Boundedness and stability for the damped and forced single well Duffing equation

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Résumé. A l'aide d'inégalités différentielles, on améliore des estimations de W.S. LOUD concernant la borne ultime et la stabilité asymptotique des solutions de l'équation de Duffing $u'' + cu' + g(u) = f(t)$ où $c > 0$, f est mesurable essentiellement bornée et g est de classe C^1 avec $g' \geq b > 0$.

Abstract. By using differential inequalities we improve some estimates of W.S. LOUD for the ultimate bound and asymptotic stability of the solutions to the Duffing equation $u'' + cu' + g(u) = f(t)$ where $c > 0$, f is measurable and essentially bounded, and g is continuously differentiable with $g' \geq b > 0$.

Keywords: Duffing equation, periodic solutions

AMS classification numbers: 34C15,34C25, 34C27, 34D05, 34D030

Introduction

In this paper we consider the second order ODE

$$u'' + cu' + g(u) = f(t) \tag{1}$$

where $c > 0$, $f \in L^\infty([t_0, +\infty))$ and $g \in C^1(\mathbb{R})$ satisfies some sign hypotheses. The typical case is

$$g(u) = bu + a|u|^p u \tag{2}$$

More generally we shall assume that $g(0) = 0$ and for some $b > 0$

$$\forall s \in \mathbb{R}, \quad g'(s) \geq b \tag{3}$$

Under this condition, W.S. Loud [9, 10] established that all solutions of (1) are ultimately bounded and more precisely

$$\overline{\lim_{t\to+\infty}} |u(t)| \leq \min\{\frac{1}{b} + \frac{4}{c^2}, \frac{1}{b} + \frac{4}{c\sqrt{b}}\}\|f\|_\infty \tag{4}$$

$$\overline{\lim_{t\to+\infty}} |u'(t)| \leq \frac{4}{c}\|f\|_\infty \tag{5}$$

where $\|f\|_\infty$ stands for $\|f\|_{L^\infty([t_0,+\infty))}$. The estimate (4) is rather sharp and its proof relies on a delicate geometrical argument in the phase space. The question naturally arises of a purely analytical proof of (4), which would be extendable to more complicated situations such as second order systems or even hyperbolic problems.

This paper is devoted to a partial realization of this program. However our method, unlike the geometrical approach of [9-10], introduces a distiction between the weakly damped case corresponding to the condition $c \leq 2\sqrt{b}$ and the strongly damped case $c \geq 2\sqrt{b}$. The analytical approach provides a better estimate for u itself but, for a reason which remains obscure, we do not recover (5) in the strongly damped case. Moreover our proof of (4) for $c \geq 2\sqrt{b}$ requires an additional assumption on g.

The plan of the paper is as follows: Sections 1 and 2 are devoted to obtaining an improved version of of (4) by a purely analytical method. Section 3 deals with asymptotic stability and Section 4 contains existence and uniqueness results for bounded solutions on the whole line under a smallness condition on f. These results improve a theorem of the same nature obtained in [10] by a more complicated method.

1- Ultimate bound for c small.

The main result of this section is the following

Theorem 1.1. *Under the condition*

$$c \le 2\sqrt{b} \tag{1.1}$$

any solution of equation (1) on $J = [t_0, +\infty)$ *satisfies the estimates*

$$\overline{\lim_{t\to+\infty}} |u(t)| \le \frac{2}{c\sqrt{b}} \overline{\lim_{t\to+\infty}} |f(t)| \tag{1.2}$$

$$\overline{\lim_{t\to+\infty}} |u'(t)| \le \left(\frac{2}{c} + \frac{1}{\sqrt{b}}\right) \overline{\lim_{t\to+\infty}} |f(t)| \tag{1.3}$$

with

$$\overline{\lim_{t\to+\infty}} |f(t)| := \inf_{T\ge t_0} \|f\|_{L^\infty([T,+\infty)} = \lim_{T\to+\infty} \|f\|_{L^\infty([T,+\infty)}$$

Proof. First we notice that since $g(u) - bu$ is a non-decreasing function of u, the primitive

$$\varphi(u) = G(u) - \frac{b}{2}u^2$$

is convex, hence

$$\varphi(u) \le \varphi'(u)u = (g(u) - bu)u$$

so that

$$\forall u \in \mathbb{R}, \quad g(u)u \ge \varphi(u) + bu^2 = G(u) + \frac{b}{2}u^2$$

For any solution u of (1) we have for all $t \in J$

$$\frac{d}{dt}(u'^2 + 2G(u) + cuu') = 2(u'' + g(u))u' + cu'^2 + cuu''$$

$$= 2u'(f - cu') + cu'^2 + cu(f - cu' - g(u)) = -c(u'^2 + ug(u) + cuu') + f(2u' + cu)$$

By the above remark we have

$$(u'^2 + ug(u) + cuu') \ge \frac{1}{2}(u'^2 + 2G(u) + cuu') + \frac{1}{2}(u'^2 + bu^2 + cuu')$$

Introducing

$$\Phi(t) := (u'^2 + 2G(u) + cuu')(t)$$

we obtain the inequality

$$\Phi'(t) \le -\frac{c}{2}\Phi(t) - \frac{c}{2}(u'^2 + bu^2 + cuu') + f(2u' + cu)$$

On the other hand the condition $c \le 2\sqrt{b}$ yields the inequality

$$(2u' + cu)^2 = 4u'^2 + c^2u^2 + 4cuu' \le 4(u'^2 + bu^2 + cuu')$$

and we deduce

$$f(2u' + cu) \le \frac{c}{8}(2u' + cu)^2 + \frac{2}{c}f^2 \le \frac{c}{2}(u'^2 + bu^2 + cuu') + \frac{2}{c}f^2$$

hence

$$\Phi'(t) \le -\frac{c}{2}\Phi(t) + \frac{2}{c}f^2 \tag{1.4}$$

This inequality classically implies

$$\overline{\lim_{t\to+\infty}}\, \Phi(t) \le \frac{4}{c^2}\|f\|_\infty^2$$

where $\|f\|_\infty$ stands for $\|f\|_{L^\infty(J)}$. By letting $t_0 \to +\infty$ we find the better estimate

$$\overline{\lim_{t\to+\infty}}\, \Phi(t) \le \frac{4}{c^2}F^2$$

where

$$F := \overline{\lim_{t\to+\infty}}\, |f(t)|$$

By Condition (3) and since $g(0) = 0$ the following inequality is valid

$$\forall s \in \mathbb{R}, \quad G(s) \ge \frac{b}{2}s^2 \tag{1.5}$$

In particular for any $\varepsilon > 0$ we have for t large enough

$$cuu' + bu^2 \le \Phi(t) \le \frac{4}{c^2}F^2 + \varepsilon$$

Solving this differential inequality for u^2 we deduce

$$u^2(t) \le \frac{1}{b}\left(\frac{4}{c^2}F^2 + 2\varepsilon\right)$$

for $t \ge T(\varepsilon)$. By letting $\varepsilon \to 0$ we obtain (1.2). For the proof of (1.3) we notice that

$$\Phi(t) \geq u'^2 + cuu' + bu^2 \geq (u' + \frac{c}{2}u)^2$$

Hence for t large enough we have

$$|u' + \frac{c}{2}u| \leq \frac{2}{c}F + \varepsilon$$

Then (1.3) becomes an immediate consequence of (1.2).

Remark 1.2.

a) Under condition (1.1) we have

$$\frac{4}{c\sqrt{b}} \leq \min\{\frac{4}{c^2} + \frac{1}{b}, \frac{4}{c\sqrt{b}} + \frac{1}{b}\}$$

and therefore (1.2) improves the estimate (4) by a factor at least 2. In addition (1.2) is optimal when $g(u) = bu$.

b) In the same way, condition (1.1) implies

$$\frac{2}{c} + \frac{1}{\sqrt{b}} \leq \frac{4}{c}$$

so that (1.3) is always better than (5) in this case. The two inequalities coincide precisely in the limiting case $c = 2\sqrt{b}$.

c) Our method improves Loud's result in another direction. Actually we have

Proposition 1.3. *Under the hypotheses of Theorem 1.1, we have the stronger estimate*

$$\overline{\lim_{t\to+\infty}}\, G(u(t)) \leq \frac{2}{c^2} \overline{\lim_{t\to+\infty}}\, |f(t)|^2$$

Proof. This follows immediately from the inequality valid for large t:

$$cuu' + 2G(u(t)) \leq \Phi(t) \leq \frac{4}{c^2}F^2 + \varepsilon$$

and from the observation that along a maximizing sequence t_n for $G(u(t))$ the quantity

$$|(uu')(t_n)| \leq \frac{1}{b}|g(u(t_n))u'(t_n)| = \frac{1}{b}|[G(u)]'(t_n)|$$

tends to zero.

2- Ultimate bound for c large.

In this section we keep the notation of Section 1. Our main result, in addition to (3) requires the following assumption which was not necessary for Loud [9-10]

$$\forall s \in \mathbb{R}, \quad g(s)s \geq 2G(s) \tag{2.1}$$

with

$$G(s) = \int_0^s g(u)du$$

Theorem 2.1. *In addition to (3) we assume (2.1) and*

$$c \geq 2\sqrt{b} \tag{2.2}$$

Then any solution of equation (1) on $J = [t_0, +\infty)$ satisfies the estimate

$$\overline{\lim_{t\to+\infty}} |u(t)| \leq \frac{1}{b} \overline{\lim_{t\to+\infty}} |f(t)| \tag{2.3}$$

Proof. Introducing as previously

$$\Phi(t) := (u'^2 + 2G(u) + cuu')(t)$$

we obtain here

$$\Phi'(t) = -c(u'^2 + ug(u) + cuu') + f(2u' + cu) \leq -c\Phi(t) + f(2u' + cu)$$

On the other hand since $G(u) \geq \frac{b}{2}u^2$ we have

$$(2u'+cu)^2 = 4u'^2+c^2u^2+4cuu' \leq 4(u'^2+cuu'+2G(u))+(c^2-4b)u^2 = 4\Phi+(c^2-4b)u^2$$

and therefore

$$f(2u' + cu) \leq \frac{b}{2c}(4\Phi + (c^2 - 4b)u^2) + \frac{c}{2b}f^2$$

yielding the differential inequality

$$\Phi'(t) \leq -(c - 2\frac{b}{c})\Phi(t) + \frac{b(c^2 - 4b)}{2c}u^2 + \frac{c}{2b}f^2 \tag{2.4}$$

Here the function $\Phi(t)$ is not necessarily nonegative. However introducing

$$U = \overline{\lim_{t\to+\infty}} |u(t)|, \quad F = \overline{\lim_{t\to+\infty}} |f(t)|$$

and

$$M = \overline{\lim_{t\to+\infty}} G(u(t)$$

we obviously have

$$M \geq \frac{b}{2}U^2$$

The differential inequality (2.4) classically yields

$$(c - 2\frac{b}{c}) \overline{\lim_{t\to+\infty}} \Phi(t) \leq \frac{b(c^2-4b)}{2c}U^2 + \frac{c}{2b}F^2$$

which in turn implies

$$2(c - 2\frac{b}{c})M \leq \frac{b(c^2-4b)}{2c}U^2 + \frac{c}{2b}F^2 \leq \frac{(c^2-4b)}{c}M + \frac{c}{2b}F^2$$

After reduction we find

$$cM \leq \frac{c}{2b}F^2 \Rightarrow M \leq \frac{1}{2b}F^2 \Rightarrow U^2 \leq \frac{1}{b^2}F^2$$

which is exactly (2.3).

Remark 2.2.

a) It is clear that (2.3) is better than (4), however for c large (4) is almost equivalent to (2.3). Moreover (2.3) is optimal when $g(u) = bu$. In the limiting case $c = 2\sqrt{b}$ we have

$$\frac{2}{b} = \frac{4}{c^2} + \frac{1}{b} = \min\{\frac{4}{c^2} + \frac{1}{b}, \frac{4}{c\sqrt{b}} + \frac{1}{b}\}$$

and therefore (2.3) improves the estimate (4) by a factor 2.

b) Strangely enough, here we do not recover the estimate (5) on u'. A weaker estimate can be obtained as follows. writing the equation (1) as

$$u'' + cu' + (\frac{c^2}{4} - b)u + g(u) = f(t) + (\frac{c^2}{4} - b)u$$

we deduce from (1.3)

$$\overline{\lim_{t\to+\infty}} |u'(t)| \leq \frac{4}{c}\{F + (\frac{c^2}{4} - b) \overline{\lim_{t\to+\infty}} |u(t)|\}$$

and by using (2.3) we obtain

$$\overline{\lim_{t\to+\infty}} |u'(t)| \leq \frac{c}{b} \overline{\lim_{t\to+\infty}} |f(t)|$$

It would be of interest to recover (5) or a better estimate by means of a purely differential technique.

c) As noted previously we have also

Proposition 2.3. *Under the hypotheses of Theorem 2.1, we have the stronger estimate*

$$\overline{\lim_{t\to+\infty}} G(u(t)) \leq \frac{1}{2b} \overline{\lim_{t\to+\infty}} |f(t)|^2$$

3- Asymptotic stability.

In [10] a sufficient condition is given by W.S. Loud in order for any two solutions of (1) to asymptote each other at $+\infty$. In this section we derive such a result by a simpler method based on the precise knowledge of the bound of an associated affine problem. More precisely we shall establish the following

Theorem 3.1. *Let u, v two solutions of (1)on $J = [t_0, +\infty)$ and let us set*

$$M = \max\{\overline{\lim_{t\to+\infty}} |u(t)|, \overline{\lim_{t\to+\infty}} |v(t)|\} \tag{3.1}$$

$$A = \sup_{s\in[0,M]} \{g'(s) - b\} \tag{3.2}$$

Then assuming

$$A \leq \frac{c^2}{4} + c\sqrt{b + \frac{c^2}{16}} \tag{3.3}$$

we have

$$\lim_{t\to+\infty} |u(t) - v(t)| = \lim_{t\to+\infty} |u'(t) - v'(t)| = 0 \tag{3.4}$$

This result will follow as a consequence of the following lemma

Lemma 3.2. *Let $a(t)$ be measurable on $\mathbb{R}$ with*

$$0 \leq a(t) \leq \frac{c^2}{4} + c\sqrt{b + \frac{c^2}{16}} \quad a.e. \ on \mathbb{R} \tag{3.5}$$

Then if w is a bounded solution on $\mathbb{R}$ of

$$w'' + cw' + (b + a(t))w = 0 \tag{3.6}$$

we have $w \equiv 0$.

Proof. Let $C := \frac{c^2}{4} + c\sqrt{b + \frac{c^2}{16}}$. We rewrite (3.6) in the form

$$w'' + cw' + (b + \frac{C}{2})w = (\frac{C}{2} - a(t))w$$

and we observe that as a consequence of (3.5) we have almost everywhere in t:

$$-\frac{C}{2} \le \frac{C}{2} - a(t) \le \frac{C}{2}$$

Now setting

$$f(t) := (\frac{C}{2} - a(t))w$$

w is a bounded solution on $\mathbb{R}$ of

$$w'' + cw' + (b + \frac{C}{2})w = f(t)$$

Hence by [7] we find, assuming $w \not\equiv 0$

$$\|w\|_\infty < \frac{2}{c\sqrt{(b + \frac{C}{2})}}\|f\|_\infty \le \frac{C}{c\sqrt{(b + \frac{C}{2})}}\|w\|_\infty$$

the strict inequality coming from the exact formula given in [7] since

$$b + \frac{C}{2} > \frac{c^2}{4}$$

as soon as $b > 0$. On the other hand it is readily verified that

$$C^2 = c^2(b + \frac{C}{2})$$

so that we obtain

$$\|w\|_\infty < \|w\|_\infty$$

and this contradiction shows that $w \equiv 0$.

Proof of Theorem 3.1. Let u, v be two solutions of (1) satisfying (3.3) and assume that (u, v) do not asymptote at $+\infty$. Then there is a sequence $t_n \to +\infty$ such that

$$\inf_n |u(t_n) - v(t_n)| = \alpha > 0$$

Because both solutions are bounded on the right we may assume, replacing if necessary t_n by a subsequence, that $u(t+t_n)$ and $v(t+t_n)$ converge uniformly on compacta of $\mathbb{R}$

as well as their derivatives to some limits u^* and v^*, the second derivatives remaining essentially bounded. Setting $w = u - v$ we have

$$w'' + cw' + g'(\zeta(t))w(t) = 0$$

where $\zeta(t)$ lies between $u(t)$ and $v(t)$, by passing to the limit we find that $w^* = u^* - v^*$ is a non zero bounded solution of an equation of type (3.6) where a satisfies (3.5). This contradiction proves that u, v are asymptotic to each other. Then the equation for w shows that w' and even w'' tends to 0 as t goes to infinity.

Corollary 3.3. *In the typical case*

$$g(u) = bu + a|u|^p u \tag{2}$$

the convergence result is obtained as soon as

$$\overline{\lim_{t\to+\infty}} |f(t)| \leq \min\{1, \frac{c}{2\sqrt{b}}\} \frac{b}{(a(p+1))^{\frac{1}{p}}} \Big(\frac{c^2}{4} + c\big(b + \frac{c^2}{16}\big)^{\frac{1}{2}} \Big)^{\frac{1}{p}} \tag{3.7}$$

Proof. We use again the notation

$$F = \overline{\lim_{t\to+\infty}} |f(t)|$$

If $c \geq 2\sqrt{b}$ we have

$$A = (p+1)a \sup\{|s|^p, \quad 0 \leq s \leq \frac{1}{b}F\} = (p+1)a\frac{1}{b^p}F^p$$

so that (3.3) reduces to

$$(p+1)a\frac{1}{b^p}F^p \leq \frac{c^2}{4} + c\sqrt{b + \frac{c^2}{16}}$$

or equivalently

$$F \leq \frac{b}{(a(p+1))^{\frac{1}{p}}} \Big(\frac{c^2}{4} + c\big(b + \frac{c^2}{16}\big)^{\frac{1}{2}} \Big)^{\frac{1}{p}}$$

If $c \leq 2\sqrt{b}$ the same calculation with $\dfrac{1}{b}$ replaced by $\dfrac{2}{cb^{\frac{1}{2}}}$ yields the condition

$$F \leq \frac{cb^{\frac{1}{2}}}{2(a(p+1))^{\frac{1}{p}}} \Big(\frac{c^2}{4} + c\big(b + \frac{c^2}{16}\big)^{\frac{1}{2}} \Big)^{\frac{1}{p}}$$

Remark 3.4. The convergence result obtained by Loud in [11] is valid under the condition

$$b + A < \frac{c^2}{2}$$

which is always more restrictive than (3.3) and implies in particular $b < \frac{c^2}{2}$. In the basic example above, assuming $c \geq 2\sqrt{b}$, Loud's condition is equivalent to

$$b + (p+1)a\frac{1}{b^p}F^p < \frac{c^2}{2}$$

For instance if $a = b = 1; c = p = 2$ our condition reduces to

$$\overline{\lim_{t\to+\infty}} |f(t)| = F \leq \sqrt{\frac{1+\sqrt{5}}{3}}$$

while Loud's condition gives

$$\|f\|_\infty < \sqrt{\frac{1}{3}}$$

Remark 3.5. A careful inspection of the proof of Loud's convergence result shows that $u - v$ tends to 0 exponentially fast at infinity. On the other hand Theorem 3.1 does not give any information on the decay rate of $u - v$. In this direction the following intermediate result is therefore interesting.

Theorem 3.6. *Let u, v two solutions of (1) and let M, A be defined by (3.1)-(3.2) Then assuming*

$$A < c\sqrt{b} \tag{3.8}$$

we have for some $K, \delta > 0$

$$|u(t) - v(t)| + |u'(t) - v'(t)| \leq K\exp(-\delta t) \tag{3.9}$$

The proof of Theorem 3.6 relies on the following

Lemma 3.7. *Let $a(t)$ be measurable on $J = [t_0, +\infty)$ with*

$$0 \leq a(t) \leq C < c\sqrt{b} \quad a.e.\ on\, J \tag{3.10}$$

Then if w is any solution on J of

$$w'' + cw' + (b + a(t))w = 0 \tag{3.11}$$

we have for some $K, \delta > 0$

$$|w(t)| + |w'(t)| \leq K \exp(-\delta t) \tag{3.12}$$

Proof. Let $t > t_0$: multiplying (3.11) by w' and integrating on (t, T) we find

$$c \int_t^T w'^2(s)ds + \Big[\frac{1}{2}w'^2 + \frac{b}{2}w^2\Big]_t^T = -\int_t^T a(s)ww'(s)ds \leq C \int_t^T |ww'(s)|ds$$

Hence introducing

$$\psi(t) := w'^2(t) + w^2(t)$$

we find

$$c \int_t^T w'^2(s)ds \leq C \int_t^T |ww'(s)|ds + C_1\psi(t)$$

hence

$$\frac{c}{2} \int_t^T w'^2(s)ds \leq \frac{C^2}{2c} \int_t^T w^2(s)ds + C_1\psi(t) \tag{3.13}$$

then multiplying (3.11) by w and integrating on (t, T) we find

$$b \int_t^T w^2(s)ds + \Big[\frac{c}{2}w^2 + ww'\Big]_t^T \leq \int_t^T w'^2(s)ds$$

hence

$$b \int_t^T w^2(s)ds \leq \int_t^T w'^2(s)ds + C_2\psi(t) \tag{3.14}$$

By combining (3.13) and (3.14) we obtain first

$$\frac{c}{2} \int_t^T w'^2(s)ds \leq \frac{C^2}{2bc} \int_t^T w'^2(s)ds + C_3\psi(t)$$

or

$$(1 - \frac{C^2}{bc^2}) \int_t^T w'^2(s)ds \leq C_4\psi(t)$$

Since $1 - \dfrac{C^2}{bc^2} > 0$, we deduce

$$\int_t^T w'^2(s)ds \leq C_5\psi(t)$$

By letting $T \to +\infty$ we find

$$\int_t^{+\infty} w'^2(s)ds \le C_5\psi(t) \tag{3.15}$$

and then by letting $T \to +\infty$ in (3.14)

$$b\int_t^{+\infty} w^2(s)ds \le \int_t^{+\infty} w'^2(s)ds + C_2\psi(t) \tag{3.16}$$

Finally the combination of (3.15) and (3.16) provides

$$\int_t^{+\infty} \psi(s)ds \le C_6\psi(t) \tag{3.17}$$

It is immediate that this implies

$$\int_t^{+\infty} \psi(s)ds \le M\exp(-\frac{1}{C_6}t)$$

We conclude by using

$$c\int_t^T w'^2(s)ds + \Big[\frac{1}{2}w'^2 + \frac{b}{2}w^2\Big]_t^T = -\int_t^T a(s)ww'(s)ds \le C\int_t^T |ww'(s)|ds$$

which implies

$$\frac{1}{2}w'^2(t) + \frac{b}{2}w^2(t) = \int_t^{+\infty} a(s)ww'(s)ds + c\int_t^{+\infty} w'^2(s)ds \le M'\int_t^{+\infty} \psi(s)ds$$

This concludes the proof.

Remark 3.8. By combining Remark 3.5 and Theorem 3.6 we obtain that (3.9) is satisfied whenever

$$A < \max\{c\sqrt{b}, \quad \frac{c^2}{2} - b\} \tag{3.18}$$

If $c > (1+\sqrt{3})\sqrt{b}$, Loud's condition is better than ours. It is hard to believe that (3.18) can be optimal. Together with remark 2.2, b, this suggests that the techniques of the present paper can probably be improved.

4 - Bounded solutions on the line.

When f is defined and bounded on $\mathbb{R}$, the classical translation method of Amerio-Biroli [1, 2] allows to construct bounded solutions on $\mathbb{R}$. More precisely we can state the following existence result

Theorem 4.1. *Under the hypotheses of either Theorem 1.1 or Theorem 2.1, assuming* $f \in L^\infty(\mathbb{R})$ *equation (1) has at least one solution.* $u \in W^{2,\infty}(\mathbb{R})$.

Proof. Standard application of the classical translation method of Amerio-Biroli [1, 2]. Using inequality (1.4) for Theorem 1.1 or (2.4) for Theorem 2.1, applied with t_0 replaced bu $\theta_n = -n$ to the solution u_n of (1) on $J_n = [-n, +\infty)$ with $u_n(0) = u_n'(0) = 0$ the methods of proof of these theorems show that u_n is uniformly bounded. The result follows by passing to the limit for a suitable subsequence of u_n as n tends to infinity.

The following uniqueness result is an easy consequence of Lemma 3.2.

Theorem 4.2. *Assume that any bounded solution u of (1) satisfies*

$$\text{For all } s \text{ with } 0 \le s \le \|u(t)\|_\infty, \quad g'(s) - b \le \frac{c^2}{4} + c\sqrt{b + \frac{c^2}{16}} \tag{4.1}$$

Then (1) has at most one bounded solution on $\mathbb{R}$.

Combining Theorem 4.1 and Theorem 4.2 we find

Corollary 4.3. *Assume that* $g(0) = 0$, *(3) is fullfilled,* $f \in L^\infty(\mathbb{R})$ *and*

$$\forall s \in \big[0, \max\{\frac{1}{b}, \frac{2}{c\sqrt{b}}\}\|f\|_\infty\big], \quad g'(s) - b \le \frac{c^2}{4} + c\sqrt{b + \frac{c^2}{16}} \tag{4.2}$$

Then (1) has exactly one bounded solution u on $\mathbb{R}$. *In addition , if f is almost periodic, u is also almost periodic and the module containment property is satisfied.*

Proof. The existence is given by Theorem 4.1 and uniqueness by Theorem 4.2, since an easy adaptation of the proofs of Theorems 1.1 and 2.1 show that any bounded solution u of (1) on $\mathbb{R}$ obeys the global estimate

$$\|u\|_\infty \le \max\{\frac{1}{b}, \frac{2}{c\sqrt{b}}\}\|f\|_\infty$$

The almost periodicity and the module containment property then follow from a classical result of Dafermos [4].

Corollary 4.4. *In the typical case* $g(u) = bu + a|u|^p u$ *if we assume*

$$\|f\|_\infty \le \min\{1, \frac{c}{2\sqrt{b}}\} \frac{b}{(a(p+1))^{\frac{1}{p}}} \Big(\frac{c^2}{4} + c\big(b + \frac{c^2}{16}\big)^{\frac{1}{2}}\Big)^{\frac{1}{p}}$$

(1) has exactly one bounded solution u on $\mathbb{R}$. In addition, if f is almost periodic, u is also almost periodic and the module containment property is satisfied.

Remark 4.5. It is known that even periodic solutions are not always unique when f is large, cf. Loud [11] and Souplet [12].

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