A large deformation, viscoelastic, thin rod model: derivation and analysis

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Abstract

We present a Cosserat-based three-dimensional to one dimensional reduction in the case of a thin rod, of the viscoelastic finite strain model introduced by P. Neff. This model is a coupled minimization/evolution problem. We prove the existence and uniqueness of the solution of the reduced minimization problem. We also show a few regularity results for this solution which allow us to establish the well-posedness of the evolution problem. Finally, the reduced model preserves observer invariance.

Résumé

Nous présentons une réduction monodimensionnelle dans le cas d'un fil mince du modèle viscoélastique en grandes déformations introduit par P. Neff. Cette réduction est effectuée à l'aide d'un Ansatz de Cosserat. Le problème réduit est un problème couplé minimisation/évolution qui satisfait le principe d'indifférence matérielle. Nous montrons l'existence et l'unicité de la solution du problème de minimisation. Des résultats de régularité pour cette solution ont été établis permettant de montrer que le problème d'évolution est bien posé.

1 Introduction

The dimensional reduction of a three-dimensional mechanical model to a one-dimensional model can be performed in different ways. Let us here just recall the direct approach and the asymptotic methods. In the direct approach, introduced by François and Eugène Cosserat [11] (1908-1909), and then used by many other authors such as Green, Naghdi, Laws, Cohen and Wang ([10], [14], [15]), the rod

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is considered from the onset as a one-dimensional object equipped with directors. For the asymptotic methods, the reduced one-dimensional model can either be deduced from a formal expansion of the three-dimensional solution in powers of the thickness considered as a small parameter (see Aganović and Tutek [2], Aganović, Tambača and Tutek [1], Rigolot [23], Bermudez and Viaño [4], Cimetière *et al.* [9], Trabucho de Campos and Viaño [24], and Ciarlet [7, 8] for three- to two-dimensional reduction), or via a rigorous convergence analysis when the thickness goes to 0 (see Le Dret [17], Murat and Sili [18]).

In [22], Neff proposed a derivation approach that does not exactly fit in any of the categories introduced above. In fact, his approach consists in reducing a given three-dimensional model via (physically) reasonable constitutive assumptions to a two-dimensional model.

Our approach to reducing the viscoelastic finite strain three-dimensional model introduced by Neff in [20] to a one-dimensional model is similar to the so-called derivation approach. In fact, we use the special Cosserat theory of rods based on the introduction of directors. In [22, 21], Neff presented a two-dimensional reduction of his three-dimensional model. Our present one-dimensional reduction follows comparable but not entirely similar lines.

Note that in the case without external surface tractions, Neff showed existence and uniqueness of the solution of the reduced two-dimensional minimization problem in $H^1(\omega,\mathbb{R}^3)$ and a unique local solution in time of the coupled problem ([21]).

An outline of this article follows. We start in Section 2 by recalling some definitions and notations. In Section 3, we briefly describe Neff's three-dimensional viscoelastic model. In Section 4, we reduce this model to a one-dimensional model by using a Cosserat kinematic Ansatz for the minimization problem, and by integrating through the thickness over the cross section of the rod for the evolution problem. In Section 5, we prove the existence and uniqueness of the solution of the reduced minimization problem. Section 6 is devoted to proving $W^{2,p}$-regularity for the deformation of the central line of the rod. The functions ρ_α appearing in the Ansatz below, which describe Poisson effects in the cross section, are shown to belong to $W^{2,p/2}$. Finally in Section 7, we establish the well-posedness of the nonlinear evolution equation coupled with the minimization problem.

2 Notation

Throughout this article, we use the Einstein summation convention, unless otherwise specified. Latin indices take their values in the set $\{1,2,3\}$ and Greek indices in the set $\{1,2\}$.

Let (e_1,e_2,e_3) be the canonical orthonormal basis of the Euclidean space $\mathbb{R}^3$. We note $u\cdot v$ the scalar product of $\mathbb{R}^3$, $|u|=\sqrt{u\cdot u}$ the associated Euclidean norm and $u\wedge v$ the vector product of u and v. Let M_3 be the space of real 3×3 matrices. The notation $A=(u|v|w)$ is meant to show the three column-vectors u, v and w of the matrix A. The standard Euclidean scalar product on M_3 is denoted by $A:B=$

$\mathrm{tr}(A^TB)$, and the associated norm is $\|A\| = \sqrt{\mathrm{tr}(A^TA)}$. Any square matrix A may be uniquely decomposed as the sum of a symmetric matrix denoted by $\mathrm{sym}(A)$ and a skew-symmetric matrix denoted by $\mathrm{skew}(A)$, where $\mathrm{sym}(A) = 1/2(A+A^T)$ and $\mathrm{skew}(A) = 1/2(A-A^T)$. Finally, we denote by M_3^{sk} the space of skew-symmetric matrices.

For any nonsingular matrix $F \in M_3$, we write the polar decomposition in the form $F = R_pU_p$, where R_p is the orthogonal part of F and U_p a positive definite symmetric matrix. Note that if $\det F > 0$, then $R_p \in \mathrm{SO}(3)$ where $\mathrm{SO}(3) = \{R \in \mathrm{O}(3);\, \det R = 1\}$ is the rotation group and $\mathrm{O}(3) = \{R \in M_3;\, R^TR = RR^T = I\}$ is the orthogonal group.

3 Neff's three-dimensional viscoelastic model

Let us start by briefly recalling the coupled minimization/evolution problem introduced by P. Neff. This problem reads: Find a deformation φ and a microrotation R (see below for a mechanical interpretation of the latter), such that at each instant t,

$$J(\varphi,R) = \inf_{\psi\in\Phi} J(\psi,R) \tag{3.1}$$

and that solves the Cauchy problem for the viscoelastic evolution

$$\frac{dR}{dt} = \frac{1}{\eta}\nu^+\big(\mathrm{skew}(FR^T)\big)\,\mathrm{skew}(FR^T)R,\; R(0) = R_0. \tag{3.2}$$

Here,

$$J(\varphi,R) = \int_\Omega W(F,R)\,dx - \int_\Omega f\cdot\varphi\,dx - \int_\Gamma N\cdot\varphi\,dS - \int_{\Gamma^1} g\cdot\varphi\,da - \int_{\Gamma^0} e\cdot\varphi\,da$$

is the total energy. The stored energy function is of the form

$$W(F,R) = \frac{\mu}{4}\|R^TF + F^TR - 2I\|^2 + \frac{\lambda}{8}\big(\mathrm{tr}(R^TF + F^TR - 2I)\big)^2$$

where $F = \nabla\varphi$ is the deformation gradient, $\mu > 0$ and $\lambda \geq 0$ are the Lamé constants of the material (see [6]) and Φ is a space of admissible deformations including boundary conditions. The functions f are dead loading body forces and N are dead loading surface tractions on a part of the boundary Γ of $\Omega = \omega\times]0,1[$ (where ω is a two-dimensional bounded domain). We also apply dead loading force densities g and e over $\Gamma^1 = \omega\times\{1\}$ and $\Gamma^0 = \omega\times\{0\}$ respectively. The term $\frac{1}{\eta}\nu^+$ in the evolution equation for the microrotation is a scalar-valued function representing elastic viscosity (see [22]) and η is a relaxation time.

Note that Neff also introduces another so-called thermodynamical model (see [20]) corresponding to another evolution equation, which is more complicated than (3.2), but with the same minimization problem. In fact, other differential equations can be chosen as long as they assure that $R(t)$ is a rotation and the energy is non-increasing.

Let us now give an interpretation of the microrotation R. Since we consider a Cosserat theory, we can then consider the following multiplicative decomposition of the deformation gradient tensor

$$F = RU, \tag{3.3}$$

where U, the first Cosserat deformation tensor, is a nonsymmetric invertible matrix so that (3.3) is not in general the polar decomposition of F, and R is an independent microrotation field. Note that R can be interpreted as a macroscopic homogenized quantity. Indeed, experimental evidence, for instance in polycrystalline aluminum samples, (see [13], [16], [19]) shows that the rotations of individual grains may deviate considerably from the continuum rotation, which is the polar part of the macroscopic deformation. The microrotation can be identified with an average of these individual grains rotations over an intermediate scale volume element.

4 The one-dimensional reduced model

Let ω be a bounded open subset of $\mathbb{R}^2$ with Lipschitz boundary and diameter equal to 1 (without loss of generality). We consider a nonlinear viscoelastic homogeneous body described by Neff's three-dimensional model, occupying the reference configuration $\Omega_h = h\omega \times]0,1[$, where $0 < h \ll 1$ is the diameter of the cross section $h\omega$. We denote by (φ_h, R_h) the solution of the three-dimensional viscoelastic problem.

We assume that the coordinate system is chosen in such way that

$$\int_\omega x_1 \, dx_1 dx_2 = \int_\omega x_2 \, dx_1 dx_2 = \int_\omega x_1 x_2 \, dx_1 dx_2 = 0, \tag{4.4}$$

which is always possible by placing the origin at the centroid of ω and choosing the principal axes of inertia of ω as coordinate axes.

Our aim is to define a good one-dimensional approximation $(\widetilde{\varphi}, \widetilde{R})$ of (φ_h, R_h), when h is small enough. As we already mentioned, we use a variant of the special Cosserat theory of rods [3]: We assume that the approximate deformation $\widetilde{\varphi}\colon \Omega_h \to \mathbb{R}^3$ obeys the following kinematic Ansatz:

$$\widetilde{\varphi}(t,x_1,x_2,x_3) = m(t,x_3) + \rho_1(t,x_3) d_1(t,x_3) x_1 + \rho_2(t,x_3) d_2(t,x_3) x_2, \tag{4.5}$$

where the scalar-valued functions $\rho_\alpha\colon [0,T] \times]0,1[\to \mathbb{R}$ take into account Poisson effects in the cross section and d_α are two orthonormal vectors, called directors in Cosserat theory. Then, the spatial deformation of the rod is composed of the motion of the central line $m\colon [0,T] \times]0,1[\to \mathbb{R}^3$ and of the motion of the directors $d_\alpha\colon [0,T] \times]0,1[\to S^2$, $\alpha = 1,2$ and $d_\alpha \cdot d_\beta = \delta_{\alpha\beta}$, which describe the motion of the cross-section.

We introduce a second Ansatz concerning the rotations:

$$\widetilde{R}(t,x_1,x_2,x_3) = R(t,x_3), \tag{4.6}$$

which is consistent with our one-dimensional reduction goal. In order to simplify the notation, from now on, we will denote respectively by R and $\nabla\phi = F$ the reduced rotation and deformation gradient tensor, which depend only on time and on the spatial variable x_3, as opposed to the notation used in Section 3. In addition, we assume that the first two columns of the matrix R and the directors are coupled via the relation

$$R_\alpha = d_\alpha, \text{ for } \alpha = 1,2. \tag{4.7}$$

This is reasonable for small h, as we now proceed to show. In fact, we have

$$\nabla\phi(0,0,x_3) = \big(\rho_1 d_1 \,|\, \rho_2 d_2 \,|\, m'\big).$$

Then the right Cauchy-Green deformation tensor becomes

$$\big(\nabla\phi(0,0,x_3)\big)^T \nabla\phi(0,0,x_3) \approx \begin{pmatrix} \rho_1^2 & 0 & 0 \\ 0 & \rho_2^2 & 0 \\ 0 & 0 & |m'|^2 \end{pmatrix} \tag{4.8}$$

under negligible shear stress, namely for small $\rho_\alpha d_\alpha \cdot m'$. Now, the polar decomposition of $\nabla\phi(0,0,x_3)$ reads

$$\nabla\phi = R_p U_p,\, R_p \in \mathrm{SO}(3),\, U_p \text{ symmetric positive definite.} \tag{4.9}$$

This is equivalent to having

$$R_p = \nabla\phi\, U_p^{-1} = \Big(\frac{\rho_1}{|\rho_1|} d_1 \Big| \frac{\rho_2}{|\rho_2|} d_2 \Big| \frac{1}{|m'|} m'\Big). \tag{4.10}$$

It is implicit in Neff's models that the microrotation R should be close to the continous rotation R_p. The choice $d_\alpha = R_\alpha$ is then reasonable. This choice can be further justified by the fact that it is reasonable to relate the microrotation to the macroscopic deformation in the thin directions, for reasons of scale.

Using the two Ansätze for the deformation and the rotation, we can now reduce the coupled problem. Let us start with the minimization problem.

4.1 Reduction of the minimization problem

By a tedious but straightforward computation, we see that the reduced internal energy of the minimization problem takes the form:

$$\begin{aligned}
&\int_0^1 \int_{h\omega} W(F,R)\, dx \\
&= \int_0^1 a_h\, \mu\, \| \mathrm{sym}\big(R^T(\rho_1 R_1 \,|\, \rho_2 R_2 \,|\, m')\big) - I \|^2\, dx_3 \\
&\quad + \int_0^1 a_h\, \frac{\lambda}{2} \Big(\mathrm{tr}\big(R^T(\rho_1 R_1 \,|\, \rho_2 \bar{R}_2 \,|\, m') - I\big)\Big)^2 dx_3 \\
&\quad + \int_0^1 J_\alpha^h \Big(\mu \| \mathrm{sym}(R^T(0|0|(\rho_\alpha R_\alpha)'))\|^2 + \frac{\lambda}{2}\big(\mathrm{tr}(R^T(0|0|(\rho_\alpha R_\alpha)'))\big)^2\Big)\, dx_3,
\end{aligned} \tag{4.11}$$

due to (4.4), where a_h and J^h_α are the area and the principal moments of inertia of the cross section $h\omega$ respectively.

We now similarly reduce the work of external forces in the total energy. Using the Ansätze (4.5)-(4.7) and Fubini's theorem, we see that the resultant body forces and surface tractions are given by

$$\bar{f}_h(x_3) = \int_{h\omega} f(x_1,x_2,x_3)\,da + \int_{\partial(h\omega)} N(x_1,x_2,x_3)\,dS, \tag{4.12}$$

the resultant of g over the cross section corresponding to $x_3 = 1$ is

$$\bar{g}_h = \int_{\Gamma^1_h} g\,da, \tag{4.13}$$

and the resultant moments are

$$\bar{k}_h(x_3) = \int_{h\omega} x_\alpha f(x_1,x_2,x_3)\,da + \int_{\partial(h\omega)} x_\alpha N(x_1,x_2,x_3)\,dS, \tag{4.14}$$

$$\bar{l}_h = \int_{\Gamma^1_h} x_\alpha\, g(x_1,x_2)\,da \tag{4.15}$$

and

$$\bar{e}_h = \int_{\Gamma^0_h} x_\alpha\, e(x_1,x_2)\,da. \tag{4.16}$$

For brevity, we will decompose the deformation gradient tensor as

$$F = \nabla\phi = A_c + A_\alpha x_\alpha, \tag{4.17}$$

with

$$A_c = \big((\rho_1 R_1)(x_3)\,|\,(\rho_2 R_2)(x_3)\,|\,m'(x_3)\big),\; A_\alpha = \big(0|0|\,(\rho_\alpha R_\alpha)'(x_3)\big), \tag{4.18}$$

(without summation). Thus, the reduced total energy becomes

$$\begin{aligned} &J(m,\rho_1,\rho_2,R)\\ &= \int_0^1 a_h\Big(\mu\|\,\mathrm{sym}(R^T A_c) - I\|^2 + \frac{\lambda}{2}\big(\mathrm{tr}(R^T A_c - I)\big)^2\Big)dx_3\\ &\quad + \int_0^1 \Big(J^h_\alpha\big(\mu\|\,\mathrm{sym}(R^T A_\alpha)\|^2 + \frac{\lambda}{2}(\mathrm{tr}(R^T A_\alpha))^2\big) - \bar{f}_h\cdot m - \rho_\alpha\,\bar{k}_h\cdot R_\alpha\Big)\,dx_3\\ &\qquad - \bar{g}_h\cdot m(1) - \rho_\alpha(1)\,\bar{l}_h\cdot R_\alpha(1) - \rho_\alpha(0)\,\bar{e}_h\cdot R_\alpha(0). \end{aligned} \tag{4.19}$$

4.2 Reduction of the evolution problem

We proceed to reduce the three-dimensional evolution equation (3.2). We start by replacing F_h with $F = A_c + A_\alpha x_\alpha$ and R_h by R, which only depends on the spatial variable x_3 and on the time t. Then, we average FR over the cross section $h\omega$ and we obtain

$$\frac{dR}{dt}(t,x_3) = \frac{1}{\eta}\nu^+\big(\mathrm{skew}(A_c(t,x_3)R^T(t,x_3))\big)\,\mathrm{skew}\,\big(A_c(t,x_3)R^T(t,x_3)\big)R(t,x_3), \tag{4.20}$$

since $\int_\omega x_\alpha\,dx_1 dx_2 = 0$.

5 Existence and uniqueness of the solution of the reduced minimization problem

In this section, time is frozen and R only depends on x_3. We thus minimize the reduced total energy (4.19) with respect to the deformation of the central line m of the rod and the coefficients ρ_1 and ρ_2 on the function space $\Phi \times \Theta$ where

$$\Phi = \{m \in H^1(]0,1[;\mathbb{R}^3), m(0) = (0,0,0)\}$$

and

$$\Theta = \{\rho = (\rho_1, \rho_2) \in H^1(]0,1[;\mathbb{R}^2)\},$$

which we will show are appropriate for the problem at hand. We also drop the dependence on R in the notation for the time being.

In terms of boundary conditions, we just fix the extremity $x_3 = 0$ of the rod for simplicity. The case when both extremities are fixed is treated in a similar way.

It should be noted that there are no Dirichlet boundary conditions for ρ_α. This is due to the fact that, in the model, the functions ρ_α and the rotation R appear together in the kinematical Ansatz. It will be seen later on that imposing Dirichlet boundary conditions on R is incompatible with the evolution problem. Therefore, it does not make sense in terms of modeling to impose Dirichlet boundary conditions on ρ_α. Adding them would of course present no difficulty for the ensuing mathematical analysis.

Theorem 5.1. *Assume that the rotations R belong to $H^1([0,1],\mathrm{SO}(3))$, the forces $\bar{f}_h$, $\bar{k}_h$ are in $L^2([0,1];\mathbb{R}^3)$ and let $\bar{g}_h$, $\bar{l}_h$, and $\bar{e}_h$ be vectors in $\mathbb{R}^3$. Then, $J : \Phi \times \Theta \to \mathbb{R}$ and there exists $(m, \rho_1, \rho_2) \in \Phi \times \Theta$ that minimizes the functional J over $\Phi \times \Theta$.*

Proof. First of all, to see that J is real-valued, observe on the one hand that $R^T A_c = R^T(\rho_1 R_1 | \rho_2 R_2 | m')$ belongs to L^2. This is due to the fact that rotations are in L^∞. On the other hand, since H^1 is an algebra in one dimension (see [5]), we see that $\rho_\alpha R_\alpha \in H^1$ and therefore $R^T A_\alpha = (0|0|R^T(\rho_\alpha R_\alpha)')$ is also in L^2. This yields the desired result since the other terms appearing in the integrals are trivially in L^1.

Let us now prove the existence of a minimizer of the energy J. We use the direct method in the calculus of variations (see [12]) by showing that J is strongly continuous, coercive and convex with respect to m and ρ_α. We concentrate on the two main points of the proof: the coercivity and strong continuity of J since convexity is obvious. We start by showing the latter point. Let m_n and $\rho_{\alpha,n}$ be two sequences such that

$$m_n \to m \text{ in } H^1 \tag{5.21}$$

and

$$\rho_{\alpha,n} \to \rho_\alpha \text{ in } H^1. \tag{5.22}$$

We wish to prove that,

$$J(m_n, \rho_{1,n}, \rho_{2,n}) \to J(m, \rho_1, \rho_2),$$

where J is given by (4.19).

We first have

$$\|R^T(A_{c,n}-A_c)\|_{L^2}^2 = \|\big((\rho_{1,n}-\rho_1)e_1|(\rho_{2,n}-\rho_2)e_2|R^T(m_n'-m')\big)\|_{L^2}^2 \\ = \sum_{\alpha=1}^{2}\|\rho_{\alpha,n}-\rho_\alpha\|_{L^2}^2+\|m_n'-m'\|_{L^2}^2, \tag{5.23}$$

since $R^TR_\alpha = e_\alpha$ and the Euclidean norm is invariant under orthogonal transformations. Using (5.21), (5.22) and (5.23), we obtain

$$\|\operatorname{sym}(R^TA_{c,n})-I\|_{L^2}^2 \longrightarrow \|\operatorname{sym}(R^TA_c)-I\|_{L^2}^2.$$

Secondly, we have

$$\|R^T(A_{\alpha,n}-A_\alpha)\|_{L^2} = \|\big(0|0|R^T\big((\rho_{\alpha,n}-\rho_\alpha)R_\alpha'+(\rho_{\alpha,n}'-\rho_\alpha')R_\alpha\big)\|_{L^2} \\ \leq \|\rho_{\alpha,n}-\rho_\alpha\|_{L^\infty}\|R_\alpha'\|_{L^2}+\|\rho_{\alpha,n}'-\rho_\alpha'\|_{L^2}. \tag{5.24}$$

Applying the Sobolev embedding $H^1 \hookrightarrow L^\infty$ in one dimension, we find that the right-hand side of estimate (5.24) tends to zero and thus the desired result:

$$\|\operatorname{sym}(R^TA_{\alpha,n})\|_{L^2} \longrightarrow \|\operatorname{sym}(R^TA_\alpha)\|_{L^2}.$$

The trace terms are treated in a similar way and the force terms converge trivially, hence the strong continuity.

Let us now check the coercivity of J. First, observe that by (4.19) and the Cauchy-Schwarz inequality, we have

$$J(m,\rho_1,\rho_2) \geq \mu\int_0^1\Big(a_h\|\operatorname{sym}(R^TA_c)-I\|^2+\sum_{\alpha=1}^{2}J_\alpha^h\|\operatorname{sym}(R^TA_\alpha)\|^2\Big)\,dx_3 \\ -\|\bar{f}_h\|_{L^2}\|m\|_{L^2}-\|\bar{k}_h\|_{L^2}\sum_{\alpha=1}^{2}\|\rho_\alpha\|_{L^2}-|\bar{g}_h||m(1)| \\ -\sum_{\alpha=1}^{2}\big(|\bar{l}_h||\rho_\alpha(1)|+|\bar{e}_h||\rho_\alpha(0)|\big). \tag{5.25}$$

We first proceed to give a lower bound for $\|\operatorname{sym}(R^TA_c)-I\|^2$. We have

$$R^TA_c = R^T(\rho_1R_1|\rho_2R_2|m') = (\rho_1e_1|\rho_2e_2|R^Tm'),$$

so that

$$\operatorname{sym}(R^TA_c) = \begin{pmatrix}\rho_1 & 0 & \frac{1}{2}R_1\cdot m' \\ 0 & \rho_2 & \frac{1}{2}R_2\cdot m' \\ \frac{1}{2}R_1\cdot m' & \frac{1}{2}R_2\cdot m' & R_3\cdot m'\end{pmatrix}.$$

Hence, we obtain

$$\|\operatorname{sym}(R^TA_c)-I\|^2 = (\rho_1-1)^2+(\rho_2-1)^2+\frac{1}{2}[(R_1\cdot m')^2+(R_2\cdot m')^2]+(R_3\cdot m'-1)^2. \tag{5.26}$$

Therefore, we see that

$$\|\operatorname{sym}(R^T A_c) - I\|^2 \geq \rho_1^2 + \rho_2^2 + \frac{1}{2}|m'|^2 - 2\rho_1 - 2\rho_2 - 2R_3 \cdot m' + 3. \tag{5.27}$$

Next, for the second term in (5.25), a similar computation yields

$$\|\operatorname{sym}(R^T A_\alpha)\|^2 = \frac{1}{2}\rho_\alpha'^2 + \frac{1}{2}\rho_\alpha^2 (R_\beta \cdot R_\alpha')^2 + \rho_\alpha^2 (R_3 \cdot R_\alpha')^2 \geq \frac{1}{2}\rho_\alpha'^2, \tag{5.28}$$

without summation and with $\beta \neq \alpha$. At this point, it is fairly clear from estimates (5.27) and (5.28) that there are constants $C_h > 0$ and C_h' such that

$$J(m, \rho_1, \rho_2) \geq C_h(\|m'^2\|_{L^2}^2 + \|\rho_1^2\|_{H^1}^2 + \|\rho_2^2\|_{H^1}^2) - C_h'.$$

The coercivity of J (in the norm of $H^1(I;\mathbb{R}^5)$) on $\Phi \times \Theta$ now follows from the Poincaré inequality. □

Note that Korn's inequality is not needed in the above proof.

Proposition 5.2. *The minimizer of J is unique.*

Proof. It follows from (5.26) that the functional

$$I_0(m, \rho_1, \rho_2) = \mu a_h \int_0^1 \|\operatorname{sym}(R^T A_c) - I\|^2 \, dx_3$$

is strictly convex. Since J is the sum of I_0 and other convex terms, we see that J is also strictly convex. □

The rest of the paper is devoted to establishing the well-posedness of the evolution problem. In order to show this, we first need to prove a few regularity results for the solution (m, ρ_1, ρ_2) of the minimization problem for R fixed.

6 Regularity of the solution of the minimization problem

6.1 The Euler-Lagrange equation

Let us first derive the Euler-Lagrange equation. Since the energy is the sum of quadratic and affine terms and is real-valued and continuous, it is obvious that it is differentiable. Hence, the minimizer satisfies an Euler-Lagrange equation. Writing down the equation is merely a matter of rather tedious, but elementary calculations.

Proposition 6.1. *The minimizer (m, ρ_1, ρ_2) satisfies the Euler-Lagrange equation*

$$\begin{aligned}
&\int_0^1 a_h\big(\mu(2\rho_\alpha\theta_\alpha + m'\cdot n' + (R_3\cdot m')(R_3\cdot n')) + \lambda(\rho_1+\rho_2+R_3\cdot m')(\theta_1+\theta_2 \\
&\qquad + R_3\cdot n')\big)\,dx_3 + \int_0^1 J_\alpha^h\big(\mu\rho_\alpha'\theta_\alpha' + \rho_\alpha\theta_\alpha\big[\mu|R_\alpha'|^2 + (\mu+\lambda)(R_3\cdot R_\alpha')^2\big]\big)\,dx_3 \\
&= \int_0^1 a_h(2\mu+3\lambda)(\theta_1+\theta_2+R_3\cdot n')\,dx_3 + \int_0^1 \big(\bar{f}_h\cdot n + \bar{k}_h\cdot\theta_\alpha R_\alpha\big)\,dx_3 + \bar{g}_h\cdot n(1) \\
&\qquad + \theta_\alpha(1)\,\bar{l}_h\cdot R_\alpha(1) + \theta_\alpha(0)\,\bar{e}_h\cdot R_\alpha(0),
\end{aligned} \tag{6.29}$$

for all $\theta_\alpha \in H^1(]0,1[;\mathbb{R})$ *and* $n \in H^1(]0,1[;\mathbb{R}^3)$, $n(0) = 0$.

In what follows, we assume $\bar{f}_h, \bar{k}_h \in L^p(]0,1[;\mathbb{R}^3)$, $R \in W^{1,p}(]0,1[;\mathrm{SO}(3))$ with $p \geq 2$. For brevity, we let (without summation)

$$K_\alpha^h = J_\alpha^h\big(\mu|R'_\alpha|^2 + (\mu+\lambda)(R_3\cdot R'_\alpha)^2\big). \tag{6.30}$$

Proposition 6.2. *The solution* (m,ρ) *of the minimization problem satisfies*

$$\begin{cases} -\mu a_h\big(m'' + ((R_3\cdot m')R_3)'\big) - \lambda a_h((\rho_1+\rho_2+R_3\cdot m')R_3)' = -(2\mu+3\lambda)a_h R'_3 + \bar{f}_h, \\ -\mu J_\alpha^h \rho''_\alpha + (2\mu a_h + K_\alpha^h)\rho_\alpha + \lambda a_h(\rho_1+\rho_2+R_3\cdot m') = a_h(2\mu+3\lambda) + \bar{k}_h\cdot R_\alpha, \end{cases} \tag{6.31}$$

(without summation) in the sense of distributions on $]0,1[$.

Proof. Most terms in (6.29) are straightforward. Since R_3 is in L^∞, $R_3\cdot m'$ is in L^2. Similarly, K_α^h is in $L^{p/2}$. Thus, taking first $\theta_\alpha = 0$ and $n \in \mathscr{D}(]0,1[;\mathbb{R}^3)$, and second $n = 0$ and $\rho_\alpha \in \mathscr{D}(]0,1[)$, we obtain equations (6.31). □

We now proceed to prove that the weak solution of the system (6.31) is indeed a strong one. For this purpose, we need some regularity results for (m,ρ_1,ρ_2).

6.2 Regularity results

We first give a general purpose lemma.

Lemma 6.3. *Let* $f \in W^{1,p}(I)$, $p \geq 2$, *and* $\psi \in H^{-1}(I)$ *where* I *is an open interval in* $\mathbb{R}$. *Then the product* $f\psi$ *is well-defined, belongs to* $H^{-1}(I)$ *and depends continuously on* (f,ψ).

Proof. Let us define a linear form $f\psi$ on $\mathscr{D}(I)$ by

$$\langle f\psi, \phi\rangle = \langle \psi, f\phi\rangle_{H^{-1},H_0^1}.$$

This is well-defined since f is in $H^1(I)$ and ϕ vanishes in the boundary. Observe that

$$|\langle f\psi,\phi\rangle| = |\langle \psi, f\phi\rangle_{H^{-1},H_0^1}| \leq \|\psi\|_{H^{-1}}\|(f\phi)'\|_{L^2} \leq C\|\psi\|_{H^{-1}}\|f\|_{H^1}\|\phi\|_{H^1},$$

since H^1 is an algebra. Therefore, the linear form extends by continuity to $H_0^1(I)$ and is thus an element of $H^{-1}(I)$. Moreover $\|f\psi\|_{H^{-1}} \leq C'\|f\|_{W^{1,p}}\|\psi\|_{H^{-1}}$, hence the continuous dependence. □

Proposition 6.4. *The first equation of system* (6.31) *can be rewritten as*

$$m'' = -\frac{1}{\mu}\Big(\big[\lambda(\rho_1+\rho_2) + (\mu+\lambda)R_3\cdot m' - (2\mu+3\lambda)\big]R'_3\cdot R_\alpha + \frac{\bar{f}_h\cdot R_\alpha}{a_h}\Big)R_\alpha \\ - \frac{1}{2\mu+\lambda}\Big(\lambda(\rho'_1+\rho'_2) + (\mu+\lambda)R'_3\cdot m' + \frac{\bar{f}_h\cdot R_3}{a_h}\Big)R_3, \tag{6.32}$$

in the sense of $H^{-1}(]0,1[)$.

Proof. We expand the left-hand side of the first equation of system (6.31) in order to identify m''. First, remark that we have the following identity in H^{-1},

$$\big((R_3 \cdot m')R_3\big)' = (R_3' \cdot m')R_3 + (R_3 \cdot m'')R_3 + (R_3 \cdot m')R_3'. \tag{6.33}$$

Note that each term of (6.33) is well defined in H^{-1} by Hölder's inequality and the Sobolev embedding in one dimension. We use a density argument. In the case when $R_3 \in C^\infty(]0,1[;\mathbb{R}^3)$, the identity is obviously true. Now, let h_n be a C^∞-sequence such that $h_n \to R_3$ in $W^{1,p}$. Since h_n converges uniformly by the Sobolev embedding, we have $(h_n \cdot m')h_n \to (R_3 \cdot m')R_3$ in L^2, hence $((h_n \cdot m')h_n)' \to ((R_3 \cdot m')R_3)'$ in H^{-1}. Similarly, by Hölder's inequality, $(h_n' \cdot m')h_n \to (R_3' \cdot m')R_3$ and $(h_n \cdot m')h_n' \to (R_3 \cdot m')R_3'$ in $L^{\frac{2p}{p+2}} \hookrightarrow H^{-1}$. Finally, $(h_n \cdot m'')h_n \to (R_3 \cdot m'')R_3$ in H^{-1} by Lemma 6.3, hence identity (6.33).

Let us rewrite the first equation in (6.31) using (6.33) as

$$-\mu m'' - (\mu+\lambda)(R_3 \cdot m'')R_3 = b, \tag{6.34}$$

where

$$b = (\mu+\lambda)[(R_3' \cdot m')R_3 + (R_3 \cdot m')R_3'] + \lambda((\rho_1+\rho_2)R_3)' - (2\mu+3\lambda)R_3' + \frac{1}{a_h}\bar{f}_h.$$

We now take the scalar product of both sides of equation (6.34) with R_α and R_3 using Lemma 6.3 , $R_i \cdot R_j = \delta_{ij}$ and $R_3 \cdot R_3' = 0$ and we conclude by observing that

$$m'' = (R_i \cdot m'')R_i, \tag{6.35}$$

which is again a consequence of Lemma 6.3 used component-wise. □

We now turn to the main result of this section.

Theorem 6.5. *Assume that the rotations R belong to $W^{1,p}(]0,1[;\mathrm{SO}(3))$ and $\bar{f}_h$, $\bar{k}_h$ are in $L^p(0,1;\mathbb{R}^3)$ with $p \geq 2$. Let (m,ρ) be the solution of the minimization problem where $\rho = (\rho_1,\rho_2)$. We have $m \in W^{2,p}(]0,1[;\mathbb{R}^3)$ and $\rho \in W^{2,\frac{p}{2}}(]0,1[;\mathbb{R}^2)$.*

Proof. Let us first deal with ρ. The second equation in system (6.31) reads

$$-\mu J_\alpha^h \rho_\alpha'' = -(2\mu a_h + K_\alpha^h)\rho_\alpha - \lambda a_h(\rho_1+\rho_2+R_3 \cdot m') + a_h(2\mu+3\lambda) + \bar{k}_h \cdot R_\alpha,$$

where K_α^h is given by (6.30). Clearly $K_\alpha^h \rho_\alpha \in L^{\frac{p}{2}}$, $R_3 \cdot m' \in L^2$ and $\bar{k}_h \cdot R_\alpha \in L^p$. Therefore, since all the other terms in the right-hand side are in L^∞, we see that for $p \leq 4$, ρ_α belongs to $W^{2,\frac{p}{2}}$, and for $p \geq 4$, ρ_α is in H^2.

Next, we use a bootstrap argument. In terms of integrability, the worst terms in the right-hand side of (6.32) are $(R_3 \cdot m')(R_3' \cdot R_\alpha)R_\alpha$ and $(R_3' \cdot m')R_3$ that both belong to $L^{\frac{2p}{p+2}}$. Therefore, m belongs to $W^{2,\frac{2p}{p+2}}$ and consequently, $m' \in L^\infty$ by the Sobolev embedding. It follows that $(R_3 \cdot m')(R_3' \cdot R_\alpha)R_\alpha$ and $(R_3' \cdot m')R_3$ actually belong to L^p, as do all the other terms in the right-hand side of (6.32), since we already know that ρ_α' is in L^∞. We conclude that m belongs to $W^{2,p}$.

Finally, we go back to the equation for ρ_α'', the right-hand side of which is now seen to belong to $L^{\frac{p}{2}}$. □

Remark 6.6. The regularity of ρ_α is governed by that of K^h_α, which accounts for the $\frac{p}{2}$ exponent instead of p as could be expected.

Proposition 6.7. *The weak solution* (m, ρ_1, ρ_2) *of system* (6.31) *is a strong solution satisfying the following Dirichlet and Neumann boundary conditions:*

$$m(0) = 0,$$

$$\mu m'(1) + (\mu+\lambda)\big(R_3(1)\cdot m'(1)\big)R_3(1) + \lambda\big(\rho_1(1)+\rho_2(1)\big)R_3(1) = (2\mu+3\lambda)R_3(1) + \frac{\bar{g}_h}{a_h},$$

and

$$\rho'_\alpha(0) = -\frac{\bar{e}_h \cdot R_\alpha(0)}{\mu J^h_\alpha} \text{ and } \rho'_\alpha(1) = \frac{\bar{l}_h \cdot R_\alpha(1)}{\mu J^h_\alpha}$$

without summation.

Proof. The regularity afforded by Theorem 6.5 enables us to integrate the Euler-Lagrange equation (6.29) by parts with arbitrary test-functions $\theta_\alpha, n \in H^1$. □

7 Well-posedness of the coupled minimization-evolution problem

Let us first recall the reduced evolution problem

$$\frac{dR}{dt}(t,x_3) = \frac{1}{\eta}\nu^+\big(\operatorname{skew}(A_c(t,x_3)R^T(t,x_3))\big)\operatorname{skew}\big(A_c(t,x_3)R^T(t,x_3)\big)R(t,x_3), \tag{7.36}$$

where $A_c = (\rho_1 R_1|\rho_2 R_2|m')$ and (ρ_1,ρ_2,m) is the unique solution of the minimization problem corresponding to the rotation R.

A direct component-wise calculation gives

$$\operatorname{skew}(A_c R^T) = \frac{1}{2}\begin{pmatrix} 0 & -(R_3\wedge m')_3 & (R_3\wedge m')_2 \\ (R_3\wedge m')_3 & 0 & -(R_3\wedge m')_1 \\ -(R_3\wedge m')_2 & (R_3\wedge m')_1 & 0 \end{pmatrix}. \tag{7.37}$$

This shows that the right-hand side of the ordinary differential equation (7.36) only depends on m' and R and not on (ρ_1,ρ_2). Thus, the well-posedness of the evolution problem hinges on the central line m being locally Lipschitz with respect to the rotation R in an appropriate norm. The coefficients ρ_α nonetheless come into play since they are coupled with m in the minimization problem.

Formula (7.37) also shows that boundary conditions of Dirichlet type on R are incompatible with the evolution problem. Indeed, assume we wished to impose $R(0,t) = I$, for instance. This would require that the right-hand side of (7.37) should vanish for all t at $x_3 = 0$. Since $m'(0,t)$ derives from R at instant t via the minimization problem, this is clearly not a reasonable expectation.

7.1 Local Lipschitz dependence of the deformation on the rotation

Our aim now is to show that m, considered as a function of R, satisfies a local Lipschitz condition in the $W^{2,p}$-norm, $p \geq 2$, with respect to R in $W^{1,p}$. The argument is again a boostrapping argument. We begin with an H^1-estimate following from the variational formulation of the minimization problem. Let

$$V = H^1(]0,1[;\mathbb{R}^2) \times \{n \in H^1(]0,1[;\mathbb{R}^3), n(0) = 0\}.$$

Lemma 7.1. *Let $Y = (\rho_\alpha, m)$, $\tilde{Y} = (\tilde{\rho}_\alpha, \tilde{m}) \in V$ be two solutions of equation* (6.29) *corresponding to $R, \tilde{R} \in W^{1,p}(]0,1[;\mathrm{SO}(3))$ respectively. Then, the following estimate holds*

$$\|Y - \tilde{Y}\|_V \leq c\|R - \tilde{R}\|_{W^{1,p}},$$

where the constant c depends on $R', \tilde{R}', \tilde{\rho}_\alpha, \tilde{m}'$ and is locally bounded with respect to R in $W^{1,p}$.

Proof. In the proof, we use a generic constant c, the value of which may vary from line to line. First of all, we rewrite equation (6.29) as follows:

$$a_R(Y,Z) = l_R(Z), \quad a_{\tilde{R}}(\tilde{Y},Z) = l_{\tilde{R}}(Z),$$

for all $Z = (\theta_\alpha, n)$ in V, where a_R is the obvious symmetric elliptic bilinear form on V corresponding to the left-hand side of (6.29) with rotation R and l_R the linear form corresponding to the right-hand side. To begin with, observe that

$$a_R(Y - \tilde{Y}, Z) = a_{\tilde{R}}(\tilde{Y}, Z) - a_R(\tilde{Y}, Z) + (l_R - l_{\tilde{R}})(Z).$$

Hence, using the ellipticity of a_R and the linearity of l_R, we obtain

$$\beta\|Y - \tilde{Y}\|_V \leq \sup_{Z\in V} \frac{|a_R(\tilde{Y},Z) - a_{\tilde{R}}(\tilde{Y},Z)|}{\|Z\|_V} + \|l_R - l_{\tilde{R}}\|_{V'}, \tag{7.38}$$

where β is the ellipticity constant, which does not depend on R. For the latter term, we have

$$\|l_R - l_{\tilde{R}}\|_{V'} \leq c\|R_3 - \tilde{R}_3\|_{W^{1,p}}, \tag{7.39}$$

by the Cauchy-Schwarz inequality, the Sobolev embedding $W^{1,p} \hookrightarrow L^\infty$ and the definition of the dual norm.

The first term is slightly more complicated. We have

$$\begin{aligned}
|a_R(\tilde{Y},Z) - a_{\tilde{R}}(\tilde{Y},Z)| \leq c &\int_0^1 |R_3 - \tilde{R}_3||n'(\tilde{\rho}_1 + \tilde{\rho}_2) + \tilde{m}'(\theta_1 + \theta_2)|\,dx_3 \\
&+ c\int_0^1 \big|(R_3 \cdot \tilde{m}')(R_3 \cdot n') - (\tilde{R}_3 \cdot \tilde{m}')(\tilde{R}_3 \cdot n')\big|\,dx_3 \\
&+ c\int_0^1 \big||R'_\alpha|^2 - |\tilde{R}'_\alpha|^2\big||\tilde{\rho}_\alpha||\theta_\alpha|\,dx_3 \\
&+ c\int_0^1 \big|(R_3 \cdot R'_\alpha)^2 - (\tilde{R}_3 \cdot \tilde{R}'_\alpha)^2\big||\tilde{\rho}_\alpha||\theta_\alpha|\,dx_3.
\end{aligned} \tag{7.40}$$

Let us show in detail how we deal with the last term in the right-hand side of estimate (7.40). We decompose it as follows

$$(R_3\cdot R'_\alpha)^2-(\tilde R_3\cdot\tilde R'_\alpha)^2=\big((R_3-\tilde R_3)\cdot R'_\alpha+\tilde R_3\cdot(R'_\alpha-\tilde R'_\alpha)\big)\big(R_3\cdot R'_\alpha+\tilde R_3\cdot\tilde R'_\alpha\big). \tag{7.41}$$

We apply the Cauchy-Schwarz inequality in $\mathbb{R}^3$, integrate over $]0,1[$, use the Sobolev embeddings $H^1, W^{1,p}\hookrightarrow L^\infty$ and the fact that $p\geq 2$, and we obtain

$$\int_0^1 \big|(R_3\cdot R'_\alpha)^2-(\tilde R_3\cdot\tilde R'_\alpha)^2\big||\tilde\rho_\alpha||\theta_\alpha|\,dx_3\leq c\|R-\tilde R\|_{W^{1,p}}\|\theta_\alpha\|_{H^1}, \tag{7.42}$$

where the constant c depends on R', $\tilde R'$ and $\tilde\rho_\alpha$ and is locally bounded when $\tilde R_\alpha$, $\tilde\rho_\alpha$ are in $W^{1,p}\subset H^1$ and L^∞ respectively.

We use similar arguments to control all the other terms in the right-hand side of estimate (7.40), which concludes the proof of the lemma. □

Let us proceed with the boostrap argument. For convenience, we use here the equivalent norm

$$\|m-\tilde m\|_{W^{2,q}}=\|m-\tilde m\|_{W^{1,q}}+\|m''-\tilde m''\|_{L^q}. \tag{7.43}$$

Lemma 7.2. *Let m and $\tilde m$ be as above. Then, we have the following estimate:*

$$\|m-\tilde m\|_{W^{2,\frac{2p}{p+2}}}\leq c\|R-\tilde R\|_{W^{1,p}}, \tag{7.44}$$

where the constant c depends on $R',\tilde R',\rho_\alpha,\tilde\rho_\alpha,\tilde\rho'_\alpha,m',\tilde m'$ and $\bar f_h$ and is locally bounded.

Proof. Since $\frac{2p}{p+2}\leq 2$, we have the continuous embedding $H^1\hookrightarrow W^{1,\frac{2p}{p+2}}$, thus the following inequality

$$\begin{aligned}\|m-\tilde m\|_{W^{1,\frac{2p}{p+2}}}&\leq c\|m-\tilde m\|_{H^1}\\&\leq c\|R-\tilde R\|_{W^{1,p}}\end{aligned}$$

by Lemma 7.1.

Let us estimate the remaining quantity $\|m''-\tilde m''\|_{L^{\frac{2p}{p+2}}}$. We write equation (6.32) for m'' and $\tilde m''$, subtract the results and obtain a rather long expression, the terms of which all belong to L^p by Theorem 6.5. The difficulty however, is that at this point, we do not have a Lipschitz estimate in L^p for all these terms. For brevity, we just write down the worst terms, those for which we can only get a Lipschitz estimate in the $L^{\frac{2p}{p+2}}$ norm for now. All the other terms are dealt with in a similar fashion.

There actually are two such terms and we proceed as before using the following decompositions:

$$\begin{aligned}(R_3\cdot m')(R'_3\cdot R_\alpha)R_\alpha-(\tilde R_3\cdot\tilde m')(\tilde R'_3\cdot\tilde R_\alpha)\tilde R_\alpha&=\big((R_3-\tilde R_3)\cdot m'\big)(R'_3\cdot R_\alpha)R_\alpha\\&\quad+\big(\tilde R_3\cdot(m'-\tilde m')\big)(R'_3\cdot R_\alpha)R_\alpha\\+(\tilde R_3\cdot\tilde m')\big[\big((R'_3-\tilde R'_3)\cdot R_\alpha\big)R_\alpha&+\big(\tilde R'_3\cdot(R_\alpha-\tilde R_\alpha)\big)R_\alpha+(\tilde R'_3\cdot\tilde R_\alpha)(R_\alpha-\tilde R_\alpha)\big],\end{aligned} \tag{7.45}$$

and

$$
\begin{aligned}
&(R_3' \cdot m')R_3 - (\tilde{R}_3' \cdot \tilde{m}')\tilde{R}_3 \\
&\qquad = \big((R_3' - \tilde{R}_3') \cdot m'\big)R_3 + \big(\tilde{R}_3' \cdot (m' - \tilde{m}')\big)R_3 + \tilde{R}_3' \cdot \tilde{m}'(R_3 - \tilde{R}_3)
\end{aligned} \tag{7.46}
$$

Consider first decomposition (7.45). Applying the Cauchy-Schwarz inequality in $\mathbb{R}^3$, then Hölder's inequality (which shows that $\|m'\|\|R_3'\|$ is in $L^{\frac{2p}{p+2}}$) and using the fact that $(R_3 - \tilde{R}_3) \in L^\infty$ and then the continuous Sobolev embedding $W^{1,p} \hookrightarrow L^\infty$, we obtain

$$
\|\big((R_3 - \tilde{R}_3) \cdot m'\big)(R_3' \cdot R_\alpha)R_\alpha\|_{L^{\frac{2p}{p+2}}} \leq c(R_3', m')\|R - \tilde{R}\|_{W^{1,p}}.
$$

With the same arguments, we find

$$
\begin{aligned}
\|\tilde{R}_3 \cdot (m' - \tilde{m}')\big)(R_3' \cdot R_\alpha)R_\alpha\|_{L^{\frac{2p}{p+2}}} &\leq \|m' - \tilde{m}'\|_{L^2}\|\tilde{R}_3'\|_{L^p} \\
&\leq c(R', \tilde{R}', \tilde{\rho}_\alpha, \tilde{m}')\|R - \tilde{R}\|_{W^{1,p}},
\end{aligned}
$$

by Lemma 7.1. The next three terms in decomposition (7.45) are similar to the first term, hence we get the desired result:

$$
\begin{aligned}
&\|(R_3 \cdot m')(R_3' \cdot R_\alpha)R_\alpha - (\tilde{R}_3 \cdot \tilde{m}')(\tilde{R}_3' \cdot \tilde{R}_\alpha)\tilde{R}_\alpha\|_{L^{\frac{2p}{p+2}}} \\
&\qquad\qquad \leq c(R', \tilde{R}', \tilde{\rho}_\alpha, m', \tilde{m}')\|R - \tilde{R}\|_{W^{1,p}}.
\end{aligned}
$$

Decomposition (7.46), as well as the other unwritten decompositions, are treated the same way as decomposition (7.45). Note that all the Lipschitz constants are bounded when $R_i, \tilde{R}_i$ belongs to a ball of $W^{1,p}$ and $m, \tilde{m}, \rho_\alpha, \tilde{\rho}_\alpha$ belong to a ball of H^1. This completes the proof of Lemma 7.2. □

Corollary 7.3. *Under the same assumptions as above, we have the following estimate:*

$$
\|m - \tilde{m}\|_{W^{1,p}} \leq c\|R - \tilde{R}\|_{W^{1,p}}. \tag{7.47}
$$

Proof. It suffices to remark that $W^{2,\frac{2p}{p+2}} \hookrightarrow W^{1,\infty}$ and that $W^{1,\infty} \hookrightarrow W^{1,p}$. □

To estimate $\|m - \tilde{m}\|_{W^{2,p}}$, we still need to bound $\|m'' - \tilde{m}''\|_{L^p}$ from above. This upper bound requires the following intermediate result.

Lemma 7.4. *We have*

$$
\|\rho_\alpha - \tilde{\rho}_\alpha\|_{W^{2,\frac{2p}{p+2}}} \leq c\|R - \tilde{R}\|_{W^{1,p}}. \tag{7.48}
$$

Proof. By Lemma 7.1, we first have

$$
\|\rho_\alpha - \tilde{\rho}_\alpha\|_{W^{1,\frac{2p}{p+2}}} < c\|R - \tilde{R}\|_{W^{1,p}}. \tag{7.49}
$$

Secondly, we observe that the second equation in system (6.31) implies that

$$\rho''_\alpha - \tilde{\rho}''_\alpha = \frac{2a_h}{J^h_\alpha}(\rho_\alpha - \tilde{\rho}_\alpha) + \Big(|R'_\alpha|^2 + \Big(1+\frac{\lambda}{\mu}\Big)(R_3\cdot R'_\alpha)^2\Big)\rho_\alpha - \Big(|\tilde{R}'_\alpha|^2 + \Big(1+\frac{\lambda}{\mu}\Big)(\tilde{R}_3\cdot \tilde{R}'_\alpha)^2\Big)\tilde{\rho}_\alpha + \frac{\lambda a_h}{\mu J^h_\alpha}\big(\rho_1+\rho_2-\tilde{\rho}_1-\tilde{\rho}_2+R_3\cdot m' - \tilde{R}_3\cdot\tilde{m}'\big) - \frac{1}{\mu J^h_\alpha}\bar{k}_h\cdot(R_\alpha - \tilde{R}_\alpha). \quad (7.50)$$

This expression is quite similar to the expressions used in the previous lemma. It is clear that the same kind of decompositions yield the local Lipschitz estimate of Lemma 7.4. □

We can now estimate $\|m''-\tilde{m}''\|_{L^p}$ and we have the following result.

Proposition 7.5. *We have*

$$\|m''-\tilde{m}''\|_{L^p} \le c\|R-\tilde{R}\|_{W^{1,p}}, \quad (7.51)$$

where the constant is locally bounded as a function of $R', \tilde{R}', \rho_\alpha, \tilde{\rho}_\alpha, \tilde{\rho}'_\alpha, m', \tilde{m}'$ *and* $\bar{f}_h$.

Proof. We reuse the decompositions of Lemma 7.2 that we can now estimate in L^p rather than in $L^{\frac{2p}{p+2}}$, based on the additional information obtained in Lemmas 7.2 and 7.4. □

To sum things up, we have proved the following result:

Theorem 7.6. *The mapping*

$$S\colon W^{1,p}(]0,1[;\mathrm{SO}(3)) \to W^{2,p}(]0,1[;\mathbb{R}^3)$$
$$R \mapsto m$$

is locally Lipschitz in the above norms.

Remark 7.7. It is fairly clear that the dependence of ρ_α on R is also locally Lipschitz in the $W^{2,\frac{p}{2}}$-norm. We do not pursue in this direction since this result is not needed for our subsequent purposes.

7.2 Locally Lipschitz character of a few auxiliary functions

First, let us rewrite the reduced evolution problem (4.20) as follows:

$$\frac{dR}{dt} = G(m',R)R \quad \text{with} \quad G(m',R) = (H\circ B)(m',R)$$

where

$$H\colon W^{1,p}(]0,1[;M_3^{sk}) \to W^{1,p}(]0,1[;M_3^{sk})$$
$$X \mapsto \frac{1}{\eta}(1+\|X\|^{r+1})^k\|X\|^{r-1}X$$

and

$$B\colon W^{2,p}(]0,1[;\mathbb{R}^3)\times W^{1,p}(]0,1[;\mathrm{SO}(3)) \to W^{1,p}(]0,1[;M_3^{sk})$$
$$(\tau,R) \mapsto \mathrm{skew}(A(\tau,R)R^T),$$

with $A(\tau,R) = (\rho_1 R_1|\rho_2 R_2|\tau')$. It will follow from the ensuing proofs that all these mappings are well-defined between the above spaces. It is implicit in the sequel that m denotes the central line deformation corresponding to the rotation R.

Remark 7.8. Note that $B(\tau,R)$ only depends on τ and R and not on ρ_α (see identity (7.37)). In addition, the matrix $\mathrm{skew}(A(\tau,R)R^T)$ is well defined in $W^{1,p}$ since $W^{1,p}$ is an algebra in one dimension. Indeed, we have $R\in W^{1,p}$ and $\tau\in W^{2,p}$ so that $\tau'\in W^{1,p}$.

Lemma 7.9. *The mapping H is locally Lipschitz.*

Proof. Let us introduce a mapping $h\colon M_3^{sk}\to M_3^{sk}$

$$h(F) = \frac{1}{(k+1)(r+1)\eta}\nabla(1+\|F\|^{r+1})^{k+1}.$$

Then H is thus the Nemytsky operator associated with h.

For $r,k\geq 1$, we see that h is of class $C^2(M_3^{sk};M_3^{sk})$ and so it is locally Lipschitz on M_3^{sk}. Hence, there exists a constant $k(r)$ over each ball of M_3 with center 0 and radius r (denoted by $B(0,r)$) such that the following inequality holds for all $F,\tilde{F}\in B(0,r)$

$$\|h(F)-h(\tilde{F})\| \leq k(r)\|F-\tilde{F}\|.$$

Thus, we obtain

$$\|H(X)-H(\tilde{X})\|_{L^p} \leq c(r)\|X-\tilde{X}\|_{L^p} \leq c(r)\|X-\tilde{X}\|_{W^{1,p}}, \tag{7.52}$$

where we have assumed that $\|X\|_{W^{1,p}(]0,1[;M_3)}, \|\tilde{X}\|_{W^{1,p}(]0,1[;M_3)} \leq r$.

In addition, since h is differentiable and X is in $W^{1,p}$, we have $H_1(X)' = Dh(X)X'$ in the sense of distributions, where the mapping $F\mapsto Dh(F)$ is at least of class C^1. This gives

$$\begin{aligned}\|H(X)'-H(\tilde{X})'\|_{L^p} &\leq \|Dh(X)-Dh(\tilde{X})\|_{L^\infty}\|\tilde{X}'\|_{L^p} + \|Dh(\tilde{X})\|_{L^\infty}\|X'-\tilde{X}'\|_{L^p}\\ &\leq rc(r)\|X-\tilde{X}\|_{W^{1,p}} + M(r)\|X-\tilde{X}\|_{W^{1,p}}\\ &\leq C(r)\|X-\tilde{X}\|_{W^{1,p}},\end{aligned} \tag{7.53}$$

where we have also used the Sobolev embedding $W^{1,p}\hookrightarrow L^\infty$ and the fact that Dh is locally bounded by $M(r)$ (since it is continuous on the compact $B(0,r)$), and where $C(r) = \max(rc(r),M(r))$.

The desired result is now obtained from inequalities (7.52) and (7.53). □

Lemma 7.10. *The mapping B is locally Lipschitz.*

Proof. This follows directly from formula (7.37) and the algebra property of $W^{1,p}$ in one dimension. $\square$

7.3 Existence and uniqueness for the coupled minimization-evolution problem

We are now ready to state the main result of this section.

Theorem 7.11. *Let $I = [0,1]$. Assume that $\bar{f}_h$ and $\bar{k}_h$ are in $C^0([0,+\infty[;L^p(I;\mathbb{R}^3))$ with the initial condition $R_0 \in W^{1,p}(I;\mathrm{SO}(3))$ for some $p \geq 2$. Then there exists a maximum time $T^\star$ such that the reduced evolution problem has a unique solution, $R \in C^1([0,T^\star[;W^{1,p}(I;\mathrm{SO}(3))$.*

Proof. We use the Cauchy-Lipschitz theorem in $W^{1,p}(I;M_3)$. The group SO(3) is a compact C^∞-submanifold of M_3. We can thus take an open tubular neighborhood $\mathscr{N}$ of SO(3) and a C^∞-mapping P from $\mathscr{N}$ to SO(3) such that P is the identity on SO(3). Let

$$U = \{F \in W^{1,p}(I;M_3); F(x_3) \in \mathscr{N} \text{ for all } x_3 \in I\}.$$

Due to the Sobolev embedding $W^{1,p} \hookrightarrow C^0$, the set U is open. Moreover, for all $F \in U$, $P(F)$ belongs to $W^{1,p}(I;\mathrm{SO}(3))$ since this is another Nemytsky operator, and it is easy to show that it is locally Lipschitz with respect to F.

Let us thus consider the Cauchy problem :

$$\frac{dF}{dt} = G(m', P(F))F, \quad F(0) = R_0, \tag{7.54}$$

where m denotes the central line deformation corresponding to the rotation $P(F)$.

To apply the Cauchy-Lipschitz theorem, we thus need to prove that the mapping G is locally Lipschitz from U into $W^{1,p}(I;M_3)$. Indeed, the additional right factor F only contributes to another Lipschitz estimate due to the algebra property.

We have written G as a composite mapping. Therefore, Lemmas 7.9 and 7.10 show that locally

$$\|G(m',P(F)) - G(\tilde{m}',P(\tilde{F}))\|_{W^{1,p}} \leq c\big(\|P(F) - P(\tilde{F})\|_{W^{1,p}} + \|m' - \tilde{m}'\|_{W^{1,p}}\big),$$

and we conclude by Theorem 7.6 that problem (7.54) has a maximal solution.

Our next step is to prove that $F(t,x_3)$ belongs to SO(3) for x_3 in $[0,1]$ and t in $[0,T^\star[$. This will ensure that $P(F) = F$ and that we have actually solved the original problem.

The preceding analysis shows that we may rewrite $G(m',P(F)) = W_F$, where W_F is in $W^{1,p}(I;M_3^{sk})$. Let $B = FF^T \in W^{1,p}(I;M_3)$. We have

$$\frac{dB}{dt} = W_F B - BW_F, \quad B(0) = I. \tag{7.55}$$

We consider this Cauchy problem as a linear ODE in B that, as such, has one obvious solution which is $B(t) = I$. Now, Cauchy-Lipschitz uniqueness implies that $FF^T = I$, that is to say that F is O(3)-valued.

Finally, we remark that for x_3 fixed, the mapping $t \mapsto \det(F(.,x_3))$ is continuous and such that $\det(F(.,x_3)) = \pm 1$. Since $\det(F(0,x_3)) = 1$, it follows that that $\det(F(.,x_3)) = 1$ for every t. This completes the proof. $\square$

Let us conclude this article with a proof of the material indifference of the reduced minimization-evolution model. Here, we ignore boundary conditions so that the minimization problem has a unique solution up to a constant translation.

Proposition 7.12. *The reduced viscoelastic model preserves observer invariance.*

Proof. Let us first establish the invariance of the elastic energy. We have for all $F \in M_3$ and $Q \in \mathrm{SO}(3)$

$$\begin{aligned} W(QF,QR) = \frac{\mu}{4}\|R^TQ^TQF + F^TQ^TQR - 2I\|^2 \\ + \frac{\lambda}{8}\big(\mathrm{tr}(R^TQ^TQF + F^TQ^TQR - 2I)\big)^2 \\ = W(F,R). \end{aligned}$$

Now, for $F = (\rho_1 R_1|\rho_2 R_2|m') + (0|0|(\rho_\alpha R_\alpha)')x_\alpha$, we also have

$$QF = (\rho_1(QR)_1|\rho_2(QR)_2|(Qm)') + (0|0|(\rho_\alpha(QR)_\alpha)')x_\alpha$$

since Q is a constant matrix. Therefore, the uniqueness of the minimization problem with rotated forces $Q\bar{f}_h$ and so forth, shows that the deformations of the central line associated with the rotation QR are given by $Qm + a$ for some translation vector $a \in \mathbb{R}^3$, and conversely.

We now turn to the evolution equation. In the notation introduced earlier, $A_c = A(m',R)$ so that

$$\begin{aligned} Q\,\mathrm{skew}(A_cR^T) &= Q\,\mathrm{skew}(A_cR^T)Q^TQ \\ &= \mathrm{skew}(QA_cR^TQ^T)Q \\ &= \mathrm{skew}(A(Qm',QR)(QR)^T)Q. \end{aligned}$$

It follows first that $\|\,\mathrm{skew}(A(Qm',QR)(QR)^T)\| = \|\,\mathrm{skew}(A_cR^T)\|$ and that

$$\begin{aligned} \frac{d(QR)}{dt} &= Q\frac{1}{\eta}\nu^+\big(\mathrm{skew}(A_cR^T)\big)\,\mathrm{skew}(A_cR^T)R \\ &= \frac{1}{\eta}\nu^+\big(\mathrm{skew}(A(Qm',QR)(QR)^T))\big)\,\mathrm{skew}(A(Qm',QR)(QR)^T)QR. \end{aligned}$$

Combining this with $(QR)(0) = QR_0$ and the Cauchy-Lipschitz uniqueness, we see that the solution of the evolution problem with rotated forces and rotated initial condition is given by QR, hence the material indifference of the model. $\square$

8 Conclusions

In this paper, we have derived a one-dimensional, geometrically exact, viscoelastic model starting from the three-dimensional model introduced by P. Neff. We establish the existence and uniqueness of the solution of the corresponding reduced coupled minimization/evolution problem.

In a forthcoming paper, we will describe the numerical approximation of our one-dimensional model.

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