ASYMPTOTIC BEHAVIOR OF STRUCTURES MADE OF CURVED RODS

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Abstract. In this paper we study the asymptotic behavior of a structure made of curved rods of thickness 2δ when $\delta \to 0$. This study is carried on within the frame of linear elasticity by using the unfolding method. It is based on several decompositions of the structure displacements and on the passing to the limit in fixed domains.

We show that any displacement of a structure is the sum of an elementary rods-structure displacement (e.r.s.d.) concerning the rods cross sections and a residual one related to the deformation of the cross-section. The e.r.s.d. coincide with rigid body displacements in the junctions. Any e.r.s.d. is given by two functions belonging to $H^1(\mathcal{S};\mathbb{R}^3)$ where $\mathcal{S}$ is the skeleton structure (i.e. the set of the rods middle lines). One of this function $\mathcal{U}$ is the skeleton displacement, the other $\mathcal{R}$ gives the cross-sections rotation. We show that $\mathcal{U}$ is the sum of an extensional displacement and an inextensional one. We establish a priori estimates and then we characterize the unfolded limits of the rods-structure displacements.

Eventually we pass to the limit in the linearized elasticity system and using all results in [5], on the one hand we obtain a variational problem that is satisfied by the limit extensional displacement, and on the other hand, a variational problem coupling the limit of inextensional displacement and the limit of the rods torsion angles.

Résumé. Dans cet article on étudie le comportement asymptotique d'une structure formée de poutres courbes d'épaisseur 2δ quand $\delta \to 0$. Cette étude se place dans le cadre de l'élasticité linéaire et utilise la méthode de l'éclatement. Cette méthode est basée sur plusieurs décompositions des déplacements et sur le passage à la limite dans des domaines fixes.

On montre que tout déplacement d'une structure formée de poutres est la somme d'un déplacement élémentaire de structure-poutres (d.e.s.p.) concernant les sections droites et d'un déplacement résiduel rendant compte du gauchissement des sections droites. Le d.e.s.p. coïncide avec un déplacement rigide dans les jonctions . Tout d.e.s.p. est donné par deux fonctions appartenant à $H^1(\mathcal{S};\mathbb{R}^3)$ où $\mathcal{S}$ est le squelette de la structure (i.e. l'ensemble des lignes moyennes des poutres). L'une de ces fonctions $\mathcal{U}$ est le déplacement du squelette, l'autre $\mathcal{R}$ donne les rotations des sections droites. On montre que $\mathcal{U}$ est la somme d'un déplacement extensionnel et d'un déplacement inextensionnel. On donne des estimations a priori et ensuite on caractérise les limites des éclatés des déplacements de la structure.

Finalement on passe à la limite dans les équations linéarisées de l'élasticité en utilisant tous les résultats de [5]. On obtient d'une part un problème variationnel vérifié par le déplacement extensionnel limite et d'autre part un problème couplant le déplacement inextensionnel limite et les limites des angles de torsion des poutres.

KEY WORDS: linear elasticity, junctions, beams, curved rods, unfolding method.

1. Introduction

This paper follows the one entitled “Asymptotic behavior of curved rods by the unfolding method”. Here we are interested in the asymptotic behavior of a structure made of curved rods in the framework of the linear elasticity according to the unfolding method. It consists in obtaining some displacements decompositions and then in passing to the limit in fixed domains.

The structure is made of curved rods whose cross sections are discs of radius δ. The middle lines of the rods are regular arcs (the skeleton structure $\mathcal{S}$). Let us take a displacement of the structure. According to the results obtained in [6] we write its restriction to each rod as the sum of a rod elementary displacement and a residual displacement. But this family of rod elementary displacements is not necessarily the restriction of a H^1 displacement of the structure because such displacements do not inevitably coincide in the junctions. This is the reason why we had to change the rod elementary displacements and to replace them by rigid body displacements in the junctions. Hence we obtain an elementary rods-structure displacement (e.r.s.d.) which has become an admissible displacement of the whole domain. Then, any displacement of the structure is the sum of an e.r.s.d. and a residual one. An e.r.s.d. is linear in the cross section and is given by two functions belonging to $H^1(\mathcal{S};\mathbb{R}^3)$. The first one $\mathcal{U}$ is the skeleton displacement and the second one gives the rotation of the cross section.

In order to account for the asymptotic behavior of the deformations tensor and the strains tensor we shall decompose the first component $\mathcal{U}$ of the elementary rods-structure displacement into the sum of an extensional displacement and an inextensional one. An extensional displacement modifies the length of the middle lines while an inextensional displacement does not change this length in a first approximation.

Very few papers have been dedicated to the study of the junction of two elastic bodies. The case of the junction of a three dimensional domain and a two dimensional one is explored in [1] by P.G. Ciarlet et al. The junction of two plates is studied in [4,5] and [9], and [12] deals with the junction of beams and plates. Concerning the elastic curved rods, we gave in [6] a thorough study. The asymptotic behavior is under the form of a coupling of the inextensional displacements and the rods torsion angles. Le Dret gave a first study of the junction of two straight rods [8]. He starts his study from the three-dimensional problem of linearized elasticity and uses a standard thin domain technique (the rods are transformed in fixed domain). Le Dret obtains the variational problem coupling the flexion displacements in both rods and some junction conditions.

This paper is organized as follows : in Section 2 we desciribe the skeleton and the structure. In Section 3 we recall the definition of an elementary rod displacement and we give the definition of an elementary rods-structure displacement (e.r.s.d.). Lemma 3.2 is technical result which allows us to associate an e.r.s.d. U_e to any displacement u of the structure. Then, Lemme 3.4 gives this e.r.s.d. estimates and those of the difference $u - U_e$ for appropriate norms. This estimates have an essential importance in ouer study : they replace the Korn inequality. In Section 4 we introduce the inextensional displacements and the extensional displacements. All these decompositions are introduced tofacilitate the study of the asymptotic behaviors of displacements sequences (Section 5) and of the strain and stress tensors sequences (see [5]). In Section 5 we also give the set of the limit inextensional displacements. In Section 6 we pose the problem of elasticity. Thanks to theorems in [6] we deduce Theorem 6.2 which on the one hand gives us the variational problem verified by the extensional displacements limit and, on the other hand, the variational problem coupling the inextensional displacements limit and the rods torsion angles limit.

In this work we use the same notation as in [5]. The constants appearing in the estimates will always be independent from δ. As a rule the Latin index i takes values in $\{1,\dots,N\}$ and the Latin indices h, j, k and l take values in $\{1,2,3\}$. We also use the Einstein convention of summation over repeated indices.

2. The structure made of curved rods

The Euclidian space $\mathbb{R}^3$ is related to the frame $(O;\vec{e}_1,\vec{e}_2,\vec{e}_3)$. We denote by x the running point of $\mathbb{R}^3$, by $\|\cdot\|_{\mathbb{R}^3}$ the euclidian norm and by $\cdot$ the scalar product in $\mathbb{R}^3$.

For any open set ω in $\mathbb{R}^3$, and any displacement u belonging to $H^1(\omega;\mathbb{R}^3)$, we put

$$\mathcal{E}(u,\omega)=\int_\omega \gamma_{kl}(u)\gamma_{kl}(u),\quad \gamma_{kl}(u)=\frac{1}{2}\Big\{\frac{\partial u_k}{\partial x_l}+\frac{\partial u_l}{\partial x_k}\Big\},\quad \mathcal{D}(u,\omega)=\int_\omega \frac{\partial u_k}{\partial x_l}\frac{\partial u_k}{\partial x_l}.$$

Let there be given a family γ_i, $i\in\{1,\ldots,N\}$ of arcs. Each curve is parametrized by its arc length s_i, $\overrightarrow{OM}(s_i)=\phi_i(s_i)$, $i\in\{1,\ldots,N\}$. The mapping ϕ_i belongs to $\mathcal{C}^3([0,L_i];\mathbb{R}^3)$ and the arc γ_i is the range of ϕ_i. The restriction of ϕ_i to interval $]0,L_i[$ is imbedded, the arcs can be closed.
The structure skeleton $\mathcal{S}$ is the curves set γ_i.

Hypotheses 2.1. *We assume the following hypotheses on $\mathcal{S}$:*

- $\mathcal{S}$ is connected,
- two arcs of $\mathcal{S}$ have their intersection reduced to a finite number of points, common points to two arcs are called knots and the set of knots is denoted $\mathcal{N}$,
- the arcs are not tangent in one knot,
- Frenet frames $(M_i(s);\overrightarrow{T}_i(s),\overrightarrow{N}_i(s),\overrightarrow{B}_i(s))$ are defined at each point of $[0,L_i]$.

For any arc γ_i the curvature c_i is given by Frenet formulae (see [5])

$$(1)\qquad \begin{cases}\dfrac{d\overrightarrow{OM}_i}{ds_i}=\overrightarrow{T}_i & ||\overrightarrow{T}_i||_{\mathbb{R}^3}=1\\[2mm] \dfrac{d\overrightarrow{T}_i}{ds_i}=c_i\overrightarrow{N}_i & ||\overrightarrow{N}_i||_{\mathbb{R}^3}=1 \qquad \overrightarrow{B}_i=\overrightarrow{T}_i\wedge\overrightarrow{N}_i\end{cases}$$

Let us now introduce the mapping $\Phi_i:[0,L_i]\times\mathbb{R}^2\longrightarrow\mathbb{R}^3$ defined by

$$(2)\qquad \Phi_i(s_i,y_2,y_3)=\overrightarrow{OM}_i(s_i)+y_2\overrightarrow{N}_i(s_i)+y_3\overrightarrow{B}_i(s_i)$$

There exists $\delta_0>0$ depending only on $\mathcal{S}$, such that the restriction of Φ_i to the compact set $[0,L_i]\times\overline{D}(O;\delta_0)$ is a $\mathcal{C}^1$-diffeomorphism of that set onto its range (see [5]).

Definition 2.2. *The curved rod $\mathcal{P}_{\delta,i}$ with rod center line γ_i is defined as follows:*

$$(3)\qquad \mathcal{P}_{\delta,i}=\Phi_i\big(]0,L_i[\times D(O;\delta)\big),\qquad \textit{for } \delta\in]0,\delta_0].$$

The whole structure $\mathcal{S}_\delta$ is

$$(4)\qquad \mathcal{S}_\delta=\bigcup_{i=1}^N\mathcal{P}_{\delta,i}$$

The running point of the cylinder $\omega_{\delta,i}=]0,L_i[\times D(O;\delta)$ is denoted (s_i,y_2,y_3). The reference domain linked to $\mathcal{P}_{\delta,i}$ is the open set $\omega_i=]0,L_i[\times D(O;1)$ obtained by transforming $\omega_{\delta,i}=]0,L_i[\times D(O;\delta)$ by orthogonal affinity of ratio $1/\delta$. The running point of the reference domain ω_i is denoted (s_i,Y_2,Y_3).

3. Elementary rods-structure displacements.

The H^1 class displacements of $\mathcal{S}$ make up a space denoted

$$H^1(\mathcal{S};\mathbb{R}^3)=\Big\{V\in\prod_{i=1}^N H^1(0,L_i;\mathbb{R}^3)\mid \forall A\in\mathcal{N},\ A\in\gamma_i\cap\gamma_j,\ V_i(a_i)=V_j(a_j)\Big\}\tag{5}$$

where a_i is the arc length of A onto γ_i and where $V_i=V_{|\gamma_i}$, $V_i(a_i)$ is denoted $V(A)$.

In the neighborhood of a knot two rods or more can join together. We call junction the union of the parts that are common to two rods at least.

There exists a real ρ greater than or equal to 1, which depends only on $\mathcal{S}$, such that for any knot A. The rods junction at A be contained in the domain

$$A^\rho_\delta=\bigcup_{i=1}^N \Phi_i\big(]a_i-\rho\delta,a_i+\rho\delta[\times D(O;\delta)\big)$$

We recall that an elementary displacement of the rod $\mathcal{P}_{\delta,i}$ (see [5]) is an element η_i of $H^1(\omega_{\delta,i};\mathbb{R}^3)$ that is written in the form

$$\eta_i(s_i,y_2,y_3)=\mathcal{A}(s_i)+\mathcal{B}(s_i)\wedge\big(y_2\overrightarrow{N}_i(s_i)+y_3\overrightarrow{B}_i(s_i)\big),\quad (s_i,y_2,y_3)\in\omega_{\delta,i},$$

where $\mathcal{A}$ and $\mathcal{B}$ are elements of $H^1(0,L_i;\mathbb{R}^3)$.

Let u be a displacement belonging to $H^1(\mathcal{S}_\delta;\mathbb{R}^3)$. We have shown (Theorem 4.3. in [5]) that there exist elementary rod displacements $U^{'}_{e,i}$ such that

$$\begin{cases}U^{'}_{e,i}(s_i,y_2,y_3)=\mathcal{U}^{'}_i(s_i)+\mathcal{R}^{'}_i(s_i)\wedge\big(y_2\overrightarrow{N}_i(s_i)+y_3\overrightarrow{B}_i(s_i)\big),\quad (s_i,y_2,y_3)\in\omega_{\delta,i},\\ \mathcal{D}(u-U^{'}_{e,i},\omega_{\delta,i})\le C\ \mathcal{E}(u,\mathcal{P}_{\delta,i}),\qquad ||u-U^{'}_{e,i}||^2_{L^2(\omega_{\delta,i};\mathbb{R}^3)}\le C\ \delta^2\mathcal{E}(u,\mathcal{P}_{\delta,i}),\\ \delta^2\Big\|\dfrac{d\mathcal{R}^{'}_i}{ds_i}\Big\|^2_{L^2(0,L_i;\mathbb{R}^3)}+\Big\|\dfrac{d\mathcal{U}^{'}_i}{ds_i}-\mathcal{R}_i\wedge\overrightarrow{T}_i\Big\|^2_{L^2(0,L_i;\mathbb{R}^3)}\le\dfrac{C}{\delta^2}\ \mathcal{E}(u,\mathcal{P}_{\delta,i}).\end{cases}\tag{6}$$

The constants depend only on the mid-lines γ_i, $i\in\{1,\dots,N\}$.

Definition 3.1. *An* **elementary rods-structure displacement** *is a displacement belonging to $H^1(\mathcal{S}_\delta;\mathbb{R}^3)$ whose restriction to each rod is an elementary displacement and whose restriction to each junction is a rigid body displacement.*

An elementary rods-structure displacement depends of two functions belonging to $H^1(\mathcal{S};\mathbb{R}^3)$.

In Lemma 3.3 we show that any displacement of $H^1(\mathcal{S}_\delta;\mathbb{R}^3)$ can be approximated by an elementary rods-structure displacement. The Lemma 3.2 will allow us to construct such displacement. In this lemma we consider a curved rod $\mathcal{P}_{\delta,i}$ of the structure $\mathcal{S}_\delta$, wich we denote without index i to simplify.

Lemma 3.2. *Let u be a displacement of $H^1(\mathcal{P}_\delta;\mathbb{R}^3)$, A be a point of the center line of arc length a, and let ρ be a real greater or equal to 1, fixed. There exists an elementary rod displacement U_e, rigid in the domain $\mathcal{P}_{a,\delta}=\Phi\big((]a-\rho\delta,a+\rho\delta[\times D(O;\delta)\big)$, verifying*

$$\begin{cases}U_e(s,y_2,y_3)=\mathcal{U}(s)+\mathcal{R}(s)\wedge\big(y_2\overrightarrow{N}(s)+y_3\overrightarrow{B}(s)\big),\ \ (s,y_2,y_3)\in]0,L[\times D(O;\delta),\\ \mathcal{D}(u-U_e,\mathcal{P}_\delta)\le C\mathcal{E}(u,\mathcal{P}_\delta)\qquad ||u-U_e||^2_{L^2(\mathcal{P}_\delta;\mathbb{R}^3)}\le C\delta^2\mathcal{E}(u,\mathcal{P}_\delta)\\ \delta^2\Big\|\dfrac{d\mathcal{R}}{ds}\Big\|^2_{L^2(0,L;\mathbb{R}^3)}+\Big\|\dfrac{d\mathcal{U}}{ds}-\mathcal{R}\wedge\overrightarrow{T}\Big\|^2_{L^2(0,L;\mathbb{R}^3)}\le\dfrac{C}{\delta^2}\mathcal{E}(u,\mathcal{P}_\delta)\end{cases}\tag{7}$$

Proof : Let $\mathcal{P}^*_{a,\delta}$ the curved rod portion $\Phi\big(]a-(\rho+1)\delta, a+(\rho+1)\delta[\times D(O;\delta)\big)$ and r be the rigid body displacement defined by using the means of $u(x)$ and $x\wedge u(x)$ in the ball $B(A;\delta/5)$, (see Lemma 3.1 of [5]). The domain $\mathcal{P}^*_{a,\delta}$ is the union of a finite number (which depends on ρ and the center line) of bounded open sets, starshaped with respect to a ball. From various implementations of Lemma 3.1 of [6] we derive the following estimates:

$$\mathcal{D}(u-r;\mathcal{P}^*_{a,\delta})\le C\mathcal{E}(u;\mathcal{P}^*_{a,\delta})\qquad ||u-r||^2_{L^2(\mathcal{P}^*_{a,\delta};\mathbb{R}^3)}\le C\mathcal{E}(u;\mathcal{P}^*_{a,\delta})\tag{8}$$

where constant C depends only on the center line and ρ.

We modify now the elementary displacement given by (6). The new elementary displacement U_e is defined in the rod portion $\mathcal{P}_{a,\delta}$, so as to coincide, within this domain, with the rigid body displacement r. We put $(M=\Phi(s,y_2,y_3)\in\mathcal{P}_{a,\delta})$

$$\begin{cases}U_e(s,y_2,y_3)=r(M)=\vec{a}+\vec{b}\wedge\overrightarrow{AM}\\ \qquad\qquad\qquad=\vec{a}+\vec{b}\wedge\overrightarrow{M(a)M(s)}+\vec{b}\wedge\big(y_2\overrightarrow{N}(s)+y_3\overrightarrow{B}(s)\big)\\ (s,y_2,y_3)\in]a-\rho\delta,a+\rho\delta[\times D(O;\delta)\end{cases}\tag{9}$$

Let m be an even function m belonging to $\mathcal{C}^\infty(\mathbb{R};[0,1])$, which satisfies

$$m(t)=0\qquad\forall t\in[0,\rho],\qquad m(t)=1\qquad\forall t\in[\rho+1,+\infty[\tag{10}$$

Let us take up the rod elementary displacement defined by (6). The new elementary displacement U_e is defined by

$$\begin{cases}\mathcal{U}(s)=m\big(\dfrac{s-a}{\delta}\big)\mathcal{U}'(s)+\Big(1-m\big(\dfrac{s-a}{\delta}\big)\Big)\Big\{\vec{a}+\vec{b}\wedge\overrightarrow{M(a)M(s)}\Big\}\\ \mathcal{R}(s)=m\big(\dfrac{s-a}{\delta}\big)\mathcal{R}'(s)+\Big(1-m\big(\dfrac{s-a}{\delta}\big)\Big)\vec{b}\\ U_e(s,y_2,y_3)=\mathcal{U}(s)+\mathcal{R}(s)\wedge\big(y_2\overrightarrow{N}(s)+y_3\overrightarrow{B}(s)\big)\quad(s,y_2,y_3)\in\omega_\delta\end{cases}\tag{11}$$

Estimates (6) and (8) give us (7). □

Remark 3.3. : The former rod elementary displacement U_e' and the new one U_e satisfy the estimates

$$||\mathcal{U}-\mathcal{U}'||^2_{L^2(0,L;\mathbb{R}^3)}+\delta^2\Big\|\frac{d\mathcal{U}}{ds}-\frac{d\mathcal{U}'}{ds}\Big\|^2_{L^2(0,L;\mathbb{R}^3)}+\delta^2||\mathcal{R}-\mathcal{R}'||^2_{L^2(0,L;\mathbb{R}^3)}\le C\mathcal{E}(u,\mathcal{P}^*_{a,\delta})\tag{12}$$

The constant depends only on the rod center line and on ρ. □

Lemma 3.4. *Let u be a displacement of $H^1(\mathcal{S}_\delta;\mathbb{R}^3)$. There exists an elementary rods-structure displacement U_e which is written in $\omega_{\delta,i}$*

$$U_{e,i}(s_i,y_2,y_3)=\mathcal{U}_i(s_i)+\mathcal{R}_i(s_i)\wedge\big(y_2\overrightarrow{N}_i(s_i)+y_3\overrightarrow{B}_i(s_i)\big)$$

where $\mathcal{U}$ and $\mathcal{R}$ belongs to $H^1(\mathcal{S};\mathbb{R}^3)$, such that

$$\begin{cases}\mathcal{D}(u-U_e,\mathcal{S}_\delta)\le C\mathcal{E}(u,\mathcal{S}_\delta)\qquad ||u-U_e||^2_{L^2(\mathcal{S}_\delta;\mathbb{R}^3)}\le C\delta^2\mathcal{E}(u,\mathcal{S}_\delta)\\ \displaystyle\sum_{i=1}^N\delta^2\Big\|\frac{d\mathcal{R}_i}{ds_i}\Big\|^2_{L^2(0,L_i;\mathbb{R}^3)}+\sum_{i=1}^N\Big\|\frac{d\mathcal{U}_i}{ds_i}-\mathcal{R}_i\wedge\overrightarrow{T}_i\Big\|^2_{L^2(0,L_i;\mathbb{R}^3)}\le\frac{C}{\delta^2}\mathcal{E}(u,\mathcal{S}_\delta)\end{cases}\tag{13}$$

Proof : Let us take a knot A. The ball centered in A and of radius $\delta/5$ is included into the junction A^ρ_δ. Lemma 3.2 gives us then an elementary displacement of each rod joining in A, rigid in A^ρ_δ. The estimates of Lemma 3.4. are immediate consequences of (7). □

The skeleton $\mathcal{S}$ is clamped at some arcs ends. Let Γ_0 be the set of $\mathcal{S}$ clamping points. The whole structure $\mathcal{S}_\delta$ is then clamped onto domains that are discs. We denote $\Gamma_{0\delta}$ the set of clamped points of $\mathcal{S}_\delta$. The space of admissible displacements of $\mathcal{S}$ (respectively $\mathcal{S}_\delta$) is denoted $H^1_{\Gamma_0}(\mathcal{S};\mathbb{R}^3)$ (respectively $H^1_{\Gamma_0}(\mathcal{S}_\delta;\mathbb{R}^3)$).

$$(13)\qquad \begin{cases} H^1_{\Gamma_0}(\mathcal{S};\mathbb{R}^3)=\Big\{u\in H^1(\mathcal{S};\mathbb{R}^3)\ \mid\ u=0 \quad \text{on } \Gamma_0\Big\}\\ H^1_{\Gamma_0}(\mathcal{S}_\delta;\mathbb{R}^3)=\Big\{u\in H^1(\mathcal{S}_\delta;\mathbb{R}^3)\ \mid\ u=0 \quad \text{on } \Gamma_{0\delta}\Big\}\end{cases}$$

If a displacement u belongs to $H^1_{\Gamma_0}(\mathcal{S}_\delta;\mathbb{R}^3)$, the elementary rods-structure displacement U_e given by Lemma 3.4 is also an admissible displacement which belongs to $H^1_{\Gamma_0}(\mathcal{S}_\delta;\mathbb{R}^3)$.

4. The inextensional and extensional displacements

The space of admissible displacements of $\mathcal{S}$ is equipped with the inner product

$$(14)\qquad <U,V>=\sum_{i=1}^N\int_0^{L_i}\frac{dU_i}{ds_i}\cdot\frac{dV_i}{ds_i}$$

This inner product turns $H^1_{\Gamma_0}(\mathcal{S};\mathbb{R}^3)$ into a Hilbert space.

Definition 4.1. *An* **inextensional displacement** *is defined as in the case of a single curved rod. It is an element of the space $H^1_{\Gamma_0}(\mathcal{S};\mathbb{R}^3)$ such that derivative tangential components vanish.*

We put

$$(15)\qquad D_I=\Big\{U\in H^1_{\Gamma_0}(\mathcal{S};\mathbb{R}^3)\ \mid\ \frac{dU_i}{ds_i}\cdot\overrightarrow{T}_i=0,\quad i\in\{1,\dots,N\}\Big\}$$

Definition 4.2. *An* **extensional displacement** *is a displacement belonging to the orthogonal of D_I in $H^1_{\Gamma_0}(\mathcal{S};\mathbb{R}^3)$.*

The set of all extensional displacements is denoted D_E. We equip this space with the semi-norm

$$(16)\qquad ||U||_E=\sqrt{\sum_{i=1}^N\int_0^{L_i}\Big|\frac{dU_i}{ds_i}\cdot\overrightarrow{T}_i\Big|^2}$$

Lemma 4.3. *The semi-norm $||.||_E$ is a norm equivalent to the $H^1_{\Gamma_0}(\mathcal{S};\mathbb{R}^3)$-norm.*

Proof : It is clear that $||.||_E$ is a norm. For each $i\in\{1,\dots,N\}$, the range of the operator

$$U\in D_E\longmapsto\Big(\frac{dU_i}{ds_i}\cdot\overrightarrow{N}_i\,,\ \frac{dU_i}{ds_i}\cdot\overrightarrow{B}_i\Big)\in L^2(0,L_i;\mathbb{R}^2)$$

is a finite sub-space of $L^2(0,L_i;\mathbb{R}^2)$. Then we prove by contradiction that $||.||_E$ is a norm equivalent to the $H^1_{\Gamma_0}(\mathcal{S};\mathbb{R}^3)$-norm. □

Let it be given a displacement u belonging to $H^1_{\Gamma_0}(\mathcal{S}_\delta;\mathbb{R}^3)$. The first component $\mathcal{U}$ of the rods-structure elementary displacement U_e associated to u is decomposed into a unique sum of an inextensional displacement and an extensional one.

$$(17)\qquad \mathcal{U}=U_I+U_E\qquad U_I\in D_I,\quad U_E\in D_E$$

From Lemma 3.4 we get the following estimates:

$$||U_E||^2_{H^1(\mathcal{S};\mathbb{R}^3)} + \delta^2||U_I||^2_{H^1(\mathcal{S};\mathbb{R}^3)} + \sum_{i=1}^N \Big\|\frac{dU_{Ii}}{ds_i} - \mathcal{R}_i \wedge \overrightarrow{T}_i\Big\|^2_{L^2(0,L_i;\mathbb{R}^3)} \le \frac{C}{\delta^2}\mathcal{E}(u,\mathcal{S}_\delta) \tag{18}$$

5. Properties of the limit of a sequence of displacements

We recall the definition of the unfolding operator (see [5]). Let w be in $L^1(\omega_{\delta,i})$. We denote $\mathcal{T}_\delta(w)$ the function belonging to $L^1(\omega_i)$ defined as follows:

$$\mathcal{T}_\delta(w)(s_i, Y_2, Y_3) = w(s_i, \delta Y_2, \delta Y_3), \qquad \forall (s_i, Y_2, Y_3) \in \omega_i.$$

Let u^δ be a sequence of displacements of $H^1_{\Gamma_0}(\mathcal{S}_\delta;\mathbb{R}^3)$ verifying

$$\mathcal{E}(u^\delta, \mathcal{S}_\delta) \le C\delta^2 \tag{19}$$

Displacement u^δ is decomposed into the sum of an elementary rods-structure displacement and a residual one. The first component of the elementary displacement is decomposed into the sum of an inextensional displacement and an extensional one. From (19) and estimates (18), we deduce that it is possible to extract from these various sequences some subsequences, still denoted in the same way, such that (see Proposition 7.1 of [5])

$$\begin{cases} \delta\mathcal{U}^\delta, \;\; \delta U_I^\delta \rightharpoonup U_I \quad \text{weakly in} \quad H^1(\mathcal{S};\mathbb{R}^3)\\ \delta\mathcal{R}^\delta \rightharpoonup \mathcal{R} \quad \text{weakly in} \quad H^1(\mathcal{S};\mathbb{R}^3)\\ U_E^\delta \rightharpoonup U_E \quad \text{weakly in} \quad H^1(\mathcal{S};\mathbb{R}^3)\\ \delta\mathcal{T}_\delta(u^\delta_{|\mathcal{P}_{\delta,i}}) \rightharpoonup U_{Ii} \quad \text{weakly in} \quad H^1(\omega_i;\mathbb{R}^3)\\ \mathcal{T}_\delta(u^\delta_{|\mathcal{P}_{\delta,i}}) - U^\delta_{Ii} \rightharpoonup U_{Ei} - \Big(Y_2\frac{dU_{Ii}}{ds_i}\cdot\overrightarrow{N}_i + Y_3\frac{dU_{Ii}}{ds_i}\cdot\overrightarrow{B}_i\Big)\overrightarrow{T}_i\\ \qquad + \Theta_{U_I,i}\Big(-Y_3\overrightarrow{N}_i + Y_2\overrightarrow{B}_i\Big) \quad \text{weakly in} \quad H^1(\omega_i;\mathbb{R}^3) \end{cases} \tag{20}$$

where $\Theta_{U_I,i} = \mathcal{R}_i\cdot\overrightarrow{T}_i$ is the limit rod torsion angle of $\mathcal{P}_{\delta,i}$. From estimates (18) we derive also

$$\frac{dU_{Ii}}{ds_i} = \mathcal{R}_i \wedge \overrightarrow{T}_i, \qquad i \in \{1,\dots,N\} \tag{21}$$

After passing to the limit there is a coupling between the inextensional displacement U_I and the rod torsion angles $\Theta_{U_I,i}$. They shall be reunited in a single functional space since we have in any knot A

$$\frac{dU_{Ii}}{ds_i}(a_i) = \mathcal{R}(A) \wedge \overrightarrow{T}_i(a_i) \qquad i \in \{1,\dots,N\} \tag{22}$$

The condition obtained in A expresses the rigidity of the junction in the neighborhood of A. The angles between two arcs joining in A remain constant.

We consider now a new and last space containing the couple $(U_I, \mathcal{R})$. We set

$$\mathcal{D}_I = \Big\{(V,\mathcal{A}) \in H^1_{\Gamma_0}(\mathcal{S};\mathbb{R}^3) \times H^1_{\Gamma_0}(\mathcal{S};\mathbb{R}^3) \mid \frac{dV_i}{ds_i} = \mathcal{A}_i \wedge \overrightarrow{T}_i,\; i \in \{1,\dots,N\}\Big\} \tag{23}$$

where $\mathcal{D}_I$ is equipped with the norm

$$||(V,\mathcal{A})|| = \sqrt{\sum_{i=1}^{N} \Big\| \frac{d\mathcal{A}_i}{ds_i} \Big\|^2_{L^2(0,L_i;\mathbb{R}^3)}} \tag{24}$$

This norm turns $\mathcal{D}_I$ into a Hilbert space. We denote $\Theta_{V,i} = \mathcal{A}_i \cdot \overrightarrow{T}_i$ the rod torsion angles associated to the couple $(V,\mathcal{A})$.

6. Asymptotic behavior of structures made of curved rods

The curved rods material is homogeneous and isotropic. Let in $\mathcal{S}_\delta$ be the elasticity system

$$\begin{cases} -\dfrac{\partial}{\partial x_j}\{\, a_{ljkh}\dfrac{\partial u^\delta_k}{\partial x_h}\} = F^\delta_l \quad \text{in} \quad \mathcal{S}_\delta, \\ u^\delta = 0 \qquad \text{on} \quad \Gamma_{0\delta}, \\ a_{ljkh}\dfrac{\partial u^\delta_k}{\partial x_h} n_j = 0 \quad \text{on} \quad \Gamma_\delta, \qquad \Gamma_\delta = \partial\mathcal{S}_\delta \setminus \Gamma_{0\delta}, \end{cases} \tag{25}$$

where the elasticity coefficients a_{ljkh} are defined by

$$a_{ljkl} = \lambda\delta_{lj}\delta_{kh} + \mu(\delta_{lk}\delta_{jh} + \delta_{lh}\delta_{jk}).$$

The constants λ and μ are the material Lamé coefficients. Obviously, the coefficients a_{ljkh} satisfy the hypothesis of coerciveness, i.e., there exists a constant $C_0 > 0$ such that

$$a_{ljkh}\ \beta_{lj}\ \beta_{kh} \geq C_0\ \beta_{lj}\ \beta_{lj}, \qquad \text{for any} \ \ \beta = (\beta_{lj})_{l,j} \ \ \text{with} \ \beta_{lj} = \beta_{jl}.$$

The applied forces F_δ are the sum of two different kinds of forces,

- forces applied on each rod

$$\delta F_I + F_E, \qquad F_I,\ F_E \in L^2(\mathcal{S};\mathbb{R}^3) \tag{26}$$

- forces applied to the knots

$$\big(\delta f_I(A) + f_E(A)\big)\frac{3}{4\delta}\mathbf{1}_{B(A;\delta)} \qquad \forall A \in \mathcal{N} \tag{27}$$

This force (27) is constant in the ball centered in A and of radius δ. As in the case of a single rod, forces F_E and $f_{A,E}$ must not be concerned by the inextensional displacements in order to be able to work on the extensional ones. Hence these forces satisfy the orthogonality condition

$$\int_{\mathcal{S}} F_E \cdot V + \sum_{A\in\mathcal{N}} f_E(A)\cdot V(A) = 0 \quad \text{ for any } \quad V \in D_I. \tag{28}$$

According to estimates (13) and (18) we obtain

$$\begin{cases} \Big|\dfrac{1}{\pi\delta^2}\displaystyle\int_{\mathcal{S}_\delta} F^\delta\cdot u - \int_{\mathcal{S}} F_E\cdot U_E - \int_{\mathcal{S}} \delta F_I \cdot U_I - \sum_{A\in\mathcal{N}} \delta f_I(A)\cdot U_I(A) \\ -\displaystyle\sum_{A\in\mathcal{N}} f_E(A)\cdot U_E(A)\Big| \leq C\sqrt{\mathcal{E}(u,\mathcal{S}_\delta)}, \qquad \forall u \in H^1_{\Gamma_0}(\mathcal{S}_\delta;\mathbb{R}^3). \end{cases} \tag{29}$$

The variational formulation of problem (25) is

$$
\begin{cases} u^\delta \in H^1_{\Gamma_0}(\mathcal{S}_\delta;\mathbb{R}^3) \\ \displaystyle\int_{\mathcal{S}_\delta}\sigma_{kh}(u^\delta)\gamma_{kh}(v) = \int_{\mathcal{S}_\delta} F^\delta\cdot v \qquad\qquad \forall v\in H^1_{\Gamma_0}(\mathcal{S}_\delta;\mathbb{R}^3)\end{cases} \tag{30}
$$

where $\sigma_{kh}(u^\delta) = a_{ljkh}\gamma_{lj}(u^\delta)$ are the stress tensor components.

We can now estimate the solution u^δ to problem (30). Due to (29) we obtain $\mathcal{E}(u^\delta,\mathcal{S}_\delta)\le C\delta^2$. Hence we can pass to the limit within the linearized elasticity problem (30).

Theorem 6.2. *Extensional displacement U_E is solution to the variational problem*

$$
\begin{cases} U_E\in D_E \\ \displaystyle E\sum_{i=1}^N\int_0^{L_i}\frac{dU_{Ei}}{ds_i}\cdot\overrightarrow{T}_i\frac{dV_i}{ds_i}\cdot\overrightarrow{T}_i = \int_{\mathcal{S}} F_E\cdot V + \sum_{A\in\mathcal{N}} f_E(A)\cdot V(A) \qquad \forall V\in D_E\end{cases} \tag{31}
$$

The couple $(U_I,\mathcal{R})$ is solution to the variational problem

$$
\begin{cases} (U_I,\mathcal{R})\in\mathcal{D}_I, \\ \displaystyle \frac{E}{3}\sum_{i=1}^N\int_0^{L_i}\Big\{\frac{d^2U_{Ii}}{ds_i^2}\cdot\overrightarrow{N}_i\frac{d^2V_i}{ds_i^2}\cdot\overrightarrow{N}_i + \Big[\frac{d^2U_{Ii}}{ds_i^2}\cdot\overrightarrow{B}_i - c_i\Theta_{U_I,i}\Big]\Big[\frac{d^2V_i}{ds_i^2}\cdot\overrightarrow{B}_i - c_i\Theta_{V,i}\Big]\Big\} \\ \displaystyle +\frac{\mu}{3}\sum_{i=1}^N\int_0^{L_i}\Big[\frac{d\Theta_{U_I,i}}{ds_i} + c_i\frac{dU_{Ii}}{ds_i}\cdot\overrightarrow{B}_i\Big]\Big[\frac{d\Theta_{V,i}}{ds_i} + c_i\frac{dV_i}{ds_i}\cdot\overrightarrow{B}_i\Big] \\ \displaystyle =\int_{\mathcal{S}} F_I\cdot V + \sum_{A\in\mathcal{N}} f_I(A)\cdot V(A) \qquad\qquad \forall (V,\mathcal{A})\in\mathcal{D}_I.\end{cases} \tag{32}
$$

Proof : The convergence results obtained in [6] can easily be transposed here since the elementary rods-structure displacement given in Lemma 3.4 and the elementary rod displacements defined by (6) are very close according to estimate (12). Hence we obtain the following limits of the stress-tensor components of sequence u^δ in each reference domain ω_i (see [5]):

$$
\mathcal{T}_\delta(\sigma_{kh}(u^\delta)_{|\mathcal{P}_{\delta,i}}) \rightharpoonup \sigma_{i,kh} \quad \text{weakly in } L^2(\omega_i),
$$

where

$$
\begin{aligned}
\sigma_{i,11} &= E\Big[\frac{dU_{E,i}}{ds_i}\cdot\overrightarrow{T}_i - Y_2\Big(\frac{d^2U_{I,i}}{ds_i^2}\cdot\overrightarrow{N}_i\Big) - Y_3\Big(\frac{d^2U_{I,i}}{ds_i^2}\cdot\overrightarrow{B}_i - c_i\,\Theta_{U_I,i}\Big)\Big],\\
\sigma_{i,12} &= \sigma_{i,21} = -\frac{\mu Y_3}{2}\Big[c_i\Big(\frac{dU_{I,i}}{ds_i}\cdot\overrightarrow{B}_i\Big) + \frac{d\Theta_{U_I,i}}{ds_i}\Big],\\
\sigma_{i,13} &= \sigma_{i,31} = \ \frac{\mu Y_2}{2}\Big[c_i\Big(\frac{dU_{I,i}}{ds_i}\cdot\overrightarrow{B}_i\Big) + \frac{d\Theta_{U_I,i}}{ds_i}\Big],\\
\sigma_{i,22} &= \sigma_{i,33} = \sigma_{i,23} = \sigma_{i,32} = 0,
\end{aligned}
$$

and where $E = \dfrac{\mu(3\lambda+2\mu)}{\lambda+\mu}$ is the Young modulus.

Let V be an extensional displacement. We construct an admissible displacement V^δ of the whole structure $\mathcal{S}_\delta$, by modifying Vat the neighborhood of the junctions. Let us consider a knot A. This knot belongs to curve γ_i and its arc length is a_i. We modify V in the neighborhood of A by posing

$$
V_i^\delta(s_i) = V_i(s_i)m\Big(\frac{s_i-a_i}{\delta}\Big) + V(A)\Big(1-m\Big(\frac{s_i-a_i}{\delta}\Big)\Big)
$$

Function m is defined in (10). Displacement V^δ is an admissible displacement of the whole structure. Moreover

$$V^\delta \longrightarrow V \qquad \text{strongly in} \quad H^1(\mathcal{S};\mathbb{R}^3).$$

We take the test-displacement V^δ in (30). We pass to the limit and we obtain (31).

Let us take now a couple $(V,\mathcal{A}) \in \mathcal{D}_I$ and let us show (32). We shall first construct a rods-structure elementary displacement v^δ whose both components strongly converge in $H^1_{\Gamma_0}(\mathcal{S};\mathbb{R}^3) \times H^1_{\Gamma_0}(\mathcal{S};\mathbb{R}^3)$ towards $(V,\mathcal{A})$. Displacement v^δ is defined outside the junctions by

$$v_i^\delta(s_i,y_2,y_3) = \frac{1}{\delta}V_i(s_i) + \mathcal{A}_i(s_i) \wedge \big(\frac{y_2}{\delta}\overrightarrow{N}_i(s_i) + \frac{y_3}{\delta}\overrightarrow{B}_i(s_i)\big\}$$

Let A be a knot belonging to curve γ_i whose arc length is a_i. In the curve rod portion $\Phi_i\big(]a_i-\rho\delta, a_i+\rho\delta[\times D(O;\delta)\big)$, the test-displacement v^δ is chosen rigid and is given by

$$v^\delta(M) = \frac{1}{\delta}V(A) + \frac{1}{\delta}\mathcal{A}(A) \wedge \overrightarrow{AM} \qquad M \in \Phi_i\big(]a_i-\rho\delta, a_i+\rho\delta[\times D(O;\delta)\big)$$

In the domains $\Phi\big([a_i-2\rho\delta, a_i-\rho\delta]\times D(O;\delta)\big)$ and $\Phi\big([a_i+\rho\delta, a_i+2\rho\delta]\times D(O;\delta)\big)$ displacement v^δ is defined as the rod elementary displacement of Lemma 3.4 has been defined. Hence we have constructed an admissible displacement of the structure $\mathcal{S}_\delta$. The unfolded strain tensor components of sequence v^δ converge strongly in $L^2(\omega_i)$ (see the proof of Theorems 7.1 and 7.2 of [5]). Lastly we consider v^δ as test-displacement in problem (30). After having passed to the limit we obtain (32) with the couple $(V,\mathcal{A})$. $\square$

Corollary : Variational problems (31) and (32) are coercive. Hence the whole sequences converge toward their limit. Following the same pattern as in Remark 7.4 of [6] we prove that the convergences (20) are strong. $\square$

7. Complements

The extensional displacement U_E, solution to problem (31), is H^2 class between two knots of a single arc. Hence, at each knot A we have

$$(33) \qquad E\sum_{i=1}^{N}\big[\big(\frac{dU_{Ei}}{ds_i}(a_i+) - \frac{dU_{Ei}}{ds_i}(a_i-)\big)\cdot\overrightarrow{T}_i(a_i)\big]\overrightarrow{T}_i(a_i) = f_E(A)$$

This equality is the knots equilibrium law for a structure made of curved rods.

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