ON THE DERIVATION OF NONLINEAR SHELL MODELS FROM THREE-DIMENSIONAL ELASTICITY

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Abstract. A nonlinearly elastic shell is modeled either by the nonlinear three-dimensional shell model or by a nonlinear two-dimensional shell model. We show how such two-dimensional shell models can be derived from the minimization problem associated with the nonlinear three-dimensional shell model. For shells made of a Saint Venant-Kirchhoff material, we obtain in particular the nonlinear shell models of Naghdi, of Koiter, and of Ciarlet-Koiter. Finally, we justify these shell models for small deformations.

Resumé. Pour modéliser le comportement d'une coque non linéairement élastique, on peut utiliser soit le modèle non linéaire de l'élasticité tridimensionnelle, soit un modèle bidimensionnel nonlinéaire de coques. Nous montrons comment de tels modèles bidimensionnels de coques peuvent être déduits du problème de minimisation associé au modèle non linéaire de l'élasticité tridimensionnelle. Pour une coque constituée d'un matériau de Saint Venant-Kirchhoff, nous obtenons en particulier les models de Naghdi, de Koiter, and de Ciarlet-Koiter. Nous justifions enfin ces modèles de coques en petites déformations.

2000 *Mathematics Subject Classification.* 74K25, 74G10, 74G65.
Key words and phrases. Elasticity, shells, energy minimization, Koiter.

1. **INTRODUCTION**

Three-dimensional elasticity predicts the stresses and displacements arising in an elastic body in response to applied body and surface forces by means of a system of nonlinear partial differential equations defined over a three-dimensional domain, say $\mathcal{M}$, representing the configuration of the body in its stress-free state. This system is called *three-dimensional model of nonlinear elasticity.*

The body is called shell when there is a surface $S \subset \mathbb{R}^3$ and a number $\varepsilon > 0$, which is small compared with the characteristic dimensions of S, such that $\mathcal{M} \subset \{x \in \mathbb{R}^3;\ \mathrm{dist}\,(x, S) < \varepsilon\}$. This means that the domain $\mathcal{M}$ lies within a thin neighborhood of the surface S. In this case, the three-dimensional model of nonlinear elasticity is also called *nonlinear 3D shell model.* For small ε, it is possible to replace the nonlinear 3D shell model by a *nonlinear 2D shell model*, i.e., by a system of partial differential equations defined over the surface S. Thus a 2D shell model is only an approximation of the 3D shell model. The accuracy of this approximation varies from one shell to another and no 2D model is valid for all shells. However, shell theory provides a multitude of 2D models that approaches the nonlinear 3D shell model for a large class of shells.

Nonlinear 2D shell models can be derived from the nonlinear 3D shell model either by finding the limit as ε goes to zero of the the nonlinear 3D shell model, or by restricting the range of admissible deformations and stresses used in the nonlinear 3D shell model by means of ad hoc a priori assumptions that are supposed to take into account the smallness of the thickness (e.g., the Cosserat assumptions, the Kirchhoff-Love assumptions, etc.).

As shown by Le Dret & Raoult [10] and Friesecke, James, Mora & Müller [6] (see Pantz [15] for an earlier attempt), the first approach yields *two* limiting nonlinear 2D shell models, valid under two *distinct* sets of assumptions. One of these limiting models is called "nonlinear membrane shell model" and governs shells whose middle surface cannot be deformed without changing their metrics, while the other one is called "nonlinear flexural shell model" and governs shells whose middle surface can be deformed without changing their metrics.

The second approach yields a variety of nonlinear 2D shell models, as, e.g., those of Koiter, Naghdi, etc. As of now, the nonlinear 2D shell models given by the second approach are not rigorously justified, but they are conjectured to govern all shells, that is, irrespectively if their middle surface can, or cannot, be deformed without changing its metric. This alleged property makes these nonlinear 2D shell models more useful than those given by the first approach, since they could be used for instance in computer simulations without first analyzing the "rigidity" of the middle surface of the shell under consideration, which could be a difficult task.

The paper studies the second approach. It provides a simple procedure to derive nonlinear 2D shell models from the nonlinear 3D shell model recast as a minimization problem. This procedure is inspired by the "general shell elements" used in computer simulations, which in effect are discrete shell models obtained from the nonlinear 3D shell model; see Chapelle [1] and the references therein.

By applying this procedure to a shell made of a Saint Venant-Kirchhoff material, we show that the "nonlinear Ciarlet-Koiter shell model", introduced in Ciarlet [3] and justified by means of formal series expansions in Ciarlet & Roquefort [5], is a nonlinear model of Cosserat type, up to some negligible terms. More specifically, it is obtained by minimizing the total energy of the nonlinear 3D shell model over the set of all stresses and deformation fields that satisfy the constraints

$$\Sigma^{33} = 0 \text{ and } \boldsymbol{\Phi}(\cdot, x_3) = \boldsymbol{\varphi} + x_3\boldsymbol{\eta},$$

where Σ^{33} is the normal component of the stress tensor, x_3 is the coordinate along the normal fibers to S, $\boldsymbol{\varphi} : S \to \mathbb{R}^3$, $\boldsymbol{\eta} = \dfrac{\partial_1\boldsymbol{\varphi} \wedge \partial_2\boldsymbol{\varphi}}{|\partial_1\boldsymbol{\theta} \wedge \partial_2\boldsymbol{\theta}|}$, and $\boldsymbol{\theta} : \omega \subset \mathbb{R}^2 \to S \subset \mathbb{R}^3$ is any chart describing locally the surface S (the above expression of $\boldsymbol{\eta}$ is independent of $\boldsymbol{\theta}$).

Finally, we show that the linear part with respect to the displacement field $\boldsymbol{\zeta} := \boldsymbol{\varphi} - \boldsymbol{\theta}$ of the nonlinear Ciarlet-Koiter shell model is a modified version of the linear Koiter model.

The paper is organized as follows. Notation and basic definitions are introduced in the next section. Section 3 introduces the three-dimensional model of nonlinear elasticity in the form of a boundary value problem, then as a minimization problem. In Section 4, we describe a general approach to the derivation of 2D shell models from the three-dimensional model of nonlinear elasticity. Section 5 introduces the nonlinear Koiter shell model together with its variant proposed by Ciarlet [3]. By applying the method described in Section 4 to a shell made of a St Venant-Kirchhoff material, we obtain in Section 6 several 2D shell models that generalize the nonlinear shell models of Naghdi, of Koiter, and of Ciarlet-Koiter. In Section 7, we identify the linear part of the (generalized) Ciarlet-Koiter shell model. In the last section, we summarizes the relationships between the shell models defined throughout the paper.

2. NOTATION AND DEFINITIONS

The space $\mathbb{R}^3$ is equipped with the Euclidean inner product $\boldsymbol{u} \cdot \boldsymbol{v}$ and with the Euclidean norm $|\boldsymbol{u}|$, where $\boldsymbol{u}, \boldsymbol{v}$ denote vectors in $\mathbb{R}^3$. The exterior product of two vectors $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^3$ is denoted $\boldsymbol{u} \wedge \boldsymbol{v}$.

For any integer $n \geq 2$, the symbols $\mathbb{M}^n$, $\mathbb{S}^n$, $\mathbb{M}^n_+$ and $\mathbb{S}^n_>$ respectively designate the space of all square matrices of order n, the space of all symmetric matrices, the set of all matrices $\boldsymbol{A} \in \mathbb{M}^n$ with $\det \boldsymbol{A} > 0$ and the set of all positive-definite symmetric matrices. The notation (a_{ij}) designates the matrix in $\mathbb{M}^n$ with a_{ij} as its element at the i-th row and j-th column. The identity matrix in $\mathbb{M}^n$ is denoted $\boldsymbol{I} := (\delta^i_j)$. The space $\mathbb{M}^n$ and its subspace $\mathbb{S}^n$ are equipped with the inner product $\boldsymbol{A} : \boldsymbol{B} := \sum_{i,j} a_{ij}b_{ij}$, the Frobenius norm $\|\boldsymbol{A}\| := \sqrt{\boldsymbol{A} : \boldsymbol{A}}$, and the spectral norm $|\boldsymbol{A}| := \sup\{|\boldsymbol{A}\boldsymbol{v}|; \boldsymbol{v} \in \mathbb{R}^n, |\boldsymbol{v}| \leq 1\}$, where $\boldsymbol{A} = (a_{ij})$ and $\boldsymbol{B} = (b_{ij})$ denote matrices in $\mathbb{M}^n$. The determinant and the trace of a matrix $\boldsymbol{A} = (a_{ij})$ are denoted $\det \boldsymbol{A}$ and $\operatorname{tr}\boldsymbol{A}$.

Let Ω be an open subset of $\mathbb{R}^n$ and let $x = (x_i)$ denote a generic point in Ω. The gradient of a function $f : \Omega \to \mathbb{R}$ is the vector field $\boldsymbol{\nabla} f := (\partial f/\partial x_i)$, where i is the row index. The gradient of a vector field $\boldsymbol{v} = (v_i) : \Omega \to \mathbb{R}^n$ is the matrix field

$\boldsymbol{\nabla v} := (\partial v_i/\partial x_j)$, where i is the row index, and the divergence of the same vector field is the function $\operatorname{div} \boldsymbol{v} := \sum_i \partial v_i/\partial x_i$. Finally, the divergence of a matrix field $\boldsymbol{T} = (t_{ij}) : \Omega \to \mathbb{M}^n$ is the vector field $\operatorname{\mathbf{div}} \boldsymbol{T} := \left(\sum_{j=1}^n \partial t_{ij}/\partial x_j\right)_{i\in\{1,\dots,n\}}$.

The space of all indefinitely derivable functions $\varphi : \Omega \to \mathbb{R}$ with compact support contained in Ω is denoted $\mathcal{D}(\Omega)$ and the space of all distributions over Ω is denoted $\mathcal{D}'(\Omega)$.

The usual Lebesgue and Sobolev spaces are respectively denoted $L^p(\Omega)$ and $W^{m,p}(\Omega)$ for any integer $m \geq 1$ and any $p \geq 1$. The space $W_0^{m,p}(\Omega)$ is the closure of $\mathcal{D}(\Omega)$ in $W^{m,p}(\Omega)$ and the dual of the space $W_0^{m,p}(\Omega)$ is denoted $W^{-m,p'}(\Omega)$, where $p' = \frac{p}{p-1}$. If the boundary of Ω is Lipschitz-continuous and if $\Gamma_0 \subset \partial\Omega$ is a relatively open subset of the boundary of Ω, we let

$$W_{\Gamma_0}^{1,p}(\Omega) := \{f \in W^{1,p}(\Omega);\ f = 0 \text{ on } \Gamma_0\},$$
$$W_{\Gamma_0}^{2,p}(\Omega) := \{f \in W^{2,p}(\Omega);\ f = \partial_{\boldsymbol{n}} f = 0 \text{ on } \Gamma_0\},$$

where $\partial_{\boldsymbol{n}}$ denote the outer normal derivative operator along $\partial\Omega$ (since Ω is Lipschitz-continuous, a unit outer normal vector $\boldsymbol{n} = (n_i)$ exists $\partial\Omega$-almost everywhere along $\partial\Omega$, and thus $\partial_{\boldsymbol{n}} = \sum_i n_i \frac{\partial}{\partial x_i}$). If $p = 2$, we use the notation $H^m(\Omega) = W^{m,2}(\Omega)$, $H^1_{\Gamma_0}(\Omega) = W^{1,2}_{\Gamma_0}(\Omega)$ and $H^2_{\Gamma_0}(\Omega) = W^{2,2}_{\Gamma_0}(\Omega)$.

For any integer $m \geq 1$ and any open set $\Omega \subset \mathbb{R}^n$, the space of all real-valued functions that are m times continuously differentiable over Ω is denoted $\mathcal{C}^m(\Omega)$. The space $\mathcal{C}^m(\overline{\Omega})$, $m \geq 1$, is defined as that consisting of all vector-valued functions $f \in \mathcal{C}^m(\Omega)$ that, together with all their partial derivatives of order $\leq m$, possess continuous extentions to the closure $\overline{\Omega}$ of Ω. The space of all continuous functions from a topological space X into a finite dimensional vectorial space Y (such as $\mathbb{R}^n$, $\mathbb{M}^n$, etc.) is denoted $\mathcal{C}^0(X;Y)$, or simply $\mathcal{C}^0(X)$ if $Y = \mathbb{R}$. The notation $\mathcal{C}^m(\Omega;Y)$, $\mathcal{C}^m(\overline{\Omega};Y)$, $L^p(\Omega;Y)$ and $W^{m,p}(\Omega;Y)$ designates the spaces of all mappings from Ω into Y whose components in Y are respectively in $\mathcal{C}^m(\Omega)$, $\mathcal{C}^m(\overline{\Omega})$, $L^p(\Omega)$ and $W^{m,p}(\Omega)$.

3. **THE THREE-DIMENSIONAL MODEL OF NONLINEAR ELASTICITY**

In this section, we present the equations governing the deformation of elastic bodies. We follow Ciarlet [2].

Consider an elastic body defined by a reference configuration $\mathcal{M} \subset \mathbb{R}^3$. Assume that the body is kept fixed on a relative open subset $(\partial\mathcal{M})_0 \subset \partial\mathcal{M}$ of its boundary and is subjected to applied forces inside the body and on the part $(\partial\mathcal{M})_1 := \partial\mathcal{M} \setminus (\partial\mathcal{M})_0$ of its lateral boundary. The response of the body to these applied forces is characterized by a *deformation field* $\boldsymbol{\Phi} : \mathcal{M} \to \mathbb{R}^3$ and a *stress field* $\boldsymbol{\Sigma} : \mathcal{M} \to \mathbb{S}^3$ (also known as the second Piola-Kirchhoff stress field) that satisfies the equation

$$\boldsymbol{\Sigma}(m) = \hat{\boldsymbol{\Sigma}}(m, \boldsymbol{\nabla\Phi}(m)) \quad \text{for all } m \in \mathcal{M}, \tag{3.1}$$

where $\hat{\boldsymbol{\Sigma}} : \mathcal{M} \times \mathbb{M}^3_+ \to \mathbb{S}^3$ is the response function of the material. The equation (3.1) is called the constitutive equation of the elastic material.

The applied forces acting on the deformed configuration $\mathbf{\Phi}(\mathcal{M})$ are given by their densities

$$\begin{aligned} \boldsymbol{f}(m) &= \hat{\boldsymbol{f}}(m, \mathbf{\Phi}), \ m \in \mathcal{M}, \\ \boldsymbol{h}(m) &= \hat{\boldsymbol{h}}(m, \mathbf{\Phi}), \ m \in (\partial\mathcal{M})_1 \end{aligned} \tag{3.2}$$

per unit volume and per unit area on the reference configuration, where $\hat{\boldsymbol{f}}$ and $\hat{\boldsymbol{h}}$ are given functions.

One of the main objectives of elasticity theory is to determine the couples $(\mathbf{\Phi}, \mathbf{\Sigma})$ corresponding to static equilibriums of the body in presence of applied forces. Thanks to the stress principle of Euler and Cauchy and Cauchy's theorem, this amounts to solving the following boundary value problem

$$\begin{aligned} -\mathbf{div}\,(\boldsymbol{\nabla}\mathbf{\Phi}\mathbf{\Sigma}) &= \boldsymbol{f} \quad \text{ in } \mathcal{M}, \\ (\boldsymbol{\nabla}\mathbf{\Phi}\mathbf{\Sigma})\boldsymbol{n} &= \boldsymbol{h} \quad \text{ on } (\partial\mathcal{M})_1 \end{aligned} \tag{3.3}$$

where $\boldsymbol{n}$ denotes the exterior unit normal vector field to the boundary of $\mathcal{M}$. The system formed by the equations (3.1)–(3.3) constitutes the *three-dimensional model of nonlinear elasticity.*

If the material constituting the shell is *hyperelastic* and the applied forces are *conservative*, the nonlinear model of three-dimensional elasticity can be recast as a minimization problem. These assumptions imply that there exist functionals W, F and H such that

$$\begin{aligned} W'(\mathbf{\Phi})(\boldsymbol{v}) &= \int_{\mathcal{M}} (\boldsymbol{\nabla}\mathbf{\Phi}\mathbf{\Sigma}) : \boldsymbol{\nabla}\boldsymbol{v}, \\ F'(\mathbf{\Phi})(\boldsymbol{v}) &= \int_{\mathcal{M}} \boldsymbol{f}\cdot\boldsymbol{v}, \\ H'(\mathbf{\Phi})(\boldsymbol{v}) &= \int_{(\partial\mathcal{M})_1} \boldsymbol{h}\cdot\boldsymbol{v} \end{aligned}$$

for all sufficiently smooth vector fields $\boldsymbol{v} : \mathcal{M} \to \mathbb{R}^3$ such that $\boldsymbol{v} = \mathbf{0}$ on $(\partial\mathcal{M})_0$. The notation W', F', H' designates the Gâteaux derivative. The functional W is defined in terms of a given function $\hat{W} : \mathcal{M} \times \mathbb{M}^3_+ \to \mathbb{R}$, called the *stored energy function*, by the relation

$$W(\mathbf{\Phi}) = \int_{\mathcal{M}} \hat{W}(m, \boldsymbol{\nabla}\mathbf{\Phi}(m))dm$$

for all admissible deformations $\mathbf{\Phi}$.

Under these assumptions, the boundary value problem (3.3) is formally equivalent to the Euler equations $J'(\mathbf{\Phi})(\boldsymbol{v}) = \mathbf{0}$, where

$$J(\mathbf{\Phi}) := \int_{\mathcal{M}} \hat{W}(m, \boldsymbol{\nabla}\mathbf{\Phi}(m))dm - F(\mathbf{\Phi}) - H(\mathbf{\Phi}). \tag{3.4}$$

The functional J is called the *total energy* of the body.

The minimization problem associated with the nonlinear model of three dimensional elasticity thus consists in finding the minimizers of the total energy J within the class of all admissible deformations $\mathbf{\Phi}$. The definition of an admissible deformation depends on the function $\hat{W}$ and notably on its coerciveness (an explicit example is given in Section 6).

To sum up, we found that the couples $(\mathbf{\Phi}, \mathbf{\Sigma})$ formed by a minimizer $\mathbf{\Phi}$ of the total energy J and $\mathbf{\Sigma}(m) = \hat{\mathbf{\Sigma}}(m, \boldsymbol{\nabla}\mathbf{\Phi}(m))$, $m \in \mathcal{M}$, are solutions to the three-dimensional problem of nonlinear elasticity.

The total energy J and the constitutive equation (3.1) of the hyperelastic material can be recast in a form more convenient for our purpose by taking into account the principle of material frame-indifference. This principle implies that there exist functions $\tilde{W} : \mathcal{M} \times \mathbb{S}^3_> \to \mathbb{R}$ and $\tilde{\boldsymbol{\Sigma}} : \mathcal{M} \times \mathbb{S}^3_> \to \mathbb{S}^3$ such that

$$\hat{W}(m, \boldsymbol{\nabla}\boldsymbol{\Phi}(m)) = \tilde{W}(m, \boldsymbol{C}(m)), \quad m \in \mathcal{M},$$
$$\boldsymbol{\Sigma}(m) = \hat{\boldsymbol{\Sigma}}(m, \boldsymbol{\nabla}\boldsymbol{\Phi}(m)) = \tilde{\boldsymbol{\Sigma}}(m, \boldsymbol{C}(m)), \quad m \in \mathcal{M},$$

with $\boldsymbol{C} = \boldsymbol{\nabla}\boldsymbol{\Phi}^T \boldsymbol{\nabla}\boldsymbol{\Phi}$, for all admissible deformation $\boldsymbol{\Phi} : \mathcal{M} \to \mathbb{R}^3$. This means that J and $\boldsymbol{\Sigma}$ depend only on the strain field $\boldsymbol{C}$ (also known as the metric tensor). In fact, it is more convenient for our purpose to recast the above relations as

$$\hat{W}(m, \boldsymbol{\nabla}\boldsymbol{\Phi}(m)) = \mathcal{W}(m, \boldsymbol{C}(m), \boldsymbol{\Sigma}(m)), \quad m \in \mathcal{M},$$
$$\boldsymbol{\Sigma}(m) = \tilde{\boldsymbol{\Sigma}}(m, \boldsymbol{C}(m)), \quad m \in \mathcal{M},$$

for some function $\mathcal{W} : \mathcal{M} \times \mathbb{S}^3_> \times \mathbb{S}^3 \to \mathbb{R}$ (this is always possible since one can for instance define $\mathcal{W}(\cdot, \cdot, \boldsymbol{\Sigma}) := \hat{W}$ for all $\boldsymbol{\Sigma} \in \mathbb{S}^3$). Thus the total energy J can be expressed as a functional of $\boldsymbol{\Phi}$, $\boldsymbol{C}$ and $\boldsymbol{\Sigma}$ by letting

$$J(\boldsymbol{\Phi}) = \mathcal{J}(\boldsymbol{\Phi}, \boldsymbol{C}, \boldsymbol{\Sigma}),$$

where

$$\mathcal{J}(\boldsymbol{\Phi}, \boldsymbol{C}, \boldsymbol{\Sigma}) := \int_{\mathcal{M}} \mathcal{W}(m, \boldsymbol{C}(m), \boldsymbol{\Sigma}(m)) dm - F(\boldsymbol{\Phi}) - H(\boldsymbol{\Phi}). \tag{3.5}$$

In this setting, solving the nonlinear model of three-dimensional elasticity consists in finding the minimizers of the total energy $\mathcal{J}$ within the set

$$\begin{aligned} \mathcal{A}(\mathcal{M}) := \{&(\boldsymbol{\Phi}, \boldsymbol{C}, \boldsymbol{\Sigma}) \in V(\mathcal{M}; \mathbb{R}^3 \times \mathbb{S}^3_> \times \mathbb{S}^3); \\ &\boldsymbol{C} = \boldsymbol{\nabla}\boldsymbol{\Phi}^T \boldsymbol{\nabla}\boldsymbol{\Phi},\ \boldsymbol{\Sigma} = \tilde{\boldsymbol{\Sigma}}(\cdot, \boldsymbol{C})\}, \end{aligned} \tag{3.6}$$

where the set $V(\mathcal{M}; \mathbb{R}^3 \times \mathbb{S}^3_> \times \mathbb{S}^3)$ is determined by the condition that $\mathcal{W}(\cdot, \boldsymbol{C}, \boldsymbol{\Sigma}) \in L^1(\mathcal{M})$ and by the boundary condition of $\boldsymbol{\Phi}$ on $(\partial\mathcal{M})_0$. An example is given in Section 6.

4. A GENERAL APPROACH TO THE DERIVATION OF SHELL MODELS

This section shows how to construct 2D shell models. We follow an idea that is used in the engineering practice and numerical computations to devise "general shell elements" (see, e.g, Chapelle [1]), though we apply it to the minimization problem, rather than variational problem, of shells.

We consider an elastic body that in its stress-free state occupies a domain of the form

$$\mathcal{M} := \{s + x_3 \boldsymbol{\nu}(s);\ s \in S,\ -\varepsilon \le x_3 \le \varepsilon\},$$

where S is a surface with boundary immersed in $\mathbb{R}^3$, $\boldsymbol{\nu} : S \to \mathbb{R}^3$ is a continuous unit vector field normal to S, and $\varepsilon > 0$ is a parameter. Such a body is called *shell* with middle surface S and thickness 2ε. Note that the results of this section also apply to shells with variable thickness, i.e., when $\mathcal{M} := \{s + x_3 \boldsymbol{\nu}(s);\ s \in S,\ -\varepsilon t(s) \le x_3 \le \varepsilon t(s)\}$ for some given function $t : S \to [0, \infty)$ that is sufficiently smooth.

The minimization problem introduced in the previous section, which models the behavior of the shell $\mathcal{M}$ in presence of applied forces, will henceforth be called *nonlinear 3D shell model*. When the thickness 2ε is small with respect to the characteristic dimensions of S, the nonlinear 3D shell model can be approximated with another minimization problem that is defined solely on the middle surface S, instead of the three-dimensional domain $\mathcal{M}$. Such a minimization problem will be called *nonlinear 2D shell model*. This section shows how to derive nonlinear 2D shell models from the nonlinear 3D shell model.

The idea is to minimize the total energy $\mathcal{J}$ over a subset of $\mathcal{A}(\mathcal{M})$ whose elements have a prescribed dependence in the transverse variable x_3. Since x_3 is close to zero, the Taylor formula suggests that a polynomial dependence would be appropriate. Given any integer $N \geq 1$, we define the subset $\mathcal{A}^N(\mathcal{M}) \subset \mathcal{A}(\mathcal{M})$ as follows: a triple $(\boldsymbol{\Phi}, \boldsymbol{C}, \boldsymbol{\Sigma}) \in V(\mathcal{M}; \mathbb{R}^3 \times \mathbb{S}^3_> \times \mathbb{S}^3)$ belongs to $\mathcal{A}^N(\mathcal{M})$ if

$$\begin{aligned}
&\boldsymbol{\Phi}(s + x_3\boldsymbol{\nu}(s)) = \sum_{k=0}^{N} (x_3)^k \boldsymbol{\varphi}_k(s), \quad \text{with } \boldsymbol{\varphi}_k : S \to \mathbb{R}^3,\\
&\boldsymbol{C} = \tilde{\boldsymbol{C}}^N(\boldsymbol{\Phi}),\\
&\boldsymbol{\Sigma} = \tilde{\boldsymbol{\Sigma}}(\cdot, \boldsymbol{C}),
\end{aligned}$$

where $\tilde{\boldsymbol{\Sigma}}$ is the function appearing in the definition (3.6) of $\mathcal{A}(\mathcal{M})$ and $\tilde{\boldsymbol{C}}^N$ is defined in terms of $\boldsymbol{\Phi}$ by a specific formula depending on N. For $N \geq 2$, this formula is the same as that appearing in the definition of $\mathcal{A}(\mathcal{M})$, i.e., $\tilde{\boldsymbol{C}}^N(\boldsymbol{\Phi}) := \boldsymbol{\nabla}\boldsymbol{\Phi}^T\boldsymbol{\nabla}\boldsymbol{\Phi}$. By contrast, for $N = 1$, this formula differs from that appearing in the definition of $\mathcal{A}(\mathcal{M})$ since $\tilde{\boldsymbol{C}}^1$ must be defined in such a way that the normal component of $\tilde{\boldsymbol{\Sigma}}(\cdot, \tilde{\boldsymbol{C}}^1(\boldsymbol{\Phi}))$ vanishes (this condition is commented upon in the last paragraph of the section). The precise definition is the following:

Definition 4.1. The *nonlinear 2D shell model* corresponding to $N \geq 1$ consists in minimizing the total energy $\mathcal{J}$ defined by (3.5) over the set

$$\begin{aligned}
\mathcal{A}^N(\mathcal{M}) := \{&(\boldsymbol{\Phi}, \boldsymbol{C}, \boldsymbol{\Sigma}) \in V(\mathcal{M}; \mathbb{R}^3 \times \mathbb{S}^3_> \times \mathbb{S}^3);\\
&\boldsymbol{\Phi}(\cdot, x_3) = \sum_{k=0}^{N} (x_3)^k \boldsymbol{\varphi}_k,\ \boldsymbol{\varphi}_k : S \to \mathbb{R}^3,\\
&\boldsymbol{C} = \tilde{\boldsymbol{C}}^N(\boldsymbol{\Phi}),\ \boldsymbol{\Sigma} = \tilde{\boldsymbol{\Sigma}}(\cdot, \boldsymbol{C})\},
\end{aligned} \tag{4.1}$$

where $\tilde{\boldsymbol{C}}^N(\boldsymbol{\Phi}) := \boldsymbol{\nabla}\boldsymbol{\Phi}^T\boldsymbol{\nabla}\boldsymbol{\Phi}$ if $N \geq 2$ and $\tilde{\boldsymbol{C}}^1(\boldsymbol{\Phi}) = \tilde{C}^1_{ij}\mathbf{g}^i \otimes \mathbf{g}^j$ is the tensor field whose coefficients $\tilde{C}^1_{ij}$ satisfy the system

$$\begin{aligned}
&\tilde{C}^1_{i\alpha} = \tilde{C}^1_{\alpha i} = [\boldsymbol{\nabla}^T\boldsymbol{\Phi}\boldsymbol{\nabla}\boldsymbol{\Phi}]_{i\alpha}\ ,\\
&[\tilde{\Sigma}(\cdot, \tilde{C}^1_{ij}\mathbf{g}^i \otimes \mathbf{g}^j]_{33} = 0.
\end{aligned} \tag{4.2}$$

Here, $\mathbf{g}^i : \mathcal{M} \to \mathbb{R}^3$, $i \in \{1, 2, 3\}$, denote any linearly independent smooth vector fields that satisfy $\mathbf{g}^3(m) = \boldsymbol{\nu}(s)$ and $\mathbf{g}^\alpha(m) \cdot \boldsymbol{\nu}(s) = 0$ for all $m = s + x_3\boldsymbol{\nu}(s) \in \mathcal{M}$, and $[\boldsymbol{T}]_{ij}$ denote the components of the matrix field $\boldsymbol{T} : \mathcal{M} \to \mathbb{M}^3$ appearing in the decomposition $\boldsymbol{T} = [T]_{ij}\mathbf{g}^i \otimes \mathbf{g}^j$.

Note that the components $\tilde{C}^1_{i\alpha} = \tilde{C}^1_{\alpha i}$ are defined explicitly by the first equation of (4.2), while the component $\tilde{C}^1_{33}$ is a function of $\tilde{C}^1_{i\alpha}$ defined implicitly by the second equation of (4.2). Note also that $\tilde{\boldsymbol{C}}^1(\boldsymbol{\Phi}) \neq \boldsymbol{\nabla\Phi}^T\boldsymbol{\nabla\Phi}$.

Remarks 4.2. a) The distinction between $N \geq 2$ and $N = 1$ in Definition 4.1 is motivated by the theory of linearly elastic shells. For if j denotes the total energy associated with the *linear* 3D shell model, then minimizing j over the linear counterpart of the set $\mathcal{A}^N(\mathcal{M})$, as defined in Definition 4.1, yields a correct linear 2D shell model for all $N \geq 1$. By contrast, if one changes the definition of $\mathcal{A}^N(\mathcal{M})$ by letting $\tilde{\boldsymbol{C}}^N(\boldsymbol{\Phi}) = \boldsymbol{\nabla}^T\boldsymbol{\Phi}\boldsymbol{\nabla\Phi}$ for *all* $N \geq 1$, then the corresponding linear 2D shell model is incorrect for $N = 1$.

b) The case $N = 0$ is excluded in Definition 4.1 because whatever the definition of $\boldsymbol{C}$ and $\boldsymbol{\Sigma}$ in terms of $\boldsymbol{\Phi}$, the corresponding minimization problem fails to approach the nonlinear 3D shell model for an arbitrary shell. Specifically, $\boldsymbol{C}$ and $\boldsymbol{\Sigma}$ can be defined in such a way that the corresponding 2D shell model does approach the nonlinear 3D shell model in the case of a "nonlinearly membrane shell", but fails to do so in the case of a "nonlinearly flexural shell".

c) The equation $\boldsymbol{\Sigma} = \tilde{\boldsymbol{\Sigma}}(\cdot, \tilde{\boldsymbol{C}}^1(\boldsymbol{\Phi}))$ can be viewed as a "two-dimensional constitutive equation" relating the stress tensor field and the deformation field $\boldsymbol{\Phi}$.

The 2D shell model defined by Definition 4.1 with $N = 1$, which can be viewed as a shell model of Cosserat or Reissner-Mindlin type, is based on two a priori assumptions. The first one asserts that any point situated on a normal fiber to S remains on a straight line, not necessarily normal, passing through the deformed middle surface after the deformation has taken place. The second assumption asserts that the normal component of the stress tensor $\tilde{\boldsymbol{\Sigma}} : \mathcal{M} \to \mathbb{S}^3$ inside the shell vanishes. This assumption is justified by estimates, due to John [7, 8], showing that the ratio of the normal stress tensor over the tangent stress tensor approaches zero as the thickness of the shell goes to zero.

5. THE NONLINEAR SHELL MODELS OF KOITER AND CIARLET-KOITER

In preparation of the next section, we recall the definition of the nonlinear shell models of Koiter and Ciarlet-Koiter. For ease of exposition, we restrict our presentation to shells whose middle surface can be described by a single chart. This means that there exists an embedding $\boldsymbol{\theta} : \overline{\omega} \to \mathbb{R}^3$ of class $\mathcal{C}^3$ such that $S = \boldsymbol{\theta}(\overline{\omega})$. We assume that $\boldsymbol{\theta}$ is of class $\mathcal{C}^3$ in $\overline{\omega}$ and that $\omega \subset \mathbb{R}^3$ is bounded, connected, with a Lipschitz-continuous boundary. A generic point in $\overline{\omega}$ is denoted $y = (y_1, y_2)$ and partial derivatives are denoted $\partial_\alpha := \partial/\partial y_\alpha$. In this setting, the reference configuration of the shell is the image $\mathcal{M} = \boldsymbol{\Theta}(\overline{\Omega})$, where $\Omega := \omega \times (-\varepsilon, \varepsilon)$, $\boldsymbol{\Theta} : \overline{\Omega} \to \mathbb{R}^3$ is defined by

$$\boldsymbol{\Theta}(y, x_3) = \boldsymbol{\theta}(y) + x_3\mathbf{a}_3(y) \quad \text{for all } (y, x_3) \in \overline{\omega} \times [-\varepsilon, \varepsilon],$$

and

$$\mathbf{a}_3(y) := \frac{\mathbf{a}_1(y) \wedge \mathbf{a}_2(y)}{|\mathbf{a}_1(y) \wedge \mathbf{a}_2(y)|}, \ \mathbf{a}_\alpha(y) = \partial_\alpha\boldsymbol{\theta}(y).$$

Note that the vector field $\mathbf{a}_3(\boldsymbol{\theta}(y)) := \mathbf{a}_3(y)$ is a smooth unit normal vector field to the middle surface S of the shell. Since the vector field $\boldsymbol{\nu}$ introduced in Section 4 is another smooth unit normal vector field to the same surface, we have either $\boldsymbol{\nu} = \mathbf{a}_3$ or $\boldsymbol{\nu} = -\mathbf{a}_3$. Hence there is no loss in generality if we assume henceforth that $\boldsymbol{\nu} = \mathbf{a}_3$ on S.

We assume that the shell is kept fixed on a part $(\partial\mathcal{M})_0 = \boldsymbol{\Theta}(\Gamma_0)$ of its lateral boundary, where $\Gamma_0 = \gamma_0 \times (\varepsilon, \varepsilon)$ and $\gamma_0 \subset \partial\omega$ is a non empty relative open set of the boundary of ω. Finally, we assume that the shell is subjected to applied body forces of density $\boldsymbol{f} : \Omega \to \mathbb{R}^3$ per unit volume and that no surface forces act on the boundary of the shell (i.e., $\boldsymbol{h} = \mathbf{0}$). Note that the density $\boldsymbol{f}$ does not depend on the deformation field $\boldsymbol{\Phi}$, which means that we only consider dead forces.

The nonlinear partial differential equations proposed by Koiter [9] for modeling such an elastic shell is based on the assumptions that the stress inside the shell is planar and parallel to the middle surface and that the deformation of the shell satisfies the Kirchhoff-Love assumption. Using these assumptions to simplify the nonlinear 3D shell model and neglecting some of the terms of a lesser order of magnitude as the principal ones, W.T. Koiter reached the conclusion that the deformation field $\boldsymbol{\varphi} : \overline{\omega} \to \mathbb{R}^3$ of the middle surface of the shell should be a minimizer, over a set of smooth enough vector fields $\boldsymbol{\psi} : \overline{\omega} \to \mathbb{R}^3$ satisfying appropriate boundary conditions on the boundary of S, of the functional J_K defined by (cf. Koiter [9, eqn. (4.2),(8.1) and (8.3)]):

$$J_K(\boldsymbol{\psi}) = \frac{1}{2}\int_\omega a^{\alpha\beta\sigma\tau}\Big\{\varepsilon G_{\sigma\tau}(\boldsymbol{\psi})G_{\alpha\beta}(\boldsymbol{\psi}) + \frac{\varepsilon^3}{3}R_{\sigma\tau}(\boldsymbol{\psi})R_{\alpha\beta}(\boldsymbol{\psi})\Big\}\sqrt{a}dy - \int_\omega\Big\{\int_{-\varepsilon}^{\varepsilon}\boldsymbol{f}(\cdot, x_3)dx_3\Big\}\cdot\boldsymbol{\psi}\sqrt{a}dy. \tag{5.1}$$

In this formula, Greek indices and exponents range in the set $\{1, 2\}$ and the summation convention with respect to repeated indices and exponents is used. The functions $G_{\alpha\beta}(\boldsymbol{\psi})$ and $R_{\alpha\beta}(\boldsymbol{\psi})$, denoting respectively the covariant components of the change of metric tensor field and the covariant components of the change of curvature tensor field associated with the deformation field $\boldsymbol{\psi}$, are defined by

$$\begin{aligned} G_{\alpha\beta}(\boldsymbol{\psi}) &:= \frac{1}{2}(a_{\alpha\beta}(\boldsymbol{\psi}) - a_{\alpha\beta}), \\ R_{\alpha\beta}(\boldsymbol{\psi}) &:= b_{\alpha\beta}(\boldsymbol{\psi}) - b_{\alpha\beta}, \end{aligned} \tag{5.2}$$

where $a_{\alpha\beta} := a_{\alpha\beta}(\boldsymbol{\theta})$, $b_{\alpha\beta} := b_{\alpha\beta}(\boldsymbol{\theta})$, and

$$a_{\alpha\beta}(\boldsymbol{\psi}) := \partial_\alpha\boldsymbol{\psi}\cdot\partial_\beta\boldsymbol{\psi} \quad\text{and}\quad b_{\alpha\beta}(\boldsymbol{\psi}) := \partial_{\alpha\beta}\boldsymbol{\psi}\cdot\frac{\partial_1\boldsymbol{\psi}\wedge\partial_2\boldsymbol{\psi}}{|\partial_1\boldsymbol{\psi}\wedge\partial_2\boldsymbol{\psi}|} \tag{5.3}$$

(the functions $a_{\alpha\beta}(\boldsymbol{\psi})$ and $b_{\alpha\beta}(\boldsymbol{\psi})$ respectively denote the covariant components of the metric tensor and the covariant components of the second fundamental form of the surface $\boldsymbol{\psi}(\overline{\omega})$). The functions $a^{\alpha\beta\sigma\tau}$ denote the contravariant components of the two-dimensional elasticity tensor. They are defined in terms of the Lamé constants $\lambda > 0$ and $\mu > 0$ characterizing the St Venant-Kirchhoff material constituting the shell by the expressions

$$a^{\alpha\beta\sigma\tau} := \frac{4\lambda\mu}{\lambda + 2\mu}a^{\alpha\beta}a^{\sigma\tau} + 2\mu(a^{\alpha\sigma}a^{\beta\tau} + a^{\alpha\tau}a^{\beta\sigma}). \tag{5.4}$$

Finally, $\sqrt{a}dy$ denotes the area element along the surface $S = \boldsymbol{\theta}(\overline{\omega})$.

From the definition of $b_{\alpha\beta}(\boldsymbol{\psi})$, one can see that the nonlinear Koiter model is well defined only for those deformations fields $\boldsymbol{\psi}$ that satisfy

$$\partial_1\boldsymbol{\psi}\wedge\partial_2\boldsymbol{\psi}\neq\mathbf{0}\ \text{ a.e. in } \omega.$$

To surmount this difficulty, Ciarlet [3] introduced a new nonlinear shell model by replacing the functions $R_{\alpha\beta}(\boldsymbol{\psi})$ appearing in the nonlinear Koiter's model with

$$\begin{aligned} R^\sharp_{\alpha\beta}(\boldsymbol{\psi}) &:= b^\sharp_{\alpha\beta}(\boldsymbol{\psi}) - b_{\alpha\beta},\\ b^\sharp_{\alpha\beta}(\boldsymbol{\psi}) &:= \partial_{\alpha\beta}\boldsymbol{\psi}\cdot\frac{\partial_1\boldsymbol{\psi}\wedge\partial_2\boldsymbol{\psi}}{|\partial_1\boldsymbol{\theta}\wedge\partial_2\boldsymbol{\theta}|}. \end{aligned} \tag{5.5}$$

Thus the nonlinear shell model of Ciarlet-Koiter consists in minimizing the functional

$$\begin{aligned} J^\sharp_K(\boldsymbol{\psi}) = \frac{1}{2}\int_\omega a^{\alpha\beta\sigma\tau}\Big\{\varepsilon G_{\sigma\tau}(\boldsymbol{\psi})G_{\alpha\beta}(\boldsymbol{\psi}) + \frac{\varepsilon^3}{3}R^\sharp_{\sigma\tau}(\boldsymbol{\psi})R^\sharp_{\alpha\beta}(\boldsymbol{\psi})\Big\}\sqrt{a}dy\\ - \int_\omega\Big\{\int_{-\varepsilon}^{\varepsilon}\boldsymbol{f}(\cdot,x_3)dx_3\Big\}\cdot\boldsymbol{\psi}\sqrt{a}dy. \end{aligned} \tag{5.6}$$

This model was justified in Ciarlet & Roquefort [5] by means of formal series expansions. In the next section, we show that this model is the "principal part" of a nonlinear 2D shell model found by the method described in Section 4.

6. THE GENERALIZED NONLINEAR SHELL MODELS OF NAGHDI, OF KOITER, AND OF CIARLET-KOITER

In this section, we apply Definition 4.1 to a shell made of a St Venant-Kirchhoff material. We show that the model obtained for $N = 1$ generalizes the nonlinear shell models of Naghdi, of Koiter, and of Ciarlet-Koiter.

To begin with, we explicit the nonlinear 2D shell model introduced in Definition 4.1. Since the shell is made of a St Venant-Kirchhoff material, the functions $\hat{\boldsymbol{\Sigma}}$ and $\mathcal{W}$ appearing in (3.5) and (4.1) are defined by

$$\begin{aligned} \tilde{\boldsymbol{\Sigma}}(\cdot,\boldsymbol{C}) &= \frac{\lambda}{2}\big\{\mathrm{tr}(\boldsymbol{C}-\boldsymbol{I})\big\}\boldsymbol{I} + \mu(\boldsymbol{C}-\boldsymbol{I}) \ \text{ for all } \boldsymbol{C}\in\mathbb{S}^3_>,\\ \mathcal{W}(\cdot,\boldsymbol{C},\boldsymbol{\Sigma}) &= \frac{1}{4}\boldsymbol{\Sigma}:(\boldsymbol{C}-\boldsymbol{I}), \end{aligned}$$

where $\boldsymbol{I}$ is the identity matrix in $\mathbb{M}^3$, tr$\mathbf{E}$ denotes the trace of the matrix $\mathbf{E}\in\mathbb{M}^3$, and $\lambda > 0$ and $\mu > 0$ are the Lamé constants of the St Venant-Kirchhoff material. These formulas imply that the space $V(\mathcal{M};\mathbb{R}^3\times\mathbb{S}^3_>\times\mathbb{S}^3)$ appearing in the definitions (3.6), (4.1) and (4.1) is defined by

$$\begin{aligned} V(\mathcal{M};\mathbb{R}^3\times\mathbb{S}^3_>\times\mathbb{S}^3) := \{&\boldsymbol{\Phi}\in W^{1,4}(\mathcal{M};\mathbb{R}^3),\ \boldsymbol{C}\in L^2(\mathcal{M};\mathbb{S}^3_>),\\ &\boldsymbol{\Sigma}\in L^2(\mathcal{M};\mathbb{S}^3);\ \boldsymbol{\Phi}(m) = m \text{ for all } m\in(\partial\mathcal{M})_0\}. \end{aligned}$$

Indeed, if $\boldsymbol{\Phi}:\mathcal{M}\to\mathbb{R}^3$ is a generic deformation field and $\boldsymbol{C} = \boldsymbol{\nabla\Phi}^T\boldsymbol{\nabla\Phi}$ and $\boldsymbol{\Sigma} = \hat{\boldsymbol{\Sigma}}(\cdot,\boldsymbol{C})$, then

$$\begin{aligned} \mathcal{W}(\cdot,\boldsymbol{C},\boldsymbol{\Sigma}) &= \frac{\lambda}{8}\big\{\mathrm{tr}(\boldsymbol{C}-\boldsymbol{I})\big\}^2 + \frac{\mu}{4}\|\boldsymbol{C}-\boldsymbol{I}\|^2\\ &\geq \frac{\mu}{4}\Big(\frac{1}{2}\|\boldsymbol{C}\|^2 - \|\boldsymbol{I}\|^2\Big) \geq \frac{\mu}{8}|\boldsymbol{\nabla\Phi}|^4 - \frac{3\mu}{4}. \end{aligned}$$

Combined with the fact that the set $\mathcal{M}$ is bounded, this shows that $\mathcal{W}(\cdot, \boldsymbol{C}, \boldsymbol{\Sigma})$ belongs to the space $L^1(\mathcal{M})$ if, and only if, $\boldsymbol{\Phi} \in W^{1,4}(\mathcal{M})$.

Now we explicit the definition of the space $\mathcal{A}^1(\mathcal{M})$ defined by (4.1). It suffices to compute the solution $\tilde{\boldsymbol{C}}^1(\boldsymbol{\Phi})$ of the system (4.2).

Since $\boldsymbol{\theta}$ is an embedding, the vector fields $\mathbf{g}_\alpha := \partial_\alpha \boldsymbol{\Theta}$ and $\mathbf{g}_3 = \mathbf{a}_3$ are linearly independent for sufficiently small ε. Thus they form a basis in $\mathbb{R}^3$ at all $m \in \mathcal{M}$ and the vector fields $\mathbf{g}^i : \mathcal{M} \to \mathbb{R}^3$ defined by $\mathbf{g}^i \cdot \mathbf{g}_j = \delta^i_j$ also form a basis there. Moreover $\mathbf{g}_3 = \mathbf{g}^3 = \mathbf{a}_3 = \boldsymbol{\nu}$ and $\mathbf{g}^\alpha \cdot \boldsymbol{\nu} = 0$ at every point of $\mathcal{M}$. Let $g_{ij} := \mathbf{g}_i \cdot \mathbf{g}_j$, $g^{ij} := \mathbf{g}^i \cdot \mathbf{g}^j$, and $g := \det(g_{ij})$. Then the tensor field $\tilde{\boldsymbol{C}}^1(\boldsymbol{\Phi}) = \tilde{C}^1_{ij} \mathbf{g}^i \otimes \mathbf{g}^j$ appearing in the definition of $\mathcal{A}^1(\mathcal{M})$ is defined by the solution $\tilde{C}^1_{ij}$ to the system

$$\begin{aligned} &\tilde{C}^1_{i\alpha} = \tilde{C}^1_{\alpha i} = \partial_i \boldsymbol{\Phi} \cdot \partial_\alpha \boldsymbol{\Phi}, \\ &\frac{\lambda}{2}\big\{\operatorname{tr}(\tilde{\boldsymbol{C}}^1(\boldsymbol{\Phi}) - \boldsymbol{I})\big\} + \mu(\tilde{C}^1_{33} - 1) = 0. \end{aligned}$$

The last relation being equivalent to

$$\lambda(g^{\alpha\beta}\tilde{C}^1_{\alpha\beta} + \tilde{C}^1_{33} - 3) + 2\mu(\tilde{C}^1_{33} - 1) = 0,$$

we have

$$\begin{aligned} &\tilde{C}^1_{i\alpha} = \tilde{C}^1_{\alpha i} = \partial_i \boldsymbol{\Phi} \cdot \partial_\alpha \boldsymbol{\Phi}, \\ &\tilde{C}^1_{33} = 1 - \frac{\lambda}{\lambda + 2\mu}\Big(g^{\alpha\beta}\partial_\alpha \boldsymbol{\Phi} \cdot \partial_\beta \boldsymbol{\Phi} - 2\Big). \end{aligned} \tag{6.1}$$

The above relations show that the space $\mathcal{A}^1(\mathcal{M})$ defined by (4.1) is the following

$$\begin{aligned} \mathcal{A}^1(\mathcal{M}) := \{(\boldsymbol{\Phi}, \boldsymbol{C}, \boldsymbol{\Sigma});\ &\boldsymbol{\Phi}(\cdot, x_3) = \boldsymbol{\varphi}_0 + x_3\boldsymbol{\varphi}_1,\ \boldsymbol{\varphi}_0, \boldsymbol{\varphi}_1 \in W^{1,4}_{\gamma_0}(\omega; \mathbb{R}^3), \\ &\boldsymbol{C} = \tilde{C}^1_{ij}\mathbf{g}^i \otimes \mathbf{g}^j,\ \boldsymbol{\Sigma} = \frac{\lambda}{2}\big\{\operatorname{tr}(\boldsymbol{C} - \boldsymbol{I})\big\}\boldsymbol{I} + \mu(\boldsymbol{C} - \boldsymbol{I})\}, \end{aligned} \tag{6.2}$$

where $\tilde{C}^1_{ij}$ are the functions defined by (6.1). A straightforward calculation shows that the above expression of the tensor field $\boldsymbol{\Sigma}$ is equivalent to $\boldsymbol{\Sigma} = \tilde{\Sigma}^{1,ij}\mathbf{g}_i \otimes \mathbf{g}_j$, where

$$\begin{aligned} &\tilde{\Sigma}^{1,\alpha\beta} = \frac{\lambda\mu}{\lambda + 2\mu}(g^{\sigma\tau}\partial_\sigma \boldsymbol{\Phi} \cdot \partial_\tau \boldsymbol{\Phi} - 2)g^{\alpha\beta} + \mu g^{\alpha\sigma}g^{\beta\tau}(\partial_\sigma \boldsymbol{\Phi} \cdot \partial_\tau \boldsymbol{\Phi} - g_{\sigma\tau}), \\ &\tilde{\Sigma}^{1,\alpha 3} = \tilde{\Sigma}^{1,3\alpha} = \mu g^{\alpha\beta}\partial_\beta \boldsymbol{\Phi} \cdot \partial_3 \boldsymbol{\Phi}, \\ &\tilde{\Sigma}^{1,33} = 0. \end{aligned}$$

The first equation of the above system can be written in the simpler form

$$\tilde{\Sigma}^{1,\alpha\beta} = \frac{1}{2}A^{\alpha\beta\sigma\tau}_{2D}(\partial_\sigma \boldsymbol{\Phi} \cdot \partial_\tau \boldsymbol{\Phi} - g_{\sigma\tau}),$$

where

$$A^{\alpha\beta\sigma\tau}_{2D} := \frac{2\lambda\mu}{\lambda + 2\mu}g^{\sigma\tau}g^{\alpha\beta} + \mu(g^{\alpha\sigma}g^{\beta\tau} + g^{\alpha\tau}g^{\beta\sigma}).$$

The above argument can be summarized in the following

Theorem 6.1. *The nonlinear 2D shell model obtained by applying* Definition 4.1 *with $N = 1$ to a St Venant-Kirchhoff material consists in minimizing the functional*

$$\begin{aligned}\mathcal{J}(\mathbf{\Phi}, \boldsymbol{C}, \mathbf{\Sigma}) &= \int_{\mathcal{M}} \frac{1}{4}\mathbf{\Sigma} : (\boldsymbol{C} - \boldsymbol{I})\ dm - \int_{\mathcal{M}} \boldsymbol{f} \cdot \mathbf{\Phi}\ dm \\ &= \frac{1}{8}\int_{\Omega} \Big\{ A_{2D}^{\alpha\beta\sigma\tau}(\tilde{C}^1_{\sigma\tau} - g_{\sigma\tau})(\tilde{C}^1_{\alpha\beta} - g_{\alpha\beta}) \\ &\quad + 4\mu g^{\alpha\beta}\tilde{C}^1_{\beta 3}\tilde{C}^1_{\alpha 3}\Big\}\sqrt{g}dx - \int_{\Omega} \boldsymbol{f}\cdot\mathbf{\Phi}\sqrt{g}dx\end{aligned}$$

over the set of all triples $(\mathbf{\Phi}, \boldsymbol{C}, \mathbf{\Sigma})$ satisfying the conditions (note that $\mathbf{\Sigma}$ and $\tilde{C}^1_{33}$ do not appear in the last expression of $\mathcal{J}(\mathbf{\Phi}, \boldsymbol{C}, \mathbf{\Sigma})$)

$$\begin{aligned}\mathbf{\Phi}(\cdot, x_3) &= \boldsymbol{\varphi}_0 + x_3\boldsymbol{\varphi}_1,\ \ \boldsymbol{\varphi}_0, \boldsymbol{\varphi}_1 \in W^{1,4}_{\gamma_0}(\omega; \mathbb{R}^3), \\ \tilde{C}^1_{\alpha 3} = \tilde{C}^1_{3\alpha} &= \partial_\alpha\boldsymbol{\varphi}_0\cdot\boldsymbol{\varphi}_1 + x_3\partial_\alpha\boldsymbol{\varphi}_1\cdot\boldsymbol{\varphi}_1, \\ \tilde{C}^1_{\alpha\beta} &= g_{\alpha\beta} + (a_{\alpha\beta}(\boldsymbol{\varphi}_0) - a_{\alpha\beta}) + x_3(\partial_\alpha\boldsymbol{\varphi}_0\cdot\partial_\beta\boldsymbol{\varphi}_1 + \partial_\beta\boldsymbol{\varphi}_0\cdot\partial_\alpha\boldsymbol{\varphi}_1 + 2b_{\alpha\beta}) \\ &\quad + (x_3)^2(\partial_\alpha\boldsymbol{\varphi}_1\cdot\partial_\beta\boldsymbol{\varphi}_1 - c_{\alpha\beta}),\end{aligned}$$

where $c_{\alpha\beta} := \partial_\alpha\mathbf{a}_3\cdot\partial_\beta\mathbf{a}_3$ is the third fundamental form of the surface $S = \boldsymbol{\theta}(\omega)$.

Since $\Omega = \omega \times (-\varepsilon, \varepsilon)$ depends on a parameter ε that approaches zero, the above shell model can be simplified by dropping from the total energy $\mathcal{J}$ those terms that are of a lesser order of magnitude with respect to ε. A first step in this simplification is to replace $A_{2D}^{\alpha\beta\sigma\tau}$ and $\sqrt{g}$ by the first term of their Taylor expansions in the variable x_3, namely $\frac{1}{2}a^{\alpha\beta\sigma\tau}$ (see (5.4)) and $\sqrt{a}$. Noting that

$$\frac{1}{2}(\tilde{C}^1_{\alpha\beta} - g_{\alpha\beta}) = G_{\alpha\beta}(\boldsymbol{\varphi}_0) - x_3R_{\alpha\beta}(\boldsymbol{\varphi}_0, \boldsymbol{\varphi}_1) + (x_3)^2P_{\alpha\beta}(\boldsymbol{\varphi}_0),$$

where

$$\begin{aligned}G_{\alpha\beta}(\boldsymbol{\varphi}_0) &:= \frac{1}{2}(a_{\alpha\beta}(\boldsymbol{\varphi}_0) - a_{\alpha\beta}), \\ R_{\alpha\beta}(\boldsymbol{\varphi}_0, \boldsymbol{\varphi}_1) &:= -\frac{1}{2}(\partial_\alpha\boldsymbol{\varphi}_0\cdot\partial_\beta\boldsymbol{\varphi}_1 + \partial_\beta\boldsymbol{\varphi}_0\cdot\partial_\alpha\boldsymbol{\varphi}_1) - b_{\alpha\beta}, \\ P_{\alpha\beta}(\boldsymbol{\varphi}_1) &:= \frac{1}{2}(\partial_\alpha\boldsymbol{\varphi}_1\cdot\partial_\beta\boldsymbol{\varphi}_1 - c_{\alpha\beta}),\end{aligned}$$

and noting that these expressions do not depend on x_3, we deduce that

$$\mathcal{J}(\mathbf{\Phi}, \boldsymbol{C}, \mathbf{\Sigma}) = J_{2D}(\boldsymbol{\varphi}_0, \boldsymbol{\varphi}_1) + \mathcal{O}(\varepsilon),$$

where

(6.3)

$$\begin{aligned} J_{2D}(\boldsymbol{\varphi}_0,\boldsymbol{\varphi}_1) =& \frac{1}{2}\int_\omega a^{\alpha\beta\sigma\tau}\Big\{\varepsilon G_{\sigma\tau}(\boldsymbol{\varphi}_0)G_{\alpha\beta}(\boldsymbol{\varphi}_0) + \frac{\varepsilon^3}{3}R_{\sigma\tau}(\boldsymbol{\varphi}_0,\boldsymbol{\varphi}_1)R_{\alpha\beta}(\boldsymbol{\varphi}_0,\boldsymbol{\varphi}_1)\Big\}\sqrt{a}dy \\ &+ \frac{1}{2}\int_\omega 2\mu a^{\alpha\beta}\varepsilon(\partial_\beta\boldsymbol{\varphi}_0\cdot\boldsymbol{\varphi}_1)(\partial_\alpha\boldsymbol{\varphi}_0\cdot\boldsymbol{\varphi}_1)\sqrt{a}dy \\ &- \int_\omega\Big\{\int_{-\varepsilon}^{\varepsilon}\boldsymbol{f}dx_3\Big\}\cdot\boldsymbol{\varphi}_0\sqrt{a}dy - \int_\omega\Big\{\int_{-\varepsilon}^{\varepsilon}x_3\boldsymbol{f}dx_3\Big\}\cdot\boldsymbol{\varphi}_1\sqrt{a}dy \\ &+ \frac{1}{2}\int_\omega a^{\alpha\beta\sigma\tau}\frac{\varepsilon^3}{3}\Big(G_{\sigma\tau}(\boldsymbol{\varphi}_0)P_{\alpha\beta}(\boldsymbol{\varphi}_1) + G_{\alpha\beta}(\boldsymbol{\varphi}_0)P_{\sigma\tau}(\boldsymbol{\varphi}_1)\Big)\sqrt{a}dy \\ &+ \frac{1}{2}\int_\omega a^{\alpha\beta\sigma\tau}\frac{\varepsilon^5}{5}P_{\sigma\tau}(\boldsymbol{\varphi}_1)P_{\alpha\beta}(\boldsymbol{\varphi}_1)\sqrt{a}dy \\ &+ \frac{1}{2}\int_\omega 2\mu a^{\alpha\beta}\frac{\varepsilon^3}{3}(\partial_\beta\boldsymbol{\varphi}_1\cdot\boldsymbol{\varphi}_1)(\partial_\alpha\boldsymbol{\varphi}_1\cdot\boldsymbol{\varphi}_1)\sqrt{a}dy. \end{aligned}$$

As a consequence, the 2D shell model defined in Theorem 6.1 can be approached with a simpler one, namely:

Definition 6.2. The *generalized nonlinear Naghdi model* consist in finding vector fields $\boldsymbol{\varphi}_0, \boldsymbol{\varphi}_1 \in W^{1,4}_{\gamma_0}(\omega)$ that minimize the "two-dimensional total energy" J_{2D}.

Remark 6.3. The generalized nonlinear Naghdi model is defined in terms of the differential operators ($G_{\alpha\beta}$, $R_{\alpha\beta}$, $P_{\alpha\beta}$) of order one (only $\boldsymbol{\varphi}_0, \boldsymbol{\varphi}_1$ and their gradients are involved). The unknown $\boldsymbol{\varphi}_0$ represents the deformation field of the middle surface of the shell. The second unknown $\boldsymbol{\varphi}_1$ can be seen as auxiliary, but cannot be dropped altogether because the curvature of the middle surface of the middle shell cannot be described only by $\boldsymbol{\varphi}_0$ and $\boldsymbol{\nabla}\boldsymbol{\varphi}_0$.

The generalized nonlinear Naghdi model can be further simplified if the vector field $\boldsymbol{\varphi}_1$ is chosen a priori as a function of $\boldsymbol{\varphi}_0$. In particular, we obtain by choosing $\boldsymbol{\varphi}_1$ to be perpendicular on the surface $\boldsymbol{\varphi}_0(\omega)$ the following result:

Theorem 6.4. a) *If* $\boldsymbol{\varphi}_1 = \dfrac{\partial_1\boldsymbol{\varphi}_0\wedge\partial_2\boldsymbol{\varphi}_0}{|\partial_1\boldsymbol{\varphi}_0\wedge\partial_2\boldsymbol{\varphi}_0|}$, *then*

$$J_{2D}(\boldsymbol{\varphi}_0,\boldsymbol{\varphi}_1) = J_K(\boldsymbol{\varphi}_0) - \int_\omega\Big\{\int_{-\varepsilon}^{\varepsilon}x_3\boldsymbol{f}dx_3\Big\}\cdot\frac{\partial_1\boldsymbol{\varphi}_0\wedge\partial_2\boldsymbol{\varphi}_0}{|\partial_1\boldsymbol{\varphi}_0\wedge\partial_2\boldsymbol{\varphi}_0|}\sqrt{a}dy + R_K(\boldsymbol{\varphi}_0), \tag{6.4}$$

where J_K is the functional defined by (5.1) *and R_K is defined by*

$$\begin{aligned} R_K(\boldsymbol{\varphi}_0) =& \frac{\varepsilon^3}{3}\int_\omega a^{\alpha\beta\sigma\tau}G_{\sigma\tau}(\boldsymbol{\varphi}_0)(c_{\alpha\beta}(\boldsymbol{\varphi}_0) - c_{\alpha\beta})\sqrt{a}dy \\ &+ \frac{\varepsilon^5}{10}\int_\omega a^{\alpha\beta\sigma\tau}(c_{\sigma\tau}(\boldsymbol{\varphi}_0) - c_{\sigma\tau})(c_{\alpha\beta}(\boldsymbol{\varphi}_0) - c_{\alpha\beta})\sqrt{a}dy. \end{aligned}$$

The functions $c_{\alpha\beta}(\boldsymbol{\varphi}_0)$ and $c_{\alpha\beta}$ respectively denote the third fundamental forms of the surfaces $\boldsymbol{\varphi}_0(\omega)$ and $\boldsymbol{\theta}(\omega)$.

b) *If* $\boldsymbol{\varphi}_1 = \frac{1}{\sqrt{a}}(\partial_1\boldsymbol{\varphi}_0\wedge\partial_2\boldsymbol{\varphi}_0)$, *then*

$$J_{2D}(\boldsymbol{\varphi}_0,\boldsymbol{\varphi}_1) = J^\sharp_K(\boldsymbol{\varphi}_0) - \int_\omega\Big\{\int_{-\varepsilon}^{\varepsilon}x_3\boldsymbol{f}dx_3\Big\}\cdot(\partial_1\boldsymbol{\varphi}_0\wedge\partial_2\boldsymbol{\varphi}_0)dy + R^\sharp_K(\boldsymbol{\varphi}_0), \tag{6.5}$$

where $J^\sharp_K$ is the functional defined by (5.6) and $R^\sharp_K$ is defined by

$$\begin{aligned}R^\sharp_K(\boldsymbol{\varphi}_0) =&\frac{\varepsilon^3}{3}\int_\omega a^{\alpha\beta\sigma\tau}G_{\sigma\tau}(\boldsymbol{\varphi}_0)P^\sharp_{\alpha\beta}(\boldsymbol{\varphi}_0)\sqrt{a}dy\\ &+\frac{\varepsilon^3}{12}\int_\omega \mu a^{\alpha\beta}\partial_\beta\Big(\frac{a(\boldsymbol{\varphi}_0)}{a}\Big)\partial_\alpha\Big(\frac{a(\boldsymbol{\varphi}_0)}{a}\Big)\sqrt{a}dy\\ &+\frac{\varepsilon^5}{10}\int_\omega a^{\alpha\beta\sigma\tau}P^\sharp_{\sigma\tau}(\boldsymbol{\varphi}_0)P^\sharp_{\alpha\beta}(\boldsymbol{\varphi}_0)\sqrt{a}dy,\end{aligned}$$

with

$$\begin{aligned}P^\sharp_{\alpha\beta}(\boldsymbol{\varphi}_0) &:= \frac{1}{2}\Big\{\partial_\alpha\Big(\frac{\partial_1\boldsymbol{\varphi}_0\wedge\partial_2\boldsymbol{\varphi}_0}{\sqrt{a}}\Big)\cdot\partial_\beta\Big(\frac{\partial_1\boldsymbol{\varphi}_0\wedge\partial_2\boldsymbol{\varphi}_0}{\sqrt{a}}\Big)-c_{\alpha\beta}\Big\}\\ a(\boldsymbol{\varphi}_0) &:= \det(a_{\alpha\beta}(\boldsymbol{\varphi}_0)).\end{aligned}$$

Proof. a) If $\boldsymbol{\varphi}_1 = \frac{\partial_1\boldsymbol{\varphi}_0\wedge\partial_2\boldsymbol{\varphi}_0}{|\partial_1\boldsymbol{\varphi}_0\wedge\partial_2\boldsymbol{\varphi}_0|}$, then $\boldsymbol{\varphi}_1$ is orthogonal on $\partial_\alpha\boldsymbol{\varphi}_0$ and $|\boldsymbol{\varphi}_1| = 1$. This implies that the second and seventh terms of the right-hand side of $J_{2D}(\boldsymbol{\varphi}_0,\boldsymbol{\varphi}_1)$ (see (6.3)) vanish and

$$\begin{aligned}R_{\alpha\beta}(\boldsymbol{\varphi}_0,\boldsymbol{\varphi}_1) &= \partial_{\alpha\beta}\boldsymbol{\varphi}_0\cdot\Big(\frac{\partial_1\boldsymbol{\varphi}_0\wedge\partial_2\boldsymbol{\varphi}_0}{|\partial_1\boldsymbol{\varphi}_0\wedge\partial_2\boldsymbol{\varphi}_0|}\Big)-b_{\alpha\beta} = R_{\alpha\beta}(\boldsymbol{\varphi}_0)\\ P_{\alpha\beta}(\boldsymbol{\varphi}_1) &= \frac{1}{2}\Big\{\partial_\alpha\Big(\frac{\partial_1\boldsymbol{\varphi}_0\wedge\partial_2\boldsymbol{\varphi}_0}{|\partial_1\boldsymbol{\varphi}_0\wedge\partial_2\boldsymbol{\varphi}_0|}\Big)\cdot\partial_\beta\Big(\frac{\partial_1\boldsymbol{\varphi}_0\wedge\partial_2\boldsymbol{\varphi}_0}{|\partial_1\boldsymbol{\varphi}_0\wedge\partial_2\boldsymbol{\varphi}_0|}\Big)-c_{\alpha\beta}\Big\}\\ &= \frac{1}{2}(c_{\alpha\beta}(\boldsymbol{\varphi}_0)-c_{\alpha\beta}).\end{aligned}$$

Using these formulas in the relation (6.3) shows that $J_{2D}(\boldsymbol{\varphi}_0,\boldsymbol{\varphi}_1)$ satisfies the equation (6.4).

b) If $\boldsymbol{\varphi}_1 = \frac{1}{\sqrt{a}}(\partial_1\boldsymbol{\varphi}_0\wedge\partial_2\boldsymbol{\varphi}_0)$, then $\boldsymbol{\varphi}_1$ is orthogonal on $\partial_\alpha\boldsymbol{\varphi}_0$. This implies that the second term of the right-hand side of $J_{2D}(\boldsymbol{\varphi}_0,\boldsymbol{\varphi}_1)$ (see (6.3)) vanish and

$$\begin{aligned}&R_{\alpha\beta}(\boldsymbol{\varphi}_0,\boldsymbol{\varphi}_1) = \partial_{\alpha\beta}\boldsymbol{\varphi}_0\cdot\Big(\frac{\partial_1\boldsymbol{\varphi}_0\wedge\partial_2\boldsymbol{\varphi}_0}{\sqrt{a}}\Big)-b_{\alpha\beta} = R^\sharp_{\alpha\beta}(\boldsymbol{\varphi}_0)\\ &P_{\alpha\beta}(\boldsymbol{\varphi}_1) = \frac{1}{2}\Big\{\partial_\alpha\Big(\frac{\partial_1\boldsymbol{\varphi}_0\wedge\partial_2\boldsymbol{\varphi}_0}{\sqrt{a}}\Big)\cdot\partial_\beta\Big(\frac{\partial_1\boldsymbol{\varphi}_0\wedge\partial_2\boldsymbol{\varphi}_0}{\sqrt{a}}\Big)-c_{\alpha\beta}\Big\} = P^\sharp_{\alpha\beta}(\boldsymbol{\varphi}_0)\\ &\partial_\alpha\boldsymbol{\varphi}_1\cdot\boldsymbol{\varphi}_1 = \frac{1}{2}\partial_\alpha(|\boldsymbol{\varphi}_1|^2) = \frac{1}{2}\partial_\alpha\Big(\frac{|\partial_1\boldsymbol{\varphi}_0\wedge\partial_2\boldsymbol{\varphi}_0|^2}{a}\Big) = \frac{1}{2}\partial_\alpha\Big(\frac{a(\boldsymbol{\varphi}_0)}{a}\Big).\end{aligned}$$

Using these formulas in the relation (6.3) shows that $J_{2D}(\boldsymbol{\varphi}_0,\boldsymbol{\varphi}_1)$ satisfies the equation (6.5).

□

The above theorem motivates the introduction of the following shell models:

Definition 6.5. a) The *generalized nonlinear Koiter model* consists in minimizing the functional J_{2D} over the set of all pairs $(\boldsymbol{\varphi}_0,\boldsymbol{\varphi}^K_1)\in[W^{1,4}_{\gamma_0}(\omega)]^2$ with

$$\boldsymbol{\varphi}^K_1 = \frac{\partial_1\boldsymbol{\varphi}_0\wedge\partial_2\boldsymbol{\varphi}_0}{|\partial_1\boldsymbol{\varphi}_0\wedge\partial_2\boldsymbol{\varphi}_0|}.$$

b) The *generalized nonlinear Ciarlet-Koiter model* consists in minimizing the functional J_{2D} over the set of all pairs $(\boldsymbol{\varphi}_0,\boldsymbol{\varphi}^{K\sharp}_1)\in[W^{1,4}_{\gamma_0}(\omega)]^2$ with

$$\boldsymbol{\varphi}^{K\sharp}_1 = \frac{\partial_1\boldsymbol{\varphi}_0\wedge\partial_2\boldsymbol{\varphi}_0}{\sqrt{a}}.$$

Remarks 6.6. a) Theorem 6.4 shows that the nonlinear Koiter and Ciarlet-Koiter shell models respectively arise from the generalized nonlinear Koiter and Ciarlet-Koiter models by neglecting the terms $R_K(\boldsymbol{\varphi}_0)$ and $R_K^\sharp(\boldsymbol{\varphi}_0)$ appearing in the expression of the functional J_{2D}. Taking into account the expressions of these terms, this amounts to neglecting the norm

$$\varepsilon^2 \sum_{\alpha,\beta} \|c_{\alpha\beta}(\boldsymbol{\varphi}_0) - c_{\alpha\beta}\|_{L^2(\omega)}$$

in the case of Koiter's model, and the norm

$$\varepsilon^2 \sum_{\alpha,\beta} \|P^\sharp_{\alpha\beta}(\boldsymbol{\varphi}_0)\|_{L^2(\omega)}$$

in the case of Ciarlet-Koiter's model. This operation would be justified if these norms were of a lesser order of magnitude when $\varepsilon \to 0$ than respectively

$$\sum_{\alpha,\beta} \|G_{\alpha\beta}(\boldsymbol{\varphi}_0)\|_{L^2(\omega)} + \varepsilon \sum_{\alpha,\beta} \|R_{\alpha\beta}(\boldsymbol{\varphi}_0)\|_{L^2(\omega)}$$

and

$$\sum_{\alpha,\beta} \|G_{\alpha\beta}(\boldsymbol{\varphi}_0)\|_{L^2(\omega)} + \varepsilon \sum_{\alpha,\beta} \|R^\sharp_{\alpha\beta}(\boldsymbol{\varphi}_0)\|_{L^2(\omega)}.$$

While the powers of ε appearing in front of the $L^2(\omega)$-norms suggest that this is indeed the case, the regularity of $\boldsymbol{\varphi}_0$ given by the coerciveness of J_K and $J_K^\sharp$, respectively, is insufficient to control the $L^2(\omega)$-norms of $(c_{\alpha\beta}(\boldsymbol{\varphi}_0) - c_{\alpha\beta})$ and $P^\sharp_{\alpha\beta}(\boldsymbol{\varphi}_0)$. This observation suggest that these "small" terms in fact may be useful in the existence theory of nonlinear 2D shell models. Incidentally, the $L^2(\omega)$-norm of $(c_{\alpha\beta}(\boldsymbol{\varphi}_0) - c_{\alpha\beta})$ also appear in the nonlinear Korn inequality on a surface established in Ciarlet, Gratie & Mardare [4]; this enforces the argument for keeping these terms in the total energy of Koiter and Ciarlet-Koiter's models.

b) The definition of the vector field $\boldsymbol{\varphi}_1^K$ shows that the unknown $\boldsymbol{\varphi}_0$ of (generalized) Koiter's model must satisfy $\partial_1\boldsymbol{\varphi}_0 \wedge \partial_2\boldsymbol{\varphi}_0 \neq 0$ in ω. By contrast, the unknown $\boldsymbol{\varphi}_0$ of (generalized) Ciarlet-Koiter's model is not subjected to this restriction since the vector field $\boldsymbol{\varphi}_1^{K\sharp}$ is always well-defined.

7. LINEARIZATION

In this section we justify the nonlinear 2D shell models introduced in Sections 5 and 6 by showing that they are good approximations of the nonlinear 3D shell model at least "for small deformations". This means that the linear part, with respect to the displacement field of the middle surface of the shell, of these nonlinear 2D shell models are linear 2D shell models whose solutions are known to approach the solution of the linear 3D shell model when $\varepsilon \to 0$; cf. Lods & Mardare [12, 13, 14].

We begin by linearizing the generalized nonlinear Naghdi model (Definition 6.2). If $\boldsymbol{\varphi}_0 := \boldsymbol{\theta} + \boldsymbol{\zeta}_0$ and $\boldsymbol{\varphi}_1 := \mathbf{a}_3 + \boldsymbol{\zeta}_1$, then

$$\begin{aligned} G_{\alpha\beta}(\boldsymbol{\varphi}_0) &= \gamma_{\alpha\beta}(\boldsymbol{\zeta}_0) + \text{h.o.t.} \\ R_{\alpha\beta}(\boldsymbol{\varphi}_0, \boldsymbol{\varphi}_1) &= \rho_{\alpha\beta}(\boldsymbol{\zeta}_0, \boldsymbol{\zeta}_1) + \text{h.o.t.} \\ \frac{1}{2}\partial_\alpha \boldsymbol{\varphi}_0 \cdot \boldsymbol{\varphi}_1 &= \delta_{\alpha 3}(\boldsymbol{\zeta}_0, \boldsymbol{\zeta}_1) + \text{h.o.t.} \\ P_{\alpha\beta}(\boldsymbol{\varphi}_1) &= \pi_{\alpha\beta}(\boldsymbol{\zeta}_1) + \text{h.o.t.}, \end{aligned}$$

where h.o.t. denotes higher order terms (at least quadratic) in $(\boldsymbol{\zeta}_0, \boldsymbol{\zeta}_1)$ and

$$\begin{aligned} \gamma_{\alpha\beta}(\boldsymbol{\zeta}_0) &:= \frac{1}{2}(\partial_\alpha \boldsymbol{\zeta}_0 \cdot \mathbf{a}_\beta + \partial_\beta \boldsymbol{\zeta}_0 \cdot \mathbf{a}_\alpha) \\ \rho_{\alpha\beta}(\boldsymbol{\zeta}_0, \boldsymbol{\zeta}_1) &:= -\frac{1}{2}(\partial_\alpha \boldsymbol{\zeta}_0 \cdot \partial_\beta \mathbf{a}_3 + \partial_\beta \boldsymbol{\zeta}_0 \cdot \partial_\alpha \mathbf{a}_3 + \mathbf{a}_\alpha \cdot \partial_\beta \boldsymbol{\zeta}_1 + \mathbf{a}_\beta \cdot \partial_\alpha \boldsymbol{\zeta}_1) \\ \delta_{\alpha 3}(\boldsymbol{\zeta}_0, \boldsymbol{\zeta}_1) &:= \frac{1}{2}(\partial_\alpha \boldsymbol{\zeta}_0 \cdot \mathbf{a}_3 + \boldsymbol{\zeta}_1 \cdot \mathbf{a}_\alpha) \\ \pi_{\alpha\beta}(\boldsymbol{\zeta}_1) &:= \frac{1}{2}(\partial_\alpha \boldsymbol{\zeta}_1 \cdot \partial_\beta \mathbf{a}_3 + \partial_\alpha \mathbf{a}_3 \cdot \partial_\beta \boldsymbol{\zeta}_1). \end{aligned}$$

Therefore

$$\begin{aligned} J_{2D}(\boldsymbol{\varphi}_0, \boldsymbol{\varphi}_1) = &- \int_\omega \Big\{ \int_{-\varepsilon}^{\varepsilon} \boldsymbol{f} dx_3 \Big\} \cdot \boldsymbol{\theta} \sqrt{a} dy - \int_\omega \Big\{ \int_{-\varepsilon}^{\varepsilon} x_3 \boldsymbol{f} dx_3 \Big\} \cdot \mathbf{a}_3 \sqrt{a} dy \\ &+ j_{2D}(\boldsymbol{\zeta}_0, \boldsymbol{\zeta}_1) + \text{h.o.t}, \end{aligned}$$

where

$$\begin{aligned} j_{2D}(\boldsymbol{\zeta}_0, \boldsymbol{\zeta}_1) = &\frac{1}{2} \int_\omega a^{\alpha\beta\sigma\tau} \Big\{ \varepsilon \gamma_{\sigma\tau}(\boldsymbol{\zeta}_0) \gamma_{\alpha\beta}(\boldsymbol{\zeta}_0) + \frac{\varepsilon^3}{3} \rho_{\sigma\tau}(\boldsymbol{\zeta}_0, \boldsymbol{\zeta}_1) \rho_{\alpha\beta}(\boldsymbol{\zeta}_0, \boldsymbol{\zeta}_1) \Big\} \sqrt{a} dy \\ &+ \frac{1}{2} \int_\omega 8\mu a^{\alpha\beta} \varepsilon \delta_{\beta 3}(\boldsymbol{\zeta}_0, \boldsymbol{\zeta}_1) \delta_{\alpha 3}(\boldsymbol{\zeta}_0, \boldsymbol{\zeta}_1) \sqrt{a} dy \\ &- \int_\omega \Big\{ \int_{-\varepsilon}^{\varepsilon} \boldsymbol{f} dx_3 \Big\} \cdot \boldsymbol{\zeta}_0 \sqrt{a} dy - \int_\omega \Big\{ \int_{-\varepsilon}^{\varepsilon} x_3 \boldsymbol{f} dx_3 \Big\} \cdot \boldsymbol{\zeta}_1 \sqrt{a} dy \\ &+ \frac{\varepsilon^3}{6} \int_\omega a^{\alpha\beta\sigma\tau} \Big(\gamma_{\sigma\tau}(\boldsymbol{\zeta}_0) \pi_{\alpha\beta}(\boldsymbol{\zeta}_1) + \gamma_{\alpha\beta}(\boldsymbol{\zeta}_0) \pi_{\sigma\tau}(\boldsymbol{\zeta}_1) \Big) \sqrt{a} dy \\ &+ \frac{\varepsilon^5}{10} \int_\omega a^{\alpha\beta\sigma\tau} \pi_{\sigma\tau}(\boldsymbol{\zeta}_1) \pi_{\alpha\beta}(\boldsymbol{\zeta}_1) \sqrt{a} dy \\ &+ \frac{\varepsilon^3}{3} \int_\omega \mu a^{\alpha\beta} \partial_\beta(\boldsymbol{\zeta}_1 \cdot \mathbf{a}_3) \partial_\alpha(\boldsymbol{\zeta}_1 \cdot \mathbf{a}_3) \sqrt{a} dy. \end{aligned}$$

The above relations show the following

Theorem 7.1. *The* linear 2D shell model *obtained by linearizing the generalized nonlinear Naghdi model consist in finding vector fields* $\boldsymbol{\zeta}_0, \boldsymbol{\zeta}_1 \in H^1_{\gamma_0}(\omega)$ *that minimize the "two-dimensional total energy"* j_{2D}.

Remark 7.2. The linear 2D shell model defined in Theorem 7.1 can be viewed as a "generalized Naghdi's model" since the linear Naghdi model is obtained from the generalized one by dropping last three terms of the functional j_{2D} and by imposing $\boldsymbol{\zeta}_1 \cdot \mathbf{a}_3 = 0$.

Next we identify the linear shell models obtained by linearizing around the deformation field $\boldsymbol{\theta}$ the nonlinear shell models of Koiter and Ciarlet-Koiter:

Theorem 7.3. a) *The* linear Koiter shell model *consists in finding a vector field* $\boldsymbol{\zeta} = \zeta_i \mathbf{a}^i$ *with* $\zeta_\alpha \in H^1_{\gamma_0}(\omega)$ *and* $\zeta_3 \in H^2_{\gamma_0}(\omega)$ *that minimizes the functional*

$$(7.1)\quad \begin{aligned} j_K(\boldsymbol{\zeta}_0) = & \frac{1}{2}\int_\omega a^{\alpha\beta\sigma\tau}\Big\{\varepsilon\gamma_{\sigma\tau}(\boldsymbol{\zeta})\gamma_{\alpha\beta}(\boldsymbol{\zeta}) + \frac{\varepsilon^3}{3}\rho_{\sigma\tau}(\boldsymbol{\zeta})\rho_{\alpha\beta}(\boldsymbol{\zeta})\Big\}\sqrt{a}dy \\ & - \int_\omega \Big\{\int_{-\varepsilon}^{\varepsilon} \boldsymbol{f} dx_3\Big\}\cdot\boldsymbol{\zeta}\sqrt{a}dy, \end{aligned}$$

where

$$\gamma_{\alpha\beta}(\boldsymbol{\zeta}) := \frac{1}{2}(\partial_\alpha\boldsymbol{\zeta}\cdot\mathbf{a}_\beta + \partial_\beta\boldsymbol{\zeta}\cdot\mathbf{a}_\alpha),$$
$$\rho_{\alpha\beta}(\boldsymbol{\zeta}) := \partial_{\alpha\beta}\boldsymbol{\zeta}\cdot\mathbf{a}_3 - (\partial_{\alpha\beta}\boldsymbol{\theta}\cdot\mathbf{a}^\sigma)(\partial_\sigma\boldsymbol{\zeta}\cdot\mathbf{a}_3).$$

b) *The* linear Ciarlet-Koiter shell model *consists in finding a vector field* $\boldsymbol{\zeta} = \zeta_i\mathbf{a}^i$ *with* $\zeta_\alpha \in H^1_{\gamma_0}(\omega)$ *and* $\zeta_3 \in H^2_{\gamma_0}(\omega)$ *that minimizes the functional*

$$(7.2)\quad \begin{aligned} j_K^\sharp(\boldsymbol{\zeta}_0) = & \frac{1}{2}\int_\omega a^{\alpha\beta\sigma\tau}\Big\{\varepsilon\gamma_{\sigma\tau}(\boldsymbol{\zeta})\gamma_{\alpha\beta}(\boldsymbol{\zeta}) + \frac{\varepsilon^3}{3}\rho^\sharp_{\sigma\tau}(\boldsymbol{\zeta})\rho^\sharp_{\alpha\beta}(\boldsymbol{\zeta})\Big\}\sqrt{a}dy \\ & - \int_\omega \Big\{\int_{-\varepsilon}^{\varepsilon} \boldsymbol{f} dx_3\Big\}\cdot\boldsymbol{\zeta}\sqrt{a}dy, \end{aligned}$$

where

$$\rho^\sharp_{\alpha\beta}(\boldsymbol{\zeta}) = \rho_{\alpha\beta}(\boldsymbol{\zeta}) + b_{\alpha\beta}a^{\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{\zeta}).$$

Proof. a) This result is well-known.

b) It suffices to compute the linear part with respect to $\boldsymbol{\zeta}$ of the operator $R^\sharp_{\alpha\beta}(\boldsymbol{\varphi})$, where $\boldsymbol{\varphi} = \boldsymbol{\theta} + \boldsymbol{\zeta}$. Since

$$\partial_1\boldsymbol{\varphi}\wedge\partial_2\boldsymbol{\varphi} = \sqrt{a}\,\mathbf{a}_3 + \partial_1\boldsymbol{\zeta}\wedge\mathbf{a}_2 + \mathbf{a}_1\wedge\partial_2\boldsymbol{\zeta} + \partial_1\boldsymbol{\zeta}\wedge\partial_2\boldsymbol{\zeta},$$

we infer from the formula (5.5) that

$$\begin{aligned} R^\sharp_{\alpha\beta}(\boldsymbol{\varphi}) &= (\partial_{\alpha\beta}\boldsymbol{\theta} + \partial_{\alpha\beta}\boldsymbol{\zeta})\cdot\Big\{\mathbf{a}_3 + \frac{1}{\sqrt{a}}(\partial_1\boldsymbol{\zeta}\wedge\mathbf{a}_2 + \mathbf{a}_1\wedge\partial_2\boldsymbol{\zeta} + \partial_1\boldsymbol{\zeta}\wedge\partial_2\boldsymbol{\zeta})\Big\} - b_{\alpha\beta} \\ &= \partial_{\alpha\beta}\boldsymbol{\zeta}\cdot\mathbf{a}_3 + \frac{1}{\sqrt{a}}\Big\{\partial_1\boldsymbol{\zeta}\cdot(\mathbf{a}_2\wedge\partial_{\alpha\beta}\boldsymbol{\theta}) + \partial_2\boldsymbol{\zeta}\cdot(\partial_{\alpha\beta}\boldsymbol{\theta}\wedge\mathbf{a}_1)\Big\} + \text{h.o.t.} \\ &= \partial_{\alpha\beta}\boldsymbol{\zeta}\cdot\mathbf{a}_3 + \Big\{-(\partial_{\alpha\beta}\boldsymbol{\theta}\cdot\mathbf{a}^\sigma)(\partial_\sigma\boldsymbol{\zeta}\cdot\mathbf{a}_3) + b_{\alpha\beta}(\partial_\sigma\boldsymbol{\zeta}\cdot\mathbf{a}^\sigma)\Big\} + \text{h.o.t.} \\ &= \rho_{\alpha\beta}(\boldsymbol{\zeta}) + b_{\alpha\beta}a^{\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{\zeta}) + \text{h.o.t.}, \end{aligned}$$

where h.o.t. denotes higher order terms (at least quadratic) in $\boldsymbol{\zeta}$. □

Now let us identify the linear shell models obtained by linearizing the *generalized* nonlinear shell models of Koiter and Ciarlet-Koiter around the deformation field $\boldsymbol{\theta}$. In view of Definition 6.5 and Theorem 7.1, it is clear that these linear shell models are obtained from the linear 2D shell model defined in Theorem 7.1 by replacing the vector field $\boldsymbol{\zeta}_1$ respectively with $\boldsymbol{\zeta}_1^K$ and $\boldsymbol{\zeta}_1^{K\sharp}$, where $\boldsymbol{\zeta}_1^K$ and $\boldsymbol{\zeta}_1^{K\sharp}$ respectively denote the linear part of $\boldsymbol{\varphi}_1^K$ and $\boldsymbol{\varphi}_1^{K\sharp}$ with respect to the displacement field $\boldsymbol{\zeta}_0 = \boldsymbol{\varphi}_0 - \boldsymbol{\theta}$.

Since

$$\partial_1\boldsymbol{\varphi}_0\wedge\partial_2\boldsymbol{\varphi}_0 = \sqrt{a}\mathbf{a}_3 + \partial_1\boldsymbol{\zeta}_0\wedge\mathbf{a}_2 + \mathbf{a}_1\wedge\partial_2\boldsymbol{\zeta}_0 + \text{h.o.t.}$$

and

$$\frac{1}{|\partial_1\boldsymbol{\varphi}_0\wedge\partial_2\boldsymbol{\varphi}_0|} = \frac{1}{\sqrt{a}}(1 - \partial_\sigma\boldsymbol{\zeta}_0\cdot\mathbf{a}^\sigma) + \text{h.o.t.},$$

where h.o.t. denotes higher order terms (at least quadratic) in $\boldsymbol{\zeta}_0$, the Taylor series expansions of the vector fields $\boldsymbol{\varphi}_1^K$ and $\boldsymbol{\varphi}_1^{K\sharp}$ (see Definition 6.5) are

$$\begin{aligned}\boldsymbol{\varphi}_1^K &= \mathbf{a}_3 - (\partial_\sigma\boldsymbol{\zeta}_0\cdot\mathbf{a}_3)\mathbf{a}^\sigma + \text{h.o.t.},\\ \boldsymbol{\varphi}_1^{K\sharp} &= \mathbf{a}_3 - (\partial_\sigma\boldsymbol{\zeta}_0\cdot\mathbf{a}_3)\mathbf{a}^\sigma + (\partial_\sigma\boldsymbol{\zeta}_0\cdot\mathbf{a}^\sigma)\mathbf{a}_3 + \text{h.o.t.}\\ &= \mathbf{a}_3 - (\partial_\sigma\boldsymbol{\zeta}_0\cdot\mathbf{a}_3)\mathbf{a}^\sigma + a^{\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{\zeta}_0)\mathbf{a}_3 + \text{h.o.t..}\end{aligned}$$

An immediate consequence of these relations is the following

Theorem 7.4. *a) The* generalized linear Koiter model *consists in minimizing the functional j_{2D} over the set of all pairs $(\boldsymbol{\zeta}_0,\boldsymbol{\zeta}_1^K)$, where*

$$\begin{aligned}\boldsymbol{\zeta}_1^K &= -(\partial_\sigma\boldsymbol{\zeta}_0\cdot\mathbf{a}_3)\mathbf{a}^\sigma,\\ \boldsymbol{\zeta}_0 &= \zeta_{0,i}\mathbf{a}^i,\ \zeta_{0,\alpha}\in H^1_{\gamma_0}(\omega),\ \zeta_{0,3}\in H^2_{\gamma_0}(\omega).\end{aligned}$$

b) The generalized linear Ciarlet-Koiter model *consists in minimizing the functional j_{2D} over the set of all pairs $(\boldsymbol{\zeta}_0,\boldsymbol{\zeta}_1^{K\sharp})$ where*

$$\begin{aligned}\boldsymbol{\zeta}_1^{K\sharp} &= -(\partial_\sigma\boldsymbol{\zeta}_0\cdot\mathbf{a}_3)\mathbf{a}^\sigma + a^{\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{\zeta}_0)\mathbf{a}_3,\\ \boldsymbol{\zeta}_0 &= \zeta_{0,i}\mathbf{a}^i,\ \zeta_{0,\alpha}\in H^1_{\gamma_0}(\omega),\ \zeta_{0,3}\in H^2_{\gamma_0}(\omega),\ a^{\alpha\beta}\gamma_{\alpha\beta}(\boldsymbol{\zeta}_0)\in H^1_{\gamma_0}(\omega).\end{aligned}$$

Proof. The functional spaces to which the unknown $\boldsymbol{\zeta}_0$ belongs are determined by imposing $\boldsymbol{\zeta}_0,\boldsymbol{\zeta}_1^K\in H^1_{\gamma_0}(\omega)$, respectively $\boldsymbol{\zeta}_0,\boldsymbol{\zeta}_1^{K\sharp}\in H^1_{\gamma_0}(\omega)$ (see Theorem 7.1). It is also a consequence of the observation made in Remark 7.6. □

The following theorem, which is the counterpart in linearized elasticity of Theorem 6.4, compares the generalized versus the usual linear shell models of Koiter and Ciarlet-Koiter:

Theorem 7.5. a) *Let $\boldsymbol{\zeta}_1^K$ be defined as in Theorem* 7.4. *Then*

$$j_{2D}(\boldsymbol{\zeta}_0,\boldsymbol{\zeta}_1^K) = j_K(\boldsymbol{\zeta}_0) + \int_\omega\Big\{\int_{-\varepsilon}^{\varepsilon} x_3\boldsymbol{f}\cdot\mathbf{a}^\sigma dx_3\Big\}(\partial_\sigma\boldsymbol{\zeta}_0\cdot\mathbf{a}_3)\sqrt{a}dy + r_K(\boldsymbol{\zeta}_0), \tag{7.3}$$

where j_K is the functional defined by (7.1) *and r_K is defined by*

$$\begin{aligned}r_K(\boldsymbol{\varphi}_0) =& \frac{\varepsilon^3}{6}\int_\omega a^{\alpha\beta\sigma\tau}\Big\{\gamma_{\sigma\tau}(\boldsymbol{\zeta}_0)\pi^K_{\alpha\beta}(\boldsymbol{\zeta}_0) + \gamma_{\alpha\beta}(\boldsymbol{\zeta}_0)\pi^K_{\sigma\tau}(\boldsymbol{\zeta}_0)\Big\}\sqrt{a}dy\\ &+ \frac{\varepsilon^5}{10}\int_\omega a^{\alpha\beta\sigma\tau}\pi^K_{\sigma\tau}(\boldsymbol{\zeta}_0)\pi^K_{\alpha\beta}(\boldsymbol{\zeta}_0)\sqrt{a}dy\end{aligned}$$

and

$$\pi^K_{\alpha\beta}(\boldsymbol{\zeta}_0) := \frac{1}{2}\Big\{b^\sigma_\alpha\rho_{\beta\tau}(\boldsymbol{\zeta}_0) + b^\sigma_\beta\rho_{\alpha\tau}(\boldsymbol{\zeta}_0)\Big\} - b^\sigma_\alpha b^\tau_\beta\gamma_{\sigma\tau}(\boldsymbol{\zeta}_0).$$

b) *Let $\boldsymbol{\zeta}_1^{K\sharp}$ be defined as in Theorem* 7.4. *Then*

$$\begin{aligned}j_{2D}(\boldsymbol{\zeta}_0,\boldsymbol{\zeta}_1^{K\sharp}) =& j^\sharp_K(\boldsymbol{\zeta}_0) + \int_\omega\Big\{\Big(\int_{-\varepsilon}^{\varepsilon} x_3\boldsymbol{f}\cdot\mathbf{a}^\sigma dx_3\Big)(\partial_\sigma\boldsymbol{\zeta}_0\cdot\mathbf{a}_3)\\ &- \Big(\int_{-\varepsilon}^{\varepsilon} x_3\boldsymbol{f}\cdot\mathbf{a}_3 dx_3\Big)a^{\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{\zeta}_0)\Big\}\sqrt{a}dy + r^\sharp_K(\boldsymbol{\zeta}_0),\end{aligned} \tag{7.4}$$

where $j_K^\sharp$ is the functional defined by (7.2) and $r_K^\sharp$ is defined by

$$\begin{aligned} r_K^\sharp(\boldsymbol{\zeta}_0) =& \frac{\varepsilon^3}{6}\int_\omega a^{\alpha\beta\sigma\tau}\Big\{\gamma_{\sigma\tau}(\boldsymbol{\zeta}_0)\pi^\sharp_{\alpha\beta}(\boldsymbol{\zeta}_0)+\gamma_{\alpha\beta}(\boldsymbol{\zeta}_0)\pi^\sharp_{\sigma\tau}(\boldsymbol{\zeta}_0)\Big\}\sqrt{a}dy \\ &+\frac{\varepsilon^5}{10}\int_\omega a^{\alpha\beta\sigma\tau}\pi^\sharp_{\sigma\tau}(\boldsymbol{\zeta}_0)\pi^\sharp_{\alpha\beta}(\boldsymbol{\zeta}_0)\sqrt{a}dy \\ &+\frac{\varepsilon^3}{3}\int_\omega \mu a^{\alpha\beta}\partial_\beta(a^{\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{\zeta}_0))\,\partial_\alpha(a^{\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{\zeta}_0))\sqrt{a}dy \end{aligned}$$

and

$$\pi^\sharp_{\alpha\beta}(\boldsymbol{\zeta}_0)=\frac{1}{2}\Big\{b^\sigma_\alpha\rho^\sharp_{\beta\sigma}(\boldsymbol{\zeta}_0)+b^\sigma_\beta\rho^\sharp_{\alpha\sigma}(\boldsymbol{\zeta}_0)\Big\}-b^\sigma_\alpha b^\tau_\beta\gamma_{\sigma\tau}(\boldsymbol{\zeta}_0).$$

Proof. a) It suffices to compute the operators $\rho_{\alpha\beta}(\boldsymbol{\zeta}_0,\boldsymbol{\zeta}_1^K)$, $\delta_{\alpha3}(\boldsymbol{\zeta}_0,\boldsymbol{\zeta}_1^K)$, $\pi_{\alpha\beta}(\boldsymbol{\zeta}_0,\boldsymbol{\zeta}_1^K)$ and $\boldsymbol{\zeta}_1^K\cdot\mathbf{a}_3$. After straightforward computations, we obtain

$$\begin{aligned} \rho_{\alpha\beta}(\boldsymbol{\zeta}_0,\boldsymbol{\zeta}_1^K) =& -\frac{1}{2}\Big\{\partial_\alpha\boldsymbol{\zeta}_0\cdot\partial_\beta\boldsymbol{\zeta}_0+\partial_\beta\boldsymbol{\zeta}_0\cdot\partial_\alpha\boldsymbol{\zeta}_0-\partial_\beta(\partial_\alpha\boldsymbol{\zeta}_0\cdot\mathbf{a}_3)-\partial_\alpha(\partial_\beta\boldsymbol{\zeta}_0\cdot\mathbf{a}_3)\Big\} \\ & -\partial_\alpha\mathbf{a}_\beta\cdot\mathbf{a}^\sigma(\partial_\sigma\boldsymbol{\zeta}_0\cdot\mathbf{a}_3)=\partial_{\alpha\beta}\boldsymbol{\zeta}_0\cdot\mathbf{a}_3-(\partial_{\alpha\beta}\boldsymbol{\theta}\cdot\mathbf{a}^\sigma)(\partial_\sigma\boldsymbol{\zeta}_0\cdot\mathbf{a}_3) \\ =&\rho_{\alpha\beta}(\boldsymbol{\zeta}_0). \end{aligned}$$

and $\delta_{\alpha3}(\boldsymbol{\zeta}_0,\boldsymbol{\zeta}_1^K)=0$ and $\boldsymbol{\zeta}_1^K\cdot\mathbf{a}_3=0$. Since

$$\begin{aligned} \partial_\alpha\boldsymbol{\zeta}_1^K\cdot\partial_\beta\,\mathbf{a}_3 &= -\partial_\alpha((\partial_\sigma\boldsymbol{\zeta}_0\cdot\mathbf{a}_3)\mathbf{a}^\sigma)\cdot\partial_\beta\mathbf{a}_3 \\ &= -(\partial_{\alpha\sigma}\boldsymbol{\zeta}_0\cdot\mathbf{a}_3+\partial_\sigma\boldsymbol{\zeta}_0\cdot\partial_\alpha\mathbf{a}_3)(\mathbf{a}^\sigma\cdot\partial_\beta\mathbf{a}_3)-(\partial_\tau\boldsymbol{\zeta}_0\cdot\mathbf{a}_3)\partial_\alpha\mathbf{a}^\tau\cdot\partial_\beta\mathbf{a}_3 \\ &= -(\mathbf{a}^\sigma\cdot\partial_\beta\mathbf{a}_3)(\partial_{\alpha\sigma}\boldsymbol{\zeta}_0\cdot\mathbf{a}_3-b^\tau_\alpha\partial_\sigma\boldsymbol{\zeta}_0\cdot\mathbf{a}_\tau)+(\partial_\tau\boldsymbol{\zeta}_0\cdot\mathbf{a}_3)\Gamma^\tau_{\alpha\sigma}(\mathbf{a}^\sigma\cdot\partial_\beta\mathbf{a}_3) \\ &= b^\sigma_\beta\{\partial_{\alpha\sigma}\boldsymbol{\zeta}_0\cdot\mathbf{a}_3-\Gamma^\tau_{\alpha\sigma}(\partial_\tau\boldsymbol{\zeta}_0\cdot\mathbf{a}_3)\}-b^\sigma_\beta b^\tau_\alpha\partial_\sigma\boldsymbol{\zeta}_0\cdot\mathbf{a}_\tau \\ &= b^\sigma_\beta\rho_{\alpha\sigma}(\boldsymbol{\zeta}_0)-b^\sigma_\beta b^\tau_\alpha(\partial_\sigma\boldsymbol{\zeta}_0\cdot\mathbf{a}_\tau), \end{aligned}$$

we next obtain

$$\begin{aligned} \pi_{\alpha\beta}(\boldsymbol{\zeta}_0,\boldsymbol{\zeta}_1^K) &= \frac{1}{2}\Big\{b^\sigma_\beta\rho_{\alpha\sigma}(\boldsymbol{\zeta}_0)-b^\sigma_\beta b^\tau_\alpha\partial_\sigma\boldsymbol{\zeta}_0\cdot\mathbf{a}_\tau+b^\sigma_\alpha\rho_{\beta\sigma}(\boldsymbol{\zeta}_0)-b^\sigma_\alpha b^\tau_\beta\partial_\sigma\boldsymbol{\zeta}_0\cdot\mathbf{a}_\tau\Big\} \\ &= \frac{1}{2}\Big\{b^\sigma_\beta\rho_{\alpha\sigma}(\boldsymbol{\zeta}_0)+b^\sigma_\alpha\rho_{\beta\sigma}(\boldsymbol{\zeta}_0)\Big\}-b^\sigma_\alpha b^\tau_\beta\gamma_{\sigma\tau}(\boldsymbol{\zeta}_0) \\ &= \pi^K_{\alpha\beta}(\boldsymbol{\zeta}_0). \end{aligned}$$

b) It suffices to compute the operators $\rho_{\alpha\beta}(\boldsymbol{\zeta}_0,\boldsymbol{\zeta}_1^{K\sharp})$, $\delta_{\alpha3}(\boldsymbol{\zeta}_0,\boldsymbol{\zeta}_1^{K\sharp})$, $\pi_{\alpha\beta}(\boldsymbol{\zeta}_0,\boldsymbol{\zeta}_1^{K\sharp})$ and $\boldsymbol{\zeta}_1^{K\sharp}\cdot\mathbf{a}_3$. Using in particular the above computations, we obtain

$$\begin{aligned} \rho_{\alpha\beta}(\boldsymbol{\zeta}_0,\boldsymbol{\zeta}_1^{K\sharp}) &= \rho_{\alpha\beta}(\boldsymbol{\zeta}_0)-\frac{1}{2}\Big\{\mathbf{a}_\alpha\cdot\partial_\beta(a^{\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{\zeta}_0)\mathbf{a}_3)+\mathbf{a}_\beta\cdot\partial_\alpha(a^{\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{\zeta}_0)\mathbf{a}_3)\Big\} \\ &= \rho_{\alpha\beta}(\boldsymbol{\zeta}_0)+\partial_\beta\mathbf{a}_\alpha\cdot\{a^{\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{\zeta}_0)\mathbf{a}_3\} \\ &= \rho_{\alpha\beta}(\boldsymbol{\zeta}_0)+b_{\alpha\beta}a^{\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{\zeta}_0)=\rho^\sharp_{\alpha\beta}(\boldsymbol{\zeta}_0) \end{aligned}$$

and $\delta_{\alpha 3}(\boldsymbol{\zeta}_0, \boldsymbol{\zeta}_1^{K\sharp}) = 0$ and $\boldsymbol{\zeta}_1^{K\sharp} \cdot \mathbf{a}_3 = a^{\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{\zeta}_0)$ and

$$\begin{aligned}
\pi_{\alpha\beta}(\boldsymbol{\zeta}_0, \boldsymbol{\zeta}_1^{K\sharp}) &= \pi^K_{\alpha\beta}(\boldsymbol{\zeta}_0) + \frac{1}{2}\Big\{\partial_\alpha \mathbf{a}_3 \cdot \partial_\beta(a^{\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{\zeta}_0)\mathbf{a}_3) + \partial_\beta \mathbf{a}_3 \cdot \partial_\alpha(a^{\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{\zeta}_0)\mathbf{a}_3)\Big\} \\
&= \pi^K_{\alpha\beta}(\boldsymbol{\zeta}_0) + (\partial_\alpha \mathbf{a}_3 \cdot \partial_\beta \mathbf{a}_3) a^{\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{\zeta}_0) \\
&= \pi^K_{\alpha\beta}(\boldsymbol{\zeta}_0) + c_{\alpha\beta} a^{\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{\zeta}_0) \\
&= \frac{1}{2}\Big\{b^\sigma_\alpha \rho^\sharp_{\beta\tau}(\boldsymbol{\zeta}_0) - b^\sigma_\alpha b_{\beta\tau} a^{\varphi\psi}\gamma_{\varphi\psi}(\boldsymbol{\zeta}_0) + b^\sigma_\beta \rho^\sharp_{\alpha\tau}(\boldsymbol{\zeta}_0) - b^\sigma_\beta b_{\alpha\tau} a^{\varphi\psi}\gamma_{\varphi\psi}(\boldsymbol{\zeta}_0)\Big\} \\
&\quad - b^\sigma_\alpha b^\tau_\beta \gamma_{\sigma\tau}(\boldsymbol{\zeta}_0) + c_{\alpha\beta} a^{\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{\zeta}_0) \\
&= \frac{1}{2}\Big\{b^\sigma_\alpha \rho^\sharp_{\beta\tau}(\boldsymbol{\zeta}_0) + b^\sigma_\beta \rho^\sharp_{\alpha\tau}(\boldsymbol{\zeta}_0)\Big\} - b^\sigma_\alpha b^\tau_\beta \gamma_{\sigma\tau}(\boldsymbol{\zeta}_0) \\
&= \pi^\sharp_{\alpha\beta}(\boldsymbol{\zeta}_0).
\end{aligned}$$

□

Remarks 7.6. a) Theorem 7.5 shows that the linear Koiter and Ciarlet-Koiter shell models respectively arise from the generalized linear Koiter and Ciarlet-Koiter shell models by neglecting the terms $r_K(\boldsymbol{\varphi}_0)$ and $r^\sharp_K(\boldsymbol{\varphi}_0)$ appearing in the expression of the functional j_{2D}. From the definition of $\pi^K_{\alpha\beta}(\boldsymbol{\zeta}_0)$ and $\pi^\sharp_{\alpha\beta}(\boldsymbol{\zeta}_0)$, one can see that there exists a constant independent of ε such that

$$\varepsilon^2 \sum_{\alpha,\beta} \|\pi^K_{\alpha\beta}(\boldsymbol{\zeta}_0)\|_{L^2(\omega)} \le C \sum_{\alpha,\beta} \Big\{\|\gamma_{\alpha\beta}(\boldsymbol{\varphi}_0)\|_{L^2(\omega)} + \varepsilon\|\rho_{\alpha\beta}(\boldsymbol{\varphi}_0)\|_{L^2(\omega)}\Big\}$$

and

$$\varepsilon^2 \sum_{\alpha,\beta} \|\pi^\sharp_{\alpha\beta}(\boldsymbol{\zeta}_0)\|_{L^2(\omega)} \le C \sum_{\alpha,\beta} \Big\{\|\gamma_{\alpha\beta}(\boldsymbol{\varphi}_0)\|_{L^2(\omega)} + \varepsilon\|\rho^\sharp_{\alpha\beta}(\boldsymbol{\varphi}_0)\|_{L^2(\omega)}\Big\}.$$

This show that $r_K(\boldsymbol{\varphi}_0)$ is indeed negligible and that $r^\sharp_K(\boldsymbol{\varphi}_0)$ can be replaced (the error is negligible) with

$$\frac{\varepsilon^3}{3}\int_\omega \mu a^{\alpha\beta}\partial_\beta(a^{\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{\zeta}_0))\,\partial_\alpha(a^{\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{\zeta}_0))\sqrt{a}dy.$$

This term is not negligible since the norm

$$\varepsilon \sum_\alpha \|\partial_\alpha(a^{\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{\zeta}_0))\|_{L^2(\omega)}$$

depends on higher derivatives of $\boldsymbol{\zeta}$ than the norm defined by the coerciveness of $j^\sharp_K(\boldsymbol{\zeta}_0)$.

b) While $r_K(\boldsymbol{\zeta}_0)$ is negligible when compared with $j_K(\boldsymbol{\zeta}_0)$, the integral appearing in the expression (7.3) of $j_{2D}(\boldsymbol{\zeta}_0)$ is not (for arbitrary densities $\boldsymbol{f}$ of forces). Therefore this integral should be kept in the energy functional j_K of the Koiter's model *at least* when the quotient

$$\frac{\int_\omega \Big\{\int_{-\varepsilon}^{\varepsilon} x_3 \boldsymbol{f} \cdot \mathbf{a}^\sigma dx_3\Big\}(\partial_\sigma \boldsymbol{\zeta}_0 \cdot \mathbf{a}_3)\sqrt{a}dy}{\sum_{\alpha,\beta}\Big\{\varepsilon\|\gamma_{\alpha\beta}(\boldsymbol{\varphi}_0)\|_{L^2(\omega)} + \varepsilon^2\|\rho_{\alpha\beta}(\boldsymbol{\varphi}_0)\|_{L^2(\omega)}\Big\}}$$

does not converge to zero when $\varepsilon \to 0$. A similar observation holds for the integrals appearing in the expression (7.4) of $j^\sharp_K$. This remark explains why we did not include these integrals in the definition of the remainders $r_K(\boldsymbol{\zeta}_0)$ and $r^\sharp_K(\boldsymbol{\zeta}_0)$.

8. **CONCLUSION**

We summarize the derivation of 2D shell models from the nonlinear 3D shell model in the following figure. The notation should be self-explanatory.

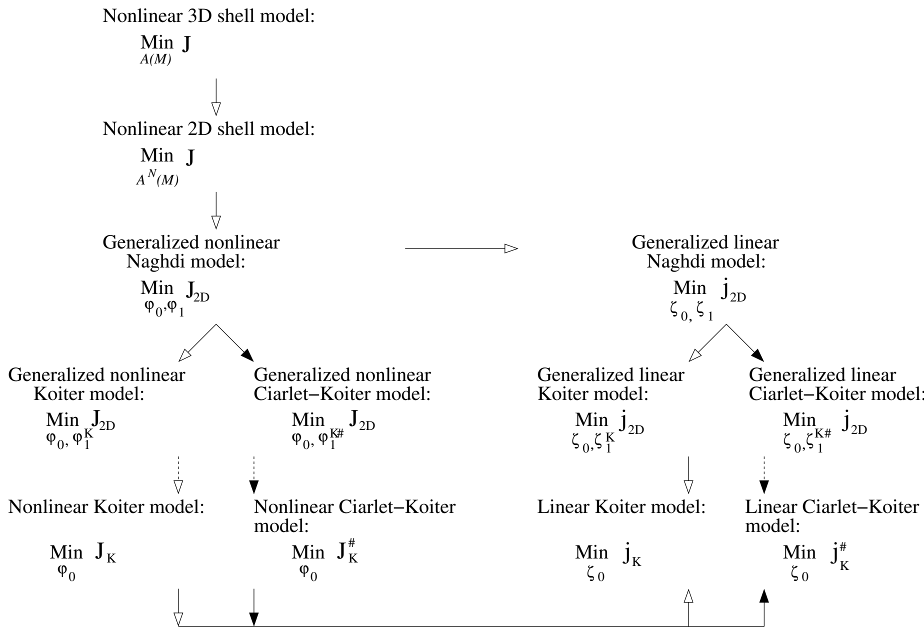

FIGURE 1. *Derivation of 2D shell models from the nonlinear 3D shell model.* An arrow between two models indicates that the model at the head of the arrow is derived from the model at the nock of the arrow. Dotted arrows indicates that the derivation is not fully justified; see Remarks 6.6 and 7.6. The accuracy of the shell models decreases from top to bottom.

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