A new finite element discretization of the Stokes problem coupled with Darcy equations

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Abstract: The flow in a rigid porous medium with a crack is usually modeled by Darcy equations coupled with the Stokes problem. We first propose a new variational formulation of the Stokes system, where the unknowns are the vorticity, the velocity and the pressure, and describe the corresponding finite element discretization. We extend this discretization to the case where Darcy and Stokes equations are coupled and prove optimal a priori and a posteriori error estimates. We conclude with some numerical experiments.

Résumé: Un modèle usuel pour l'écoulement dans un milieu poreux rigide avec une fracture consiste à coupler les équations de Darcy et de Stokes. Nous proposons tout d'abord une nouvelle formulation variationnelle des équations de Stokes où les inconnues sont le tourbillon, la vitesse et la pression et décrivons la discrétisation par éléments finis correspondante. Nous étendons cette discrétisation au cas où les systèmes de Darcy et de Stokes sont couplés et prouvons des estimations d'erreur a priori et a posteriori optimales. Nous présentons quelques expériences numériques.

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1. Introduction.

The treatment of cracks in porous media plays an important role in the numerical simulation of underground flows. Indeed, the flow of a viscous incompressible fluid in a porous medium is usually modeled by Darcy equations and, when the thickness of the crack is too large to be neglected, the Stokes system must be considered in the crack and coupled with these equations. So, we are led to consider the following system, already studied in [7]: If Ω_P and Ω_F denote the domains occupied by the porous medium and the fluid, respectively, and Γ is the interface $\partial\Omega_P \cap \partial\Omega_F$,

$$\begin{cases} \mu\, \boldsymbol{u} + \mathbf{grad}\, p = \boldsymbol{f} & \text{in } \Omega_P, \\ -\nu\, \Delta\boldsymbol{u} + \mathbf{grad}\, p = \boldsymbol{f} & \text{in } \Omega_F, \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega_P \text{ and } \Omega_F, \\ \boldsymbol{u} \cdot \boldsymbol{n} = k & \text{on } \partial\Omega_P \setminus \Gamma, \\ (\boldsymbol{u}_{|\Omega_P} - \boldsymbol{u}_{|\Omega_F}) \cdot \boldsymbol{n} = 0 & \text{on } \Gamma, \\ p_{|\Omega_P} - p_{|\Omega_F} = 0 & \text{on } \Gamma, \\ \mathbf{curl}\, \boldsymbol{u}_{|\Omega_F} \times \boldsymbol{n} = \mathbf{0} & \text{on } \Gamma. \end{cases} \tag{1.1}$$

In this system, the unknowns are the velocity $\boldsymbol{u}$ and the pressure p of the fluid. The data are a density of body forces $\boldsymbol{f}$ and a boundary normal velocity k. The parameters ν and μ are assumed to be positive constants: ν is the viscosity of the fluid and μ depends on this viscosity and on the permeability of the medium which is supposed to be homogeneous (we refer to [6] for a recent work concerning the treatment of inhomogeneous media).

The basic idea for handling the last interface condition, namely $\mathbf{curl}\, \boldsymbol{u}_{|\Omega_F} \times \boldsymbol{n} = \mathbf{0}$ on Γ, is due to [21], see also [15] and [16]: It consists in introducing the vorticity $\boldsymbol{\omega} = \mathbf{curl}\, \boldsymbol{u}$ as a new unknown on the fluid domain Ω_F. This approach was already considered in [7] for coupling Stokes and Darcy systems. However we have rather use another variational formulation of these equations, which leads to a different finite element discretization. Indeed, this new formulation seems more appropriate for the coupling with Darcy equations. It is introduced in Section 2, together with the finite element discrete problem associated with the Stokes system.

In the next step, we write the variational formulation of the full system (1.1) and prove its well-posedness. The discretization of this problem that we propose relies
- on the Lagrange finite elements of degree 1 in dimension $d = 2$ and the Nédélec finite elements [20] of order 1 in dimension $d = 3$ for the vorticity,
- on piecewise constant functions for the velocity,
- on the Crouzeix-Raviart elements [14] for the pressure.

This discretization is nonconforming, however the interest of using such elements for Darcy equations has been brought to light in [1]: It leads to an exactly divergence-free discrete velocity. We prove optimal a priori and a posteriori error estimates for the discrete problem. Some numerical experiments are in good agreement with the analysis and justify our choice of discretization.

The outline of the paper is as follows.

- In Section 2, we write the new variational formulation of the Stokes problem and propose a discretization of it.
- In Section 3, we write the variational formulation of the full system (1.1) and prove its well-posedness.
- Section 4 is devoted to the description of the finite element discrete problem. We also check its well-posedness.
- A priori and a posteriori error estimates for this problem are proved in Section 5 and 6, respectively.
- In Section 7, we present some numerical experiments which confirm the interest of the discretization.

2. A new formulation of the Stokes problem.

Let Ω_F be a bounded connected domain in $\mathbb{R}^d$, $d = 2$ or 3, with a Lipschitz–continuous boundary $\partial\Omega_F$. For simplicity, we assume that Ω_F has a connected boundary and is simply-connected. We introduce the unit outward normal vector $\boldsymbol{n}$ to Ω_F on $\partial\Omega_F$ and we consider the Stokes problem

$$\begin{cases} -\nu\, \Delta \boldsymbol{u} + \mathbf{grad}\, p = \boldsymbol{f} & \text{in } \Omega_F, \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega_F, \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0 & \text{on } \partial\Omega_F, \\ \gamma_t(\mathbf{curl}\, \boldsymbol{u}) = \mathbf{0} & \text{on } \partial\Omega_F. \end{cases} \tag{2.1}$$

To make precise the sense of the operator γ_t, we recall that

- in dimension $d = 2$, for any vector field $\boldsymbol{v}$ with components v_x and v_y, $\mathbf{curl}\, \boldsymbol{v}$ stands for the scalar function $\partial_x v_y - \partial_y v_x$, so that the operator γ_t is the trace operator on $\partial\Omega$,
- in dimension $d = 3$, for any vector field $\boldsymbol{v}$ with components v_x, v_y and v_z, $\mathbf{curl}\, \boldsymbol{v}$ stands for the vector field with components $\partial_y v_z - \partial_z v_y$, $\partial_z v_x - \partial_x v_z$ and $\partial_x v_y - \partial_y v_x$, and the operator γ_t is the tangential trace operator on $\partial\Omega$, defined by: $\gamma_t(\boldsymbol{w}) = \boldsymbol{w} \times \boldsymbol{n}$.

Of course, the operator γ_t is only defined on smooth enough functions as will be made precise later on.

We now introduce the vorticity $\boldsymbol{\omega} = \mathbf{curl}\, \boldsymbol{u}$ as a new unknown and we recall from the lines above that it is a scalar function in dimension $d = 2$, a vector field in dimension $d = 3$. So we are led to consider a modified curl operator, now denoted by **Curl**:

- In dimension $d = 2$ and for a scalar function θ, $\mathbf{Curl}\, \theta$ denotes the vector field with components $\partial_y \theta$ and $-\partial_x \theta$,
- In dimension $d = 3$ and for any vector field $\boldsymbol{\vartheta}$, $\mathbf{Curl}\, \boldsymbol{\vartheta}$ simply coincides with $\mathbf{curl}\, \boldsymbol{\vartheta}$.

Then, it is readily checked that system (2.1) can be written equivalently

$$\begin{cases} \nu\, \mathbf{Curl}\, \boldsymbol{\omega} + \mathbf{grad}\, p = \boldsymbol{f} & \text{in } \Omega_F, \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega_F, \\ \boldsymbol{\omega} = \mathbf{curl}\, \boldsymbol{u} & \text{in } \Omega_F, \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0 & \text{on } \partial\Omega_F, \\ \gamma_t \boldsymbol{\omega} = \mathbf{0} & \text{on } \partial\Omega_F. \end{cases} \tag{2.2}$$

For any bounded connected domain Ω in $\mathbb{R}^d$, $d = 2$ or 3, with a Lipschitz–continuous boundary $\partial\Omega$, we use the full scales of Hilbertian Sobolev spaces $H^s(\Omega)$ and $H^s(\partial\Omega)$, $s \geq 0$. Let $H^1_0(\Omega)$ denote the closure in $H^1(\Omega)$ of the space $\mathscr{D}(\Omega)$ of infinitely differentiable functions with a compact support in Ω, and $H^{-1}(\Omega)$ stand for its dual space. All these spaces are provided with the usual norms. As standard, $L^2_0(\Omega)$ denotes the space

$$L^2_0(\Omega) = \Big\{ q \in L^2(\Omega);\ \int_\Omega q(\boldsymbol{x})\, d\boldsymbol{x} = 0 \Big\}. \tag{2.3}$$

In dimension $d = 3$, we also need the space

$$H(\mathbf{Curl}, \Omega) = \Big\{ \boldsymbol{\vartheta} \in L^2(\Omega)^3;\ \mathbf{Curl}\, \boldsymbol{\vartheta} \in L^2(\Omega)^3 \Big\}, \tag{2.4}$$

equipped with the graph norm, and its subspace

$$H_0(\mathbf{Curl}, \Omega) = \Big\{ \boldsymbol{\vartheta} \in H(\mathbf{Curl}, \Omega);\ \gamma_t(\boldsymbol{\vartheta}) = \mathbf{0} \text{ on } \partial\Omega \Big\}. \tag{2.5}$$

In order to have a common notation for both dimensions $d = 2$ and $d = 3$, we set

$$\mathbb{C}(\Omega_F) = \begin{cases} H^1_0(\Omega_F) & \text{if } d = 2, \\ H_0(\mathbf{Curl}, \Omega_F) & \text{if } d = 3. \end{cases} \tag{2.6}$$

We now consider the variational problem

Find $(\boldsymbol{\omega}, \boldsymbol{u}, p)$ *in* $\mathbb{C}(\Omega_F) \times L^2(\Omega_F)^d \times \big(H^1(\Omega_F) \cap L^2_0(\Omega_F)\big)$ *such that*

$$\begin{aligned} &\forall \boldsymbol{v} \in L^2(\Omega_F)^d, \quad \nu \int_{\Omega_F} (\mathbf{Curl}\, \boldsymbol{\omega})(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x})\, d\boldsymbol{x} + \int_{\Omega_F} \boldsymbol{v}(\boldsymbol{x}) \cdot (\mathbf{grad}\, p)(\boldsymbol{x})\, d\boldsymbol{x} \\ &\qquad\qquad = \int_{\Omega_F} \boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x})\, d\boldsymbol{x}, \\ &\forall q \in H^1(\Omega_F) \cap L^2_0(\Omega_F), \quad \int_{\Omega_F} \boldsymbol{u}(\boldsymbol{x}) \cdot (\mathbf{grad}\, q)(\boldsymbol{x})\, d\boldsymbol{x} = 0, \\ &\forall \boldsymbol{\varphi} \in \mathbb{C}(\Omega_F), \quad \int_{\Omega_F} \boldsymbol{\omega}(\boldsymbol{x}) \cdot \boldsymbol{\varphi}(\boldsymbol{x})\, d\boldsymbol{x} - \int_{\Omega_F} \boldsymbol{u}(\boldsymbol{x}) \cdot (\mathbf{Curl}\, \boldsymbol{\varphi})(\boldsymbol{x})\, d\boldsymbol{x} = 0. \end{aligned} \tag{2.7}$$

We first check its equivalence with system (2.2).

Proposition 2.1. *For any data* $\boldsymbol{f}$ *in* $L^2(\Omega_F)^d$, *a triple* $(\boldsymbol{\omega}, \boldsymbol{u}, p)$ *in* $\mathbb{C}(\Omega_F) \times L^2(\Omega_F)^d \times \big(H^1(\Omega_F) \cap L^2_0(\Omega_F)\big)$ *is a solution of problem* (2.2) *in the distribution sense if and only if it is a solution of problem* (2.7).

Proof: The result is established in four steps:
1) The first equation in (2.7) obviously implies the first equation in (2.2) and the converse property follows from the density of $\mathscr{D}(\Omega_F)^d$ in $L^2(\Omega_F)^d$.
2) The second equation in (2.7) is obviously satisfied when q is constant, hence for all functions q in $H^1(\Omega_F)$. So, it implies the second equation of (2.2). The boundary condition

in the fourth equation of (2.2) is then derived by integration by parts (note that it is satisfied in the dual space of $H^{\frac{1}{2}}(\partial\Omega_F)$). Conversely, the second and fourth equations of (2.2) imply the second equation of (2.7).
3) Due to the density of $\mathscr{D}(\Omega_F)$ in $H^1_0(\Omega_F)$ and of $\mathscr{D}(\Omega_F)^3$ in $H_0(\mathbf{Curl},\Omega_F)$, see [17, Chap. I, Thm 2.12] or [4, Thm 1.2.29] for instance, the third equation of (2.7) is fully equivalent to the third equation of (2.2).
4) And finally the boundary condition in the fifth equation of (2.2) is explicitly taken into account since $\boldsymbol{\omega}$ belongs to $\mathbb{C}(\Omega_F)$.

Remark 2.2. In the more standard variational formulation of system (2.2), see [21] or [4, §2.5], the solution $(\boldsymbol{\omega},\boldsymbol{u},p)$ is sought for in $H_0(\mathbf{Curl},\Omega_F)\times H_0(\mathrm{div},\Omega_F)\times L^2_0(\Omega_F)$, with

$$H_0(\mathrm{div},\Omega_F)=\Big\{\boldsymbol{v}\in L^2(\Omega_F)^d;\ \mathrm{div}\,\boldsymbol{v}\in L^2(\Omega_F)\ \text{ and }\ \boldsymbol{v}\cdot\boldsymbol{n}=0\text{ on }\partial\Omega_F\Big\}.$$

and the bilinear form: $(\boldsymbol{v},q)\mapsto\int_{\Omega_F}\boldsymbol{v}(\boldsymbol{x})\cdot(\mathbf{grad}\,q)(\boldsymbol{x})\,d\boldsymbol{x}$ is replaced by

$$(\boldsymbol{v},q)\ \mapsto\ -\int_{\Omega_F}(\mathrm{div}\,\boldsymbol{v})(\boldsymbol{x})\,q(\boldsymbol{x})\,d\boldsymbol{x}.$$

So it follows from [4, Prop. 2.5.1] and Proposition 2.1 that it is fully equivalent to problem (2.7). However it leads to rather different discretizations.

In order to perform the analysis of problem (2.7), we first note that the bilinear forms:

$$(\boldsymbol{\vartheta},\boldsymbol{w};\boldsymbol{v})\ \mapsto\ \nu\int_{\Omega_F}(\mathbf{Curl}\,\boldsymbol{\vartheta})(\boldsymbol{x})\cdot\boldsymbol{v}(\boldsymbol{x})\,d\boldsymbol{x},\quad(\boldsymbol{v},q)\ \mapsto\int_{\Omega_F}\boldsymbol{v}(\boldsymbol{x})\cdot(\mathbf{grad}\,q)(\boldsymbol{x})\,d\boldsymbol{x},$$
$$(\boldsymbol{\vartheta},\boldsymbol{w};\boldsymbol{\varphi})\ \mapsto\int_{\Omega_F}\boldsymbol{\vartheta}(\boldsymbol{x})\cdot\boldsymbol{\varphi}(\boldsymbol{x})\,d\boldsymbol{x}-\int_{\Omega_F}\boldsymbol{w}(\boldsymbol{x})\cdot(\mathbf{Curl}\,\boldsymbol{\varphi})(\boldsymbol{x})\,d\boldsymbol{x},$$

are continuous on

$$\big(\mathbb{C}(\Omega_F)\times L^2(\Omega_F)^d\big)\times L^2(\Omega_F)^d,\quad L^2(\Omega_F)^d\times H^1(\Omega_F)\quad\text{and}\quad\big(\mathbb{C}(\Omega_F)\times L^2(\Omega_F)^d\big)\times\mathbb{C}(\Omega_F),$$

respectively. So the kernel

$$V(\Omega_F)=\Big\{\boldsymbol{v}\in L^2(\Omega_F)^d;\ \forall q\in H^1(\Omega_F)\cap L^2_0(\Omega_F),\ \int_{\Omega_F}\boldsymbol{v}(\boldsymbol{x})\cdot(\mathbf{grad}\,q)(\boldsymbol{x})\,d\boldsymbol{x}=0\Big\},\tag{2.8}$$

is a Hilbert space. Moreover, the next characterization is readily checked

$$V(\Omega_F)=\Big\{\boldsymbol{v}\in L^2(\Omega_F)^d;\ \mathrm{div}\,\boldsymbol{v}=0\text{ in }\Omega_F\ \text{ and }\ \boldsymbol{v}\cdot\boldsymbol{n}=0\text{ on }\partial\Omega_F\Big\}.\tag{2.9}$$

Similarly, we consider the kernel

$$\begin{aligned}\mathcal{W}(\Omega_F)=\Big\{&(\boldsymbol{\vartheta},\boldsymbol{w})\in\mathbb{C}(\Omega_F)\times V(\Omega_F);\\&\forall\boldsymbol{\varphi}\in\mathbb{C}(\Omega_F),\ \int_{\Omega_F}\boldsymbol{\vartheta}(\boldsymbol{x})\cdot\boldsymbol{\varphi}(\boldsymbol{x})\,d\boldsymbol{x}-\int_{\Omega_F}\boldsymbol{w}(\boldsymbol{x})\cdot(\mathbf{Curl}\,\boldsymbol{\varphi})(\boldsymbol{x})\,d\boldsymbol{x}=0\Big\}.\end{aligned}\tag{2.10}$$

It is a Hilbert space and coincides with the space of pairs $(\boldsymbol{\vartheta}, \boldsymbol{w})$ in $\mathbb{C}(\Omega_F) \times V(\Omega_F)$ such that $\boldsymbol{\vartheta}$ is equal to $\mathbf{curl}\,\boldsymbol{w}$ in the distribution sense.

The reason for introducing these kernels is that, for any solution $(\boldsymbol{\omega}, \boldsymbol{u}, p)$ of problem (2.7), the pair $(\boldsymbol{\omega}, \boldsymbol{u})$ is a solution of the next reduced problem

Find $(\boldsymbol{\omega}, \boldsymbol{u})$ in $\mathcal{W}(\Omega_F)$ such that

$$\forall \boldsymbol{v} \in V(\Omega_F), \quad \nu \int_{\Omega_F} (\mathbf{Curl}\,\boldsymbol{\omega})(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x})\, d\boldsymbol{x} = \int_{\Omega_F} \boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x})\, d\boldsymbol{x}. \tag{2.11}$$

To prove the well-posedness of this problem, we observe from (2.9) that the space $V(\Omega_F)$ and consequently $\mathcal{W}(\Omega_F)$ coincide with their analogues for the other formulation of the problem hinted in Remark 2.2 and are provided with the same norms. So we refer to [4, Prop. 2.5.3 & 2.5.4] for the proof of the following lemma.

Lemma 2.3. *The following positivity property holds*

$$\forall \boldsymbol{v} \in V(\Omega_F),\ \boldsymbol{v} \neq \boldsymbol{0}, \qquad \sup_{(\boldsymbol{\vartheta}, \boldsymbol{w}) \in \mathcal{W}(\Omega_F)} \nu \int_{\Omega_F} (\mathbf{Curl}\,\boldsymbol{\vartheta})(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x})\, d\boldsymbol{x} > 0. \tag{2.12}$$

There exists a constant $\alpha > 0$ such that the following inf-sup condition holds

$$\forall (\boldsymbol{\vartheta}, \boldsymbol{w}) \in \mathcal{W}(\Omega_F),$$
$$\sup_{\boldsymbol{v} \in V(\Omega_F)} \frac{\nu \int_{\Omega_F} (\mathbf{Curl}\,\boldsymbol{\vartheta})(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x})\, d\boldsymbol{x}}{\|\boldsymbol{v}\|_{L^2(\Omega_F)^d}} > \alpha \left(\|\boldsymbol{\vartheta}\|_{\mathbb{C}(\Omega_F)} + \|\boldsymbol{w}\|_{L^2(\Omega_F)^d} \right). \tag{2.13}$$

The well-posedness of problem (2.11) is now a direct consequence of Lemma 2.3, see [4, Thm 1.3.7]. Conversely, if $(\boldsymbol{\omega}, \boldsymbol{u})$ is a solution of problem (2.11), it follows from (2.8) and (2.10) that it satisfies the second and third equations in (2.7). The existence of a pressure p such that the first equation is satisfied is a consequence of the following lemma.

Lemma 2.4. *There exists a constant $\beta > 0$ such that the following inf-sup condition holds*

$$\forall q \in H^1(\Omega_F) \cap L^2_0(\Omega_F), \qquad \sup_{\boldsymbol{v} \in L^2(\Omega_F)^d} \frac{\int_{\Omega_F} \boldsymbol{v}(\boldsymbol{x}) \cdot (\mathbf{grad}\, q)(\boldsymbol{x})\, d\boldsymbol{x}}{\|\boldsymbol{v}\|_{L^2(\Omega_F)^d}} > \beta\, \|q\|_{H^1(\Omega_F)}. \tag{2.14}$$

Proof: Taking $\boldsymbol{v} = \mathbf{grad}\, q$ yields

$$\int_{\Omega_F} \boldsymbol{v}(\boldsymbol{x}) \cdot (\mathbf{grad}\, q)(\boldsymbol{x})\, d\boldsymbol{x} = |q|^2_{H^1(\Omega_F)} = \|\boldsymbol{v}\|_{L^2(\Omega_F)^d}\, |q|_{H^1(\Omega_F)}.$$

Thus, applying a Bramble-Hilbert inequality, see [17, Chap. I, Thm 1.9] or [4, Prop. 1.2.9] for instance, gives (2.14).

By combining all this, we derive the well-posedness of problem (2.7).

Theorem 2.5. *For any data $\boldsymbol{f}$ in $L^2(\Omega_F)^d$, problem* (2.7) *has a unique solution $(\boldsymbol{\omega}, \boldsymbol{u}, p)$ in $\mathbb{C}(\Omega_F) \times L^2(\Omega_F)^d \times \big(H^1(\Omega_F) \cap L^2_0(\Omega_F)\big)$. Moreover, this solution satisfies*

$$\|\boldsymbol{\omega}\|_{\mathbb{C}(\Omega_F)} + \|\boldsymbol{u}\|_{L^2(\Omega_F)^d} + \|p\|_{H^1(\Omega_F)} \le c\, \|\boldsymbol{f}\|_{L^2(\Omega_F)^d}. \tag{2.15}$$

To conclude, we briefly describe the discretization of problem (2.7) that we use in the next sections. Let $\mathcal{T}_h^F$ be a triangulation of Ω_F by triangles ($d = 2$) or tetrahedra ($d = 3$). For each K in $\mathcal{T}_h^F$ and any nonnegative integer k, we denote by $\mathcal{P}_k(K)$ the space of restrictions to K of polynomials with d variables and total degree $\le k$. In dimension $d = 3$, we also need the space $\widetilde{\mathcal{P}}(K)$ of restrictions to K of polynomials of the form $\boldsymbol{a}+\boldsymbol{b}\times\boldsymbol{x}$, $\boldsymbol{a} \in \mathbb{R}^3$, $\boldsymbol{b} \in \mathbb{R}^3$. Next, we introduce the discrete spaces:

• The space $\mathbb{C}_h(\Omega_F)$ of discrete vorticities which approximates $\mathbb{C}(\Omega_F)$ is defined, in dimension $d = 2$, by

$$\mathbb{C}_h(\Omega_F) = \big\{\boldsymbol{\vartheta}_h \in H^1_0(\Omega_F);\ \forall K \in \mathcal{T}_h^F,\ \boldsymbol{\vartheta}_{h|K} \in \mathcal{P}_1(K)\big\}, \tag{2.16}$$

and, in dimension $d = 3$, by

$$\mathbb{C}_h(\Omega_F) = \big\{\boldsymbol{\vartheta}_h \in H_0(\mathbf{Curl}, \Omega_F);\ \forall K \in \mathcal{T}_h^F,\ \boldsymbol{\vartheta}_{h|K} \in \widetilde{\mathcal{P}}(K)\big\}. \tag{2.17}$$

• The space $\mathbb{X}_h(\Omega_F)$ of discrete velocities which approximates $L^2(\Omega_F)^d$ is defined by

$$\mathbb{X}_h(\Omega_F) = \big\{\boldsymbol{v}_h \in L^2(\Omega_F)^d;\ \forall K \in \mathcal{T}_h^F,\ \boldsymbol{v}_{h|K} \in \mathcal{P}_0(K)^d\big\}, \tag{2.18}$$

• And finally the space $\mathbb{M}_h(\Omega_F)$ of discrete pressures which approximates $H^1(\Omega_F)\cap L^2_0(\Omega_F)$ is the space of functions q_h in $L^2_0(\Omega_F)$ such that their restrictions to all elements K of $\mathcal{T}_h^F$ belong to $\mathcal{P}_1(K)$ and which are continuous at the midpoint of each edge ($d = 2$) or barycenter of each face ($d = 3$) of these elements.

The nonconformity of the discretization is due to the fact that this last space is not contained in $H^1(\Omega_F)$. With these choices, the discrete problem now reads

Find $(\boldsymbol{\omega}_h, \boldsymbol{u}_h, p_h)$ in $\mathbb{C}_h(\Omega_F) \times \mathbb{X}_h(\Omega_F) \times \mathbb{M}_h(\Omega_F)$ such that

$$\forall \boldsymbol{v}_h \in \mathbb{X}_h(\Omega_F), \quad \nu \int_{\Omega_F} (\mathbf{Curl}\, \boldsymbol{\omega}_h)(\boldsymbol{x}) \cdot \boldsymbol{v}_h(\boldsymbol{x})\, d\boldsymbol{x} + \sum_{K\in\mathcal{T}_h^F} \int_K \boldsymbol{v}_h(\boldsymbol{x}) \cdot (\mathbf{grad}\, p_h)(\boldsymbol{x})\, d\boldsymbol{x} = \int_{\Omega_F} \boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{v}_h(\boldsymbol{x})\, d\boldsymbol{x},$$

$$\forall q_h \in \mathbb{M}_h(\Omega_F), \quad \sum_{K\in\mathcal{T}_h^F} \int_K \boldsymbol{u}_h(\boldsymbol{x}) \cdot (\mathbf{grad}\, q_h)(\boldsymbol{x})\, d\boldsymbol{x} = 0,$$

$$\forall \boldsymbol{\varphi}_h \in \mathbb{C}_h(\Omega_F) \quad \int_{\Omega_F} \boldsymbol{\omega}_h(\boldsymbol{x}) \cdot \boldsymbol{\varphi}_h(\boldsymbol{x})\, d\boldsymbol{x} - \int_{\Omega_F} \boldsymbol{u}_h(\boldsymbol{x}) \cdot (\mathbf{Curl}\, \boldsymbol{\varphi}_h)(\boldsymbol{x})\, d\boldsymbol{x} = 0.$$

We prefer to postpone its analysis to the more complex case of the coupled problem (1.1).

3. The coupled problem.

Let Ω and Ω_F be two bounded connected open sets in $\mathbb{R}^d$, $d = 2$ or 3, with Lipschitz–continuous boundaries such that $\overline{\Omega}_F \subset \Omega$ (this is only for simplicity since the next study easily extends to the case where Ω_F is not connected, i.e. to a finite number of cracks, and also to the case where $\partial\Omega_F \cap \partial\Omega$ is not empty, i.e., where the crack goes up to the surface for instance). As previously, we assume that Ω_F has a connected boundary and is simply-connected. Thus, when setting $\Omega_P = \Omega \setminus \overline{\Omega}_F$, we observe that the interface $\Gamma = \partial\Omega_F \cap \partial\Omega_P$ coincides with $\partial\Omega_F$ and also that $\partial\Omega_P$ is equal to $\partial\Omega \cup \Gamma$, see Figure 1.

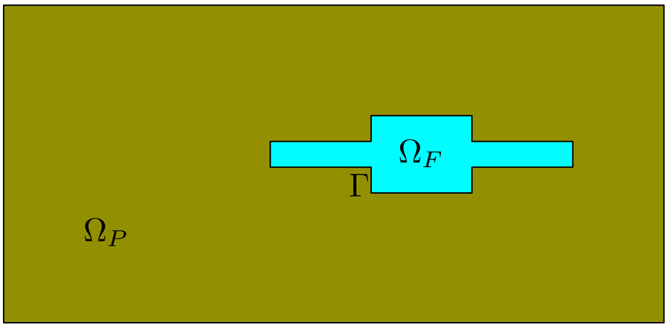

Figure 1: A porous medium with a crack

With the notation of Section 2, we observe that system (1.1) can be written equivalently

$$\begin{cases} \mu\,\boldsymbol{u} + \mathbf{grad}\, p = \boldsymbol{f} & \text{in } \Omega_P, \\ \nu\, \mathbf{Curl}\, \boldsymbol{\omega} + \mathbf{grad}\, p = \boldsymbol{f} & \text{in } \Omega_F, \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega_P \text{ and } \Omega_F, \\ \boldsymbol{\omega} = \mathbf{curl}\, \boldsymbol{u} & \text{in } \Omega_F, \\ \boldsymbol{u} \cdot \boldsymbol{n} = k & \text{on } \partial\Omega, \\ (\boldsymbol{u}_{|\Omega_P} - \boldsymbol{u}_{|\Omega_F}) \cdot \boldsymbol{n} = 0 & \text{on } \Gamma, \\ p_{|\Omega_P} - p_{|\Omega_F} = 0 & \text{on } \Gamma, \\ \gamma_t \boldsymbol{\omega} = \mathbf{0} & \text{on } \Gamma. \end{cases} \tag{3.1}$$

When integrating the third equation of this system on Ω and using the sixth equation, it can be checked that the existence of a solution requires the following compatibility condition on the datum k

$$\langle k, 1 \rangle_{\partial\Omega} = 0, \tag{3.2}$$

where $\langle \cdot, \cdot \rangle_{\partial\Omega}$ denotes the duality pairing between $H^{\frac{1}{2}}(\partial\Omega)$ and its dual space $H^{-\frac{1}{2}}(\partial\Omega)$.

All this leads to consider the following variational problem

Find $(\boldsymbol{\omega}, \boldsymbol{u}, p)$ *in* $\mathbb{C}(\Omega_F) \times L^2(\Omega)^d \times \big(H^1(\Omega) \cap L^2_0(\Omega)\big)$ *such that*

$$\begin{aligned}
&\forall \boldsymbol{v} \in L^2(\Omega)^d, \quad a(\boldsymbol{\omega}, \boldsymbol{u}; \boldsymbol{v}) + b(\boldsymbol{v}, p) = \int_\Omega \boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x})\, d\boldsymbol{x}, \\
&\forall q \in H^1(\Omega) \cap L^2_0(\Omega), \quad b(\boldsymbol{u}, q) = \langle k, q \rangle_{\partial\Omega}, \\
&\forall \boldsymbol{\varphi} \in \mathbb{C}(\Omega_F), \quad c(\boldsymbol{\omega}, \boldsymbol{u}; \boldsymbol{\varphi}) = 0,
\end{aligned} \tag{3.3}$$

where the bilinear forms $a(\cdot, \cdot; \cdot)$, $b(\cdot, \cdot)$ and $c(\cdot, \cdot; \cdot)$ are defined by

$$\begin{aligned}
a(\boldsymbol{\vartheta}, \boldsymbol{w}; \boldsymbol{v}) &= \mu \int_{\Omega_P} \boldsymbol{w}(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x})\, d\boldsymbol{x} + \nu \int_{\Omega_F} (\mathbf{Curl}\, \boldsymbol{\vartheta})(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x})\, d\boldsymbol{x}, \\
b(\boldsymbol{v}, q) &= \int_\Omega \boldsymbol{v}(\boldsymbol{x}) \cdot (\mathbf{grad}\, q)(\boldsymbol{x})\, d\boldsymbol{x}, \\
c(\boldsymbol{\vartheta}, \boldsymbol{w}; \boldsymbol{\varphi}) &= \int_{\Omega_F} \boldsymbol{\vartheta}(\boldsymbol{x}) \cdot \boldsymbol{\varphi}(\boldsymbol{x})\, d\boldsymbol{x} - \int_{\Omega_F} \boldsymbol{w}(\boldsymbol{x}) \cdot (\mathbf{Curl}\, \boldsymbol{\varphi})(\boldsymbol{x})\, d\boldsymbol{x}.
\end{aligned} \tag{3.4}$$

Proving the equivalence of problems (3.1) and (3.3) relies on similar arguments as for Proposition 2.1 (note that the second equation of (3.3) is now equivalent to the third, fifth and sixth equations of problem (3.1)). So we state without proof the following result.

Proposition 3.1. *For any data* $\boldsymbol{f}$ *in* $L^2(\Omega)^d$ *and* k *in* $H^{-\frac{1}{2}}(\partial\Omega)$ *satisfying* (3.2), *a triple* $(\boldsymbol{\omega}, \boldsymbol{u}, p)$ *in* $\mathbb{C}(\Omega_F) \times L^2(\Omega)^d \times \big(H^1(\Omega) \cap L^2_0(\Omega)\big)$ *is a solution of problem* (3.1) *in the distribution sense if and only if it is a solution of problem* (3.3).

As in Section 2, the continuity of the forms $a(\cdot, \cdot; \cdot)$, $b(\cdot, \cdot)$ and $c(\cdot, \cdot; \cdot)$ on the appropriate spaces is readily checked. So, proving the well-posedness of problem (3.3) relies on the analogues of Lemmas 2.3 and 2.4. The new statements require the kernels

$$V(\Omega) = \Big\{ \boldsymbol{v} \in L^2(\Omega)^d;\ \forall q \in H^1(\Omega) \cap L^2_0(\Omega),\ b(\boldsymbol{v}, q) = 0 \Big\}, \tag{3.5}$$

and

$$\mathcal{W}(\Omega) = \Big\{ (\boldsymbol{\vartheta}, \boldsymbol{w}) \in \mathbb{C}(\Omega_F) \times V(\Omega);\ \forall \boldsymbol{\varphi} \in \mathbb{C}(\Omega_F),\ c(\boldsymbol{\vartheta}, \boldsymbol{w}; \boldsymbol{\varphi}) = 0 \Big\}. \tag{3.6}$$

There also, the following characterization holds

$$V(\Omega) = \Big\{ \boldsymbol{v} \in L^2(\Omega)^d;\ \mathrm{div}\, \boldsymbol{v} = 0 \text{ in } \Omega \text{ and } \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \partial\Omega \Big\}, \tag{3.7}$$

while $\mathcal{W}(\Omega)$ coincides with the space of pairs $(\boldsymbol{\vartheta}, \boldsymbol{w})$ in $\mathbb{C}(\Omega_F) \times V(\Omega)$ such that $\boldsymbol{\vartheta}$ is equal to $\mathbf{curl}\, \boldsymbol{w}$ in the distribution sense on Ω_F.

Lemma 3.2. *The following positivity property holds*

$$\forall \boldsymbol{v} \in V(\Omega),\ \boldsymbol{v} \neq \boldsymbol{0}, \qquad \sup_{(\boldsymbol{\vartheta},\boldsymbol{w})\in\mathcal{W}(\Omega)} a(\boldsymbol{\vartheta},\boldsymbol{w};\boldsymbol{v}) > 0. \tag{3.8}$$

Proof: Let $\boldsymbol{v}$ be any element of $V(\Omega)$, $\boldsymbol{v} \neq \boldsymbol{0}$. We consider successively two cases.
1) When $\boldsymbol{v}$ is not identically zero on Ω_P, we consider the problem

$$\begin{cases} -\Delta\chi = 0 & \text{in } \Omega_F, \\ \partial_n\chi = \boldsymbol{v}\cdot\boldsymbol{n} & \text{on } \Gamma. \end{cases}$$

It follows from the compatibility condition

$$\int_\Gamma (\boldsymbol{v}\cdot\boldsymbol{n})(\boldsymbol{\tau})\,d\boldsymbol{\tau} = \int_{\partial\Omega_P} (\boldsymbol{v}\cdot\boldsymbol{n})(\boldsymbol{\tau})\,d\boldsymbol{\tau} = \int_{\Omega_P} (\operatorname{div}\boldsymbol{v})(\boldsymbol{x})\,d\boldsymbol{x} = 0,$$

that this problem admits a unique solution in $H^1(\Omega_F)$, up to an additive constant. The pair $(\boldsymbol{\vartheta},\boldsymbol{w})$, with $\boldsymbol{\vartheta} = \boldsymbol{0}$ and $\boldsymbol{w}$ equal to $\boldsymbol{v}$ on Ω_P and to $\mathbf{grad}\,\chi$ on Ω_F, thus belongs to $\mathcal{W}(\Omega)$ and

$$a(\boldsymbol{\vartheta},\boldsymbol{w};\boldsymbol{v}) = \mu\,\|\boldsymbol{v}\|^2_{L^2(\Omega_P)^d} > 0.$$

2) When $\boldsymbol{v}$ is zero on Ω_P, it is not identically zero on Ω_F. Moreover, the continuity of the normal trace of $\boldsymbol{v}$ through Γ implies that $\boldsymbol{v}_{|\Omega_F}$ belongs to the space $V(\Omega_F)$ defined in (2.8) (see (2.9)). So, the desired result is a direct consequence of property (2.12).

Lemma 3.3. *There exists a constant $\alpha > 0$ such that the following inf-sup condition holds*

$$\forall(\boldsymbol{\vartheta},\boldsymbol{w}) \in \mathcal{W}(\Omega), \qquad \sup_{\boldsymbol{v}\in V(\Omega)} \frac{a(\boldsymbol{\vartheta},\boldsymbol{w};\boldsymbol{v})}{\|\boldsymbol{v}\|_{L^2(\Omega)^d}} \geq \alpha\,\big(\|\boldsymbol{\vartheta}\|_{\mathbb{C}(\Omega_F)} + \|\boldsymbol{w}\|_{L^2(\Omega)^d}\big). \tag{3.9}$$

Proof: We observe that, for any $(\boldsymbol{\vartheta},\boldsymbol{w})$ in $\mathcal{W}(\Omega)$, the function $\overline{\boldsymbol{v}}$ defined by

$$\overline{\boldsymbol{v}} = \begin{cases} \boldsymbol{0} & \text{on } \Omega_P, \\ \mathbf{Curl}\,\boldsymbol{\vartheta} & \text{on } \Omega_F, \end{cases}$$

belongs to $V(\Omega)$. So taking $\boldsymbol{v}$ equal to $\boldsymbol{w} + \overline{\boldsymbol{v}}$ gives

$$a(\boldsymbol{\vartheta},\boldsymbol{w};\boldsymbol{v}) = \mu\,\|\boldsymbol{w}\|^2_{L^2(\Omega_P)^d} + \nu \int_{\Omega_F} (\mathbf{Curl}\,\boldsymbol{\vartheta})(\boldsymbol{x})\cdot\boldsymbol{w}(\boldsymbol{x})\,d\boldsymbol{x} + \nu\,\|\mathbf{Curl}\,\boldsymbol{\vartheta}\|^2_{L^2(\Omega_F)^d}.$$

Using the definition of $\mathcal{W}(\Omega)$ to evaluate the second term in the right-hand side leads to

$$a(\boldsymbol{\vartheta},\boldsymbol{w};\boldsymbol{v}) = \mu\,\|\boldsymbol{w}\|^2_{L^2(\Omega_P)^d} + \nu\,\|\boldsymbol{\vartheta}\|^2_{\mathbb{C}(\Omega_F)}. \tag{3.10}$$

On the other hand, as a simple extension of [2, Cor. 3.16], we have

$$\|\boldsymbol{w}\|_{L^2(\Omega_F)^d} \leq c\left(\|\boldsymbol{\vartheta}\|_{L^2(\Omega_F)^d} + \|\boldsymbol{w}_{|\Omega_F} \cdot \boldsymbol{n}\|_{H^{-\frac{1}{2}}(\Gamma)}\right).$$

It follows from the continuity of the normal trace of $\boldsymbol{w}$ through Γ together with the continuity of this normal trace on the domain of the divergence operator (see [17, Chap. I, Thm 2.5] or [4, Thm 1.2.26] for instance) that

$$\|\boldsymbol{w}_{|\Omega_F} \cdot \boldsymbol{n}\|_{H^{-\frac{1}{2}}(\Gamma)} \leq c\,\|\boldsymbol{w}\|_{L^2(\Omega_P)^d}.$$

So combining the last two lines with (3.10) yields

$$a(\boldsymbol{\vartheta}, \boldsymbol{w}; \boldsymbol{v}) \geq c\left(\|\boldsymbol{\vartheta}\|^2_{\mathbb{C}(\Omega_F)} + \|\boldsymbol{w}\|^2_{L^2(\Omega)^d}\right).$$

On the other hand, we obviously have

$$\|\boldsymbol{v}\|_{L^2(\Omega)^d} \leq \|\boldsymbol{\vartheta}\|_{\mathbb{C}(\Omega_F)} + \|\boldsymbol{w}\|_{L^2(\Omega)^d}.$$

This gives the desired result.

We skip the proof of the next lemma, since it is exactly the same as for Lemma 2.4, with Ω_F replaced by Ω.

Lemma 3.4. *There exists a constant $\beta > 0$ such that the following inf-sup condition holds*

$$\forall q \in H^1(\Omega) \cap L^2_0(\Omega), \qquad \sup_{v \in L^2(\Omega)^d} \frac{b(\boldsymbol{v}, q)}{\|\boldsymbol{v}\|_{L^2(\Omega)^d}} \geq \beta\,\|q\|_{H^1(\Omega)}. \tag{3.11}$$

We are now in a position to prove the main result of this section.

Theorem 3.5. *For any data $\boldsymbol{f}$ in $L^2(\Omega)^d$ and k in $H^{-\frac{1}{2}}(\partial\Omega)$ satisfying (3.2), problem (3.3) has a unique solution $(\boldsymbol{\omega}, \boldsymbol{u}, p)$ in $\mathbb{C}(\Omega_F) \times L^2(\Omega)^d \times \left(H^1(\Omega) \cap L^2_0(\Omega)\right)$. Moreover, this solution satisfies*

$$\|\boldsymbol{\omega}\|_{\mathbb{C}(\Omega_F)} + \|\boldsymbol{u}\|_{L^2(\Omega)^d} + \|p\|_{H^1(\Omega)} \leq c\left(\|\boldsymbol{f}\|_{L^2(\Omega)^d} + \|k\|_{H^{-\frac{1}{2}}(\partial\Omega)}\right). \tag{3.12}$$

Proof: We establish first the uniqueness, second the existence of the solution.
1) Let us consider problem (3.3) with zero data $\boldsymbol{f}$ and k. The part $(\boldsymbol{\omega}, \boldsymbol{u})$ of any solution of this problem belongs to $\mathcal{W}(\Omega)$ and satisfies

$$\forall \boldsymbol{v} \in V(\Omega), \quad a(\boldsymbol{\omega}, \boldsymbol{u}; \boldsymbol{v}) = 0.$$

Applying Lemma 3.3 yields that it is zero. Then, the pressure p satisfies

$$\forall \boldsymbol{v} \in L^2(\Omega)^d, \quad b(\boldsymbol{v}, p) = 0,$$

and it follows from Lemma 3.4 that it is zero. This yields the uniqueness of the solution of the linear problem (3.3).
2) Note that Lemma 3.4 still holds with Ω replaced by Ω_P and the integral on Ω in the definition of the form $b(\cdot,\cdot)$ replaced by the integral on Ω_P. Due to this modified result, there exists a function $\boldsymbol{u}_b$ in $L^2(\Omega_P)^d$ such that

$$\forall q \in H^1(\Omega_P) \cap L^2_0(\Omega_P), \quad \int_{\Omega_P} \boldsymbol{u}_b(\boldsymbol{x}) \cdot (\mathbf{grad}\, q)(\boldsymbol{x})\, d\boldsymbol{x} = \langle k, q \rangle_{\partial\Omega},$$

and

$$\|\boldsymbol{u}_b\|_{L^2(\Omega_P)^d} \leq \beta^{-1} \|k\|_{H^{-\frac{1}{2}}(\partial\Omega)},$$

see [17, Chap. I, Lemma 4.1] or [4, Thm 1.3.4] for instance. Thanks to (3.2), the previous equation holds for any q in $H^1(\Omega_P)$. We denote by $\overline{\boldsymbol{u}}_b$ the extension of $\boldsymbol{u}_b$ by zero on Ω_F and observe that, since the normal trace of $\boldsymbol{u}_b$ vanishes on Γ, the next two properties hold

$$\forall q \in H^1(\Omega) \cap L^2_0(\Omega), \quad b(\overline{\boldsymbol{u}}_b, q) = \langle k, q \rangle_{\partial\Omega},$$

and

$$\|\overline{\boldsymbol{u}}_b\|_{L^2(\Omega)^d} \leq \beta^{-1} \|k\|_{H^{-\frac{1}{2}}(\partial\Omega)}. \tag{3.13}$$

Next, combining Lemmas 3.2 and 3.3 with [4, Lemma 1.3.13] yields that there exists a unique solution $(\boldsymbol{\omega}, \boldsymbol{u}_0)$ in $\mathcal{W}(\Omega)$ of the problem

$$\forall \boldsymbol{v} \in V(\Omega), \quad a(\boldsymbol{\omega}, \boldsymbol{u}_0; \boldsymbol{v}) = \int_\Omega \boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x})\, d\boldsymbol{x} - a(\mathbf{0}, \overline{\boldsymbol{u}}_b; \boldsymbol{v}),$$

which satisfies

$$\|\boldsymbol{\omega}\|_{\mathbb{C}(\Omega_F)} + \|\boldsymbol{u}_0\|_{L^2(\Omega)^d} \leq \alpha^{-1} \big(\|\boldsymbol{f}\|_{L^2(\Omega)^d} + \mu \|\overline{\boldsymbol{u}}_b\|_{L^2(\Omega)^d} \big). \tag{3.14}$$

Finally, the linear form

$$\boldsymbol{v} \quad \mapsto \quad \int_\Omega \boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x})\, d\boldsymbol{x} - a(\boldsymbol{\omega}, \boldsymbol{u}_0; \boldsymbol{v}) - a(\mathbf{0}, \overline{\boldsymbol{u}}_b; \boldsymbol{v}),$$

belongs to the polar set of $V(\Omega)$, so that applying once more Lemma 3.4 gives the existence of a pressure p in $H^1(\Omega) \cap L^2_0(\Omega)$ such that

$$\forall \boldsymbol{v} \in L^2(\Omega)^d, \quad b(\boldsymbol{v}, p) = \int_\Omega \boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x})\, d\boldsymbol{x} - a(\boldsymbol{\omega}, \boldsymbol{u}_0; \boldsymbol{v}) - a(\mathbf{0}, \overline{\boldsymbol{u}}_b; \boldsymbol{v}),$$

and which satisfies

$$\|p\|_{H^1(\Omega)} \leq \beta^{-1} \big(\|\boldsymbol{f}\|_{L^2(\Omega)^d} + \mu\, \|\overline{\boldsymbol{u}}_b\|_{L^2(\Omega)^d} + \mu\, \|\boldsymbol{u}_0\|_{L^2(\Omega)^d} + \|\boldsymbol{\omega}\|_{\mathbb{C}(\Omega_F)} \big). \tag{3.15}$$

When setting $\boldsymbol{u} = \boldsymbol{u}_b + \boldsymbol{u}_0$, we observe that the triple $(\boldsymbol{\omega}, \boldsymbol{u}, p)$ is a solution of problem (3.3). Moreover estimate (3.12) follows from (3.13) to (3.15).

Remark 3.6. A modified proof of the previous existence result consists in replacing the pair $(\boldsymbol{0}, \overline{\boldsymbol{u}}_b)$ by $(\boldsymbol{\omega}_b, \boldsymbol{u}_b^1 + \boldsymbol{u}_b^2)$ constructed in the following way:
(i) From Lemma 3.4, there exists a function $\boldsymbol{u}_b^1$ in $L^2(\Omega)^d$ such that

$$\forall q \in H^1(\Omega) \cap L^2_0(\Omega), \quad b(\boldsymbol{u}_b^1, q) = \langle k, q \rangle_{\partial\Omega}.$$

(ii) The following inf-sup condition is easily derived by taking $(\boldsymbol{\vartheta}, \boldsymbol{w})$ equal to $(\boldsymbol{\varphi}, -\mathbf{Curl}\,\boldsymbol{\varphi})$

$$\forall \boldsymbol{\varphi} \in \mathbb{C}(\Omega_F), \qquad \sup_{(\boldsymbol{\vartheta}, \boldsymbol{w}) \in \mathbb{C}(\Omega_F) \times V(\Omega_F)} \frac{c(\boldsymbol{\vartheta}, \boldsymbol{w}; \boldsymbol{\varphi})}{\|\boldsymbol{\vartheta}\|_{\mathbb{C}(\Omega_F)} + \|\boldsymbol{w}\|_{L^2(\Omega_F)^d}} > \gamma\, \|\boldsymbol{\varphi}\|_{\mathbb{C}(\Omega_F)}. \tag{3.16}$$

(iii) The previous result yields the existence of a pair $(\boldsymbol{\omega}_b, \boldsymbol{u}_b^2)$ in $\mathbb{C}(\Omega_F) \times V(\Omega_F)$ such that

$$\forall \boldsymbol{\varphi} \in \mathbb{C}(\Omega_F), \quad c(\boldsymbol{\omega}_b, \boldsymbol{u}_b^2; \boldsymbol{\varphi}) = -c(\boldsymbol{0}, \boldsymbol{u}_b^1; \boldsymbol{\varphi}),$$

and the extension of $\boldsymbol{u}_b^2$ by zero to Ω_P, still denoted by $\boldsymbol{u}_b^2$, belongs to $V(\Omega)$.
The interest of this new proof is that, in contrast with the first one, it does not require the compatibility condition (3.2) and can easily be applied in the discrete case.

Remark 3.7. When $\mathbf{curl}\,\boldsymbol{f}$ is zero (this includes the physical case where $\boldsymbol{f}$ is zero), taking $\boldsymbol{v}$ in (3.3) equal to $\mathbf{Curl}\,\boldsymbol{\omega}$ on Ω_F and to $\boldsymbol{0}$ elsewhere yields that the vorticity $\boldsymbol{\omega}$ is zero. So, in this case, a simpler formulation of the problem can be used. However we have rather work with problem (3.3) for the sake of generality.

4. The discrete problem and its well-posedness.

We introduce a regular family $(\mathcal{T}_h)_h$ of triangulations of Ω, in the usual sense that:
- For each h, $\overline{\Omega}$ is the union of all elements of $\mathcal{T}_h$;
- The intersection of two different elements of $\mathcal{T}_h$, if not empty, is a vertex or a whole edge or a whole face of both of them;
- The ratio of the diameter h_K of any element K of $\mathcal{T}_h$ to the diameter of its inscribed circle or sphere is smaller than a constant σ independent of h.

As standard, h stands for the maximum of the diameters h_K, $K \in \mathcal{T}_h$. We make the further and non restrictive assumption that Γ is contained in the boundaries of elements K of $\mathcal{T}_h$, and we agree to denote
- by $\mathcal{T}_h^P$ and $\mathcal{T}_h^F$ the set of elements K of $\mathcal{T}_h$ that are contained in $\overline{\Omega}_P$ and $\overline{\Omega}_F$, respectively,
- by h_P the maximum of the diameters h_K, $K \in \mathcal{T}_h^P$, and by h_F the maximum of the diameters h_K, $K \in \mathcal{T}_h^F$.

In what follows, c, c', ..., stand for generic constants which may vary from a line to the next one but are always independent of h, h_P and h_F.

The discrete spaces are defined in a very similar way as in Section 2, more precisely
- the space $\mathbb{C}_h(\Omega_F)$ of discrete vorticities is still defined by (2.16) or (2.17), according to the dimension d;
- the space $\mathbb{X}_h(\Omega)$ of discrete velocities is defined by

$$\mathbb{X}_h(\Omega) = \left\{ \boldsymbol{v}_h \in L^2(\Omega)^d;\ \forall K \in \mathcal{T}_h,\ \boldsymbol{v}_{h|K} \in \mathcal{P}_0(K)^d \right\}; \tag{4.1}$$

- the space $\mathbb{M}_h(\Omega)$ of discrete pressures is the space of functions q_h in $L^2_0(\Omega)$ such that their restrictions to all elements K of $\mathcal{T}_h$ belong to $\mathcal{P}_1(K)$ and which are continuous at the midpoint of each edge ($d = 2$) or barycenter of each face ($d = 3$) of these elements.

For simplicity, we now assume that $(\boldsymbol{f}, k)$ belongs to $L^2(\Omega)^d \times H^{-\sigma}(\Gamma)$, $0 \leq \sigma < \frac{1}{2}$. For reasons that are explained later on, denoting by $\mathcal{E}_h^b$ the set of edges ($d = 2$) or faces ($d = 3$) of elements of $\mathcal{T}_h$ which are contained in $\partial\Omega$, we introduce the piecewise constant approximation k_h of k defined by

$$\forall e \in \mathcal{E}_h^b, \quad k_{h|e} = \frac{1}{\text{meas}(e)} \int_e k(\boldsymbol{\tau})\, d\boldsymbol{\tau}, \tag{4.2}$$

where the previous integral is replaced by a duality pairing when k is not integrable on $\partial\Omega$. Note that, if k satisfies (3.2), the same property holds for k_h.

The discrete problem is then constructed from (3.3) by the Galerkin method. It reads

Find $(\boldsymbol{\omega}_h, \boldsymbol{u}_h, p_h)$ *in* $\mathbb{C}_h(\Omega_F) \times \mathbb{X}_h(\Omega) \times \mathbb{M}_h(\Omega)$ *such that*

$$\begin{aligned}
&\forall \boldsymbol{v}_h \in \mathbb{X}_h(\Omega), \quad a(\boldsymbol{\omega}_h, \boldsymbol{u}_h; \boldsymbol{v}_h) + b_h(\boldsymbol{v}_h, p_h) = \int_\Omega \boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{v}_h(\boldsymbol{x})\, d\boldsymbol{x}, \\
&\forall q_h \in \mathbb{M}_h(\Omega), \quad b_h(\boldsymbol{u}_h, q_h) = \int_{\partial\Omega} k_h(\boldsymbol{\tau}) q_h(\boldsymbol{\tau})\, d\boldsymbol{\tau}, \\
&\forall \boldsymbol{\varphi}_h \in \mathbb{C}_h(\Omega), \quad c(\boldsymbol{\omega}_h, \boldsymbol{u}_h; \boldsymbol{\varphi}_h) = 0,
\end{aligned} \tag{4.3}$$

where the bilinear forms $a(\cdot,\cdot;\cdot)$ and $c(\cdot,\cdot;\cdot)$ are introduced in (3.4) while the modified form $b_h(\cdot,\cdot)$ is defined by

$$b_h(\boldsymbol{v},q) = \sum_{K\in\mathcal{T}_h} \int_K \boldsymbol{v}(\boldsymbol{x}) \cdot (\mathbf{grad}\, q)(\boldsymbol{x})\, d\boldsymbol{x}. \tag{4.4}$$

Note that it coincides with $b(\cdot,\cdot)$ on $L^2(\Omega)^d \times H^1(\Omega)$.

In order to perform the analysis of problem (4.3) and in analogy with the continuous case, we introduce the discrete kernels

$$V_h(\Omega) = \Big\{\boldsymbol{v}_h \in \mathbb{X}_h(\Omega);\ \forall q_h \in \mathbb{M}_h(\Omega),\ b_h(\boldsymbol{v}_h,q_h)=0\Big\}, \tag{4.5}$$

and

$$\mathcal{W}_h(\Omega) = \Big\{(\boldsymbol{\vartheta}_h,\boldsymbol{w}_h) \in \mathbb{C}_h(\Omega_F)\times V_h(\Omega);\ \forall \boldsymbol{\varphi}_h \in \mathbb{C}_h(\Omega_F),\ c(\boldsymbol{\vartheta}_h,\boldsymbol{w}_h;\boldsymbol{\varphi}_h)=0\Big\}. \tag{4.6}$$

We first recall some standard properties of these kernels. Part of the next results are proved in [7, §4] or in [10, §5.2.4] in a different framework.

Lemma 4.1. *(i) The kernel $V_h(\Omega)$ coincides with $V(\Omega)\cap\mathbb{X}_h(\Omega)$.*
(ii) The operator

$$P_h:\quad \boldsymbol{\varphi}_h \ \mapsto\ P_h\boldsymbol{\varphi}_h = \begin{cases} 0 & \text{in } \Omega_P,\\ \mathbf{Curl}\,\boldsymbol{\varphi}_h & \text{in } \Omega_F,\end{cases} \tag{4.7}$$

maps $\mathbb{C}_h(\Omega_F)$ into $V_h(\Omega)$ and satisfies

$$\forall \boldsymbol{\varphi}_h \in \mathbb{C}_h(\Omega_F),\quad \|P_h\boldsymbol{\varphi}_h\|_{L^2(\Omega)^d} \le \|\boldsymbol{\varphi}_h\|_{\mathbb{C}(\Omega_F)}. \tag{4.8}$$

(iii) The operator P_h is onto the space $V_h(\Omega_F)$ defined as follows

$$V_h(\Omega_F) = \Big\{\boldsymbol{v}_h \in V_h(\Omega);\ \boldsymbol{v}_h = \mathbf{0} \text{ on } \Omega_P\Big\}. \tag{4.9}$$

Moreover, it admits a right inverse Q_h which satisfies

$$\forall \boldsymbol{v}_h \in V_h(\Omega_F),\quad \|Q_h\boldsymbol{v}_h\|_{\mathbb{C}(\Omega_F)} \le c\,\|\boldsymbol{v}_h\|_{L^2(\Omega)^d}. \tag{4.10}$$

Proof: The three assertions are proved successively.
1) Any function $\boldsymbol{v}_h$ in $V_h(\Omega)$ is constant on each element K of $\mathcal{T}_h$, hence has a zero-divergence on this K. Moreover, the equation in the definition of $V_h(\Omega)$ is obviously satisfied when q_h is a constant, hence for all functions q_h in the sum $\overline{\mathbb{M}}_h(\Omega)$ of $\mathbb{M}_h(\Omega)$ and of the constants. This gives

$$\forall q_h \in \overline{\mathbb{M}}_h(\Omega),\quad \sum_{K\in\mathcal{T}_h} \int_{\partial K} (\boldsymbol{v}_h \cdot \boldsymbol{n})(\boldsymbol{\tau}) q_h(\boldsymbol{\tau})\, d\boldsymbol{\tau} = 0.$$

On the other hand, if $\mathcal{E}_h$ denotes the set of all edges ($d=2$) or faces ($d=3$) of elements K of $\mathcal{T}_h$, standard properties of the Crouzeix–Raviart element, see [14], yield that there exist functions q_e in $\overline{\mathbb{M}}_h(\Omega)$, $e \in \mathcal{E}_h$, such that

$$\forall e' \in \mathcal{E}_h, \quad \int_{e'} q_e(\boldsymbol{\tau})\,d\boldsymbol{\tau} = \begin{cases} 1 & \text{if } e' = e, \\ 0 & \text{otherwise,} \end{cases}$$

and that the q_e, $e \in \mathcal{E}_h$, form a basis of $\overline{\mathbb{M}}_h(\Omega)$. Letting q_h run through this basis in the previous equation implies that, on each edge or face e shared by two elements of $\mathcal{T}_h$, the jump of $\boldsymbol{v}_h \cdot \boldsymbol{n}$ through e is zero and, on each edge or face e contained in $\partial\Omega$, $\boldsymbol{v}_h \cdot \boldsymbol{n}$ is zero. So $\boldsymbol{v}_h$ belongs to the space $H_0(\mathrm{div},\Omega)$ defined as in Remark 2.2, hence to $V(\Omega)$.
2) The properties of the operator P_h are derived from the imbedding of $\mathbb{C}_h(\Omega_F)$ into $\mathbb{C}(\Omega_F)$ and the fact that the **Curl** of any function of $\mathbb{C}_h(\Omega_F)$ is constant on each element K of $\mathcal{T}_h^F$.
3) The same arguments as in [18] yield that the operator P_h is onto $V_h(\Omega_F)$. So any function $\boldsymbol{v}_h$ in $V_h(\Omega_F)$ is equal to $\mathbf{Curl}\,\boldsymbol{\psi}_h^0$ for a $\boldsymbol{\psi}_h^0$ in $\mathbb{C}_h(\Omega_F)$. In dimension $d=2$, we simply take $Q_h\boldsymbol{v}_h$ equal to $\boldsymbol{\psi}_h^0$, and estimate (4.10) is derived from a Poincaré-Friedrichs inequality. In dimension $d=3$, we introduce the space

$$\mathbb{M}_h^0(\Omega_F) = \big\{\mu_h \in H_0^1(\Omega_F);\ \forall K \in \mathcal{T}_h,\ \mu_{h|K} \in \mathcal{P}_1(K)\big\},$$

and consider the problem

Find $(\boldsymbol{\psi}_h, \lambda_h)$ in $\mathbb{C}_h(\Omega_F) \times \mathbb{M}_h^0(\Omega_F)$ such that

$$\begin{aligned}\forall \boldsymbol{\varphi}_h \in \mathbb{C}_h(\Omega_F), \quad & \int_{\Omega_F} (\mathbf{Curl}\,\boldsymbol{\psi}_h)(\boldsymbol{x}) \cdot (\mathbf{Curl}\,\boldsymbol{\varphi}_h)(\boldsymbol{x})\,d\boldsymbol{x} \\ & + \int_{\Omega_F} \boldsymbol{\varphi}_h(\boldsymbol{x}) \cdot (\mathbf{grad}\,\lambda_h)(\boldsymbol{x})\,d\boldsymbol{x} = \int_{\Omega_F} \boldsymbol{v}_h(\boldsymbol{x}) \cdot (\mathbf{Curl}\,\boldsymbol{\varphi}_h)(\boldsymbol{x})\,d\boldsymbol{x}, \\ \forall \mu_h \in \mathbb{M}_h(\Omega), \quad & \int_{\Omega_F} \boldsymbol{\psi}_h(\boldsymbol{x}) \cdot (\mathbf{grad}\,\mu_h)(\boldsymbol{x})\,d\boldsymbol{x} = 0.\end{aligned}$$

Since Ω_F has a connected boundary and is simply-connected, it follows from [2, Prop. 4.11 & 4.12] that this problem has a unique solution $(\boldsymbol{\psi}_h, \lambda_h)$, with $\lambda_h = 0$, and that the following estimate holds

$$\|\boldsymbol{\psi}_h\|_{\mathbb{C}(\Omega_F)} \le c\,\|\boldsymbol{v}_h\|_{L^2(\Omega_F)^d}.$$

Replacing $\boldsymbol{v}_h$ by $\mathbf{Curl}\,\boldsymbol{\psi}_h^0$ and setting $\lambda_h = 0$ and $\boldsymbol{\varphi}_h = \boldsymbol{\psi}_h - \boldsymbol{\psi}_h^0$ in the first line of this problem yields that $\mathbf{Curl}\,\boldsymbol{\psi}_h$ is equal to $\mathbf{Curl}\,\boldsymbol{\psi}_h^0$, hence to $\boldsymbol{v}_h$. So, taking $Q_h\boldsymbol{v}_h$ equal to $\boldsymbol{\psi}_h$ leads to the desired result.

Remark 4.2. Part (i) of Lemma 4.1 states a very important property of the discretization that we consider: In the case where the boundary datum k is zero, so is k_h and the discrete velocity $\boldsymbol{u}_h$ in the solution $(\boldsymbol{\omega}_h, \boldsymbol{u}_h, p_h)$ of problem (4.2) is exactly divergence-free on Ω.

Moreover the analogous property holds in the general case: The part $\boldsymbol{u}_h$ of any solution $(\boldsymbol{\omega}_h, \boldsymbol{u}_h, p_h)$ of problem (4.3) satisfies

$$\operatorname{div} \boldsymbol{u}_h = 0 \quad \text{in } \Omega \qquad \text{and} \qquad \boldsymbol{u}_h \cdot \boldsymbol{n} = k_h \quad \text{on } \partial\Omega. \tag{4.11}$$

But even the divergence-free property would not hold if k were not replaced by k_h in this problem.

We are now in a position to prove the inf-sup condition on the form $a(\cdot, \cdot; \cdot)$ between the discrete kernels.

Lemma 4.3. *There exists a constant $\alpha_* > 0$ independent of h such that the following inf-sup condition holds*

$$\forall (\boldsymbol{\vartheta}_h, \boldsymbol{w}_h) \in \mathcal{W}_h(\Omega), \qquad \sup_{\boldsymbol{v}_h \in V_h(\Omega)} \frac{a(\boldsymbol{\vartheta}_h, \boldsymbol{w}_h; \boldsymbol{v}_h)}{\|\boldsymbol{v}_h\|_{L^2(\Omega)^d}} \geq \alpha_* \big(\|\boldsymbol{\vartheta}_h\|_{\mathbb{C}(\Omega_F)} + \|\boldsymbol{w}_h\|_{L^2(\Omega)^d} \big). \tag{4.12}$$

Proof: It follows from Part (ii) of Lemma 4.1 that, for any $(\boldsymbol{\vartheta}_h, \boldsymbol{w}_h)$ in $\mathcal{W}_h(\Omega)$, the function $\boldsymbol{v}_h = \boldsymbol{w}_h + P_h \boldsymbol{\vartheta}_h$ belongs to $V_h(\Omega)$. Then, we have

$$a(\boldsymbol{\vartheta}_h, \boldsymbol{w}_h; \boldsymbol{v}_h) = \mu \|\boldsymbol{w}_h\|^2_{L^2(\Omega_P)^d} + \nu \int_{\Omega_F} (\mathbf{Curl}\, \boldsymbol{\vartheta}_h)(\boldsymbol{x}) \cdot \boldsymbol{w}_h(\boldsymbol{x})\, d\boldsymbol{x} + \nu \|\mathbf{Curl}\, \boldsymbol{\vartheta}_h\|^2_{L^2(\Omega_F)^d},$$

whence, thanks to the definition (4.6) of $\mathcal{W}_h(\Omega)$,

$$a(\boldsymbol{\vartheta}_h, \boldsymbol{w}_h; \boldsymbol{v}_h) = \mu \|\boldsymbol{w}_h\|^2_{L^2(\Omega_P)^d} + \nu \|\boldsymbol{\vartheta}_h\|^2_{\mathbb{C}_h(\Omega_F)}. \tag{4.13}$$

On the other hand, using once more the definition (4.6) of $\mathcal{W}_h(\Omega)$, now combined with Part (iii) of Lemma 4.1, yields

$$\begin{aligned} \|\boldsymbol{w}_h\|^2_{L^2(\Omega_F)^d} &= \int_{\Omega_F} (\mathbf{Curl}\, Q_h \boldsymbol{w}_h)(\boldsymbol{x}) \cdot \boldsymbol{w}_h(\boldsymbol{x})\, d\boldsymbol{x} \\ &= \int_{\Omega_F} (Q_h \boldsymbol{w}_h)(\boldsymbol{x}) \cdot \boldsymbol{\vartheta}_h(\boldsymbol{x})\, d\boldsymbol{x} \leq \|Q_h \boldsymbol{w}_h\|_{L^2(\Omega_F)^{\frac{d(d-1)}{2}}} \|\boldsymbol{\vartheta}_h\|_{L^2(\Omega_F)^{\frac{d(d-1)}{2}}}. \end{aligned}$$

Hence, owing to (4.10), we derive

$$\|\boldsymbol{w}_h\|_{L^2(\Omega_F)^d} \leq c\, \|\boldsymbol{\vartheta}_h\|_{L^2(\Omega_F)^{\frac{d(d-1)}{2}}}. \tag{4.14}$$

Finally, it is readily checked from (4.8) that

$$\|\boldsymbol{v}_h\|_{L^2(\Omega)^d} \leq \|\boldsymbol{w}_h\|_{L^2(\Omega)^d} + \|\boldsymbol{\vartheta}_h\|_{\mathbb{C}(\Omega_F)}. \tag{4.15}$$

The desired inf-sup condition is easily derived from (4.13) to (4.15).

The next discrete inf-sup condition on the form $c(\cdot,\cdot;\cdot)$ is also derived from Lemma 4.1 by taking $(\boldsymbol{\vartheta}_h, \boldsymbol{w}_h)$ equal to $(\boldsymbol{\varphi}_h, -P_h\boldsymbol{\varphi}_h)$.

Lemma 4.4. *There exists a constant $\gamma_* > 0$ independent of h such that the following inf-sup condition holds*

$$\forall \boldsymbol{\varphi}_h \in \mathbb{C}_h(\Omega_F), \qquad \sup_{(\boldsymbol{\vartheta}_h, \boldsymbol{w}_h)\in \mathbb{C}_h(\Omega_F)\times V_h(\Omega)} \frac{c(\boldsymbol{\vartheta}_h, \boldsymbol{w}_h; \boldsymbol{\varphi}_h)}{\|\boldsymbol{\vartheta}_h\|_{\mathbb{C}(\Omega_F)} + \|\boldsymbol{w}_h\|_{L^2(\Omega)^d}} \geq \gamma_* \, \|\boldsymbol{\varphi}_h\|_{\mathbb{C}(\Omega_F)}. \quad (4.16)$$

To investigate the properties of the form $b_h(\cdot,\cdot)$, we need further results on the space $\mathbb{M}_h(\Omega)$. They involve the broken semi-norm and norm

$$|q_h|_{H^1_h(\Omega)} = \Big(\sum_{K\in\mathcal{T}_h} |q_h|^2_{H^1(K)}\Big)^{\frac{1}{2}}, \qquad \|q_h\|_{H^1_h(\Omega)} = \Big(\sum_{K\in\mathcal{T}_h} \|q_h\|^2_{H^1(K)}\Big)^{\frac{1}{2}}. \quad (4.17)$$

We recall the next results from [1, Lemma 4.2] and [3, Th. 1], respectively.

Lemma 4.5. *(i) The semi-norm $|\cdot|_{H^1_h(\Omega)}$ is a norm on $\mathbb{M}_h(\Omega)$, and there exists a constant $c > 0$ independent of h such that the following property holds*

$$\forall q_h \in \mathbb{M}_h(\Omega), \quad \|q_h\|_{H^1_h(\Omega)} \leq c\,|q_h|_{H^1_h(\Omega)}. \quad (4.18)$$

(ii) For any real number σ, $0 \leq \sigma < \frac{1}{2}$, there exists a constant $c' > 0$ independent of h such that the following trace result holds

$$\forall q_h \in \mathbb{M}_h(\Omega), \quad \|q_h\|_{H^\sigma(\partial\Omega)} \leq c'\,|q_h|_{H^1_h(\Omega)}. \quad (4.19)$$

Lemma 4.6. *The form $b_h(\cdot,\cdot)$ satisfies the following continuity property*

$$\forall \boldsymbol{v}_h \in \mathbb{X}_h(\Omega), \forall q_h \in \mathbb{M}_h(\Omega), \quad |b_h(\boldsymbol{v}_h, q_h)| \leq \|\boldsymbol{v}_h\|_{L^2(\Omega)^d} |q_h|_{H^1_h(\Omega)}. \quad (4.20)$$

Moreover, there exists a constant $\beta_* > 0$ such that the following inf-sup condition holds

$$\forall q_h \in \mathbb{M}_h(\Omega), \qquad \sup_{\boldsymbol{v}_h\in\mathbb{X}_h(\Omega)} \frac{b_h(\boldsymbol{v}_h, q_h)}{\|\boldsymbol{v}_h\|_{L^2(\Omega)^d}} > \beta_* \, \|q\|_{H^1_h(\Omega)}. \quad (4.21)$$

Proof: The first estimate follows from local Cauchy-Schwarz inequalities. To prove the second one, we take $\boldsymbol{v}_h$ such that, for each K in $\mathcal{T}_h$, $\boldsymbol{v}_{h|K} = \mathbf{grad}\, q_{h|K}$, observe that it belongs to $\mathbb{X}_h(\Omega)$ and use (4.18).

The well-posedness of problem (4.3) can now be derived from the previous properties.

Theorem 4.7. *For any data* $\boldsymbol{f}$ *in* $L^2(\Omega)^d$ *and* k *in* $H^{-\sigma}(\partial\Omega)$, $0 \le \sigma < \frac{1}{2}$, *satisfying* (3.2), *problem* (4.3) *has a unique solution* $(\boldsymbol{\omega}_h, \boldsymbol{u}_h, p_h)$ *in* $\mathbb{C}_h(\Omega_F) \times \mathbb{X}_h(\Omega) \times \mathbb{M}_h(\Omega)$. *Moreover, this solution satisfies*

$$\|\boldsymbol{\omega}_h\|_{\mathbb{C}(\Omega_F)} + \|\boldsymbol{u}_h\|_{L^2(\Omega)^d} + \|p_h\|_{H^1_h(\Omega)} \le c\,\big(\|\boldsymbol{f}\|_{L^2(\Omega)^d} + \|k_h\|_{H^{-\sigma}(\partial\Omega)}\big). \tag{4.22}$$

Proof: We prove successively the existence and uniqueness, next estimate (4.22).
1) Since problem (4.3) is equivalent to a square linear system, the existence and uniqueness of its solution are a consequence of the fact that its only solution for data $\boldsymbol{f}$ and k_h equal to zero is also zero. When taking $\boldsymbol{f}$ and k_h equal to zero, we observe that $(\boldsymbol{\omega}_h, \boldsymbol{u}_h)$ belongs to $\mathcal{W}_h(\Omega)$ and satisfies

$$\forall \boldsymbol{v}_h \in V_h(\Omega), \quad a(\boldsymbol{\omega}_h, \boldsymbol{u}_h; \boldsymbol{v}_h) = 0.$$

So, it follows from the inf-sup condition (4.12) that $(\boldsymbol{\omega}_h, \boldsymbol{u}_h)$ is zero. The pressure p_h then satisfies

$$\forall \boldsymbol{v}_h \in \mathbb{X}_h(\Omega), \quad b_h(\boldsymbol{v}_h, q_h) = 0.$$

Then the inf-sup condition (4.21) yields that it is zero.
2) It follows from the inf-sup condition (4.21) that there exists a function $\boldsymbol{u}^1_{hb}$ in $\mathbb{X}_h(\Omega)$ such that

$$\forall q_h \in \mathbb{M}_h(\Omega), \quad b(\boldsymbol{u}^1_{hb}, q_h) = \int_{\partial\Omega} k_h(\boldsymbol{\tau}) q_h(\boldsymbol{\tau})\, d\boldsymbol{\tau},$$

and

$$\|\boldsymbol{u}^1_{hb}\|_{L^2(\Omega)^d} \le \beta_*^{-1} \sup_{q_h \in \mathbb{M}_h(\Omega)} \frac{\int_{\partial\Omega} k_h(\boldsymbol{\tau}) q_h(\boldsymbol{\tau})\, d\boldsymbol{\tau}}{\|q_h\|_{H^1_h(\Omega)}}.$$

Thanks to the trace result (4.19), this implies

$$\|\boldsymbol{u}^1_{hb}\|_{L^2(\Omega)^d} \le c\, \|k_h\|_{H^{-\sigma}(\partial\Omega)}. \tag{4.23}$$

Then, it follows from the inf-sup condition (4.16) that there exists a pair $(\boldsymbol{\omega}_{hb}, \boldsymbol{u}^2_{hb})$ in $\mathbb{C}_h(\Omega_F) \times V_h(\Omega)$ such that

$$\forall \boldsymbol{\varphi}_h \in \mathbb{C}_h(\Omega_F), \quad c(\boldsymbol{\omega}_{hb}, \boldsymbol{u}^2_{hb}; \boldsymbol{\varphi}_h) = -c(\boldsymbol{0}, \boldsymbol{u}^1_{hb}; \boldsymbol{\varphi}_h),$$

and

$$\|\boldsymbol{\omega}_{hb}\|_{\mathbb{C}(\Omega_F)} + \|\boldsymbol{u}^2_{hb}\|_{L^2(\Omega)^d} \le c\, \|\boldsymbol{u}^1_{hb}\|_{L^2(\Omega)^d}. \tag{4.24}$$

When setting $\boldsymbol{\omega}_{h0} = \boldsymbol{\omega}_h - \boldsymbol{\omega}_{hb}$ and $\boldsymbol{u}_{h0} = \boldsymbol{u}_h - \boldsymbol{u}^1_{hb} - \boldsymbol{u}^2_{hb}$, we oberve that the pair $(\boldsymbol{\omega}_{h0}, \boldsymbol{u}_{h0})$ belongs to $\mathcal{W}_h(\Omega)$ and satisfies

$$\forall \boldsymbol{v}_h \in V_h(\Omega), \quad a(\boldsymbol{\omega}_{h0}, \boldsymbol{u}_{h0}; \boldsymbol{v}_h) = \int_\Omega \boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{v}_h(\boldsymbol{x})\, d\boldsymbol{x} - a(\boldsymbol{\omega}_{hb}, \boldsymbol{u}^1_{hb} + \boldsymbol{u}^2_{hb}; \boldsymbol{v}_h).$$

Thus, the inf-sup condition (4.12) yields

$$\begin{aligned}
&\|\boldsymbol{\omega}_{h0}\|_{\mathbb{C}(\Omega_F)} + \|\boldsymbol{u}_{h0}\|_{L^2(\Omega)^d} \\
&\qquad\qquad \le c\,\big(\|\boldsymbol{f}\|_{L^2(\Omega_F)^d} + \|\boldsymbol{\omega}_{hb}\|_{\mathbb{C}(\Omega_F)} + \|\boldsymbol{u}^1_{hb}\|_{L^2(\Omega)^d} + \|\boldsymbol{u}^2_{hb}\|_{L^2(\Omega)^d}\big).
\end{aligned} \tag{4.25}$$

Combining (4.23) to (4.25) leads to the desired estimate for $\boldsymbol{\omega}_h$ and $\boldsymbol{u}_h$. To conclude, we observe that the pressure p_h satisfies

$$\forall \boldsymbol{v}_h \in \mathbb{X}_h(\Omega), \quad b_h(\boldsymbol{v}_h, p_h) = \int_\Omega \boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{v}_h(\boldsymbol{x})\, d\boldsymbol{x} - a(\boldsymbol{\omega}_h, \boldsymbol{u}_h; \boldsymbol{v}_h),$$

and we derive the estimate for p_h from the inf-sup condition (4.21) and the previous estimates.

5. A priori error analysis.

We now intend to prove an a priori error estimate between the solutions of problems (3.3) and (4.3). In analogy with (4.5) and (4.6), we introduce the affine subspaces of $\mathbb{X}_h(\Omega)$ and $\mathbb{C}_h(\Omega_F) \times \mathbb{X}_h(\Omega)$, defined by

$$V_h^k(\Omega) = \Big\{ \boldsymbol{v}_h \in \mathbb{X}_h(\Omega);\ \forall q_h \in \mathbb{M}_h(\Omega),\ b_h(\boldsymbol{v}_h, q_h) = \int_{\partial\Omega} k_h(\boldsymbol{\tau}) q_h(\boldsymbol{\tau})\, d\boldsymbol{\tau} \Big\}, \tag{5.1}$$

and

$$\mathcal{W}_h^k(\Omega) = \Big\{ (\boldsymbol{\vartheta}_h, \boldsymbol{w}_h) \in \mathbb{C}_h(\Omega_F) \times V_h^k(\Omega);\ \forall \boldsymbol{\varphi}_h \in \mathbb{C}_h(\Omega_F),\ c(\boldsymbol{\vartheta}_h, \boldsymbol{w}_h; \boldsymbol{\varphi}_h) = 0 \Big\}. \tag{5.2}$$

Lemma 5.1. *The following error estimate holds between the solution $(\boldsymbol{\omega}, \boldsymbol{u}, p)$ of problem (3.3) and the solution $(\boldsymbol{\omega}_h, \boldsymbol{u}_h, p_h)$ of problem (4.3)*

$$\begin{aligned} &\|\boldsymbol{\omega} - \boldsymbol{\omega}_h\|_{\mathbb{C}(\Omega_F)} + \|\boldsymbol{u} - \boldsymbol{u}_h\|_{L^2(\Omega)^d} \\ &\qquad \le c \inf_{(\boldsymbol{\vartheta}_h, \boldsymbol{w}_h) \in \mathcal{W}_h^k(\Omega)} \big(\|\boldsymbol{\omega} - \boldsymbol{\vartheta}_h\|_{\mathbb{C}(\Omega_F)} + \|\boldsymbol{u} - \boldsymbol{w}_h\|_{L^2(\Omega)^d} \big). \end{aligned} \tag{5.3}$$

Proof: For any $(\boldsymbol{\vartheta}_h, \boldsymbol{w}_h)$ in $\mathcal{W}_h^k(\Omega)$, the pair $(\boldsymbol{\omega}_h - \boldsymbol{\vartheta}_h, \boldsymbol{u}_h - \boldsymbol{w}_h)$ belongs to $\mathcal{W}_h(\Omega)$ and satisfies

$$\forall \boldsymbol{v}_h \in V_h(\Omega), \quad a(\boldsymbol{\omega}_h - \boldsymbol{\vartheta}_h, \boldsymbol{u}_h - \boldsymbol{w}_h; \boldsymbol{v}_h) = \int_\Omega \boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{v}_h\, d\boldsymbol{x} - a(\boldsymbol{\vartheta}_h, \boldsymbol{w}_h; \boldsymbol{v}_h).$$

It follows from Lemma 4.1 that $V_h(\Omega)$ is contained in $V(\Omega)$, whence

$$\forall \boldsymbol{v}_h \in V_h(\Omega), \quad a(\boldsymbol{\omega}_h - \boldsymbol{\vartheta}_h, \boldsymbol{u}_h - \boldsymbol{w}_h; \boldsymbol{v}_h) = a(\boldsymbol{\omega} - \boldsymbol{\vartheta}_h, \boldsymbol{u} - \boldsymbol{w}_h; \boldsymbol{v}_h).$$

By combining the inf-sup condition (4.12) with the continuity of $a(\cdot, \cdot; \cdot)$ on $\mathbb{C}(\Omega_F) \times L^2(\Omega)^d$, we derive

$$\|\boldsymbol{\omega}_h - \boldsymbol{\vartheta}_h\|_{\mathbb{C}(\Omega_F)} + \|\boldsymbol{u}_h - \boldsymbol{w}_h\|_{L^2(\Omega)^d} \le c \big\| \boldsymbol{\omega} - \boldsymbol{\vartheta}_h\|_{\mathbb{C}(\Omega_F)} + \|\boldsymbol{u} - \boldsymbol{w}_h\|_{L^2(\Omega)^d} \big).$$

We conclude thanks to a triangle inequality.

In the next lemma, we evaluate the distance of $(\boldsymbol{\omega}, \boldsymbol{u})$ to $\mathcal{W}_h^k(\Omega)$. This requires the set $\mathcal{E}_h^0$ of all edges ($d = 2$) or faces ($d = 3$) of elements K of $\mathcal{T}_h$ which are not contained in $\partial\Omega$. On each e of $\mathcal{E}_h^0$, $[\cdot]_e$ denotes the jump through e with the appropriate sign.

Lemma 5.2. *Assume that the datum k belongs to $H^{-\sigma}(\partial\Omega)$, $0 \le \sigma < \frac{1}{2}$. The following error estimate holds for the solution $(\boldsymbol{\omega}, \boldsymbol{u}, p)$ of problem (3.3)*

$$\begin{aligned} &\inf_{(\boldsymbol{\vartheta}_h, \boldsymbol{w}_h) \in \mathcal{W}_h^k(\Omega)} \big(\|\boldsymbol{\omega} - \boldsymbol{\vartheta}_h\|_{\mathbb{C}(\Omega_F)} + \|\boldsymbol{u} - \boldsymbol{w}_h\|_{L^2(\Omega)^d} \big) \\ &\qquad \le c \Big(\inf_{\boldsymbol{\zeta}_h \in \mathbb{C}_h(\Omega_F)} \|\boldsymbol{\omega} - \boldsymbol{\zeta}_h\|_{\mathbb{C}(\Omega_F)} + \inf_{\boldsymbol{z}_h \in \mathbb{X}_h(\Omega)} \|\boldsymbol{u} - \boldsymbol{z}_h\|_{L^2(\Omega)^d} \\ &\qquad\qquad + \|k - k_h\|_{H^{-\sigma}(\partial\Omega)} + E_c(\boldsymbol{u}) \Big), \end{aligned} \tag{5.4}$$

where the quantity $E_c(\boldsymbol{u})$ is defined by

$$E_c(\boldsymbol{u}) = \sup_{q_h \in \mathbb{M}_h(\Omega)} \frac{\sum_{e \in \mathcal{E}_h^0} \int_e (\boldsymbol{u} \cdot \boldsymbol{n})(\boldsymbol{\tau})[q_h(\boldsymbol{\tau})]_e \, d\boldsymbol{\tau}}{\|q_h\|_{H_h^1(\Omega)}}. \tag{5.5}$$

Proof: Let $(\boldsymbol{\zeta}_h, \boldsymbol{z}_h)$ be any pair in $\mathbb{C}_h(\Omega_F) \times \mathbb{X}_h(\Omega)$. We proceed in two steps.
1) Owing to the inf-sup condition (4.21), there exists a function $\boldsymbol{z}_h^\sharp$ in $\mathbb{X}_h(\Omega)$ such that

$$\forall q_h \in \mathbb{M}_h(\Omega), \quad b_h(\boldsymbol{z}_h^\sharp, q_h) = \int_{\partial\Omega} k_h(\boldsymbol{\tau}) q_h(\boldsymbol{\tau}) \, d\boldsymbol{\tau} - b_h(\boldsymbol{z}_h, q_h),$$

and

$$\|\boldsymbol{z}_h^\sharp\|_{L^2(\Omega)^d} \le \beta_*^{-1} \sup_{q_h \in \mathbb{M}_h(\Omega)} \frac{\int_{\partial\Omega} k_h(\boldsymbol{\tau}) q_h(\boldsymbol{\tau}) \, d\boldsymbol{\tau} - b_h(\boldsymbol{z}_h, q_h)}{\|q_h\|_{H_h^1(\Omega)}}.$$

On the other hand, integrating by parts on each K yields, for all q_h in $\mathbb{M}_h(\Omega)$,

$$b_h(\boldsymbol{u}, q_h) = \int_{\partial\Omega} k(\boldsymbol{\tau}) q_h(\boldsymbol{\tau}) \, d\boldsymbol{\tau} + \sum_{e \in \mathcal{E}_h^0} \int_e (\boldsymbol{u} \cdot \boldsymbol{n})(\boldsymbol{\tau})[q_h(\boldsymbol{\tau})]_e \, d\boldsymbol{\tau}.$$

Thus, we have

$$\begin{aligned} \int_{\partial\Omega} k_h(\boldsymbol{\tau}) q_h(\boldsymbol{\tau}) \, d\boldsymbol{\tau} - b_h(\boldsymbol{z}_h, q_h) = - \int_{\partial\Omega} (k - k_h)(\boldsymbol{\tau}) q_h(\boldsymbol{\tau}) \, d\boldsymbol{\tau} + b_h(\boldsymbol{u} - \boldsymbol{z}_h, q_h) \\ - \sum_{e \in \mathcal{E}_h^0} \int_e (\boldsymbol{u} \cdot \boldsymbol{n})(\boldsymbol{\tau})[q_h(\boldsymbol{\tau})]_e \, d\boldsymbol{\tau}, \end{aligned}$$

and, by combining the previous estimate with the trace result (4.19) and the continuity of $b_h(\cdot,\cdot)$, see Lemma 4.6, we obtain

$$\|\boldsymbol{z}_h^\sharp\|_{L^2(\Omega)^d} \le c \left(\|k - k_h\|_{H^{-\sigma}(\partial\Omega)} + \|\boldsymbol{u} - \boldsymbol{z}_h\|_{L^2(\Omega)^d} + E_c(\boldsymbol{u}) \right). \tag{5.6}$$

2) It follows from the inf-sup condition (4.16) that there exists a pair $(\boldsymbol{\zeta}_h^\flat, \boldsymbol{z}_h^\flat)$ in $\mathbb{C}_h(\Omega_F) \times V_h(\Omega)$ such that

$$\forall \boldsymbol{\varphi}_h \in \mathbb{M}_h(\Omega), \quad c(\boldsymbol{\zeta}_h^\flat, \boldsymbol{z}_h^\flat; \boldsymbol{\varphi}_h) = -c(\boldsymbol{\zeta}_h, \boldsymbol{z}_h + \boldsymbol{z}_h^\sharp; \boldsymbol{\varphi}_h)$$

and

$$\|\boldsymbol{\zeta}_h^\flat\|_{\mathbb{C}(\Omega_F)} + \|\boldsymbol{z}_h^\flat\|_{L^2(\Omega)^d} \le \gamma_*^{-1} \sup_{\boldsymbol{\varphi}_h \in \mathbb{C}_h(\Omega_F)} \frac{-c(\boldsymbol{\zeta}_h, \boldsymbol{z}_h + \boldsymbol{z}_h^\sharp; \boldsymbol{\varphi}_h)}{\|\boldsymbol{\varphi}_h\|_{\mathbb{C}(\Omega_F)}}.$$

Noting that $-c(\boldsymbol{\omega}, \boldsymbol{u}; \boldsymbol{\varphi}_h)$ vanish for all $\boldsymbol{\varphi}_h$ in $\mathbb{C}_h(\Omega_F)$, we have

$$-c(\boldsymbol{\zeta}_h, \boldsymbol{z}_h + \boldsymbol{z}_h^\sharp; \boldsymbol{\varphi}_h) = c(\boldsymbol{\omega} - \boldsymbol{\zeta}_h, \boldsymbol{u} - \boldsymbol{z}_h - \boldsymbol{z}_h^\sharp; \boldsymbol{\varphi}_h),$$

so that

$$\|\boldsymbol{\zeta}_h^\flat\|_{\mathbb{C}(\Omega_F)} + \|\boldsymbol{z}_h^\flat\|_{L^2(\Omega)^d} \le c\,\big(\|\boldsymbol{\omega} - \boldsymbol{\zeta}_h\|_{\mathbb{C}(\Omega_F)} + \|\boldsymbol{u} - \boldsymbol{z}_h\|_{L^2(\Omega)^d} + \|\boldsymbol{z}_h^\sharp\|_{L^2(\Omega)^d}\big). \qquad (5.7)$$

To conclude, we observe that the pair $(\boldsymbol{\vartheta}_h, \boldsymbol{w}_h)$, with $\boldsymbol{\vartheta}_h = \boldsymbol{\zeta}_h + \boldsymbol{\zeta}_h^\flat$ and $\boldsymbol{w}_h = \boldsymbol{z}_h + \boldsymbol{z}_h^\sharp + \boldsymbol{z}_h^\flat$, belongs to $\mathcal{W}_h^k(\Omega)$, so that estimate (5.4) follows from a triangle inequality combined with (5.6) and (5.7).

We now bound the consistency error $E_c(\boldsymbol{u})$ which is due to the nonconformity of the method.

Lemma 5.3. *The following estimate holds for any function $\boldsymbol{u}$ in $H^s(\Omega)^d$, $\frac{1}{2} < s \le 1$,*

$$|E_c(\boldsymbol{u})| \le c\,h^s\,\|\boldsymbol{u}\|_{H^s(\Omega)^d}. \qquad (5.8)$$

Proof: Each e in $\mathcal{E}_h^0$ is shared by two elements K and K' of $\mathcal{T}_h$. We recall that $q_{h|K}$ and $q_{h|K'}$ have the same mean value on e that we denote by $\overline{q}_e$. Denoting by w the function $\boldsymbol{u}\cdot\boldsymbol{n}$, where $\boldsymbol{n}$ is the fixed unit vector normal to e, and by $\overline{w}_e$ its meanvalue on e, we derive in an obvious way

$$\int_e (\boldsymbol{u}\cdot\boldsymbol{n})(\boldsymbol{\tau})[q_h(\boldsymbol{\tau})]_e\,d\boldsymbol{\tau} = \int_e (w-\overline{w}_e)(\boldsymbol{\tau})(q_{h|K}-\overline{q}_e)(\boldsymbol{\tau})\,d\boldsymbol{\tau} + \int_e (w-\overline{w}_e)(\boldsymbol{\tau})(q_{h|K'}-\overline{q}_e)(\boldsymbol{\tau})\,d\boldsymbol{\tau}.$$

By using the affine transformation which maps a reference element $\hat{K}$ onto K and one of its edges or faces $\hat{e}$ onto e, we have with standard notation

$$\Big|\int_e (w-\overline{w}_e)(\boldsymbol{\tau})(q_{h|K}-\overline{q}_e)(\boldsymbol{\tau})\,d\boldsymbol{\tau}\Big| \le c\,h_K^{d-1}\|\hat{w}-\overline{w}_e\|_{H^s(\hat{K})}\|\hat{q}_h-\overline{q}_e\|_{H^1(\hat{K})}.$$

When using a generalized Bramble–Hilbert inequality for each term, this yields

$$\Big|\int_e (w-\overline{w}_e)(\boldsymbol{\tau})(q_{h|K}-\overline{q}_e)(\boldsymbol{\tau})\,d\boldsymbol{\tau}\Big| \le c\,h_K^{d-1}|\hat{w}|_{H^s(\hat{K})}|\hat{q}_h|_{\hat{H}^1(K)}.$$

Thus, switching back to K gives

$$\Big|\int_e (w-\overline{w}_e)(\boldsymbol{\tau})(q_{h|K}-\overline{q}_e)(\boldsymbol{\tau})\,d\boldsymbol{\tau}\Big| \le c\,h_K^s\,\|w\|_{H^s(K)}\|q\|_{H^1(K)}.$$

By applying the same argument on K', next summing these estimates on the e we obtain the desired result.

The error on the pressure can now be estimated from the previous results.

Lemma 5.4. *The following error estimate holds between the solution* $(\boldsymbol{\omega}, \boldsymbol{u}, p)$ *of problem* (3.3) *and the solution* $(\boldsymbol{\omega}_h, \boldsymbol{u}_h, p_h)$ *of problem* (4.3)

$$\|p - p_h\|_{H^1_h(\Omega)} \leq c\big(\inf_{q_h \in \mathbb{M}_h(\Omega)} \|p - q_h\|_{H^1_h(\Omega)} + \|\boldsymbol{\omega} - \boldsymbol{\omega}_h\|_{\mathbb{C}(\Omega_F)} + \|\boldsymbol{u} - \boldsymbol{u}_h\|_{L^2(\Omega)^d}\big). \tag{5.9}$$

Proof: Let q_h be any function in $\mathbb{M}_h(\Omega)$. Using the fact that the forms $b(\cdot,\cdot)$ and $b_h(\cdot,\cdot)$ coincide on $L^2(\Omega)^d \times H^1(\Omega)$, we derive from problems (3.3) and (4.3) that

$$\forall \boldsymbol{v}_h \in \mathbb{X}_h(\Omega), \quad b_h(\boldsymbol{v}_h, p_h - q_h) = a(\boldsymbol{\omega} - \boldsymbol{\omega}_h, \boldsymbol{u} - \boldsymbol{u}_h; \boldsymbol{v}_h) + b_h(\boldsymbol{v}_h, p - q_h).$$

Thus applying the inf-sup condition (4.21) gives

$$\|p_h - q_h\|_{H^1_h(\Omega)} \leq c\big(\|\boldsymbol{\omega} - \boldsymbol{\omega}_h\|_{\mathbb{C}(\Omega_F)} + \|\boldsymbol{u} - \boldsymbol{u}_h\|_{L^2(\Omega)^d} + \|p - q_h\|_{H^1_h(\Omega)}\big).$$

We conclude thanks to a triangle inequality.

Bounding the different error approximation terms that appear in (4.29) and (4.32) rely on already known results:
(i) In dimension $d = 2$, the estimate for the distance of $\boldsymbol{\omega}$ to $\mathbb{C}_h(\Omega_F)$ can be derived for instance by using a Clément type regularization operator, see [17, Chap. I, Thm A.4]: If $\boldsymbol{\omega}$ belongs to $H^{s+1}(\Omega)$, $0 \leq s \leq 1$,

$$\inf_{\boldsymbol{\zeta}_h \in \mathbb{C}_h(\Omega_F)} \|\boldsymbol{\omega} - \boldsymbol{\zeta}_h\|_{\mathbb{C}(\Omega_F)} \leq c\, h_F^s\, \|\boldsymbol{\omega}\|_{H^{s+1}(\Omega_F)}. \tag{5.10}$$

The analogous property in dimension $d = 3$ can be found in [5, Prop. 1] (see also [11, Lemmas 3.2 & 3.3] or [12, §4]): For any function $\boldsymbol{\omega}$ in $H^s(\Omega_F)^3$ such that $\mathbf{Curl}\,\boldsymbol{\omega}$ belongs to $H^s(\Omega_F)^3$, $\frac{1}{2} < s \leq 1$,

$$\inf_{\boldsymbol{\zeta}_h \in \mathbb{C}_h(\Omega_F)} \|\boldsymbol{\omega} - \boldsymbol{\zeta}_h\|_{\mathbb{C}(\Omega_F)} \leq c\, h_F^s\, \big(\|\boldsymbol{\omega}\|_{H^s(\Omega_F)^3} + \|\mathbf{Curl}\,\boldsymbol{\omega}\|_{H^s(\Omega_F)^3}\big). \tag{5.11}$$

(ii) The distance of $\boldsymbol{u}$ to $\mathbb{X}_h(\Omega)$ is easily evaluated by switching to the reference element and using Poincaré–Friedrichs inequality, see also [17, Chap. I, Lemma A.5]: If $\boldsymbol{u}$ belongs to $H^s(\Omega)^d$, $0 \leq s \leq 1$,

$$\inf_{\boldsymbol{z}_h \in \mathbb{X}_h(\Omega)} \|\boldsymbol{u} - \boldsymbol{z}_h\|_{L^2(\Omega)^d} \leq c\, h^s\, \|\boldsymbol{u}\|_{H^s(\Omega)^d}. \tag{5.12}$$

(iii) Since the space $\mathbb{M}_h(\Omega)$ contains $\mathbb{M}_h(\Omega) \cap H^1(\Omega)$, the next estimate is derived by the same arguments as in part (i) of these remarks but now in dimensions $d = 2$ and $d = 3$: If p belongs to $H^{s+1}(\Omega)$, $0 \leq s \leq 1$,

$$\inf_{q_h \in \mathbb{M}_h(\Omega)} \|p - q_h\|_{H^1_h(\Omega)} \leq c\, h^s\, \|p\|_{H^{s+1}(\Omega)}. \tag{5.13}$$

(iv) Finally, if π_h denotes the orthogonal projection from $L^2(\partial\Omega)$ onto the space of functions which are constant on each e in $\mathcal{E}_h^b$ and k belongs to $L^2(\partial\Omega)$, it is readily checked that the function k_h defined in (4.2) is equal to $\pi_h k$. So a standard duality argument, combined with the $(d-1)$-analogue of (5.12) yields

$$\|k - k_h\|_{H^{-\sigma}(\partial\Omega)} \leq c\, h_b^\sigma\, \|k - k_h\|_{L^2(\partial\Omega)},$$

where h_b denotes the maximal diameter of the edges ($d = 2$) or faces ($d = 3$) of elements of $\mathcal{T}_h^P$ contained in $\partial\Omega$. Thus, if the function k belongs to $H^\tau(\partial\Omega)$, $0 \leq \tau \leq 1$,

$$\|k - k_h\|_{H^{-\sigma}(\partial\Omega)} \leq c\, h_b^{\sigma+\tau}\, \|k\|_{H^\tau(\partial\Omega)}. \tag{5.14}$$

By combining (5.10) to (5.14) with Lemmas 5.1 to 5.4, we are in a position to derive the final error estimate. In order to have a unified notation for dimensions $d = 2$ and $d = 3$, we introduce, for all $s \geq 0$, the spaces

$$\mathbb{C}^s(\Omega_F) = \begin{cases} H_0^1(\Omega_F) \cap H^{s+1}(\Omega_F) & \text{if } d = 2, \\ \left\{\boldsymbol{\varphi} \in H_0(\mathbf{Curl}, \Omega_F) \cap H^s(\Omega)^3;\ \mathbf{Curl}\, \boldsymbol{\varphi} \in H^s(\Omega)^3\right\} & \text{if } d = 3. \end{cases} \tag{5.15}$$

Theorem 5.5. *Assume that*
(i) the data $(\boldsymbol{f}, k)$ belong to $L^2(\Omega)^3 \times H^\tau(\partial\Omega)$, $0 \leq \tau \leq 1$,
(ii) the solution $(\boldsymbol{\omega}, \boldsymbol{u}, p)$ of problem (3.3) belong to $\mathbb{C}^{s_1}(\Omega_F) \times H^{s_2}(\Omega)^d \times H^{s_3+1}(\Omega)$, with

$$\frac{d-2}{2} < s_1 \leq 1, \qquad \frac{1}{2} < s_2 \leq 1, \qquad 0 \leq s_3 \leq 1. \tag{5.16}$$

Then the following a priori error estimates hold between this solution and the solution $(\boldsymbol{\omega}_h, \boldsymbol{u}_h, p_h)$ of problem (4.3), for any $\sigma < \frac{1}{2}$,

$$\begin{aligned} &\|\boldsymbol{\omega} - \boldsymbol{\omega}_h\|_{\mathbb{C}(\Omega_F)} + \|\boldsymbol{u} - \boldsymbol{u}_h\|_{L^2(\Omega)^d} \\ &\qquad \leq c\,\Big(h_F^{s_1}\, \|\boldsymbol{\omega}\|_{\mathbb{C}^{s_1}(\Omega_F)} + h^{s_2}\, \|\boldsymbol{u}\|_{H^{s_2}(\Omega)^d} + h_b^{\sigma+\tau}\, \|k\|_{H^\tau(\partial\Omega)}\Big), \end{aligned} \tag{5.17}$$

and

$$\begin{aligned} &\|p - p_h\|_{H_h^1(\Omega)} \\ &\quad \leq c\,\Big(h_F^{s_1}\, \|\boldsymbol{\omega}\|_{\mathbb{C}^{s_1}(\Omega_F)} + h^{s_2}\, \|\boldsymbol{u}\|_{H^{s_2}(\Omega)^d} + h^{s_3}\, \|p\|_{H^{s_3+1}(\Omega)} + h_b^{\sigma+\tau}\, \|k\|_{H^\tau(\partial\Omega)}\Big). \end{aligned} \tag{5.18}$$

Estimates (5.17) and (5.18) are fully optimal and indicate a convergence order equal to 1 for smooth solutions and data. Moreover the regularity property that is required on $\boldsymbol{\omega}$ in dimension $d = 3$ is not at all restrictive: Indeed it follows from [2, §2] that, since $\boldsymbol{\omega} = \mathbf{Curl}\, \boldsymbol{u}$ belongs to $H_0(\mathbf{Curl}, \Omega_F)$ and is divergence-free, it belongs to $H^s(\Omega_F)^3$ for a real number $s > \frac{1}{2}$ and the same property holds for $\mathbf{Curl}\, \boldsymbol{\omega}$ whenever the restriction of $\boldsymbol{f}$ to Ω_F belongs to $H(\mathbf{Curl}, \Omega_F)$.

6. A posteriori error analysis.

The a posteriori error estimates that we intend to prove require two families of error indicators.
(i) Divergence error indicators
Let $\mathcal{E}_h^0$ denote the set of edges ($d=2$) or faces ($d=3$) of elements of $\mathcal{T}_h$ which are not contained in $\partial\Omega$. With each e in $\mathcal{E}_h^0$, we associate the error indicator

$$\eta_e = h_e^{-\frac{1}{2}} \, \|[p_h]_e\|_{L^2(e)}, \tag{6.1}$$

where h_e denotes the length ($d=2$) or diameter ($d=3$) of e. As in Section 5, $[\cdot]_e$ stands for the jump through e. An efficient algorithm for computing the η_e with low cost is given in [1, Rem. 4.6].
(ii) Curl error indicators
For each element K of $\mathcal{T}_h^F$, we denote by $\mathcal{E}_K$ the set of edges ($d=2$) or faces ($d=3$) of K which are not contained in Γ. We also introduce the modified trace operator $\tilde{\gamma}_t$: for any vector field $\boldsymbol{v}$, $\tilde{\gamma}_t \boldsymbol{v}$ denotes the tangential trace of $\boldsymbol{v}$ in dimension $d=2$ and coincides with $\gamma_t \boldsymbol{v}$ in dimension $d=3$. With each K in $\mathcal{T}_h^F$, we associate the error indicator

$$\begin{aligned}\eta_K = h_K \, \|\boldsymbol{\omega}_h\|_{L^2(K)^{\frac{d(d-1)}{2}}} \\ + \sum_{e\in\mathcal{E}_K} \Big(h_e^{\frac{1}{2}} \, \|[\tilde{\gamma}_t(\boldsymbol{u}_h)]_e\|_{L^2(e)^{\frac{d(d-1)}{2}}} + \delta_{3d}\, h_e^{\frac{1}{2}} \, \|[\boldsymbol{\omega}_h \cdot \boldsymbol{n}]_e\|_{L^2(e)} \Big),\end{aligned} \tag{6.2}$$

where $\delta_{\cdot\cdot}$ stands for the Kronecker's symbol.

We now write the residual equations associated with problems (3.3) and (4.3). Due to the nonconformity of the discretization, we are led to introduce a "conforming" approximation of the discrete pressure, more precisely an approximation in the space $\mathbb{M}_h(\Omega)\cap H^1(\Omega)$. We recall the next result from [1, Thm 4.7] (see also [6, Lemma 24]).

Lemma 6.1. *For each function p_h in $\mathbb{M}_h(\Omega)$, there exists a function p_h^* in $\mathbb{M}_h(\Omega)\cap H^1(\Omega)$ such that*

$$\|p_h - p_h^*\|_{H_h^1(\Omega)} \le c \Big(\sum_{e\in\mathcal{E}_h^0} h_e^{-1} \, \|[p_h]_e\|^2_{L^2(e)} \Big)^{\frac{1}{2}}. \tag{6.3}$$

We now introduce the orthogonal projection of $\boldsymbol{f}$ onto $\mathbb{X}_h(\Omega)$, defined by

$$\forall K\in\mathcal{T}_h, \quad \boldsymbol{f}_{h|K} = \frac{1}{\text{meas}(K)} \int_K \boldsymbol{f}(\boldsymbol{x})\,d\boldsymbol{x}. \tag{6.4}$$

To go further, we note that $\mu\,\boldsymbol{u}_h + \mathbf{grad}\,p_h - \boldsymbol{f}_h$ is constant on each element K of $\mathcal{T}_h^P$ and similarly that $\nu\,\mathbf{Curl}\,\boldsymbol{\omega}_h + \mathbf{grad}\,p_h - \boldsymbol{f}_h$ is constant on each element K of $\mathcal{T}_h^F$. Thus, when taking each component of $\boldsymbol{v}_h$ equal to the characteristic function of K in the first equation of (4.3), we obtain

$$\begin{aligned} &\forall K \in \mathcal{T}_h^P, \quad \mu\, \boldsymbol{u}_h + \mathbf{grad}\, p_h = \boldsymbol{f}_h \quad \text{on } K, \\ &\qquad\qquad \forall K \in \mathcal{T}_h^F, \quad \nu\, \mathbf{Curl}\, \boldsymbol{\omega}_h + \mathbf{grad}\, p_h = \boldsymbol{f}_h \quad \text{on } K, \end{aligned} \tag{6.5}$$

This leads to the residual equation

$$\forall \boldsymbol{v} \in L^2(\Omega)^d, \quad a(\boldsymbol{\omega} - \boldsymbol{\omega}_h, \boldsymbol{u} - \boldsymbol{u}_h; \boldsymbol{v}) + b_h(\boldsymbol{v}, p - p_h) = \int_\Omega (\boldsymbol{f} - \boldsymbol{f}_h)(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x})\, d\boldsymbol{x}. \tag{6.6}$$

However, in order to exploit the properties of the continuous bilinear forms, more precisely the inf-sup condition (3.11), we have rather write it as, for the function p_h^* introduced in Lemma 6.1,

$$\begin{aligned} &\forall \boldsymbol{v} \in L^2(\Omega)^d, \quad a(\boldsymbol{\omega} - \boldsymbol{\omega}_h, \boldsymbol{u} - \boldsymbol{u}_h; \boldsymbol{v}) + b(\boldsymbol{v}, p - p_h^*) \\ &\qquad\qquad = \int_\Omega (\boldsymbol{f} - \boldsymbol{f}_h)(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x})\, d\boldsymbol{x} + b_h(\boldsymbol{v}, p_h - p_h^*). \end{aligned} \tag{6.7}$$

Similarly, it follows from (4.11) by integration by parts that

$$\forall q \in H^1(\Omega) \cap L^2_0(\Omega), \quad b(\boldsymbol{u} - \boldsymbol{u}_h, q) = \langle k - k_h, q \rangle_{\partial\Omega}. \tag{6.8}$$

To write the third residual equation, we treat separately the cases of dimension $d = 2$ and $d = 3$.
1) In dimension $d = 2$, with any $\boldsymbol{\varphi}$ in $\mathbb{C}(\Omega_F)$ and $\boldsymbol{\varphi}_h$ in $\mathbb{C}_h(\Omega_F)$, we have

$$c(\boldsymbol{\omega} - \boldsymbol{\omega}_h, \boldsymbol{u} - \boldsymbol{u}_h; \boldsymbol{\varphi}) = -c(\boldsymbol{\omega}_h, \boldsymbol{u}_h; \boldsymbol{\varphi} - \boldsymbol{\varphi}_h),$$

whence, by integrating by parts on each K in $\mathcal{T}_h^F$ and using the fact that $\mathbf{curl}\, \boldsymbol{u}_h$ is zero on K,

$$\begin{aligned} &\forall \boldsymbol{\varphi} \in \mathbb{C}(\Omega_F), \forall \boldsymbol{\varphi}_h \in \mathbb{C}_h(\Omega_F), \\ &\quad c(\boldsymbol{\omega} - \boldsymbol{\omega}_h, \boldsymbol{u} - \boldsymbol{u}_h; \boldsymbol{\varphi}) = \sum_{K \in \mathcal{T}_h^F} \Big(- \int_K \boldsymbol{\omega}_h(\boldsymbol{x}) \cdot (\boldsymbol{\varphi} - \boldsymbol{\varphi}_h)(\boldsymbol{x})\, d\boldsymbol{x} \\ &\qquad\qquad - \frac{1}{2} \sum_{e \in \mathcal{E}_K} \int_e [\tilde{\gamma}_t(\boldsymbol{u}_h)]_e(\boldsymbol{\tau}) \cdot (\boldsymbol{\varphi} - \boldsymbol{\varphi}_h)(\boldsymbol{\tau})\, d\boldsymbol{\tau} \Big). \end{aligned} \tag{6.9}$$

2) A further argument is needed in dimension $d = 3$. With each $\boldsymbol{\varphi}$ in $\mathbb{C}(\Omega_F)$, we associate the solution $\tilde{\mu}$ in $H^1_0(\Omega_F)$ of the problem

$$\forall \rho \in H^1_0(\Omega_F), \quad \int_{\Omega_F} (\mathbf{grad}\, \tilde{\mu})(\boldsymbol{x}) \cdot (\mathbf{grad}\, \rho)(\boldsymbol{x})\, d\boldsymbol{x} = \int_{\Omega_F} \boldsymbol{\varphi}(\boldsymbol{x}) \cdot (\mathbf{grad}\, \rho)(\boldsymbol{x})\, d\boldsymbol{x}.$$

The function $\tilde{\boldsymbol{\varphi}} = \boldsymbol{\varphi} - \mathbf{grad}\,\tilde{\mu}$ is now divergence-free in the distribution sense. Thus it follows from [13] (see also [9]) that $\tilde{\varphi}$ can be written as the sum of a function φ^* in $H^1(\Omega_F)^3$ and of the gradient of a function in $H_0^1(\Omega_F)$; moreover, when setting $\mathbf{grad}\,\mu^* = \boldsymbol{\varphi} - \boldsymbol{\varphi}^*$, the following estimate holds

$$\|\boldsymbol{\varphi}^*\|_{H^1(\Omega_F)^3} + \|\mu^*\|_{H^1(\Omega_F)} \leq c\,\|\boldsymbol{\varphi}\|_{\mathbb{C}(\Omega_F)}. \tag{6.10}$$

Let us introduce the space

$$\mathbb{M}_h^0(\Omega_F) = \big\{\rho_h \in H_0^1(\Omega_F);\ \forall K \in \mathcal{T}_h^F,\ \rho_{h|K} \in \mathcal{P}_1(K)\big\}; \tag{6.11}$$

It can be observed that, for any μ_h in $\mathbb{M}_h^0(\Omega_F)$, $\mathbf{grad}\,\mu_h$ belongs to $\mathbb{C}_h(\Omega_F)$. Thus, for any $\boldsymbol{\varphi}_h$ in $\mathbb{C}_h(\Omega_F)$, we have the equation

$$c(\boldsymbol{\omega} - \boldsymbol{\omega}_h, \boldsymbol{u} - \boldsymbol{u}_h; \boldsymbol{\varphi}) = -c(\boldsymbol{\omega}_h, \boldsymbol{u}_h; \boldsymbol{\varphi}^* - \boldsymbol{\varphi}_h) - c(\boldsymbol{\omega}_h, \boldsymbol{u}_h; \mathbf{grad}\,(\mu^* - \mu_h)).$$

Integrating by parts on each K and noting that both $\mathbf{curl}\,\boldsymbol{u}_h$ and $\operatorname{div}\boldsymbol{\omega}_h$ are zero on K lead to

$$\begin{aligned} c(\boldsymbol{\omega} - \boldsymbol{\omega}_h, \boldsymbol{u} - \boldsymbol{u}_h; \boldsymbol{\varphi}) = \sum_{K \in \mathcal{T}_h^F} \Big(-\int_K \boldsymbol{\omega}_h(\boldsymbol{x}) \cdot (\boldsymbol{\varphi}^* - \boldsymbol{\varphi}_h)(\boldsymbol{x})\,d\boldsymbol{x} \\ - \frac{1}{2} \sum_{e \in \mathcal{E}_K} \Big(\int_e [\tilde{\gamma}_t(\boldsymbol{u}_h)]_e(\boldsymbol{\tau}) \cdot (\boldsymbol{\varphi}^* - \boldsymbol{\varphi}_h)(\boldsymbol{\tau})\,d\boldsymbol{\tau} + \int_e [\boldsymbol{\omega}_h \cdot \boldsymbol{n}]_e(\boldsymbol{\tau}) \cdot (\mu^* - \mu_h)(\boldsymbol{\tau})\,d\boldsymbol{\tau} \Big) \Big). \end{aligned} \tag{6.12}$$

Due to the definition (2.16) or (2.17) of $\mathbb{C}_h(\Omega_F)$, estimating the right-hand side of (6.9) and the two first terms in the right-hand side of (6.12) is easy: Indeed, taking $\boldsymbol{\varphi}_h$ equal to the image of $\boldsymbol{\varphi}$ by a Clément type regularization operator (see [8, Thm IX.3.11 & Cor. IX.3.12] in dimension $d = 2$ and [5, Prop. 2] or [12, §4.3] in dimension $d = 3$) leads to, for any triangle K of $\mathcal{T}_h^F$ and any edge e of $\mathcal{E}_K$,

$$\begin{aligned} \|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{L^2(K)^{\frac{d(d-1)}{2}}} \leq c\,h_K\,\|\boldsymbol{\varphi}\|_{H^1(\Delta_K)^{\frac{d(d-1)}{2}}}, \\ \|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{L^2(e)^{\frac{d(d-1)}{2}}} \leq c\,h_e^{\frac{1}{2}}\,\|\boldsymbol{\varphi}\|_{H^1(\Delta_e)^{\frac{d(d-1)}{2}}}, \end{aligned} \tag{6.13}$$

where Δ_K and Δ_e stand for the union of elements of $\mathcal{T}_h^F$ that intersects K and e, respectively. The same arguments allow for evaluating the last term in the right-hand side of (6.12): There exists a function μ_h in $\mathbb{M}_h^0(\Omega_F)$ such that, for any tetrahedron K of $\mathcal{T}_h^F$ and any face e of $\mathcal{E}_K$,

$$\|\mu^* - \mu_h\|_{L^2(e)} \leq c\,h_e^{\frac{1}{2}}\,\|\mu^*\|_{H^1(\Delta_e)}, \tag{6.14}$$

So, we are is a position to state the a posteriori estimate of the error.

Theorem 6.2. *The following a posteriori error estimate holds between the solution* $(\boldsymbol{\omega}, \boldsymbol{u}, p)$ *of problem* (3.3) *and the solution* $(\boldsymbol{\omega}_h, \boldsymbol{u}_h, p_h)$ *of problem* (4.3),

$$\|\boldsymbol{\omega} - \boldsymbol{\omega}_h\|_{\mathbb{C}(\Omega_F)} + \|\boldsymbol{u} - \boldsymbol{u}_h\|_{L^2(\Omega)^d} + \|p - p_h\|_{H^1_h(\Omega)} \\ \leq c\Big(\big(\sum_{e \in \mathcal{E}_h^0} \eta_e^2\big)^{\frac{1}{2}} + \big(\sum_{K \in \mathcal{T}_h^F} \eta_K^2\big)^{\frac{1}{2}} + \|\boldsymbol{f} - \boldsymbol{f}_h\|_{L^2(\Omega)^d} + \|k - k_h\|_{H^{-\frac{1}{2}}(\partial\Omega)} \Big). \tag{6.15}$$

Proof: We use the same arguments as in Remark 3.6.
1) It follows from the inf-sup condition (3.11) and equation (6.8) that there exists a function $\boldsymbol{u}_\sharp^1$ in $L^2(\Omega)^d$ such that

$$\forall q \in H^1(\Omega) \cap L^2_0(\Omega), \quad b(\boldsymbol{u}_\sharp^1, q) = \langle k - k_h, q \rangle_{\partial\Omega},$$

and

$$\|\boldsymbol{u}_\sharp^1\|_{L^2(\Omega)^d} \leq c\, \|k - k_h\|_{H^{-\frac{1}{2}}(\partial\Omega)}. \tag{6.16}$$

2) We denote by $\langle R, \boldsymbol{\varphi} \rangle$ the righ-hand sides of equations (6.9) or (6.12) according to the dimension d. Indeed, it follows from these equations and the inf-sup condition (3.16) that there exists a pair $(\boldsymbol{\omega}_\sharp, \boldsymbol{u}_\sharp^2)$ in $\mathbb{C}(\Omega_F) \times V(\Omega_F)$ such that

$$\forall \boldsymbol{\varphi} \in \mathbb{C}(\Omega_F), \quad c(\boldsymbol{\omega}_\sharp, \boldsymbol{u}_\sharp^2; \boldsymbol{\varphi}) = \langle R, \boldsymbol{\varphi} \rangle - c(\boldsymbol{0}, \boldsymbol{u}_\sharp^1; \boldsymbol{\varphi}),$$

and

$$\|\boldsymbol{\omega}_\sharp\|_{\mathbb{C}(\Omega_F)} + \|\boldsymbol{u}_\sharp^2\|_{L^2(\Omega)^d} \leq \Big(\sup_{\boldsymbol{\varphi} \in \mathbb{C}(\Omega_F)} \frac{\langle R, \boldsymbol{\varphi} \rangle}{\|\boldsymbol{\varphi}\|_{\mathbb{C}(\Omega_F)}} + \|\boldsymbol{u}_\sharp^1\|_{L^2(\Omega)^d} \Big).$$

Evaluating $\langle R, \boldsymbol{\varphi} \rangle$ in dimension $d = 2$ follows from (6.13) and the fact that $\mathbb{C}(\Omega_F)$ is equal to $H^1_0(\Omega_F)$. Evaluating it in dimension $d = 3$ follows on one hand from (6.13) and the bound for $\boldsymbol{\varphi}^*$ in (6.10), on the other hand from (6.14) and the bound for μ^* in (6.10). Thus, we derive

$$\|\boldsymbol{\omega}_\sharp\|_{\mathbb{C}(\Omega_F)} + \|\boldsymbol{u}_\sharp^2\|_{L^2(\Omega)^d} \leq \Big(\big(\sum_{K \in \mathcal{T}_h^F} \eta_K^2 \big)^{\frac{1}{2}} + \|\boldsymbol{u}_\sharp^1\|_{L^2(\Omega)^d} \Big). \tag{6.17}$$

3) Using the extension of $\boldsymbol{u}_\sharp^2$ by zero to Ω without change of notation, we observe that the pair $(\boldsymbol{\omega}_0, \boldsymbol{u}_0)$, with $\boldsymbol{\omega}_0 = \boldsymbol{\omega} - \boldsymbol{\omega}_h - \boldsymbol{\omega}_\sharp$ and $\boldsymbol{u}_0 = \boldsymbol{u} - \boldsymbol{u}_h - \boldsymbol{u}_\sharp^1 - \boldsymbol{u}_\sharp^2$, belongs to $\mathcal{W}(\Omega)$ and satisfies the equation

$$\forall \boldsymbol{v} \in V(\Omega), \quad a(\boldsymbol{\omega}_0, \boldsymbol{u}_0; \boldsymbol{v}) \\ = \int_\Omega (\boldsymbol{f} - \boldsymbol{f}_h)(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x})\, d\boldsymbol{x} + b_h(\boldsymbol{v}, p_h - p_h^*) - a(\boldsymbol{\omega}_\sharp, \boldsymbol{u}_\sharp^1 + \boldsymbol{u}_\sharp^2 . \boldsymbol{v}).$$

Thus applying Lemma 3.3 implies

$$\begin{aligned}&\|\boldsymbol{\omega}_0\|_{\mathbb{C}(\Omega_F)} + \|\boldsymbol{u}_0\|_{L^2(\Omega)^d}\\ &\qquad \le c\,\big(\|\boldsymbol{f}-\boldsymbol{f}_h\|_{L^2(\Omega)^d} + \|p_h - p_h^*\|_{H_h^1(\Omega)} + \|\boldsymbol{\omega}_\sharp\|_{\mathbb{C}(\Omega_F)} + \|\boldsymbol{u}_\sharp^1\|_{L^2(\Omega)^d} + \|\boldsymbol{u}_\sharp^2\|_{L^2(\Omega)^d}\big).\end{aligned}$$

A further triangle inequality, combined with Lemma 6.1, leads to

$$\begin{aligned}&\|\boldsymbol{\omega}-\boldsymbol{\omega}_h\|_{\mathbb{C}(\Omega_F)} + \|\boldsymbol{u}-\boldsymbol{u}_h\|_{L^2(\Omega)^d}\\ &\qquad \le c\,\big(\|\boldsymbol{f}-\boldsymbol{f}_h\|_{L^2(\Omega)^d} + \big(\sum_{e\in\mathcal{E}_h^0}\eta_e^2\big)^{\frac{1}{2}} + \|\boldsymbol{\omega}_\sharp\|_{\mathbb{C}(\Omega_F)} + \|\boldsymbol{u}_\sharp^1\|_{L^2(\Omega)^d} + \|\boldsymbol{u}_\sharp^2\|_{L^2(\Omega)^d}.\end{aligned}$$

The desired estimate for $\|\boldsymbol{\omega}-\boldsymbol{\omega}_h\|_{\mathbb{C}(\Omega_F)}$ and $\|\boldsymbol{u}-\boldsymbol{u}_h\|_{L^2(\Omega)^d}$ follows by inserting (6.16) and (6.17) in this last inequality.
4) When applying the inf-sup condition (3.11) in (6.7), we obtain

$$\|p-p_h^*\|_{H^1(\Omega)} \le c\,\big(\|\boldsymbol{\omega}-\boldsymbol{\omega}_h\|_{\mathbb{C}(\Omega_F)} + \|\boldsymbol{u}-\boldsymbol{u}_h\|_{L^2(\Omega)^d} + \|\boldsymbol{f}-\boldsymbol{f}_h\|_{L^2(\Omega)^d} + \|p_h-p_h^*\|_{H_h^1(\Omega)}\big).$$

A further triangle inequality, combined with Lemma 6.1 and the previous estimates, leads to the desired estimate for $\|p-p_h\|_{H_h^1(\Omega)}$.

Note that estimate (6.15) is fully optimal both in dimension $d = 2$ and in dimension $d = 3$. We now prove an upper bound for each indicator η_e and η_F. We skip the proof for the η_e since it is exactly the same as for [1, Thm 4.8] (indeed, this indicator is issued from the nonconformity of the discretization).

Proposition 6.3. *The following estimate holds for each indicator η_e, $e \in \mathcal{E}_h^0$,*

$$\eta_e \le c \sum_{K\in\mathcal{T}_e} |p-p_h|_{H^1(K)}, \tag{6.18}$$

where $\mathcal{T}_e$ denotes the set of elements of $\mathcal{T}_h$ that contain e.

We only give an abridged proof of the next proposition and refer to [7, Prop. 5.5] for more details.

Proposition 6.4. *The following estimate holds for each indicator η_K, $e \in \mathcal{T}_h^F$,*

$$\eta_K \le c\,\big(\|\boldsymbol{\omega}-\boldsymbol{\omega}_h\|_{\mathbb{C}(\Delta_K)} + \|\boldsymbol{u}-\boldsymbol{u}_h\|_{L^2(\Delta_K)^d}\big), \tag{6.19}$$

where Δ_K denotes the union of elements of $\mathcal{T}_h$ that share at least an edge ($d = 2$) or a face ($d = 3$) with K.

Proof: We treat separately the cases of dimension $d = 2$ and $d = 3$.
1) In dimension $d = 2$, we consider equation (6.9) with $\boldsymbol{\varphi}_h$ equal to zero and take successively $\boldsymbol{\varphi}$ equal to

$$\boldsymbol{\varphi}_K = \begin{cases} \boldsymbol{\omega}_h\,\psi_K & \text{on } K,\\ 0 & \text{on } \Omega_F\setminus K,\end{cases}, \qquad \boldsymbol{\varphi}_e = \begin{cases} \mathcal{L}_e([\tilde{\gamma}(\boldsymbol{u}_h)]_e)\,\psi_e & \text{on } K\cup K',\\ 0 & \text{on } \Omega_F\setminus (K\cup K'),\end{cases}$$

where ψ_K and ψ_e stand for the bubble functions on K and e respectively, K' is the other triangle of $\mathcal{T}_h^F$ that contains e and $\mathcal{L}$ is a lifting operator from e onto $K \cup K'$ constructed from a fixed lifting operator on two reference triangles. Thus, standard inverse inequalities, see [22, §1.2], lead to the desired bound for η_K.
2) In dimension $d = 3$, we consider equation (6.12) with $\boldsymbol{\varphi}_h$ and μ_h equal to zero and observe that it is valid for all $\boldsymbol{\varphi}^*$ in $\mathbb{C}(\Omega_F)$ and μ^* in $H_0^1(\Omega_F)$. The same choices for $\boldsymbol{\varphi}^*$ as in dimension $d = 2$, namely $\boldsymbol{\varphi}^* = \boldsymbol{\varphi}_K$ and $\boldsymbol{\varphi}^* = \boldsymbol{\varphi}_e$ (with now $\mu = 0$) give the estimate for the first two terms in η_K. To bound the third one, we take $\boldsymbol{\varphi}^*$ equal to zero and μ^* equal to, with the same notation as previously,

$$\mu^* = \begin{cases} \mathcal{L}_e([\boldsymbol{\omega}_h \cdot \boldsymbol{n}_e]_e)\, \psi_e & \text{on } K \cup K', \\ 0 & \text{on } \Omega_F \setminus (K \cup K'). \end{cases}$$

There also the estimate is derived from inverse inequalities, see [7, form. (5.21)].

Estimates (6.18) and (6.19) are fully optimal whatever the geometry of Ω is. Moreover, since both of them are local, it can be thought that η_e and η_K provide a good representation of the local error.

7. Some numerical experiments.

The numerical simulations that follow have been performed on the finite element code FreeFem++, see [19]. We present two kinds of numerical experiments. In both cases, relying on the results of Section 6, mesh adaptivity is performed according to a very simple strategy, see [6, §6] for instance: The method is coarsened and refined on the whole domain Ω according to the ratios $\eta_e/\overline{\eta}$, where $\overline{\eta}$ is the mean value of the η_e, next on Ω_F, according to the ratios $\eta_K/\overline{\eta}^F$, where $\overline{\eta}^F$ is the mean value of the η_K. A final process is needed in order to ensure the conformity of the triangulation on Γ.

The vertical cracks

We work with the domains

$$\Omega =]-1, 1[\times]-1, 0[, \qquad \Omega_F =]-\frac{\tau}{2}, \frac{\tau}{2}[\times]-\frac{1}{2}, -\frac{1}{10}[, \tag{7.1}$$

where τ represents the thickness of the crack, $0 < \tau \leq 1$. The next data ν and μ correspond to the flow of water in calcalenite mixed with sand:

$$\nu = 10^{-2}, \qquad \mu = 10^3. \tag{7.2}$$

Finally, we work with data $\boldsymbol{f} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ (which is the gravity) and k equal to $\boldsymbol{u}_e \cdot \boldsymbol{n}$, with $\boldsymbol{u}_e = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$. Indeed, $\boldsymbol{u}_e$ represents a rainfall rate: The water enters the domain in the top part Γ_u of the boundary and goes out in the bottom part Γ_d, with

$$\Gamma_u =]-1, 1[\times\{0\}, \qquad \Gamma_d =]-1, 1[\times\{-1\}. \tag{7.3}$$

Here, the indices u and d means upstream and downstream, respectively. Note that, for the reasons explained in Remark 3.7, the discrete vorticity is zero.

Figures 2, 3 and 4 present the final adapted mesh, the isovalues of the stream-function associated with the discrete velocity and the isovalues of the pressure, for the three values of τ:

$$\tau = \frac{1}{10}, \quad \tau = \frac{1}{50}, \quad \tau = \frac{1}{100}. \tag{7.4}$$

It can be noted that, with the same data and when the crack is empty, the curves of isovalues of the stream-function are simply parallel vertical lines, which is rather different from their analogues in Figures 2, 3 and 4. This of course justifies the interest of using a coupled model.

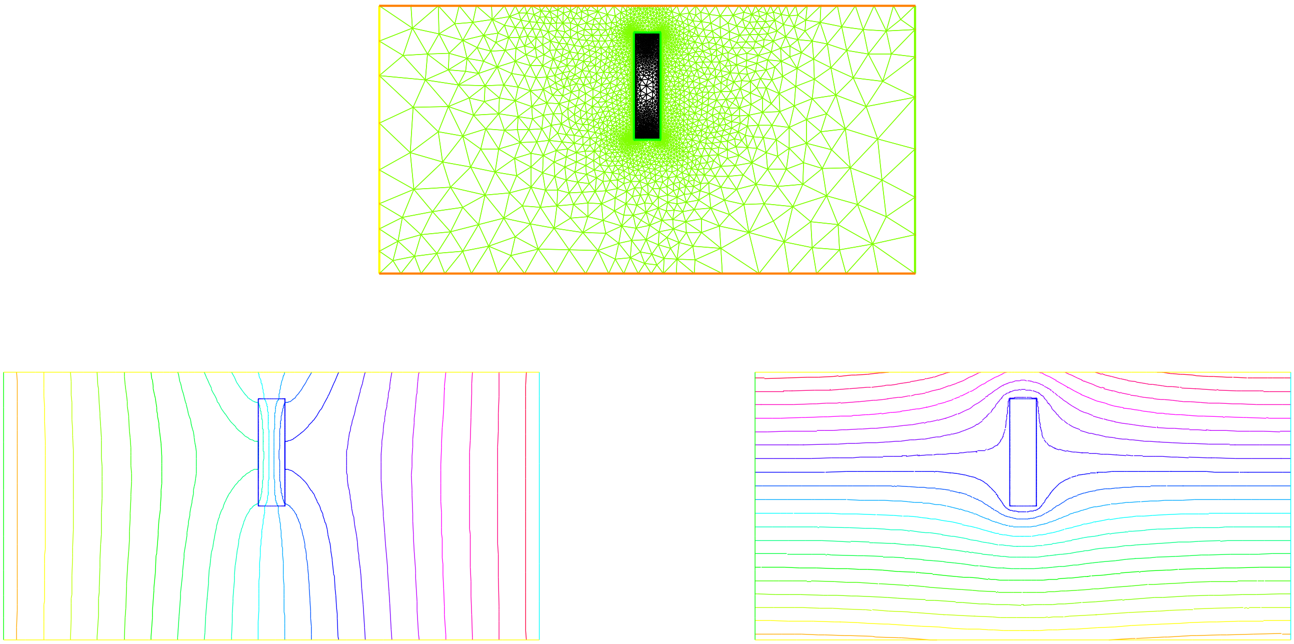

Figure 2: The crack with thickness $\tau = \frac{1}{10}$

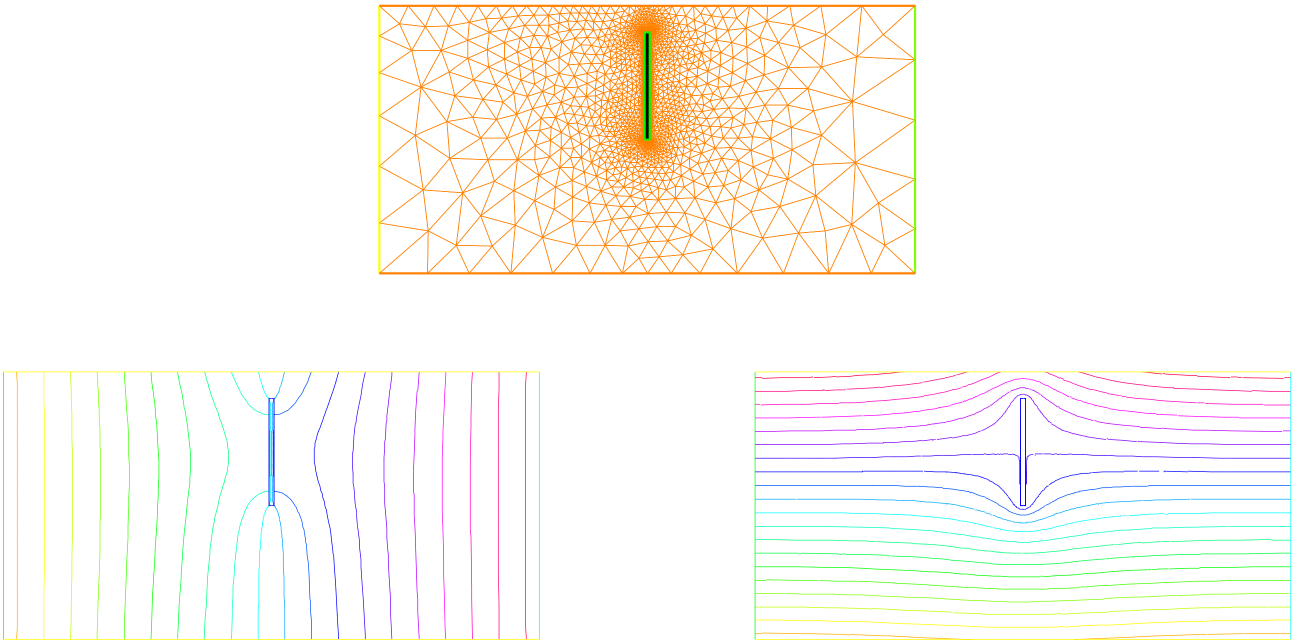

Figure 3: The crack with thickness $\tau = \frac{1}{50}$

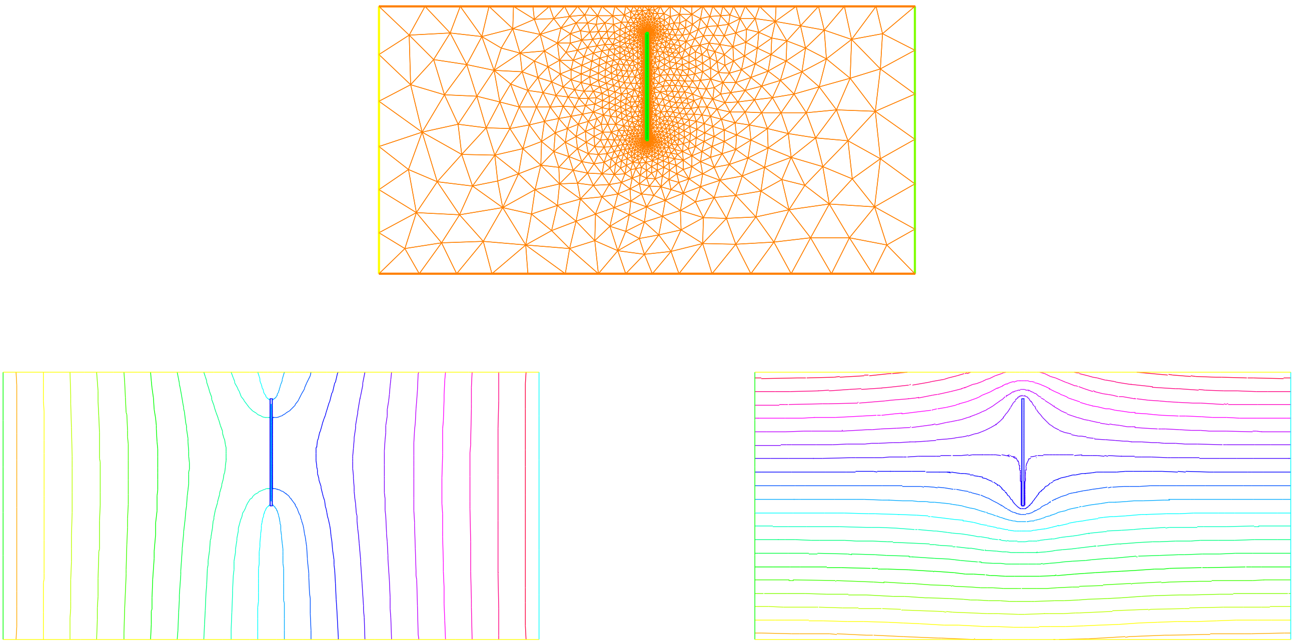

Figure 4: The crack with thickness $\tau = \frac{1}{100}$

THE V–CRACK

The domain Ω is the same as previously, but now the crack is V-shaped:

$$\Omega =]-1,1[\times]-1,0[, \qquad \Omega_F = \left\{(x,z) \in \Omega;\ -\frac{1}{2} < z < 0 \text{ and } |x| < \frac{z}{2} + \frac{1}{4}\right\}. \tag{7.5}$$

The data ν and μ are still given in (7.2) and the datum $\boldsymbol{f}$ is still equal to $\boldsymbol{f} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. However, we now work with the slightly modified boundary conditions

$$\boldsymbol{u} \cdot \boldsymbol{n} = k \quad \text{on } \partial\Omega \qquad \text{and} \qquad \boldsymbol{\omega} = \boldsymbol{\omega}_w \quad \text{on } \Gamma_F, \tag{7.6}$$

where $\Gamma_F = \partial\Omega \cap \partial\Omega_F$ is the segment $[-\frac{1}{4}, \frac{1}{4}] \times \{0\}$. The datum k is exactly the same as previously, i.e., $k = \boldsymbol{u}_e \cdot \boldsymbol{n}$, with $\boldsymbol{u}_e = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ and the datum $\boldsymbol{\omega}_w$ which is linked to the wind on the surface is taken equal

$$\omega_w(x,0) = \frac{1}{16} - x^2, \qquad -\frac{1}{4} \le x \le \frac{1}{4}. \tag{7.7}$$

Figure 5 presents from left to right and from top to bottom the final adapted mesh, the isovalues of the vorticity, of the stream-function associated with the discrete velocity and of the pressure.

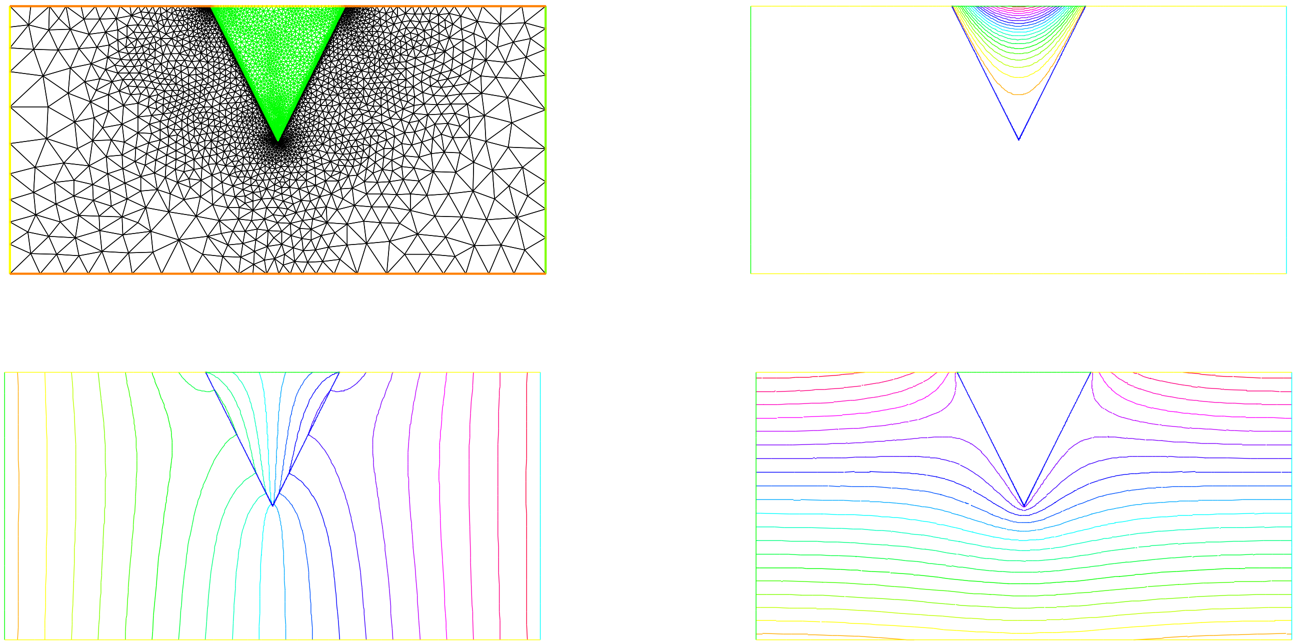

Figure 5: The $V-$crack

From the previous experiments, we observe that the geometry of the crack does not affect the accuracy of the simulation, since mesh adaptation enables us to handle this geometry in a natural way.

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