A penalty algorithm for the spectral element discretization of the Stokes problem

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Abstract: The penalty method when applied to the Stokes problem provides a very efficient algorithm for solving any discretization of this problem since it gives rise to a system of two equations where the unknowns are uncoupled. For a spectral or spectral element discretization of the Stokes problem, we prove a posteriori estimates that allow us to optimize the penalty parameter as a function of the discretization parameter. Numerical experiments confirm the interest of this technique.

Résumé: La méthode de pénalisation appliquée au problème de Stokes fournit un algorithme très efficace pour résoudre n'importe quelle discrétisation de ce problème car il réduit la résolution du problème discret à celle d'un système de deux équations où les inconnues sont découplées. Pour une discrétisation spectrale ou par éléments spectraux du problème de Stokes, nous prouvons des estimations a posteriori qui permettent d'optimiser le choix du paramètre de pénalisation en fonction du paramètre de discrétisation. Des expériences numériques confirment l'intérêt de cette technique.

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1. Introduction.

The Stokes problem models the laminar flow of a viscous incompressible flow in a two- or three-dimensional domain. Its unknowns are the velocity and the pressure of the fluid. Any discretization of this problem by Galerkin type methods results into a linear system of two coupled equations. A large number of algorithms exist to uncouple the two unknowns, see [14], [15], and the references therein. In this work, we are interested in the penalty method for spectral element discretizations.

Indeed, the penalty method, as described in [14, Chap. I, §4.3] in an abstract framework, has been extensively used in the case of finite element discretizations, see [2][3] and [16] for the first a priori error analysis and numerical experiments, and [11][12][13] for complementary results. However, up to our knowledge, this method has not so far been considered in the context of spectral and spectral element discretizations. The main reason can be expressed as follows: The high accuracy of spectral methods and the convergence of order 1 with respect to the penalty parameter would lead to choose a very small penalty parameter in order to equilibrate the two types of errors and, as a consequence, the condition number of the matrix that must be inverted would be very high. Nevertheless, we think that the penalty method is very interesting in the framework of spectral methods for the two next reasons:
(i) It is well-known [16] in the case of finite element discretizations that the addition of a penalty term stabilizes the discrete problem when the constant on the inf-sup condition for the pressure is not independent of the discretization parameter, which is the case for most spectral methods, see [7, §24–26];
(ii) Still in the case of finite elements, it has been recently proved in [5] that the construction of appropriate error indicators leads to optimizing the choice of the penalty parameter for a fixed discretization, more precisely to choose this parameter such that the penalty error and the discretization error are of the same order. The main interest of this optimization is a high reduction of the computation cost for solving the discrete problem.

So, this paper is aimed to the a posteriori analysis of the penalized spectral element discretization of the Stokes problem. Numerical experiments confirm the interest of such an algorithm and also allow us to compare the cases of discretizations with optimal or non optimal inf-sup constants.

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An outline of the paper is as follows.
- In Section 2, we describe the continuous, penalized and discrete Stokes problems and recall their main properties.
- Section 3 is devoted to the a posteriori analysis of the penalized discrete problem.
- In Section 4, we describe the strategy that is used in order to optimize the choice of the penalty parameter and present some numerical experiments concerning the penalty spectral element method.

2. The continuous, penalized and discrete problems.

Let Ω be a bounded connected open set in $\mathbb{R}^d$, $d = 2$ or 3, with a Lipschitz–continuous boundary $\partial\Omega$. The Stokes problem in this domain reads

$$\begin{cases} -\nu\,\Delta \boldsymbol{u} + \mathbf{grad}\, p = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{0} & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where the unknowns are the velocity $\boldsymbol{u}$ and the pressure p. The data $\boldsymbol{f}$ represent a density of body forces and the viscosity ν is a positive constant.

We use the standard notation for the Sobolev spaces $H^s(\Omega)$, $s \in \mathbb{R}$, and $H^s_0(\Omega)$, $s \geq 0$, provided with the corresponding norms. We denote by $L^2_0(\Omega)$ the space of functions in $L^2(\Omega)$ with a null integral on Ω. Thus, for any data $\boldsymbol{f}$ in $H^{-1}(\Omega)^d$, problem (2.1) admits the equivalent variational formulation:

Find $(\boldsymbol{u}, p)$ *in* $H^1_0(\Omega)^d \times L^2_0(\Omega)$ *such that*

$$\begin{aligned} &\forall \boldsymbol{v} \in H^1_0(\Omega)^d, \quad a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) = \langle \boldsymbol{f}, \boldsymbol{v} \rangle, \\ &\forall q \in L^2_0(\Omega), \quad b(\boldsymbol{u}, q) = 0, \end{aligned} \tag{2.2}$$

where the bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are defined by

$$a(\boldsymbol{u}, \boldsymbol{v}) = \nu \int_\Omega \mathbf{grad}\, \boldsymbol{u} : \mathbf{grad}\, \boldsymbol{v}\, d\boldsymbol{x}, \qquad b(\boldsymbol{v}, q) = -\int_\Omega (\operatorname{div} \boldsymbol{v})(\boldsymbol{x}) q(\boldsymbol{x})\, d\boldsymbol{x}, \tag{2.3}$$

while $\langle\cdot,\cdot\rangle$ stands for the duality pairing between $H^{-1}(\Omega)$ and $H^1_0(\Omega)$.

We recall the following properties, see [14, Chap. I] for instance:
(i) There exists a constant $\alpha > 0$ such that the following ellipticity property holds

$$\forall \boldsymbol{v} \in H^1_0(\Omega)^d, \quad a(\boldsymbol{v}, \boldsymbol{v}) \geq \alpha\, \|\boldsymbol{v}\|^2_{H^1(\Omega)^d}; \tag{2.4}$$

(ii) There exists a constant $\beta > 0$ such that the following inf-sup condition holds

$$\forall q \in L^2_0(\Omega), \qquad \sup_{v \in H^1_0(\Omega)^d} \frac{b(\boldsymbol{v}, q)}{\|\boldsymbol{v}\|_{H^1(\Omega)^d}} \geq \beta\, \|q\|_{L^2(\Omega)}. \tag{2.5}$$

Thus, it is readily checked [14, Chap. I, Cor. 4.1] that problem (2.2) admits a unique solution $(\boldsymbol{u}, p)$ in $H^1_0(\Omega)^d \times L^2_0(\Omega)$ which moreover satisfies

$$\|\boldsymbol{u}\|_{H^1(\Omega)^d} + \|p\|_{L^2(\Omega)} \leq c\, \|\boldsymbol{f}\|_{H^{-1}(\Omega)^d}. \tag{2.6}$$

Let now ε be a penalty parameter, $0 < \varepsilon \leq 1$. We consider the penalized problem

Find $(\boldsymbol{u}^\varepsilon, p^\varepsilon)$ *in* $H^1_0(\Omega)^d \times L^2_0(\Omega)$ *such that*

$$\begin{aligned} &\forall \boldsymbol{v} \in H^1_0(\Omega)^d, \quad a(\boldsymbol{u}^\varepsilon, \boldsymbol{v}) + b(\boldsymbol{v}, p^\varepsilon) = \langle \boldsymbol{f}, \boldsymbol{v} \rangle, \\ &\forall q \in L^2_0(\Omega), \quad b(\boldsymbol{u}^\varepsilon, q) = \varepsilon \int_\Omega p^\varepsilon(\boldsymbol{x}) q(\boldsymbol{x})\, d\boldsymbol{x}. \end{aligned} \tag{2.7}$$

We recall from [14, Chap. I, Thm 4.3] the following result.

Proposition 2.1. *For any data $\boldsymbol{f}$ in $H^{-1}(\Omega)^d$, problem* (2.7) *has a unique solution $(\boldsymbol{u}^\varepsilon, p^\varepsilon)$ in $H^1_0(\Omega)^d \times L^2_0(\Omega)$. Moreover the following estimate holds between this solution and the solution $(\boldsymbol{u}, p)$ of problem* (2.2)

$$\|\boldsymbol{u} - \boldsymbol{u}^\varepsilon\|_{H^1(\Omega)^d} + \|p - p^\varepsilon\|_{L^2(\Omega)} \le c\,\varepsilon\, \|\boldsymbol{f}\|_{H^{-1}(\Omega)^d}. \tag{2.8}$$

To describe the discrete problem, we introduce a partition of the domain Ω without overlap:

$$\overline{\Omega} = \cup_{k=1}^K \overline{\Omega}_k \qquad \text{and} \qquad \Omega_k \cap \Omega_{k'} = \emptyset, \quad 1 \le k < k' \le K,$$

where each Ω_k is a rectangle in dimension $d = 2$, a rectangular parallelepiped in dimension $d = 3$. We make the further assumption that the intersection of two different $\overline{\Omega}_k$, if not empty, is either a vertex or a whole edge or a whole face of the two subdomains. We take without restriction the edges of the Ω_k parallel to the coordinate axes.

For each nonnegative real number s, let $\mathbb{P}_s(\Omega_k)$ be the space of restrictions to Ω_k of polynomials with d variables and degree smaller than the integer part of s with respect to each variable. Let now N be an integer, $N \ge 2$. We introduce the discrete spaces

$$\mathbb{X}_N = \big\{\boldsymbol{v}_N \in H^1_0(\Omega)^d;\ \boldsymbol{v}_{N|\Omega_k} \in \mathbb{P}_N(\Omega_k)^d,\ 1 \le k \le K\big\}, \tag{2.9}$$

and, for a fixed real number λ, $0 < \lambda \le 1$,

$$\mathbb{M}_N = \big\{q_N \in L^2_0(\Omega);\ q_{N|\Omega_k} \in \mathbb{P}_{N-2}(\Omega_k) \cap \mathbb{P}_{\lambda N}(\Omega_k),\ 1 \le k \le K\big\}. \tag{2.10}$$

The reason for this choice is that, even for $\lambda = 1$, the space $\mathbb{M}_N$ does not contain spurious modes. But the constant of the discrete inf-sup condition on the form $b(\cdot,\cdot)$ is independent of N only for $\lambda < 1$, see [7, §24–26] and [8, Prop. 3.1].

We recall the standard properties of the Gauss-Lobatto formula on $]-1,1[$: With $\xi_0 = -1$ and $\xi_N = 1$, there exist a unique set of nodes ξ_j, $1 \le j \le N-1$, in $]-1,1[$ and a unique set of weights ρ_j, $0 \le j \le N$, such that

$$\forall \Phi \in \mathbb{P}_{2N-1}(-1,1), \qquad \int_{-1}^1 \Phi(\zeta)\,d\zeta = \sum_{j=0}^N \Phi(\xi_j)\,\rho_j. \tag{2.11}$$

Moreover, the ρ_j are positive and the following property holds, see [7, form. (13.20)]:

$$\forall \varphi_N \in \mathbb{P}_N(-1,1), \quad \|\varphi_N\|^2_{L^2(-1,1)} \le \sum_{j=0}^N \varphi_N^2(\xi_j)\,\rho_j \le 3\,\|\varphi_N\|^2_{L^2(-1,1)}. \tag{2.12}$$

Denoting by F_k one of the mappings which send the square or cube $]-1,1[^d$ onto Ω_k, we define the discrete product, for any functions u and v continuous on $\overline{\Omega}$,

$$(u,v)_N = \begin{cases} \sum_{k=1}^K \frac{\text{meas}(\Omega_k)}{4} \sum_{i=0}^N \sum_{j=0}^N u \circ F_k(\xi_i,\xi_j) v \circ F_k(\xi_i,\xi_j)\,\rho_i\rho_j & \text{if } d=2, \\ \sum_{k=1}^K \frac{\text{meas}(\Omega_k)}{8} \sum_{i=0}^N \sum_{j=0}^N \sum_{p=0}^N u \circ F_k(\xi_i,\xi_j,\xi_p) v \circ F_k(\xi_i,\xi_j,\xi_p)\,\rho_i\rho_j\rho_p & \text{if } d=3. \end{cases}$$

We also introduce the associated Lagrange interpolation operator $\mathcal{I}_N$: For any continuous function g on $\overline{\Omega}$, $\mathcal{I}_N g_{|\Omega_k}$ belongs to $\mathbb{P}_N(\Omega_k)$ and is equal to g at all nodes $F_k(\xi_i,\xi_j)$, $0 \le i,j \le N$, in dimension $d=2$, at all nodes $F_k(\xi_i,\xi_j,\xi_p)$, $0 \le i,j,p \le N$, in dimension $d=3$.

We are now in a position to write the discrete problem (where we skip the exponent ε for simplicity): For any data $\boldsymbol{f}$ continuous on $\overline{\Omega}$,

Find $(\boldsymbol{u}_N, p_N)$ in $\mathbb{X}_N \times \mathbb{M}_N$ such that

$$\begin{aligned} &\forall \boldsymbol{v}_N \in \mathbb{X}_N, \quad a_N(\boldsymbol{u}_N,\boldsymbol{v}_N) + b_N(\boldsymbol{v}_N,p_N) = (\boldsymbol{f},\boldsymbol{v}_N)_N, \\ &\forall q_N \in \mathbb{M}_N, \quad b_N(\boldsymbol{u}_N,q_N) = \varepsilon\,(p_N,q_N)_N, \end{aligned} \tag{2.13}$$

where the bilinear forms $a_N(\cdot,\cdot)$ and $b_N(\cdot,\cdot)$ are defined by

$$a_N(\boldsymbol{u}_N,\boldsymbol{v}_N) = \nu\,(\mathbf{grad}\,\boldsymbol{u}_N, \mathbf{grad}\,\boldsymbol{v}_N)_N, \qquad b_N(\boldsymbol{v}_N,q_N) = -(\mathrm{div}\,\boldsymbol{v}_N, q_N)_N. \tag{2.14}$$

Note that, thanks to the exactness property (2.11) and the choice of $\mathbb{M}_N$, $b_N(\cdot,\cdot)$ can be replaced by $b(\cdot,\cdot)$ in this problem; similarly, in the second line of this problem, the discrete product $(\cdot,\cdot)_N$ can be replaced by the scalar product of $L^2(\Omega)$.

The main advantage of this discrete problem with respect to the standard non penalized discrete Stokes problem can be written as follows. Let Π_N denote the orthogonal projection operator from $L^2_0(\Omega)$ onto $\mathbb{M}_N$. Then problem (2.13) is fully equivalent to the system

$$\forall \boldsymbol{v}_N \in \mathbb{X}_N, \quad a_N(\boldsymbol{u}_N,\boldsymbol{v}_N) + \varepsilon^{-1}\big(\Pi_N(\mathrm{div}\,\boldsymbol{u}_N), \Pi_N(\mathrm{div}\,\boldsymbol{v}_N)\big)_N = (\boldsymbol{f},\boldsymbol{v}_N)_N, \tag{2.15}$$

and

$$p_N = -\varepsilon^{-1}\,\Pi_N(\mathrm{div}\,\boldsymbol{u}_N). \tag{2.16}$$

The only unknown of equation (2.15) is the discrete velocity $\boldsymbol{u}_N$ and equation (2.16) provides an explicit formula for the discrete pressure p_N. So solving problem (2.13) is not at all expensive.

Proposition 2.2. *For any data $\boldsymbol{f}$ continuous on $\overline{\Omega}$, problem* (2.13) *has a unique solution* $(\boldsymbol{u}_N, p_N)$.

Proof: The ellipticity of the form $a_N(\cdot,\cdot)$ (with an ellipticity constant independent of N) follows from (2.12) and a Poincaré–Friedrichs inequality. Next, when setting

$$\tilde{a}_N(\boldsymbol{u}_N,\boldsymbol{v}_N) = a_N(\boldsymbol{u}_N,\boldsymbol{v}_N) + \varepsilon^{-1}\big(\Pi_N(\mathrm{div}\,\boldsymbol{u}_N), \Pi_N(\mathrm{div}\,\boldsymbol{v}_N)\big)_N, \tag{2.17}$$

we observe that $\tilde{a}_N(\boldsymbol{v}_N,\boldsymbol{v}_N) \ge a_N(\boldsymbol{v}_N,\boldsymbol{v}_N)$, whence the ellipticity of $\tilde{a}_N(\cdot,\cdot)$. Thus, problem (2.15) has a unique solution in $\mathbb{X}_N$. Thus, the pair $(\boldsymbol{u}_N,p_N)$, with p_N given by (2.16), is the unique solution of problem (2.13).

To go further, we recall from [1, Prop. 5.1] the following inf-sup condition:

$$\begin{aligned} &\forall q_N \in \mathbb{M}_N, \quad \sup_{\boldsymbol{v}_N \in \mathbb{X}_N} \frac{b_N(\boldsymbol{v}_N,q_N)}{\|\boldsymbol{v}_N\|_{H^1(\Omega)^d}} \ge \beta_N(\lambda)\,\|q_N\|_{L^2(\Omega)}, \\ &\qquad\qquad \text{with} \quad \beta_N(\lambda) = \begin{cases} \beta_1\,N^{-\frac{d-1}{2}} & \text{if } \lambda = 1, \\ \beta_2 & \text{if } \lambda < 1, \end{cases} \end{aligned} \tag{2.18}$$

where β_1 and β_2 are positive constants independent of N. We now establish the a priori error estimate.

Proposition 2.3. *Assume that the data $\boldsymbol{f}$ belong to $H^\sigma(\Omega)^d$, $\sigma > \frac{d}{2}$, and that the solutions $(\boldsymbol{u}, p)$ of problem (2.2) and $(\boldsymbol{u}^\varepsilon, p^\varepsilon)$ of problem (2.7) belongs to $H^{s+1}(\Omega)^d \times H^s(\Omega)$, $s \geq 0$. There exists a constant c only depending on these data and solutions such that the following error estimate holds between the solutions $(\boldsymbol{u}, p)$ of problem (2.2) and $(\boldsymbol{u}_N, p_N)$ of problem (2.13)*

$$\|\boldsymbol{u} - \boldsymbol{u}_N\|_{H^1(\Omega)^d} + \mu\, \|p - p_N\|_{L^2(\Omega)} \leq c\Big(\varepsilon + (1 + \mu^{-1})\, N^{-s} + N^{-\sigma}\Big), \tag{2.19}$$

with $\mu = \max\{\beta_N(\lambda), \varepsilon\}$.

Proof: We proceed in two steps.
1) Estimate (2.19) with $\mu = \beta_N(\lambda)$ is derived first by bounding the error between $(\boldsymbol{u}, p)$ and the solution of problem (2.13) for $\varepsilon = 0$ as performed in [1, Thm 5.3] in a more general framework and second by bounding the error between this solution and $(\boldsymbol{u}_N, p_N)$ thanks to [14, Chap. I, Thm 4.3].
2) Estimate (2.19) with $\mu = \varepsilon$ is derived first by using (2.8), second by writing the same formulation as in (2.15)−(2.16) for the solution $(\boldsymbol{u}^\varepsilon, p^\varepsilon)$ of problem (2.7), third by bounding successively the errors $\|\boldsymbol{u}^\varepsilon - \boldsymbol{u}_N\|_{H^1(\Omega)^d}$ and $\|p^\varepsilon - p_N\|_{L^2(\Omega)}$ from this last formulation.

Note that, in practical situations, ε is most often smaller than N^{-1}, so that estimate (2.19) with $\mu = \beta_N(\lambda)$ is the best one. Moreover, it only involves the regularity of the solution $(\boldsymbol{u}, p)$.

3. A posteriori estimate of the penalty and discretization errors.

We wish to prove an upper bound of the error between the solutions $(\boldsymbol{u}, p)$ of problem (2.2) and $(\boldsymbol{u}_N, p_N)$ of problem (2.13), by a quantity which can be computed explicitly once the discrete solution is known. As now standard, the main idea for this is to use the triangle inequalities

$$\begin{aligned} \|\boldsymbol{u}-\boldsymbol{u}_N\|_{H^1(\Omega)^d} &\le \|\boldsymbol{u}-\boldsymbol{u}^\varepsilon\|_{H^1(\Omega)^d} + \|\boldsymbol{u}^\varepsilon-\boldsymbol{u}_N\|_{H^1(\Omega)^d}, \\ \|p-p_N\|_{L^2(\Omega)} &\le \|p-p^\varepsilon\|_{L^2(\Omega)} + \|p^\varepsilon-p_N\|_{L^2(\Omega)}, \end{aligned} \tag{3.1}$$

and to evaluate separately the errors issued from the penalization and the discretization.

Indeed, when subtracting problem (2.7) from problem (2.2), we obtain the following system of residual equations

$$\begin{aligned} &\forall \boldsymbol{v} \in H_0^1(\Omega)^d, \quad a(\boldsymbol{u}-\boldsymbol{u}^\varepsilon, \boldsymbol{v}) + b(\boldsymbol{v}, p-p^\varepsilon) = 0, \\ &\forall q \in L_0^2(\Omega), \quad b(\boldsymbol{u}-\boldsymbol{u}^\varepsilon, q) = -\varepsilon \int_\Omega p^\varepsilon(\boldsymbol{x}) q(\boldsymbol{x})\, d\boldsymbol{x}. \end{aligned} \tag{3.2}$$

Thus, standard arguments (see [14, Chap. I, Cor. 4.1]), combined with the ellipticity property (2.4) and the inf-sup condition (2.5), yield the bound

$$\|\boldsymbol{u}-\boldsymbol{u}^\varepsilon\|_{H^1(\Omega)^d} + \|p-p^\varepsilon\|_{L^2(\Omega)} \le c\,\varepsilon\, \|p^\varepsilon\|_{L^2(\Omega)}. \tag{3.3}$$

However, in view of the implementation, we wish to define error indicators which only depend on the discrete solution $(\boldsymbol{u}_N, p_N)$. So we introduce the error indicator

$$\eta^\varepsilon = \varepsilon\, \|p_N\|_{L^2(\Omega)}. \tag{3.4}$$

We are now in a position to state the first a posteriori estimates.

Theorem 3.1. *There exists a constant c independent of ε and N such that the following error estimate holds between the solutions $(\boldsymbol{u}, p)$ of problem* (2.2) *and $(\boldsymbol{u}^\varepsilon, p^\varepsilon)$ of problem* (2.7)

$$\|\boldsymbol{u}-\boldsymbol{u}^\varepsilon\|_{H^1(\Omega)^d} + \|p-p^\varepsilon\|_{L^2(\Omega)} \le c\,\big(\eta^\varepsilon + \varepsilon\, \|p^\varepsilon - p_N\|_{L^2(\Omega)}\big). \tag{3.5}$$

The following bound holds for the indicator η^ε defined in (3.4)

$$\eta^\varepsilon \le |\boldsymbol{u}-\boldsymbol{u}^\varepsilon|_{H^1(\Omega)^d} + \varepsilon\, \|p^\varepsilon - p_N\|_{L^2(\Omega)}. \tag{3.6}$$

Proof: Estimate (3.5) follows from (3.3) and a triangle inequality. On the other hand, when taking q equal to p^ε in the second line of (3.2) and using the formula

$$\forall \boldsymbol{v} \in H_0^1(\Omega)^d, \quad |\boldsymbol{v}|^2_{H^1(\Omega)^d} = \|\operatorname{div} \boldsymbol{v}\|^2_{L^2(\Omega)} + \|\mathbf{curl}\, \boldsymbol{v}\|^2_{L^2(\Omega)^{\frac{d(d-1)}{2}}},$$

we derive

$$\varepsilon\, \|p^\varepsilon\|_{L^2(\Omega)} \le |\boldsymbol{u}-\boldsymbol{u}^\varepsilon|_{H^1(\Omega)^d}.$$

Combining this with a further triangle inequality gives (3.6).

To estimate the discretization error, we follow the approach in [10, §4] and [5, §3.3], combined with the arguments in [4, §2]. Indeed, let us set, for all $U = (\boldsymbol{u}, p)$ and $V = (\boldsymbol{v}, q)$,

$$\mathcal{A}_\varepsilon(U, V) = a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) + b(\boldsymbol{u}, q) - \varepsilon \int_\Omega p(\boldsymbol{x}) q(\boldsymbol{x})\, d\boldsymbol{x}. \tag{3.7}$$

The form $\mathcal{A}_\varepsilon(\cdot, \cdot)$ is bilinear and continuous on $\mathcal{X}(\Omega) \times \mathcal{X}(\Omega)$, with

$$\mathcal{X}(\Omega) = H_0^1(\Omega)^d \times L_0^2(\Omega).$$

Moreover, the following inf-sup condition is proved in [5, Lemma 3.5] as a consequence of (2.4) and (2.5) and with obvious definition of the norm $\|\cdot\|_{\mathcal{X}(\Omega)}$: There exists a constant $\beta_* > 0$ independent of ε such that

$$\forall U \in \mathcal{X}(\Omega), \qquad \sup_{V \in \mathcal{X}(\Omega)} \frac{\mathcal{A}_\varepsilon(U, V)}{\|V\|_{\mathcal{X}(\Omega)}} \geq \beta_* \, \|U\|_{\mathcal{X}(\Omega)}. \tag{3.8}$$

So we are led to evaluate the residual $\mathcal{A}_\varepsilon(U^\varepsilon - U_N, V)$, with $U^\varepsilon = (\boldsymbol{u}^\varepsilon, p^\varepsilon)$ and $U_N = (\boldsymbol{u}_N, p_N)$.

We first observe from problem (2.13) and the exactness property (2.11) that, for any $V_{N-1} = (\boldsymbol{v}_{N-1}, 0)$ with $\boldsymbol{v}_{N-1}$ in $\mathbb{X}_{N-1}$,

$$\mathcal{A}_\varepsilon(U_N, V_{N-1}) = \int_\Omega (\mathcal{I}_N \boldsymbol{f})(\boldsymbol{x}) \cdot \boldsymbol{v}_{N-1}(\boldsymbol{x})\, d\boldsymbol{x}.$$

Thus, we derive from (2.7) and the previous line

$$\mathcal{A}_\varepsilon(U^\varepsilon - U_N, V) = \mathcal{A}_\varepsilon(U^\varepsilon - U_N, V - V_{N-1}) + \int_\Omega (\boldsymbol{f} - \mathcal{I}_N \boldsymbol{f})(\boldsymbol{x}) \cdot \boldsymbol{v}_{N-1}(\boldsymbol{x})\, d\boldsymbol{x},$$

or equivalently

$$\begin{aligned} \mathcal{A}_\varepsilon(U^\varepsilon - U_N, V) = \int_\Omega (\mathcal{I}_N \boldsymbol{f})(\boldsymbol{x}) \cdot (\boldsymbol{v} - \boldsymbol{v}_{N-1})(\boldsymbol{x})\, d\boldsymbol{x} - \mathcal{A}_\varepsilon(U_N, V - V_{N-1}) \\ + \int_\Omega (\boldsymbol{f} - \mathcal{I}_N \boldsymbol{f})(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x})\, d\boldsymbol{x}. \end{aligned} \tag{3.9}$$

Integrating by parts on each Ω_k, we can also write

$$\begin{aligned} &\int_\Omega (\mathcal{I}_N \boldsymbol{f})(\boldsymbol{x}) \cdot (\boldsymbol{v} - \boldsymbol{v}_{N-1})(\boldsymbol{x})\, d\boldsymbol{x} - \mathcal{A}_\varepsilon(U_N, V - V_{N-1}) \\ &\quad = \sum_{k=1}^K \Big(\int_{\Omega_k} (\mathcal{I}_N \boldsymbol{f} + \nu\, \Delta \boldsymbol{u}_N - \mathbf{grad}\, p_N)(\boldsymbol{x}) \cdot (\boldsymbol{v} - \boldsymbol{v}_{N-1})(\boldsymbol{x})\, d\boldsymbol{x} \\ &\qquad - \int_{\partial\Omega_k} (\nu\, \partial_n \boldsymbol{u}_N - p_N\, \boldsymbol{n})(\boldsymbol{\tau}) \cdot (\boldsymbol{v} - \boldsymbol{v}_{N-1})(\boldsymbol{\tau})\, d\boldsymbol{\tau} \\ &\qquad + \int_{\Omega_k} (\operatorname{div} \boldsymbol{u}_N)(\boldsymbol{x}) q(\boldsymbol{x})\, d\boldsymbol{x} + \varepsilon \int_{\Omega_k} p_N(\boldsymbol{x}) q(\boldsymbol{x})\, d\boldsymbol{x} \Big). \end{aligned} \tag{3.10}$$

To go further, we need some notation.

Notation 3.2. For each k, let $\Gamma_{k\ell}$, $1 \leq \ell \leq L(k)$, be the edges ($d=2$) or faces ($d=3$) of Ω_k which are not contained in $\partial\Omega$. We denote by $[\cdot]_{k\ell}$ the jump through each $\Gamma_{k\ell}$.

This leads to the following definition of the error indicators: For $1 \leq k \leq K$,

$$\begin{aligned}\eta_k = N^{-1} \, \|\mathcal{I}_N \boldsymbol{f} + \nu\, \Delta \boldsymbol{u}_N - \mathbf{grad}\, p_N\|_{L^2(\Omega_k)^d} \\ + \sum_{\ell=1}^{L(k)} N^{-\frac{1}{2}} \, \| \left[\nu\, \partial_n \boldsymbol{u}_N - p_N\, \boldsymbol{n}\right]_{k\ell} \|_{L^2(\Gamma_{k\ell})^d} + \|\mathrm{div}\, \boldsymbol{u}_N\|_{L^2(\Omega_k)}.\end{aligned} \tag{3.11}$$

The following result deals with approximation error estimates which are derived from duality arguments. Let $\Pi_N^{1,0}$ denote the orthogonal projection operator from $H_0^1(\Omega)$ onto $\mathbb{X}_N$ for the scalar product associated with the norm $|\cdot|_{H^1(\Omega)}$.

Lemma 3.3. *The following estimate is derived for any function v in $H_0^1(\Omega)$*

$$\|v - \Pi_N^{1,0} v\|_{L^2(\Omega)} \leq c\, \rho_\Omega \, N^{-1} \, \|v\|_{H^1(\Omega)}, \tag{3.12}$$

where ρ_Ω is equal
(i) to 1 in dimension $d=2$ or if Ω is convex,
(ii) to $N^{\frac{1}{2}}$ in dimension $d=3$ and when Ω is not convex.

Proof: We have

$$\|v - \Pi_N^{1,0} v\|_{L^2(\Omega)} = \sup_{\chi \in L^2(\Omega)} \frac{\int_\Omega (v - \Pi_N^{1,0} v)(\boldsymbol{x}) \chi(\boldsymbol{x})\, d\boldsymbol{x}}{\|\chi\|_{L^2(\Omega)}}.$$

For any χ in $L^2(\Omega)$, the problem

$$-\Delta\varphi = \chi \quad \text{in } \Omega, \qquad \varphi = 0 \quad \text{on } \partial\Omega,$$

has a unique solution φ in $H_0^1(\Omega)$. Moreover, we note that

$$\begin{aligned}\int_\Omega (v - \Pi_N^{1,0} v)(\boldsymbol{x}) \chi(\boldsymbol{x})\, d\boldsymbol{x} &= \int_\Omega \big(\mathbf{grad}\,(v - \Pi_N^{1,0} v)\big)(\boldsymbol{x}) \cdot (\mathbf{grad}\,\varphi)(\boldsymbol{x})\, d\boldsymbol{x} \\ &= \int_\Omega (\mathbf{grad}\, v)(\boldsymbol{x}) \cdot \big(\mathbf{grad}\,(\varphi - \Pi_N^{1,0} \varphi)\big)(\boldsymbol{x})\, d\boldsymbol{x},\end{aligned}$$

so that

$$\int_\Omega (v - \Pi_N^{1,0} v)(\boldsymbol{x}) \chi(\boldsymbol{x})\, d\boldsymbol{x} \leq \|v\|_{H^1(\Omega)} \|\varphi - \Pi_N^{1,0} \varphi\|_{H^1(\Omega)}.$$

Moreover, the following result is easily derived from [9, Lemma VI.2.5] thanks to an interpolation argument, for any real number $s \geq 0$,

$$\|\varphi - \Pi_N^{1,0} \varphi\|_{H^1(\Omega)} \leq c\, N^{-s} \, \|\varphi\|_{H^{s+1}(\Omega)}. \tag{3.13}$$

To conclude, we recall that the mapping: $\chi \mapsto \varphi$ is continuous from $L^2(\Omega)$ into $H^{s+1}(\Omega)$, with $s \geq \frac{1}{2}$ in the general case and $s \geq 1$ when Ω is convex. In dimension $d=2$ and when Ω is not convex, we are led to use a more complex argument: φ is the sum of a function φ_r

in $H^2(\Omega)$ and of a singular function S with support in a neighbourhood of the nonconvex corners of Ω. The approximation properties of this last function are established in [6, §3].

We omit the proof of the next statement since the result is established in [4, Cor. 2.6] in dimension $d = 2$, and the arguments can easily be extended to the case of dimension $d = 3$.

Lemma 3.4. *The following estimate is derived for any function v in $H^1_0(\Omega)$, for all Ω_k, $1 \le k \le K$,*

$$\|v - \Pi_N^{1,0} v\|_{L^2(\partial\Omega_k)} \le c\, N^{-\frac{1}{2}}\, \|v\|_{H^1(\Omega)}. \tag{3.14}$$

By applying (3.8) with U equal to $U^\varepsilon - U_N$ and using (3.9) combined with (3.10), Cauchy–Schwarz inequalities and Lemmas 3.3 and 3.4, we derive the final estimate.

Theorem 3.5. *There exists a constant c independent of ε and N such that the following a posteriori error estimate holds between the solutions $(\boldsymbol{u}^\varepsilon, p^\varepsilon)$ of problem (2.7) and $(\boldsymbol{u}_N, p_N)$ of problem (2.13)*

$$\|\boldsymbol{u}^\varepsilon - \boldsymbol{u}_N\|_{H^1(\Omega)^d} + \|p^\varepsilon - p_N\|_{L^2(\Omega)} \le c\,\Big(\eta^\varepsilon + \rho_\Omega\,\big(\sum_{k=1}^{K} \eta_k^2\big)^{\frac{1}{2}} + \|\boldsymbol{f} - \mathcal{I}_N \boldsymbol{f}\|_{L^2(\Omega)^d}\Big), \tag{3.15}$$

where ρ_Ω is equal
(i) to 1 in dimension $d = 2$ or if Ω is convex,
(ii) to $N^{\frac{1}{2}}$ in dimension $d = 3$ and when Ω is not convex.

In dimension $d = 2$ or when Ω is convex, estimate (3.15) is fully optimal and leads to an explicit upper bound for the error. The converse estimate (i.e. the upper bound of each η_k as a function of the error) would likely be not optimal, see [4, Thm 2.9]. We do not present it because we do not intend to perform adaptivity with respect to N.

4. Optimization strategy and numerical experiments.

This section is devoted to a numerical comparison of the discretizations with and without penalization and also, in the penalization case, with and without optimization of the penalty parameter. So, we first describe the strategy that is used for this optimization.

4.1. The optimization strategy

Assuming that the data $\boldsymbol{f}$ are smooth, we work with sufficiently large N for the quantity $\|\boldsymbol{f} - \mathcal{I}_N \boldsymbol{f}\|_{L^2(\Omega)^d}$ which appears in (3.15) to be neglectable with respect to the other terms. We fix a real number ρ, $0 < \rho < 1$, and an initial value ε^0 of ε. Next, we apply iteratively the following process.

OPTIMIZATION STEP. For a given value ε^m of ε, we compute the solution $(\boldsymbol{u}_N, p_N)$ of the corresponding problem (2.15) – (2.16), and the associated error indicators η^{ε^m} defined in (3.4) and η_k defined in (3.11). We also set:

$$\eta_{(N)} = \Big(\sum_{k=1}^{K} \eta_k^2\Big)^{\frac{1}{2}}. \tag{4.1}$$

Next, when

$$\rho\, \eta_{(N)} \leq \eta^{\varepsilon^m} \leq \frac{1}{\rho}\, \eta_{(N)}, \tag{4.2}$$

we stop the process. Otherwise, we take ε^{m+1} equal to a constant times $\varepsilon^m\, \eta_{(N)}/\eta^{\varepsilon^m}$.

The optimization step is iterated until condition (4.2) is realized (when possible) or only a limited number of times $M_{\max}$.

Remark 4.1. The algorithm for computing the operator: $\boldsymbol{v}_N \mapsto \Pi_N(\operatorname{div} \boldsymbol{v}_N)$ plays a key role in the implementation of the penalized discrete problem. So we now describe it, in the case $d = 2$ for simplicity. We first note from the definition (2.10) that the projection operator Π_N reduces to local ones: More precisely, if Π_N^k denotes the orthogonal projection operator from $L^2(\Omega_k)$ onto $\mathbb{P}_{N-2}(\Omega_k) \cap \mathbb{P}_{\lambda N}(\Omega_k)$, for any φ in $L_0^2(\Omega)$, $(\Pi_N \varphi)_{|\Omega_k}$ coincides with $\Pi_N^k(\varphi_{|\Omega_k})$. So, without restriction, we only work on the reference square $\hat{\Omega} =]-1, 1[^2$ and we denote by $\widehat{\Pi}_N$ the corresponding projection operator. Let φ_i be the Lagrange polynomials associated with the nodes ξ_j: For each i, $0 \leq i \leq N$, φ_i belongs to $\mathbb{P}_N(-1, 1)$ and satisfies: $\varphi_i(\xi_j) = \delta_{ij}$. Any function $\boldsymbol{v}_N$ in $\mathbb{P}_N(\widehat{\Omega})^2$ can be written as

$$\boldsymbol{v}_N(\zeta, \xi) = \sum_{i=0}^{N} \sum_{j=0}^{N} \boldsymbol{v}^{ij}\, \varphi_i(\zeta) \varphi_j(\xi), \tag{4.3}$$

where each $\boldsymbol{v}^{ij} = (v_1^{ij}, v_2^{ij})$ is equal to $\boldsymbol{v}_N(\xi_i, \xi_j)$. Thus, we have

$$(\operatorname{div} \boldsymbol{v}_N)(\zeta, \xi) = \sum_{i=0}^{N} \sum_{j=0}^{N} \big(v_1^{ij}\, \varphi_i'(\zeta) \varphi_j(\xi) + v_2^{ij}\, \varphi_i(\zeta) \varphi_j'(\xi)\big). \tag{4.4}$$

We also have the expansion

$$(\operatorname{div} \boldsymbol{v}_N)(\zeta, \xi) = \sum_{\ell=0}^{N} \sum_{n=0}^{N} d_{\ell n}\, L_\ell(\zeta) L_n(\xi), \tag{4.5}$$

so that

$$\widehat{\Pi}_N(\operatorname{div} \boldsymbol{v}_N)(\zeta,\xi) = (\operatorname{div} \boldsymbol{v}_N)(\zeta,\xi) - \sum_{(\ell,n)\in\mathcal{N}} d_{\ell n}\, L_\ell(\zeta) L_n(\xi), \tag{4.6}$$

where $\mathcal{N}$ stands for the set of pair of indices (ℓ, n), $0 \le \ell, n \le N$, such that either ℓ or n is larger than $\min\{N-2, \lambda N\}$. It remains to compute the $d_{\ell n}$ as a function of the $\boldsymbol{v}^{ij}$. It follows from (4.4) that

$$d_{\ell n} = \sum_{i=0}^{N} \sum_{j=0}^{N} \big(v_1^{ij}\, \beta_\ell^i \alpha_n^j + v_2^{ij}\, \alpha_\ell^i \beta_n^j\big), \tag{4.7}$$

with

$$\alpha_n^j = \frac{1}{\|L_n\|^2_{L^2(-1,1)}} \int_{-1}^{1} \varphi_j(\zeta) L_n(\zeta)\, d\zeta, \qquad \beta_n^j = \frac{1}{\|L_n\|^2_{L^2(-1,1)}} \int_{-1}^{1} \varphi_j'(\zeta) L_n(\zeta)\, d\zeta. \tag{4.8}$$

From the formulas (see [7, Thm 3.2 & form. (13.19)])

$$\|L_n\|^2_{L^2(-1,1)} = \frac{1}{n+\frac{1}{2}}, \qquad \sum_{i=0}^{N} L_N^2(\xi_i)\, \rho_i = \frac{2}{N},$$

combined with (2.11), we derive

$$\alpha_n^j = c(n)\, L_n(\xi_j)\, \rho_j \qquad \text{with} \quad c(n) = \begin{cases} n+\frac{1}{2} & \text{if } 0 \le n < N, \\ \frac{N}{2} & \text{if } n = N. \end{cases} \tag{4.9}$$

Evaluating the β_n^j requires a further integration by parts:

$$\beta_n^j = (n+\frac{1}{2})\, \big(-L_n'(\xi_j)\, \rho_j + \delta_{jN} - (-1)^n \delta_{j0}\big). \tag{4.10}$$

4.2. First computations

We conclude with several types of numerical experiments, in dimension $d = 2$. We first take the data $\boldsymbol{f}$ equal to zero and the boundary condition $\boldsymbol{u} = \boldsymbol{0}$ replaced by

$$\boldsymbol{u} = \boldsymbol{g} \quad \text{on } \partial\Omega, \tag{4.11}$$

for a continuous function $\boldsymbol{g}$ in $H^{\frac{1}{2}}(\partial\Omega)^2$, satisfying

$$\int_{\partial\Omega} (\boldsymbol{g} \cdot \boldsymbol{n})(\tau)\, d\tau = 0.$$

The corresponding discrete condition reads

$$\boldsymbol{u}_N = \boldsymbol{g}_N \quad \text{on } \partial\Omega, \tag{4.12}$$

where $\boldsymbol{g}_N$ is the interpolate of $\boldsymbol{g}$ at all nodes $F_k(\xi_i, \xi_j)$ which belong to $\partial\Omega$, with values in the trace space of $\mathbb{X}_N$. Note that the previous analysis easily extends to this new situation.

The numerical experiments deal either with the square $\Omega =]-1,1[^2$ without domain decomposition or with the L-shaped domain $\Omega =]-1,1[^2 \backslash [0,1[^2$ divided into three equal squares in an obvious way. These domains and the corresponding Gauss–Lobatto grids for $N = 20$ are illustrated in Figure 1.

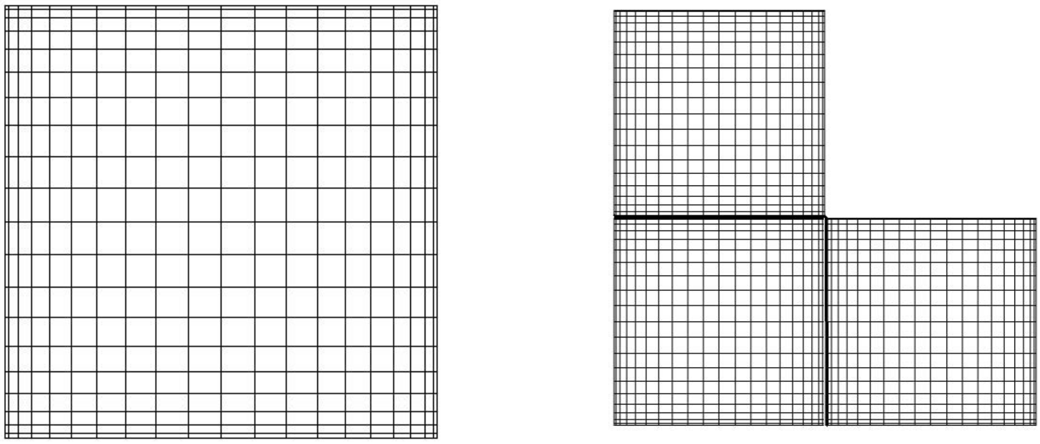

Figure 1. The computation domains and examples of Gauss–Lobatto grids

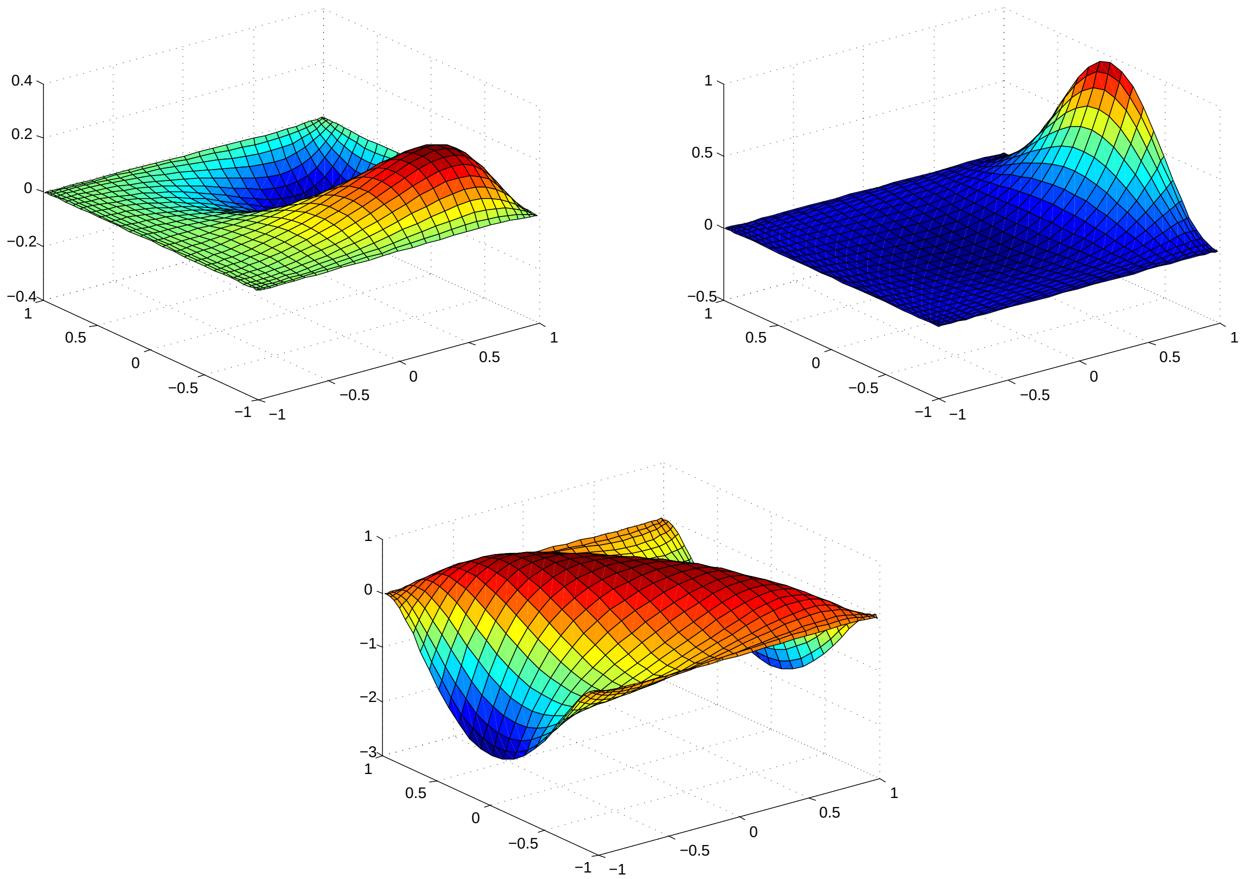

Figure 2. The two components of the velocity and the pressure for data in (4.13)

We take the viscosity ν equal to 10^{-2} and the data $\boldsymbol{g}$ corresponding

- when Ω is the square, to the regularized driven cavity problem

$$\boldsymbol{g}(-1,y)=\boldsymbol{g}(x,-1)=\boldsymbol{g}(x,1)=\boldsymbol{0}, \qquad \boldsymbol{g}(1,y)=\begin{pmatrix}0\\(1-x^2)^{\frac{5}{2}}\end{pmatrix}, \quad -1\le x,y\le 1, \quad (4.13)$$

- when Ω is the L-shaped domain, to a Poiseuille type flow

$$\begin{aligned}\boldsymbol{g}(-1,y)&=\begin{pmatrix}1-y^2\\0\end{pmatrix}, \quad -1\le y\le 1,\\ \boldsymbol{g}(1,y)&=\begin{pmatrix}-8y(1+y)\\0\end{pmatrix}, \quad -1\le y\le 0,\\ \boldsymbol{g}&=\boldsymbol{0} \quad \text{elsewhere.}\end{aligned} \quad (4.14)$$

Figures 2 and 3 present from top to bottom the curves of isovalues of the two components of the velocity and of the pressure for these two problems, obtained with $N=30$ and $\varepsilon=10^{-5}$.

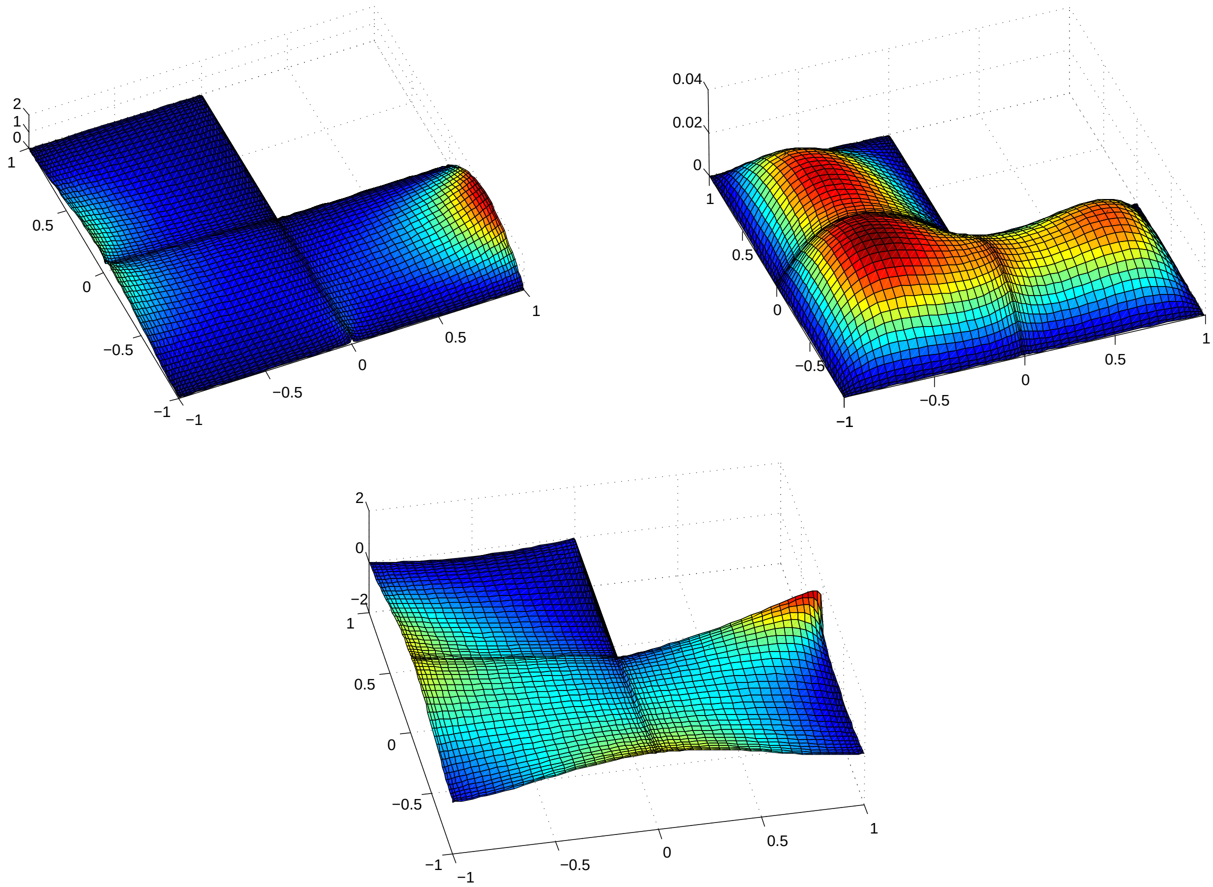

Figure 3. The two components of the velocity and the pressure for data in (4.14)

4.3. Optimization of the penalty parameter

We now work on the L-shaped domain, again with $\nu=10^{-2}$. We consider the solution $(\boldsymbol{u}=\mathbf{curl}\,\psi,p)$ given by

$$\psi(x,y)=(1-x^2)^{\frac{5}{2}}(1-y^2)^{\frac{5}{2}}\,\sin(\pi x)\sin(\pi y), \qquad p(x,y)=xy+\frac{1}{12}. \quad (4.15)$$

We first study the influence of ε and N on the indicators η^ε and $\eta_{(N)}$. For N fixed equal to 30 and ε varying between 10^{-1} and 10^{-5}, Figure 4 presents the error $\|\boldsymbol{u} - \boldsymbol{u}_N\|_{H^1(\Omega)^d}$ (plain red line), the error indicators η^ε (dashed dotted blue line) and $\eta_{(N)}$ (dashed black line). It can thus be checked that the $\eta_{(N)}$ are fully independent of ε (we refer to [5, §5] for similar results in the finite element case). Moreover the error and η^ε decrease with exactly the same slope until the discretization error becomes larger than the penalization error.

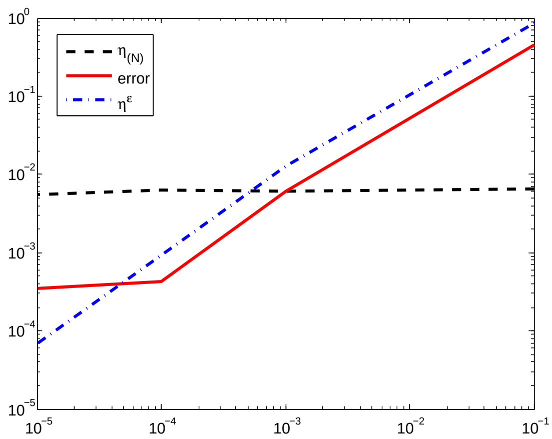

Figure 4. The error and the indicators η^ε and $\eta_{(N)}$ for a fixed N

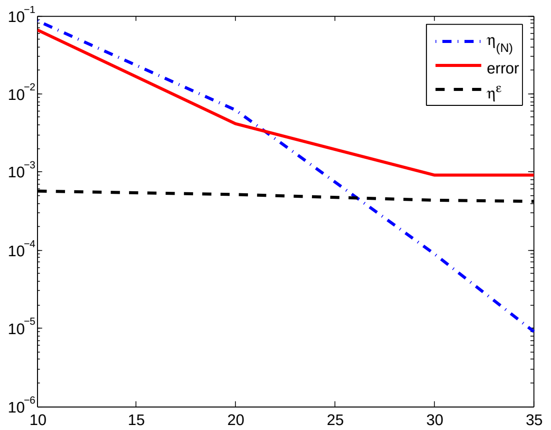

Figure 5. The error and the indicators η^ε and $\eta_{(N)}$ for a fixed ε

Similarly, for ε fixed equal to 10^{-5} and N varying between 10 and 30, Figure 5 presents the error $\|\boldsymbol{u} - \boldsymbol{u}_N\|_{H^1(\Omega)^d}$ (plain red line), the error indicators η^ε (dashed black line) and $\eta_{(N)}$ (dashed dotted blue line). Here, the η^ε are completely independent of N.

We still work with the solution $(\boldsymbol{u}, p)$ defined from (4.15). We apply the optimization strategy for ε described in Section 4.1, with $\rho = 0.8$ and $M_{\max} = 5$. Table 1 presents for five values of N the different values of the optimized ε (where "optimized" means that (4.2) holds or that $M_{\max} = 5$ iterations have been performed), denoted by ε_{opt}. It can be observed that ε_{opt} quickly decreases when N increases, which seems in good coherence with the previous analysis. Indeed, since the solution $(\boldsymbol{u}, p)$ is smooth, the error $\|\boldsymbol{u}^\varepsilon - \boldsymbol{u}_N\|_{H^1(\Omega)^d}$ also quickly decreases when N increases.

N	5	10	15	20	30
ε_{opt}	0.0160	0.0088	0.0063	0.0016	0.0009

Table 1: Values of ε_{opt} as a function of N

4.4. Comparison of the discretizations with and without penalty

In order to check the efficiency of our algorithm, we first compare three algorithms: The Uzawa method (which is another well-known algorithm for uncoupling the two unknowns, see [15, §3.1] for instance) combined with Conjugate Gradient iterations, the penalty method with $\varepsilon = 10^{-5}$ and the penalty method with optimized ε. Table 2 presents the CPU time needed on the computer based on Intel Pentium (4 CPU 3.06 GHz) to invert the final system resulting from the three methods.

N	5	10	15	20	30
Uzawa	3.9211	6.0961	45.722	83.184	131.53
$\varepsilon = 10^{-5}$	3.1607	5.9128	25.201	70.901	99.432
ε_{opt}	1.9131	4.5512	11.294	40.706	98.841

Table 2: Comparison of the CPU times for the three algorithms

As well-known, the penalty method with any reasonable choice of ε is less expensive than the Uzawa algorithm. Moreover, optimizing ε allows us to reduce the computation cost at least for low values of N (since ε_{opt} becomes closer to 10^{-5} when N increases). Note also that the cost of the optimization process is neglectable with respect to the final computation.

4.5. About the choice of the pressure space

We compare the convergence for three choices of discrete pressure spaces:
(i) The space

$$\mathbb{M}_N^+ = \big\{q_N \in L_0^2(\Omega);\ q_{N|\Omega_k} \in \mathbb{P}_{N-1}(\Omega_k),\ 1 \le k \le K\big\}. \tag{4.16}$$

(ii) The space $\mathbb{M}_N$ defined in (2.10) with $\lambda = 1$.
(iii) The space $\mathbb{M}_N$ defined in (2.10) with $\lambda = 0.9$.
It is well-known [7, §24] that the space $\mathbb{M}_N^+$ contains at least one spurious mode for the pressure, so that the non-penalized discrete problem is a priori not well-posed. In constrast, the same arguments as for Proposition 2.2 yield that the penalized discrete problem has a unique solution (but no convergence can be established).

We again work with the square $\Omega =]-1, 1[^2$, for the exact solution $(\boldsymbol{u} = \mathbf{curl}\,\psi, p)$ now given by

$$\psi(x, y) = (1 - x^2)^{\frac{5}{2}}(1 - y^2)^{\frac{5}{2}}, \qquad p(x, y) = xy. \tag{4.17}$$

We first deal with the space $\mathbb{M}_N^+$ defined in (4.16). For N varying from 5 to 40, Figure 6 presents the curves for the errors $\|\boldsymbol{u} - \boldsymbol{u}_N\|_{H^1(\Omega)^2}$ (plain red line) and $\|p - p_N\|_{L^2(\Omega)}$ (dashed dotted blue line), obtained either with the Uzawa method (left part) and the penalization method with $\varepsilon = 10^{-5}$ (right part). As standard, there is no convergence for the Uzawa algorithm and the convergence for the penalization algorithm is of low order.

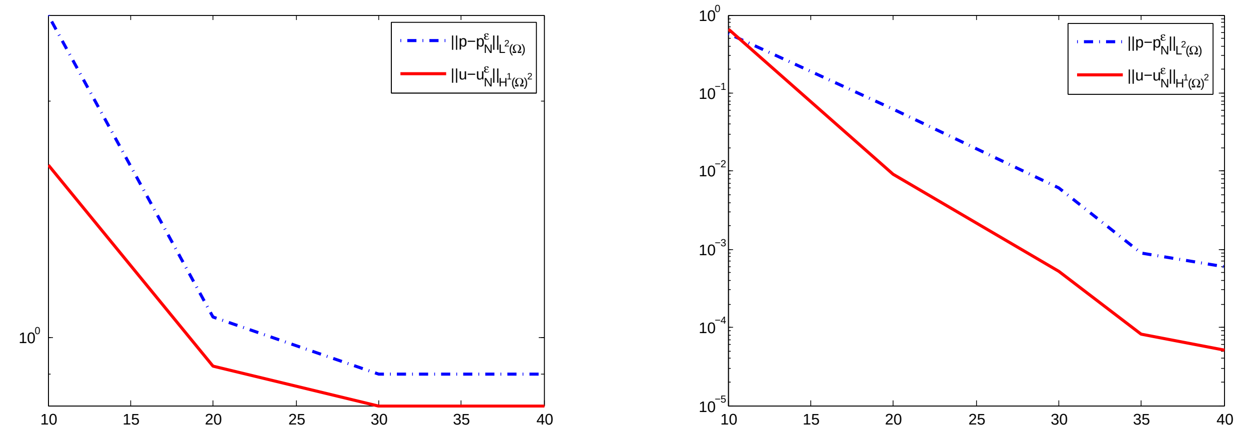

Figure 6. The errors for the space $\mathbb{M}_N^+$ (Uzawa and penalization algorithms)

Figure 7 presents the same curves for the errors, now obtained with the penalization method with $\varepsilon = 10^{-5}$ for the spaces $\mathbb{M}_N$ with $\lambda = 1$ (left part) and $\lambda = 0.9$ (right part). In both cases, the convergence is of spectral type, i.e., of order only limited by the regularity of the exact solution as appears in (2.19). The convergence order also slightly increases when $\lambda = 0.9$, in good agreement with the fact that the parameter μ in (2.19) is equal to 1 in this case.

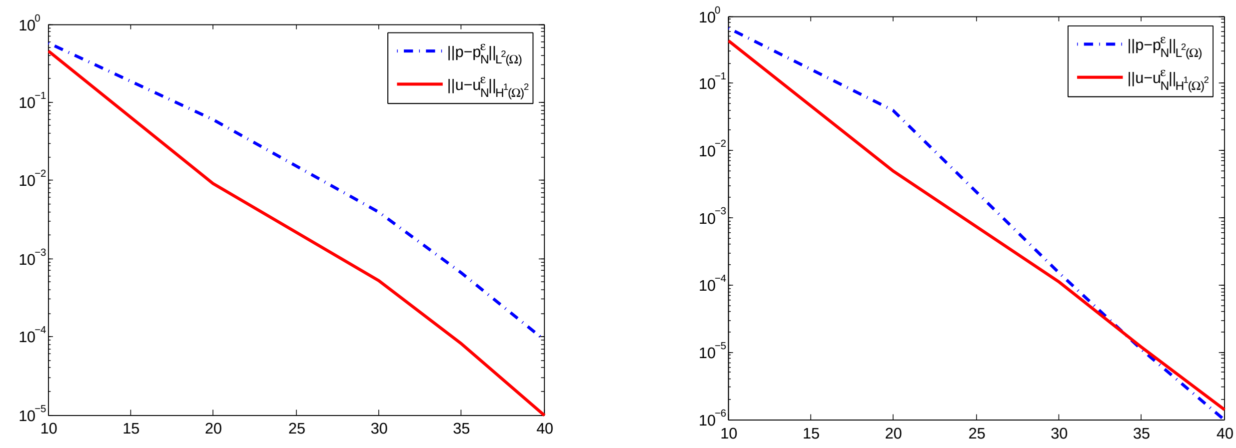

Figure 7. The errors for two types of spaces $\mathbb{M}_N$

To conclude, we present in the following Table 3 the values of $\eta_{(N)}$ obtained with ε fixed equal to 10^{-5} and of ε_{opt} (computed by the optimization strategy described in Section 4.1, with $\varepsilon^0 = 1$, $\rho = 0.8$ and $M_{\max} = 5$) for the three choices of spaces of pressure. When compared with the previous figures, this table indicates that the $\eta_{(N)}$ provide a good representation of the discretization error. But, when starting with $\varepsilon^0 = 1$, $M_{\max} = 5$ iterations do not seem sufficient to optimize ε, i.e., to obtain that the penalization and discretization errors are of the same order.

	N	5	10	20	30	40
Choice (i)	$\eta_{(N)}$	0.857	0.620	0.91×10^{-2}	0.13×10^{-3}	0.80×10^{-4}
	$\varepsilon_{\rm opt}$	0.521	0.585	0.75×10^{-1}	0.29×10^{-1}	0.85×10^{-2}
Choice (ii)	$\eta_{(N)}$	0.18×10^{-1}	0.93×10^{-3}	0.81×10^{-4}	0.64×10^{-5}	0.31×10^{-6}
	$\varepsilon_{\rm opt}$	0.98×10^{-2}	0.72×10^{-2}	0.15×10^{-2}	0.87×10^{-3}	0.24×10^{-3}
Choice (iii)	$\eta_{(N)}$	0.18×10^{-1}	0.93×10^{-3}	0.81×10^{-4}	0.13×10^{-5}	10^{-8}
	$\varepsilon_{\rm opt}$	0.98×10^{-2}	0.72×10^{-2}	0.15×10^{-2}	0.25×10^{-3}	10^{-4}

Table 3: Comparison of $\eta_{(N)}$ and of $\varepsilon_{\rm opt}$ for the different spaces of pressures

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