varepsilon_n}(a_k^n)} (\rho_n^2 - 1) \nabla\varphi \cdot \nabla h_n dx\,. \end{aligned} \tag{4.18}$$

In view of the compact embedding $(C_0^0(\mathcal{D}))^* \hookrightarrow (C_0^{0,1}(\mathcal{D}))^*$, $\hat{\mu}_n \to \mu$ strongly in $(C_0^{0,1}(\mathcal{D}))^*$. Hence we can estimate using (4.13),

$$\begin{aligned} |\langle \mu_n - \mu, \varphi \rangle| &\le \big\| \hat{\mu}_n - \mu \big\|_{(C_0^{0,1}(\mathcal{D}))^*} + \omega_n^{-1} \big(\mathcal{L}^2(\cup_k B_{2\varepsilon_n}(a_k^n)) \big)^{1/2} \|\nabla h_n\|_{L^2(\mathcal{D})} \le \\ &\le \big\| \hat{\mu}_n - \mu \big\|_{(C_0^{0,1}(\mathcal{D}))^*} + C\varepsilon_n |\ln \varepsilon_n|^{1/2} \underset{n\to+\infty}{\longrightarrow} 0\,, \end{aligned} \tag{4.19}$$

and (4.17) is proved.

We now compute the energy $F_{\varepsilon_n}(u_n)$. We infer from (4.13) and (4.16) that

$$\begin{aligned} E_{\varepsilon_n}(u_n, \mathcal{D}) &= E_{\varepsilon_n}(\rho_n, \mathcal{D}) + \frac{1}{2} \int_{\mathcal{D} \setminus \cup_k \overline{B}_{\varepsilon_n}(a_k^n)} \rho_n^2 |\nabla\phi_n|^2 dx \le \\ &\le \frac{1}{2} \int_{\mathcal{D}} |\nabla h_n|^2 + O(\omega_n) \le \pi D_n |\ln \varepsilon_n| + \omega_n^2 I(\mu) + o(\omega_n^2)\,, \end{aligned} \tag{4.20}$$

and it remains to evaluate the interaction with the rotation potential. First (4.17) yields

$$\begin{aligned} \Omega_n \int_{\mathcal{D}} \nabla^\perp\zeta \cdot j(u_n)\, dx &= \frac{|\ln \varepsilon_n|}{2\zeta_{\max}} \int_{\mathcal{D}} \nabla^\perp\zeta \cdot j(u_n)\, dx - \omega_n^2 \langle \mu_n, \zeta \rangle = \\ &= \frac{|\ln \varepsilon_n|}{2\zeta_{\max}} \int_{\mathcal{D}} \nabla^\perp\zeta \cdot j(u_n)\, dx - \zeta_{\max} \mu(\mathcal{D}) \omega_n^2 + o(\omega_n^2)\,. \end{aligned}$$

Arguing as in (4.18)-(4.19) we derive

$$\begin{aligned} \int_{\mathcal{D}} \nabla^\perp\zeta \cdot j(u_n)\, dx &= -\omega_n \langle \hat{\mu}_n, \zeta \rangle + O(\varepsilon_n \omega_n |\ln \varepsilon_n|^{1/2}) = \\ &= -2\pi \sum_{k=1}^{D_n} \frac{1}{\pi\varepsilon_n^2} \int_{B_{\varepsilon_n}(a_k^n)} \zeta(x) dx + o(\varepsilon_n |\ln \varepsilon_n|^{3/2}) = -2\pi D_n \zeta_{\max} + o(\varepsilon_n |\ln \varepsilon_n|^{3/2})\,, \end{aligned}$$

and consequently,

$$\Omega_n \int_{\mathcal{D}} \nabla^\perp\zeta \cdot j(u_n)\, dx = -\pi D_n |\ln \varepsilon_n| - \zeta_{\max} \mu(\mathcal{D}) \omega_n^2 + o(\omega_n^2)\,. \tag{4.21}$$

Combining (4.20) with (4.21) finally leads to

$$F_{\varepsilon_n}(u_n) \leq I(\mu) - \zeta_{\max}\mu(\mathcal{D}) + o(1)\,.$$

In view of Theorem 3.1, the conclusion follows taking the lim sup as $n \to +\infty$ in the previous inequality.

Step 4. We now consider the case where the density f is allowed to vanish. Let $\{\delta_k\} \subset \mathbb{R}$ be a sequence decreasing to 0 as $k \to +\infty$. Then for $k \in \mathbb{N}$, we consider the measure

$$\mu_k := \mu + \delta_k \mathcal{H}^1 \llcorner \Sigma = \big(f(x) + \delta_k\big)\mathcal{H}^1 \llcorner \Sigma\,.$$

By monotone convergence, one has

$$I(\mu_k) \underset{k\to+\infty}{\longrightarrow} I(\mu) \qquad \text{and} \qquad \mu_k(\mathcal{D}) \underset{k\to+\infty}{\longrightarrow} \mu(\mathcal{D})\,. \tag{4.22}$$

Obviously μ_k also converges to μ strongly in $(C_0^{0,1}(\mathcal{D}))^*$. Applying Step 1 to Step 3, we find for every $k \in \mathbb{N}$ a sequence $\{v_n^k\}_{n\in\mathbb{N}} \subset H^1(\mathcal{D};\mathbb{C})$ such that $\omega_n^{-1}\mathrm{curl}\, j(v_n^k) \to \mu_k$ strongly in $(C_0^{0,1}(\mathcal{D}))^*$ and $F_{\varepsilon_n}(v_n^k) \to I(\mu_k) - \zeta_{\max}\mu_k(\mathcal{D})$ as $n \to +\infty$. Hence for every $k \in \mathbb{N}$, we can find $N_k \in \mathbb{N}$ such that for every $n \geq N_k$,

$$\|\omega_n^{-1}\mathrm{curl}\, j(v_n^k) - \mu\|_{(C_0^{0,1}(\mathcal{D}))^*} \leq 2^{-k}$$

and

$$\big|F_{\varepsilon_n}(v_n^k) - I(\mu) + \zeta_{\max}\mu(\mathcal{D})\big| \leq 2^{-k}\,.$$

Moreover we can assume without loss of generality that the sequence of integers $\{N_k\}_{k\in\mathbb{N}}$ is strictly increasing. Therefore given any integer n large enough, there is a unique $k_n \in \mathbb{N}$ such that $N_{k_n} \leq n < N_{k_n+1}$, and $k_n \to +\infty$ as $n \to +\infty$. We may then define $u_n := v_n^{k_n}$. By construction, the sequence $\{u_n\}$ satisfies the required properties. □

To consider the case of a general measure μ in $H^{-1}(\mathcal{D})$, we shall need the following continuity lemma.

Lemma 4.2. *Let μ be a nonnegative Radon measure in $H^{-1}(\mathcal{D})$ such that* $\mathrm{supp}\,\mu \subset\subset \mathcal{D}$. *For $\xi \in \mathbb{R}^2$, let $\tau_\xi\mu$ be the translated measure defined by*

$$\tau_\xi\mu(B) = \mu(-\xi + B) \quad \textit{for any Borel set } B \subset \mathbb{R}^2\,.$$

Then there exists $0 < \delta < \mathrm{dist}\,(\mathrm{supp}\,\mu, \partial\mathcal{D})$ such that $\tau_\xi\mu \in H^{-1}(\mathcal{D})$ for every $\xi \in B_\delta(0)$, and the mapping $\xi \mapsto h_{\tau_\xi\mu} \in H^1(\mathcal{D})$ is strongly continuous on $B_\delta(0)$.

Proof. For $\delta > 0$ we set $\widetilde{\mathcal{D}}_\delta := \{x \in \mathbb{R}^2\,,\ \mathrm{dist}\,(x,\mathcal{D}) < 2\delta\}$. Then choose $\delta > 0$ such that $\partial\widetilde{\mathcal{D}}_\delta$ is smooth and $2\delta < \mathrm{dist}\,(\mathrm{supp}\,\mu, \partial\mathcal{D})$. For every $\xi \in B_\delta(0)$ we have $\mathcal{D} \subset\subset \xi + \widetilde{\mathcal{D}}_\delta$, $\mathrm{supp}\,\tau_\xi\mu = \xi + \mathrm{supp}\,\mu \subset \mathcal{D}$ and $\mathrm{dist}\,(\xi + \mathrm{supp}\,\mu, \partial\mathcal{D}) > \delta$.

Obviously $\mu \in H^{-1}(\widetilde{\mathcal{D}}_\delta)$ and we can set $\bar{h} \in H^1(\widetilde{\mathcal{D}}_\delta)$ to be the unique solution of

$$\begin{cases} -\Delta\bar{h} = \mu & \text{in } \widetilde{\mathcal{D}}_\delta\,,\\ \bar{h} = 0 & \text{on } \widetilde{\mathcal{D}}_\delta\,. \end{cases}$$

By our choice of δ, the function $\bar{h}$ is smooth in the δ-neighborhood of $\partial\mathcal{D}$. Next, for $\xi \in B_\delta(0)$ we denote by $\bar{h}_\xi \in H^1(\mathcal{D})$ the function defined by $\bar{h}_\xi := \bar{h}(x-\xi)$ for $x \in \mathcal{D}$. Observe that $\bar{h}_\xi \in H^1(\mathcal{D})$ and $-\Delta\bar{h}_\xi = \tau_\xi\mu$ in $\mathcal{D}$. Hence $\tau_\xi\mu \in H^{-1}(\mathcal{D})$.

Now consider a sequence $\{\xi_n\} \subset B_\delta(0)$ such that $\xi_n \to \xi \in B_\delta(0)$ as $n \to +\infty$. Denote $h_n := h_{\tau_{\xi_n}\mu}$. We have $\Delta(\bar{h}_{\xi_n} - h_n) = 0$ in $\mathcal{D}$ and $(\bar{h}_{\xi_n} - h_n) = \bar{h}(x-\xi_n)$ on $\partial\mathcal{D}$. By standard elliptic estimates, $(\bar{h}_{\xi_n} - h_n)$ strongly converges in $H^1(\mathcal{D})$ to the harmonic function in $\mathcal{D}$ equal to $\bar{h}(x-\xi)$ on $\partial\mathcal{D}$, that is $\bar{h}_\xi - h_{\tau_\xi\mu}$. On the other hand, $\bar{h}_{\xi_n} \to \bar{h}_\xi$ strongly in $H^1(\mathcal{D})$ by strong continuity of translations in H^1. Therefore $h_n \to h_{\tau_\xi}$ strongly in $H^1(\mathcal{D})$, and the proof is complete. □

Theorem 4.1. *Assume that* (H1), (H2) *and* (1.2) *hold. Let $\varepsilon_n \to 0^+$ be an arbitrary sequence. For every nonnegative Radon measure $\mu \in H^{-1}(\mathcal{D})$ supported by Σ, there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subset H^1(\mathcal{D};\mathbb{R}^2)$ such that*

$$\frac{1}{\omega_n}\,\mathrm{curl}\; j(u_n) \underset{n\to+\infty}{\longrightarrow} \mu \quad \textit{strongly in } (C_0^{0,1}(\mathcal{D}))^*\,,$$

and

$$\lim_{n\to+\infty} F_{\varepsilon_n}(u_n) = I(\mu) - \zeta_{\max}\mu(\mathcal{D})\,.$$

Proof. We shall prove that for any nonnegative Radon measure $\mu \in H^{-1}(\mathcal{D})$ supported by Σ, there exists a sequence of nonnegative Radon measures $\{\mu_k\}_{k\in\mathbb{N}}$ of the form (4.2) such that $\mu_k \to \mu$ strongly in $(C_0^{0,1}(\mathcal{D}))^*$, $\mu_k(\mathcal{D}) \to \mu(\mathcal{D})$ and

$$I(\mu_k) \to I(\mu) \quad \text{as } k \to +\infty\,. \tag{4.23}$$

Assuming that such a sequence exists, Proposition 4.1 yields for each k a sequence $\{v_n^k\}_{n\in\mathbb{N}} \subset H^1(\mathcal{D};\mathbb{R}^2)$ such that $\omega_n^{-1}\mathrm{curl}\, j(v_n^k) \to \mu_k$ strongly in $(C_0^{0,1}(\mathcal{D}))^*$ and $F_{\varepsilon_n}(v_n^k) \to I(\mu_k) - \zeta_{\max}\mu_k(\mathcal{D})$ as $n \to +\infty$. It then suffices to apply the diagonal argument used in the proof of Proposition 4.1, Step 4, to construct the required sequence.

Step 1. We first consider the case where Σ is a segment in $\mathcal{D}$. Without loss of generality we may assume that $\Sigma = [a,b]\times\{0\} \subset\subset \mathcal{D}$ for some $a,b\in\mathbb{R}$ with $a<b$. Assume in addition that $\Sigma' := \mathrm{supp}\,\mu \subset\subset]a,b[\times\{0\}$. We shall regularize the measure μ using the following standard procedure. Consider a smooth function $\varrho \in C^\infty(\mathbb{R})$ such that $\varrho \geq 0$, $\mathrm{supp}\,\varrho \subset [-1,1]$ and $\int_{\mathbb{R}} \varrho = 1$. For a positive integer k and $x=(x_1,x_2)\in\mathbb{R}^2$, we introduce $\varrho_k(x) := k\varrho(kx_1)$, and we define

$$g_k(x) := \int_{\Sigma'} \varrho_k(x-y)\,d\mu(y)\,.$$

By construction, the function g_k is nonnegative, smooth and supported by $[a,b]\times\mathbb{R}$ for k large enough. Next we define for k large the measure

$$\mu_k := g_k(x)\,\mathcal{H}^1 \llcorner \Sigma\,.$$

One may easily check that $\mu_k \in H^{-1}(\mathcal{D})$, $\mu_k(\mathcal{D}) \to \mu(\mathcal{D})$ and that $\mu_k \rightharpoonup \mu$ weakly* in the sense of measures on $\mathcal{D}$ as $k\to+\infty$. In particular, $\mu_k \to \mu$ strongly in $(C_0^{0,1}(\mathcal{D}))^*$.

We claim that (4.23) holds. Indeed, using Fubini's theorem we first derive that

$$I(\mu_k) = \frac{1}{2}\iint_{\Sigma\times\Sigma} G(z,z')\,d\mu_k(z)d\mu_k(z') =$$
$$= \frac{1}{2}\iint_{\Sigma'\times\Sigma'}\left(\iint_{(\Sigma\cap B_{\frac{1}{k}}(x))\times(\Sigma\cap B_{\frac{1}{k}}(y))} G(z,z')\varrho_k(z-x)\varrho_k(z'-y)\,d\mathcal{H}^1_z d\mathcal{H}^1_{z'}\right) d\mu(x)d\mu(y)\,.$$

Next we observe that for k large enough, we have $\Sigma\cap B_{\frac{1}{k}}(x)=(x_1,0)+J_k$ with $J_k:=(\frac{-1}{k},\frac{1}{k})\times\{0\}$ for every $x=(x_1,x_2)\in\Sigma'$. Changing variables in (z,z') and using Fubini's theorem again, we obtain

$$\begin{aligned}I(\mu_k)&=\frac{1}{2}\iint_{\Sigma'\times\Sigma'}\left(\iint_{J_k\times J_k}G(x+\xi,y+\xi')\varrho_k(\xi)\varrho_k(\xi')\,d\mathcal{H}^1_\xi d\mathcal{H}^1_{\xi'}\right)d\mu(x)d\mu(y)\\&=\frac{1}{2}\iint_{J_k\times J_k}\left(\iint_{\Sigma'\times\Sigma'}G(x+\xi,y+\xi')\,d\mu(x)d\mu(y)\right)\varrho_k(\xi)\varrho_k(\xi')\,d\mathcal{H}^1_\xi d\mathcal{H}^1_{\xi'}\\&=\frac{1}{2}\iint_{J_k\times J_k}\left(\iint_{\mathcal{D}\times\mathcal{D}}G(x,y)\,d(\tau_\xi\mu)(x)d(\tau_{\xi'}\mu)(y)\right)\varrho_k(\xi)\varrho_k(\xi')\,d\mathcal{H}^1_\xi d\mathcal{H}^1_{\xi'}\,.\end{aligned}$$

From the Green representation of $h_{\tau_\xi\mu}$ we infer that for every $(\xi,\xi')\in J_k\times J_k$,

$$\iint_{\mathcal{D}\times\mathcal{D}}G(x,y)\,d(\tau_\xi\mu)(x)d(\tau_{\xi'}\mu)(y)=\int_{\mathcal{D}}\big(\nabla h_{\tau_\xi\mu}\big)\cdot\big(\nabla h_{\tau_{\xi'}\mu}\big)\,dx\,.$$

Then from Lemma 4.2 we deduce that the function

$$\Theta:(\xi,\xi')\mapsto\int_{\mathcal{D}}\big(\nabla h_{\tau_\xi\mu}\big)\cdot\big(\nabla h_{\tau_{\xi'}\mu}\big)\,dx$$

is continuous on $B_\delta(0)\times B_\delta(0)$ for some $0<\delta<\operatorname{dist}(\Sigma,\partial\mathcal{D})$. Therefore,

$$\lim_{k\to+\infty}I(\mu_k)=\lim_{k\to+\infty}\frac{1}{2}\iint_{J_k\times J_k}\Theta(\xi,\xi')\varrho_k(\xi)\varrho_k(\xi')\,d\mathcal{H}^1_\xi d\mathcal{H}^1_{\xi'}=\frac{1}{2}\Theta(0,0)=I(\mu)\,,$$

and (4.23) is proved.

Step 2. We now consider the case where Σ is a smooth embedded arc. We further assume that there exists a C^1-diffeomorphism $\Phi:\mathcal{D}\to\mathcal{D}$ such that $\Phi(x)=x$ in a neighborhood of $\partial\mathcal{D}$ and $\bar\Sigma:=\Phi(\Sigma)$ is a segment compactly included in $\mathcal{D}$. Let μ be a nonnegative Radon measure in $H^{-1}(\mathcal{D})$ whose support is compactly included in the relative interior of Σ. Denote by $\bar\mu$ the push-forward of μ through Φ, *i.e.*, $\bar\mu:=\Phi_\#\mu$. Then $\operatorname{supp}\bar\mu$ is compactly included in the relative interior of $\bar\Sigma$ and $\bar\mu\in H^{-1}(\mathcal{D})$. Indeed, we easily check that

$$\begin{aligned}I(\bar\mu)=\frac{1}{2}\iint_{\bar\Sigma\times\bar\Sigma}G(x,y)\,d\bar\mu(x)d\bar\mu(y)&=\frac{1}{2}\iint_{\Sigma\times\Sigma}G\big(\Phi(x),\Phi(y)\big)\,d\mu(x)d\mu(y)\\&\le\frac{1}{4\pi}\iint_{\Sigma\times\Sigma}\ln\big|\Phi(x)-\Phi(y)\big|\,d\mu(x)d\mu(y)+C\\&\le\frac{1}{4\pi}\iint_{\Sigma\times\Sigma}\ln|x-y|\,d\mu(x)d\mu(y)+C<+\infty\,,\end{aligned}$$

where we have used (4.3) and the constant C only depends on Σ, $\mu(\Sigma)$ and $\|\nabla\Phi\|_{L^\infty(\mathcal{D})}$.

Therefore we can apply Step 1 to $\bar\mu$ to find a sequence of measures $\{\bar\mu_k\}_{k\in\mathbb{N}}$ of the form (4.2) such that $\operatorname{supp}\bar\mu_k\subset\bar\Sigma$, $\bar\mu_k(\mathcal{D})\to\bar\mu(\mathcal{D})$, $\bar\mu_k\rightharpoonup\bar\mu$ weakly* as measures on $\mathcal{D}$, and $I(\bar\mu_k)\to I(\bar\mu)$ as $k\to+\infty$. Then we set $\mu_k:=(\Phi^{-1})_\#\bar\mu_k$ for every integer k. Observe that μ_k is of the form (4.2). Indeed, writing $\bar\mu_k=\bar g_k(x)\,\mathcal{H}^1\llcorner\bar\Sigma$ with $\bar g_k\in C^0(\bar\Sigma)$, the area formula (see *e.g.* [5]) yields

$$\mu_k=\bar g_k\circ\Phi(x)|\nabla_\tau\Phi(x)|\,\mathcal{H}^1\llcorner\Sigma\,,$$

where $\nabla_\tau\Phi$ denotes the tangential gradient of Φ along Σ. Then one may check that $\mu_k(\mathcal{D})\to\mu(\mathcal{D})$, $\mu_k\rightharpoonup\mu$ weakly* as measures on $\mathcal{D}$ as $k\to+\infty$.

We claim that (4.23) holds. First write

$$
\begin{aligned}
I(\mu_k) &= \frac{1}{2}\iint_{\bar\Sigma\times\bar\Sigma} G\big(\Phi^{-1}(x),\Phi^{-1}(y)\big)\,d\bar\mu_k(x)d\bar\mu_k(y)\\
&= \frac{1}{2}\iint_{\bar\Sigma\times\bar\Sigma} S\big(\Phi^{-1}(x),\Phi^{-1}(y)\big)\,d\bar\mu_k(x)d\bar\mu_k(y)\\
&\quad + \frac{1}{4\pi}\iint_{\bar\Sigma\times\bar\Sigma} \ln\left(\frac{1}{|\Phi^{-1}(x)-\Phi^{-1}(y)|}\right) d\bar\mu_k(x)d\bar\mu_k(y)\\
&=: I_k + II_k\,, \qquad (4.24)
\end{aligned}
$$

where S denotes the *regular part* of the Green function G, *i.e.*,

$$
S(x,y) := G(x,y) + \frac{1}{2\pi}\ln|x-y|
$$

(which is a locally smooth function on $\mathcal{D}\times\mathcal{D}$). Since $\bar\mu_k$ converges weakly* as measures to $\bar\mu$, we have

$$
\bar\mu_k\otimes\bar\mu_k \rightharpoonup \bar\mu\otimes\bar\mu \quad \text{weakly* as measures on } \mathcal{D}\times\mathcal{D}\,, \qquad (4.25)
$$

and we deduce that

$$
\lim_{k\to+\infty} I_k = \frac{1}{2}\iint_{\bar\Sigma\times\bar\Sigma} S\big(\Phi^{-1}(x),\Phi^{-1}(y)\big)\,d\bar\mu(x)d\bar\mu(y) = \frac{1}{2}\iint_{\Sigma\times\Sigma} S(x,y)\,d\mu(x)d\mu(y)\,. \qquad (4.26)
$$

Next we consider a decreasing sequence $\alpha_n\to 0$. For every integer n, we introduce a smooth cut-off function $\chi_n\in C^\infty(\overline{\mathcal{D}}\times\overline{\mathcal{D}})$ such that $0\le\chi_n\le 1$, $\chi_n(x,y)=0$ if $|\Phi^{-1}(x)-\Phi^{-1}(y)|\ge\alpha_n$, and $\chi_n(x,y)=1$ if $|\Phi^{-1}(x)-\Phi^{-1}(y)|\le\alpha_{n+1}$. Note since $\bar\mu\in H^{-1}(\mathcal{D})$, the measure $\bar\mu$ has no atoms, and hence $\bar\mu\otimes\bar\mu$ does not charge the diagonal $\{x=y\}\cap\mathcal{D}\times\mathcal{D}$. Consequently, $\chi_n\to 0$ $\bar\mu\otimes\bar\mu$–a.e. in $\mathcal{D}\times\mathcal{D}$. Then write

$$
\begin{aligned}
II_k &= \frac{1}{4\pi}\iint_{\bar\Sigma\times\bar\Sigma} \chi_n(x,y)\ln\left(\frac{1}{|\Phi^{-1}(x)-\Phi^{-1}(y)|}\right) d\bar\mu_k(x)d\bar\mu_k(y) + \\
&+ \frac{1}{4\pi}\iint_{\bar\Sigma\times\bar\Sigma} (1-\chi_n(x,y))\ln\left(\frac{1}{|\Phi^{-1}(x)-\Phi^{-1}(y)|}\right) d\bar\mu_k(x)d\bar\mu_k(y) =: III_k^n + IV_k^n\,. \qquad (4.27)
\end{aligned}
$$

By the choice of χ_n and (4.25), we have for every n,

$$
\lim_{k\to+\infty} IV_k^n = \frac{1}{4\pi}\iint_{\bar\Sigma\times\bar\Sigma} (1-\chi_n(x,y))\ln\left(\frac{1}{|\Phi^{-1}(x)-\Phi^{-1}(y)|}\right) d\bar\mu(x)d\bar\mu(y)\,.
$$

Next observe that

$$
\frac{C_1}{|x-y|} \le \frac{1}{\big|\Phi^{-1}(x)-\Phi^{-1}(y)\big|} \le \frac{C_2}{|x-y|} \quad \text{for every } (x,y)\in\mathcal{D}\times\mathcal{D},\ x\neq y\,, \qquad (4.28)
$$

for some constants $C_1>0$ and $C_2>0$ independent of x and y. Since $I(\bar\mu)<+\infty$, estimate (4.3) tells us that the function $\ln|x-y|$ belongs to $L^1(\mathcal{D}\times\mathcal{D},\bar\mu\otimes\bar\mu)$. Therefore we may apply the dominated convergence theorem to derive

$$
\lim_{n\to+\infty}\lim_{k\to+\infty} IV_k^n = \frac{1}{4\pi}\iint_{\bar\Sigma\times\bar\Sigma} \ln\left(\frac{1}{|\Phi^{-1}(x)-\Phi^{-1}(y)|}\right) d\bar\mu(x)d\bar\mu(y)\,. \qquad (4.29)
$$

Let us now treat the term III_k^n. We first deduce from (4.28) that

$$\frac{1}{4\pi}\iint_{\bar\Sigma\times\bar\Sigma}\chi_n(x,y)\ln\left(\frac{C_1}{|x-y|}\right)d\bar\mu_k(x)d\bar\mu_k(y)\leq III_k^n\leq$$
$$\leq\frac{1}{4\pi}\iint_{\bar\Sigma\times\bar\Sigma}\chi_n(x,y)\ln\left(\frac{C_2}{|x-y|}\right)d\bar\mu_k(x)d\bar\mu_k(y)\,. \quad (4.30)$$

Since the function

$$G(x,y)-\frac{\chi_n(x,y)}{2\pi}\ln\left(\frac{C_i}{|x-y|}\right)=S(x,y)+\frac{1-\chi_n(x,y)}{2\pi}\ln|x-y|-\frac{\chi_n(x,y)}{2\pi}\ln(C_i)$$

is locally smooth in $\mathcal{D}\times\mathcal{D}$ and $I(\bar\mu_k)\to I(\bar\mu)$, we infer from (4.25) that

$$\lim_{k\to+\infty}\frac{1}{4\pi}\iint_{\bar\Sigma\times\bar\Sigma}\chi_n(x,y)\ln\left(\frac{C_i}{|x-y|}\right)d\bar\mu_k(x)d\bar\mu_k(y)=$$
$$\frac{1}{4\pi}\iint_{\bar\Sigma\times\bar\Sigma}\chi_n(x,y)\ln\left(\frac{C_i}{|x-y|}\right)d\bar\mu(x)d\bar\mu(y)\quad\text{for } i=1,2\,. \quad (4.31)$$

Using that $\chi_n\to 0$ $\bar\mu\otimes\bar\mu$–a.e. and $\ln|x-y|$ belongs to $L^1(\mathcal{D}\times\mathcal{D},\bar\mu\otimes\bar\mu)$, we infer as previously that

$$\lim_{n\to+\infty}\frac{1}{4\pi}\iint_{\bar\Sigma\times\bar\Sigma}\chi_n(x,y)\ln\left(\frac{C_i}{|x-y|}\right)d\bar\mu(x)d\bar\mu(y)=0\quad\text{for } i=1,2\,. \quad (4.32)$$

Combining (4.30), (4.31) and (4.32) we derive

$$\lim_{n\to+\infty}\liminf_{k\to+\infty}III_k^n=\lim_{n\to+\infty}\limsup_{k\to+\infty}III_k^n=0\,,$$

which yields together with (4.27) and (4.29),

$$\lim_{k\to+\infty}II_k=\frac{1}{4\pi}\iint_{\bar\Sigma\times\bar\Sigma}\ln\left(\frac{1}{|\Phi^{-1}(x)-\Phi^{-1}(y)|}\right)d\bar\mu(x)d\bar\mu(y)$$
$$=\frac{1}{4\pi}\iint_{\Sigma\times\Sigma}\ln\left(\frac{1}{|x-y|}\right)d\mu(x)d\mu(y)\,. \quad (4.33)$$

Then (4.23) follows gathering (4.24), (4.26) and (4.33).

Step 3. We now consider the general Σ case. If Σ is an embedded arc, we may assume without loss of generality that $\Sigma\subset\Sigma'$ for some C^2–Jordan curve Σ' compactly included in $\mathcal{D}$. Hence it suffices to consider the case where Σ is a Jordan curve. We shall use the following lemma. Its proof is postponed at the end of the section.

Lemma 4.3. *Assume that Σ is a C^2-Jordan curve. Then there exists $\delta_1>0$ such that for every $x_0\in\Sigma$, there exists a C^1-diffeomorphism $\Phi:\mathcal{D}\to\mathcal{D}$ satisfying $\Phi(x)=x$ in a neighborhood of $\partial\mathcal{D}$ and such that $\Phi(\Sigma\cap\overline{B}_{\delta_1}(x_0))$ is a segment compactly included in $\mathcal{D}$.*

Now let $\gamma:[0,1]\to\Sigma$ be a constant speed parametrization of Σ. Let N be a positive integer to be chosen and set $t_n=n/N$ for $n=0,\dots,N$, and

$$\Sigma_n:=\gamma([t_{n-1},t_n])\quad\text{for } n=1,\dots,N.$$

We choose N in such a way that $\mathrm{diam}(\Sigma_n)\leq\delta_1$ for each n, where the constant δ_1 is given by Lemma 4.3. Setting $x_n=\gamma((t_{n-1}+t_n)/2)$ for $n=1,\dots,N$, we can apply Lemma 4.3 to each x_n to find a C^1-diffeomorphism $\Phi_n:\mathcal{D}\to\mathcal{D}$ such that $\Phi_n(\Sigma_n)$ is a segment compactly included in $\mathcal{D}$, and $\Phi_n(x)=x$ in a neighborhood of $\partial\mathcal{D}$.

Let μ be an arbitrary nonnegative Radon measure in $H^{-1}(\mathcal{D})$ supported by Σ. Consider a decreasing sequence $\alpha_k \to 0$ and define for k large enough,

$$\Sigma_n^k := \gamma([t_{n-1}+\alpha_k, t_n-\alpha_k])\,, \quad \mu_n^k := \mu \llcorner \Sigma_n^k \quad \text{for } n = 1, \ldots, N.$$

Oviously $\mu_n^k \in H^{-1}(\mathcal{D})$ with $\operatorname{supp}\mu_n^k \subset \Sigma_n^k$. Applying Step 2 for each n and k, we find a sequence of measures $\{\mu_{n,m}^k\}_{m\in\mathbb{N}}$ of the form (4.2) such that $\operatorname{supp}\mu_{n,m}^k \subset \Sigma_n$, $\mu_{n,m}^k(\mathcal{D}) \to \mu_n^k(\mathcal{D})$, $\mu_{n,m}^k \rightharpoonup \mu_n^k$ weakly* as measures on $\mathcal{D}$, and $I(\mu_{n,m}^k) \to I(\mu_n^k)$ as $m \to +\infty$. Define the measures

$$\mu_m^k := \sum_{n=1}^N \mu_{n,m}^k \quad \text{and} \quad \mu^k := \sum_{n=1}^N \mu_n^k = \mu \llcorner (\cup_n \Sigma_n^k)\,.$$

Then $\mu_m^k(\mathcal{D}) \to \mu^k(\mathcal{D})$ and $\mu_m^k \rightharpoonup \mu^k$ weakly* as measures on $\mathcal{D}$ as $m \to +\infty$. In addition, from Lemma 4.1 we infer that $h_{\mu_{n,m}^k} \to h_{\mu_n^k}$ strongly in $H^1(\mathcal{D})$ for every integers n and k. Hence

$$\begin{aligned} I(\mu_m^k) &= \frac{1}{2}\sum_{i,j=1}^N \iint_{\mathcal{D}\times\mathcal{D}} G(x,y)\,d\mu_{i,m}^k d\mu_{j,m}^k = \frac{1}{2}\sum_{i,j=1}^N \int_{\mathcal{D}} \nabla h_{\mu_{i,m}^k}\cdot\nabla h_{\mu_{j,m}^k}\,dx \\ &\underset{m\to+\infty}{\longrightarrow} \frac{1}{2}\sum_{i,j=1}^N \int_{\mathcal{D}} \nabla h_{\mu_i^k}\cdot\nabla h_{\mu_j^k}\,dx = \frac{1}{2}\sum_{i,j=1}^N \iint_{\mathcal{D}\times\mathcal{D}} G(x,y)\,d\mu_i^k d\mu_j^k = I(\mu^k)\,. \end{aligned}$$

Next recall that μ is atomless. Hence, by monotone convergence we have $\mu_k(\mathcal{D}) \to \mu(\mathcal{D})$ and $I(\mu^k) \to I(\mu)$ as $k \to +\infty$, as well as the weak* convergence of μ^k to μ. Consequently,

$$\begin{aligned} \lim_{k\to+\infty}\lim_{m\to+\infty} |\mu_m^k(\mathcal{D}) - \mu(\mathcal{D})| &= \lim_{k\to+\infty}\lim_{m\to+\infty} |I(\mu_m^k) - I(\mu)| \\ &= \lim_{k\to+\infty}\lim_{m\to+\infty} \|\mu_m^k - \mu\|_{(C_0^{0,1}(\mathcal{D}))^*} = 0 \end{aligned}$$

(here we use again the compact embedding $(C_0^0(\mathcal{D}))^* \hookrightarrow (C_0^{0,1}(\mathcal{D}))^*$), and the conclusion follows for a suitable diagonal sequence $\mu_k = \mu_{m_k}^k$. ☐

Proof of Lemma 4.3. By assumption on Σ, there exists $\delta_0 > 0$ such that for every $x_0 \in \Sigma$, $\Sigma \cap \overline{B}_{2\delta_0}(x_0)$ is the graph of a C^2-function and $\overline{B}_{2\delta_0}(x_0) \subset \mathcal{D}$. Now fix $x_0 \in \Sigma$ and write every $x \in \mathcal{D}$ as $x = x_0 + s\tau + t\tau^\perp$ where τ denotes a unit tangent vector to Σ at x_0. Then $\Sigma \cap \overline{B}_{2\delta_0}(x_0) = \{x_0 + s\tau + f(s)\tau^\perp\,,\ s \in [s_{\min}, s_{\max}]\}$ for some $0 > s_{\min} \geq -2\delta_0$, $0 < s_{\max} \leq 2\delta_0$, and a C^2-function $f : [s_{\min}, s_{\max}] \to \mathbb{R}$ satisfying $f(0) = f'(0) = 0$. Since Σ is C^2, there exists a constant $\kappa > 0$ which only depends on Σ such that $|f''(s)| \leq \kappa$ for every $s \in [s_{\min}, s_{\max}]$. Hence we may choose δ_0 smaller if necessary (uniformly with respect to x_0) in such a way that $|f'| \leq 1$. Then $s_{\min} \leq -\delta_0$, $s_{\max} \geq \delta_0$ and $\Sigma \cap \overline{B}_\delta(x_0)$ is still a connected arc for any $\delta \leq 2\delta_0$.

Set $\delta_1 := \delta_0/(2+\kappa)$. We claim that $\Sigma \cap \overline{B}_{\delta_1}(x_0)$ satisfies the requirement. Indeed, we may construct a C^1-diffeomorphism $\Phi : \mathcal{D} \to \mathcal{D}$ as follows. Consider a smooth cut-off function $\chi : \mathbb{R}^2 \to \mathbb{R}$ such that $0 \leq \chi \leq 1$, $\chi(x) = 1$ if $|x| \leq \delta_1$, $\chi(x) = 0$ if $|x| \geq 2\delta_0$ and $|\nabla\chi| \leq \delta_0^{-1}$. Then we set for $x \in \mathcal{D}$,

$$\Phi(x) := x - \chi(x - x_0) f\big((x - x_0)\cdot\tau\big)\tau^\perp\,.$$

The reader may check that Φ maps $\mathcal{D}$ into $\mathcal{D}$, Φ is one-to-one and defines a C^1-diffeomorphism. Moreover $\Phi(\Sigma \cap \overline{B}_{\delta_1}(x_0)) = \{x_0 + s\tau\,,\ -\delta_1 \leq s \leq \delta_1\}$ is a segment compactly included in $\mathcal{D}$. ☐

Proof of Corollary 1.1. *Step 1.* For any nonnegative Radon measure μ supported by Σ we have

$$I(\mu) - \zeta_{\max}\mu(\mathcal{D}) \geq I_*(\mu(\mathcal{D}))^2 - \zeta_{\max}\mu(\mathcal{D})\,, \tag{4.34}$$

and equality holds if and only if $\mu = \lambda\mu_*$ for some constant $\lambda \geq 0$. We recall that μ_* is the unique minimizer of I among all probability measures supported by Σ and that $I_* := I(\mu_*)$. The existence and uniqueness of μ_* is classical, and we refer to [14] for further details. Optimizing (4.34) with respect to λ for measures of the form $\mu = \lambda\mu_*$, we derive that $\frac{\zeta_{\max}}{2I_*}\mu_*$ is the unique minimizer of $\mu \mapsto I(\mu) - \zeta_{\max}\mu(\mathcal{D})$ over all nonnegative Radon measures supported by Σ.

Step 2. Let $\varepsilon_n \to 0^+$ be an arbitrary sequence. The existence of a minimizer u_n of F_{ε_n} is classical and follows from standard arguments based on coercivity and lower semi-continuity properties of F_{ε_n}. We first observe that $F_{\varepsilon_n}(u_n) \leq F_{\varepsilon_n}(1) = 0$. Hence, by Theorem 1.1, there exists a subsequence $\{\varepsilon_{n_k}\}$ such that

$$\frac{1}{\omega_{n_k}} j(u_{n_k}) \to \mu_0 \quad \text{strongly in } (C_0^{0,1}(\mathcal{D}))^* \text{ as } k \to +\infty,$$

for some nonnegative Radon measure $\mu_0 \in H^{-1}(\mathcal{D})$ supported by Σ. Moreover,

$$\liminf_{k\to+\infty} F_{\varepsilon_{n_k}}(u_{n_k}) \geq I(\mu_0) - \zeta_{\max}\mu_0(\mathcal{D})\,. \tag{4.35}$$

On the other hand, by Theorem 1.1, any nonnegative Radon measure $\mu \in H^{-1}(\mathcal{D})$ supported by Σ can be strongly approximated in $(C_0^{0,1}(\mathcal{D}))^*$ by some sequence $\{\omega_{n_k}^{-1} j(v_k)\}$ with $\{v_k\} \subset H^1(\mathcal{D};\mathbb{C})$ satisfying

$$\lim_{k\to+\infty} F_{\varepsilon_{n_k}}(v_k) = I(\mu) - \zeta_{\max}\mu(\mathcal{D})\,.$$

Since $F_{\varepsilon_{n_k}}(u_{n_k}) \leq F_{\varepsilon_{n_k}}(v_k)$ we infer that μ_0 minimizes $\mu \mapsto I(\mu) - \zeta_{\max}\mu(\mathcal{D})$ over all nonnegative Radon measures supported by Σ. Consequently, $\mu_0 = \frac{\zeta_{\max}}{2I_*}\mu_*$ and the lim inf in (4.35) is actually a limit. Then the result along the full sequence $\{\varepsilon_n\}$ follows from a standard argument on the uniqueness of the limit. □

5. Γ–convergence analysis for annular domains

In this section we briefly show how to extend the above techniques to the case of a multiply connected domain. The method we outline here may be applied for any finite number of holes (see [2]), but for simplicity we restrict to domains which are topological annuli. Let $\mathcal{D}$ denote a simply connected domain in $\mathbb{R}^2$ with smooth boundary, and $\mathcal{B} \subset\subset \mathcal{D}$ a smooth, simply connected domain compactly contained inside $\mathcal{D}$. Then let $\mathcal{A} := \mathcal{D} \setminus \mathcal{B}$. For $u \in H^1(\mathcal{A};\mathbb{C})$ we define the functional

$$J_\varepsilon(u) := \int_{\mathcal{A}} \left\{ \frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}(|u|^2-1)^2 - \Omega_\varepsilon V(x)\cdot j(u) \right\} dx\,.$$

Here the given vector field $V : \mathbb{R}^2 \to \mathbb{R}^2$ is assumed (for simplicity) to be locally Lipschitz continuous. We are interested in the asymptotic behavior of J_ε as $\varepsilon \to 0$, with an angular speed Ω_ε as in (1.2).

5.1. *Asymptotic vorticity of the hole*

For multiply connected domains, the highest order term in an expansion of the minimal energy is partially due to the turning of the phase of a minimizer around the holes. The first step in studying vortices in the interior is to identify the *asymptotic vorticity of the hole*, and then split the energy into contributions from the hole and from the interior. To this purpose we first study the minimization of the functional J_ε over $\mathbb{S}^1$–valued maps. Observe that for $\mathbb{S}^1$–valued maps, the functional J_ε only depends on the angular speed and not anymore on ε itself, *i.e.*, for every $u \in H^1(\mathcal{A};\mathbb{S}^1)$,

$$J_\varepsilon(u) = \mathcal{H}_\Omega(u) := \int_{\mathcal{A}} \left\{ \frac{1}{2}|\nabla u|^2 - \Omega V(x)\cdot j(u) \right\} dx\,,$$

with $\Omega = \Omega_\varepsilon$. We are therefore interested in minimizing $\mathcal{H}_\Omega$ over the class $H^1(\mathcal{A};\mathbb{S}^1)$, and here $\Omega > 0$ could be any positive parameter. It is well known that maps in $H^1(\mathcal{A};\mathbb{S}^1)$ are classified by their topological degree, *i.e.*, their winding number around the hole $\mathcal{B}$. Hence, minimizing first in each homotopy class and then choosing the lowest energy level, one reaches the minimum of the energy of $\mathcal{H}_\Omega$, *i.e.*,

$$\min \mathcal{H}_\Omega = \min_{d\in\mathbb{Z}} g(d,\Omega)\,, \tag{5.1}$$

where

$$g(d,\Omega) := \min\left\{ \mathcal{H}_\Omega(u) \;:\; u \in H^1(\mathcal{A};\mathbb{S}^1)\,,\ \deg u = d \right\}. \tag{5.2}$$

Concerning the minimization problem (5.2), we have the following result.

Proposition 5.1. *For every $d \in \mathbb{Z}$, the minimization problem* (5.2) *admits a unique solution u_d up to a (complex) multiplicative constant of modulus one. Moreover,*

$$g(d,\Omega) = \frac{1}{2}\int_{\mathcal{A}} \left\{ |\nabla\Phi_d|^2 - \Omega^2|V|^2 \right\} dx\,, \tag{5.3}$$

where Φ_d is the unique solution of the linear equation

$$\begin{cases} -\Delta\Phi_d = \Omega\,\mathrm{curl}\,V & \text{in } \mathcal{A}\,,\\ \Phi_d = 0 & \text{on } \partial\mathcal{D}\,,\\ \Phi_d = \text{const.} & \text{on } \partial\mathcal{B}\,,\\ \displaystyle\int_{\partial\mathcal{B}} \frac{\partial\Phi_d}{\partial\nu} = 2\pi d - \Omega\int_{\partial\mathcal{B}} V\cdot\tau & \end{cases} \tag{5.4}$$

Proof. We follow here some of the arguments in [6, Chap. 1], and we provide some details for the reader conveniance.

Step 1. We claim that for any $u \in H^1(\mathcal{A};\mathbb{S}^1)$ such that $\deg u = d$, we have

$$\mathcal{H}_\Omega(u) \geq \frac{1}{2}\int_{\mathcal{A}} \left\{ |\nabla\Phi_d|^2 - \Omega^2|V|^2 \right\} dx\,.$$

Indeed, we first observe that $\mathrm{curl}\, j(u) = 0$ since u is $\mathbb{S}^1$-valued. On the other hand, $\nabla^\perp\Phi_d + \Omega V$ is also curl-free and

$$\int_{\partial\mathcal{B}} \big(j(u) - \nabla^\perp\Phi_d - \Omega V\big)\cdot\tau = 2\pi d - \int_{\partial\mathcal{B}} \frac{\partial\Phi_d}{\partial\nu} - \Omega\int_{\partial\mathcal{B}} V\cdot\tau = 0\,,$$

so that we can find a scalar function $H \in H^1(\mathcal{A})$ such that $j(u) = \nabla H + \nabla^\perp\Phi_d + \Omega V$. Since u is $\mathbb{S}^1$-valued, we have $|\nabla u|^2 = |j(u)|^2$, and thus

$$\begin{aligned} \mathcal{H}_\Omega(u) &= \frac{1}{2}\int_{\mathcal{A}} \left\{ |j(u) - \Omega V|^2 - \Omega^2|V|^2 \right\} dx \\ &= \frac{1}{2}\int_{\mathcal{A}} \left\{ |\nabla\Phi_d|^2 - \Omega^2|V|^2 \right\} dx + \frac{1}{2}\int_{\mathcal{A}} |\nabla H|^2 dx + \int_{\mathcal{A}} \nabla^\perp\Phi_d\cdot\nabla H\, dx\,. \end{aligned}$$

Then, using the fact that the function Φ_d is constant on $\partial\mathcal{A}$, an integration by parts yields $\int_{\mathcal{A}} \nabla^\perp \Phi_d \cdot \nabla H \, dx = 0$ and the claim follows.

Step 2. We claim that there exists $u_d \in H^1(\mathcal{A}; \mathbb{S}^1)$ such that $\deg u_d = d$ and

$$j(u_d) = \nabla^\perp \Phi_d + \Omega V \,.$$

Indeed, since $\operatorname{curl}(\nabla^\perp \Phi_d + \Omega V) = 0$ and

$$\frac{1}{2\pi} \int_{\partial\mathcal{B}} (\nabla^\perp \Phi_d + \Omega V) \cdot \tau \, = d \in \mathbb{Z} \,,$$

we may locally define a scalar function ψ in $\mathcal{A}$ such that

$$\nabla\psi = \nabla^\perp \Phi_d + \Omega V \,.$$

Then $u_d := \exp(i\psi)$ is well defined and satisfies the required properties. Clearly the construction of u_d is unique modulo a constant phase, and the proof is complete. □

In order to solve problem (5.1), it now suffices to express (5.3) explicitely in terms of the integer d. To this purpose, we first introduce the solution ξ of the linear problem

$$\begin{cases} \Delta\xi = 0 & \text{in } \mathcal{A}\,, \\ \xi = 0 & \text{on } \partial\mathcal{D}\,, \\ \xi = 1 & \text{on } \partial\mathcal{B}\,. \end{cases} \tag{5.5}$$

The function ξ is smooth in $\overline{\mathcal{A}}$ and $0 \leq \xi \leq 1$ by the maximum principle. Moreover, the Dirichlet energy of ξ is the so-called H^1-capacity of $\mathcal{B}$ inside $\mathcal{D}$ which we denote by $\operatorname{cap}(\mathcal{B})$, *i.e.*,

$$\operatorname{cap}(\mathcal{B}) := \int_{\mathcal{A}} |\nabla\xi|^2 \, dx = -\int_{\partial\mathcal{B}} \frac{\partial\xi}{\partial\nu} \, > 0 \,. \tag{5.6}$$

Next we consider the unique solution ζ of

$$\begin{cases} -\Delta\zeta = \operatorname{curl} V & \text{in } \mathcal{A}\,, \\ \zeta = 0 & \text{on } \partial\mathcal{A}\,. \end{cases} \tag{5.7}$$

From the Lipschitz assumption on V and standard elliptic regularity, we infer that ζ belongs to $C^{1,\alpha}_0(\overline{\mathcal{A}})$ for every $0 \leq \alpha < 1$. We set

$$\gamma_V := \int_{\partial\mathcal{D}} \left\{ \frac{\partial\zeta}{\partial\nu} + V \cdot \tau \right\} .$$

Observing that (5.7) implies

$$\int_{\partial\mathcal{B}} \frac{\partial\zeta}{\partial\nu} = \gamma_V - \int_{\partial\mathcal{B}} V \cdot \tau \,,$$

we find that for every integer d, the function Φ_d determined by (5.4) is explicitly given by

$$\Phi_d = \left(\frac{\gamma_V \Omega - 2\pi d}{\operatorname{cap}(\mathcal{B})} \right) \xi + \Omega\,\zeta \,. \tag{5.8}$$

Moreover, using (5.5), (5.6) and (5.7) we readily obtain that for every $d \in \mathbb{Z}$,

$$\frac{1}{2} \int_{\mathcal{A}} \left\{ |\nabla\Phi_d|^2 - \Omega^2 |V|^2 \right\} \, dx = \frac{|\gamma_V \Omega - 2\pi d|^2}{2\operatorname{cap}(\mathcal{B})} - \frac{\Omega^2}{2} \int_{\mathcal{A}} \left\{ |V|^2 - |\nabla\zeta|^2 \right\} \, dx \,.$$

As a consequence, an integer d_Ω is a minimizer in (5.1) if and only if d_Ω minimizes the function $d \in \mathbb{Z} \mapsto |\gamma_V \Omega - 2\pi d|$.

We may now state our result concerning problem (5.1).

Theorem 5.1. *Up to multiplicative constants of modulus one, the minimization problem* (5.1) *admits exactly two solutions (of distinct topological degree) if $\gamma_V\Omega/\pi$ is an odd integer, and a unique solution otherwise. Moreover, if $d_\Omega \in \mathbb{Z}$ is a minimizer in* (5.1), *then $d_\Omega \in \big\{[\frac{\gamma_V\Omega}{2\pi}], [\frac{\gamma_V\Omega}{2\pi}]+1\big\}$ where $[\cdot]$ denotes the integer part, and*

$$\min \mathcal{H}_\Omega = -\frac{\Omega^2}{2}\int_{\mathcal{A}} \big\{|V|^2 - |\nabla\zeta|^2\big\}\,dx + O(1) \quad \text{as } \Omega \to +\infty\,. \tag{5.9}$$

5.2. *The* Γ*−convergence result*

To state the parallel Γ-convergence result for the anular domain case we must give more specific hypotheses on the potential V and the angular speed Ω_ε. In addition to the Lipschitz regularity, we assume in the sequel that V satisfies the following assumptions:

(H1') the solution ζ of (5.7) is such that $\zeta_{\max} := \max_{x\in\overline{\mathcal{A}}} \zeta(x) = \max_{x\in\overline{\mathcal{A}}} |\zeta(x)| > 0$;

(H2') the set $\Sigma := \{x \in \mathcal{A} :\ \zeta(x) = \zeta_{\max}\} \subset\subset \mathcal{A}$ is a Jordan curve or a simple embedded arc of class C^2.

We note that in (H1'), the assumption that $\zeta_{\max}$ is achieved at positive values of ζ is not restrictive. Indeed, considering the complex conjugate of an admissible function replaces V by $-V$ in the energy and hence ζ by $-\zeta$.

As for the simply connected domain case, we assume that Ω_ε is near the critical value needed for the presence of vortices which again reads

$$\Omega_\varepsilon = \frac{|\ln\varepsilon|}{2\zeta_{\max}} + \omega(\varepsilon)\,, \tag{5.10}$$

for some positive function ω satisfying $\omega(\varepsilon) \to +\infty$ with $\omega(\varepsilon) \leq o(|\ln\varepsilon|)$ as $\varepsilon \to 0^+$, exactly as in (1.2).

In the sequel, for an arbitrary sequence $\varepsilon_n \to 0^+$, we will denote by $u_n^\star$ a minimizer of $\mathcal{H}_{\Omega_n}$, *i.e.*, a solution of (5.1), and its corresponding topological degree will be denoted by d_n. For brievety we shall also write (5.8) as

$$\Phi_n := \Phi_{d_n} = \alpha_n\xi + \Omega_n\zeta \quad \text{with} \quad \alpha_n := \frac{\gamma_V\Omega_n - 2\pi d_n}{\mathrm{cap}(\mathcal{B})}\,.$$

We emphasize that $\alpha_n = O(1)$ as $n \to +\infty$ thanks to Theorem 5.1.

For $v \in H^1(\mathcal{A};\mathbb{C})$, we now define

$$\overline{F}_{\varepsilon_n}(v) := \omega_n^{-2}\int_{\mathcal{A}} \left\{\frac{|\nabla v|^2}{2} + \frac{(1-|v|^2)^2}{4\varepsilon_n^2} + \nabla^\perp\Phi_n \cdot j(v)\right\} dx\,.$$

The following proposition shows that the functional $\overline{F}_{\varepsilon_n}(\bar{u}_n^\star u)$ captures the energy induced by interior vorticity of a given configuration u.

Proposition 5.2 (Energy decomposition). *Assume that* (5.10) *holds. Let $\varepsilon_n \to 0^+$ be an arbitrary sequence and $\{u_n\}_{n\in\mathbb{N}} \subset H^1(\mathcal{A};\mathbb{C})$. Then, setting $v_n = \bar{u}_n^\star u_n \in H^1(\mathcal{A};\mathbb{C})$, we have*

$$\sup_n \Omega_n^{-2} J_{\varepsilon_n}(u_n) < +\infty \quad \textit{if and only if} \quad \sup_n \Omega_n^{-2}\omega_n^2\overline{F}_{\varepsilon_n}(v_n) < +\infty\,. \tag{5.11}$$

Moreover, if one of the conditions in (5.11) *holds, then*

$$J_{\varepsilon_n}(u_n) = \min \mathcal{H}_{\Omega_n} + \omega_n^2 \overline{F}_{\varepsilon_n}(v_n) + o(1) \quad \text{as } n \to +\infty . \tag{5.12}$$

Proof. Straightforward computations yield

$$\frac{|\nabla u_n|^2}{2} = |u_n|^2 \frac{|\nabla u_n^\star|^2}{2} + \frac{|\nabla v_n|^2}{2} + j(u_n^\star) \cdot j(v_n) ,$$

and

$$j(u_n) = |u_n|^2 j(u_n^\star) + j(v_n) . \tag{5.13}$$

By the proof of Proposition 5.1, we have $j(u_n^\star) = \nabla^\perp \Phi_n + \Omega_n V$. Hence,

$$J_{\varepsilon_n}(u_n) = \min \mathcal{H}_{\Omega_n} + \omega_n^2 \overline{F}_{\varepsilon_n}(v_n) + \int_{\mathcal{A}} (|u_n|^2 - 1) \left\{ \frac{|\nabla u_n^\star|^2}{2} - \Omega_n V \cdot j(u_n^\star) \right\} dx .$$

Then we observe that

$$\|\nabla u_n^\star\|_{L^\infty(\mathcal{A})} = \|j(u_n^\star)\|_{L^\infty(\mathcal{A})} = \|\nabla^\perp \Phi_n + \Omega_n V\|_{L^\infty(\mathcal{A})} = O(\Omega_n) .$$

Therefore,

$$\left| \int_{\mathcal{A}} (|u_n|^2 - 1) \left\{ \frac{|\nabla u_n^\star|^2}{2} - \Omega_n V \cdot j(u_n^\star) \right\} dx \right| \leq O\left(\varepsilon_n \Omega_n^2 \sqrt{E_{\varepsilon_n}(u_n, \mathcal{A})} \right) .$$

Since $|u_n| = |v_n|$, we also have the same estimate as above with $E_{\varepsilon_n}(v_n, \mathcal{A})$ instead of $E_{\varepsilon_n}(u_n, \mathcal{A})$. Assuming that one of the conditions in (5.11) holds and arguing as in (2.2), we derive that either $\sqrt{E_{\varepsilon_n}(u_n, \mathcal{A})} \leq O(\Omega_n)$, or $\sqrt{E_{\varepsilon_n}(v_n, \mathcal{A})} \leq O(\Omega_n)$. Consequently, if one of the conditions in (5.11) is satisfied, (5.12) holds and the conclusion follows combining (5.12) with (5.9). □

For a nonnegative Radon measure μ on $\mathcal{A}$, we define

$$\bar{I}(\mu) := \frac{1}{2} \iint_{\mathcal{A} \times \mathcal{A}} \bar{G}(x, y) \, d\mu(x) \, d\mu(y) ,$$

where the function $\bar{G}$ denotes the Dirichlet Green's function of the domain $\mathcal{A}$, *i.e.*, for every $y \in \mathcal{A}$, $\bar{G}(\cdot, y)$ is the solution of

$$\begin{cases} -\Delta \bar{G}(\cdot, y) = \delta_y & \text{in } \mathscr{D}'(\mathcal{A}) , \\ \bar{G}(\cdot, y) = 0 & \text{on } \partial \mathcal{A} . \end{cases} \tag{5.14}$$

We may now state the Γ-convergence result for annular domains which involves the family of *"reduced"* functionals $\{\overline{F}_\varepsilon\}_{\varepsilon > 0}$.

Theorem 5.2. *Assume that* (H1'), (H2') *and* (5.10) *hold. Let $\varepsilon_n \to 0^+$ be an arbitrary sequence. Then,*

(i) for any $\{v_n\}_{n \in \mathbb{N}} \subset H^1(\mathcal{D}; \mathbb{C})$ satisfying $\sup_n \overline{F}_{\varepsilon_n}(v_n) < +\infty$, there exist a subsequence (not relabelled) and a nonnegative Radon measure μ in $H^{-1}(\mathcal{A})$ supported by Σ such that

$$\frac{1}{\omega(\varepsilon_n)} \operatorname{curl} j(v_n) \underset{n \to +\infty}{\longrightarrow} \mu \quad \text{strongly in } (C_0^{0,1}(\mathcal{A}))^* ; \tag{5.15}$$

(ii) for any $\{v_n\}_{n\in\mathbb{N}} \subset H^1(\mathcal{D};\mathbb{C})$ *such that* (5.15) *holds for some nonnegative Radon measure* μ *in* $H^{-1}(\mathcal{A})$ *supported by* Σ*, we have*

$$\liminf_{n\to+\infty} \overline{F}_{\varepsilon_n}(v_n) \geq \bar{I}(\mu) - \zeta_{\max}\,\mu(\mathcal{A})\,;$$

(iii) for any nonnegative Radon measure μ *in* $H^{-1}(\mathcal{A})$ *supported by* Σ*, there exists a sequence* $\{v_n\}_{n\in\mathbb{N}} \subset H^1(\mathcal{A};\mathbb{C})$ *such that* (5.15) *holds and*

$$\lim_{n\to+\infty} \overline{F}_{\varepsilon_n}(v_n) = \bar{I}(\mu) - \zeta_{\max}\,\mu(\mathcal{A})\,.$$

As in the simply connected case this Γ-convergence result could lead to the asymptotic description of the vorticity in $\overline{F}_\varepsilon$-global minimizers. Actually Theorem 5.2 combined with Proposition 5.2 also gives the asymptotic behavior of vorticity in J_ε-global minimizers. The key observation here is that minimizers for J_ε yield quasi-minimizers for $\overline{F}_\varepsilon$, and conversely.

Corollary 5.1. *Assume that* (H1'), (H2') *and* (5.10) *hold. Let* $\varepsilon_n \to 0^+$ *be an arbitrary sequence. For every integer* $n \in \mathbb{N}$*, let* $u_n \in H^1(\mathcal{A};\mathbb{C})$ *be a minimizer of* $J_{\varepsilon_n}(\cdot)$*. Then,*

$$\frac{1}{\omega(\varepsilon_n)}\,\mathrm{curl}\, j(u_n) \underset{n\to+\infty}{\longrightarrow} \frac{\zeta_{\max}}{2\bar{I}_*}\,\bar{\mu}_* \quad \textit{strongly in } (C_0^{0,1}(\mathcal{A}))^*\,,$$

where $\bar{\mu}_*$ *is the unique minimizer of* $\bar{I}(\cdot)$ *over all probability measures supported by* Σ*, and* $\bar{I}_* := \bar{I}(\bar{\mu}_*)$*. In addition,*

$$J_{\varepsilon_n}(u_n) = -\frac{\Omega_n^2}{2}\int_{\mathcal{A}} \left\{|V|^2 - |\nabla\zeta|^2\right\} dx - \frac{\zeta_{\max}^2}{4\bar{I}_*}\omega_n^2 + o(\omega_n^2)\,. \tag{5.16}$$

We conclude this subsection with an elementary example motivated by [1,2,3].

Example 5.1. Assume that $\mathcal{D} = B_1(0)$, $\mathcal{B} = B_\rho(0)$ for some $0 < \rho < R$ and $V(x) = x^\perp$. Then the solution ζ of (5.7) is given by

$$\zeta(x) = -\frac{|x|^2}{2} + \frac{R^2-\rho^2}{2\ln(R/\rho)}\ln|x| + \frac{\rho^2\ln R - R^2\ln\rho}{2\ln(R/\rho)}\,.$$

In particular, the set Σ is given by the concentric circle $B_{r_*}(0)$ with

$$r_* = \sqrt{\frac{R^2-\rho^2}{2\ln(R/\rho)}} \in (\rho, R)\,.$$

Here again, the uniform measure $\mu_* = (2\pi r_*)^{-1} d\mathcal{H}^1 \llcorner \Sigma$ turns out to be the Green equilibrium measure for Σ in $\mathcal{A}$, *i.e.*, $\bar{I}(\mu_*) = \bar{I}_*$. Indeed, one may easily check that the function

$$h_*(x) = \begin{cases} \dfrac{\ln(R/r_*)}{2\pi(\ln(R/r_*)+\ln(r_*/\rho))}\ln(|x|/\rho) & \text{if } \rho \leq |x| \leq r_*\,, \\ \dfrac{\ln(r_*/\rho)}{2\pi(\ln(R/r_*)+\ln(r_*/\rho))}\ln(R/|x|) & \text{if } r_* \leq |x| \leq R\,, \end{cases}$$

solves $-\Delta h_* = \mu_*$ in $\mathcal{A}$ with $h_{*|\partial\mathcal{A}} = 0$. Hence $h_*(x) = \int_{\mathcal{A}} \bar{G}(x,y)\,d\mu_*(y)$, and since h_* is constant on Σ the conclusion follows from Theorem II.5.12 in [14].

5.3. *Compactness of normalised weak Jacobians*

In this subsection we shall be concerned with the proof of claim *(i)* in Theorem 5.2. We consider an arbitrary sequence $\varepsilon_n \to 0^+$. For any $\{v_n\}_{n\in\mathbb{N}} \subset H^1(\mathcal{D};\mathbb{C})$ satisfying $\sup_n \overline{F}_{\varepsilon_n}(v_n) < +\infty$, we first derive exactly as in Lemma 2.1 the estimates

$$E_{\varepsilon_n}(v_n, \mathcal{A}) \leq O(|\ln \varepsilon_n|^2) \quad \text{and} \quad \|v_n\|_{L^4(\mathcal{A})} \leq O(1)\,. \tag{5.17}$$

Hence, assuming in addition that $\{v_n\}_{n\in\mathbb{N}} \subset C^1(\mathcal{A})$, we can apply the vortex ball construction in Proposition 2.1 with $\mathcal{A}$ in place of $\mathcal{D}$, and $\mathcal{A}_\varepsilon := \{x \in \mathcal{A} : \operatorname{dist}(x, \partial\mathcal{A}) > \varepsilon\}$ in place of $\mathcal{D}_\varepsilon$. We choose again $r = r_n := |\ln \varepsilon_n|^{-4}$, thus obtaining a finite collection of disjoint closed balls $\{\overline{B}(a_i^n, \rho_{i,n})\}_{i\in I_n}$ (written $B_i^n := \overline{B}(a_i^n, \rho_{i,n})$), with associated degrees $d_{i,n}$ and total approximate vorticity

$$D_n := \sum_{i\in I_n} |d_{i,n}|\,,$$

as in Proposition 2.1.

The first difference with the simply connected case arises in an estimate analogue to (2.3), as we must take into account an additional contribution to the potential term due to the boundary $\partial\mathcal{B}$. Since one or more of the vortex balls B_i^n may intersect $\partial\mathcal{A}_\varepsilon \setminus \partial\mathcal{D}_\varepsilon$, we will need to perturb this boundary slightly. When calculating the boundary term it will be convenient to choose a level set of ξ. For $0 < s < t < 1$, denote by

$$\sigma_t := \{x \in \mathcal{A} : \xi(x) = t\}, \quad \text{and} \quad A_{s,t} := \{x \in \mathcal{A} : \; s < \xi < t\}\,. \tag{5.18}$$

As an easy consequence of the Maximum Principle and the Hopf boundary lemma, each curve σ_t is smooth and the family $\{\sigma_t\}_{0<t<1}$ realizes a foliation of $\mathcal{A}$. Then, for every $t \in (0,1)$ the curve σ_t is diffeomorphic to $\partial\mathcal{B}$, and the set $A_{t,1}$ is a neighborhood of $\partial\mathcal{B}$ in $\mathcal{A}$. Now we shall choose an appropriate level set of ξ. Define

$$J_n = \big\{t \in (0,1) \,:\, \sigma_t \cap (\cup_{i\in I_n} B_i^n) = \emptyset\big\}\,. \tag{5.19}$$

We note that the measure of the complement $(0,1) \setminus J_n$ is of the same order as $r_n = |\ln \varepsilon_n|^{-4}$. Hence we can find $t_n \in J_n$ such that the level curve $\gamma_n := \sigma_{t_n}$ satisfies

$$\varepsilon_n < \operatorname{dist}(\gamma_n, \partial\mathcal{B}) \leq O(|\ln \varepsilon_n|^{-3})\,,$$

and consequently $t_n = 1 + O(|\ln \varepsilon_n|^{-3})$.

This construction allows us to define to topological degree of v_n around γ_n since v_n does not vanish on γ_n, *i.e.*,

$$\delta_n := \deg\left(\frac{v_n}{|v_n|}, \gamma_n\right).$$

Then an approximate total vorticity in $\mathcal{A}$ of the configuration v_n is given by $|\delta_n| + D_n$.

We may now state the following proposition which parallels Proposition 2.2.

Proposition 5.3. *Assume that* (H1') *and* (5.10) *hold. Let* $\{v_n\}_{n\in\mathbb{N}} \subset H^1(\mathcal{D};\mathbb{C}) \cap C^1(\mathcal{A})$, D_n *and* δ_n *as above. Then* $|\delta_n| + D_n \leq O(\omega_n)$.

Proof. *Step 1.* Arguing exactly as in the proof of (2.3), we first derive

$$\Omega_n \int_{\mathcal{A}} \nabla^\perp \zeta \cdot j(v_n) = -2\pi\Omega_n \sum_{i\in I_n} d_{i,n}\zeta(a_i^n) + o(1)\,. \tag{5.20}$$

Then following essentially the proof of Lemma 3.4 in [2], we obtain

$$\int_{\mathcal{A}} \nabla^{\perp}\xi \cdot j(v_n) = -2\pi \sum_{i\in I_n} d_{i,n}\xi(a_i^n) - 2\pi t_n\delta_n + o(1)\,. \tag{5.21}$$

Here the fact that γ_n is a level set of ξ is essential in obtaining the degree δ_n from the boundary term when integrating by parts.

Step 2. As in the proof of Proposition 2.2, we may assume that $\omega_n \leq O(D_n + |\delta_n|)$. From (5.17) and claim *(iv)* in Proposition 2.1, we have $D_n \leq O(|\ln\varepsilon_n|)$. Then we infer from (2.1), (5.20) and (5.21) that

$$O(\omega_n^2) \geq \omega_n^2\overline{F}_{\varepsilon_n}(v_n) \geq \pi D_n\big(|\ln\varepsilon_n| - C\ln|\ln\varepsilon_n|\big) - 2\pi\Omega_n\sum_{i\in I_n} d_{i,n}\zeta(a_i^n) \; - 2\pi|\alpha_n|D_n \\ - 2\pi|\alpha_n||\delta_n| + \int_{\mathcal{A}\setminus\cup_{i\in I_n}B_i^n} |\nabla v_n|^2\,dx + o(1)\,, \tag{5.22}$$

where we have used the fact that $0 \leq \xi \leq 1$ and $0 < t_n < 1$.

Next we consider a sequence $\eta_n \to 0$ as in (2.6), and we group the vortex balls into different classes as in the proof of Proposition 2.2 (we refer to it for the notation). Exactly as in (2.10) and (2.12), we derive that

$$\pi D_n^*|\ln\varepsilon_n| - 2\pi\Omega_n\sum_{i\in I_n^*} d_{i,n}\zeta(a_i^n) \geq -2\pi\omega_n\zeta_{\max}D_n^*\,, \tag{5.23}$$

and

$$\pi D_n^+|\ln\varepsilon_n| - 2\pi\Omega_n\sum_{i\in I_n^+} d_{i,n}\zeta(a_i^n) \geq C\Omega_n\eta_n D_n^+\,. \tag{5.24}$$

For negative degrees, we observe that $\{\zeta \leq 0\}\cap\Sigma = \emptyset$ since $\zeta_{\max} = |\zeta|_{\max} > 0$. Hence we can estimate as for the class I_n^+,

$$\pi D_n^-|\ln\varepsilon_n| - 2\pi\Omega_n\sum_{i\in I_n^-} d_{i,n}\zeta(a_i^n) \geq \pi D_n^-|\ln\varepsilon_n| - 2\pi\Omega_n\sum_{i\in I_n^-,\,\zeta(a_i^n)\leq 0} d_{i,n}\zeta(a_i^n) \\ \geq C\Omega_n\eta_n D_n^-\,. \tag{5.25}$$

Inserting (5.23), (5.24) and (5.25) in (5.22) yields

$$O(\omega_n^2) \geq -\pi C_1 D_n\ln|\ln\varepsilon_n| - 2\pi\omega_n\zeta_{\max}D_n + C\eta_n\Omega_n(D_n^+ + D_n^-) \\ - 2\pi|\alpha_n||\delta_n| + \frac{1}{2}\int_{\mathcal{A}\setminus\cup B_i^n}|\nabla v_n|^2dx\,. \tag{5.26}$$

Using the fact that $|\alpha_n| = O(1)$, we easily deduce the estimate

$$D_n^+ + D_n^- \leq C\,\frac{\max\{\omega_n, \ln|\ln\varepsilon_n|\}(D_n + |\delta_n|)}{\eta_n|\ln\varepsilon_n|}\,, \tag{5.27}$$

for a constant $C > 0$ independent of n.

We claim that

$$\int_{\mathcal{A}\setminus\cup B_i^n}|\nabla v_n|^2dx \geq C(D_n + |\delta_n|)^2\,. \tag{5.28}$$

Accepting (5.28), we infer from (5.26) that

$$(D_n + |\delta_n|)^2 - C\max\{\omega_n, \ln|\ln\varepsilon_n|\}(D_n + |\delta_n|) \leq O(\omega_n^2)\,,$$

which clearly implies

$$D_n + |\delta_n| \le O(\max\{\omega_n, \ln|\ln\varepsilon_n|\})\,. \tag{5.29}$$

To prove (5.28) we introduce

$$t_0 := 1/2\min\{t \in (0,1) : \sigma_t \cap \Sigma \neq \emptyset\}\,, \quad t_1 := 1/2(1+\max\{t \in (0,1) : \sigma_t \cap \Sigma \neq \emptyset\})\,,$$

and

$$J_n^1 := \big\{t \in (\varepsilon_n, t_0) \,:\, \sigma_t \cap (\cup_{i\in I_n} B_i^n) = \emptyset\big\}\,, \quad J_n^2 := \big\{t \in (t_1, t_n) \,:\, \sigma_t \cap (\cup_{i\in I_n} B_i^n) = \emptyset\big\}\,.$$

Then $\operatorname{dist}(\sigma_t, \Sigma) \ge C > 0$ for every $t \in J_n^1 \cup J_n^2$ and $\min(\mathcal{L}^1(J_n^1), \mathcal{L}^1(J_n^2)) \ge C > 0$ for a constant C independent of n. Next we consider

$$D_n(t) := \deg\bigg(\frac{v_n}{|v_n|}, \sigma_t\bigg)\,.$$

If $2|\delta_n| \ge D_n$ and n large enough, we estimate using (5.27),

$$|D_n(t)| \ge \big||\delta_n| - D_n^+ - D_n^-\big| \ge \frac{1}{2}|\delta_n| \ge C(D_n + |\delta_n|) \quad \text{for every } t \in J_n^2\,. \tag{5.30}$$

In the opposite case $2|\delta_n| < D_n$ (and n large), we have in view of (5.27),

$$|D_n(t)| \ge \big|D_n - D_n^+ - 2D_n^- - |\delta_n|\big| \ge \frac{1}{4}D_n \ge C(D_n + |\delta_n|) \quad \text{for every } t \in J_n^1\,. \tag{5.31}$$

Set $\tilde v_n := v_n/|v_n|$. Using claim *(ii)* in Proposition 2.1, the Coarea Formula and Jensen Inequality, we derive from (5.30) and (5.31) that

$$\begin{aligned}\int_{\mathcal{A}\setminus\cup B_i^n} |\nabla v_n|^2\,dx &\ge C\int_{\mathcal{A}\setminus\cup B_i^n} |\nabla \tilde v_n|^2 |\nabla\xi|\,dx \ge C\int_{J_n^1\cup J_n^2}\bigg(\int_{\sigma_t} |\nabla\tilde v_n|^2\bigg)dt\\ &\ge C\int_{J_n^1\cup J_n^2}\bigg(\int_{\sigma_t} |\tilde v_n \wedge \nabla_\tau \tilde v_n|^2 d\mathcal{H}^1\bigg)dt \ge C\int_{J_n^1\cup J_n^2} \frac{|D_n(t)|^2}{\mathcal{H}^1(\sigma_t)}\,dt \ge C(D_n + |\delta_n|)^2\,,\end{aligned}$$

and (5.28) is proved (here we have also used the fact that $|\nabla\xi|$ does not vanish in $\overline{\mathcal{A}}$).

Step 3. If $\ln|\ln\varepsilon_n| = o(\omega_n)$, the conclusion follows from (5.27) and (5.29). If $\omega_n = O(\ln|\ln\varepsilon_n|)$, we refine the lower bound by growing the vortex balls as already performed in Step 3 of the proof of Proposition 2.2. Following the same arguments with minor modifications yields the announced result, so we omit the details. □

Proof of Theorem 5.2, claim *(i)*. In view of Proposition 5.3 and estimate (5.27), we can follow the proof of Theorem 2.1, considering first a sequence $\{v_n\}_{n\in\mathbb{N}} \subset H^1(\mathcal{D};\mathbb{C}) \cap C^1(\mathcal{A})$ and then the general case. □

5.4. *The lower bound inequality*

Proof of Theorem 5.2, claim *(ii)*. We shall use the notations of Subsection 5.3. Without loss of generality, we may assume that

$$\liminf_{n\to+\infty} \overline{F}_{\varepsilon_n}(v_n) = \lim_{n\to+\infty} \overline{F}_{\varepsilon_n}(v_n) < +\infty\,. \tag{5.32}$$

As in the proof of Theorem 3.1, we may also assume that $\{u_n\}_{n\in\mathbb{N}} \subset H^1(\mathcal{D};\mathbb{C}) \cap C^1(\mathcal{D})$. We shall only consider the case $\ln|\ln\varepsilon_n| = o(\omega_n)$ since the other case can be completed as we already pursued in the proof of Theorem 3.1.

By Proposition 5.3 and (5.21), we have

$$\left|\alpha_n \int_{\mathcal{A}} \nabla^{\perp}\xi \cdot j(v_n)\,dx\right| = O(\omega_n)\,.$$

Then we can argue exactly as in (3.6) to derive that

$$\overline{F}_{\varepsilon_n}(v_n) \geq \frac{1}{2\omega_n^2}\int_{\mathcal{A}_{\varepsilon_n}\setminus\cup B_i^n} |j(v_n)|^2 - \zeta_{\max}\mu(\mathcal{A}) + o(1)\,.$$

Setting $\tilde{j}_n(x)$ as in (3.5) (with $\mathcal{A}_{\varepsilon_n}$ in place of $\mathcal{D}_{\varepsilon_n}$), up to a subsequence we have $\tilde{j}_n \rightharpoonup j_*$ weakly in $L^2(\mathcal{A};\mathbb{R}^2)$ as $n\to+\infty$. By lower semicontinuity, we obtain

$$\liminf_{n\to+\infty} \overline{F}_{\varepsilon_n}(v_n) \geq \frac{1}{2}\int_{\mathcal{A}} |j_*|^2 - \zeta_{\max}\mu(\mathcal{A})\,. \tag{5.33}$$

In addition, arguing as in the proof of (3.8), we deduce that

$$\operatorname{curl} j_* = \mu \quad \text{in } \mathscr{D}'(\mathcal{A})\,,$$

and thus $\mu \in H^{-1}(\mathcal{D})$ since $j_* \in L^2(\mathcal{A};\mathbb{R}^2)$.

Next we introduce $h_\mu \in H_0^1(\mathcal{A})$ to be the unique solution of

$$\begin{cases} -\Delta h_\mu = \mu & \text{in } H^{-1}(\mathcal{A})\,,\\ h_\mu = 0 & \text{on } \partial\mathcal{A}\,,\end{cases}$$

and we set

$$\bar{h}_\mu := h_\mu + \frac{1}{\operatorname{cap}(\mathcal{B})}\left(\int_{\partial\mathcal{B}} j_*\cdot\tau + \frac{\partial h_\mu}{\partial\nu}\right)\xi\,.$$

By construction, we have $\operatorname{curl}(j_* + \nabla^{\perp}\bar{h}_\mu) = 0$ in $H^{-1}(\mathcal{A})$ and $\int_{\partial\mathcal{B}}(j_* + \nabla^{\perp}\bar{h}_\mu)\cdot\tau = 0$. Hence there exists $g \in H^1(\mathcal{A})$ such that $j_* + \nabla^{\perp}\bar{h}_\mu = \nabla g$. Arguing as in the proof of Proposition 5.1, we derive

$$\int_{\mathcal{A}} |j_*|^2 = \int_{\mathcal{A}} |\nabla\bar{h}_\mu|^2 + \int_{\mathcal{A}} |\nabla g|^2 \geq \int_{\mathcal{A}} |\nabla\bar{h}_\mu|^2\,. \tag{5.34}$$

Then using (5.5) and $h_{\mu|\partial\mathcal{A}} = 0$, we obtain $\int_{\mathcal{A}} \nabla h_\mu\cdot\nabla\xi = 0$, so that

$$\int_{\mathcal{A}} |\nabla\bar{h}_\mu|^2 = \int_{\mathcal{A}} |\nabla h_\mu|^2 + \frac{1}{\operatorname{cap}(\mathcal{B})}\left(\int_{\partial\mathcal{B}} j_*\cdot\tau + \frac{\partial h_\mu}{\partial\nu}\right)^2 \geq \int_{\mathcal{A}} |\nabla h_\mu|^2\,. \tag{5.35}$$

Finally, using the Green representation of h_μ we obtain

$$\frac{1}{2}\int_{\mathcal{A}} |\nabla h_\mu|^2\,dx = \frac{1}{2}\iint_{\mathcal{A}\times\mathcal{A}} G(x,y)\,d\mu(x)d\mu(y)\,, \tag{5.36}$$

and the conclusion follows gathering (5.33), (5.34), (5.35) and (5.36). □

5.5. *The upper bound inequality*

Proof of Theorem 5.2, claim *(iii)*. We present here the proof in the case where the measure $\mu \in H^{-1}(\mathcal{A})$ is absolutely continuous with respect to $\mathcal{H}^1 \llcorner \Sigma$, and more precisely for μ of the form (4.2) with a nonvanishing density function f. The general case follows by approximation as already pursued in Section 4.

For such a measure μ we first proceed exactly as in Step 1 of the proof of Proposition 4.1, and we refer to it for the notation. For each integer n, we consider the function

f_n as defined in (4.5) and we set $\hat\mu_n := f_n\mathcal{L}^2 \llcorner \mathcal{A}$. Then $\hat\mu_n \rightharpoonup \mu$ weakly* as measures on $\mathcal{A}$ as $n \to +\infty$. Next we consider the solution h_n of

$$\begin{cases} -\Delta h_n = \omega_n f_n & \text{in } \mathcal{A}\,, \\ h_n = 0 & \text{on } \partial\mathcal{A}\,. \end{cases}$$

Arguing exactly as in Step 2 of the proof of Proposition 4.1, we derive that

$$\frac{1}{2}\int_{\mathcal{A}} |\nabla h_n|^2\,dx \leq \pi D_n|\ln\varepsilon_n| + \omega_n^2 \bar I(\mu) + o(\omega_n^2)\,.$$

Next we introduce

$$\bar h_n := h_n + \frac{\kappa_n}{\mathrm{cap}(\mathcal{B})}\xi \quad \text{with } \kappa_n := \int_{\partial\mathcal{B}} \frac{\partial h_n}{\partial\nu} - 2\pi\left[\frac{1}{2\pi}\int_{\partial\mathcal{B}}\frac{\partial h_n}{\partial\nu}\right]\,,$$

where $[\cdot]$ denotes the integer part. Noticing that $\kappa_n = O(1)$ we deduce

$$\begin{aligned} \frac{1}{2}\int_{\mathcal{A}} |\nabla \bar h_n|^2\,dx = \frac{1}{2}\int_{\mathcal{A}} |\nabla h_n|^2\,dx + \frac{\kappa_n^2}{2\mathrm{cap}(\mathcal{B})^2}\int_{\mathcal{A}}|\nabla\xi|^2\,dx \\ \leq \pi D_n|\ln\varepsilon_n| + \omega_n^2\bar I(\mu) + o(\omega_n^2)\,. \end{aligned} \tag{5.37}$$

In view of (5.6) we have

$$\frac{1}{2\pi}\int_{\partial\mathcal{B}}\frac{\partial\bar h_n}{\partial\nu} = \left[\frac{1}{2\pi}\int_{\partial\mathcal{B}}\frac{\partial h_n}{\partial\nu}\right] \in \mathbb{Z}\,,$$

and since ξ is harmonic in $\mathcal{A}$,

$$\frac{1}{2\pi}\int_{\partial B_{\varepsilon_n}(a_k^n)}\frac{\partial\bar h_n}{\partial\nu} = \frac{1}{2\pi}\int_{\partial B_{\varepsilon_n}(a_k^n)}\frac{\partial h_n}{\partial\nu} = -1 \quad \text{for every } k = 1,\dots,D_n\,.$$

Hence, for any smooth Jordan curve Θ inside $\mathcal{A}\setminus\cup_k\overline{B}_{\varepsilon_n}(a_k^n)$,

$$\frac{1}{2\pi}\int_\Theta \nabla^\perp\bar h_n\cdot\tau \in \mathbb{Z}\,,$$

where $\tau : \Theta \to \mathbb{S}^1$ is any smooth vector field tangent to Θ. Consequently, we may locally define a phase ϕ_n in $\mathcal{A}\setminus\cup_k\overline{B}_{\varepsilon_n}(a_k^n)$ by

$$\nabla\phi_n(x) = -\nabla^\perp\bar h_n(x)\,, \quad x \in \mathcal{A}\setminus\cup_k\overline{B}_{\varepsilon_n}(a_k^n)\,,$$

and then the map $\exp(i\phi_n(x))$ is well defined for every $x \in \mathcal{A}\setminus\cup_k\overline{B}_{\varepsilon_n}(a_k^n)$.

Finally we consider a profile function ρ_n as defined in (4.15) and we set

$$v_n(x) := \begin{cases} \rho_n(x)e^{i\phi_n(x)} & \text{for } x \in \mathcal{A}\setminus\cup_k\overline{B}_{\varepsilon_n}(a_k^n)\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Since $-\Delta\bar h_n = \omega_n\hat\mu_n$ in $\mathcal{A}$, using (5.37) we may proceed as in the proof of Proposition 4.1 Step 3, to prove that

$$\omega_n^{-1}\mathrm{curl}\,j(v_n) \to \mu \quad \text{strongly in } (C_0^{0,1}(\mathcal{A}))^* \text{ as } n \to +\infty\,,$$

and that

$$E_{\varepsilon_n}(v_n,\mathcal{A}) \leq \pi D_n|\ln\varepsilon_n| + \omega_n^2\bar I(\mu) + o(\omega_n^2)\,. \tag{5.38}$$

To evaluate the rotation part of the energy, we first argue as for (4.21) to obtain

$$\Omega_n\int_{\mathcal{A}}\nabla\zeta^\perp\cdot j(v_n)\,dx = -\pi D_n|\ln\varepsilon_n| - \zeta_{\max}\mu(\mathcal{A})\omega_n^2 + o(\omega_n^2)\,. \tag{5.39}$$

In a similar way we derive that

$$\alpha_n \int_{\mathcal{A}} \nabla^\perp \xi \cdot j(v_n)\, dx = -\alpha_n \int_{\mathcal{A}} \nabla \bar{h}_n \cdot \nabla \xi \, dx + o(1) = -\alpha_n \kappa_n + o(1) = O(1)\,, \quad (5.40)$$

since $\alpha_n = O(1)$ and $\kappa_n = O(1)$. Then the conclusion follows gathering (5.38), (5.39) and (5.40). □

5.6. *Application to J_ε- minimizers*

Proof of Corollay 5.1. Let $\varepsilon_n \to 0^+$ be an arbitrary sequence. As for Corollary 1.1, the existence of minimizers for J_{ε_n} and $\overline{F}_{\varepsilon_n}$ is classical, we omit the details. Then for every $n \in \mathbb{N}$, let $u_n \in H^1(\mathcal{A};\mathbb{C})$ be a minimizer of J_{ε_n}. Since $J_{\varepsilon_n}(u_n) \leq \min \mathcal{H}_{\Omega_n} = O(\Omega_n^2)$, we can apply Proposition 5.2 to the sequence $\{u_n\}_{n\in\mathbb{N}}$ to infer that

$$\min \mathcal{H}_{\Omega_n} \geq J_{\varepsilon_n}(u_n) = \min \mathcal{H}_{\Omega_n} + \omega_n^2 \overline{F}_{\varepsilon_n}(v_n) + o(1)\,, \quad (5.41)$$

with $v_n := \bar{u}_n^\star u_n$. On the other hand, for any minimizer $\tilde{v}_n \in H^1(\mathcal{A};\mathbb{C})$ of $\overline{F}_{\varepsilon_n}$, we have $\overline{F}_{\varepsilon_n}(\tilde{v}_n) \leq \overline{F}_{\varepsilon_n}(1) = 0$. Hence Proposition 5.2 yields

$$J_{\varepsilon_n}(u_n) \leq J_{\varepsilon_n}(u_n^\star \tilde{v}_n) = \min \mathcal{H}_{\Omega_n} + \omega_n^2 \min \overline{F}_{\varepsilon_n} + o(1)\,. \quad (5.42)$$

Combining (5.41) with (5.42), we infer that $\omega_n^2 \overline{F}_{\varepsilon_n}(v_n) \leq o(1)$ and that $\{v_n\}$ is a sequence of quasi-minimizers for $\{\overline{F}_{\varepsilon_n}\}$, *i.e.*,

$$\overline{F}_{\varepsilon_n}(v_n) = \min \overline{F}_{\varepsilon_n} + o(1) \quad (5.43)$$

as $n \to +\infty$. By Theorem 5.2 there exists a subsequence $\{\varepsilon_{n_k}\}$ such that

$$\omega_{n_k}^{-1} j(v_{n_k}) \to \mu_0$$

strongly in $(C_0^{0,1}(\mathcal{A}))^*$ for some nonnegative Radon measure $\mu_0 \in H^{-1}(\mathcal{A})$ supprted by Σ. Using (5.43) together with claims *(ii)* and *(iii)* in Theorem 5.2, we deduce as in the proof of Corollary 1.1 that μ_0 minimizes $\mu \mapsto \bar{I}(\mu) - \zeta_{\max}\mu(\mathcal{A})$ over all nonnegative Radon measures supported by Σ. The minimizer is again unique and given by

$$\mu_0 = \frac{\zeta_{\max}}{2\bar{I}_*}\, \bar{\mu}_*\,,$$

whence the convergence of $\omega_n^{-1} j(v_n)$ along the full sequence (recall that $\bar{I}_* = \bar{I}(\mu_*)$ and that $\bar{\mu}_*$ is the minimizer of $\bar{I}$ over all probality measures supported by Σ, see [14]). In addition,

$$\lim_{n\to+\infty} \overline{F}_{\varepsilon_n}(v_n) = \bar{I}(\mu_0) - \zeta_{\max}\mu_0(\mathcal{A}) = -\frac{\zeta_{\max}^2}{4\bar{I}_*}\,,$$

which combined with (5.41) and (5.9) yields (5.16).

Next it remains to prove that

$$\omega_n^{-1} \operatorname{curl} j(u_n) \underset{n\to+\infty}{\longrightarrow} \mu_0$$

strongly in $(C_0^{0,1}(\mathcal{A}))^*$. Indeed, given an arbitrary function $\varphi \in C_0^{0,1}(\mathcal{A})$, we have

$$\begin{aligned} \frac{1}{\omega_n} \int_{\mathcal{A}} j(u_n) \cdot \nabla^\perp \varphi &= \frac{1}{\omega_n} \int_{\mathcal{A}} j(v_n) \cdot \nabla^\perp \varphi + \frac{1}{\omega_n} \int_{\mathcal{A}} |v_n|^2 j(u_n^\star) \cdot \nabla^\perp \varphi \\ &= \langle \mu_0, \varphi \rangle + \frac{1}{\omega_n} \int_{\mathcal{A}} (|v_n|^2 - 1) j(u_n^\star) \cdot \nabla^\perp \varphi + o(1)\,, \end{aligned}$$

where we have used (5.13) and the fact that $\operatorname{curl} j(u_n^\star) = 0$ in $\mathcal{A}$. Arguing as in the proof of Proposition 5.2 we estimate

$$\left| \frac{1}{\omega_n} \int_{\mathcal{A}} (|v_n|^2 - 1) j(u_n^\star) \cdot \nabla^\perp \varphi \right| \leq C \frac{\varepsilon_n}{\omega_n} \|j(u_n^\star)\|_{L^\infty(\mathcal{A})} \sqrt{E_{\varepsilon_n}(v_n)} \leq C \frac{\varepsilon_n \Omega_n^2}{\omega_n} = o(1)\,,$$

where the constant C only depends on φ, and the proof is complete. □

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