Microscopic effects in the homogenization of the junction of rods and a thin plate

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Abstract

This paper is devoted to investigate a few microscopic effects in the homogenization process of the junction of a periodic family of rods with a thin plate in elasticity. We focus on the case where the thickness of the plate tends to zero faster than the periodicity. As a consequence of the studied microscopic effects, the elastic coefficients of the membrane and bending limit problems for the plate are modified. Moreover, we observe a torsion in the homogenized "continuum" of rods which depends on the curl of the membrane displacement of the plate.

Résumé

Cet article est destiné à mettre en évidence des effets microscopiques dans le processus d'homogénéisation de la jonction d'une famille périodique de poutres avec une plaque mince en élasticité. Nous examinons la situation où l'épaisseur de la plaque tend vers zéro plus vite que la période. Les effets microscopiques observés se traduisent par une modification des coefficients élastiques dans les problèmes limites de membrane et de flexion de la plaque. De plus, on met en évidence une torsion dans le "continuum" homogénéisé de poutres qui dépend du rotationnel du champ de déplacement membranaire de la plaque.

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1. Introduction

This paper pertains to the general problem of modeling the asymptotic behavior of a multistructure made of a ε-periodic set of elastic rods, with radius $r = k\varepsilon$ ($k < 1/2$), in junction with an elastic plate whose thickness δ can tend to zero as the periodicity ε of the rods vanishes. The two critical cases $\delta = 1$ and $\delta \sim \varepsilon^{2/3}$ are investigated in [1] and [2], and they come under the situation where $\varepsilon/\delta \to 0$ for which there is no microscopic effect induced by the rods on the limit problem for the plate which consists in the standard membrane $2d$ model and the usual bending model. Here we investigate the case where $\delta/\varepsilon \to 0$ which corresponds to a "very" thin plate (with respect to the periodicity) and where such microscopic effects occur. Let us briefly describe the results obtained in the present paper. The main result of this work is that, even for a homogeneous and isotropic elastic material, the homogenization of the junction of the rods with the plate leads to a membrane $2d$ model and a bending model for the plate which have different coefficients that the ones of the standard models. These new coefficients are derived through solving elastic local problems. Loosely speaking the microscopic effects are due to the fact that the displacement is asymptotically rigid in the small cylinders of the plate which are below the rods.

As far as the rods are concerned, the limit model is a continuum of rods (indexed by (x_1, x_2)). In this continuum, each rod has a rigid body displacement which is given by the rigid displacement of the fiber of the plate which is below this specific rod. Moreover we show that, if $\dfrac{\sqrt{\delta}}{\varepsilon}$ is bounded then each rod has a constant rotation around its axis; the angle of this rotation is equal to the curl of the $2d$ membrane displacement of the limit plate model. We also show that, if $\dfrac{\sqrt{\delta}}{\varepsilon} \to +\infty$ then each small cylinder included in the plate and below each rod has a constant rotation around its axis; the angle of this rotation is also equal to the curl of the $2d$ membrane displacement. The first tool used to carry out the analysis is two decompositions of the displacement field in each rod and in the plate (see [19], [20], [1], [2]). The second main tool is the periodic unfolding operator in homogenization (see [10], [12], [1], [2]).

For the general theory of elasticity, we refer e.g. to [7] and for the mathematical justification of elastic plates model to [8] and [9] (and to the references quoted in these works). A general introduction to the mathematical modeling of elastic rods models can be found in [24].

The paper is organized as follows. Section 2 is devoted to describe the geometry and the equations of the problem. In Section 3 we first recall the decompositions of the displacement in the rods and in the plate and we give estimates on the terms of these decompositions in term of the elastic energy. A special care is dedicated to these estimates in the small cylinders below the rods. Section 4 contains a few recall on the unfolding operators in the rods and in the plate. In Section 5 we show that the limit displacement in each rod (i.e. for a.e. (x_1, x_2) is a rigid displacement and that the unfold strain of the rods converges to 0. In Section 6 we identify the weak limits of the unfold displacement and of the unfold strain. Section 7 is concerned with the derivation of the kinematic transmission condition between the rods and the plate in the limit model. In Section 8, we introduce the local membrane correctors in order to derive the membrane limit problem for the plate and the torsion angle in each rod. Section 9 is devoted to introduce the local bending correctors to be a position to obtain the limit bending problem in the plate. At last, in Section 10 we prove the strong convergence of the $3d$ energy to the energy of the limit problem as ε tends to 0.

2. The geometry and the problem

Throughout the paper $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ denotes the standard basis of $\mathbb{R}^3$. Let δ be a sequence of positive real numbers which tends to zero and let ε be a sequence of positive real numbers which depends on δ and tends to zero with δ and such that $\dfrac{\delta}{\varepsilon} \to 0$.

Let us consider a bounded connected regular domain ω contained in the (x_1, x_2) coordinate plane. The set $\mathcal{N}_\varepsilon$ is defined for ε small enough as the following subset of $\mathbb{Z}^2$

$$\mathcal{N}_\varepsilon = \left\{(p,q) \in \mathbb{Z}^2 \; ; \;]\varepsilon p - \varepsilon/2, \varepsilon p + \varepsilon/2[\times]\varepsilon q - \varepsilon/2, \varepsilon q + \varepsilon/2[\subset \omega \right\}.$$

Fix $L > 0$. For each $(p,q) \in \mathbb{Z}^2$, $\varepsilon > 0$ and $r = k\varepsilon$ $(k < 1/2)$, we consider a rod $\mathcal{P}^\varepsilon_{p,q}$ whose cross section is the disk of center $(\varepsilon p, \varepsilon q)$ and radius r, and whose axis is x_3 and with a height equal to L

$$\mathcal{D}^\varepsilon_{p,q} = \left\{(x_1, x_2) \in \mathbb{R}^2 \; ; \; (x_1 - \varepsilon p)^2 + (x_2 - \varepsilon q)^2 < r^2 \right\},$$

$$\mathcal{P}^\varepsilon_{p,q} = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 \; ; \; (x_1, x_2) \in \mathcal{D}^{\varepsilon,r}_{p,q}, \; 0 < x_3 < L \right\}.$$

Then, we denote by Ω^+_ε the set of all the rods defined as above

$$\Omega^+_\varepsilon = \bigcup_{(p,q)\in\mathcal{N}_\varepsilon} \mathcal{P}^\varepsilon_{p,q}.$$

The lower cross sections of all the rods is denoted by ω_ε

$$\omega_\varepsilon = \bigcup_{(p,q)\in\mathcal{N}_\varepsilon} \mathcal{D}^\varepsilon_{p,q} \times \{0\} \subset \omega.$$

In order to shorten the notation, we set

$$\widetilde{\omega}_\varepsilon = \bigcup_{(p,q)\in\mathcal{N}_\varepsilon} \left(\left]\varepsilon p - \frac{\varepsilon}{2}, \varepsilon p + \frac{\varepsilon}{2}\right[\times \left]\varepsilon q - \frac{\varepsilon}{2}, \varepsilon q + \frac{\varepsilon}{2}\right[\right) \subset \omega.$$

We have assumed that $k < 1/2$ in order to avoid the contact between two different rods (recall that $r = k\varepsilon$). The domain filled by the oscillating part Ω^+_ε (as ε tends to zero) is denoted by Ω^+

$$\Omega^+ = \omega \times]0, L[.$$

Moreover, we set

$$\Omega^- = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 \; ; \; (x_1, x_2) \in \omega, \; -1 < x_3 < 0 \right\},$$

$$\Omega = \omega \times] - 1, L[.$$

The $3d$-plate Ω^-_δ is defined, for $\delta > 0$, by

$$\Omega^-_\delta = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 \; ; \; (x_1, x_2) \in \omega, \; -\delta < x_3 < 0 \right\}.$$

The elastic body under consideration is

$$\Omega_{\varepsilon,\delta} = \Omega^+_\varepsilon \cup \omega_\varepsilon \cup \Omega^-_\delta.$$

In order to derive the estimates that lead to the junction conditions between the rods and the plate, we introduce below the following subsets of $\Omega_{\varepsilon,\delta}$

$$\mathcal{C}^{\varepsilon\delta} = \bigcup_{(p,q)\in\mathcal{N}_\varepsilon} \mathcal{D}^\varepsilon_{p,q} \times] - \delta, 0[,$$

$$\Omega^+_{\varepsilon,\delta} = \Omega^+_\varepsilon \cup \mathcal{C}^{\varepsilon\delta}.$$

Remark that $\Omega^+_{\varepsilon,\delta}$ is actually made of rods of length $L+\delta$, each one being a rod of Ω^+_ε which is extended for $-\delta < x_3 < 0$ into the plate.

We consider the standard linear equations of elasticity in $\Omega_{\varepsilon,\delta}$.

The displacement field in $\Omega_{\varepsilon,\delta}$ is denoted by

$$u^\delta \,:\, \Omega_{\varepsilon,\delta} \to \mathbb{R}^3.$$

The linearized strain field in $\Omega_{\varepsilon,\delta}$ is defined by

$$\gamma(u^\delta) = \frac{1}{2}\left(Du^\delta + (Du^\delta)^T\right),$$

or equivalently by its components

$$\gamma_{ij}(u^\delta) = \frac{1}{2}\left(\partial_i u^\delta_j + \partial_j u^\delta_i\right), \quad i,j = 1,2,3.$$

The Cauchy stress tensor in $\Omega_{\varepsilon,\delta}$ is linked to $\gamma(u^\delta)$ through the standard Hooke's law

$$\sigma^\delta_{ij} = \lambda\left(\sum_{k=1}^{3}\gamma_{kk}(u^\delta)\right)\delta_{ij} + 2\mu\gamma_{ij}(u^\delta), \quad i,j = 1,2,3, \tag{2.1}$$

where λ and μ denotes the Lamé's coefficients of the elastic material, and where $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ij} = 1$ if $i = j$. The equation of equilibrium in $\Omega_{\varepsilon,\delta}$ is

$$-\sum_{j=1}^{3}\partial_j \sigma^\delta_{ij} = f^\delta_i \text{ in } \Omega_{\varepsilon,\delta}, \quad i = 1,2,3, \tag{2.2}$$

where $f^\delta : \Omega_{\varepsilon,\delta} \to \mathbb{R}^3$ denotes the applied force.

In order to specify the boundary conditions on $\partial\Omega_{\varepsilon,\delta}$, we will assume that the $3d$ plate is clamped on its lateral boundary $\partial\omega\times]-\delta,0[= \Gamma_\delta$

$$u^\delta = 0 \text{ on } \Gamma_\delta, \tag{2.3}$$

and that the boundary $\partial\Omega_{\varepsilon,\delta}\setminus\Gamma_\delta$ is free

$$\sigma^\delta\nu = 0 \text{ on } \partial\Omega_{\varepsilon,\delta}\setminus\Gamma_\delta, \tag{2.4}$$

where ν denotes the exterior unit normal to $\Omega_{\varepsilon,\delta}$.

Remark 2.1 The boundary condition (2.4) means that the density of applied surface forces on the boundary $\partial\Omega_\varepsilon\setminus\Gamma_\delta$ is zero. This assumption is not necessary to carry on the analysis, but it is a bit natural as far as the fast oscillating boundary $\partial\Omega^+_\varepsilon$ is concerned.

The variational formulation of (2.2)-(2.3)-(2.4) is very standard. If $V_{\varepsilon,\delta}$ denotes the space:

$$V_{\varepsilon,\delta} = \left\{v \in \left(H^1(\Omega_{\varepsilon,\delta})\right)^3 \;;\; v = 0 \text{ on } \Gamma_\delta\right\},$$

the variational formulation is

$$(2.5)\qquad \begin{cases} u^\delta \in V_{\varepsilon,\delta}, \\ \displaystyle\int_{\Omega_{\varepsilon,\delta}} \sum_{i,j=1}^3 \sigma^\delta_{ij}\gamma_{ij}(v)dx = \int_{\Omega_{\varepsilon,\delta}} \sum_{i=1}^3 f_i^\delta v_i dx, \qquad \forall v \in V_{\varepsilon,\delta}. \end{cases}$$

Throughout the paper and for any $v \in V_{\varepsilon,\delta}$ we denote by

$$\mathcal{E}(v) = \int_{\Omega_{\varepsilon,\delta}} \left[\lambda \left(\sum_{k=1}^3 \gamma_{kk}(v) \right)^2 + 2\mu \sum_{i,j=1}^3 (\gamma_{ij}(v))^2 \right] dx$$

the total elastic energy of the displacement v and we set

$$|v|_{\mathcal{E}} = \sqrt{\mathcal{E}(v)}.$$

Indeed choosing $v = u^\delta$ in (2.5) leads to the usual energy relation

$$(2.6)\qquad \mathcal{E}(u^\delta) = \int_{\Omega_{\varepsilon,\delta}} \sum_{i=1}^3 f_i^\delta u_i^\delta dx.$$

3. Decompositions of the displacement and estimates

In this section, we consider two decompositions of the displacement field in each rod and in the plate as this was the case in [2]. These types of decompositions have been introduced in [17]-[20] to describe the asymptotic behavior of elastic multistructures. Remark that in the present paper and in order to describe the junction conditions between the rods and the plate, the decomposition of the displacement u^δ is twofold. In the small cylinders of the plate below each rod, u^δ is split as a rod type displacement and as a plate type displacement. This is the object of Subsection 3.1. Estimates of the terms of the decompositions of u^δ are given in Subsection 3.2. This leads to natural assumptions on the forces f_i^δ in Subsection 3.3. In Subsection 3.4 we derive estimates in the small junction cylinders $\mathcal{C}^{\varepsilon,\delta}$.

3.1 Decompositions of the displacement

The displacement field in the rods $\Omega_\varepsilon^+ \cup \mathcal{C}^{\varepsilon,\delta}$ is decomposed following [17] (see also [18]) as below:

$$(3.1)\qquad \begin{cases} u^\delta(x) = \mathcal{U}^{\delta+}(\varepsilon p, \varepsilon q, x_3) + \mathcal{R}^{\delta+}(\varepsilon p, \varepsilon q, x_3) \wedge \begin{pmatrix} x_1 - \varepsilon p \\ x_2 - \varepsilon q \\ 0 \end{pmatrix} + \overline{u}^{\delta+}(x) \\ \qquad = U_e^{\delta+}(\varepsilon p, \varepsilon q, x_3) + \overline{u}^{\delta+}(x) \end{cases} \qquad x \in \mathcal{D}^\varepsilon_{p,q} \times]-\delta, L[,$$

where the field $\overline{u}^{\delta+}$ satisfies

$$\int_{\mathcal{D}^\varepsilon_{p,q}} \overline{u}^{\delta+}(x_1, x_2, x_3) dx_1 dx_2 = 0,$$

$$\int_{\mathcal{D}^\varepsilon_{p,q}} (x_1 - \varepsilon p) \overline{u}_3^{\delta+}(x_1, x_2, x_3) dx_1 dx_2 = \int_{\mathcal{D}^\varepsilon_{p,q}} (x_2 - \varepsilon q) \overline{u}_3^{\delta+}(x_1, x_2, x_3) dx_1 dx_2 = 0,$$

$$\int_{\mathcal{D}^\varepsilon_{p,q}} \big\{ (x_1 - \varepsilon p) \overline{u}_2^{\delta+}(x_1, x_2, x_3) - (x_2 - \varepsilon q) \overline{u}_1^{\delta+}(x_1, x_2, x_3) \big\} dx_1 dx_2 = 0,$$

for almost any x_3 in $]-\delta, L[$.
The functions $\mathcal{U}^{\delta+}$ and $\mathcal{R}^{\delta+}$ are extended to the whole domain $\omega\times]-\delta, L[$ through

$$
(3.2)\qquad \begin{cases} \mathcal{U}^{\delta+}(x_1,x_2,x_3) = \mathcal{U}^{\delta+}(\varepsilon p,\varepsilon q,x_3) \quad \text{if} \quad (x_1,x_2)\in \Big]\varepsilon p-\dfrac{\varepsilon}{2},\varepsilon p+\dfrac{\varepsilon}{2}\Big[\times\Big]\varepsilon q-\dfrac{\varepsilon}{2},\varepsilon q+\dfrac{\varepsilon}{2}\Big[\\ \mathcal{R}^{\delta+}(x_1,x_2,x_3) = \mathcal{R}^{\delta+}(\varepsilon p,\varepsilon q,x_3) \quad \text{if} \quad (x_1,x_2)\in \Big]\varepsilon p-\dfrac{\varepsilon}{2},\varepsilon p+\dfrac{\varepsilon}{2}\Big[\times\Big]\varepsilon q-\dfrac{\varepsilon}{2},\varepsilon q+\dfrac{\varepsilon}{2}\Big[\\ \mathcal{U}^{\delta+}(x_1,x_2,x_3) = \mathcal{R}^{\delta+}(x_1,x_2,x_3) = 0 \quad \text{if} \quad (x_1,x_2)\in\omega\setminus\widetilde{\omega}_\varepsilon. \end{cases}
$$

In the plate, we use the decomposition introduced in [19]

$$
(3.3)\qquad \begin{cases} u^\delta(x) = \mathcal{U}^{\delta-}(x_1,x_2)+\mathcal{R}^{\delta-}(x_1,x_2)\wedge\begin{pmatrix}0\\0\\x_3+\delta/2\end{pmatrix}+\overline{u}^{\delta-}(x) \qquad x\in\Omega^-_\delta,\\ \qquad = U_e^{\delta-}(x)+\overline{u}^{\delta-}(x) \end{cases}
$$

where the field $\overline{u}^{\delta-}$ satisfies

$$
(3.4)\qquad \int_{-\delta}^0 \overline{u}^{\delta-}(x_1,x_2,x_3)dx_3 = 0 \text{ and } \int_{-\delta}^0\Big(x_3+\frac{\delta}{2}\Big)\overline{u}^{\delta-}_\alpha(x_1,x_2,x_3)dx_3 = 0 \quad \text{for} \quad \alpha=1,2,
$$

for almost any (x_1,x_2) in ω. Remark that the boundary condition (2.3) implies that $\mathcal{U}^{\delta-}$ and $\mathcal{R}^{\delta-}$ belong to $H^1_0(\omega)$.

3.2 Estimates in term of the elastic energy

Firstly, as a consequence of [19], by setting $\mathcal{U}^{\delta-}_m = \mathcal{U}^{\delta-}_1\mathbf{e}_1+\mathcal{U}^{\delta-}_2\mathbf{e}_2$ we have

$$
(3.5)\qquad \begin{cases} \delta\Big\|\dfrac{\partial\mathcal{R}^{\delta-}}{\partial x_\alpha}\Big\|_{(L^2(\omega))^2}+\Big\|\dfrac{\partial\mathcal{U}^{\delta-}_3}{\partial x_1}+\mathcal{R}^{\delta-}_2\Big\|_{L^2(\omega)}\\ \quad +\Big\|\dfrac{\partial\mathcal{U}^{\delta-}_3}{\partial x_2}-\mathcal{R}^{\delta-}_1\Big\|_{L^2(\omega)}+\big\|\gamma_{\alpha\beta}(\mathcal{U}^{\delta-}_m)\big\|_{L^2(\omega)}\le C\dfrac{|u^\delta|_{\mathcal{E}}}{\sqrt{\delta}},\\ ||\overline{u}^{\delta-}||_{(L^2(\Omega^-_\delta))^3}\le C\delta|u^\delta|_{\mathcal{E}}, \qquad ||\nabla\overline{u}^{\delta-}||_{(L^2(\Omega^-_\delta))^9}\le C|u^\delta|_{\mathcal{E}}. \end{cases}
$$

It follows that

$$
(3.6)\qquad \begin{cases} \delta||\mathcal{R}^{\delta-}_\alpha||_{H^1(\omega)}+\delta||\mathcal{U}^{\delta-}_3||_{H^1(\omega)}+||\mathcal{U}^{\delta-}_\alpha||_{H^1(\omega)}\le C\dfrac{|u^\delta|_{\mathcal{E}}}{\sqrt{\delta}},\\ ||u^\delta_\alpha||_{L^2(\Omega^-_\delta)}\le C|u^\delta|_{\mathcal{E}}, \qquad ||u^\delta_3||_{L^2(\Omega^-_\delta)}\le\dfrac{C}{\delta}|u^\delta|_{\mathcal{E}}. \end{cases}
$$

Secondly, using Lemma 3.1 of [18] and proceeding as Section 4 of [2] and since $r=k\varepsilon$, we obtain the following estimates:

$$
(3.7)\qquad \begin{cases} \Big\|\dfrac{\partial\mathcal{R}^{\delta+}}{\partial x_3}\Big\|_{(L^2(\omega\times]-\delta,L[))^2}\le C\dfrac{|u^\delta|_{\mathcal{E}}}{\varepsilon}, \qquad \Big\|\dfrac{\partial\mathcal{U}^{\delta+}}{\partial x_3}-\mathcal{R}^{\delta+}\wedge\mathbf{e}_3\Big\|_{(L^2(\omega\times]-\delta,L[))^3}\le C|u^\delta|_{\mathcal{E}},\\ ||\overline{u}^{\delta+}||_{(L^2(\Omega^+_{\varepsilon,\delta}))^3}\le C\varepsilon|u^\delta|_{\mathcal{E}}, \qquad ||\nabla\overline{u}^{\delta+}||_{(L^2(\Omega^+_{\varepsilon,\delta}))^9}\le C|u^\delta|_{\mathcal{E}}. \end{cases}
$$

At last, the estimates of $\mathcal{U}^{\delta+}(x_1,x_2,0)$ and of $\mathcal{R}^{\delta+}(x_1,x_2,0)$ are given in Section 4.1 of [2] (using $r=k\varepsilon$ and δ/ε bounded)

$$
(3.8)\qquad \begin{cases} ||\mathcal{U}^{\delta+}_\alpha(.,.,0)||_{L^2(\omega)}\le C\dfrac{|u^\delta|_{\mathcal{E}}}{\sqrt{\delta}}, \qquad ||\mathcal{U}^{\delta+}_3(.,.,0)||_{L^2(\omega)}\le\dfrac{C}{\delta}\dfrac{|u^\delta|_{\mathcal{E}}}{\sqrt{\delta}},\\ ||\mathcal{R}^{\delta+}_\alpha(.,.,0)||_{L^2(\omega)}\le\dfrac{C}{\delta}\dfrac{|u^\delta|_{\mathcal{E}}}{\sqrt{\delta}}, \qquad ||\mathcal{R}^{\delta+}_3(.,.,0)||_{L^2(\omega)}\le C\dfrac{|u^\delta|_{\mathcal{E}}}{\sqrt{\delta}}. \end{cases}
$$

From (3.7) and (3.8) we deduce that (if needed the reader is referred to [2])

$$
(3.9)\qquad \begin{cases}
\displaystyle ||\mathcal{U}_3^{\delta+}||_{L^2(\omega\times]-\delta,L[)} \le \frac{C}{\delta}\frac{|u^\delta|_\mathcal{E}}{\sqrt{\delta}}, & \displaystyle \Big\|\frac{\partial \mathcal{U}_3^{\delta+}}{\partial x_3}\Big\|_{L^2(\omega\times]-\delta,L[)} \le C|u^\delta|_\mathcal{E},\\
\displaystyle ||\mathcal{R}_\alpha^{\delta+}||_{L^2(\omega\times]-\delta,L[)} \le \frac{C}{\delta}\frac{|u^\delta|_\mathcal{E}}{\sqrt{\delta}}, & \displaystyle \Big\|\frac{\partial \mathcal{R}_\alpha^{\delta+}}{\partial x_3}\Big\|_{L^2(\omega\times]-\delta,L[)} \le C\frac{|u^\delta|_\mathcal{E}}{\varepsilon},\\
\displaystyle ||\mathcal{R}_3^{\delta+}||_{L^2(\omega\times]-\delta,L[)} \le C\Big\{\frac{1}{\sqrt{\delta}}+\frac{1}{\varepsilon}\Big\}|u^\delta|_\mathcal{E}, & \displaystyle \Big\|\frac{\partial \mathcal{R}_3^{\delta+}}{\partial x_3}\Big\|_{L^2(\omega\times]-\delta,L[)} \le C\frac{|u^\delta|_\mathcal{E}}{\varepsilon},\\
\displaystyle ||\mathcal{U}_\alpha^{\delta+}||_{L^2(\omega\times]-\delta,L[)} \le \frac{C}{\delta}\frac{|u^\delta|_\mathcal{E}}{\sqrt{\delta}}, & \displaystyle \Big\|\frac{\partial \mathcal{U}_\alpha^{\delta+}}{\partial x_3}\Big\|_{L^2(\omega\times]-\delta,L[)} \le \frac{C}{\delta}\frac{|u^\delta|_\mathcal{E}}{\sqrt{\delta}}.
\end{cases}
$$

From the estimates (3.7), (3.9) and the decomposition (3.1), it follows that

$$
(3.10)\qquad ||u^\delta||_{(L^2(\Omega_\varepsilon^+))^3} \le \frac{C}{\delta}\frac{|u^\delta|_\mathcal{E}}{\sqrt{\delta}}.
$$

3.3 Assumption on the forces

In view of the energy relation (2.6), estimates (3.6) and (3.10), we assume throughout the paper

$$
(3.11)\qquad \begin{cases}
f_i^\delta = \delta^2 f_i \ \text{in}\ \Omega_\varepsilon^+, \ \text{for}\ i=1,2,3,\\
f_\alpha^\delta(x) = f_\alpha\big(x_1,x_2,\dfrac{x_3}{\delta}\big) \ \text{in}\ \Omega_\delta^-, \ \text{for}\ \alpha=1,2,\\
f_3^\delta(x) = \delta f_3\big(x_1,x_2,\dfrac{x_3}{\delta}\big) \ \text{in}\ \Omega_\delta^-,
\end{cases}
$$

where $f \in \big(L^2(\Omega)\big)^3$ is given. As a consequence, we obtain the following bound on the energy

$$
(3.12)\qquad |u^\delta|_\mathcal{E} \le C\sqrt{\delta},
$$

where C is a constant independent of ε and δ.

Remark 3.1. Actually and due to the analysis developped in Section 4.3 of [2] the order specified in (3.11) for the applied forces permits to obtain estimate (3.12) as soon as $\dfrac{\delta^3}{\varepsilon^2}$ remains bounded.

3.4 Estimates in the domain $\mathcal{C}^{\varepsilon,\delta}$

In this subsection we show that in each small cylinder of $\mathcal{C}^{\varepsilon,\delta}$ the displacement u^δ is asymptotically a rigid displacement. Due to (3.7) we have

$$
(3.13)\qquad \begin{cases}
||\mathcal{R}^{\delta+} - \mathcal{R}^{\delta+}(.,.,0)||_{(L^2(\omega\times]-\delta,0[))^3} \le C\dfrac{\delta^{3/2}}{\varepsilon},\\
||\mathcal{U}^{\delta+} - \mathcal{U}^{\delta+}(.,.,0) - x_3\mathcal{R}^{\delta+}(.,.,0)\wedge \mathbf{e}_3||_{(L^2(\omega\times]-\delta,0[))^3} \le C\delta^{3/2}.
\end{cases}
$$

Then let us define in each small cylinder $\mathcal{D}_{p,q}^\varepsilon\times]-\delta,0[$ the rigid body displacement

$$
\mathbf{R}^\delta(x_1,x_2,x_3) = \mathcal{U}^{\delta+}(p\varepsilon,q\varepsilon,0) + \mathcal{R}^{\delta+}(p\varepsilon,q\varepsilon,0)\wedge\big((x_1-\varepsilon p)\mathbf{e}_1 + (x_2-\varepsilon q)\mathbf{e}_2 + x_3\mathbf{e}_3\big).
$$

In view of the definitions (3.2) of $\mathcal{U}^{\delta+}$ and $\mathcal{R}^{\delta+}$ and thanks to (3.13) we obtain

$$
(3.14)\qquad ||U_e^{\delta+} - \mathbf{R}^\delta||^2_{(L^2(\mathcal{C}^{\varepsilon,\delta}))^3} \le C\delta^3, \qquad ||\nabla(U_e^{\delta+} - \mathbf{R}^\delta)||^2_{(L^2(\mathcal{C}^{\varepsilon,\delta}))^9} \le C\delta.
$$

From the estimates (3.7) of $\overline{u}^{\delta+}$ we deduce that (using the fact that each cylinder of $\mathcal{C}^{\varepsilon,\delta}$ has a height equal to δ)

$$||\overline{u}^{\delta+}||^2_{(L^2(\mathcal{C}^{\varepsilon,\delta}))^3} \le C\varepsilon\delta^2, \qquad ||\nabla\overline{u}^{\delta+}||^2_{(L^2(\mathcal{C}^{\varepsilon,\delta}))^9} \le C\delta.$$

Then from (3.14) and the above estimates we obtain

$$||u^\delta - \mathbf{R}^\delta||^2_{(L^2(\mathcal{C}^{\varepsilon,\delta}))^3} \le C\varepsilon\delta^2, \qquad ||\nabla(u^\delta - \mathbf{R}^\delta)||^2_{(L^2(\mathcal{C}^{\varepsilon,\delta}))^9} \le C\delta. \tag{3.15}$$

Indeed using the decomposition (3.3) and estimates (3.5) of $\overline{u}^{\delta-}$ leads to

$$||U_e^{\delta-} - \mathbf{R}^\delta||^2_{(L^2(\mathcal{C}^{\varepsilon,\delta}))^3} \le C\varepsilon\delta^2, \qquad ||\nabla(U_e^{\delta-} - \mathbf{R}^\delta)||^2_{(L^2(\mathcal{C}^{\varepsilon,\delta}))^9} \le C\delta. \tag{3.16}$$

4. Rescaling of Ω_δ^- and unfolding operators in Ω_ε^+ and Ω^-

We denote by D the unit disk of $\mathbb{R}^2$ and by Y the unit cell $(]-1/2,1/2[)^2$. We first recall the definition of the unfolding operator $\mathcal{T}^\varepsilon$ given in Section 5 of [1] which is defined for any $v \in L^2(\Omega_\varepsilon^+)$ by, for almost $(x_1,x_2,x_3) \in \Omega^+$ and $(X_1,X_2) \in D$,

$$\mathcal{T}^\varepsilon(v)(x_1,x_2,x_3,X_1,X_2) = \begin{cases} v(p\varepsilon + r_\varepsilon X_1, q\varepsilon + r_\varepsilon X_2, x_3), \\ \ \text{if } (x_1,x_2,x_3) \in \left]\varepsilon p - \dfrac{\varepsilon}{2}, \varepsilon p + \dfrac{\varepsilon}{2}\right[\times \left]\varepsilon q - \dfrac{\varepsilon}{2}, \varepsilon q + \dfrac{\varepsilon}{2}\right[\times]0,L[, \ \text{ and } (p,q) \in \mathcal{N}_\varepsilon, \\ 0, \quad \text{if } (x_1,x_2,x_3) \in \big(\omega \setminus \widetilde{\omega}_\varepsilon\big) \times]0,L[. \end{cases}$$

The above definition of the operator $\mathcal{T}^\varepsilon$ is an adaptation to the specific geometry considered here of the notion of unfolding operators introduced in [9] and [10] (see also the definition of the operator $\mathcal{T}_\varepsilon$ below). We refer to Lemma 5.1 of [1] for the properties of this operator. Then, in order to take into account the necessary rescaling of Ω_δ^-, we introduce the following operator Π_δ defined for any function $v \in L^2(\Omega_\delta^-)$

$$\Pi_\delta(v)(x_1,x_2,X_3) = v(x_1,x_2,\delta X_3) \quad \text{ for } (x_1,x_2,X_3) \in \Omega^- = \omega\times]-1,0[.$$

Remark that $\Pi_\delta(v) \in L^2(\Omega^-)$. Indeed we have for any $v \in L^2(\Omega_\delta^-)$ and any $w \in L^2(\Omega_\delta^-)$

$$\int_{\Omega^-} \Pi_\delta(v)\Pi_\delta(w)dx_1dx_2dX_3 = \frac{1}{\delta}\int_{\Omega_\delta^-} vwdx_1dx_2dx_3, \tag{4.1}$$

$$\frac{\partial \Pi_\delta(v)}{\partial x_\alpha} = \Pi_\delta\left(\frac{\partial v}{\partial x_\alpha}\right), \qquad \text{for } \alpha = 1,2, \tag{4.2}$$

$$\frac{\partial \Pi_\delta(v)}{\partial X_3} = \delta\Pi_\delta\left(\frac{\partial v}{\partial x_3}\right). \tag{4.3}$$

At last since we will use a few oscillating test functions in Ω^- in Section 6, we also introduce the usual unfolding operator in homogenization theory (see [9] and [10]). The operator $\mathcal{T}_\varepsilon$ is defined for any $v \in L^2(\Omega^-)$ by, for almost $(x_1,x_2,X_3) \in \Omega^-$ and $(X_1,X_2) \in Y$

$$\mathcal{T}_\varepsilon(v)(x_1,x_2,X_3,X_1,X_2) = \begin{cases} v(p\varepsilon + \varepsilon X_1, q\varepsilon + \varepsilon X_2, X_3), \\ \ \text{if } (x_1,x_2,X_3) \in \left]\varepsilon p - \dfrac{\varepsilon}{2}, \varepsilon p + \dfrac{\varepsilon}{2}\right[\times \left]\varepsilon q - \dfrac{\varepsilon}{2}, \varepsilon q + \dfrac{\varepsilon}{2}\right[\times]-1,0[\ \text{ and } (p,q) \in \mathcal{N}_\varepsilon, \\ 0 \quad \text{if } (x_1,x_2,X_3) \in (\omega \setminus \widetilde{\omega}_\varepsilon)\times]-1,0[. \end{cases}$$

Indeed $\mathcal{T}_\varepsilon(v)$ belongs to $L^2(\Omega^- \times Y)$.

The main properties of $\mathcal{T}_\varepsilon$ that we will use in this paper are recalled in Appendix A and Appendix B of [2], and we refer to [9] and [10] for the proofs and various applications in homogenization.

Through application of the operators $\mathcal{T}^\varepsilon$, $\mathcal{T}_\varepsilon$ and Π_δ, the weak formulation (2.5) gives that for any $v \in V_{\varepsilon,\delta}$ such that $v = 0$ in $(\omega \setminus \widetilde{\omega}_\varepsilon)\times]-\delta, 0[$ (after deviding (2.5) by δ)

$$
\left\{
\begin{aligned}
&\frac{k^2}{\delta}\int_{\Omega^+\times D} \mathcal{T}^\varepsilon(\sigma^\delta_{ij})\mathcal{T}^\varepsilon(\gamma_{ij}(v))dx_1dx_2dx_3dX_1dX_2\\
&+\int_{\Omega^-\times Y} \mathcal{T}_\varepsilon\circ\Pi_\delta(\sigma^\delta_{ij})\mathcal{T}_\varepsilon\circ\Pi_\delta(\gamma_{ij}(v))dx_1dx_2dX_3dX_1dX_2\\
&=k^2\delta\int_{\Omega^+\times D}\mathcal{T}^\varepsilon(f)\cdot\mathcal{T}^\varepsilon(v)dx_1dx_2dx_3dX_1dX_2\\
&+\int_{\Omega^-}\Pi_\delta(f_\alpha)\Pi_\delta(v_\alpha)dx_1dx_2dX_3+\delta\int_{\Omega^-}\Pi_\delta(f_3)\Pi_\delta(v_3)dx_1dx_2dX_3.
\end{aligned}
\right.
\tag{4.4}
$$

Actually (4.4) holds true for any $v \in V_{\varepsilon,\delta}$ such that $v = 0$ in $(\omega \setminus \widetilde{\omega}_\varepsilon)\times]-\delta, 0[$ because for such a v which is zero on a neightborhood of the lateral surface Γ_δ of the plate, the definition of $\mathcal{T}_\varepsilon$ shows that

$$
\frac{1}{\delta}\int_{\Omega^-_\delta}\sum_{i,j=1}^{3}\sigma^\delta_{ij}\gamma_{ij}(v) = \int_{\Omega^-\times Y}\mathcal{T}_\varepsilon\circ\Pi_\delta(\sigma^\delta_{ij})\mathcal{T}_\varepsilon\circ\Pi_\delta(\gamma_{ij}(v))dx_1dx_2dX_3dX_1dX_2.
$$

5. Weak convergences of the displacement field and of the strain field in the rods

In this section, we introduce the weak limits of the terms of the displacement's decomposition in the rods. In particular we show that (upon the correct scaling) the limit torsion in the rods is independent of x_3. At last, we prove that the limit strain in the rods is null.

5.1 Weak convergences of the displacement field in Ω^+

In view of estimates (3.9), (3.10) and (3.12), there exist subsequences (still indexed by δ) such that

$$
\left\{
\begin{aligned}
&\delta\mathcal{T}^\varepsilon(u^\delta_i) \rightharpoonup u^+_i && \text{weakly in} \quad L^2(\Omega^+\times D),\\
&\delta\mathcal{U}^{\delta+} \rightharpoonup \mathcal{U}^+ && \text{weakly in} \quad L^2(\omega; H^1(0,L;\mathbb{R}^3)),\\
&\delta\mathcal{R}^{\delta+}_\alpha \rightharpoonup \mathcal{R}^+_\alpha && \text{weakly in} \quad L^2(\omega; H^1(0,L)).
\end{aligned}
\right.
\tag{5.1}
$$

According to (3.9) the functions $\mathcal{U}^+_3$, $\mathcal{R}^+_1$ and $\mathcal{R}^+_2$ do not depend on x_3. Then by (3.7)

$$
\frac{\partial\mathcal{U}^+}{\partial x_3} - \mathcal{R}^+\wedge\mathbf{e}_3 = 0,
$$

which together with (3.8) leads to

$$
\mathcal{U}^+(x_1,x_2,x_3) = x_3\mathcal{R}^+_2(x_1,x_2)\,\mathbf{e}_1 - x_3\mathcal{R}^+_1(x_1,x_2)\,\mathbf{e}_2 + \mathcal{U}^+_3(x_1,x_2)\,\mathbf{e}_3.
\tag{5.2}
$$

Moreover due to the estimate (3.7) on $\overline{u}^{\delta+}$ and to (3.9) and (5.1) we have

$$
u^+ = \mathcal{U}^+.
\tag{5.3}
$$

As far as $\mathcal{R}_3^{\delta+}$ is concerned, we first consider the case where $\dfrac{\sqrt{\delta}}{\varepsilon}$ is bounded. In this case estimate (3.9) shows that

$$\mathcal{R}_3^{\delta+} \rightharpoonup \mathcal{R}_3^{+} \qquad \text{weakly in} \quad L^2(\omega; H^1(0,L)).$$

More precisely, if $\dfrac{\sqrt{\delta}}{\varepsilon} \to 0$ then $\mathcal{R}_3^{+}$ does not depend on x_3 and belongs to $L^2(\omega)$. If $\dfrac{\sqrt{\delta}}{\varepsilon} \to c$, $c \in \mathbb{R}_+^*$, we will show in Section 6 that again the function $\mathcal{R}_3^{+}$ is independent of x_3.
In the case where $\dfrac{\sqrt{\delta}}{\varepsilon} \to +\infty$ we have from the same estimates

$$\frac{\varepsilon}{\sqrt{\delta}}\mathcal{R}_3^{\delta+} \rightharpoonup \mathcal{R}_3^{+} \qquad \text{weakly in} \quad L^2(\omega; H^1(0,L)) \tag{5.4}$$

and from (3.8) we get $\mathcal{R}_3^{+}(.,.,0) = 0$. We will show in Section 6 that the function $\mathcal{R}_3^{+}$ is equal to zero.

In order to analyse the asymptotic behavior of the strain tensor $\gamma(u^\delta)$ in Ω_ε^+ in the next subsection, we first introduce the following quantities

$$\begin{cases} \mathcal{R}^{'\delta+} = \mathcal{R}^{\delta+} - \mathcal{R}^{\delta+}(.,.,0) \\ \mathcal{U}^{'\delta+} = \mathcal{U}^{\delta+} - \mathcal{U}^{\delta+}(.,.,0) - x_3\, \mathcal{R}^{\delta+}(.,.,0) \wedge \mathbf{e}_3 \end{cases} \qquad \text{in} \quad \Omega^+. \tag{5.5}$$

Due to estimates (3.7) and (3.12) we obtain

$$\begin{cases} \|\mathcal{R}^{'\delta+}\|_{L^2(\omega;H^1(0,L;\mathbb{R}^3))} \le C\dfrac{\sqrt{\delta}}{\varepsilon}, \qquad \Big\|\dfrac{\partial \mathcal{U}^{'\delta+}}{\partial x_3} - \mathcal{R}^{'\delta+} \wedge \mathbf{e}_3\Big\|_{(L^2(\omega\times]0,L[))^3} \le C\sqrt{\delta}, \\ \|\mathcal{U}_\alpha^{'\delta+}\|_{L^2(\omega;H^1(0,L))} \le C\dfrac{\sqrt{\delta}}{\varepsilon}, \qquad \|\mathcal{U}_3^{'\delta+}\|_{L^2(\omega;H^1(0,L))} \le C\sqrt{\delta}. \end{cases} \tag{5.6}$$

The constants do not depend on ε and δ.

We prove the following lemma:

Lemma 5.1 : *The following weak concergences hold true (up to subsequences still indexed by δ):*

$$\begin{cases} \dfrac{\varepsilon}{\sqrt{\delta}}\mathcal{R}_i^{'\delta+} \rightharpoonup \mathcal{R}_i^{'+} & \text{weakly in} \quad L^2(\omega; H^1(0,L)), \\ \dfrac{\varepsilon}{\sqrt{\delta}}\mathcal{U}_\alpha^{'\delta+} \rightharpoonup \mathcal{U}_\alpha^{'+} & \text{weakly in} \quad L^2(\omega; H^1(0,L)), \\ \dfrac{1}{\sqrt{\delta}}\mathcal{U}_3^{'\delta+} \rightharpoonup \mathcal{U}_3^{'+} & \text{weakly in} \quad L^2(\omega; H^1(0,L)), \\ \dfrac{1}{\sqrt{\delta}}\Big(\dfrac{\partial \mathcal{U}_1^{'\delta+}}{\partial x_3} - \mathcal{R}_2^{'\delta+}\Big) \rightharpoonup \widehat{\mathcal{Z}}_1^{+} & \text{weakly in} \quad L^2(\Omega^+), \\ \dfrac{1}{\sqrt{\delta}}\Big(\dfrac{\partial \mathcal{U}_2^{'\delta+}}{\partial x_1} + \mathcal{R}_1^{'\delta+}\Big) \rightharpoonup \widehat{\mathcal{Z}}_2^{+} & \text{weakly in} \quad L^2(\Omega^+), \\ \dfrac{1}{\varepsilon\sqrt{\delta}}\mathcal{T}^\varepsilon(\overline{u}^{\delta+}) \rightharpoonup \overline{u}^{+} & \text{weakly in} \quad L^2(\Omega^+; H^1(D;\mathbb{R}^3)), \\ \dfrac{1}{\sqrt{\delta}}\mathcal{T}^\varepsilon\Big(\dfrac{\partial \overline{u}^{\delta+}}{\partial x_3}\Big) \rightharpoonup 0 & \text{weakly in} \quad L^2(\Omega^+ \times D;\mathbb{R}^3), \end{cases} \tag{5.7}$$

where $\mathcal{U}^{'+} \in L^2(\omega; H^1(0,L;\mathbb{R}^3))$, $\widehat{\mathcal{Z}}_\alpha^{+} \in L^2(\Omega^+ \times D)$ and $\overline{u}^{+} \in L^2(\Omega^+; H^1(D;\mathbb{R}^3))$.

Proof of lemma 5.1 : Convergences (5.7) mainly follow from the properties of the operator $\mathcal{T}^\varepsilon$ together with (5.6). □

Convergences (5.7) imply that

$$\frac{\varepsilon}{\sqrt{\delta}}\left(\frac{\partial \mathcal{U}_1^{'\delta+}}{\partial x_3}-\mathcal{R}_2^{'\delta+}\right)\to 0 \text{ strongly in } L^2(\Omega^+),$$
$$\frac{\varepsilon}{\sqrt{\delta}}\left(\frac{\partial \mathcal{U}_2^{'\delta+}}{\partial x_3}+\mathcal{R}_1^{'\delta+}\right)\to 0 \text{ strongly in } L^2(\Omega^+),$$

as δ tends to 0, from which we deduce that

$$\frac{\partial \mathcal{U}_1^{'+}}{\partial x_3}=\mathcal{R}_2^{'+}\qquad \frac{\partial \mathcal{U}_2^{'+}}{\partial x_3}=-\mathcal{R}_1^{'+} \text{ in } \quad \Omega^+. \tag{5.8}$$

It follows that $\mathcal{U}_\alpha^{'+}\in L^2\big(\omega,H^2(0,L)\big)$, for $\alpha=1,2$. Moreover due to the definition (5.5) of $\mathcal{R}^{'\delta+}$ and $\mathcal{U}^{'\delta+}$ and to (5.8), we have

$$\mathcal{R}^{'+}(.,.,0)=\mathcal{U}^{'+}(.,.,0)=0\qquad \frac{\partial \mathcal{U}_1^{'+}}{\partial x_3}=\frac{\partial \mathcal{U}_2^{'+}}{\partial x_3}=0 \quad \text{a.e. in} \quad \omega. \tag{5.9}$$

5.2 Weak convergences of the strain field in Ω^+

Although we have assumed throughout the paper that $\dfrac{\delta}{\varepsilon}\to 0$, we show now that the weak limit of the strain in Ω^+ is null as soon as the ratio $\dfrac{\delta^{3/2}}{\varepsilon}$ tends to zero (see also remark 3.1). This is the reason why all the estimates and convergences derived in this section involve the two parameters ε and δ. In view of the expression of the strain tensor $\gamma(u^\delta)$ and of Lemma 5.1, we obtain the following weak convergences in $L^2(\Omega^+\times D)$:

$$\begin{cases}\dfrac{1}{\sqrt{\delta}}\mathcal{T}^\varepsilon\big(\gamma_{\alpha\beta}(u^\delta)\big)\rightharpoonup \dfrac{1}{2k}\Big(\dfrac{\partial \overline{u}_\alpha^+}{\partial X_\beta}+\dfrac{\partial \overline{u}_\beta^+}{\partial X_\alpha}\Big),\\ \dfrac{1}{\sqrt{\delta}}\mathcal{T}^\varepsilon\big(\gamma_{13}(u^\delta)\big)\rightharpoonup \dfrac{1}{2}\Big\{\widehat{\mathcal{Z}}_1^+ + kX_2\dfrac{\partial \mathcal{R}_3^{'+}}{\partial x_3}+\dfrac{1}{k}\dfrac{\partial \overline{u}_3^+}{\partial X_1}\Big\},\\ \dfrac{1}{\sqrt{\delta}}\mathcal{T}^\varepsilon\big(\gamma_{23}(u^\delta)\big)\rightharpoonup \dfrac{1}{2}\Big\{\widehat{\mathcal{Z}}_2^+ + kX_1\dfrac{\partial \mathcal{R}_3^{'+}}{\partial x_3}+\dfrac{1}{k}\dfrac{\partial \overline{u}_3^+}{\partial X_2}\Big\},\\ \dfrac{1}{\sqrt{\delta}}\mathcal{T}^\varepsilon\big(\gamma_{33}(u^\delta)\big)\rightharpoonup \dfrac{\partial \mathcal{U}_3^{'+}}{\partial x_3}-kX_1\dfrac{\partial^2 \mathcal{U}_1^{'+}}{\partial x_3^2}-kX_2\dfrac{\partial^2 \mathcal{U}_2^{'+}}{\partial x_3^2}.\end{cases} \tag{5.10}$$

We denote by Σ^+ the weak limit of the unfold stress $\dfrac{1}{\sqrt{\delta}}\mathcal{T}^\varepsilon(\sigma^\delta)$

$$\frac{1}{\sqrt{\delta}}\mathcal{T}^\varepsilon(\sigma^\delta_{ij})\rightharpoonup \Sigma^+_{ij}\quad \text{weakly in} \quad L^2(\Omega^+\times D). \tag{5.11}$$

Proceeding exactly as in Section 6.1 of [1] and Section 8.1 of [2], we first derive $\overline{u}^+$ and this gives

$$\begin{cases}\overline{u}_1{}^+=\nu\left\{-kX_1\dfrac{\partial \mathcal{U}_3^{'+}}{\partial x_3}+k^2\dfrac{X_1^2-X_2^2}{2}\dfrac{\partial^2 \mathcal{U}_1^{'+}}{\partial x_3^2}+k^2X_1X_2\dfrac{\partial^2 \mathcal{U}_2^{'+}}{\partial x_3^2}\right\},\\ \overline{u}_2{}^+=\nu\left\{-kX_2\dfrac{\partial \mathcal{U}_3^{'+}}{\partial x_3}+k^2X_1X_2\dfrac{\partial^2 \mathcal{U}_1^{'+}}{\partial x_3^2}+k^2\dfrac{X_2^2-X_1^2}{2}\dfrac{\partial^2 \mathcal{U}_2^{'+}}{\partial x_3^2}\right\},\end{cases} \tag{5.12}$$

where $\nu = \dfrac{\lambda}{2(\lambda+\mu)}$ is the Poisson's coefficient of the material. Similarly, the same computations as in Section 6.1 of [1] leads to $\widehat{\mathcal{Z}}_\alpha^+ = 0$ and $\overline{u}_3{}^+ = 0$. As a consequence (5.10), (5.11) and (5.12) we obtain

$$
\begin{cases}
\Sigma_{11}^+ = \Sigma_{22}^+ = \Sigma_{12}^+ = 0, \\
\Sigma_{13}^+ = -\mu k X_2 \dfrac{\partial \mathcal{R}_3^{'+}}{\partial x_3}, \qquad \Sigma_{23}^+ = \mu k X_1 \dfrac{\partial \mathcal{R}_3^{'+}}{\partial x_3}, \\
\Sigma_{33}^+ = E\left(\dfrac{\partial \mathcal{U}_3^{'+}}{\partial x_3} - kX_1 \dfrac{\partial^2 \mathcal{U}_1^{'+}}{\partial x_3^2} - kX_2 \dfrac{\partial^2 \mathcal{U}_2^{'+}}{\partial x_3^2} \right).
\end{cases}
\tag{5.13}
$$

where $E = \dfrac{\mu(3\lambda+2\mu)}{\lambda+\mu}$ is the Young's modulus of the elastic material.

In order to determine $\mathcal{R}_3^{'}$ and $\mathcal{U}^{'}$ let us now introduce the test displacement that we use in (4.4). Let $\varphi \in C_0^\infty(\omega)$, $(\mathcal{V}_1, \mathcal{V}_2)$ be in $C^\infty([0,L])$ such that $\mathcal{V}_1(0) = \mathcal{V}_2(0) = \mathcal{V}_1'(0) = \mathcal{V}_2'(0) = 0$, $\mathcal{V}_3$, $\mathcal{A}_3$ be in $C^\infty([0,L])$ such that $\mathcal{A}_3(0) = \mathcal{V}_3(0) = 0$.

The test displacement is defined in Ω_ε^+ by

$$
\begin{aligned}
v^\delta(x_1,x_2,x_3) =& \sqrt{\delta}\, \varphi(\varepsilon p, \varepsilon q) \Bigg[\left(\frac{1}{\varepsilon} \mathcal{V}_1(x_3) - \frac{x_2 - \varepsilon q}{\varepsilon} \mathcal{A}_3(x_3) \right) \mathbf{e}_1 \\
&+ \left(\frac{1}{\varepsilon} \mathcal{V}_2(x_3) + \frac{x_1 - \varepsilon p}{\varepsilon} \mathcal{A}_3(x_3) \right) \mathbf{e}_2 + \left(\mathcal{V}_3 - \frac{x_1 - \varepsilon p}{\varepsilon} \mathcal{V}_1'(x_3) - \frac{x_2 - \varepsilon q}{\varepsilon} \mathcal{V}_2'(x_3) \right) \mathbf{e}_3 \Bigg],
\end{aligned}
$$

if $(x_1,x_2) \in \mathcal{D}_{p,q}^\varepsilon$, $x_3 \in]0,L[$, for $(p,q) \in \mathcal{N}_\varepsilon$, and $v^\delta = 0$ in Ω_δ^-. Remark that the boundary conditions on $\mathcal{V}_1$, $\mathcal{V}_2$, $\mathcal{V}_3$ and $\mathcal{A}_3$ at $x_3 = 0$ imply that $v^\delta \in V_{\varepsilon,\delta}$.

Then in Ω_ε^+ we have

$$
\begin{aligned}
\gamma_{11}(v^\delta) &= \gamma_{22}(v^\delta) = \gamma_{12}(v^\delta) = 0, \\
\gamma_{13}(v^\delta) &= -\sqrt{\delta}\, \varphi(\varepsilon p, \varepsilon q) \frac{x_2 - \varepsilon q}{\varepsilon} \frac{1}{2} \mathcal{A}_3'(x_3), \\
\gamma_{23}(v^\delta) &= \sqrt{\delta}\, \varphi(\varepsilon p, \varepsilon q) \frac{x_1 - \varepsilon p}{\varepsilon} \frac{1}{2} \mathcal{A}_3'(x_3), \\
\gamma_{33}(v^\delta) &= \sqrt{\delta}\, \varphi(\varepsilon p, \varepsilon q) \left(\mathcal{V}_3'(x_3) - \frac{x_1 - \varepsilon p}{\varepsilon} \mathcal{V}_1''(x_3) - \frac{x_2 - \varepsilon q}{\varepsilon} \mathcal{V}_2''(x_3) \right).
\end{aligned}
$$

Since the function φ is smooth, the above expression of the strain of v^δ lead to the following strong convergences in $L^2(\Omega^+ \times D)$:

$$
\begin{aligned}
\frac{1}{\sqrt{\delta}} \mathcal{T}^\varepsilon(\gamma_{13}(v^\delta)) &\to -\varphi X_2 \frac{1}{2} \mathcal{A}_3'(x_3), \\
\frac{1}{\sqrt{\delta}} \mathcal{T}^\varepsilon(\gamma_{23}(v^\delta)) &\to \varphi X_1 \frac{1}{2} \mathcal{A}_3'(x_3), \\
\frac{1}{\sqrt{\delta}} \mathcal{T}^\varepsilon(\gamma_{23}(v^\delta)) &\to \varphi \left[\mathcal{V}_3'(x_3) - X_1 \mathcal{V}_1''(x_3) - X_2 \mathcal{V}_2''(x_3) \right].
\end{aligned}
$$

Passing to the limit in the left hand side of (4.4) with the test displacement v^δ, using (5.11)-(5.13) and the above convergences give the following quantity:

$$
\begin{cases}
-\displaystyle\int_{\Omega^+ \times D} \varphi \Sigma_{13}^+ X_2 \mathcal{A}_3' dx_1 dx_2 dx_3 dX_1 dX_2 + \int_{\Omega^+ \times D} \varphi \Sigma_{23}^+ X_1 \mathcal{A}_3' dx_1 dx_2 dx_3 dX_1 dX_2 \\
+\displaystyle\int_{\Omega^+ \times D} \varphi \Sigma_{33}^+ \left(\mathcal{V}_3'(x_3) - X_1 \mathcal{V}_1''(x_3) - X_2 \mathcal{V}_2''(x_3) \right) dx_1 dx_2 dx_3 dX_1 dX_2.
\end{cases}
\tag{5.14}
$$

To estimate the limit of the right hand side of (4.4), we remark that

$$||v^\delta||_{L^2(\Omega_\varepsilon^+)} \le C\sqrt{\delta}\ ||\phi||_{L^\infty(\omega)}\Big\{\frac{1}{\varepsilon}||\mathcal{V}_1||_{H^1(0,L)} + \frac{1}{\varepsilon}||\mathcal{V}_2||_{H^1(0,L)} + ||\mathcal{V}_3||_{L^2(0,L)} + ||\mathcal{A}_3||_{L^2(0,L)}\Big\}.$$

Hence we obtain

$$k^2\delta\int_{\Omega^+\times D}\mathcal{T}^\varepsilon(f)\cdot\mathcal{T}^\varepsilon(v^\delta)dx_1dx_2dx_3dX_1dX_2 \le C\frac{\delta^{3/2}}{\varepsilon}$$

from which we deduce that as soon as $\dfrac{\delta^{3/2}}{\varepsilon} \to 0$

$$\lim_{\delta\to 0} k^2\delta\int_{\Omega^+\times D}\mathcal{T}^\varepsilon(f)\mathcal{T}^\varepsilon(v^\delta)dx_1dx_2dx_3dX_1dX_2 = 0. \tag{5.15}$$

Using (5.14), (5.15) we obtain

$$\begin{cases} \mu k\displaystyle\int_{\Omega^+\times D}\varphi(X_1^2+X_2^2)\frac{\partial\mathcal{R}_3^{'+}}{\partial x_3}\mathcal{A}_3' dx_1dx_2dx_3dX_1dX_2 \\ +E\displaystyle\int_{\Omega^+\times D}\varphi\left[\frac{\partial\mathcal{U}_3^{'+}}{\partial x_3} - kX_1\frac{\partial^2\mathcal{U}_1^{'+}}{\partial x_3^2} - kX_2\frac{\partial^2\mathcal{U}_2^{'+}}{\partial x_3^2}\right][\mathcal{V}_3' - X_1\mathcal{V}_1'' - X_2\mathcal{V}_2'']\,dx_1dx_2dx_3dX_1dX_2 = 0\end{cases} \tag{5.16}$$

for any $\varphi \in C_0^\infty(\omega)$, $\mathcal{V}_3,\ \mathcal{A}_3 \in C^\infty([0,L])$ such that $\mathcal{V}_3(0) = \mathcal{A}_3(0) = 0$, for $(\mathcal{V}_1,\mathcal{V}_2) \in C^\infty([0,L])$ such that $\mathcal{V}_1(0) = \mathcal{V}_2(0) = \mathcal{V}_1'(0) = \mathcal{V}_2'(0) = 0$. Since (5.16) can be localized with respect to the variable (x_1,x_2) we obtain the standard torsion problem for $\mathcal{R}_3^{'+}$, the standard bending problem for $\mathcal{U}_\alpha^{'+}$ and the standard compression problem for $\mathcal{U}_3^{'+}$ with all applied forces equal to 0 (see e.g. [1]). Taking into account the boundary conditions (5.9) (for $x_3 = 0$), we deduce that $\mathcal{R}_3^{'+} = \mathcal{U}_i^{'+} = 0$ a.e. in Ω^+. It follows from the definition of $\mathcal{R}_3^{'+}$ that the function $\mathcal{R}_3^+$ is actually independent of the variable x_3 also when $\dfrac{\sqrt{\delta}}{\varepsilon} \to c$ ($c \in \mathbb{R}_+^*$) (see (5.4) and (5.5)). In the case where $\dfrac{\sqrt{\delta}}{\varepsilon} \to +\infty$ the function $\mathcal{R}_3^+$ is equal to zero. As a conclusion of this subsection, we obtain that

$$\frac{1}{\sqrt{\delta}}\mathcal{T}^\varepsilon\big(\gamma_{ij}(u^\delta)\big) \rightharpoonup 0 \qquad \text{weakly in} \qquad L^2(\Omega^+\times D). \tag{5.17}$$

6. Weak convergences of the displacement field and of the strain field in the plate

In this section, we first identify the weak limit of the unfold strain in the plate in terms of two macroscopic membrane-flexion displacements and of microscopic correctors. Then, the weak limit of the unfold stress field is expressed as a plate stress tensor of a local Kirchhoff-Love displacement. In the whole paper, we use the following notation for any vector field V smooth enough with respect to the variables (X_1,X_2) (which can depend on others variables)

$$\Gamma_{\alpha\beta}(V) = \frac{1}{2}\Big(\frac{\partial V_\alpha}{\partial X_\beta} + \frac{\partial V_\beta}{\partial X_\alpha}\Big).$$

6.1 Weak convergences of the displacement field in Ω^-

We first prove Lemma 6.1 below.

Lemma 6.1: *The following weak concergences hold true (up to subsequences still indexed by δ) :*

$$
(6.1)\qquad \begin{cases}
\Pi_\delta(u_\alpha^\delta) \rightharpoonup u_\alpha^- \text{ weakly in } H^1(\Omega^-),\\
\delta\Pi_\delta(u_3^\delta) \to u_3^- \text{ strongly in } H^1(\Omega^-),\\
\mathcal{U}_\alpha^{\delta-} \rightharpoonup \mathcal{U}_\alpha^- \quad \text{weakly in} \quad H_0^1(\omega),\\
\mathcal{T}_\varepsilon\big(\mathcal{U}_\alpha^{\delta-}\big) \to \mathcal{U}_\alpha^- \quad \text{strongly in} \quad L^2(\omega\times Y),\\
\mathcal{T}_\varepsilon\Big(\dfrac{\partial \mathcal{U}_\alpha^{\delta-}}{\partial x_\beta}\Big) \rightharpoonup \dfrac{\partial \mathcal{U}_\alpha^-}{\partial x_\beta} + \dfrac{\partial \widehat{\mathbf{u}}_\alpha}{\partial X_\beta} \quad \text{weakly in} \quad L^2(\omega\times Y),\\
\delta\mathcal{U}_3^{\delta-} \to \mathcal{U}_3^- \quad \text{strongly in} \quad H_0^1(\omega),\\
\delta\mathcal{R}_\alpha^{\delta-} \rightharpoonup \mathcal{R}_\alpha^- \quad \text{weakly in} \quad H_0^1(\omega),\\
\delta\mathcal{T}_\varepsilon\big(\mathcal{U}_3^{\delta-}\big) \to \mathcal{U}_3^- \quad \text{strongly in} \quad L^2(\omega\times Y),\\
\delta\mathcal{T}_\varepsilon\big(\mathcal{R}_1^{\delta-}\big) \to \mathcal{R}_1^- = \dfrac{\partial \mathcal{U}_3^-}{\partial x_2} \quad \text{strongly in} \quad L^2(\omega\times Y),\\
\delta\mathcal{T}_\varepsilon\big(\mathcal{R}_2^{\delta-}\big) \to \mathcal{R}_2^- = -\dfrac{\partial \mathcal{U}_3^-}{\partial x_1} \quad \text{strongly in} \quad L^2(\omega\times Y),\\
\delta\mathcal{T}_\varepsilon\Big(\dfrac{\partial \mathcal{R}_1^{\delta-}}{\partial x_\alpha}\Big) \rightharpoonup -\dfrac{\partial^2 \mathcal{U}_3^-}{\partial x_\alpha\partial x_2} - \dfrac{\partial^2 \widehat{\mathbf{u}}_3}{\partial X_\alpha \partial X_2} \quad \text{weakly in} \quad L^2(\omega\times Y),\\
\delta\mathcal{T}_\varepsilon\Big(\dfrac{\partial \mathcal{R}_2^{\delta-}}{\partial x_\alpha}\Big) \rightharpoonup \dfrac{\partial^2 \mathcal{U}_3^-}{\partial x_1 x_\alpha} + \dfrac{\partial^2 \widehat{\mathbf{u}}_3}{\partial X_1 \partial X_\alpha} \quad \text{weakly in} \quad L^2(\omega\times Y),
\end{cases}
$$

and

$$
(6.2)\qquad \begin{cases}
\mathcal{T}_\varepsilon\Big(\dfrac{\partial \mathcal{U}_3^{\delta-}}{\partial x_1} + \mathcal{R}_2^{\delta-}\Big) \rightharpoonup \widehat{\mathcal{Z}}_1^- \quad \text{weakly in} \quad L^2(\omega\times Y),\\
\mathcal{T}_\varepsilon\Big(\dfrac{\partial \mathcal{U}_3^{\delta-}}{\partial x_2} - \mathcal{R}_1^{\delta-}\Big) \rightharpoonup \widehat{\mathcal{Z}}_2^- \quad \text{weakly in} \quad L^2(\omega\times Y),\\
\dfrac{1}{\delta}\mathcal{T}_\varepsilon\circ\Pi_\delta(\overline{u}^{\delta-}) \rightharpoonup \overline{u}^- \quad \text{weakly in} \quad L^2(\omega\times Y; H^1(-1,0;\mathbb{R}^3)),\\
\mathcal{T}_\varepsilon\circ\Pi_\delta\Big(\dfrac{\partial \overline{u}^{\delta-}}{\partial x_\alpha}\Big) \rightharpoonup 0 \quad \text{weakly in} \quad L^2(\Omega^-\times Y;\mathbb{R}^3),
\end{cases}
$$

where $\mathcal{U}_3^- \in H_0^2(\omega)$, $\mathbf{u}\in H_0^1(\omega)$, $\widehat{\mathbf{u}}_\alpha \in L^2(\omega; H^1_{per}(Y))$, $\widehat{\mathbf{u}}_3 \in L^2(\omega; H^2_{per}(Y))$, $\overline{u}^- \in L^2(\omega\times Y; H^1(-1,0;\mathbb{R}^3))$, $\widehat{\mathcal{Z}}_\alpha^- \in L^2(\omega\times Y)$.

Proof of lemma 6.1 : Estimates (6.1) mainly follow from the properties of the operator $\mathcal{T}_\varepsilon$ together with (3.6) and (3.12). Notice that the strong convergence of $\delta\mathcal{U}_3^{\delta-}$ is a direct consequence of estimate (3.5) and of the strong convergence of $\delta\mathcal{R}_\alpha^{\delta-}$ in $L^2(\omega)$. As a consequence of (3.3) and (3.5), we deduce that $\delta\Pi_\delta(u_3^\delta)$ strongly converges in $H^1(\Omega^-)$. We now detail the two last estimates of (6.1).

Due to (3.6) and (3.12) the functions $\delta\mathcal{R}_\alpha^{\delta-}$, $\alpha\in\{1,2\}$, are bounded in $H_0^1(\omega)$. Then there exist two functions $\widehat{\mathbf{r}}_1$, $\widehat{\mathbf{r}}_2 \in L^2(\omega; H^1_{per}(Y))$ (see [10-11]) such that

$$
(6.3)\qquad \begin{cases}
\delta\mathcal{R}_\alpha^{\delta-} \rightharpoonup \mathcal{R}_\alpha^- \quad \text{weakly in} \quad H_0^1(\omega),\\
\delta\mathcal{T}_\varepsilon\Big(\nabla\mathcal{R}_\alpha^{\delta-}\Big) \rightharpoonup \nabla\mathcal{R}_\alpha^- + \nabla_X\widehat{\mathbf{r}}_\alpha \quad \text{weakly in} \quad (L^2(\omega\times Y))^2,
\end{cases}
$$

where ∇_X is the gradient with respect to the variables X_1 and X_2. Let then $\mathbf{U}^\delta \in H_0^1(\omega)$ be the solution of the problem

$$
(6.4)\qquad \begin{cases}
\displaystyle\int_\omega \nabla\mathbf{U}^\delta\nabla\Phi = -\int_\omega \Big[\mathcal{R}_2^{\delta-}\frac{\partial\Phi}{\partial x_1} - \mathcal{R}_1^{\delta-}\frac{\partial\Phi}{\partial x_2}\Big]\\
\forall\Phi\in H_0^1(\omega).
\end{cases}
$$

Since $\mathbf{U}^\delta$ belongs to $H^2_{loc}(\omega)$ and due to (3.6) and (3.12) it satisfies the following estimates:

$$(6.5)\qquad ||\mathbf{U}^\delta||_{H^1(\omega)} \leq \frac{C}{\delta}, \qquad \Big\|\rho\frac{\partial^2 \mathbf{U}^\delta}{\partial x_\alpha \partial x_\beta}\Big\|_{L^2(\omega)} \leq \frac{C}{\delta},$$

where ρ is defined by

$$\rho(x) = dist(x, \partial\omega) \qquad x \in \omega.$$

Let us define the two functions

$$Z_1^\delta = \frac{\partial \mathbf{U}^\delta}{\partial x_1} + \mathcal{R}_2^{\delta-}, \qquad Z_2^\delta = \frac{\partial \mathbf{U}^\delta}{\partial x_2} - \mathcal{R}_1^{\delta-}.$$

It follows from (3.5), (3.12), (6.4) and (6.5) that

$$||\mathbf{U}^\delta - \mathcal{U}_3^{\delta-}||_{H^1(\omega)} \leq C, \qquad ||Z_\alpha^\delta||_{L^2(\omega)} \leq C \qquad ||\rho\nabla Z_\alpha^\delta||_{L^2(\omega)} \leq \frac{C}{\delta}.$$

Then since $\delta/\varepsilon \to 0$, we have (see [2], [10], [12])

$$\begin{cases} \delta\mathbf{U}^\delta \rightharpoonup \mathcal{U}_3^- \quad \text{weakly in} \quad H_0^1(\omega),\\ \dfrac{\delta}{\varepsilon} Z_\alpha^\delta \to 0 \quad \text{strongly in} \quad L^2(\omega),\\ \dfrac{\delta}{\varepsilon}\mathcal{T}_\varepsilon(Z_\alpha^\delta) \to 0 \quad \text{strongly in} \quad L^2(\omega\times Y),\\ \delta\mathcal{T}_\varepsilon(\nabla Z_\alpha^\delta) \rightharpoonup 0 \quad \text{weakly in} \quad L^2_{loc}(\omega; L^2(Y,\mathbb{R}^2)). \end{cases}$$

There exists $\widehat{\mathbf{u}}_3 \in L^2_{loc}(\omega; H^2_{per}(Y))$ such that (see [11])

$$\delta\mathcal{T}_\varepsilon\Big(\frac{\partial^2\mathbf{U}^\delta}{\partial x_\alpha\partial x_\beta}\Big) \rightharpoonup \frac{\partial^2\mathcal{U}_3^-}{\partial x_\alpha\partial x_\beta} + \frac{\partial^2\widehat{\mathbf{u}}_3}{\partial X_\alpha\partial X_\beta} \quad \text{weakly in} \quad L^2_{loc}(\omega; L^2(Y)).$$

Using (6.3) and the definition of the Z_α^δ we deduce that

$$\begin{aligned} 0 = \delta\mathcal{T}_\varepsilon\Big(\frac{\partial^2\mathbf{U}^\delta}{\partial x_1\partial x_\alpha} + \frac{\partial\mathcal{R}_2^{\delta-}}{\partial x_\alpha} - \frac{\partial Z_1^\delta}{\partial x_\alpha}\Big) &\rightharpoonup \frac{\partial^2\mathcal{U}_3^-}{\partial x_1\partial x_\alpha} + \frac{\partial^2\widehat{\mathbf{u}}_3}{\partial X_1\partial X_\alpha} + \frac{\partial\mathcal{R}_2}{\partial x_\alpha} + \frac{\partial\widehat{\mathbf{r}}_2}{\partial X_\alpha} \quad \text{weakly in} \quad L^2_{loc}(\omega; L^2(Y)),\\ 0 = \delta\mathcal{T}_\varepsilon\Big(\frac{\partial^2\mathbf{U}^\delta}{\partial x_\alpha\partial x_2} - \frac{\partial\mathcal{R}_1^{\delta-}}{\partial x_\alpha} - \frac{\partial Z_2^\delta}{\partial x_\alpha}\Big) &\rightharpoonup \frac{\partial^2\mathcal{U}_3^-}{\partial x_\alpha\partial x_2} + \frac{\partial^2\widehat{\mathbf{u}}_3}{\partial X_\alpha\partial X_2} - \frac{\partial\mathcal{R}_1}{\partial x_\alpha} - \frac{\partial\widehat{\mathbf{r}}_1}{\partial X_\alpha} \quad \text{weakly in} \quad L^2_{loc}(\omega; L^2(Y)). \end{aligned}$$

As a consequence, we have

$$(6.6)\qquad \frac{\partial^2\widehat{\mathbf{u}}_3}{\partial X_1\partial X_\alpha} = -\frac{\partial\widehat{\mathbf{r}}_2}{\partial X_\alpha}, \qquad \frac{\partial^2\widehat{\mathbf{u}}_3}{\partial X_\alpha\partial X_2} = \frac{\partial\widehat{\mathbf{r}}_1}{\partial X_\alpha},$$

and then the function $\widehat{\mathbf{u}}_3$ belongs to $L^2(\omega; H^2_{per}(Y))$ and the two last concergences of (6.1) hold true. Estimates (6.2) are direct consequences of (3.5) and (3.12). □

Using (3.3) and Lemma 6.1, we deduce that the limit displacement u^- is a Kirchhoff-Love displacement:

$$(6.7)\qquad u_3^- = \mathcal{U}_3^-, \qquad u_\alpha^- = \mathcal{U}_\alpha^- - \Big(X_3 + \frac{1}{2}\Big)\frac{\partial\mathcal{U}_3^-}{\partial x_\alpha}.$$

6.2 Weak convergences of the strain field in the plate

First in view of estimate (3.12) and of (4.1), it follows that:

$$
\begin{cases} \mathcal{T}_\varepsilon \circ \Pi_\delta(\gamma_{ij}(u^\delta)) \rightharpoonup X^-_{ij} & \text{weakly in} \quad L^2(\Omega^- \times Y), \\ \mathcal{T}_\varepsilon \circ \Pi_\delta(\sigma^\delta_{ij}) \rightharpoonup \Sigma^-_{ij} & \text{weakly in} \quad L^2(\Omega^- \times Y), \end{cases} \tag{6.8}
$$

where $X^-_{ij} \in L^2(\Omega^- \times Y)$ and $\Sigma^-_{ij} \in L^2(\Omega^- \times Y)$.

Then due to the expression of the strain $\gamma(u^\delta)$, to (3.3) and to Lemma 6.1, we obtain the following relations:

$$
\begin{cases} X^-_{\alpha\beta} = \gamma_{\alpha\beta}(\mathcal{U}^-_m) + \Gamma_{\alpha\beta}(\widehat{\mathbf{u}}_m) - \Big(X_3 + \dfrac{1}{2}\Big)\Big\{\dfrac{\partial^2 \mathcal{U}^-_3}{\partial x_\alpha \partial x_\beta} + \dfrac{\partial^2 \widehat{\mathbf{u}}_3}{\partial X_\alpha \partial X_\beta}\Big\}, \\ X^-_{\alpha 3} = \dfrac{1}{2}\Big\{\widehat{\mathcal{Z}}^-_\alpha + \dfrac{\partial \overline{u}^-_\alpha}{\partial X_3}\Big\}, \qquad X^-_{33} = \dfrac{\partial \overline{u}^-_3}{\partial X_3}, \end{cases} \tag{6.9}
$$

where the fields $\mathcal{U}^-_m$ and $\widehat{\mathbf{u}}_m$ are defined by

$$
\mathcal{U}^-_m = \mathcal{U}^-_1 \mathbf{e}_1 + \mathcal{U}^-_2 \mathbf{e}_2, \qquad \widehat{\mathbf{u}}_m = \widehat{\mathbf{u}}_1 \mathbf{e}_1 + \widehat{\mathbf{u}}_2 \mathbf{e}_2.
$$

6.3 Determination of $\overline{u}^-$ and $\widehat{\mathcal{Z}}^-_\alpha$

Let us consider the test displacement

$$
\mathbf{v}^\delta_\varepsilon(x) = \delta \phi(x_1, x_2) \Phi\Big(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}\Big) \Theta\Big(\frac{x_3}{\delta}\Big) \qquad \text{in} \quad \Omega_{\varepsilon,\delta},
$$

with $\phi \in \mathcal{C}^\infty_0(\omega)$, $\Phi \in H^1_{per}(Y)$, $\Theta \in \mathcal{C}^\infty([-1, +\infty[; \mathbb{R}^3)$ and $\Theta(x_3) = 0$ if $x_3 \geq 0$. Notice that the displacement $\mathbf{v}^\delta_\varepsilon$ is zero in Ω^+_ε and that $\Pi_\delta(\mathbf{v}^\delta_\varepsilon) \longrightarrow 0$ in $(L^2(\Omega^-))^3$.
Since δ/ε tends to 0, we have the following strong convergences in $L^2(\Omega^- \times Y)$:

$$
\begin{cases} \mathcal{T}_\varepsilon \circ \Pi_\delta\big(\gamma_{\alpha\beta}(\mathbf{v}^\delta_\varepsilon)\big) \longrightarrow 0, \\ \mathcal{T}_\varepsilon \circ \Pi_\delta\big(\gamma_{\alpha 3}(\mathbf{v}^\delta_\varepsilon)\big) \longrightarrow \dfrac{1}{2} \phi\, \Phi \dfrac{d\Theta_\alpha}{dX_3}, \\ \mathcal{T}_\varepsilon \circ \Pi_\delta\big(\gamma_{33}(\mathbf{v}^\delta_\varepsilon)\big) \longrightarrow \phi\, \Phi \dfrac{d\Theta_3}{dX_3}. \end{cases} \tag{6.10}
$$

Using the above displacement $\mathbf{v}^\delta_\varepsilon$ in the formulation (4.4) with $\Theta_3 = 0$ and passing to the limit as δ tends to zero lead to also using (6.10)

$$
\begin{cases} \dfrac{\partial^2}{\partial X_3^2}\big(\overline{u}^-_\alpha + \widehat{\mathcal{Z}}^-_\alpha\big) = 0 & \text{in} \qquad \Omega^- \times Y, \\ \dfrac{\partial}{\partial X_3}\big(\overline{u}^-_\alpha + \widehat{\mathcal{Z}}^-_\alpha\big)_{|X_3=-1} = 0 & \text{in} \qquad \omega \times Y, \end{cases}
$$

while by (3.4) and (6.2) we have

$$
\int_{-1}^0 \overline{u}^-_\alpha dX_3 = \int_{-1}^0 \Big(X_3 + \frac{1}{2}\Big) \overline{u}^-_\alpha dX_3 = 0 \quad \text{for} \quad \alpha = 1, 2.
$$

We easily deduce from the above relations that the functions $\overline{u}^-_\alpha$ and $\widehat{\mathcal{Z}}^-_\alpha$ are equal to zero (for $\alpha = 1,2$) and then $X^-_{\alpha 3} = 0$.
Now choosing the displacement $\mathbf{v}^\delta_\varepsilon$ in the formulation (4.4) with $\Theta_\alpha = 0$ and passing to the limit as δ tends to zero lead to also using (6.10)

$$\frac{\partial}{\partial X_3}\Big[\lambda\Big(\gamma_{\alpha\alpha}(\mathcal{U}^-_m) + \Gamma_{\alpha\alpha}(\widehat{\mathbf{u}}_m) - \Big(X_3 + \frac{1}{2}\Big)\Big\{\frac{\partial^2 \mathcal{U}^-_3}{\partial x_\alpha \partial x_\alpha} + \frac{\partial^2 \widehat{\mathbf{u}}_3}{\partial X_\alpha \partial X_\alpha}\Big\}\Big) + (\lambda + 2\mu)\frac{\partial \overline{u}^-_3}{\partial X_3}\Big] = 0 \quad \text{in} \quad \Omega^- \times Y,$$

$$\Big[\lambda\Big(\gamma_{\alpha\alpha}(\mathcal{U}^-_m) + \Gamma_{\alpha\alpha}(\widehat{\mathbf{u}}_m) - \Big(X_3 + \frac{1}{2}\Big)\Big\{\frac{\partial^2 \mathcal{U}^-_3}{\partial x_\alpha \partial x_\alpha} + \frac{\partial^2 \widehat{\mathbf{u}}_3}{\partial X_\alpha \partial X_\alpha}\Big\}\Big) + (\lambda + 2\mu)\frac{\partial \overline{u}^-_3}{\partial X_3}\Big]_{|X_3=-1} = 0 \quad \text{in} \quad \omega \times Y.$$

This gives

$$X^-_{33} = \frac{\partial \overline{u}^-_3}{\partial X_3} = -\frac{\lambda}{\lambda + 2\mu}\Big(\gamma_{\alpha\alpha}(\mathcal{U}^-_m) + \Gamma_{\alpha\alpha}(\widehat{\mathbf{u}}_m) - \Big(X_3 + \frac{1}{2}\Big)\Big\{\frac{\partial^2 \mathcal{U}^-_3}{\partial x_\alpha \partial x_\alpha} + \frac{\partial^2 \widehat{\mathbf{u}}_3}{\partial X_\alpha \partial X_\alpha}\Big\}\Big).$$

Since by (3.4) and (6.2) we have

$$\int_{-1}^0 \overline{u}^-_3 \, dX_3 = 0,$$

the above relation permits to derive the function $\overline{u}^-_3$ in terms of the fields $\mathcal{U}^-_m$, $\mathcal{U}^-_3$, $\widehat{\mathbf{u}}_m$ and $\widehat{\mathbf{u}}_3$. Inserting the obtained expression into (6.9) and using (2.1) lead to

$$(6.11) \qquad \begin{cases} \Sigma^-_{11} = \dfrac{E}{1-\nu^2}\Big[\gamma_{11}(\mathcal{U}^-_m) + \Gamma_{11}(\widehat{\mathbf{u}}_m) - \Big(X_3 + \dfrac{1}{2}\Big)\Big\{\dfrac{\partial^2 \mathcal{U}^-_3}{\partial x_1^2} + \dfrac{\partial^2 \widehat{\mathbf{u}}_3}{\partial X_1^2}\Big\}\Big] \\ \qquad + \dfrac{E\nu}{1-\nu^2}\Big[\gamma_{22}(\mathcal{U}^-_m) + \Gamma_{22}(\widehat{\mathbf{u}}_m) - \Big(X_3 + \dfrac{1}{2}\Big)\Big\{\dfrac{\partial^2 \mathcal{U}^-_3}{\partial x_2^2} + \dfrac{\partial^2 \widehat{\mathbf{u}}_3}{\partial X_2^2}\Big\}\Big], \\ \Sigma^-_{22} = \dfrac{E}{1-\nu^2}\Big[\gamma_{22}(\mathcal{U}^-_m) + \Gamma_{22}(\widehat{\mathbf{u}}_m) - \Big(X_3 + \dfrac{1}{2}\Big)\Big\{\dfrac{\partial^2 \mathcal{U}^-_3}{\partial x_2^2} + \dfrac{\partial^2 \widehat{\mathbf{u}}_3}{\partial X_2^2}\Big\}\Big] \\ \qquad + \dfrac{E\nu}{1-\nu^2}\Big[\gamma_{11}(\mathcal{U}^-_m) + \Gamma_{11}(\widehat{\mathbf{u}}_m) - \Big(X_3 + \dfrac{1}{2}\Big)\Big\{\dfrac{\partial^2 \mathcal{U}^-_3}{\partial x_1^2} + \dfrac{\partial^2 \widehat{\mathbf{u}}_3}{\partial X_1^2}\Big\}\Big], \\ \Sigma^-_{12} = \dfrac{E}{1+\nu}\Big[\gamma_{12}(\mathcal{U}^-_m) + \Gamma_{12}(\widehat{\mathbf{u}}_m) - \Big(X_3 + \dfrac{1}{2}\Big)\Big\{\dfrac{\partial^2 \mathcal{U}^-_3}{\partial x_1 \partial x_2} + \dfrac{\partial^2 \widehat{\mathbf{u}}_3}{\partial X_1 \partial X_2}\Big\}\Big], \\ \Sigma^-_{i3} = 0. \end{cases}$$

We now first introduce the $2d$ local displacement field $\widetilde{\mathbf{U}}_m$ which belongs to $L^2(\omega; H^1(Y;\mathbb{R}^2))$ through the formula:

$$\widetilde{\mathbf{U}}_m = \big[X_1\gamma_{11}(\mathcal{U}^-_m) + X_2\gamma_{12}(\mathcal{U}^-_m) + \widehat{\mathbf{u}}_1\big]\mathbf{e}_1 + \big[X_1\gamma_{12}(\mathcal{U}^-_m) + X_2\gamma_{22}(\mathcal{U}^-_m) + \widehat{\mathbf{u}}_2\big]\mathbf{e}_2.$$

Then we consider the local Kirchhoff-Love displacement field defined by

$$\widetilde{\mathbf{U}}_f = -\Big(X_3 + \frac{1}{2}\Big)\frac{\partial \widetilde{\mathbf{U}}_3}{\partial X_1}\mathbf{e}_1 - \Big(X_3 + \frac{1}{2}\Big)\frac{\partial \widetilde{\mathbf{U}}_3}{\partial X_2}\mathbf{e}_2 + \widetilde{\mathbf{U}}_3\mathbf{e}_3,$$

where

$$\widetilde{\mathbf{U}}_3 = \frac{1}{2}\Big[X_1^2\frac{\partial^2 \mathcal{U}^-_3}{\partial x_1^2} + 2X_1X_2\frac{\partial^2 \mathcal{U}^-_3}{\partial x_1 \partial x_2} + X_2^2\frac{\partial^2 \mathcal{U}^-_3}{\partial x_2^2}\Big] + \widehat{\mathbf{u}}_3.$$

Remark that $\widetilde{\mathbf{U}}_3 \in L^2(\omega; H^2(Y))$ (see Lemma 6.1). Hence (6.11) leads to

$$(6.12) \qquad \begin{cases} \Sigma^-_{11} = \dfrac{E}{1-\nu^2}\big[\Gamma_{11}(\widetilde{\mathbf{U}}_m + \widetilde{\mathbf{U}}_f) + \nu\Gamma_{22}(\widetilde{\mathbf{U}}_m + \widetilde{\mathbf{U}}_f)\big], \\ \Sigma^-_{22} = \dfrac{E}{1-\nu^2}\big[\Gamma_{22}(\widetilde{\mathbf{U}}_m + \widetilde{\mathbf{U}}_f) + \nu\Gamma_{11}(\widetilde{\mathbf{U}}_m + \widetilde{\mathbf{U}}_f)\big], \\ \Sigma^-_{12} = \dfrac{E}{1+\nu}\Gamma_{12}(\widetilde{\mathbf{U}}_m + \widetilde{\mathbf{U}}_f) \qquad \Sigma^-_{i3} = 0, \end{cases}$$

and with (6.9)

$$
(6.13) \qquad \begin{cases} X^-_{\alpha\beta} = \Gamma_{\alpha\beta}(\widetilde{\mathbf{U}}_m + \widetilde{\mathbf{U}}_f), \\ X^-_{33} = -\dfrac{\lambda}{\lambda+2\mu}\big[\Gamma_{11}(\widetilde{\mathbf{U}}_m + \widetilde{\mathbf{U}}_f) + \Gamma_{22}(\widetilde{\mathbf{U}}_m + \widetilde{\mathbf{U}}_f)\big], \\ X^-_{\alpha 3} = 0. \end{cases}
$$

7. Kinematic conditions

In this section, we first prove that under the rods, the displacement of the plate is asymptotically a rigid body displacement. Then we deduce the limit kinematic junction condition betweeen the plate and the rods from which follows the expression of the limit displacement in the rods.

Using the definition of $\mathcal{T}_\varepsilon$, we have from (3.16)

$$
\|\mathcal{T}_\varepsilon \circ \Pi_\delta\big(U^{\delta -}_e - \mathbf{R}^\delta\big)\|^2_{(L^2(\Omega^-\times D_k))^3} = \frac{1}{\delta}\|U^{\delta -}_e - \mathbf{R}^\delta\|^2_{(L^2(\mathcal{C}^{\varepsilon,\delta}))^3} \le C\varepsilon\delta
$$

(where $D_k = kD$) and $\dfrac{1}{\varepsilon}\mathcal{T}_\varepsilon \circ \Pi_\delta\big(U^{\delta -}_e - \mathbf{R}^\delta\big)$ is bounded in $(L^2(\Omega^-; H^1(Y)))^3$. It follows that

$$
\begin{aligned}
\frac{1}{\varepsilon}\mathcal{T}_\varepsilon \circ \Pi_\delta\big(U^{\delta -}_e - \mathbf{R}^\delta\big) &\longrightarrow 0 \qquad \text{strongly in} \qquad (L^2(\Omega^-\times D_k))^3,\\
\frac{1}{\varepsilon}\frac{\partial}{X_\alpha}\mathcal{T}_\varepsilon \circ \Pi_\delta\big(U^{\delta -}_e - \mathbf{R}^\delta\big) &\rightharpoonup 0 \qquad \text{weakly in} \qquad (L^2(\Omega^-\times D_k))^3.
\end{aligned}
$$

Due to estimate (3.8) of $\mathcal{R}^{\delta +}_3(.,.,0)$ we have

$$
\mathcal{R}^{\delta +}_3(.,.,0) \rightharpoonup \mathcal{R}^C_3 \quad \text{weakly in} \quad L^2(\omega).
$$

Notice that if $\dfrac{\sqrt{\delta}}{\varepsilon}$ is bounded we have $\mathcal{R}^C_3 = \mathcal{R}^+_3$ and if $\dfrac{\sqrt{\delta}}{\varepsilon} \to +\infty$ the function $\mathcal{R}^C_3$ is asymptotically the angle of rotation of the small cylinders included in the plate and below the rods.

In view of the definitions (3.3) of $U^{\delta -}_e$ and of the rigid displacement $\mathbf{R}^\delta$ (see Section 3.4) and (5.1), we deduce that

$$
\begin{aligned}
&\frac{\partial \mathcal{U}^-_\alpha}{\partial x_\alpha} + \frac{\partial \widehat{\mathbf{u}}_\alpha}{\partial X_\alpha} - \Big(X_3 + \frac{1}{2}\Big)\Big\{\frac{\partial^2 \mathcal{U}^-_3}{\partial x^2_\alpha} + \frac{\partial^2 \widehat{\mathbf{u}}_3}{\partial X^2_\alpha}\Big\} = 0 \qquad \text{in} \quad \omega\times D_k,\\
&\frac{\partial \mathcal{U}^-_1}{\partial x_2} + \frac{\partial \widehat{\mathbf{u}}_1}{\partial X_2} - \Big(X_3 + \frac{1}{2}\Big)\Big\{\frac{\partial^2 \mathcal{U}^-_3}{\partial x_1\partial x_2} + \frac{\partial^2 \widehat{\mathbf{u}}_3}{\partial X_1\partial X_2}\Big\} = -\mathcal{R}^C_3 \qquad \text{in} \quad \omega\times D_k,\\
&\frac{\partial \mathcal{U}^-_2}{\partial x_1} + \frac{\partial \widehat{\mathbf{u}}_2}{\partial X_1} - \Big(X_3 + \frac{1}{2}\Big)\Big\{\frac{\partial^2 \mathcal{U}^-_3}{\partial x_1\partial x_2} + \frac{\partial^2 \widehat{\mathbf{u}}_3}{\partial X_1\partial X_2}\Big\} = \mathcal{R}^C_3 \qquad \text{in} \quad \omega\times D_k.
\end{aligned}
$$

Since all the fields $\mathcal{U}^-$ and $\widehat{\mathbf{u}}$ are independent of X_3, the above relations lead to

$$
(7.1) \qquad \begin{cases} \widehat{\mathbf{u}}_1 = -\dfrac{\partial \mathcal{U}^-_1}{\partial x_1}X_1 - \Big\{\dfrac{\partial \mathcal{U}^-_1}{\partial x_2} + \mathcal{R}^C_3\Big\}X_2 + a_1 \qquad \text{in} \quad \omega\times D_k,\\ \widehat{\mathbf{u}}_2 = -\Big\{\dfrac{\partial \mathcal{U}^-_2}{\partial x_1} - \mathcal{R}^C_3\Big\}X_1 - \dfrac{\partial \mathcal{U}^-_2}{\partial x_2}X_2 + a_2 \qquad \text{in} \quad \omega\times D_k,\\ \widehat{\mathbf{u}}_3 = -\dfrac{1}{2}\Big[\dfrac{\partial^2 \mathcal{U}^-_3}{\partial x^2_1}X^2_1 + 2\dfrac{\partial^2 \mathcal{U}^-_3}{\partial x_1\partial x_2}X_1X_2 + \dfrac{\partial^2 \mathcal{U}^-_3}{\partial x^2_2}X^2_2\Big] + a_3 + c_1X_1 + c_2X_2 \qquad \text{in} \quad \omega\times D_k, \end{cases}
$$

where, *a priori*, the functions a_1, a_2, a_3, c_1 and c_2 belong to $L^2(\omega)$. Actually since the field $\widehat{\mathbf{u}}$ is defined up to a field depending only on (x_1, x_2), we can choose $a_1 = a_2 = a_3 = 0$.

Proposition 7.1: *We have*

$$\mathcal{R}_1^+ = \frac{\partial \mathcal{U}_3^-}{\partial x_2}, \qquad \mathcal{R}_2^+ = -\frac{\partial \mathcal{U}_3^-}{\partial x_1}. \tag{7.2}$$

Proof of proposition 7.1: We consider the two first components of the displacement $U_e^{\delta-} - \mathbf{R}^\delta$ and due to (3.16) we obtain the estimate

$$||\mathcal{R}_\alpha^{\delta-} - \mathcal{R}_\alpha^{\delta+}(.,0)||^2_{L^2(\mathcal{C}_\varepsilon)} \le C\frac{\varepsilon}{\delta}$$

where $\mathcal{C}_\varepsilon = \bigcup_{(p,q)\in\mathcal{N}_\varepsilon} \mathcal{D}^\varepsilon_{p,q}$. Then we have

$$\|\mathcal{T}_\varepsilon(\mathcal{R}_\alpha^{\delta-}) - \mathcal{R}_\alpha^{\delta+}(.,0)||^2_{L^2(\omega\times D_k)} \le C\frac{\varepsilon}{\delta}$$

and we use the convergences (5.1) and (6.1) to deduce that $\delta\mathcal{R}_\alpha^{\delta+}(.,0)$ converges strongly in $L^2(\omega)$ to $\mathcal{R}_\alpha^-$ and then $\mathcal{R}_\alpha^- = \mathcal{R}_\alpha^+(.,0)$ in $L^2(\omega)$. □

Remark 7.2 The strong convergence of $\delta\mathcal{R}_\alpha^{\delta+}(.,0)$ and the estimate on $\dfrac{\partial \mathcal{R}_\alpha^{\delta+}}{\partial x_3}$ in (3.9) show that the convergence of $\delta\mathcal{R}_\alpha^{\delta+}$ is actually strong in $L^2(\omega; H^1(0,L))$. □

Let us end this section with the continuity relation between the traction $\mathcal{U}_3^+$ in the rods and the bending $\mathcal{U}_3^-$ of the plate on ω. Indeed from (3.16) we have

$$\|\delta\mathcal{T}_\varepsilon(\mathcal{U}_3^{\delta-}) - \delta\mathcal{U}_3^{\delta+}(.,0)||^2_{L^2(\omega\times D_k)} \le C\varepsilon\delta^3$$

which implies together with estimates (3.9) that $\delta\mathcal{U}_3^{\delta+}(.,0)$ strongly converges to $\mathcal{U}_3^-$ in $L^2(\omega\times D_k)$ and moreover that

$$\mathcal{U}_3^+ = \mathcal{U}_3^- \quad \text{in} \quad \omega. \tag{7.3}$$

As a consequence of (5.2), (7.2) and (7.3), we get

$$\mathcal{U}^+(x_1,x_2,x_3) = -x_3\frac{\partial \mathcal{U}_3^-}{\partial x_1}(x_1,x_2)\,\mathbf{e}_1 - x_3\frac{\partial \mathcal{U}_3^-}{\partial x_2}(x_1,x_2)\,\mathbf{e}_2 + \mathcal{U}_3^-(x_1,x_2)\,\mathbf{e}_3. \tag{7.4}$$

Remark 7.3 The strong convergences of $\delta\mathcal{U}_3^{\delta+}(.,0)$, the fact that $\delta\mathcal{U}_\alpha^{\delta+}(.,0)\to 0$ and the estimate on $\dfrac{\partial \mathcal{U}^{\delta+}}{\partial x_3}$ in (3.9) show that the convergence of $\delta\mathcal{U}^{\delta+}$ is actually strong in $(L^2(\omega;H^1(0,L)))^3$. Let us emphazise that the strong convergences of $\delta\mathcal{U}^{\delta+}$ and of $\delta\mathcal{R}^{\delta+}$ together with estimate (3.9) imply that (see also (5.3))

$$\delta\mathcal{T}^\varepsilon(u_i^\delta) \to u_i^+ = \mathcal{U}_i^+ \qquad \text{strongly in} \quad L^2(\omega; H^1(D\times]0,L[)).$$

□

8. The limit membrane problem for the plate and the torsion in the rods

This section is devoted to express the membrane corrector $\widehat{\mathbf{u}}_m$ in terms of four basic membrane correctors. Using the properties of these basic correctors, we deduce the membrane elastic problem in the plate and we obtain the torsion in each rod as the curl of the macroscopic membrane displacement at the corresponding point of the plate.

Throughout the paper , ve denote by $\mathbb{R}_q[X_1, X_2]$ the space of polynoms of the two variables (X_1, X_2) with degree less or equal to the integer q $(q \geq 0)$ and we set for $p = 1, 2$ and $q = 1, 2$

$$\mathbb{V}^p_{per,D_k,0}(Y) = \Big\{\Phi \in H^p_{per}(Y) \mid \Phi = 0 \ \text{ on } \ D_k\Big\}$$
$$\mathbb{V}^p_{per,D_k,q}(Y) = \Big\{\Phi \in H^p_{per}(Y) \mid \Phi \in \mathbb{R}_q[X_1, X_2] \ \text{ on } \ D_k, \ \text{ and } \ \Phi(0,0) = 0\Big\}$$

Recall that due to (7.1) we have $\widehat{\mathbf{u}}_\alpha \in \mathbb{V}^1_{per,D_k,1}(Y)$ and $\widehat{\mathbf{u}}_3 \in \mathbb{V}^2_{per,D_k,2}(Y)$ for almost any $(x_1, x_2) \in \omega$.

8.1 Determination of the membrane corrector $\widehat{\mathbf{u}}_m$

In this step we derive the expression of the fields $\widehat{\mathbf{u}}_m$ in terms of $\mathcal{U}^-_m$ and of four correctors.

In what follow, we denote by χ a function of $\mathcal{C}^\infty_0(Y)$ such that $\chi = 1$ on D_k.

Let $\psi \in \mathcal{C}^\infty_0(\omega)$ and define the function ψ_ε by

$$\psi_\varepsilon(x_1, x_2) = \begin{cases} \Big(1 - \chi\Big(\dfrac{x_1}{\varepsilon}, \dfrac{x_2}{\varepsilon}\Big)\Big)\psi(x_1, x_2) + \chi\Big(\dfrac{x_1}{\varepsilon}, \dfrac{x_2}{\varepsilon}\Big)\psi(p\varepsilon, q\varepsilon), \\ \qquad \text{if } \ (x_1, x_2) \in \Big]\varepsilon p - \dfrac{\varepsilon}{2}, \varepsilon p + \dfrac{\varepsilon}{2}\Big[\times \Big]\varepsilon q - \dfrac{\varepsilon}{2}, \varepsilon q + \dfrac{\varepsilon}{2}\Big[, \quad \text{and } (p, q) \in \mathcal{N}_\varepsilon, \\ 0, \qquad \text{if } (x_1, x_2) \in \omega \setminus \widetilde{\omega}_\varepsilon. \end{cases}$$

We consider a test displacement of the following type:

$$\mathbf{v}^\delta_\varepsilon(x) = \varepsilon\psi_\varepsilon(x_1, x_2)\Big\{\Phi_1\Big(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}\Big)\mathbf{e}_1 + \Phi_2\Big(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}\Big)\mathbf{e}_2\Big\} \quad \text{in} \quad \Omega_{\varepsilon,\delta}$$

where $\Phi_\alpha \in \mathbb{V}^1_{per,D_k,0}(Y)$ and we set $\Phi_m = \Phi_1\mathbf{e}_1 + \Phi_2\mathbf{e}_2$. Remark that $\gamma_{ij}(\mathbf{v}^\delta_\varepsilon) = 0$ in Ω^+_ε and $\gamma_{i3}(\mathbf{v}^\delta_\varepsilon) = 0$ in $\Omega_{\varepsilon,\delta}$. We have the following strong convergence:

$$\mathcal{T}^\varepsilon\big(\mathbf{v}^\delta_\varepsilon\big) \longrightarrow 0 \qquad \text{in} \quad (L^2(\Omega^+ \times D))^3.$$

Since ϕ is smooth and $\psi - \psi_\varepsilon$ tends to 0 in $L^\infty(\omega)$, we easily deduce that

$$\Pi_\delta\big(\mathbf{v}^\delta_\varepsilon\big) \longrightarrow 0 \qquad \text{strongly in} \quad (L^2(\Omega^- \times Y))^2,$$
$$\mathcal{T}_\varepsilon \circ \Pi_\delta\big(\gamma_{\alpha\beta}(\mathbf{v}^\delta_\varepsilon)\big) \longrightarrow \psi\Gamma_{\alpha\beta}(\Phi_m) \qquad \text{strongly in} \quad L^2(\Omega^- \times Y).$$

We use the above displacement $\mathbf{v}^\delta_\varepsilon$ in the formulation (4.4). Since the right hand side of the obtained equation tends to zero we obtain

$$\frac{E}{1-\nu^2}\int_{\Omega^- \times Y} \psi\Big[(1-\nu)\Gamma_{\alpha\beta}(\widetilde{\mathbf{U}}_m + \widetilde{\mathbf{U}}_f)\Gamma_{\alpha\beta}(\Phi_m) + \nu\Gamma_{\alpha\alpha}(\widetilde{\mathbf{U}}_m + \widetilde{\mathbf{U}}_f)\Gamma_{\beta\beta}(\Phi_m)\Big] = 0,$$

which implies that, using also the periodicity of Φ_m

$$\text{(8.1)} \qquad \int_Y \Big[(1-\nu)\Gamma_{\alpha\beta}(\widehat{\mathbf{u}}_m)\Gamma_{\alpha\beta}(\Phi_m) + \nu\Gamma_{\alpha\alpha}(\widehat{\mathbf{u}}_m)\Gamma_{\beta\beta}(\Phi_m)\Big] = 0 \qquad \text{a.e. in } \Omega^-,$$

for any $\Phi_m \in [\mathbb{V}^1_{per,D_k,0}(Y)]^2$. Taking into account (7.1), the problem (8.1) is a $2d$ elasticity problem in $Y \setminus D_k$ with prescribed displacement given by (7.1) on ∂D_k. In order to obtain the expression of $\widehat{\mathbf{u}}_m$, let us now introduce the four correctors $\widehat{\mathbf{v}}^{(i)}_m$, $i \in \{1, 2, 3, 4\}$ solutions of the following problems:

$$\text{(8.2)} \qquad \begin{cases} \widehat{\mathbf{v}}^{(i)}_m \in [\mathbb{V}^1_{per,D_k,1}(Y)]^2 \\ \displaystyle\int_Y \Big[(1-\nu)\Gamma_{\alpha\beta}(\widehat{\mathbf{v}}^{(i)}_m)\Gamma_{\alpha\beta}(\Phi_m) + \nu\Gamma_{\alpha\alpha}(\widehat{\mathbf{v}}^{(i)}_m)\Gamma_{\beta\beta}(\Phi_m)\Big] = 0, \\ \forall\Phi_m \in [\mathbb{V}^1_{per,D_k,0}(Y)]^2, \end{cases}$$

with the boundary conditions on ∂D_k

$$\widehat{\mathbf{v}}_m^{(1)} = X_1\mathbf{e}_1, \qquad \widehat{\mathbf{v}}_m^{(2)} = X_2\mathbf{e}_1, \qquad \widehat{\mathbf{v}}_m^{(3)} = X_1\mathbf{e}_2, \qquad \widehat{\mathbf{v}}_m^{(4)} = X_2\mathbf{e}_2. \tag{8.3}$$

Hence using (7.1) we obtain

$$\widehat{\mathbf{u}}_m = -\gamma_{11}(\mathcal{U}_m^-)\widehat{\mathbf{v}}_m^{(1)} - \Big(\frac{\partial \mathcal{U}_1^-}{\partial x_2} + \mathcal{R}_3^C\Big)\widehat{\mathbf{v}}_m^{(2)} - \Big(\frac{\partial \mathcal{U}_2^-}{\partial x_1} - \mathcal{R}_3^C\Big)\widehat{\mathbf{v}}_m^{(3)} - \gamma_{22}(\mathcal{U}_m^-)\widehat{\mathbf{v}}_m^{(4)}. \tag{8.4}$$

8.2 Properties of the basic membrane correctors $\widehat{\mathbf{v}}_m^{(i)}$

Let us first deduce the equations (for $i = 1, \dots 4$) that follows from the weak formulation (8.2)

$$\begin{cases} \dfrac{\partial^2 \widehat{\mathbf{v}}_{m,1}^{(i)}}{\partial X_1^2} + \dfrac{1-\nu}{2}\dfrac{\partial^2 \widehat{\mathbf{v}}_{m,1}^{(i)}}{\partial X_2^2} + \dfrac{1+\nu}{2}\dfrac{\partial^2 \widehat{\mathbf{v}}_{m,2}^{(i)}}{\partial X_1 \partial X_2} = 0 \\ \dfrac{\partial^2 \widehat{\mathbf{v}}_{m,2}^{(i)}}{\partial X_1^2} + \dfrac{1-\nu}{2}\dfrac{\partial^2 \widehat{\mathbf{v}}_{m,2}^{(i)}}{\partial X_2^2} + \dfrac{1+\nu}{2}\dfrac{\partial^2 \widehat{\mathbf{v}}_{m,1}^{(i)}}{\partial X_1 \partial X_2} = 0 \end{cases} \quad \text{in} \quad Y \setminus D_k. \tag{8.5}$$

The symmetric characters of the unit cell Y and D_k together with the structure of equations (8.5) and the boundary conditions (8.3) permit to obtain

$$\begin{cases} \widehat{\mathbf{v}}_{m,1}^{(4)}(X_1, X_2) = \widehat{\mathbf{v}}_{m,2}^{(1)}(X_2, X_1) \\ \widehat{\mathbf{v}}_{m,2}^{(4)}(X_1, X_2) = \widehat{\mathbf{v}}_{m,1}^{(1)}(X_2, X_1) \end{cases} \quad \text{and} \quad \begin{cases} \widehat{\mathbf{v}}_{m,1}^{(3)}(X_1, X_2) = \widehat{\mathbf{v}}_{m,2}^{(2)}(X_2, X_1) \\ \widehat{\mathbf{v}}_{m,2}^{(3)}(X_1, X_2) = \widehat{\mathbf{v}}_{m,1}^{(2)}(X_2, X_1) \end{cases} \tag{8.6}$$

and the following properties of symmetry

$$\begin{cases} \widehat{\mathbf{v}}_{m,1}^{(1)} \text{ is odd with respect to } X_1 \text{ and even w.r.t. } X_2, \\ \widehat{\mathbf{v}}_{m,2}^{(1)} \text{ is even w.r.t. } X_1 \text{ and odd w.r.t. } X_2, \\ \widehat{\mathbf{v}}_{m,1}^{(2)} \text{ is even w.r.t. } X_1 \text{ and odd w.r.t. } X_2, \\ \widehat{\mathbf{v}}_{m,2}^{(2)} \text{ is odd w.r.t. } X_1 \text{ and even w.r.t. } X_2. \end{cases}$$

Let us introduce the following notation

$$< \Psi, \Phi >_m = \int_Y \Big[(1-\nu)\Gamma_{\alpha\beta}(\Psi)\Gamma_{\alpha\beta}(\Phi_m) + \nu\Gamma_{\alpha\alpha}(\Psi)\Gamma_{\beta\beta}(\Phi)\Big] \qquad \forall(\Psi, \Phi) \in [H^1(Y)]^2 \times [H^1(Y)]^2.$$

Indeed $< \cdot, \cdot >_m$ is a scalar product in $[\mathbb{V}^1_{per, D_k, 1}(Y)]^2$. The above properties of symmetry of the correctors imply that

$$\begin{cases} < \widehat{\mathbf{v}}_m^{(1)}, \widehat{\mathbf{v}}_m^{(1)} >_m = < \widehat{\mathbf{v}}_m^{(4)}, \widehat{\mathbf{v}}_m^{(4)} >_m, \\ < \widehat{\mathbf{v}}_m^{(2)}, \widehat{\mathbf{v}}_m^{(2)} >_m = < \widehat{\mathbf{v}}_m^{(3)}, \widehat{\mathbf{v}}_m^{(3)} >_m . \end{cases} \tag{8.7}$$

Through integration by parts we have

$$\begin{aligned} < \widehat{\mathbf{v}}_m^{(i)}, \widehat{\mathbf{v}}_m^{(j)} >_m = & \int_{\partial D_k} \Big[\Gamma_{11}(\widehat{\mathbf{v}}_m^{(i)}) n_1 \widehat{\mathbf{v}}_{m,1}^{(j)} + (1-\nu)\Gamma_{12}(\widehat{\mathbf{v}}_m^{(i)}) n_2 \widehat{\mathbf{v}}_{m,1}^{(j)} + \nu\Gamma_{22}(\widehat{\mathbf{v}}_m^{(i)}) n_1 \widehat{\mathbf{v}}_{m,1}^{(j)}\Big] \\ & + \int_{\partial D_k} \Big[\Gamma_{22}(\widehat{\mathbf{v}}_m^{(i)}) n_2 \widehat{\mathbf{v}}_{m,2}^{(j)} + (1-\nu)\Gamma_{12}(\widehat{\mathbf{v}}_m^{(i)}) n_1 \widehat{\mathbf{v}}_{m,2}^{(j)} + \nu\Gamma_{11}(\widehat{\mathbf{v}}_m^{(i)}) n_2 \widehat{\mathbf{v}}_{m,2}^{(j)}\Big] \end{aligned}$$

where $\mathbf{n} = n_1\mathbf{e}_1 + n_2\mathbf{e}_2 = -\dfrac{X_1\mathbf{e}_1 + X_2\mathbf{e}_2}{\sqrt{X_1^2 + X_2^2}}$. We deduce that

$$(8.8) \qquad <\widehat{\mathbf{v}}_m^{(1)}, \widehat{\mathbf{v}}_m^{(2)}>_m = <\widehat{\mathbf{v}}_m^{(1)}, \widehat{\mathbf{v}}_m^{(3)}>_m = <\widehat{\mathbf{v}}_m^{(4)}, \widehat{\mathbf{v}}_m^{(3)}>_m = <\widehat{\mathbf{v}}_m^{(4)}, \widehat{\mathbf{v}}_m^{(2)}>_m = 0.$$

Let us now define the 4×4 real matrix $\mathbf{A}_m$ by $(\mathbf{A}_m)_{ij} = <\widehat{\mathbf{v}}_m^{(i)}, \widehat{\mathbf{v}}_m^{(j)}>_m$ for $i, j = 1\ldots, 4$. As a consequence of equalities (8.7) and (8.8) we obtain the following structure for the matrix $\mathbf{A}_m$:

$$(8.9) \qquad \mathbf{A}_m = \begin{pmatrix} a & 0 & 0 & b \\ 0 & c & d & 0 \\ 0 & d & c & 0 \\ b & 0 & 0 & a \end{pmatrix} \qquad \text{where} \qquad \begin{cases} a = <\widehat{\mathbf{v}}_m^{(1)}, \widehat{\mathbf{v}}_m^{(1)}>_m, & b = <\widehat{\mathbf{v}}_m^{(1)}, \widehat{\mathbf{v}}_m^{(4)}>_m, \\ c = <\widehat{\mathbf{v}}_m^{(2)}, \widehat{\mathbf{v}}_m^{(2)}>_m, & d = <\widehat{\mathbf{v}}_m^{(2)}, \widehat{\mathbf{v}}_m^{(3)}>_m . \end{cases}$$

Indeed the very definition of $\mathbf{A}_m$ shows that this matrix is positively defined.

8.3 The membrane-torsion problem.

Let $\phi_m = \phi_1\mathbf{e}_1 + \phi_2\mathbf{e}_2 \in [\mathcal{C}_0^\infty(\omega)]^2$ and $\Theta \in \mathcal{C}_0^\infty(\omega)$. We introduce below the test-displacement which allows, after passing to the limit in (4.4), to obtain the limit problem which couples the membrane displacement $\mathcal{U}_m^-$ of the plate and the torsion angle $\mathcal{R}_3^C$ in the cylinders. We set

$$\mathbf{w}_\varepsilon^\delta(x) = \begin{cases} \Big(1 - \chi\Big(\dfrac{x_1}{\varepsilon}, \dfrac{x_2}{\varepsilon}\Big)\Big)\phi_m(x_1, x_2) + \chi\Big(\dfrac{x_1}{\varepsilon}, \dfrac{x_2}{\varepsilon}\Big)\Big[\phi_m(p\varepsilon, q\varepsilon) - (x_2 - q\varepsilon)\Theta(p\varepsilon, q\varepsilon)\mathbf{e}_1 \\ \quad + (x_1 - p\varepsilon)\Theta(p\varepsilon, q\varepsilon)\mathbf{e}_2\Big] \\ \qquad \text{if } (x_1, x_2) \in \Big]\varepsilon p - \dfrac{\varepsilon}{2}, \varepsilon p + \dfrac{\varepsilon}{2}\Big[\times \Big]\varepsilon q - \dfrac{\varepsilon}{2}, \varepsilon q + \dfrac{\varepsilon}{2}\Big[, \quad \text{and } (p, q) \in \mathcal{N}_\varepsilon, \\ 0, \quad \text{if } (x_1, x_2) \in \omega \setminus \widetilde{\omega}_\varepsilon. \end{cases}$$

This displacement belongs to $V_{\varepsilon,\delta}$ and satisfies $\mathbf{w}_\varepsilon^\delta = 0$ in $(\omega \setminus \widetilde{\omega}_\varepsilon) \times]-\delta, 0[$. Moreover $\mathbf{w}_\varepsilon^\delta$ is a rigid displacement in $\Omega_{\varepsilon,\delta}^+$. Since ϕ_m and Θ are smooth we have

$$(8.10) \qquad \begin{cases} \delta\mathcal{T}^\varepsilon(\mathbf{w}_\varepsilon^\delta) \to 0 & \text{strongly in} \quad [L^2(\Omega \times D)]^3, \\ \Pi_\delta(\mathbf{w}_\varepsilon^\delta) \to \phi_m & \text{strongly in} \quad [L^2(\Omega^-)]^3. \end{cases}$$

Passing to the limit in the right hand side of (4.4) with the help of (8.10) gives the following term

$$(8.11) \qquad \int_\omega F_\alpha\phi_\alpha, \qquad \text{with} \qquad F_\alpha(x_1, x_2) = \int_{-1}^0 f_\alpha(x_1, x_2, X_3)dX_3.$$

As far as the strain of the field $\mathbf{w}_\varepsilon^\delta$ is concerned, the strong convergences hold true in $L^2(\Omega^- \times Y)$

$$(8.12) \qquad \begin{cases} \mathcal{T}_\varepsilon \circ \Pi_\delta\big(\gamma_{11}(\mathbf{w}_\varepsilon^\delta)\big) \to \big(1 - \chi\big)\gamma_{11}(\phi_m) - \dfrac{\partial\chi}{\partial X_1}\Big[X_1\dfrac{\partial\phi_1}{\partial x_1} + X_2\Big\{\dfrac{\partial\phi_1}{\partial x_2} + \Theta\Big\}\Big], \\ \mathcal{T}_\varepsilon \circ \Pi_\delta\big(\gamma_{22}(\mathbf{w}_\varepsilon^\delta)\big) \to \big(1 - \chi\big)\gamma_{22}(\phi_m) - \dfrac{\partial\chi}{\partial X_2}\Big[X_1\Big\{\dfrac{\partial\phi_2}{\partial x_1} - \Theta\Big\} + X_2\dfrac{\partial\phi_2}{\partial x_2}\Big], \\ \mathcal{T}_\varepsilon \circ \Pi_\delta\big(\gamma_{12}(\mathbf{w}_\varepsilon^\delta)\big) \to \big(1 - \chi\big)\gamma_{12}(\phi_m) - \dfrac{1}{2}\dfrac{\partial\chi}{\partial X_2}\Big[X_1\dfrac{\partial\phi_1}{\partial x_1} + X_2\Big\{\dfrac{\partial\phi_1}{\partial x_2} + \Theta\Big\}\Big] \\ \qquad\qquad - \dfrac{1}{2}\dfrac{\partial\chi}{\partial X_1}\Big[X_1\Big\{\dfrac{\partial\phi_2}{\partial x_1} - \Theta\Big\} + X_2\dfrac{\partial\phi_2}{\partial x_2}\Big], \\ \mathcal{T}_\varepsilon \circ \Pi_\delta\big(\gamma_{i3}(\mathbf{w}_\varepsilon^\delta)\big) = 0. \end{cases}$$

Passing to the limit in (4.4) with the test-displacement $\mathbf{w}^\delta_\varepsilon$, using (6.11), (6.12) and the strong convergences (8.12) leads to

$$\frac{E}{1-\nu^2}\int_{\Omega^-\times Y}\Big[(1-\nu)\Gamma_{\alpha\beta}(\widetilde{\mathbf{U}}_m+\widetilde{\mathbf{U}}_f)\Gamma_{\alpha\beta}(\widetilde{\Phi}_m)+\nu\Gamma_{\alpha\alpha}(\widetilde{\mathbf{U}}_m+\widetilde{\mathbf{U}}_f)\Gamma_{\beta\beta}(\widetilde{\Phi}_m)\Big]=\int_\omega F_\alpha\phi_\alpha,$$

where

$$\begin{aligned}\widetilde{\Phi}_m=&\Big(\Big[X_1\gamma_{11}(\phi_m)+X_2\Big\{\frac{\partial\phi_1}{\partial x_2}+\Theta\Big\}\Big]-\chi(X)\Big[X_1\gamma_{11}(\phi_m)+X_2\Big\{\frac{\partial\phi_1}{\partial x_2}+\Theta\Big\}\Big]\Big)\mathbf{e}_1\\&+\Big(\Big[X_2\gamma_{22}(\phi_m)+X_1\Big\{\frac{\partial\phi_2}{\partial x_1}-\Theta\Big\}\Big]-\chi(X)\Big[X_2\gamma_{22}(\phi_m)+X_1\Big\{\frac{\partial\phi_2}{\partial x_1}-\Theta\Big\}\Big]\Big)\mathbf{e}_2.\end{aligned}$$

Using the expressions of $\widetilde{\mathbf{U}}_f$ and $\widetilde{\Phi}_m$, we obtain

$$\frac{E}{1-\nu^2}\int_\omega <\widetilde{\mathbf{U}}_m,\widetilde{\Phi}_m>_m=\int_\omega F_\alpha\phi_\alpha,\tag{8.13}$$

where

$$\begin{aligned}\widetilde{\Phi}_m=&\Big(\Big[X_1\gamma_{11}(\phi_m)+X_2\Big\{\frac{\partial\phi_1}{\partial x_2}+\Theta\Big\}\Big]-\chi(X)\Big[X_1\gamma_{11}(\phi_m)+X_2\Big\{\frac{\partial\phi_1}{\partial x_2}+\Theta\Big\}\Big]\Big)\mathbf{e}_1\\&+\Big(\Big[X_2\gamma_{22}(\phi_m)+X_1\Big\{\frac{\partial\phi_2}{\partial x_1}-\Theta\Big\}\Big]-\chi(X)\Big[X_2\gamma_{22}(\phi_m)+X_1\Big\{\frac{\partial\phi_2}{\partial x_1}-\Theta\Big\}\Big]\Big)\mathbf{e}_2.\end{aligned}$$

We have

$$<\widetilde{\mathbf{U}}_m,\widetilde{\Phi}_m>_m=<\widehat{\mathbf{u}}_m,\widetilde{\Phi}_m>_m+(1-\nu)\gamma_{\alpha\beta}(\mathcal{U}^-_m)\gamma_{\alpha\beta}(\phi_m)+\nu\gamma_{\alpha\alpha}(\mathcal{U}^-_m)\gamma_{\beta\beta}(\phi_m).$$

Due to the periodicity of $\widehat{\mathbf{u}}_m$, the fact that $\chi=1$ on D_k and (8.1), we obtain

$$<\widehat{\mathbf{u}}_m,\widetilde{\Phi}_m>_m=<\widehat{\mathbf{u}}_m,\widehat{\Phi}_m>_m$$

where

$$\widehat{\Phi}_m=-\gamma_{11}(\phi_m)\widehat{\mathbf{v}}^{(1)}_m-\Big(\frac{\partial\phi_1}{\partial x_2}+\Theta\Big)\widehat{\mathbf{v}}^{(2)}_m-\Big(\frac{\partial\phi_2}{\partial x_1}-\Theta\Big)\widehat{\mathbf{v}}^{(3)}_m-\gamma_{22}(\phi_m)\widehat{\mathbf{v}}^{(4)}_m.$$

Taking into account the definition of the matrix $\mathbf{A}_m$ we finally obtain

$$<\widetilde{\mathbf{U}}_m,\widetilde{\Phi}_m>_m=\mathbf{A}_m\begin{pmatrix}\gamma_{11}(\mathcal{U}^-_m)\\ \frac{\partial\mathcal{U}^-_1}{\partial x_2}+\mathcal{R}^C_3\\ \frac{\partial\mathcal{U}^-_2}{\partial x_1}-\mathcal{R}^C_3\\ \gamma_{22}(\mathcal{U}^-_m)\end{pmatrix}\cdot\begin{pmatrix}\gamma_{11}(\phi_m)\\ \frac{\partial\phi_1}{\partial x_2}+\Theta\\ \frac{\partial\phi_2}{\partial x_1}-\Theta\\ \gamma_{22}(\phi_m)\end{pmatrix}+(1-\nu)\gamma_{\alpha\beta}(\mathcal{U}^-_m)\gamma_{\alpha\beta}(\phi_m)+\nu\gamma_{\alpha\alpha}(\mathcal{U}^-_m)\gamma_{\beta\beta}(\phi_m).$$

In view of (8.13) and using standard density results, the above equality shows that $\mathcal{U}^-_m$ and $\mathcal{R}^C_3$ are solutions of the problem

$$\begin{cases}\mathcal{U}^-_m\in H^1_0(\omega;\mathbb{R}^2),\quad \mathcal{R}^C_3\in L^2(\omega)\\ \displaystyle\frac{E}{1-\nu^2}\int_\omega\Big[\mathbf{A}_m\begin{pmatrix}\gamma_{11}(\mathcal{U}^-_m)\\ \frac{\partial\mathcal{U}^-_1}{\partial x_2}+\mathcal{R}^C_3\\ \frac{\partial\mathcal{U}^-_2}{\partial x_1}-\mathcal{R}^C_3\\ \gamma_{22}(\mathcal{U}^-_m)\end{pmatrix}\cdot\begin{pmatrix}\gamma_{11}(\phi)\\ \frac{\partial\phi_1}{\partial x_2}+\Theta\\ \frac{\partial\phi_2}{\partial x_1}-\Theta\\ \gamma_{22}(\phi)\end{pmatrix}+(1-\nu)\gamma_{\alpha\beta}(\mathcal{U}^-_m)\gamma_{\alpha\beta}(\phi)+\nu\gamma_{\alpha\alpha}(\mathcal{U}^-_m)\gamma_{\beta\beta}(\phi)\Big]\\ \displaystyle=\int_\omega F_\alpha\phi_\alpha,\qquad\forall\phi\in H^1_0(\omega;\mathbb{R}^2),\quad\forall\Theta\in L^2(\omega).\end{cases}\tag{8.14}$$

In order to derive $\mathcal{R}_3^C$ in terms of $\mathcal{U}_m^-$, we first choose $\phi = 0$ in (8.14) and we obtain due to the expression (8.9) of $\mathbf{A}_m$

$$(c-d)\Big(\frac{\partial \mathcal{U}_1^-}{\partial x_2} - \frac{\partial \mathcal{U}_2^-}{\partial x_1} + 2\mathcal{R}_3^C\Big) = 0,$$

which gives since $c \neq d$

$$\mathcal{R}_3^C = -\frac{1}{2}\Big(\frac{\partial \mathcal{U}_1^-}{\partial x_2} - \frac{\partial \mathcal{U}_2^-}{\partial x_1}\Big). \tag{8.15}$$

Through elimination of the function $\mathcal{R}_3^C$ and choosing $\Theta = 0$ in (8.14) permit to obtain the following Theorem.

Theorem 8.1: *The membrane displacement* $\mathcal{U}_m^-$ *is the solution of the problem*

$$\begin{cases} \mathcal{U}_m^- \in H_0^1(\omega;\mathbb{R}^2), \\ \dfrac{E}{1-\nu^2}\displaystyle\int_\omega \Big[\mathcal{A}_m \begin{pmatrix} \gamma_{11}(\mathcal{U}_m^-) \\ \gamma_{12}(\mathcal{U}_m^-) \\ \gamma_{22}(\mathcal{U}_m^-) \end{pmatrix} \cdot \begin{pmatrix} \gamma_{11}(\phi) \\ \gamma_{12}(\phi) \\ \gamma_{22}(\phi) \end{pmatrix} + (1-\nu)\gamma_{\alpha\beta}(\mathcal{U}_m^-)\gamma_{\alpha\beta}(\phi) + \nu\gamma_{\alpha\alpha}(\mathcal{U}_m^-)\gamma_{\beta\beta}(\phi)\Big] \\ = \displaystyle\int_\omega F_\alpha \phi_\alpha, \qquad \forall \phi \in H_0^1(\omega;\mathbb{R}^2), \end{cases} \tag{8.16}$$

where

$$\mathcal{A}_m = \begin{pmatrix} a & 0 & b \\ 0 & 2(c+d) & 0 \\ b & 0 & a \end{pmatrix}$$

where the real numbers a,b and $2(c+d)$ *are given by*

$$a =< \widehat{\mathbf{v}}_m^{(1)}, \widehat{\mathbf{v}}_m^{(1)} >_m, \quad b =< \widehat{\mathbf{v}}_m^{(1)}, \widehat{\mathbf{v}}_m^{(4)} >_m, \quad 2(c+d) =< \widehat{\mathbf{v}}_m^{(2)} + \widehat{\mathbf{v}}_m^{(3)}, \widehat{\mathbf{v}}_m^{(2)} + \widehat{\mathbf{v}}_m^{(3)} >_m .$$

□

Remark 8.2 Indeed the properties of the matrix $\mathcal{A}_m$ show that (8.16) admits a unique solution. □

In the standard membrane elastic problem, the usual matrix $\mathcal{A}_m$ is equal to zero in (8.16) so that the membrane elastic matrix is given by $\begin{pmatrix} 1 & 0 & \nu \\ 0 & 2(1-\nu) & 0 \\ \nu & 0 & 1 \end{pmatrix}$. Let us emphazise that, in the present case where $\dfrac{\delta}{\varepsilon} \to 0$, the membrane elastic matrix in (8.16) is equal to $\begin{pmatrix} 1+a & 0 & \nu+b \\ 0 & 2(1-\nu+c+d) & 0 \\ \nu+b & 0 & 1+a \end{pmatrix}$ which corresponds to an anisotropic elastic material. This shows that the rods induce microscopic effects on the elastic matrix of the membrane problem. Actually, since the material is homogeneous, they are due to geometrical effects which take into account the fact that the rods force the small cylinders $\mathcal{C}^{\varepsilon\delta}$ (in the plate below the rods) become asymptotically rigid bodies. These microscopic effects do not occur in the case where $\dfrac{\delta}{\varepsilon} \to +\infty$ (see [1] and [2]).

9. The limit bending for the plate

This section is organized as the previous one. We first express the bending corrector $\widehat{\mathbf{u}}_3$ in terms of three basic bending correctors and then we deduce the bending macroscopic problem for the plate.

9.1 Determination of the bending corrector $\widehat{\mathbf{u}}_3$

In this step we derive the expression of the function $\widehat{\mathbf{u}}_3$ in terms of $\mathcal{U}_3^-$ and of three basic correctors. We consider a test displacement of the following type:

$$\mathbf{V}_\varepsilon^\delta(x) = -(x_3+\delta/2)\frac{\partial \mathbf{V}_{\varepsilon,3}^\delta}{\partial x_1}\mathbf{e}_1 - (x_3+\delta/2)\frac{\partial \mathbf{V}_{\varepsilon,3}^\delta}{\partial x_2}\mathbf{e}_2 + \mathbf{V}_{\varepsilon,3}^\delta\mathbf{e}_3,$$
$$\mathbf{V}_{\varepsilon,3}^\delta(x) = \frac{\varepsilon^2}{\delta}\psi_\varepsilon(x_1,x_2)\Phi\Big(\frac{x_1}{\varepsilon},\frac{x_2}{\varepsilon}\Big),$$

where the function ψ_ε is defined in Subsection 8.1 and $\Phi\in\mathbb{V}^2_{per,D_k,1}(Y)$. We first have

$$\|\mathbf{V}_\varepsilon^\delta\|_{(L^2(\Omega^+))^3}\le C\frac{\varepsilon}{\delta},$$

and then

$$\delta\mathcal{T}^\varepsilon(\mathbf{V}_\varepsilon^\delta)\to 0 \qquad \text{strongly in} \quad [L^2(\Omega^+\times D)]^3.$$

Secondly one has

$$\|\mathbf{V}_{\varepsilon,3}^\delta\|_{L^2(\omega)}\le C\frac{\varepsilon^2}{\delta}, \qquad \|\Pi_\delta(\mathbf{V}_{\varepsilon,\alpha}^\delta)\|_{L^2(\Omega^-)}\le C\varepsilon$$

and then

$$\begin{cases} \delta\Pi_\delta(\mathbf{V}_{\varepsilon,3}^\delta)\to 0 & \text{strongly in} \quad L^2(\Omega^-),\\ \Pi_\delta(\mathbf{V}_{\varepsilon,\alpha}^\delta)\to 0 & \text{strongly in} \quad L^2(\Omega^-).\end{cases} \tag{9.1}$$

Due to the definition of ψ_ε and to the properties of Φ, the strain tensor of the above test displacement is zero in the rods Ω_ε^+. As far as this strain tensor in Ω^- is concerned, we have

$$\begin{cases} \mathcal{T}_\varepsilon\circ\Pi_\delta\big(\gamma_{\alpha\beta}(\mathbf{V}_\varepsilon^\delta)\big) \longrightarrow -\Big(X_3+\frac{1}{2}\Big)\psi\,\frac{\partial^2\Phi}{\partial X_\alpha\partial X_\beta} & \text{strongly in} \quad L^2(\Omega^-\times Y),\\ \mathcal{T}_\varepsilon\circ\Pi_\delta\big(\gamma_{i3}(\mathbf{V}_\varepsilon^\delta)\big)=0. \end{cases} \tag{9.2}$$

Let us introduce the local Kirchhoff-Love displacement Φ_f

$$\Phi_f = -\Big(X_3+\frac{1}{2}\Big)\frac{\partial\Phi}{\partial X_1}\mathbf{e}_1 - \Big(X_3+\frac{1}{2}\Big)\frac{\partial\Phi}{\partial X_2}\mathbf{e}_2 + \Phi\mathbf{e}_3$$

so that the limit in (9.2) is nothing else that the strain tensor $\Gamma_{\alpha\beta}(\psi\Phi_f)$. Using this notation, passing to the limit in (4.4) according to (6.12), (9.1) and (9.2) leads to

$$\frac{E}{1-\nu^2}\int_{\Omega^-\times Y}\psi\Big[(1-\nu)\Gamma_{\alpha\beta}(\widetilde{\mathbf{U}}_m+\widetilde{\mathbf{U}}_f)\Gamma_{\alpha\beta}(\Phi_f)+\nu\Gamma_{\alpha\alpha}(\widetilde{\mathbf{U}}_m+\widetilde{\mathbf{U}}_f)\Gamma_{\beta\beta}(\Phi_f)\Big]=0,$$

which implies that, using also the periodicity of Φ_f

$$\int_Y\Big[(1-\nu)\frac{\partial^2\widehat{\mathbf{u}}_3}{\partial X_\alpha\partial X_\beta}\frac{\partial^2\Phi}{\partial X_\alpha\partial X_\beta}+\nu\Delta_X\widehat{\mathbf{u}}_3\Delta_X\Phi\Big]=0 \qquad \text{a.e. in } \Omega^-. \tag{9.3}$$

Let us now introduce the local correctors in order to explicit the expression of $\widehat{\mathbf{u}}_3$ in terms of $\mathcal{U}_3^-$.

The space $\mathbb{V}^2_{per,D_k,2}(Y)$ is endowed with the scalar product

$$< \Phi, \Psi >_f = \int_Y \Big[(1-\nu)\frac{\partial^2\Phi}{\partial X_\alpha \partial X_\beta}\frac{\partial^2\Psi}{\partial X_\alpha \partial X_\beta} + \nu\Delta_X\Phi\Delta_X\Psi\Big].$$

Recalling that χ is a function of $\mathcal{C}_0^\infty(Y)$ such that $\chi = 1$ on D_k, the three functions $1/2\chi(X)X_1^2$, $\chi(X)X_1X_2$ and $1/2\chi(X)X_2^2$ indeed belong to $\mathbb{V}^2_{per,D_k,2}(Y)$. Let us denote respectively by $\widehat{V}^{(1)}$, $\widehat{V}^{(2)}$ and $\widehat{V}^{(3)}$ the orthogonal projections of $1/2\chi(X)X_1^2$, $1/2\chi(X)X_2^2$ and $\chi(X)X_1X_2$ on the subspace $\big(\mathbb{V}^2_{per,D_k,1}(Y)\big)^\perp$ with respect to the scalar product $< .,. >_f$. Remark that the correctors $\widehat{V}^{(1)}$, $\widehat{V}^{(2)}$ and $\widehat{V}^{(3)}$ do not depend on the function χ since $\chi = 1$ in D_k. Now in view of (9.3), of the quadratic part of $\widehat{\mathbf{u}}_3$ given by (7.1) and the definitions of the correctors we deduce the following decomposition

$$\widehat{\mathbf{u}}_3 = -\frac{\partial^2\mathcal{U}_3^-}{\partial x_1^2}\widehat{V}^{(1)} - 2\frac{\partial^2\mathcal{U}_3^-}{\partial x_1\partial x_2}\widehat{V}^{(2)} - \frac{\partial^2\mathcal{U}_3^-}{\partial x_2^2}\widehat{V}^{(3)}. \tag{9.4}$$

9.2 Properties of the basic bending correctors $\widehat{V}^{(i)}$

Using the explicit expression of the scalar product $< .,. >_f$, the geometrical symmetry of the cell Y and of D_k and the symmetric properties of the three functions $1/2X_1^2$, X_1X_2 and $1/2X_2^2$, we have that

$$\begin{cases} \widehat{V}^{(3)}(X_1,X_2) = \widehat{V}^{(1)}(X_2,X_1) \qquad \text{and} \qquad \widehat{V}^{(2)}(X_1,X_2) = \widehat{V}^{(2)}(X_2,X_1) \\ \widehat{V}^{(1)} \text{ is even w.r.t. } X_1 \text{ and even w.r.t. } X_2 \\ \widehat{V}^{(2)} \text{ is odd w.r.t. } X_1 \text{ and odd w.r.t. } X_2 \\ \widehat{V}^{(3)} \text{ is even w.r.t. } X_1 \text{ and even w.r.t. } X_2. \end{cases} \tag{9.5}$$

From the above properties of $\widehat{V}^{(1)}$, $\widehat{V}^{(2)}$ and $\widehat{V}^{(3)}$, it follows that

$$< \widehat{V}^{(1)}, \widehat{V}^{(2)} >_f = 0 \quad \text{and} \quad < \widehat{V}^{(3)}, \widehat{V}^{(2)} >_f = 0. \tag{9.6}$$

Let us now define the 3×3 real matrix $\mathcal{A}_f$ by $(\mathcal{A}_f)_{ij} = < \widehat{V}^{(i)}, \widehat{V}^{(j)} >_f$. Indeed, $\mathcal{A}_f$ is a positively defined matrix. Moreover, as a consequence of the properties (9.5) and (9.6), we obtain the following structure of the matrix $\mathcal{A}_f$:

$$\mathcal{A}_f = \begin{pmatrix} a' & 0 & b' \\ 0 & c' & 0 \\ b' & 0 & a' \end{pmatrix} \qquad \text{where} \qquad a' = < \widehat{V}^{(1)}, \widehat{V}^{(1)} >_f, \quad b' = < \widehat{V}^{(1)}, \widehat{V}^{(3)} >_f, \quad c' = < \widehat{V}^{(2)}, \widehat{V}^{(2)} >_f .$$

9.3 The bending problem.

Let ϕ be arbitrary in $\mathcal{C}_0^\infty(\omega)$ and define the function ϕ_ε on $\omega\times]-\delta, L[$ by

$$\phi_\varepsilon(x) = \begin{cases} \phi(p\varepsilon,q\varepsilon) + (x_1 - p\varepsilon)\dfrac{\partial\phi}{\partial x_1}(p\varepsilon,q\varepsilon) + (x_2 - q\varepsilon)\dfrac{\partial\phi}{\partial x_2}(p\varepsilon,q\varepsilon) \\ \text{if} \quad x \in \Big]\varepsilon p - \dfrac{\varepsilon}{2}, \varepsilon p + \dfrac{\varepsilon}{2}\Big[\times \Big]\varepsilon q - \dfrac{\varepsilon}{2}, \varepsilon q + \dfrac{\varepsilon}{2}\Big[\times]-\delta, L[, \qquad \text{and } (p,q) \in \mathcal{N}_\varepsilon, \\ 0 \qquad \text{if } x \in (\omega\setminus\widetilde{\omega}_\varepsilon)\times]-\delta, L[. \end{cases}$$

We choose the following test displacement of $V_{\varepsilon,\delta}$

$$\mathbf{w}^\delta_{\varepsilon,\alpha}(x) = -(x_3 + \delta/2)\frac{\partial \mathbf{w}^\delta_{\varepsilon,3}}{\partial x_\alpha} \qquad \text{where} \qquad \mathbf{w}^\delta_{\varepsilon,3}(x) = \frac{1}{\delta}\Big(1 - \chi\Big(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}\Big)\Big)\phi + \frac{1}{\delta}\chi\Big(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}\Big)\phi_\varepsilon(x).$$

Let us notice that the displacement $\mathbf{w}_\varepsilon^\delta$ is a rigid displacement in each rod of Ω_ε^+ while it is a Kirchhoff-Love displacement in Ω_δ^-. Since ϕ is smooth and $\dfrac{\phi-\phi_\varepsilon}{\varepsilon}$ tends to 0 in $L^\infty(\omega)$, we have the following convergences:

$$\begin{cases} \delta\mathcal{T}^\varepsilon(\mathbf{w}^\delta_{\varepsilon,\alpha}) \to -x_3\dfrac{\partial\phi}{\partial x_\alpha} \qquad \text{strongly in} \quad L^2(\Omega^+\times D),\\ \delta\mathcal{T}^\varepsilon(\mathbf{w}^\delta_{\varepsilon,3}) \to \phi \qquad \text{strongly in} \quad L^2(\Omega^+\times D),\\ \Pi_\delta(\mathbf{w}^\delta_{\varepsilon,\alpha}) \to -\Big(X_3+\dfrac12\Big)\dfrac{\partial\phi}{\partial x_\alpha} \qquad \text{strongly in} \quad L^2(\Omega^-),\\ \delta\Pi_\delta(\mathbf{w}^\delta_{\varepsilon,3}) \to \phi \qquad \text{strongly in} \quad L^2(\Omega^-). \end{cases} \tag{9.7}$$

Now we derive the limit of the unfold strain $\gamma_{ij}(\mathbf{w}_\varepsilon^\delta)$. We only detail the computations for $\gamma_{11}(\mathbf{w}_\varepsilon^\delta)$. We have

$$\mathcal{T}_\varepsilon\circ\Pi_\delta\big(\gamma_{11}(\mathbf{w}_\varepsilon^\delta)\big) = -\Big(X_3+\frac12\Big)\Big[(1-\chi)\mathcal{T}_\varepsilon\Big(\frac{\partial^2\phi}{\partial x_1^2}\Big) - 2\frac{\mathcal{T}_\varepsilon\Big(\dfrac{\partial\phi}{\partial x_1}\Big)-\mathcal{T}_\varepsilon\Big(\dfrac{\partial\phi_\varepsilon}{\partial x_1}\Big)}{\varepsilon}\frac{\partial\chi}{\partial X_1} - \frac{\mathcal{T}_\varepsilon(\phi)-\mathcal{T}_\varepsilon(\phi_\varepsilon)}{\varepsilon^2}\frac{\partial^2\chi}{\partial X_1^2}\Big].$$

We use the following results which can be found in Lemma A1 in Appendix A of [2] and which allow to pass to the limit in $\mathcal{T}_\varepsilon\circ\Pi_\delta\big(\gamma_{11}(\mathbf{w}_\varepsilon^\delta)\big)$

$$\begin{aligned} \frac{\mathcal{T}_\varepsilon\Big(\dfrac{\partial\phi}{\partial x_1}\Big)-\mathcal{T}_\varepsilon\Big(\dfrac{\partial\phi_\varepsilon}{\partial x_1}\Big)}{\varepsilon} &\to X_1\frac{\partial^2\phi}{\partial x_1^2}+X_2\frac{\partial^2\phi}{\partial x_1\partial x_2} \qquad \text{strongly in} \qquad L^\infty(\omega\times Y),\\ \frac{\mathcal{T}_\varepsilon(\phi)-\mathcal{T}_\varepsilon(\phi_\varepsilon)}{\varepsilon^2} &\to \frac12\Big[X_1^2\frac{\partial^2\phi}{\partial x_1^2}+2X_1X_2\frac{\partial^2\phi}{\partial x_1\partial x_2}+X_2^2\frac{\partial^2\phi}{\partial x_2^2}\Big] \qquad \text{strongly in} \qquad L^\infty(\omega\times Y). \end{aligned}$$

Proceeding similarly for the other components of the strain tensor, we finally obtain:

$$\begin{cases} \mathcal{T}_\varepsilon\circ\Pi_\delta\big(\gamma_{\alpha\alpha}(\mathbf{w}_\varepsilon^\delta)\big) \longrightarrow -\Big(X_3+\dfrac12\Big)\Big((1-\chi)\dfrac{\partial^2\phi}{\partial x_\alpha^2} - 2\dfrac{\partial\chi}{\partial X_\alpha}\Big[X_1\dfrac{\partial^2\phi}{\partial x_1\partial x_\alpha}+X_2\dfrac{\partial^2\phi}{\partial x_2\partial x_\alpha}\Big]\\ \qquad\qquad -\dfrac12\dfrac{\partial^2\chi}{\partial X_\alpha^2}\Big[X_1^2\dfrac{\partial^2\phi}{\partial x_1^2}+2X_1X_2\dfrac{\partial^2\phi}{\partial x_1\partial x_2}+X_2^2\dfrac{\partial^2\phi}{\partial x_2^2}\Big]\Big),\\ \mathcal{T}_\varepsilon\circ\Pi_\delta\big(\gamma_{12}(\mathbf{w}_\varepsilon^\delta)\big) \longrightarrow -\Big(X_3+\dfrac12\Big)\Big((1-\chi)\dfrac{\partial^2\phi}{\partial x_1\partial x_2} - \dfrac{\partial\chi}{\partial X_1}\Big[X_1\dfrac{\partial^2\phi}{\partial x_1\partial x_2}+X_2\dfrac{\partial^2\phi}{\partial x_2^2}\Big]\\ \qquad -\dfrac{\partial\chi}{\partial X_2}\Big[X_1\dfrac{\partial^2\phi}{\partial x_1^2}+X_2\dfrac{\partial^2\phi}{\partial x_1\partial x_2}\Big] - \dfrac12\dfrac{\partial^2\chi}{\partial X_1\partial X_2}\Big[X_1^2\dfrac{\partial^2\phi}{\partial x_1^2}+2X_1X_2\dfrac{\partial^2\phi}{\partial x_1\partial x_2}+X_2^2\dfrac{\partial^2\phi}{\partial x_2^2}\Big]\Big),\\ \mathcal{T}_\varepsilon\circ\Pi_\delta\big(\gamma_{i3}(\mathbf{w}_\varepsilon^\delta)\big) = 0. \end{cases} \tag{9.8}$$

Let us introduce the local Kirchhoff-Love displacement $\widetilde{\Phi}_f$ (i.e. with respect to the local variables (X_1,X_2,X_3))

$$\begin{aligned} \widetilde{\Phi}_f &= -\Big(X_3+\frac12\Big)\frac{\partial\widetilde{\Phi}_3}{\partial X_1}\mathbf{e}_1 - \Big(X_3+\frac12\Big)\frac{\partial\widetilde{\Phi}_3}{\partial X_2}\mathbf{e}_2+\widetilde{\Phi}_3\mathbf{e}_3\\ \text{where}\qquad \widetilde{\Phi}_3 &= \Big[\frac{X_1^2}{2}\frac{\partial^2\phi}{\partial x_1^2}+X_1X_2\frac{\partial^2\phi}{\partial x_1\partial x_2}+\frac{X_2^2}{2}\frac{\partial^2\phi}{\partial x_2^2}\Big]-\frac{\chi}{2}\Big[X_1^2\frac{\partial^2\phi}{\partial x_1^2}+2X_1X_2\frac{\partial^2\phi}{\partial x_1\partial x_2}+X_2^2\frac{\partial^2\phi}{\partial x_2^2}\Big]. \end{aligned}$$

Passing to the limit in (4.4) with the test-displacement $\mathbf{w}_\varepsilon^\delta$, using (6.11), (6.12), (9.7) and the strong convergences (9.8) leads to the following problem:

$$\frac{E}{1-\nu^2}\int_{\Omega^-\times Y}\Big[(1-\nu)\Gamma_{\alpha\beta}(\widetilde{\mathbf{U}}_m+\widetilde{\mathbf{U}}_f)\Gamma_{\alpha\beta}(\widetilde{\Phi}_f)+\nu\Gamma_{\alpha\alpha}(\widetilde{\mathbf{U}}_m+\widetilde{\mathbf{U}}_f)\Gamma_{\beta\beta}(\widetilde{\Phi}_f)\Big] = \int_\omega F_3\phi - \int_\omega G_\alpha\frac{\partial\phi}{\partial x_\alpha}$$

where

$$(9.9) \qquad \begin{cases} F_3 = \displaystyle\int_{-1}^{0} f_3(.,.,X_3)dX_3 + k^2 \int_0^L f_3(.,.,x_3)dx_3, \\ G_\alpha = \displaystyle\int_{-1}^{0} (X_3 + \frac{1}{2}) f_\alpha(.,.,X_3)dX_3 + k^2 \int_0^L x_3 f_\alpha(.,.,x_3)dx_3. \end{cases}$$

Then using the fact that $\widetilde{\mathbf{U}}_m$ does not depend of X_3, the above equation gives

$$(9.10) \qquad \frac{E}{12(1-\nu^2)} \int_\omega < \widetilde{\mathbf{U}}_f, \widetilde{\Phi}_f >_f = \int_\omega F_3 \phi - \int_\omega G_\alpha \frac{\partial \phi}{\partial x_\alpha}.$$

Now we set

$$\widehat{\mathbf{u}}_f = -\Big(X_3 + \frac{1}{2}\Big) \frac{\partial \widehat{\mathbf{u}}_3}{\partial X_1} \mathbf{e}_1 - \Big(X_3 + \frac{1}{2}\Big) \frac{\partial \widehat{\mathbf{u}}_3}{\partial X_2} \mathbf{e}_2 + \widehat{\mathbf{u}}_3 \mathbf{e}_3$$

so that

$$< \widetilde{\mathbf{U}}_f, \widetilde{\Phi}_f >_f = < \widehat{\mathbf{u}}_f, \widetilde{\Phi}_f >_f + (1-\nu) \frac{\partial^2 \mathcal{U}_3^-}{\partial x_\alpha \partial x_\beta} \frac{\partial^2 \phi}{\partial x_\alpha \partial x_\beta} + \nu \Delta \mathcal{U}_3^- \Delta \phi.$$

Due to the periodicity of $\widehat{\mathbf{u}}_f$, the fact that $\chi = 1$ on D_k and (9.3) we obtain

$$< \widehat{\mathbf{u}}_f, \widetilde{\Phi}_f >_f = < \widehat{\mathbf{u}}_f, \widehat{\Phi}_f >_f,$$

where

$$\begin{aligned} \widehat{\Phi}_f &= -\Big(X_3 + \frac{1}{2}\Big) \frac{\partial \widehat{\Phi}_3}{\partial X_1} \mathbf{e}_1 - \Big(X_3 + \frac{1}{2}\Big) \frac{\partial \widehat{\Phi}_3}{\partial X_2} \mathbf{e}_2 + \widehat{\Phi}_3 \mathbf{e}_3, \\ \widehat{\Phi}_3 &= -\frac{\partial^2 \phi}{\partial x_1^2} \widehat{V}^{(1)} - 2 \frac{\partial^2 \phi}{\partial x_1 \partial x_2} \widehat{V}^{(2)} - \frac{\partial^2 \phi}{\partial x_2^2} \widehat{V}^{(3)}. \end{aligned}$$

Taking into account the definition of the matrix $\mathcal{A}_f$ we finally obtain

$$< \widetilde{\mathbf{U}}_f, \widetilde{\Phi}_f >_f = \mathcal{A}_f \begin{pmatrix} \frac{\partial^2 \mathcal{U}_3^-}{\partial x_1^2} \\ \frac{\partial^2 \mathcal{U}_3^-}{\partial x_1 \partial x_2} \\ \frac{\partial^2 \mathcal{U}_3^-}{\partial x_2^2} \end{pmatrix} \begin{pmatrix} \frac{\partial^2 \phi}{\partial x_1^2} \\ \frac{\partial^2 \phi}{\partial x_1 \partial x_2} \\ \frac{\partial^2 \phi}{\partial x_2^2} \end{pmatrix} + (1-\nu) \frac{\partial^2 \mathcal{U}_3^-}{\partial x_\alpha \partial x_\beta} \frac{\partial^2 \phi}{\partial x_\alpha \partial x_\beta} + \nu \Delta \mathcal{U}_3^- \Delta \phi.$$

In view of (9.10) and using standard density arguments, the above equality shows that $\mathcal{U}_3^-$ is the unique solution of the problem detailed in the following theorem.

Theorem 9.1: *The bending* $\mathcal{U}_3^-$ *is the unique solution of the problem*

$$(9.11) \qquad \begin{cases} \mathcal{U}_3^- \in H_0^2(\omega) \\ \displaystyle\frac{E}{12(1-\nu^2)} \int_\omega \Big[\mathcal{A}_f \begin{pmatrix} \frac{\partial^2 \mathcal{U}_3^-}{\partial x_1^2} \\ \frac{\partial^2 \mathcal{U}_3^-}{\partial x_1 \partial x_2} \\ \frac{\partial^2 \mathcal{U}_3^-}{\partial x_2^2} \end{pmatrix} \begin{pmatrix} \frac{\partial^2 \phi}{\partial x_1^2} \\ \frac{\partial^2 \phi}{\partial x_1 \partial x_2} \\ \frac{\partial^2 \phi}{\partial x_2^2} \end{pmatrix} + (1-\nu) \frac{\partial^2 \mathcal{U}_3^-}{\partial x_\alpha \partial x_\beta} \frac{\partial^2 \phi}{\partial x_\alpha \partial x_\beta} + \nu \Delta \mathcal{U}_3^- \Delta \phi \Big] \\ \displaystyle = \int_\omega F_3 \phi - \int_\omega G_\alpha \frac{\partial \phi}{\partial x_\alpha} \\ \forall \phi \in H_0^2(\omega) \end{cases}$$

where

$$\mathcal{A}_f=\begin{pmatrix} a' & 0 & b'\\ 0 & c' & 0\\ b' & 0 & a'\end{pmatrix}\qquad \textit{with}\qquad a'=<\widehat{V}^{(1)},\widehat{V}^{(1)}>_f,\quad b'=<\widehat{V}^{(1)},\widehat{V}^{(3)}>_f,\quad c'=<\widehat{V}^{(2)},\widehat{V}^{(2)}>_f\ .$$

□

Remark 9.2 Indeed the properties of the matrix $\mathcal{A}_f$ show that (9.11) admits a unique solution. □

Let us notice that, as for the membrane problem (8.16), there are microscopic effects of the rods on the bending elastic matrix in the problem (9.11) since the usual bending matrix for a homogeneous material is simply $\begin{pmatrix} 1 & 0 & \nu\\ 0 & 2(1-\nu) & 0\\ \nu & 0 & 1\end{pmatrix}$ while here it is equal to $\begin{pmatrix} 1+a' & 0 & \nu+b'\\ 0 & 2(1-\nu)+c' & 0\\ \nu+b' & 0 & 1+a'\end{pmatrix}$.

10. Convergence of the energies

In this section we prove that the rescaled elastic energy $\dfrac{1}{\delta}\mathcal{E}(u^\delta)$ converges to the total energy of the limit problems (8.16) and (9.11) as δ tends to zero. We take u^δ as a test displacement in (2.5) and we use the inequality

$$(10.1)\qquad \begin{cases} \dfrac{k^2}{\delta}\displaystyle\int_{\Omega^+\times D}\mathcal{T}^\varepsilon(\sigma^\delta_{ij})\mathcal{T}^\varepsilon(\gamma_{ij}(u^\delta))dx_1dx_2dx_3dX_1dX_2\\ +\displaystyle\int_{\Omega^-\times Y}\mathcal{T}_\varepsilon\circ\Pi_\delta(\sigma^\delta_{ij})\mathcal{T}_\varepsilon\circ\Pi_\delta(\gamma_{ij}(u^\delta))dx_1dx_2dX_3dX_1dX_2\le\dfrac{1}{\delta}\mathcal{E}(u^\delta)\end{cases}$$

to obtain

$$(10.2)\qquad \begin{cases} \dfrac{k^2}{\delta}\displaystyle\int_{\Omega^+\times D}\mathcal{T}^\varepsilon(\sigma^\delta_{ij})\mathcal{T}^\varepsilon(\gamma_{ij}(u^\delta))dx_1dx_2dx_3dX_1dX_2\\ +\displaystyle\int_{\Omega^-\times Y}\mathcal{T}_\varepsilon\circ\Pi_\delta(\sigma^\delta_{ij})\mathcal{T}_\varepsilon\circ\Pi_\delta(\gamma_{ij}(u^\delta))dx_1dx_2dX_3dX_1dX_2\\ \le k^2\delta\displaystyle\int_{\Omega^+\times D}\mathcal{T}^\varepsilon(f)\cdot\mathcal{T}^\varepsilon(u^\delta)dx_1dx_2dx_3dX_1dX_2\\ +\displaystyle\int_{\Omega^-}\Pi_\delta(f_\alpha)\Pi_\delta(u^\delta_\alpha)dx_1dx_2dX_3+\delta\int_{\Omega^-}\Pi_\delta(f_3)\Pi_\delta(u^\delta_3)dx_1dx_2dX_3.\end{cases}$$

Using (5.1),(5.3), (6.1), (6.7), (7.2), (8.11), (9.9) to pass to the limit in (10.2), leads to

$$(10.3)\qquad \begin{cases} A=\displaystyle\limsup_{\delta\to0}\Big(\dfrac{k^2}{\delta}\int_{\Omega^+\times D}\mathcal{T}^\varepsilon(\sigma^\delta_{ij})\mathcal{T}^\varepsilon(\gamma_{ij}(u^\delta))dx_1dx_2dx_3dX_1dX_2\\ \qquad+\displaystyle\int_{\Omega^-\times Y}\mathcal{T}_\varepsilon\circ\Pi_\delta(\sigma^\delta_{ij})\mathcal{T}_\varepsilon\circ\Pi_\delta(\gamma_{ij}(u^\delta))dx_1dx_2dX_3dX_1dX_2\Big)\\ \qquad\le\displaystyle\int_\omega F_\alpha\mathcal{U}^-_\alpha+\int_\omega F_3\mathcal{U}^-_3-\int_\omega G_\alpha\dfrac{\partial\mathcal{U}^-_3}{\partial x_\alpha}.\end{cases}$$

From (8.14) in which we choose $\phi=\mathcal{U}^-_m$ and $\Theta=\mathcal{R}^C_3$ we obtain

$$\int_\omega F_\alpha\mathcal{U}^-_\alpha=\frac{E}{1-\nu^2}\int_\omega\Big[\mathbf{A}_m\begin{pmatrix}\gamma_{11}(\mathcal{U}^-_m)\\ \gamma_{12}(\mathcal{U}^-_m)\\ \gamma_{21}(\mathcal{U}^-_m)\\ \gamma_{22}(\mathcal{U}^-_m)\end{pmatrix}\cdot\begin{pmatrix}\gamma_{11}(\mathcal{U}^-_m)\\ \gamma_{12}(\mathcal{U}^-_m)\\ \gamma_{21}(\mathcal{U}^-_m)\\ \gamma_{22}(\mathcal{U}^-_m)\end{pmatrix}+(1-\nu)\gamma_{\alpha\beta}(\mathcal{U}^-_m)\gamma_{\alpha\beta}(\mathcal{U}^-_m)+\nu\gamma_{\alpha\alpha}(\mathcal{U}^-_m)\gamma_{\beta\beta}(\mathcal{U}^-_m)\Big].$$

Now from (8.4), (8.15), the definition of the matrix $\mathbf{A}_m$ and the periodicity of $\widehat{\mathbf{u}}_m$ we deduce

$$\begin{aligned}\int_\omega F_\alpha \mathcal{U}_\alpha^- &= \frac{E}{1-\nu^2}\int_\omega \big[< \widehat{\mathbf{u}}_m, \widehat{\mathbf{u}}_m >_m + (1-\nu)\gamma_{\alpha\beta}(\mathcal{U}_m^-)\gamma_{\alpha\beta}(\mathcal{U}_m^-) + \nu\gamma_{\alpha\alpha}(\mathcal{U}_m^-)\gamma_{\beta\beta}(\mathcal{U}_m^-)\big]\\ &= \frac{E}{1-\nu^2}\int_\omega < \widetilde{\mathbf{U}}_m, \widetilde{\mathbf{U}}_m >_m .\end{aligned}$$

Proceding similarly from (9.11) we obtain

$$\int_\omega F_3\mathcal{U}_3^- - \int_\omega G_\alpha \frac{\partial \mathcal{U}_3^-}{\partial x_\alpha} = \frac{E}{12(1-\nu^2)}\int_\omega < \widetilde{\mathbf{U}}_f, \widetilde{\mathbf{U}}_f >_f .$$

Notice that

$$\begin{aligned}&\frac{E}{1-\nu^2}\int_\omega < \widetilde{\mathbf{U}}_m, \widetilde{\mathbf{U}}_m >_m + \frac{E}{12(1-\nu^2)}\int_\omega < \widetilde{\mathbf{U}}_f, \widetilde{\mathbf{U}}_f >_f\\ =&\frac{E}{1-\nu^2}\int_{\omega\times Y}\Big[(1-\nu)\Gamma_{\alpha\beta}(\widetilde{\mathbf{U}}_m)\Gamma_{\alpha\beta}(\widetilde{\mathbf{U}}_m) + \nu\Gamma_{\alpha\alpha}(\widetilde{\mathbf{U}}_m)\Gamma_{\beta\beta}(\widetilde{\mathbf{U}}_m)\Big]\\ +&\frac{E}{12(1-\nu^2)}\int_{\omega\times Y}\Big[(1-\nu)\Gamma_{\alpha\beta}(\widetilde{\mathbf{U}}_f)\Gamma_{\alpha\beta}(\widetilde{\mathbf{U}}_f) + \nu\Gamma_{\alpha\alpha}(\widetilde{\mathbf{U}}_f)\Gamma_{\beta\beta}(\widetilde{\mathbf{U}}_f)\Big]\\ =&\frac{E}{1-\nu^2}\int_{\Omega^-\times Y}\Big[(1-\nu)\Gamma_{\alpha\beta}(\widetilde{\mathbf{U}}_m+\widetilde{\mathbf{U}}_f)\Gamma_{\alpha\beta}(\widetilde{\mathbf{U}}_m+\widetilde{\mathbf{U}}_f) + \nu\Gamma_{\alpha\alpha}(\widetilde{\mathbf{U}}_m+\widetilde{\mathbf{U}}_f)\Gamma_{\beta\beta}(\widetilde{\mathbf{U}}_m+\widetilde{\mathbf{U}}_f)\Big].\end{aligned}$$

Then using (10.3), the above equalities lead to

$$A \le \frac{E}{1-\nu^2}\int_{\Omega^-\times Y}\Big[(1-\nu)\Gamma_{\alpha\beta}(\widetilde{\mathbf{U}}_m+\widetilde{\mathbf{U}}_f)\Gamma_{\alpha\beta}(\widetilde{\mathbf{U}}_m+\widetilde{\mathbf{U}}_f) + \nu\Gamma_{\alpha\alpha}(\widetilde{\mathbf{U}}_m+\widetilde{\mathbf{U}}_f)\Gamma_{\beta\beta}(\widetilde{\mathbf{U}}_m+\widetilde{\mathbf{U}}_f)\Big].$$

Now from (6.12) and (6.13), the right hand side of the above inequality can be expressed as

$$A \le \int_{\Omega^-\times Y}\Sigma_{ij}^- X_{ij}^-$$

so that the definition of A in (10.3), the linear constitutive relation (2.1) and the classical l.s.c. argument permit to conclude that

$$\text{(10.4)}\qquad \begin{cases} \dfrac{1}{\sqrt{\delta}}\mathcal{T}^\varepsilon(\sigma_{ij}^\delta) \to 0 \quad \text{strongly in} \quad L^2(\Omega^+\times D),\\ \dfrac{1}{\sqrt{\delta}}\mathcal{T}^\varepsilon(\gamma_{ij}(u^\delta)) \to 0 \quad \text{strongly in} \quad L^2(\Omega^+\times D),\end{cases}$$

and

$$\text{(10.5)}\qquad \begin{cases} \mathcal{T}_\varepsilon\circ\Pi_\delta(\gamma_{ij}(u^\delta)) \to X_{ij}^- \quad \text{strongly in} \quad L^2(\Omega^-\times Y),\\ \mathcal{T}_\varepsilon\circ\Pi_\delta(\sigma_{ij}^\delta) \to \Sigma_{ij}^- \quad \text{strongly in} \quad L^2(\Omega^-\times Y).\end{cases}$$

Indeed the above analysis and the strong convergences (10.4) and (10.5) show that the elastic energy of the problem (2.5) converges to the total energy of the limit problems (8.16) and (9.11) i.e.

$$\begin{aligned}\frac{1}{\delta}\mathcal{E}(u^\delta) \longrightarrow &\frac{E}{1-\nu^2}\int_\omega \Big[\mathcal{A}_m\begin{pmatrix}\gamma_{11}(\mathcal{U}_m^-)\\ \gamma_{12}(\mathcal{U}_m^-)\\ \gamma_{22}(\mathcal{U}_m^-)\end{pmatrix}\cdot\begin{pmatrix}\gamma_{11}(\mathcal{U}_m^-)\\ \gamma_{12}(\mathcal{U}_m^-)\\ \gamma_{22}(\mathcal{U}_m^-)\end{pmatrix} + (1-\nu)\gamma_{\alpha\beta}(\mathcal{U}_m^-)\gamma_{\alpha\beta}(\mathcal{U}_m^-) + \nu\gamma_{\alpha\alpha}(\mathcal{U}_m^-)\gamma_{\beta\beta}(\mathcal{U}_m^-)\Big]\\ &+\frac{E}{12(1-\nu^2)}\int_\omega\Big[\mathcal{A}_f\begin{pmatrix}\frac{\partial^2\mathcal{U}_3^-}{\partial x_1^2}\\ \frac{\partial^2\mathcal{U}_3^-}{\partial x_1\partial x_2}\\ \frac{\partial^2\mathcal{U}_3^-}{\partial x_2^2}\end{pmatrix}\begin{pmatrix}\frac{\partial^2\mathcal{U}_3^-}{\partial x_1^2}\\ \frac{\partial^2\mathcal{U}_3^-}{\partial x_1\partial x_2}\\ \frac{\partial^2\mathcal{U}_3^-}{\partial x_2^2}\end{pmatrix} + (1-\nu)\frac{\partial^2\mathcal{U}_3^-}{\partial x_\alpha\partial x_\beta}\frac{\partial^2\mathcal{U}_3^-}{\partial x_\alpha\partial x_\beta} + \nu\Delta\mathcal{U}_3^-\Delta\mathcal{U}_3^-\Big].\end{aligned}$$

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