

# Hyperbolic conservation laws on spacetimes. A finite volume scheme based on differential forms

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## Abstract

We consider nonlinear hyperbolic conservation laws, posed on a differential  $(n + 1)$ -manifold with boundary referred to as a *spacetime*, and in which the “flux” is defined as a *flux field of  $n$ -forms* depending on a parameter (the unknown variable). We introduce a formulation of the initial and boundary value problem which is geometric in nature and is more natural than the vector field approach recently developed for Riemannian manifolds. Our main assumption on the manifold and the flux field is a *global hyperbolicity* condition, which provides a global time-orientation as is standard in Lorentzian geometry and general relativity. Assuming that the manifold admits a foliation by compact slices, we establish the existence of a semi-group of entropy solutions. Moreover, given any two hypersurfaces with one lying in the future of the other, we establish a “contraction” property which compares two entropy solutions, in a (geometrically natural) distance equivalent to the  $L^1$  distance. To carry out the proofs, we rely on a new version of the *finite volume method*, which only requires the knowledge of the given  $n$ -volume form structure on the  $(n + 1)$ -manifold and involves the *total flux* across faces of the elements of the triangulations, only, rather than the product of a numerical flux times the measure of that face.

## 1 Introduction

The development of the mathematical theory (existence, uniqueness, qualitative behavior, approximation) of shock wave solutions to scalar conservation laws *defined on manifolds* is motivated by similar questions arising in compressible fluid dynamics. For instance, the shallow water equations of geophysical fluid dynamics (for which the background manifold is the Earth or, more generally,

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Riemannian manifold) and the Einstein-Euler equations in general relativity (for which the manifold metric is also part of the unknowns) provide important examples where the partial differential equations of interest are naturally posed on a (curved) manifold. Scalar conservation laws yield a drastically simplified, yet very challenging, mathematical model for understanding nonlinear aspects of shock wave propagation on manifolds.

In the present paper, given a (smooth) differential  $(n+1)$ -manifold  $M$  which we refer to as a *spacetime*, we consider the following class of *nonlinear conservation laws*

$$d(\omega(u)) = 0, \quad u = u(x), \quad x \in M. \quad (1.1)$$

Here, for each  $\bar{u} \in \mathbb{R}$ ,  $\omega = \omega(\bar{u})$  is a (smooth) *field of  $n$ -forms* on  $M$  which we refer to as the *flux field* of the conservation law (1.1).

Two special cases of (1.1) were recently studied in the literature. When  $M = \mathbb{R}_+ \times N$  and the  $n$ -manifold  $N$  is endowed with a *Riemannian metric*  $h$ , the conservation law (1.1) is here equivalent to

$$\partial_t u + \operatorname{div}_h(b(u)) = 0, \quad u = u(t, y), \quad t \geq 0, \quad y \in N.$$

Here,  $\operatorname{div}_h$  denotes the divergence operator associated with the metric  $h$ . In this case, the flux field is a *flux vector field*  $b = b(\bar{u})$  on the  $n$ -manifold  $N$  and does not depend on the time variable. More generally, we may suppose that  $M$  is endowed with a *Lorentzian metric*  $g$  and, then, (1.1) takes the equivalent form

$$\operatorname{div}_g(a(u)) = 0, \quad u = u(x), \quad x \in M.$$

Observe that the flux  $a = a(\bar{u})$  is now a vector field on the  $(n+1)$ -manifold  $M$ .

Recall that, in the Riemannian or Lorentzian settings, the theory of weak solutions on manifolds was initiated by Ben-Artzi and LeFloch [4] and further developed in the follow-up papers by LeFloch and his collaborators [1, 2, 21, 22, 23]. Hyperbolic equations on manifolds were also studied by Panov in [25] with a vector field standpoint. The actual implementation of a finite volume scheme on the sphere was recently realized by Ben-Artzi, Falcovitz, and LeFloch [5].

In the present paper, we propose a new approach in which the conservation law is written in the form (1.1), that is, the flux  $\omega = \omega(\bar{u})$  is defined as a *field of differential forms of degree  $n$* . Hence, no geometric structure is a priori assumed on  $M$ , and the sole knowledge of the flux field structure is required. The fact that the equation (1.1) is a “conservation law” for the unknown quantity  $u$  can be understood by expressing Stokes theorem: for sufficiently smooth solutions  $u$ , at least, the conservation law (1.1) is equivalent to saying that the total flux

$$\int_{\partial \mathcal{U}} \omega(u) = 0, \quad \mathcal{U} \subset M, \quad (1.2)$$

vanishes for every open subset  $\mathcal{U}$  with smooth boundary. By relying on the conservation law (1.1) rather than the equivalent expressions in the special cases of Riemannian or Lorentzian manifolds, we are able to develop here a theory of entropy solutions to conservation laws posed on manifolds, which is technically

and conceptually simpler but also provides a significant generalization of earlier works.

Recall that weak solutions to conservation laws contain shock waves and, for the sake of uniqueness, the class of such solutions must be restricted by an entropy condition (Lax [20]). This theory of conservation laws on manifolds is a generalization of fundamental works by Kruzkov [18], Kuznetsov [19], and DiPerna [12] who treated equations posed on the (flat) Euclidian space  $\mathbb{R}^n$ .

Our main result in the present paper is a generalization of the formulation and convergence of the finite volume method for general conservation law (1.1). In turn, we will establish the existence of a semi-group of entropy solutions which is contracting in a suitable distance.

The first difficulty is formulating the initial and boundary problem for (1.1) in the sense of distributions. A weak formulation of the boundary condition is proposed which takes into account the nonlinearity and hyperbolicity of the equation under consideration. We emphasize that our weak formulation applies to an arbitrary differential manifold. However, to proceed with the development of the well-posedness theory we then need to impose that the manifold satisfies a *global hyperbolicity condition*, which provides a global time-orientation and allow us to distinguish between “future” and “past” directions in the time-evolution. This assumption is standard in Lorentzian geometry for applications to general relativity. For simplicity in this paper, we then restrict attention to the case that the manifold is foliated by compact slices.

Second, we introduce a new version of the finite volume method (based on monotone numerical flux terms). The proposed scheme provides a natural discretization of the conservation law (1.1), which solely uses the  $n$ -volume form structure associated with the prescribed flux field  $\omega$ .

Third, we derive several stability estimates satisfied by the proposed scheme, especially discrete versions of the entropy inequalities. As a corollary, we obtain a uniform control of the entropy dissipation measure associated with the scheme, which, however, is not sufficient by itself to the compactness of the sequence of approximate solutions.

The above stability estimates are sufficient to show that the sequence of approximate solutions generated by the finite volume scheme converges to an *entropy measure-valued solution* in the sense of DiPerna. To conclude our proof, we rely on a generalization of DiPerna’s uniqueness theorem [12] and conclude with the existence of entropy solutions to the corresponding initial value problem.

In the course of this analysis, we also establish a *contraction property* for any two entropy solutions  $u, v$ , that is, given two hypersurfaces  $H, H'$  such that  $H'$  lies in the future of  $H$ ,

$$\int_{H'} \Omega(u_{H'}, v_{H'}) \leq \int_H \Omega(u_H, v_H). \quad (1.3)$$

Here, for all reals  $\bar{u}, \bar{v}$ , the  $n$ -form field  $\Omega(\bar{u}, \bar{v})$  is determined from the given flux field  $\omega(\bar{u})$  and can be seen as a generalization (to the spacetime setting) of the notion of Kruzkov entropy  $|\bar{u} - \bar{v}|$ .

Recall that DiPerna's measure-valued solutions were used to establish the convergence of schemes by Szepessy [26, 28], Coquel and LeFloch [9, 10, 11], and Cockburn, Coquel, and LeFloch [6, 7]. For many related results and a review about the convergence techniques for hyperbolic problems, we refer to Tadmor [29] and Tadmor, Rascle, and Bagneiri [30]. Further hyperbolic models including also a coupling with elliptic equations and many applications were successfully investigated in the works by Kröner [16], and Eymard, Gallouet, and Herbin [14]. For higher-order schemes, see the paper by Kröner, Noelle, and Rokyta [17]. Also, an alternative approach to the convergence of finite volume schemes was later proposed by Westdickenberg and Noelle [31]. Finally, note that Kuznetsov's error estimate [6, 8] were recently extended to conservation laws on manifolds by LeFloch, Neves, and Okutmustur [22].

An outline of the paper is as follows. In Section 2, we introduce our definition of entropy solution which includes both initial-boundary data and entropy inequalities. The finite volume method is presented in Section 3, and discrete stability properties are then established in Section 4. The main statements are given at the beginning of Section 5, together with the final step of the convergence proof.

## 2 Conservation laws posed on a spacetime

### 2.1 A notion of weak solution

In this section we assume that  $M$  is an oriented, compact, differentiable  $(n+1)$ -manifold with boundary. Given an  $(n+1)$ -form  $\alpha$ , its *modulus* is defined as the  $(n+1)$ -form

$$|\alpha| := |\bar{\alpha}| dx^0 \wedge \cdots \wedge dx^n,$$

where  $\alpha = \bar{\alpha} dx^1 \wedge \cdots \wedge dx^n$  is written in an oriented frame determined from local coordinates  $x = (x^\alpha) = (x^0, \dots, x^n)$ . If  $H$  is a hypersurface, we denote by  $i = i_H : H \rightarrow M$  the canonical injection map, and by  $i^* = i_H^*$  is the pull-back operator acting on differential forms defined on  $M$ .

On this manifold, we introduce a class of nonlinear hyperbolic equations, as follows.

**Definition 2.1.** *1. A flux field  $\omega$  on the  $(n+1)$ -manifold  $M$  is a parametrized family  $\omega(\bar{u}) \in \Lambda^n(M)$  of smooth fields of differential forms of degree  $n$ , that depends smoothly upon the real parameter  $\bar{u}$ .*

*2. The conservation law associated with a flux field  $\omega$  and with unknown  $u : M \rightarrow \mathbb{R}$  is*

$$d(\omega(u)) = 0, \tag{2.1}$$

*where  $d$  denotes the exterior derivative operator and, therefore,  $d(\omega(u))$  is a field of differential forms of degree  $(n+1)$  on  $M$ .*

*3. A flux field  $\omega$  is said to grow at most linearly if for every 1-form  $\rho$  on  $M$*

$$\sup_{\bar{u} \in \mathbb{R}} \int_M |\rho \wedge \partial_u \omega(\bar{u})| < \infty. \tag{2.2}$$

With the above notation, by introducing local coordinates  $x = (x^\alpha)$  we can write for all  $\bar{u} \in \mathbb{R}$

$$\begin{aligned}\omega(\bar{u}) &= \omega^\alpha(\bar{u}) (\widehat{dx})_\alpha, \\ (\widehat{dx})_\alpha &:= dx^0 \wedge \dots \wedge dx^{\alpha-1} \wedge dx^{\alpha+1} \wedge \dots \wedge dx^n.\end{aligned}$$

Here, the coefficients  $\omega^\alpha = \omega^\alpha(\bar{u})$  are smooth functions defined in the chosen local chart. Recall that the operator  $d$  acts on differential forms with arbitrary degree and that, given a  $p$ -form  $\rho$  and a  $p'$ -form  $\rho'$ , one has  $d(d\rho) = 0$  and  $d(\rho \wedge \rho') = d\rho \wedge \rho' + (-1)^p \rho \wedge d\rho'$ .

As it stands, the equation (2.1) makes sense for unknown functions that are, for instance, Lipschitz continuous. However, it is well-known that solutions to nonlinear hyperbolic equations need not be continuous and, consequently, we need to recast (2.1) in a weak form.

Given a *smooth* solution  $u$  of (2.1) we can apply Stokes theorem on any open subset  $\mathcal{U}$  that is compactly included in  $M$  and has smooth boundary  $\partial\mathcal{U}$ . We obtain

$$0 = \int_{\mathcal{U}} d(\omega(u)) = \int_{\partial\mathcal{U}} i^*(\omega(u)). \quad (2.3)$$

Similarly, given any smooth function  $\psi : M \rightarrow \mathbb{R}$  we can write

$$d(\psi \omega(u)) = d\psi \wedge \omega(u) + \psi d(\omega(u)),$$

where the differential  $d\psi$  is a 1-form field. Provided  $u$  satisfies (2.1), we find

$$\int_M d(\psi \omega(u)) = \int_M d\psi \wedge \omega(u)$$

and, by Stokes theorem,

$$\int_M d\psi \wedge \omega(u) = \int_{\partial M} i^*(\psi \omega(u)). \quad (2.4)$$

Note that a suitable orientation of the boundary  $\partial M$  is required for this formula to hold. This identity is satisfied by every smooth solution to (2.1) and this motivates us to reformulate (2.1) in the following weak form.

**Definition 2.2** (Weak solutions on a spacetime). *Given a flux field with at most linear growth  $\omega$ , a function  $u \in L^1(M)$  is called a weak solution to the conservation law (2.1) posed on the spacetime  $M$  if*

$$\int_M d\psi \wedge \omega(u) = 0$$

for every function  $\psi : M \rightarrow \mathbb{R}$  compactly supported in the interior  $\overset{\circ}{M}$ .

The above definition makes sense since the function  $u$  is integrable and  $\omega(\bar{u})$  has at most linear growth in  $\bar{u}$ , so that the  $(n+1)$ -form  $d\psi \wedge \omega(u)$  is integrable on the compact manifold  $M$ .

## 2.2 Entropy inequalities

As is standard for nonlinear hyperbolic problems, weak solution must be further constrained by imposing initial, boundary, as well as entropy conditions.

**Definition 2.3.** *A (smooth) field of  $n$ -forms  $\Omega = \Omega(\bar{u})$  is called a (convex) entropy flux field for the conservation law (2.1) if there exists a (convex) function  $U : \mathbb{R} \rightarrow \mathbb{R}$  such that*

$$\Omega(\bar{u}) = \int_0^{\bar{u}} \partial_u U(\bar{v}) \partial_u \omega(\bar{v}) d\bar{v}, \quad \bar{u} \in \mathbb{R}.$$

*It is said to be also admissible if, moreover,  $\sup |\partial_u U| < \infty$ .*

For instance, if one chooses the function  $U(\bar{u}, \bar{v}) := |\bar{u} - \bar{v}|$ , where  $\bar{v}$  is a real parameter, the entropy flux field reads

$$\mathbf{\Omega}(\bar{u}, \bar{v}) := \text{sgn}(\bar{u} - \bar{v}) (\omega(\bar{u}) - \omega(\bar{v})), \quad (2.5)$$

which is a generalization to a spacetime of the so-called Kruzkov's entropy-entropy flux pair.

Based on the notion of entropy flux above, we can derive entropy inequalities in the following way. Given any smooth solution  $u$  to (2.1), by multiplying (2.1) by  $\partial_u U(u)$  we obtain the additional conservation law

$$d(\Omega(u)) - (d\Omega)(u) + \partial_u U(u)(d\omega)(u) = 0.$$

However, for discontinuous solutions this identity can not be satisfied as an equality and, instead, we should impose that the entropy inequalities

$$d(\Omega(u)) - (d\Omega)(u) + \partial_u U(u)(d\omega)(u) \leq 0 \quad (2.6)$$

hold in the sense of distributions for all admissible entropy pair  $(U, \Omega)$ . These inequalities can be justified, for instance, via the vanishing viscosity method, that is by searching for weak solutions that are realizable as limits of smooth solutions to the parabolic regularization of (2.1).

It remains to prescribe initial and boundary conditions. We emphasize that, without further assumption on the flux field (to be imposed shortly below), points along the boundary  $\partial M$  can not be distinguished and it is natural to prescribe the trace of the solution along the *whole* of the boundary  $\partial M$ . This is possible provided the boundary data,  $u_B : \partial M \rightarrow \mathbb{R}$ , is assumed by the solution in a suitably *weak sense*. Following Dubois and LeFloch [13], we use the notation

$$u|_{\partial M} \in \mathcal{E}_{U, \Omega}(u_B) \quad (2.7)$$

for all convex entropy pair  $(U, \Omega)$ , where for all reals  $\bar{u}$

$$\mathcal{E}_{U, \Omega}(\bar{u}) := \left\{ \bar{v} \in \mathbb{R} \mid E(\bar{u}, \bar{v}) := \Omega(\bar{u}) + \partial_u U(\bar{u})(\omega(\bar{v}) - \omega(\bar{u})) \leq \Omega(\bar{v}) \right\}.$$

Recall that the boundary conditions for hyperbolic conservation laws (posed on the Euclidian space) were first studied by Bardos, Leroux, and Nedelec [3] in the class of solutions with bounded variation and, then, in the class of measured-valued solutions by Szepessy [27]. Later, a different approach was introduced by Cockburn, Coquel, and LeFloch [8] (see, in particular, the discussion p. 701 therein) in the course of their analysis of the finite volume methods, which was later expanded in Kondo and LeFloch [15]. An alternative and also powerful approach to the boundary conditions for conservation laws was independently introduced by Otto [24] and developed by followers. In the present paper, our proposed formulation of the initial and boundary value problem is a generalization of the works [8] and [15].

**Definition 2.4** (Entropy solutions on a spacetime with boundary). *Let  $\omega = \omega(\bar{u})$  be a flux field with at most linear growth and  $u_B \in L^1(\partial M)$  be a prescribed boundary function. A function  $u \in L^1(M)$  is called an entropy solution to the boundary value problem (2.1) and (2.7) if there exists a bounded and measurable field of  $n$ -forms  $\gamma \in L^1\Lambda^n(\partial M)$  such that*

$$\begin{aligned} & \int_M \left( d\psi \wedge \Omega(u) + \psi (d\Omega)(u) - \psi \partial_u U(u)(d\omega)(u) \right) \\ & + \int_{\partial M} \psi|_{\partial M} \left( i^* \Omega(u_B) + \partial_u U(u_B)(\gamma - i^* \omega(u_B)) \right) \geq 0 \end{aligned}$$

for every admissible convex entropy pair  $(U, \Omega)$  and every smooth function  $\psi : M \rightarrow \mathbb{R}_+$ .

Observe that the above definition makes sense since each of the terms  $d\psi \wedge \Omega(u)$ ,  $(d\Omega)(u)$ ,  $(d\omega)(u)$  belong to  $L^1(M)$ . The above definition can be generalized to encompass solutions within the much larger class of measure-valued mappings. Following DiPerna [12], we consider solutions that are no longer functions but *Young measures*, i.e, weakly measurable maps  $\nu : M \rightarrow \text{Prob}(\mathbb{R})$  taking values within is the set of probability measures  $\text{Prob}(\mathbb{R})$ . For simplicity, we assume that the support  $\text{supp } \nu$  is a compact subset of  $\mathbb{R}$ .

**Definition 2.5.** *Given a flux field  $\omega = \omega(\bar{u})$  with at most linear growth and given a boundary function  $u_B \in L^\infty(\partial M)$ , one says that a compactly supported Young measure  $\nu : M \rightarrow \text{Prob}(\mathbb{R})$  is an entropy measure-valued solution to the boundary value problem (2.1), (2.7) if there exists a bounded and measurable field of  $n$ -forms  $\gamma \in L^\infty\Lambda^n(\partial M)$  such that the inequalities*

$$\begin{aligned} & \int_M \left\langle \nu, d\psi \wedge \Omega(\cdot) + \psi (d(\Omega(\cdot)) - \partial_u U(\cdot)(d\omega)(\cdot)) \right\rangle \\ & + \int_{\partial M} \psi|_{\partial M} \left\langle \nu, \left( i^* \Omega(u_B) + \partial_u U(u_B)(\gamma - i^* \omega(u_B)) \right) \right\rangle \geq 0 \end{aligned}$$

hold for all convex entropy pair  $(U, \Omega)$  and all smooth functions  $\psi \geq 0$ .

### 2.3 Global hyperbolicity and geometric compatibility

In general relativity, it is a standard assumption that the spacetime should be globally hyperbolic. This notion must be adapted to the present setting, since we do not have a Lorentzian structure, but solely the  $n$ -volume form structure associated with the flux field  $\omega$ .

We assume here that the manifold  $M$  is foliated by hypersurfaces, say

$$M = \bigcup_{0 \leq t \leq T} H_t, \quad (2.8)$$

where each slice has the topology of a (smooth)  $n$ -manifold  $N$  with boundary. Topologically we have  $M = [0, T] \times N$ , and the boundary of  $M$  can be decomposed as

$$\begin{aligned} \partial M &= H_0 \cup H_T \cup B, \\ B &= (0, T) \times N := \bigcup_{0 < t < T} \partial H_t. \end{aligned} \quad (2.9)$$

The following definition imposes a non-degeneracy condition on the averaged flux on the hypersurfaces of the foliation.

**Definition 2.6.** *Consider a manifold  $M$  with a foliation (2.8)-(2.9) and let  $\omega = \omega(\bar{u})$  be a flux field. Then, the conservation law (2.1) on the manifold  $M$  is said to satisfy the global hyperbolicity condition if there exist constants  $0 < \underline{c} < \bar{c}$  such that for every non-empty hypersurface  $e \subset H_t$ , the integral  $\int_e i^* \partial_u \omega(0)$  is positive and the function  $\varphi_e : \mathbb{R} \rightarrow \mathbb{R}$ ,*

$$\varphi_e(\bar{u}) := \int_e i^* \omega(\bar{u}) = \frac{\int_e i^* \omega(\bar{u})}{\int_e i^* \partial_u \omega(0)}, \quad \bar{u} \in \mathbb{R}$$

satisfies

$$\underline{c} \leq \partial_u \varphi_e(\bar{u}) \leq \bar{c}, \quad \bar{u} \in \mathbb{R}. \quad (2.10)$$

The function  $\varphi_e$  represents the *averaged flux* along the hypersurface  $e$ . From now we assume that the conditions in Definition 2.6 are satisfied. It is natural to refer to  $H_0$  as an initial hypersurface and to prescribe an ‘‘initial data’’  $u_0 : H_0 \rightarrow \mathbb{R}$  on this hypersurface and, on the other hand, to impose a boundary data  $u_B$  along the submanifold  $B$ . It will be convenient here to use the standard terminology of general relativity and to refer to  $H_t$  as *spacelike hypersurfaces*.

Under the global hyperbolicity condition (2.8)–(2.10), the initial and boundary value problem now takes the following form. The boundary condition (2.7) decomposes into an initial data

$$u_{H_0} = u_0 \quad (2.11)$$

and a boundary condition

$$u|_B \in \mathcal{E}_{U, \Omega}(u_B). \quad (2.12)$$



Correspondingly, the condition in Definition 2.4 now reads

$$\begin{aligned} & \int_M \left( d\psi \wedge \Omega(u) + \psi (d\Omega)(u) - \psi \partial_u U(u)(d\omega)(u) \right) \\ & + \int_B \psi|_{\partial M} (i^* \Omega(u_B) + \partial_u U(u_B)(\gamma - i^* \omega(u_B))) \\ & + \int_{H_T} i^* \Omega(u_{H_T}) - \int_{H_0} i^* \Omega(u_0) \geq 0. \end{aligned}$$

Finally, we introduce:

**Definition 2.7.** *A flux field  $\omega$  is called geometry-compatible if it is closed for each value of the parameter,*

$$(d\omega)(\bar{u}) = 0, \quad \bar{u} \in \mathbb{R}. \quad (2.13)$$

This compatibility condition is natural since it ensures that constants are trivial solutions to the conservation law, a property shared by many models of fluid dynamics (such as the shallow water equations on a curved manifold). When (2.13) holds, then it follows from Definition 2.3 that every entropy flux field  $\Omega$  also satisfies the condition

$$(d\Omega)(\bar{u}) = 0, \quad \bar{u} \in \mathbb{R}.$$

In turn, the entropy inequalities (2.6) for a solution  $u : M \rightarrow \mathbb{R}$  simplify drastically and take the form

$$d(\Omega(u)) \leq 0. \quad (2.14)$$

### 3 Finite volume method on a spacetime

#### 3.1 Assumptions and formulation

From now on we assume that the manifold  $M = [0, T] \times N$  is foliated by slices with *compact* topology  $N$ , and the initial data  $u_0$  is taken to be a bounded function. We also assume that the global hyperbolicity condition holds and that the flux field  $\omega$  is geometry-compatible, which simplifies the presentation but is not an essential assumption.

Let  $\mathcal{T}^h = \bigcup_{K \in \mathcal{T}^h} K$  be a *triangulation* of the manifold  $M$ , that is, a collection of finitely many *cells (or elements)*, determined as the images of polyhedra of  $\mathbb{R}^{n+1}$ , satisfying the following conditions:

- The boundary  $\partial K$  of an element  $K$  is a piecewise smooth,  $n$ -manifold,  $\partial K = \bigcup_{e \subset \partial K} e$  and contains exactly two spacelike faces, denoted by  $e_K^+$  and  $e_K^-$ , and “vertical” elements

$$e^0 \in \partial^0 K := \partial K \setminus \{e_K^+, e_K^-\}.$$

- The intersection  $K \cap K'$  of two distinct elements  $K, K' \in \mathcal{T}^h$  is either a common face of  $K, K'$  or else a submanifold with dimension at most  $(n - 1)$ .
- The triangulation is compatible with the foliation (2.8)-(2.9) in the sense that there exists a sequence of times  $t_0 = 0 < t_1 < \dots < t_N = T$  such that all spacelike faces are submanifolds of  $H_n := H_{t_n}$  for some  $n = 0, \dots, N$ , and determine a triangulation of the slices. We denote by  $\mathcal{T}_0^h$  the set of all elements  $K$  which admit one face belonging to the initial hypersurface  $H_0$ .

We define the measure  $|e|$  of a hypersurface  $e \subset M$  by

$$|e| := \int_e i^* \partial_u \omega(0). \quad (3.1)$$

This quantity is positive if  $e$  is sufficiently “close” to one of the hypersurfaces along which we have assumed the hyperbolicity condition (2.10). Provided  $|e| > 0$  which is the case if  $e$  is included in one of the slices of the foliation, we associate to  $e$  the function  $\varphi_e : \mathbb{R} \rightarrow \mathbb{R}$ , as defined earlier. Recall the following hyperbolicity condition which holds along the triangulation since the spacelike elements are included in the spacelike slices:

$$\underline{c} \leq \partial_u \varphi_{e_K^\pm}(\bar{u}) \leq \bar{c}, \quad K \in \mathcal{T}^h. \quad (3.2)$$

We introduce the finite volume method by formally averaging the conservation law (2.1) over each element  $K \in \mathcal{T}^h$  of the triangulation, as follows. Applying Stokes theorem with a smooth solution  $u$  to (2.1), we get

$$0 = \int_K d(\omega(u)) = \int_{\partial K} i^* \omega(u).$$

Then, decomposing the boundary  $\partial K$  into its parts  $e_K^+, e_K^-$ , and  $\partial^0 K$  we find

$$\int_{e_K^+} i^* \omega(u) - \int_{e_K^-} i^* \omega(u) + \sum_{e^0 \in \partial^0 K} \int_{e^0} i^* \omega(u) = 0. \quad (3.3)$$

Given the averaged values  $u_K^-$  along  $e_K^-$  and  $u_{K_{e^0}}^-$  along  $e^0 \in \partial^0 K$ , we need an approximation  $u_K^+$  of the average value of the solution  $u$  along  $e_K^+$ . To this end, the second term in (3.3) can be approximated by

$$\int_{e_K^-} i^* \omega(u) \approx \int_{e_K^-} i^* \omega(u_K^-) = |e_K^-| \varphi_{e_K^-}(u_K^-)$$

and the last term by

$$\int_{e^0} i^* \omega(u) \approx q_{K, e^0}(u_K^-, u_{K_{e^0}}^-),$$

where the *total discrete flux*  $q_{K,e^0} : \mathbb{R}^2 \rightarrow \mathbb{R}$  (i.e., a scalar-valued function) must be prescribed.

Finally, the proposed version of the *finite volume method* for the conservation law (2.1) takes the form

$$\int_{e_K^+} i^* \omega(u_K^+) = \int_{e_K^-} i^* \omega(u_K^-) - \sum_{e^0 \in \partial^0 K} q_{K,e^0}(u_K^-, u_{K_{e^0}}^-) \quad (3.4)$$

or, equivalently,

$$|e_K^+| \varphi_{e_K^+}(u_K^+) = |e_K^-| \varphi_{e_K^-}(u_K^-) - \sum_{e^0 \in \partial^0 K} q_{K,e^0}(u_K^-, u_{K_{e^0}}^-). \quad (3.5)$$

We assume that the functions  $q_{K,e^0}$  satisfy the following natural assumptions for all  $\bar{u}, \bar{v} \in \mathbb{R}$  :

- *Consistency property* :

$$q_{K,e^0}(\bar{u}, \bar{u}) = \int_{e^0} i^* \omega(\bar{u}). \quad (3.6)$$

- *Conservation property* :

$$q_{K,e^0}(\bar{v}, \bar{u}) = -q_{K_{e^0},e^0}(\bar{u}, \bar{v}). \quad (3.7)$$

- *Monotonicity property* :

$$\partial_{\bar{u}} q_{K,e^0}(\bar{u}, \bar{v}) \geq 0, \quad \partial_{\bar{v}} q_{K,e^0}(\bar{u}, \bar{v}) \leq 0. \quad (3.8)$$

We note that, in our notation, there is some ambiguity with the orientation of the faces of the triangulation. To complete the definition of the scheme we need to specify the discretization of the initial data and we define constant initial values  $u_{K,0} = u_K^-$  (for  $K \in \mathcal{T}_0^h$ ) associated with the initial slice  $H_0$  by setting

$$\int_{e_K^-} i^* \omega(u_K^-) := \int_{e_K^-} i^* \omega(u_0), \quad e_K^- \subset H_0. \quad (3.9)$$

Finally, we define a piecewise constant function  $u^h : M \rightarrow \mathbb{R}$  by setting for every element  $K \in \mathcal{T}^h$

$$u^h(x) = u_K^-, \quad x \in K. \quad (3.10)$$

It will be convenient to introduce  $N_K := \#\partial^0 K$ , the total number of “vertical” neighbors of an element  $K \in \mathcal{T}^h$ , which we suppose to be uniformly bounded. For definiteness, we fix a finite family of local charts covering the manifold  $M$ , and we assume that the parameter  $h$  coincides with the largest diameter of faces  $e_K^\pm$  of elements  $K \in \mathcal{T}^h$ , where the diameter is computed with the Euclidian metric expressed in the chosen local coordinates (which are fixed once for all and, of course, overlap in certain regions of the manifold).

For the sake of stability we will need to restrict the time-evolution and impose the following version of the Courant-Friedrich-Levy condition: for all  $K \in \mathcal{T}^h$ ,

$$\frac{N_K}{|e_K^+|} \max_{e^0 \in \partial^0 K} \sup_u \left| \int_{e^0} \partial_u \omega(u) \right| < \inf_u \partial_u \varphi_{e_K^+}, \quad (3.11)$$

in which the supremum and infimum in  $u$  are taken over the range of the initial data.

We then assume the following conditions on the family of triangulations:

$$\lim_{h \rightarrow 0} \frac{\tau_{\max}^2 + h^2}{\tau_{\min}} = \lim_{h \rightarrow 0} \frac{\tau_{\max}^2}{h} = 0 \quad (3.12)$$

where  $\tau_{\max} := \max_i(t_{i+1} - t_i)$  and  $\tau_{\min} := \min_i(t_{i+1} - t_i)$ . For instance, these conditions are satisfied if  $\tau_{\max}$ ,  $\tau_{\min}$ , and  $h$  vanish at the same order.

Our main objective in the rest of this paper is to prove the convergence of the above scheme towards an entropy solution in the sense defined in the previous section.

### 3.2 A convex decomposition

Our analysis of the finite volume method relies on a decomposition of (3.5) into essentially one-dimensional schemes. This technique goes back to Tadmor [29], Coquel and LeFloch [9], and Cockburn, Coquel, and LeFloch [8].

By applying Stokes theorem to (2.13) with an arbitrary  $\bar{u} \in \mathbb{R}$ , we have

$$\begin{aligned} 0 &= \int_K d(\omega(\bar{u})) = \int_{\partial K} i^* \omega(\bar{u}) \\ &= \int_{e_K^+} i^* \omega(\bar{u}) - \int_{e_K^-} i^* \omega(\bar{u}) + \sum_{e^0 \in \partial^0 K} q_{K,e^0}(\bar{u}, \bar{u}). \end{aligned}$$

Choosing  $\bar{u} = u_K^-$ , we deduce the identity

$$|e_K^+| \varphi_{e_K^+}(u_K^-) = |e_K^-| \varphi_{e_K^-}(u_K^-) - \sum_{e^0 \in \partial^0 K} q_{K,e^0}(u_K^-, u_K^-), \quad (3.13)$$

which can be combined with (3.5) so that

$$\begin{aligned} &\varphi_{e_K^+}(u_K^+) \\ &= \varphi_{e_K^+}(u_K^-) - \sum_{e^0 \in \partial^0 K} \frac{1}{|e_K^+|} \left( q_{K,e^0}(u_K^-, u_{K_{e^0}^-}) - q_{K,e^0}(u_K^-, u_K^-) \right) \\ &= \sum_{e^0 \in \partial^0 K} \left( \frac{1}{N_K} \varphi_{e_K^+}(u_K^-) - \frac{1}{|e_K^+|} \left( q_{K,e^0}(u_K^-, u_{K_{e^0}^-}) - q_{K,e^0}(u_K^-, u_K^-) \right) \right). \end{aligned}$$

By introducing the intermediate values  $\tilde{u}_{K,e^0}^+$  given by

$$\varphi_{e_K^+}(\tilde{u}_{K,e^0}^+) := \varphi_{e_K^+}(u_K^-) - \frac{N_K}{|e_K^+|} \left( q_{K,e^0}(u_K^-, u_{K_{e^0}^-}) - q_{K,e^0}(u_K^-, u_K^-) \right), \quad (3.14)$$

we arrive at the desired *convex decomposition*

$$\varphi_{e_K^+}(u_K^+) = \frac{1}{N_K} \sum_{e^0 \in \partial^0 K} \varphi_{e_K^+}(\tilde{u}_{K,e^0}^+). \quad (3.15)$$

Given any entropy pair  $(U, \Omega)$  and any hypersurface  $e \subset M$  satisfying  $|e| > 0$  we introduce the averaged entropy flux along  $e$  defined by

$$\varphi_e^\Omega(u) := \int_e i^* \Omega(u).$$

Obviously, we have  $\varphi_e^\omega(u) = \varphi_e(u)$ .

**Lemma 3.1.** *For every convex entropy flux  $\Omega$  one has*

$$\varphi_{e_K^+}^\Omega(u_K^+) \leq \frac{1}{N_K} \sum_{e^0 \in \partial^0 K} \varphi_{e_K^+}^\Omega(\tilde{u}_{K,e^0}^+). \quad (3.16)$$

The proof below will actually show that the function  $\varphi_{e_K^+}^\Omega \circ (\varphi_{e_K^+}^\omega)^{-1}$  is convex.

*Proof.* It suffices to show the inequality for the entropy flux themselves, and then to average this inequality over  $e$ . So, we need to check:

$$\Omega(u_K^+) \leq \frac{1}{N_K} \sum_{e^0 \in \partial^0 K} \Omega(\tilde{u}_{K,e^0}^+). \quad (3.17)$$

Namely, we have

$$\begin{aligned} & \frac{1}{N_K} \sum_{e^0 \in \partial^0 K} (\Omega(\tilde{u}_{K,e^0}^+) - \Omega(u_K^+)) \\ &= \frac{1}{N_K} \sum_{e^0 \in \partial^0 K} (\omega(u_K^+) - \omega(\tilde{u}_{K,e^0}^+)) \partial_u U(u_K^+) + \frac{1}{N_K} \sum_{e^0 \in \partial^0 K} D_{K,e^0}, \end{aligned}$$

with

$$D_{K,e^0} := \int_0^1 \partial_{uu} U(u_K^+) \left( \omega(\tilde{u}_{K,e^0}^+ + a(u_K^+ - \tilde{u}_{K,e^0}^+)) - \omega(\tilde{u}_{K,e^0}^+) \right) (u_K^+ - \tilde{u}_{K,e^0}^+) da.$$

In the right-hand side of the above identity, the former term vanishes identically in view of (3.14) while the latter term is non-negative since  $U(u)$  is convex in  $u$  and  $\partial_u \omega$  is a positive  $n$ -form.  $\square$

## 4 Discrete stability estimates

### 4.1 Entropy inequalities

Using the convex decomposition (3.15), we can derive a discrete version of the entropy inequalities.

**Lemma 4.1** (Entropy inequalities for the faces). *For every convex entropy pair  $(U, \Omega)$  and all  $K \in \mathcal{T}^h$  and  $e^0 \in \partial^0 K$ , there exists a family of numerical entropy flux functions  $Q_{K,e^0} : \mathbb{R}^2 \rightarrow \mathbb{R}$  satisfying the following conditions for all  $u, v \in \mathbb{R}$ :*

- $Q_{K,e^0}$  is consistent with the entropy flux  $\Omega$ :

$$Q_{K,e^0}(u, u) = \int_{e^0} i^* \Omega(u). \quad (4.1)$$

- Conservation property:

$$Q_{K,e^0}(u, v) = -Q_{K_{e^0},e^0}(v, u). \quad (4.2)$$

- Discrete entropy inequality: with the notation introduced earlier, the finite volume scheme satisfies

$$\varphi_{e_K^+}^\Omega(\tilde{u}_{K,e^0}^+) - \varphi_{e_K^+}^\Omega(u_K^-) + \frac{N_K}{|e_K^+|} \left( Q_{K,e^0}(u_K^-, u_{K_{e^0}}^-) - Q_{K,e^0}(u_K^-, u_K^-) \right) \leq 0. \quad (4.3)$$

Combining Lemma 3.1 with the above lemma immediately implies:

**Lemma 4.2** (Entropy inequalities for the elements). *For each  $K \in \mathcal{T}^h$  one has the inequality*

$$|e_K^+| \left( \varphi_{e_K^+}^\Omega(u_K^+) - \varphi_{e_K^+}^\Omega(u_K^-) \right) + \sum_{e^0 \in \partial^0 K} \left( Q(u_K^-, u_{K_{e^0}}^-) - Q(u_K^-, u_K^-) \right) \leq 0. \quad (4.4)$$

*Proof of Lemma 4.1. Step 1.* For  $u, v \in \mathbb{R}$  and  $e^0 \in \partial^0 K$  we introduce the notation

$$H_{K,e^0}(u, v) := \varphi_{e_K^+}(u) - \frac{N_K}{|e_K^+|} \left( q_{K,e^0}(u, v) - q_{K,e^0}(u, u) \right).$$

Observe that

$$H_{K,e^0}(u, u) = \varphi_{e_K^+}(u).$$

We claim that  $H_{K,e^0}$  satisfies the following properties:

$$\frac{\partial}{\partial u} H_{K,e^0}(u, v) \geq 0, \quad \frac{\partial}{\partial v} H_{K,e^0}(u, v) \geq 0. \quad (4.5)$$

The proof of the second property is immediate by the monotonicity property (3.8), whereas, for the first one, we use the CFL condition (3.11) together with the monotonicity property (3.8). From the definition of  $H_{K,e^0}(u, v)$ , we observe that

$$H_{K,e^0}(u, u_{K_{e^0}}) = \left( 1 - \sum_{e^0 \in \partial^0 K} \alpha_{K,e^0} \right) \varphi_{e_K^+}(u) + \sum_{e^0 \in \partial^0 K} \alpha_{K,e^0} \varphi_{e_K^+}(u_{K_{e^0}}),$$

where

$$\alpha_{K,e^0} := \frac{1}{|e_K^+|} \frac{q_{K,e^0}(u, u_{K_{e^0}}) - q_{K,e^0}(u, u)}{\varphi_{e_K^+}(u) - \varphi_{e_K^+}(u_{K_{e^0}})}.$$

This gives a convex combination of  $\varphi_{e_K^+}(u)$  and  $\varphi_{e_K^+}(u_{K_{e^0}})$ . Indeed, by the monotonicity property (3.8) we have  $\sum_{e^0 \in \partial^0 K} \alpha_{K,e^0} \geq 0$  and the CFL condition (3.11) gives us

$$\sum_{e^0 \in \partial^0 K} \alpha_{K,e^0} \leq \sum_{e^0 \in \partial^0 K} \frac{1}{|e_K^+|} \left| \frac{q_{K,e^0}(u, u_{K_{e^0}}) - q_{K,e^0}(u, u)}{\varphi_{e_K^+}(u) - \varphi_{e_K^+}(u_{K_{e^0}})} \right| \leq 1.$$

*Step 2.* It is sufficient to establish the entropy inequalities for the family of Kruzkov's entropies  $\Omega$ . In connection with this choice, we introduce the numerical version of Kruzkov's entropy flux

$$\mathbf{Q}(u, v, c) := q_{K,e^0}(u \vee c, v \vee c) - q_{K,e^0}(u \wedge c, v \wedge c),$$

where  $a \vee b = \max(a, b)$  and  $a \wedge b = \min(a, b)$ . Observe that  $Q_{K,e^0}(u, v)$  satisfies the first two properties of the lemma with the entropy flux replaced by the Kruzkov's family of entropies  $\Omega = \mathbf{\Omega}$  defined in (2.5).

First, we observe

$$\begin{aligned} & H_{K,e^0}(u \vee c, v \vee c) - H_{K,e^0}(u \wedge c, v \wedge c) \\ &= \varphi_{e_K^+}(u \vee c) - \frac{N_K}{|e_K^+|} (q_{K,e^0}(u \vee c, v \vee c) - q_{K,e^0}(u \vee c, u \vee c)) \\ &\quad - \left( \varphi_{e_K^+}(u \wedge c) - \frac{N_K}{|e_K^+|} (q_{K,e^0}(u \wedge c, v \wedge c) - q_{K,e^0}(u \wedge c, u \wedge c)) \right) \\ &= \varphi_{e_K^+}^\Omega(u, c) - \frac{N_K}{|e_K^+|} (\mathbf{Q}(u, v, c) - \mathbf{Q}(u, u, c)), \end{aligned} \tag{4.6}$$

where we used

$$\varphi_{e_K^+}(u \vee c) - \varphi_{e_K^+}(u \wedge c) = \int_{e_K^+} i^* \mathbf{\Omega}(u, c) = \varphi_{e_K^+}^\Omega(u, c).$$

Second, we check that for  $u = u_{\bar{K}}$ ,  $v = u_{\bar{K}_{e^0}}$  and for any  $c \in \mathbb{R}$

$$H_{K,e^0}(u_{\bar{K}} \vee c, u_{\bar{K}_{e^0}} \vee c) - H_{K,e^0}(u_{\bar{K}} \wedge c, u_{\bar{K}_{e^0}} \wedge c) \geq \varphi_{e_K^+}^\Omega(\tilde{u}_{K,e^0}^+, c). \tag{4.7}$$

To prove (4.7) we observe that

$$\begin{aligned} H_{K,e^0}(u, v) \vee H_{K,e^0}(\lambda, \lambda) &\leq H_{K,e^0}(u \vee \lambda, v \vee \lambda), \\ H_{K,e^0}(u, v) \wedge H_{K,e^0}(\lambda, \lambda) &\geq H_{K,e^0}(u \wedge \lambda, v \wedge \lambda), \end{aligned}$$

where  $H_{K,e^0}$  is monotone in both variables. Since  $\varphi_{e_K^+}$  is monotone, we have

$$\begin{aligned}
& H_{K,e^0}(u_K^- \vee c, u_{K_{e^0}}^- \vee c) - H_{K,e^0}(u_K^- \wedge c, u_{K_{e^0}}^- \wedge c) \\
& \geq \left| H_{K,e^0}(u_K^-, u_{K_{e^0}}^-) - H_{K,e^0}(c, c) \right| = \left| \varphi_{e_K^+}(\tilde{u}_{K,e^0}^+) - \varphi_{e_K^+}(c) \right| \\
& = \operatorname{sgn}(\varphi_{e_K^+}(\tilde{u}_{K,e^0}^+) - \varphi_{e_K^+}(c)) (\varphi_{e_K^+}(\tilde{u}_{K,e^0}^+) - \varphi_{e_K^+}(c)) \\
& = \operatorname{sgn}(\tilde{u}_{K,e^0}^+ - c) (\varphi_{e_K^+}(\tilde{u}_{K,e^0}^+) - \varphi_{e_K^+}(c)) = \varphi_{e_K^+}^\Omega(\tilde{u}_{K,e^0}^+, c).
\end{aligned}$$

Combining this identity with (4.6) (with  $u = u_K^-$ ,  $v = u_{K_{e^0}}^-$ ), we obtain the following inequality for the Kruzkov's entropies

$$\varphi_{e_K^+}^\Omega(\tilde{u}_{K,e^0}^+, c) - \varphi_{e_K^+}^\Omega(u_K^-, c) + \frac{N_K}{|e_K^+|} \left( \mathbf{Q}(u, v, c) - \mathbf{Q}(u, u, c) \right) \leq 0.$$

As already noticed, this inequality implies a similar inequality for all convex entropy flux fields and this completes the proof.  $\square$

If  $V$  is a convex function, then a *modulus of convexity* for  $V$  is any positive real  $\beta < \inf V''$ , where the infimum is taken over the range of data under consideration. We have seen in the proof of Lemma 3.1 that  $\varphi_e^\Omega \circ (\varphi_e^\omega)^{-1}$  is convex for every spacelike hypersurface  $e$  and every convex function  $U$  (involved in the definition of  $\Omega$ ).

**Lemma 4.3** (Entropy balance inequality between two hypersurfaces). *For  $K \in \mathcal{T}^h$ , let  $\beta_{e_K^+}$  be a modulus of convexity for the function  $\varphi_{e_K^+}^\Omega \circ (\varphi_{e_K^+}^\omega)^{-1}$  and set  $\beta = \min_{K \in \mathcal{T}^h} \beta_{e_K^+}$ . Then, for  $i \leq j$  one has*

$$\begin{aligned}
& \sum_{K \in \mathcal{T}_{t_j}^h} |e_K^+| \varphi_{e_K^+}^\Omega(u_K^+) + \sum_{\substack{K \in \mathcal{T}_{[t_i, t_j]}^h \\ e^0 \in \partial^0 K}} \frac{\beta}{2N_K} |e_K^+| |\tilde{u}_{K,e^0}^+ - u_K^+|^2 \\
& \leq \sum_{K \in \mathcal{T}_{t_i}^h} |e_K^-| \varphi_{e_K^-}^\Omega(u_K^-),
\end{aligned} \tag{4.8}$$

where  $\mathcal{T}_{t_i}^h$  is the subset of all elements  $K$  satisfying  $e_K^- \in H_{t_i}$  while  $\mathcal{T}_{[t_i, t_j]}^h := \bigcup_{i \leq k < j} \mathcal{T}_{t_k}^h$ .

We observe that the numerical entropy flux terms no longer appear in (4.8).

*Proof.* Consider the discrete entropy inequality (4.3). Multiplying by  $|e_K^+|/N_K$  and summing in  $K \in \mathcal{T}^h$ ,  $e^0 \in \partial^0 K$  gives

$$\begin{aligned}
& \sum_{\substack{K \in \mathcal{T}^h \\ e^0 \in \partial^0 K}} \frac{|e_K^+|}{N_K} \varphi_{e_K^+}^\Omega(\tilde{u}_{K,e^0}^+) - \sum_{K \in \mathcal{T}^h} |e_K^+| \varphi_{e_K^+}^\Omega(u_K^-) \\
& + \sum_{\substack{K \in \mathcal{T}^h \\ e^0 \in \partial^0 K}} (Q_{K,e^0}(u_K^-, u_{K_{e^0}}^-) - Q_{K,e^0}(u_K^-, u_K^-)) \leq 0.
\end{aligned} \tag{4.9}$$



Next, observe that the conservation property (4.2) gives

$$\sum_{\substack{K \in \mathcal{T}^h \\ e^0 \in \partial^0 K}} Q_{K,e^0}(u_{\bar{K}}, u_{\bar{K}e^0}) = 0. \quad (4.10)$$

So (4.9) becomes

$$\begin{aligned} & \sum_{\substack{K \in \mathcal{T}^h \\ e^0 \in \partial^0 K}} \frac{|e_K^+|}{N_K} \varphi_{e_K^+}^\Omega(\tilde{u}_{K,e^0}^+) - \sum_{K \in \mathcal{T}^h} |e_K^+| \varphi_{e_K^+}^\Omega(u_{\bar{K}}) \\ & - \sum_{\substack{K \in \mathcal{T}^h \\ e^0 \in \partial^0 K}} Q_{K,e^0}(u_{\bar{K}}, u_{\bar{K}}) \leq 0. \end{aligned} \quad (4.11)$$

Now, if  $V$  is a convex function, and if  $v = \sum_j \alpha_j v_j$  is a convex combination of  $v_j$ , then an elementary result on convex functions gives

$$V(v) + \frac{\beta}{2} \sum_j \alpha_j |v_j - v|^2 \leq \sum_j \alpha_j V(v_j),$$

where  $\beta = \inf V''$ , the infimum being taken over all  $v_j$ . We apply this inequality with  $v = \varphi_{e_K^+}(u_K^+)$  and  $V = \varphi_{e_K^+}^\Omega \circ (\varphi_{e_K^+}^\omega)^{-1}$ , which is convex.

Thus, in view of the convex combination (3.15) and by multiplying the above inequality by  $|e_K^+|$  and then summing in  $K \in \mathcal{T}^h$ , we obtain

$$\begin{aligned} & \sum_{\substack{K \in \mathcal{T}^h \\ e^0 \in \partial^0 K}} |e_K^+| \varphi_{e_K^+}^\Omega(u_K^+) + \sum_{\substack{K \in \mathcal{T}^h \\ e^0 \in \partial^0 K}} \frac{\beta}{2} \frac{|e_K^+|}{N_K} |\tilde{u}_{K,e^0}^+ - u_K^+|^2 \\ & \leq \sum_{\substack{K \in \mathcal{T}^h \\ e^0 \in \partial^0 K}} \frac{|e_K^+|}{N_K} \varphi_{e_K^+}^\Omega(\tilde{u}_{K,e^0}^+). \end{aligned}$$

Combining the result with (4.11), we find

$$\begin{aligned} & \sum_{K \in \mathcal{T}^h} |e_K^+| \varphi_{e_K^+}^\Omega(u_K^+) - \sum_{K \in \mathcal{T}^h} |e_K^+| \varphi_{e_K^+}^\Omega(u_{\bar{K}}) + \sum_{\substack{K \in \mathcal{T}^h \\ e^0 \in \partial^0 K}} \frac{\beta}{2} \frac{|e_K^+|}{N_K} |\tilde{u}_{K,e^0}^+ - u_K^+|^2 \\ & \leq \sum_{\substack{K \in \mathcal{T}^h \\ e^0 \in \partial^0 K}} Q_{K,e^0}(u_{\bar{K}}, u_{\bar{K}}). \end{aligned} \quad (4.12)$$

Using finally the identity

$$\begin{aligned} 0 &= \int_K d(\Omega(u_{\bar{K}})) = \int_{\partial K} i^* \Omega(u_{\bar{K}}) \\ &= |e_K^+| \varphi_{e_K^+}^\Omega(u_{\bar{K}}) - |e_{\bar{K}}^-| \varphi_{e_{\bar{K}}^-}^\Omega(u_{\bar{K}}) + \sum_{e^0 \in \partial^0 K} Q_{K,e^0}(u_{\bar{K}}, u_{\bar{K}}), \end{aligned}$$

we obtain the desired inequality, after further summation over all of the elements  $K$  within two arbitrary hypersurfaces.  $\square$

We apply Lemma 4.3 with a specific choice of entropy function  $U$  and obtain the following uniform estimate.

**Lemma 4.4** (Global entropy dissipation estimate). *The following global estimate of the entropy dissipation holds:*

$$\sum_{\substack{K \in \mathcal{T}^h \\ e^0 \in \partial^0 K}} \frac{|e_K^+|}{N_K} |\tilde{u}_{K,e^0}^+ - u_K^+|^2 \lesssim C \int_{H_0} i^* \Omega(u_0) \quad (4.13)$$

for some uniform constant  $C > 0$ , which only depends upon the flux field and the sup-norm of the initial data, and where  $\Omega$  is the  $n$ -form entropy flux field associated with the quadratic entropy function  $U(u) = u^2/2$ .

*Proof.* We apply the inequality (4.8) with the choice  $U(u) = u^2$

$$0 \geq \sum_{K \in \mathcal{T}^h} (|e_K^+| \varphi_{e_K^+}^\Omega(u_K^+) - |e_K^-| \varphi_{e_K^-}^\Omega(u_K^-)) + \sum_{\substack{K \in \mathcal{T}^h \\ e^0 \in \partial^0 K}} \frac{\beta |e_K^+|}{2 N_K} |\tilde{u}_{K,e^0}^+ - u_K^+|^2.$$

After summing up in the “vertical” direction and keeping only the contribution of the elements  $K \in \mathcal{T}_0^h$  on the initial hypersurface  $H_0$ , we find

$$\sum_{\substack{K \in \mathcal{T}^h \\ e^0 \in \partial^0 K}} \frac{|e_K^+|}{N_K} \beta |\tilde{u}_{K,e^0}^+ - u_K^+|^2 \leq \frac{2}{\beta} \sum_{K \in \mathcal{T}_0^h} |e_K^-| \varphi_{e_K^-}^\Omega(u_{K,0}).$$

Finally, we observe that, for some uniform constant  $C > 0$ ,

$$\sum_{K \in \mathcal{T}_0^h} |e_K^-| \varphi_{e_K^-}^\Omega(u_{K,0}) \leq C \int_{H_0} i^* \Omega(u_0).$$

These expressions are essentially  $L^2$  norm of the initial data, and the above inequality can be checked by fixing a reference volume form on the initial hypersurface  $H_0$  and using the discretization (3.9) of the initial data  $u_0$ .  $\square$

## 4.2 Global form of the discrete entropy inequalities

We now derive a global version of the (local) entropy inequality (4.3), i.e. we obtain a discrete version of the entropy inequalities arising in the very definition of entropy solutions.

One additional notation is necessary to handle “vertical face” of the triangulation: we fix a reference field of non-degenerate  $n$ -forms  $\tilde{\omega}$  on  $M$  which will be used to measure the “area” of the faces  $e^0 \in \partial K^0$ . This is necessary in our

convergence proof, only, not in the formulation of the finite volume method. So, for every  $K \in \mathcal{T}^h$  we define

$$|e^0|_{\tilde{\omega}} := \int_{e^0} i^* \tilde{\omega} \quad \text{for faces } e^0 \in \partial^0 K \quad (4.14)$$

and the non-degeneracy condition means that  $|e^0|_{\tilde{\omega}} > 0$ .

Given a test-function  $\psi$  defined on  $M$  and a face  $e^0 \in \partial^0 K$  of some element, we introduce the following averages

$$\psi_{e^0} := \frac{\int_{e^0} \psi i^* \tilde{\omega}}{\int_{e^0} i^* \tilde{\omega}}, \quad \psi_{\partial^0 K} := \frac{1}{N_K} \sum_{e^0 \in \partial^0 K} \psi_{e^0},$$

where, for the first time in our analysis, we use the reference  $n$ -volume form  $\tilde{\omega}$ .

**Lemma 4.5** (Global form of the discrete entropy inequalities). *Let  $\Omega$  be a convex entropy flux field, and let  $\psi$  be a non-negative test-function supported away from the hypersurface  $t = T$ . Then, the finite volume approximations satisfy the global entropy inequality*

$$\begin{aligned} & - \sum_{K \in \mathcal{T}^h} \int_K d(\psi \Omega)(u_{\bar{K}}) - \sum_{K \in \mathcal{T}_0^h} \int_{e_{\bar{K}}} \psi i^* \Omega(u_{K,0}) \\ & \leq A^h(\psi) + B^h(\psi) + C^h(\psi), \end{aligned} \quad (4.15)$$

with

$$\begin{aligned} A^h(\psi) &:= \sum_{\substack{K \in \mathcal{T}^h \\ e^0 \in \partial^0 K}} \frac{|e_K^+|}{N_K} (\psi_{\partial^0 K} - \psi_{e^0}) (\varphi_{e_K^+}^\Omega(\tilde{u}_{K,e^0}^+) - \varphi_{e_K^+}^\Omega(u_K^+)), \\ B^h(\psi) &:= \sum_{\substack{K \in \mathcal{T}^h \\ e^0 \in \partial^0 K}} \int_{e^0} (\psi_{e^0} - \psi) i^* \Omega(u_{\bar{K}}), \\ C^h(\psi) &:= - \sum_{K \in \mathcal{T}^h} \int_{e_K^+} (\psi_{\partial^0 K} - \psi) (i^* \Omega(u_K^+) - i^* \Omega(u_{\bar{K}})). \end{aligned}$$

*Proof.* From the discrete entropy inequalities (4.3), we obtain

$$\begin{aligned} & \sum_{\substack{K \in \mathcal{T}^h \\ e^0 \in \partial^0 K}} \frac{|e_K^+|}{N_K} \psi_{e^0} (\varphi_{e_K^+}^\Omega(\tilde{u}_{K,e^0}^+) - \varphi_{e_K^+}^\Omega(u_{\bar{K}})) \\ & + \sum_{\substack{K \in \mathcal{T}^h \\ e^0 \in \partial^0 K}} \psi_{e^0} (Q_{K,e^0}(u_{\bar{K}}, u_{\bar{K}_{e^0}}) - Q_{K,e^0}(u_{\bar{K}}, u_{\bar{K}})) \leq 0. \end{aligned} \quad (4.16)$$

Thanks the conservation property (4.2), we have

$$\sum_{\substack{K \in \mathcal{T}^h \\ e^0 \in \partial^0 K}} \psi_{e^0} Q_{K,e^0}(u_{\bar{K}}, u_{\bar{K}_{e^0}}) = 0$$

and, from the consistency property (4.1),

$$\begin{aligned} \sum_{\substack{K \in \mathcal{T}^h \\ e^0 \in \partial^0 K}} \psi_{e^0} Q_{K,e^0}(u_{\bar{K}}, u_{\bar{K}}) &= \sum_{\substack{K \in \mathcal{T}^h \\ e^0 \in \partial^0 K}} \psi_{e^0} \int_{e^0} i^* \Omega(u_{\bar{K}}) \\ &= \sum_{\substack{K \in \mathcal{T}^h \\ e^0 \in \partial^0 K}} \int_{e^0} \psi i^* \Omega(u_{\bar{K}}) + \sum_{\substack{K \in \mathcal{T}^h \\ e^0 \in \partial^0 K}} \int_{e^0} (\psi_{e^0} - \psi) i^* \Omega(u_{\bar{K}}). \end{aligned}$$

Next, we observe

$$\begin{aligned} &\sum_{\substack{K \in \mathcal{T}^h \\ e^0 \in \partial^0 K}} \frac{|e_K^+|}{N_K} \psi_{e^0} \varphi_{e_K^+}^\Omega(\tilde{u}_{K,e^0}^+) \\ &= \sum_{\substack{K \in \mathcal{T}^h \\ e^0 \in \partial^0 K}} \frac{|e_K^+|}{N_K} \psi_{\partial^0 K} \varphi_{e_K^+}^\Omega(\tilde{u}_{K,e^0}^+) + \sum_{\substack{K \in \mathcal{T}^h \\ e^0 \in \partial^0 K}} \frac{|e_K^+|}{N_K} (\psi_{e^0} - \psi_{\partial^0 K}) \varphi_{e_K^+}^\Omega(\tilde{u}_{K,e^0}^+) \\ &\geq \sum_{K \in \mathcal{T}^h} |e_K^+| \psi_{\partial^0 K} \varphi_{e_K^+}^\Omega(u_K^+) + \sum_{\substack{K \in \mathcal{T}^h \\ e^0 \in \partial^0 K}} \frac{|e_K^+|}{N_K} (\psi_{e^0} - \psi_{\partial^0 K}) \varphi_{e_K^+}^\Omega(\tilde{u}_{K,e^0}^+), \end{aligned}$$

where, we used the inequality (3.16) and the convex combination (3.15). In view of

$$\sum_{\substack{K \in \mathcal{T}^h \\ e^0 \in \partial^0 K}} \frac{|e_K^+|}{N_K} \psi_{e^0} \varphi_{e_K^+}^\Omega(u_{\bar{K}}) = \sum_{K \in \mathcal{T}^h} |e_K^+| \psi_{\partial^0 K} \varphi_{e_K^+}^\Omega(u_{\bar{K}}),$$

the inequality (4.16) becomes

$$\begin{aligned} &\sum_{K \in \mathcal{T}^h} |e_K^+| \psi_{\partial^0 K} \left( \varphi_{e_K^+}^\Omega(u_K^+) - \varphi_{e_K^+}^\Omega(u_{\bar{K}}) \right) - \sum_{\substack{K \in \mathcal{T}^h \\ e^0 \in \partial^0 K}} \int_{e^0} \psi i^* \Omega(u_{\bar{K}}) \\ &\leq - \sum_{\substack{K \in \mathcal{T}^h \\ e^0 \in \partial^0 K}} \frac{|e_K^+|}{N_K} (\psi_{e^0} - \psi_{\partial^0 K}) \varphi_{e_K^+}^\Omega(\tilde{u}_{K,e^0}^+) + \sum_{\substack{K \in \mathcal{T}^h \\ e^0 \in \partial^0 K}} \int_{e^0} (\psi_{e^0} - \psi) i^* \Omega(u_{\bar{K}}). \end{aligned} \tag{4.17}$$

Note that the first term in (4.17) can be written as

$$\begin{aligned} &\sum_{K \in \mathcal{T}^h} |e_K^+| \psi_{\partial^0 K} \left( \varphi_{e_K^+}^\Omega(u_K^+) - \varphi_{e_K^+}^\Omega(u_{\bar{K}}) \right) \\ &= \sum_{K \in \mathcal{T}^h} \int_{e_K^+} \psi (i^* \Omega(u_K^+) - i^* \Omega(u_{\bar{K}})) + \sum_{K \in \mathcal{T}^h} \int_{e_K^+} (\psi_{\partial^0 K} - \psi) (i^* \Omega(u_K^+) - i^* \Omega(u_{\bar{K}})). \end{aligned}$$

We can sum up (with respect to  $K$ ) the identities

$$\begin{aligned} \int_K d(\psi\Omega)(u_K^-) &= \int_{\partial K} \psi i^* \Omega(u_K^-) \\ &= \int_{e_K^+} \psi i^* \Omega(u_K^-) - \int_{e_K^-} \psi i^* \Omega(u_K^-) + \sum_{e^0 \in \partial^0 K} \int_{e^0} \psi i^* \Omega(u_K^-) \end{aligned}$$

and combine them with the inequality (4.17). Finally, we arrive at the desired conclusion by noting that

$$\sum_{K \in \mathcal{T}^h} \left( \int_{e_K^+} \psi i^* \Omega(u_K^+) - \int_{e_K^-} \psi i^* \Omega(u_K^-) \right) = - \sum_{K \in \mathcal{T}_0^h} \int_{e_K^-} \psi i^* \Omega(u_{K,0}).$$

□

## 5 Convergence and well-posedness results

We are now in a position to establish:

**Theorem 5.1** (Convergence of the finite volume method). *Under the assumptions made in Section 3 and provided the flux field is geometry-compatible, the family of approximate solutions  $u^h$  generated by the finite volume scheme converges (as  $h \rightarrow 0$ ) to an entropy solution of the initial value problem (2.1), (2.11).*

Our proof of convergence of the finite volume method can be viewed as a generalization to spacetimes of the technique introduced by Cockburn, Coquel and LeFloch [6, 7] for the (flat) Euclidean setting and already extended to Riemannian manifolds by Amorim, Ben-Artzi, and LeFloch [1] and to Lorentzian manifolds by Amorim, LeFloch, and Okutmustur [2].

We also deduce that:

**Corollary 5.2** (Well-posedness theory on a spacetime). *Let  $M = [0, T] \times N$  be a  $(n + 1)$ -dimensional spacetime foliated by  $n$ -dimensional hypersurfaces  $H_t$  ( $t \in [0, T]$ ) with compact topology  $N$  (cf. (2.1)). Let  $\omega$  be a geometry-compatible flux field on  $M$  satisfying the global hyperbolicity condition (2.10). An initial data  $u_0$  being prescribed on  $H_0$ , the initial value problem (2.1), (2.11) admits an entropy solution  $u \in L^\infty(M)$  which, moreover, has well-defined  $L^1$  traces on any spacelike hypersurface of  $M$ . These solutions determines a (Lipschitz continuous) contracting semi-group in the sense that the inequality*

$$\int_{H'} i_{H'}^* \Omega(u_{H'}, v_{H'}) \leq \int_H i_H^* \Omega(u_H, v_H) \quad (5.1)$$

holds for any two hypersurfaces  $H, H'$  such that  $H'$  lies in the future of  $H$ , and the initial condition is assumed in the weak sense

$$\lim_{\substack{t \rightarrow 0 \\ t > 0}} \int_{H_t} i_{H_t}^* \Omega(u(t), v(t)) = \int_{H_0} i_{H_0}^* \Omega(u_0, v_0). \quad (5.2)$$

We can also extend a result originally established by DiPerna [12] (for conservation laws posed on the Euclidian space) within the broad class of entropy measure-valued solutions.

**Theorem 5.3.** *Let  $\omega$  be a geometry-compatible flux field on a spacetime  $M$  satisfying the global hyperbolicity condition (2.10). Then, any entropy measure-valued solution  $\nu$  (see Definition 2.5) to the initial value problem (2.1), (2.11) reduces to a Dirac mass at each point, more precisely*

$$\nu = \delta_u, \quad (5.3)$$

where  $u \in L^\infty(M)$  is the unique entropy solution to the same problem.

We omit the details of the proof, since it is a variant of the Riemannian proof given in [4].

It remains to provide a proof of Theorem 5.1. Recall that a Young measure  $\nu$  allows us to determine all weak-\* limits of composite functions  $a(u^h)$  for all continuous functions  $a$ , as  $h \rightarrow 0$ ,

$$a(u^h) \xrightarrow{*} \langle \nu, a \rangle = \int_{\mathbb{R}} a(\lambda) d\nu(\lambda). \quad (5.4)$$

**Lemma 5.4** (Entropy inequalities for the Young measure). *Let  $\nu$  be a Young measure associated with the finite volume approximations  $u^h$ . Then, for every convex entropy flux field  $\Omega$  and every non-negative test-function  $\psi$  supported away from the hypersurface  $t = T$ , one has*

$$\int_M \langle \nu, d\psi \wedge \Omega(\cdot) \rangle - \int_{H_0} i^* \Omega(u_0) \leq 0. \quad (5.5)$$

Based on this lemma, we are now in position to complete the proof of Theorem 5.1. Thanks to (5.5), we have for all convex entropy pairs  $(U, \Omega)$ ,

$$d\langle \nu, \Omega(\cdot) \rangle \leq 0$$

in the sense of distributions on  $M$ . On the initial hypersurface  $H_0$  the (trace of the) Young measure  $\nu$  coincides with the Dirac mass  $\delta_{u_0}$ . By Theorem 5.3 there exists a unique function  $u \in L^\infty(M)$  (the entropy solution to the initial-value problem under consideration) such that the measure  $\nu$  coincides with the Dirac mass  $\delta_u$ . Moreover, this property also implies that the approximations  $u^h$  converge strongly to  $u$ , and this concludes the proof of the convergence of the finite volume scheme.

*Proof of Lemma 5.4.* The proof is a direct passage to the limit in the inequality (4.15), by using the property (5.4) of the Young measure. First of all, we observe that, the left-hand side of the inequality (4.15) converges to the left-hand side of (5.5). Indeed, since  $\omega$  is geometry-compatible, the first term of interest

$$\sum_{K \in \mathcal{T}^h} \int_K d(\psi \Omega)(u_K^-) = \sum_{K \in \mathcal{T}^h} \int_K d\psi \wedge \Omega(u_K^-) = \int_M d\psi \wedge \Omega(u^h)$$

converges to  $\int_M \langle \nu, d\psi \wedge \Omega(\cdot) \rangle$ . On the other hand, the initial contribution

$$\sum_{K \in \mathcal{T}_0^h} \int_{e_K^-} \psi i^* \Omega(u_{K,0}) = \int_{H_0} \psi i^* \Omega(u_0^h) \rightarrow \int_{H_0} \psi i^* \Omega(u_0),$$

in which  $u_0^h$  is the initial discretization of the data  $u_0$  and converges strongly to  $u_0$  since the maximal diameter  $h$  of the element tends to zero.

It remains to check that the terms on the right-hand side of (4.15) vanish in the limit  $h \rightarrow 0$ . We begin with the first term  $A^h(\psi)$ . Taking the modulus of this expression, applying Cauchy-Schwarz inequality, and finally using the global entropy dissipation estimate (4.13), we obtain

$$\begin{aligned} |A^h(\psi)| &\leq \sum_{\substack{K \in \mathcal{T}^h \\ e^0 \in \partial^0 K}} \frac{|e_K^+|}{N_K} |\psi_{\partial^0 K} - \psi| |\tilde{u}_{K,e^0}^+ - u_K^-| \\ &\leq \left( \sum_{\substack{K \in \mathcal{T}^h \\ e^0 \in \partial^0 K}} \frac{|e_K^+|}{N_K} |\psi_{\partial^0 K} - \psi|^2 \right)^{1/2} \left( \sum_{\substack{K \in \mathcal{T}^h \\ e^0 \in \partial^0 K}} \frac{|e_K^+|}{N_K} |\tilde{u}_{K,e^0}^+ - u_K^-|^2 \right)^{1/2} \\ &\leq \left( \sum_{\substack{K \in \mathcal{T}^h \\ e^0 \in \partial^0 K}} \frac{|e_K^+|}{N_K} (C(\tau_{\max} + h))^2 \right)^{1/2} \left( \int_{H_0} i^* \Omega(u_0) \right)^{1/2}, \end{aligned}$$

hence

$$|A^h(\psi)| \leq C' (\tau_{\max} + h) \left( \sum_{K \in \mathcal{T}^h} |e_K^+| \right)^{1/2} \leq C'' \frac{\tau_{\max} + h}{(\tau_{\min})^{1/2}}.$$

Here,  $\Omega$  is associated with the quadratic entropy and have used the fact that  $|\psi_{\partial^0 K} - \psi| \leq C(\tau_{\max} + h)$ . Our conditions (3.12) imply the upper bound for  $A^h(\psi)$  tends to zero with  $h$ .

Next, we rely on the regularity of  $\psi$  and  $\Omega$  and estimate the second term on the right-hand side of (4.15). By setting

$$C_{e^0} := \frac{\int_{e^0} i^* \Omega(u_K^-)}{\int_{e^0} i^* \tilde{\omega}},$$

we obtain

$$\begin{aligned} |B^h(\psi)| &= \left| \sum_{\substack{K \in \mathcal{T}^h \\ e^0 \in \partial^0 K}} \int_{e^0} (\psi_{e^0} - \psi) \left( i^* \Omega(u_K^-) - C_{e^0} i^* \tilde{\omega} \right) \right| \\ &\leq \sum_{\substack{K \in \mathcal{T}^h \\ e^0 \in \partial^0 K}} \sup_K |\psi_{e^0} - \psi| \int_{e^0} \left| i^* \Omega(u_K^-) - C_{e^0} i^* \tilde{\omega} \right| \\ &\leq C \sum_{\substack{K \in \mathcal{T}^h \\ e^0 \in \partial^0 K}} (\tau_{\max} + h)^2 |e^0|_{\tilde{\omega}}, \end{aligned}$$

hence

$$|B^h(\psi)| \leq C \frac{(\tau_{\max} + h)^2}{h}.$$

Again, our assumptions imply the upper bound for  $B^h(\psi)$  tends to zero with  $h$ .

Finally, consider the last term in the right-hand side of (4.15)

$$|C^h(\psi)| \leq \sum_{K \in \mathcal{T}^h} |e_K^\pm| \sup_K |\psi_{\partial^0 K} - \psi| \int_{e_K^\pm} |i^* \Omega(u_K^\pm) - i^* \Omega(u_{\bar{K}})|,$$

using the modulus defined in the beginning of Section 2. In view of the inequality (3.17), we obtain

$$|C^h(\psi)| \leq C \sum_{\substack{K \in \mathcal{T}^h \\ e^0 \in \partial^0 K}} \frac{|e_K^\pm|}{N_K} |\psi_{\partial^0 K} - \psi| |\tilde{u}_{K, e^0}^+ - u_{\bar{K}}^-|,$$

and it is now clear that  $C^h(\psi)$  satisfies the same estimate as the one we derived for  $A^h(\psi)$ .  $\square$

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