An inverse inequality for some transport-diffusion equation. Application to the regional approximate controllability.

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Abstract

In this paper we prove an inverse inequality for the parabolic equation

$$u_t - \varepsilon \Delta u + M \cdot \nabla u = f \mathbb{1}_\omega$$

with Dirichlet boundary conditions. With the motivation of finding an estimate of f in terms on the trace of the solution in $\mathcal{O} \times (0,T)$ for ε small, our approach consists in studying the convergence of the solutions of this equation to the solutions of some transport equation when $\varepsilon \to 0$, and then recover some inverse inequality from the properties of the last one. Under some conditions on the open sets ω, $\mathcal{O}$ and the time T, we are able to prove that, in the particular case when $f \in H_0^1(\omega)$ and it does not depend on time, we have:

$$|f|_{L^2(\omega)} \leq C \left(|u|_{H^1(0,T;L^2(\mathcal{O}))} + \varepsilon^{1/2} |f|_{H^1(\omega)} \right).$$

On the other hand, we prove that this estimate implies a regional controllability result for the same equation but with a control acting in $\mathcal{O} \times (0,T)$ through the right hand side: for any fixed $g \in L^2(\omega)$, the L^2-norm of the control needed to have $|u(T)|_\omega - g|_{H^{-1}(\omega)} \leq \gamma$ remains bounded with respect to γ if $\varepsilon \leq C\gamma^2$.

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1 Introduction

Let $\Omega \subset I\!R^n$ be a bounded connected open set with $\Gamma := \partial\Omega \in C^2$ and let $T > 0$ be given.

We consider the following transport-diffusion equation:

$$\left\{ \begin{array}{rcll} y_t^\varepsilon + M \cdot \nabla y^\varepsilon - \varepsilon \Delta y^\varepsilon & = & g 1\!\!1_\omega & \text{in } \Omega \times (0,T), \\ y^\varepsilon & = & 0 & \text{on } \Gamma \times (0,T), \\ y^\varepsilon(0) & = & 0 & \text{in } \Omega, \end{array} \right. \tag{1}$$

where $\varepsilon > 0$ is a constant (which is intended to be small), $M \in I\!R^n$ is constant, $g = g(x)$ is a x-dependent function and $\omega \subset\subset \Omega$ is an open set.

We deal with the following problem: given an open set $\mathcal{O} \subset\subset \Omega$, find an estimate for g in terms of an observation of the solution of (1) in $\mathcal{O} \times (0,T)$, for small ε.

Let us apply the time derivative operator to system (1). Then, since $g = g(x)$, the function $z^\varepsilon = \partial_t y^\varepsilon$ satisfies

$$\left\{ \begin{array}{rcll} z_t^\varepsilon + M \cdot \nabla z^\varepsilon - \varepsilon \Delta z^\varepsilon & = & 0 & \text{in } \Omega \times (0,T), \\ z^\varepsilon & = & 0 & \text{on } \Gamma \times (0,T), \\ z^\varepsilon(0) & = & g 1\!\!1_\omega & \text{in } \Omega. \end{array} \right. \tag{2}$$

As long as this system is concerned, we would like to prove an estimate like

$$\|g\|_{Y(\omega)} \leq C \|z^\varepsilon\|_{X(\mathcal{O} \times (0,T))}, \tag{3}$$

for some positive constant C and for some spaces X and Y.

Nevertheless, if one choose $X = L^2(\mathcal{O} \times (0,T))$ and $Y = L^2(\omega)$, estimate (3) is not true: observe that, in case (3) were true, it would imply that the exact regional controllability of the adjoint system of (2) holds, that is to say, it would imply that for each $q^0 \in L^2(\Omega)$, there exists a control $v \in L^2(\mathcal{O} \times (0,T))$ such that the solution of

$$\left\{ \begin{array}{rcll} -q_t^\varepsilon - M \cdot \nabla q^\varepsilon - \varepsilon \Delta q^\varepsilon & = & v 1\!\!1_{\mathcal{O}} & \text{in } \Omega \times (0,T), \\ q^\varepsilon & = & 0 & \text{on } \Gamma \times (0,T), \\ q^\varepsilon(T) & = & 0 & \text{in } \Omega \end{array} \right. \tag{4}$$

satisfies $q^\varepsilon(0) = q^0$ in ω. But we know that such a property is not true due to the regularizing effect of the heat equation. Therefore, we have to look for a weaker estimate than (3).

Our approach in this paper consists in proving that, under some geometric assumptions, this type of inverse inequalities hold for the corresponding transport equation for $\varepsilon = 0$ (see system (11) below) and to analyze the convergence of the solutions of (2) towards those of the transport equation when ε goes to zero. As a conclusion, we recover some inequality for the equation (2) for small ε.

Precisely, let us set:

Definition

• We say that $\omega \subset_{M,T} \mathcal{O}$ if there exists $\delta_0 > 0$ such that for any $x \in \omega$, we have $|\mathcal{O}(x)| \geq \delta_0 > 0$ ($|\cdot|$ designs the Lebesgue measure), where

$$\mathcal{O}(x) = \{t \in [0,T] : x + tM \in \mathcal{O}\}.$$

We also define:

• $T_0 = \frac{d(\overline{\omega},\overline{\mathcal{O}})}{|M|}$, where $d(\cdot,\cdot)$ denotes the euclidean distance.

And finally, for each $A \subset \Omega$:

• $\gamma_{M,T}(A) = \{x + tM : x \in A, t \in [0,T] \text{ and } x + sM \in \Omega \quad \forall s \in (0,t)\}$.

Through all the paper, we will assume the following:

$$\nu(x) \cdot M \neq 0 \qquad \text{for all } x \in \gamma_{M,T}(\omega) \cap \Gamma. \tag{5}$$

Here, we have denoted $\nu = \nu(x)$ the outward unit normal vector to Γ.

Before giving our first result, we need the definition of the following space:

$$\widetilde{H}^{\delta}(\omega) = \begin{cases} H^{\delta}(\omega) & \text{if } 0 \leq \delta \leq 1/2, \\ H_0^{\delta}(\omega) & \text{if } 1 \geq \delta > 1/2. \end{cases} \tag{6}$$

The main result of this paper is presented in the following theorem:

Theorem 1 *Assume that $\omega \subset_{M,T} \mathcal{O}$. Then, there exist a positive constant C and $\varepsilon_0 > 0$ such that for any $\delta \in (0,1]$ we have:*

$$|g|_{L^2(\omega)} \leq C(|z^{\varepsilon}|_{L^2(T_0,T;L^2(\mathcal{O}))} + \varepsilon^{\delta/2}|g|_{H^{\delta}(\omega)}) \tag{7}$$

for all $g \in \widetilde{H}^{\delta}(\omega)$ and $\varepsilon < \varepsilon_0$, where z^{ε} is the solution of (2) associated to g.

Remark 1 *The property $\omega \subset_{M,T} \mathcal{O}$ means that all the trajectories defined by M starting in ω intersects $\mathcal{O}$ during a positive interval of time before T; this not only involves geometric properties of ω and $\mathcal{O}$ but, in particular, it implies that*

$$T > T_1 := \frac{\sup_{x\in\omega} d(x,\overline{\mathcal{O}})}{|M|}.$$

Remark 2 *Condition (5) means that the vector M is not tangent to the boundary of Ω at those points intersected for the first time by the trajectories defined by M starting in ω, before the time T.*

In order to explain the second result of this paper, we introduce the following system associated to our transport-diffusion equation with internal control supported in $\mathcal{O}$:

$$\begin{cases} q^\varepsilon_t + M \cdot \nabla q^\varepsilon - \varepsilon \Delta q^\varepsilon &= v \mathbb{1}_{\mathcal{O}} \quad \text{in } \Omega \times (0,T), \\ q^\varepsilon &= 0 \quad \text{on } \Gamma \times (0,T), \\ q^\varepsilon(0) &= 0 \quad \text{in } \Omega. \end{cases} \tag{8}$$

Our goal is to identify inequality (7) as an observability inequality for the adjoint system associated to (8). Let us introduce it:

$$\begin{cases} -\varphi^\varepsilon_t - M \cdot \nabla \varphi^\varepsilon - \varepsilon \Delta \varphi^\varepsilon &= 0 \quad \text{in } \Omega \times (0,T), \\ \varphi^\varepsilon &= 0 \quad \text{on } \Gamma \times (0,T), \\ \varphi^\varepsilon(T) &= \varphi_T \mathbb{1}_\omega \quad \text{in } \Omega. \end{cases} \tag{9}$$

Here, $\varphi_T \in L^2(\omega)$ is the initial condition. With the change of variables $\hat{\varphi}(t) = \varphi(T-t)$ equation (9) takes the same form as (2) (with velocity $-M$), and therefore Theorem 1 directly implies the next result:

Corollary 1 *If $\omega \subset_{-M,T} \mathcal{O}$, then there exist a positive constant $C > 0$ and $\varepsilon_0 > 0$ such that for each $\delta \in (0,1]$, we have*

$$|\varphi_T|_{L^2(\omega)} \le C \left(|\varphi^\varepsilon|_{L^2(T_0,T;L^2(\mathcal{O}))} + \varepsilon^{\delta/2} |\varphi_T|_{H^\delta(\omega)} \right) \tag{10}$$

for all $\varphi_T \in \widetilde{H}^\delta(\omega)$ and $\varepsilon < \varepsilon_0$, where φ^ε is the solution of (9) with $\varphi^\varepsilon(T) = \varphi_T \mathbb{1}_\omega$.

Now, without the second term in the right hand side of (10), (that is, with $\varepsilon = 0$), this inequality would mean that equation (8) is **regionally exactly controllable** (as it is the transport equation). We will show that with this additional term, inequality (10) implies the existence of some kind of regional exact controllability *with corrector* for equation (8), property which gives us some information about the *cost of approximate controllability* of equation (8) for small ε.

In order to explain these facts more precisely, let us do the following definition:

Definition 1 *For each $\lambda, \varepsilon, \gamma > 0$ and $f \in L^2(\omega)$, denoting q^ε as the solution of (8) with control $v \in L^2(\mathcal{O} \times (0,T))$, we define*

$$C^\lambda_\varepsilon(f,\gamma) = \inf\{ |v|_{L^2(\mathcal{O}\times(0,T))} : |q^\varepsilon(T) - f|_{H^{-\lambda}(\omega)} \le \gamma \}.$$

Then, we have:

Theorem 2 *There exists $C > 0$ such that for each $f \in L^2(\omega)$, $\gamma > 0$ and $\lambda \in (0,1]$, there exists $\varepsilon_0 > 0$ such that*

$$C^\lambda_\varepsilon(f,\gamma) \le C |f|_{L^2(\omega)} \qquad \forall \varepsilon \le \varepsilon_0.$$

Remark 3 *In the proof of Theorem 2, we will see that the explicit dependence of* ε_0 *with respect to* f, λ *and* γ *is*

$$\varepsilon_0 = \left(\frac{\gamma}{|f|_{L^2(\omega)}} \right)^{2/\lambda}.$$

This paper is organized as follows. In Section 2 we study all issues related with the transport equation; in particular, we prove there the inverse inequality for this equation. In Section 3 we analyze the convergence of the solutions of the transport-diffusion equation (2) to the transport equation (11) when the diffusion ε goes to zero; with the help of this convergence we are able to prove Theorem 1. Finally, in Section 4 we establish the duality schema and we prove Theorem 2.

2 Inverse inequality for the transport equation

We consider the following transport equation:

$$\left\{ \begin{array}{rcll} z_t + M \cdot \nabla z & = & 0 & \text{in } \Omega \times (0,T), \\ z & = & 0 & \text{on } \Gamma_- \times (0,T), \\ z(0) & = & g \mathbb{1}_\omega & \text{in } \Omega, \end{array} \right. \tag{11}$$

where $\Gamma_- := \{x \in \Gamma : M \cdot \nu(x) < 0\}$. In the sequel, we will also employ the notation $\Sigma_- = \Gamma_- \times (0,T)$ and $\Sigma_+ = \Gamma_+ \times (0,T)$.

As far as this system is concerned, we know that for all $g \in L^2(\Omega)$, system (11) has a unique solution z which belongs to $C([0,T]; L^2(\Omega))$ and which satisfies

$$|z|_{C([0,T];L^2(\Omega))} \leq C |g|_{L^2(\omega)} \tag{12}$$

for a positive constant $C > 0$.

Now, we recall two regularity results for the solutions of (11), the first one concerning its trace and the second one concerning its differentiability in space.

In the first one, we prove that, as long as L^2 initial conditions are considered, the solution of (11) has finite trace in $L^2(\Sigma_+)$ (see [1] for more general results):

Lemma 1 *Let* $g \in L^2(\omega)$*. Then, the solution of* (11) *satisfies* $z \in L^2(\Sigma_+)$ *and there exists a constant* $C > 0$ *such that*

$$|z|_{L^2(\Sigma_+)} \leq C |g|_{L^2(\omega)}.$$

Proof:

Multiplying the equation in (11) by z, integrating in $\Omega \times (0,T)$ and integrating by parts, it is not difficult to see that

$$\int_\Omega |z(T)|^2 dx + \iint_{\Sigma^+} |z|^2 |M \cdot \nu| d\sigma \, dt = \int_\omega |g|^2 dx. \tag{13}$$

Now, since we have that the initial condition of (11) has support in $\omega \subset\subset \Omega$ and Ω is bounded, by (5) we deduce that

$$|M \cdot \nu| \geq \eta > 0 \quad \text{in } \operatorname{supp(z)} \cap \Sigma \tag{14}$$

for some $\eta > 0$.

Therefore,

$$|z|_{L^2(\Sigma_+)} \leq \eta^{-1/2} |z|_{L^2(\Sigma_+, |M\cdot\nu| dS)}$$

and, with (13), the Lemma is proved. ■

In the second one, we prove that the solution belongs to $H^1(\Omega)$ when the initial data does:

Lemma 2 *If $g \in H_0^1(\omega)$ then $z \in C([0,T]; H^1(\Omega))$ and there exists a positive constant C such that*

$$|z|_{C([0,T];H^1(\Omega))} \leq C |g|_{H_0^1(\omega)}.$$

Proof:

If we take the spatial derivative in (11), we obtain

$$(\partial_t + M \cdot \nabla)(\nabla z) = 0 \ \text{ in } \Omega \times (0,T)$$

and

$$\nabla z(0) = \nabla g \in L^2(\Omega)$$

(recall that g has null trace).

Now, as $supp(g) \subset \omega \subset\subset \Omega$, from the explicit representation of the solution of (11) we deduce that $z = 0$ in a neighborhood of Γ_-. Therefore $\nabla z = 0$ in Γ_- and then ∇z solves system (11) with ∇g as initial condition, which give us the desired result by applying inequality (12).

■

Let us now prove an inverse inequality for g:

Proposition 1 *Assume that $\omega \subset_{M,T} \mathcal{O}$. Then, there exists a positive constant C such that for all $g \in L^2(\omega)$, we have:*

$$|g|_{L^2(\omega)} \leq C |z|_{L^2(T_0,T;L^2(\mathcal{O}))}, \tag{15}$$

where z is the solution of (11).

Proof:

Let us consider the following transport equation (backwards in time):

$$\begin{cases} -y_t - M \cdot \nabla y &= \mathbb{1}_{\mathcal{O}} \quad \text{in } \Omega \times (0,T), \\ y &= 0 \quad \text{on } \Gamma_+ \times (0,T), \\ y(T) &= 0 \quad \text{in } \Omega. \end{cases} \tag{16}$$

Multiplying the equation in (16) by z^2, integrating in $\Omega \times (0,T)$ and integrating by parts, it is not difficult to see that

$$\int_0^T \int_{\mathcal{O}} |z|^2 \, dx \, dt = \int_\Omega |z(0)|^2 y(0) \, dx,$$

which readily implies that

$$\int_\omega |g|^2 y(0) \, dx = \int_0^T \int_{\mathcal{O}} |z|^2 \, dx \, dt.$$

Now, thanks to the hypothesis $\omega \subset_{M,T} \mathcal{O}$, we can prove that

$$0 < \delta_0 \le y(0) \text{ in } \omega$$

(see the Definition in the Introduction) and therefore the proof of the lemma is finished.

■

Remark 4 *Observe that the solution of* (11) z *vanishes in* $(0,T_0) \times \mathcal{O}$. *That is the reason why the* L^2 *norm in the right hand side of* (15) *takes place in the time interval* (T_0,T).

3 The inverse inequality for (2) when $\varepsilon \to 0$

In this paragraph we show that there exists a convergence of the solutions of (2) to those of (11) when $\varepsilon \to 0$ as long as $g \in L^2(\omega)$. Furthermore, as a consequence of (15), we prove that this convergence implies in fact some observability inequality for the solutions of (2) when ε is small enough.

3.1 The corrector

In order to prove this convergence and following classical ideas (see, for instance, [8]), for each $g \in L^2(\omega)$ we define the *corrector* θ^ε as the solution of:

$$\begin{cases} \theta^\varepsilon_t + M \cdot \nabla \theta^\varepsilon - \varepsilon \Delta \theta^\varepsilon = 0 & \text{in } \Omega \times (0,T), \\ \theta^\varepsilon = z & \text{on } \Gamma \times (0,T), \\ \theta^\varepsilon(0) = 0 & \text{in } \Omega, \end{cases} \tag{17}$$

where z is the solution of (11) with $z(0) = g$.

With the following result, we prove that this system is well-posed:

Proposition 2 *For each* $\varepsilon \in (0,1)$ *and* $g \in L^2(\omega)$ *we have* $\theta^\varepsilon \in L^2(\Omega \times (0,T))$ *and there exists a positive constant* $C > 0$ *independent of* ε *and* g *such that*

$$|\theta^\varepsilon|_{L^2(\Omega \times (0,T))} \le C \varepsilon^{1/2} |g|_{L^2(\omega)}. \tag{18}$$

Proof:

We follow a classical argument, by transposition of system (17). Thus, for each $h \in L^2(\Omega \times (0,T))$, we consider the following adjoint problem:

$$\left\{ \begin{array}{rcll} -u_t^\varepsilon - M \cdot \nabla u^\varepsilon - \varepsilon \Delta u^\varepsilon & = & h & \text{in } \Omega \times (0,T), \\ u^\varepsilon & = & 0 & \text{on } \Gamma \times (0,T), \\ u^\varepsilon(T) & = & 0 & \text{in } \Omega. \end{array} \right. \tag{19}$$

It is well-known (see for instance [7]) that (19) possesses a unique solution belonging to the space

$$X_0 := L^2(0,T; H^2(\Omega)) \cap H^1(0,T; L^2(\Omega))$$

and which depends continuously on h, that is to say, there exists a positive constant C_ε such that

$$|u^\varepsilon|_{X_0} \leq C_\varepsilon |h|_{L^2(\Omega)}.$$

We have also the next result, which will be proved below:

Lemma 3 *For each $\eta > 0$ there exists $C > 0$ independent of ε such that*

$$\varepsilon^{1/2} \left| \frac{\partial u^\varepsilon}{\partial \nu} \right|_{L^2(0,T;L^2(\Gamma_\eta))} \leq C \, |h|_{L^2(\Omega \times (0,T))} \tag{20}$$

for all $h \in L^2(\Omega \times (0,T))$, where u^ε is the solution of (19), and $\Gamma_\eta = \{x \in \Gamma : \nu(x) \cdot M \geq \eta\}$.

Multiplying the differential equation in (19) by θ^ε, integrating in $\Omega \times (0,T)$, and integrating by parts it is not difficult to deduce that

$$\langle h, \theta^\varepsilon \rangle_{L^2(\Omega \times (0,T))} = -\varepsilon \int_0^T \int_{\Gamma_\eta} z \frac{\partial u^\varepsilon}{\partial \nu} d\sigma dt, \tag{21}$$

where $\eta > 0$ is given by (14).

Then if we define the operator

$$\begin{array}{rcl} F_\varepsilon : L^2(\Omega \times (0,T)) & \to & L^2(0,T; L^2(\Gamma_\eta)) \\ h & \mapsto & -\varepsilon^{1/2} \dfrac{\partial u^\varepsilon}{\partial \nu} \mathbb{1}_{\Gamma_\eta} \end{array} \tag{22}$$

which is well defined, from (21) we have that its dual operator is

$$\begin{array}{rcl} F_\varepsilon^* : L^2(0,T; L^2(\Gamma_\eta)) & \to & L^2(\Omega \times (0,T)) \\ z & \mapsto & \varepsilon^{-1/2} \theta^\varepsilon. \end{array}$$

Inequality (20) implies $\|F_\varepsilon\| \leq C$ and then $\|F_\varepsilon^*\| \leq C$, for all $\varepsilon \in (0,1)$ with C independent of ε. This result together with Lemma 1 implies (18). ∎

Let us now prove Lemma 3:

We first see that from classical energy estimates we obtain

$$|u^\varepsilon|_{C^0([0,T];L^2(\Omega))} + \varepsilon^{1/2}|u^\varepsilon|_{L^2(0,T;H^1(\Omega))} \le C|h|_{L^2(\omega)}, \tag{23}$$

for some $C > 0$.

On the other hand, we multiply the equation in (19) by Δu^ε and we integrate by parts in $\Omega \times (0,T)$. We have

$$-\frac{1}{2}\int_\Omega |\nabla u^\varepsilon(0)|^2 - \frac{1}{2}\iint_\Sigma \left|\frac{\partial u^\varepsilon}{\partial \nu}\right|^2 (M\cdot\nu) - \varepsilon \iint_Q |\Delta u^\varepsilon|^2 = 2\iint_Q h\Delta u^\varepsilon. \tag{24}$$

Therefore

$$\iint_\Sigma \left|\frac{\partial u^\varepsilon}{\partial \nu}\right|^2 (M\cdot\nu) + 2\varepsilon|\Delta u^\varepsilon|^2_{L^2(Q)} \le 2|h|_{L^2(Q)}|\Delta u^\varepsilon|_{L^2(Q)} \tag{25}$$

Now, we take a cut-off function $\varphi \in C^\infty(I\!R^n)$ such that

(i) $\varphi \equiv 1$ in a neighborhood of $\Gamma_{2\eta/3}$,

(ii) $\varphi \equiv 0$ in a neighborhood of $(\Gamma_{\eta/3})^c = \{x \in \Gamma : \nu(x)\cdot M < \eta/3\}$.

Let $\psi^\varepsilon = \varphi u^\varepsilon$. It follows that ψ^ε satisfies the system (19) with right hand side $h^\varepsilon := \varphi h - (\nabla\varphi\cdot M)u^\varepsilon - 2\varepsilon\nabla\varphi\cdot\nabla u^\varepsilon - \varepsilon\Delta\varphi u^\varepsilon$ instead of h.

By (23), we have

$$|h^\varepsilon|_{L^2(Q)} \le C|h|_{L^2(Q)} \tag{26}$$

and by construction $\psi^\varepsilon \equiv 0$ in a neighborhood of $\Gamma_- = \{x \in \Gamma : \nu(x)\cdot M < 0\}$.

Hence, from inequality (25) applied to ψ^ε we get

$$\int_0^T\int_{\Gamma_\eta} \left|\frac{\partial \psi^\varepsilon}{\partial \nu}\right|^2 |M\cdot\nu| + 2\varepsilon|\Delta\psi^\varepsilon|^2_{L^2(Q)} \le 2|h^\varepsilon|_{L^2(Q)}|\Delta\psi^\varepsilon|_{L^2(Q)}. \tag{27}$$

From (27) we deduce

$$|\Delta\psi^\varepsilon|_{L^2(Q)} \le \varepsilon^{-1}|h^\varepsilon|_{L^2(Q)} \tag{28}$$

and then we have

$$\int_0^T\int_{\Gamma_\eta} \left|\frac{\partial \psi^\varepsilon}{\partial \nu}\right|^2 |M\cdot\nu| \le 2\varepsilon^{-1}|h^\varepsilon|^2_{L^2(Q)}. \tag{29}$$

By definition $\nabla\psi^\varepsilon = \nabla u^\varepsilon$ and $|M\cdot\nu| \ge \eta$ on Γ_η. These facts together with (29) and (26) yields (20). Lemma 3 is proved.

■

3.2 Convergence when ε goes to zero

Thanks to the result of the previous paragraph, one naturally introduces the function $w^\varepsilon = z^\varepsilon - (z - \theta^\varepsilon)$, expecting to have 'nice' estimates on it as $\varepsilon \to 0$. Indeed, in our next result we show that we can take g only a little more regular than $L^2(\omega)$ in order to have a convergence to zero for w^ε. Of course, the velocity of convergence depends on how regular g is.

We have the following:

Proposition 3 *Let $\varepsilon \in (0,1)$. Then, for all $\delta \in [0,1]$ and for all $g \in \widetilde{H}^\delta(\omega)$, the function w^ε defined above satisfies*

$$|w^\varepsilon|_{L^2(\Omega\times(0,T))} \le C\varepsilon^{\delta/2}|g|_{H^\delta(\omega)}$$

for some $C > 0$ independent of ε and g.

Proof:

We check the two limit cases, and then apply interpolation:

- $\delta = 1$. Let us consider the function $w^\varepsilon = z^\varepsilon - (z - \theta^\varepsilon)$. It fulfills:

$$\left\{ \begin{array}{rcll} w^\varepsilon_t + M\cdot\nabla w^\varepsilon - \varepsilon\Delta w^\varepsilon &=& \varepsilon\Delta z & \text{in } \Omega\times(0,T), \\ w^\varepsilon &=& 0 & \text{on } \Gamma\times(0,T), \\ w^\varepsilon(0) &=& 0 & \text{in } \Omega. \end{array} \right. \tag{30}$$

  Now, if $g \in H^1_0(\omega)$, by Lemma 2 we have that $z \in L^2(0,T;H^1(\Omega))$ and then $\Delta z \in L^2(0,T;H^{-1}(\Omega))$.

  Then, by classical estimates (Gronwall) we have:

$$\begin{array}{rcl} |w^\varepsilon|^2_{L^\infty(0,T;L^2(\Omega))} &\le& \varepsilon e^{\varepsilon T}|\Delta z|^2_{L^2(0,T;H^{-1}(\Omega))} \\ &\le& \varepsilon e^{\varepsilon T}|z|^2_{L^2(0,T;H^1(\Omega))} \\ &\le& \varepsilon e^{\varepsilon T}|g|^2_{H^1_0(\Omega)}, \end{array} \tag{31}$$

  which gives the desired inequality for $\delta = 1$.

- $\delta = 0$. For all $g \in L^2(\omega)$, the solutions z and z^ε of the equations (11) and (2) respectively depend continuously on the L^2-norm of g. This is also true for θ^ε by Proposition 2, so we are done.

- By the last two points, we have defined the operators

$$\begin{array}{rcl} A_0 : L^2(\omega) &\to& L^2(\Omega\times(0,T)), \\ A_1 : H^1_0(\omega) &\to& L^2(\Omega\times(0,T)) \end{array}$$

  which associates w^ε solution of (30) to each g (recall that z denotes the solution of (11) associated to g). From the $\delta = 0$ and $\delta = 1$ cases, we know that $\|A_0\| \leq C$ and $\|A_1\| \leq C\varepsilon^{1/2}$, for a positive constant C independent of ε.

Applying a classical interpolation result (see, for instance, [9]), we have that $A_\delta : \widetilde{H}^\delta(\omega) \to L^2(\Omega \times (0,T))$ is well defined for $\delta \in [0,1]$, and

$$\|A_\delta\| \leq C\|A_0\|^{1-\delta}\|A_1\|^\delta \leq C\varepsilon^{\delta/2}$$

for a constant $C > 0$ independent of ε and the proposition is proved.

■

In particular, from the two previous results we can establish the convergence of the solutions of the transport-diffusion problem (2) to those of the transport problem (11):

Corollary 2 *For each $\delta \in (0,1]$, there exists $C > 0$ such that*

$$|z^\varepsilon - z|_{L^2(\Omega\times(0,T))} \leq C\varepsilon^{\delta/2}|g|_{H^\delta(\omega)}$$

for all $g \in \widetilde{H}^\delta(\omega)$ and all $\varepsilon \in (0,1)$, where z^ε and z are the solutions of (2) *and* (11) *respectively, with $z^\varepsilon(0) = z(0) = g\mathbb{1}_\omega$.*

In particular, z^ε converges to z as $\varepsilon \searrow 0^+$ in the space $L^2(\Omega \times (0,T))$ as long as $\delta \in (0,1]$.

Remark 5 *Observe that the fact that*

$$\begin{cases} |z^\varepsilon - z|_{L^2(\mathcal{O}\times(0,T))} \leq \gamma(\varepsilon)|g|_{L^2(\omega)} \quad \forall g \in L^2(\omega) \\ \gamma(\varepsilon) \to 0 \ \text{ as } \ \varepsilon \to 0 \end{cases} \tag{32}$$

does not hold can be proven directly.

In fact, from proposition 1 applied to $\mathcal{O} = \Omega$, we have that there exists $C > 0$ such that

$$|g|_{L^2(\omega)} \leq C|z|_{L^2(\Omega\times(0,T))} \leq C(|z^\varepsilon|_{L^2(\Omega\times(0,T))} + |z^\varepsilon - z|_{L^2(\Omega\times(0,T))}).$$

Assume for a moment that (32) *holds. Then,*

$$|g|_{L^2(\omega)} \leq C(|z^\varepsilon|_{L^2(\Omega\times(0,T))} + \gamma(\varepsilon)|g|_{L^2(\omega)}).$$

Taking $\varepsilon > 0$ small enough, we deduce that

$$|g|_{L^2(\omega)} \leq C|z^\varepsilon|_{L^2(\Omega\times(0,T))},$$

which contradicts the fact that the heat equation is not exactly controllable (see (3) *above).*

3.3 Proof of Theorem 1.

For each $g \in \widetilde{H}^\delta(\omega)$ (recall that $\widetilde{H}^\delta(\omega)$ was defined in (6)) and $\varepsilon > 0$ let z and z^ε be the solutions of (11) and (2) respectively, with $z(0) = z^\varepsilon(0) = g\mathbb{1}_\omega$.

We have:

$$\begin{array}{rcll} |g|_{L^2(\omega)} & \leq & C|z|_{L^2(T_0,T;L^2(\mathcal{O}))} & \text{by Proposition 1} \\ & \leq & C\left(|z - z^\varepsilon|_{L^2(\Omega\times(0,T))} + |z^\varepsilon|_{L^2(T_0,T;L^2(\mathcal{O}))}\right) & \\ & \leq & C\left(\varepsilon^{\delta/2}|g|_{H^\delta(\omega)} + |z^\varepsilon|_{L^2(T_0,T;L^2(\mathcal{O}))}\right) & \text{by Corollary 2} \end{array}$$

and Theorem 1 is proved. ■

Remark 6 *For each $g \in L^2(\omega)$, we can take $\{g_n\} \subset \widetilde{H}^\delta(\omega)$ such that $g_n \to g$ in $L^2(\omega)$. Then, applying the previous Theorem we get the following inequality for each g_n and for each $\varepsilon < \varepsilon_0$:*

$$|g_n|_{L^2(\omega)} \leq C(|z_n^\varepsilon|_{L^2(T_0,T;L^2(\mathcal{O}))} + \varepsilon^{\delta/2}|g_n|_{H^\delta(\omega)}),$$

where z_n^ε is the solution of (2) associated to g_n.

Taking $\{\varepsilon_n\}$ convergent to zero in such a way that $\varepsilon_n^{\delta/2}|g_n|_{H^\delta(\omega)}$ also converges to zero, we get the inequality for the transport equation, proved in proposition 1 (see (15)).

4 The cost of the regional approximate controllability

It is well known that equation (8) is approximately controllable, that is to say, for each $\gamma > 0$ and each $f \in L^2(\Omega)$ there exist a control $v \in L^2(\mathcal{O} \times (0,T))$ such that

$$|q^\varepsilon(T) - f|_{L^2(\Omega)} < \gamma. \tag{33}$$

Indeed, this property is equivalent to some *unique continuation property* for the equation (9):

$$\varphi^\varepsilon = 0 \text{ in } \mathcal{O} \times (0,T) \Rightarrow \varphi^\varepsilon \equiv 0.$$

This property is a consequence of the Holmgren Uniqueness Theorem (see [6] for example).

But from this qualitative property we can not obtain any information concerning the cost of the approximate controllability, that is to say, the minimal control $L^2(\mathcal{O})$-norm among all the controls which drive the solution of (8) at a L^2-distance lower than γ of the target $f \in L^2(\Omega)$.

However, we know that the cost of this approximate controllability coincides with the $L^2(\omega)$-norm of the element φ_T where the functional

$$J_{f,\gamma}(\varphi_T) = \frac{1}{2}\int_0^T \int_{\mathcal{O}} |\varphi|^2 \, dx \, dt - \int_\Omega \varphi_T \, f \, dx + \gamma|\varphi_T|_{L^2(\Omega)}$$

attains its minimum (see [3], [4]). Using this approach and with the help of an estimate of the *observability constant* (whose existence is equivalent to the *null controllability*) in terms of the time T, in [4] (see also [10]) the authors prove that given $f \in H^2(\Omega) \cap H_0^1(\Omega)$, the norm of the control needed to have (33) can be estimated by $exp(C/\gamma^{1/2})$ as γ goes to zero. Here, q^ε stands for the solution of a heat equation, that is, taking $\varepsilon = 1$ and $M = 0$ in (8). The fact that the norm of the control needed to approximate some given function goes to infinity as the error goes to zero is related to the fact that the heat equation, or equation (8), is not *exactly controllable*.

Now, inequality (12) tells us precisely that the transport equation (11) (actually, its adjoint equation) is exactly controllable -at least regionally-. Then it is natural to ask if, given $f \in L^2(\omega)$, the norm of the control needed to have the regional version of (33), that is,

$$|q^\varepsilon(T) - f|_{L^2(\omega)} < \gamma. \tag{34}$$

remains bounded when, in addition to γ, the coefficient ε goes to zero. Related to this problem, in the works [2] and [5] the authors proved that, under some natural conditions, the cost of the null controllability of equation (8) goes to zero as ε goes to zero. One of these conditions (which is in fact *necessary*) is to take the time T large enough, so is not possible to follow the approach of [4] in order to prove the analogous property for the approximate control.

Here we give a partial answer to this problem: We will prove that inequality (10), which is satisfied for the solutions of equation (9) under the usual conditions for ω, $\mathcal{O}$ and T, implies the existence of some kind of approximate regional controllability for equation (8) with error (measured in norm $H^{-\lambda}$ for $\lambda \in (0,1]$) less or equal than some power of the diffusion coefficient ε (precisely, $\varepsilon^{\lambda/2}$). Furthermore, the controls which lead to this property are uniformly bounded with respect to ε.

For each $\varepsilon > 0$ we define the space:

$$Z_\varepsilon = \{(\varepsilon^{1/2}\varphi_T, \varphi^\varepsilon|_{\mathcal{O}\times(0,T)}) : \varphi_T \in H_0^1(\omega)\}, \tag{35}$$

where φ^ε is the solution of (9) with $\varphi^\varepsilon(T) = \varphi_T \mathbb{1}_\omega$.

From the continuity of the solutions of (9) with respect to the initial condition, we have that Z_ε is a closed subspace of $H_0^1(\omega) \times L^2(\mathcal{O} \times (0,T))$, and therefore, with the scalar product induced by this space, is a Hilbert space. Let us remark that in fact Z_ε is a closed subspace of $H_0^1(\omega) \times F_\varepsilon$, where

$$F_\varepsilon = \overline{\{\varphi^\varepsilon|_{\mathcal{O}\times(0,T)} : \varphi_T \in H_0^1(\omega)\}} \subset L^2(\mathcal{O} \times (0,T)).$$

Now, for each $\varepsilon > 0$ we define the linear operator

$$\begin{array}{rcl} B_\varepsilon : & Z_\varepsilon & \to L^2(\omega) \\ & (\varepsilon^{1/2}\varphi_T, \varphi^\varepsilon|_{\mathcal{O}\times(0,T)}) & \mapsto \varphi_T. \end{array}$$

Thanks to inequality (10) for $\delta = 1$, we directly have the following result:

Lemma 4 *There exists $C > 0$ and $\varepsilon_0 > 0$ such that*

$$\|B_\varepsilon\| \leq C$$

for all $0 < \varepsilon < \varepsilon_0$.

Therefore we also have that the adjoint operator of B_ε, $B_\varepsilon^* : L^2(\omega) \longrightarrow Z_\varepsilon'$, is bounded and

$$\|B_\varepsilon^*\| \leq C \qquad \text{for each } 0 < \varepsilon < \varepsilon_0. \tag{36}$$

Since F_ε is a closed subespace of $L^2(\mathcal{O} \times (0,T))$, it can be identified (using the L^2-product) with its dual. Thus, being Z_ε a closed subspace of $H_0^1(\omega) \times F_\varepsilon$, we have (see for example [11], section 4.8) that the dual space Z_ε' is isomorphic to the quotient space $(H^{-1}(\omega) \times F_\varepsilon)/Z_\varepsilon^\perp$ where $Z_\varepsilon^\perp$ is the set

$$\left\{(h,u) \in H^{-1}(\omega) \times F_\varepsilon : \left\langle h, \varepsilon^{1/2}\varphi_T \right\rangle_{-1,1} + \int_0^T \int_{\mathcal{O}} u\, \varphi^\varepsilon \, = 0 \;\; \forall \varphi_T \in H_0^1(\omega) \right\},$$

called *the annihilator* of Z_ε.

This means that $B_\varepsilon^* y$ is an equivalence class in $H^{-1}(\omega) \times F_\varepsilon$ for each $y \in L^2(\omega)$.

In fact, we can show that for each $y \in L^2(\omega)$, $B_\varepsilon^* y$ is exactly the equivalence class of all the y-admissible pairs *corrector-control* in $H^{-1}(\omega) \times F_\varepsilon$:

Lemma 5 *Given $y \in L^2(\omega)$, we have $y = q_u(T)|_\omega + \varepsilon^{1/2} h$ for each $(h,u) \in B_\varepsilon^* y$, where q_u is the solution of equation* (8) *with control u.*

Proof:

Fix $\varepsilon > 0$ and $y \in L^2(\omega)$. For each $(h,u) \in H^{-1}(\omega) \times F_\varepsilon$ such that $B_\varepsilon^* y = (h,u) + Z_\varepsilon^\perp$, by the definition of dual operator and the construction of B_ε, we have:

$$\begin{aligned} \int_\omega y\, \varphi_T \, dx &= \int_\omega y\, B_\varepsilon(\varepsilon^{1/2}\varphi_T, \varphi|_{\mathcal{O}\times(0,T)})\, dx \\ &= \left\langle B_\varepsilon^* y, (\varepsilon^{1/2}\varphi_T, \varphi^\varepsilon|_{\mathcal{O}\times(0,T)}) \right\rangle_{Z_\varepsilon', Z_\varepsilon} \\ &= \left\langle h, \varepsilon^{1/2}\varphi_T \right\rangle_{-1,1} + \int_0^T \int_{\mathcal{O}} u\, \varphi^\varepsilon \, dx\, dt \quad \forall \varphi_T \in H_0^1(\omega). \end{aligned} \tag{37}$$

If we consider the equation (8) with control u and solution q_u, multiplying it by φ^ε and integrating by parts results:

$$\int_0^T \int_{\mathcal{O}} u\, \varphi^\varepsilon \, dx\, dt = \int_\omega q_u(T)\, \varphi_T \, dx \qquad \forall \varphi_T \in H_0^1(\omega). \tag{38}$$

From (37) and (38) we conclude that

$$\left\langle h, \varepsilon^{1/2}\varphi_T \right\rangle_{-1,1} = \int_\omega (y - q_u(T))\, \varphi_T \, dx \qquad \forall \varphi_T \in H_0^1(\omega), \tag{39}$$

that is, $\varepsilon^{1/2}h = y - q_u(T)$ in $L^2(\omega)$. ■

Remark 7 *In fact, we can describe explicitly the mentioned annihilator: It is not difficult to see that*

$$Z_\varepsilon^\perp = \{(-\varepsilon^{-1/2}q_u(T), u) : u \in F_\varepsilon\}.$$

Moreover, for each $y \in L^2(\Omega)$, a trivial pair corrector-control is $(\varepsilon^{-1/2}y, 0)$. It follows that the complete class of such pairs is given by:

$$B_\varepsilon^* y = \{(\varepsilon^{-1/2}(y - q_u(T)), u) : u \in F_\varepsilon\}.$$

On the other hand, let us recall that by definition of the norm in a quotient space we have:

$$\|B_\varepsilon^* y\|_{H^{-1}\times F_\varepsilon/H_\varepsilon} = \inf\{|h|_{H^{-1}(\omega)} + |u|_{L^2(\mathcal{O}\times(0,T))} : y = q_u(T)|_\omega + \varepsilon^{1/2}h\}. \quad (40)$$

Then, by (36), (40) and Lemma 5 we have:

Proposition 4 *For all $y \in L^2(\omega)$ and for all $\varepsilon > 0$ there exist $u^\varepsilon \in L^2(\mathcal{O} \times (0,T))$ and $h^\varepsilon \in L^2(\omega)$ such that:*

- $z^\varepsilon(T)|_\omega + \varepsilon^{1/2}h^\varepsilon = y,$
- $|u^\varepsilon|_{L^2(\mathcal{O}_T)} + |h^\varepsilon|_{H^{-1}(\omega)} \leq C|y|_{L^2(\omega)}.$

Moreover, we can prove in the same way the analogue statements of Lemmas 4 and 5 and Proposition 4 in the framework of spaces $H^{-\delta}$ and H^δ. It is clear that Theorem 2 is a direct consequence of the δ-version of Proposition 4.

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