Lowest Landau Level functional and Bargmann spaces for Bose Einstein Condensates

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Abstract

A fast rotating Bose Einstein condensate can be described by a complex valued wave function minimizing an energy restricted to the lowest Landau level or Fock-Bargmann space. Using some structures associated with this space, we study the distribution of zeroes of the minimizer and prove in particular that the number of zeroes is infinite. We relate their location to the combination of two problems: a confining problem producing an inverted parabola profile and the Abrikosov problem of minimizing an energy on a lattice, using Theta functions.

Résumé

Un condensat en rotation rapide est bien modélisé par une fonction d'onde à valeurs complexes qui minimise une énergie dans le niveau de Landau fondamental ou espace de Fock-Bargamann. En utilisant des propriétés liés à cet espace, nous étudions la distribution des zéros de cette fonction d'onde et démontrons en particulier que le nombre de ces zéros est infini. Leur position est reliée à deux problèmes : un problème avec confinement dont la solution est une parabole inversée et le problème d'Abrikosov de minimisation de l'énergie sur un réseau, utilisant des fonctions Theta.

1 Introduction

We consider complex valued functions u of the complex variable $z = x + iy$ and we are interested in minimizing the functional

$$E^h_{LLL}(u) = \int_{\mathbb{C}} |z|^2|u(z)|^2 + \frac{Na\Omega_h^2}{2}\,|u(z)|^4 \;\; L(dz), \tag{1.1}$$

under the constraint that $\int_{\mathbb{C}} |u|^2 L(dz) = 1$ and $f(z) = u(z)e^{|z|^2/2h}$ is a holomorphic function. Here, $L(dz)$ denotes the Lebesgue measure $dxdy$, h is a small parameter, N, a are prescribed constants and $\Omega_h^2 = 1 - h^2$. We would like to study some qualitative properties of the minimizers, in particular the location of its zeroes, and identify a limiting problem as h tends to 0. The holomorphy constraint is one of the delicate points to take care of. For this purpose, we will use some structures associated with the Fock-Bargmann space $\mathcal{F}_h$:

$$\mathcal{F}_h = \left\{ f \in L^2(\mathbb{C}, e^{-\frac{|z|^2}{h}} L(dz)),\ \text{s.t } f \text{ entire} \right\} \tag{1.2}$$

$$\text{with} \qquad \|f\|_{\mathcal{F}_h}^2 = \int_{\mathbb{C}} |f(z)|^2 e^{-\frac{|z|^2}{h}} L(dz)\ . \tag{1.3}$$

The problem is thus to understand the properties of the solutions of

$$\inf \left\{ E_{LLL}^h(e^{-\frac{|z|^2}{2h}} f),\ f \in \mathcal{F}_h,\ \|f\|_{\mathcal{F}_h} = 1 \right\}. \tag{1.4}$$

We may also use the notation

$$G^h(f) = E_{LLL}^h(e^{-\frac{|z|^2}{2h}} f). \tag{1.5}$$

This problem arises in the study of fast rotating Bose Einstein condensates [ARVK, Aft, ABD, Ho, WBP]. This state of matter is described by a macroscopic complex valued wave function ψ minimizing an energy, called the Gross-Pitaevskii energy depending on N the number of atoms, a the scattering length and Ω_h the rotational velocity:

$$E_{GP}(\psi) = \int_{\mathbb{C}} \frac{1}{2} |\nabla \psi - i\Omega_h \mathbf{r}^\perp \psi|^2 + \frac{1}{2}(1 - \Omega_h^2)|z|^2|\psi|^2 + \frac{1}{2} Na|\psi|^4\ L(dz),$$

under $\int_{\mathbb{C}} |\psi|^2 = 1$, with $\mathbf{r} = (x, y)$, $\mathbf{r}^\perp = (-y, x)$ and $z = x + iy$. Experimentally, condensates are confined in a harmonic trapping potential, which is modelled by the second term in the energy. The quartic term models the atomic interaction in a mean field approach. The fast rotating regime corresponds to the limit when Ω_h tends to 1, and the rotational force almost compensates the confining potential so that the condensate expands and nucleates vortices which arrange themselves on an almost triangular lattice. One of the key issues is to understand the properties of the minimizers as Ω_h tends to 1. In this limit, a simplified problem, which reduces to (1.4), is often considered. Indeed, the first eigenvalue of the operator $-(\nabla - i\Omega_h \mathbf{r}^\perp)^2$ is Ω_h and the first eigenspace is the Hilbert space generated by functions $\psi(z) = P(z)e^{-\Omega_h|z|^2/2}$ where P varies in a basis of polynomials. This space is the subspace of $L^2(\mathbb{C}, L(dz))$ made of functions $\psi(z) = f(z)e^{-\Omega_h|z|^2/2}$ with f holomorphic. It is called the Lowest Landau Level (LLL) and is equal to $\mathcal{F}_h$ up to rescaling: after the change of variables $\psi(z) = \sqrt{\Omega_h h}\ u(z\sqrt{\Omega_h h})$, we find that if $u(z)e^{|z|^2/2h}$ is in $\mathcal{F}_h$, $E_{GP}(\psi) = \Omega_h + (h/\Omega_h)E_{LLL}^h(u)$, which justifies our aim to study problem (1.4).

Characterizing the location of the zeroes of the minimizers of (1.4) is an important issue in the physics community [ABD, Ho, WBP]. It has been observed both experimentally and numerically that they lie on an almost triangular lattice, as illustrated in the numerical simulations of [ABD]. In fact, two scales emerge from the problem: one is related to the characteristic spacing of the zeroes and the other one to the scale of variations of a profile which corresponds to the average of the modulus of u over a few cells around the zeroes. This average gives rise to a slow varying envelope of $|u|$ which is close to an inverted parabola. The first scale is of order $\sqrt{h}$, and is thus small, while the second one is of order 1. In [AfBl], an upper bound for the energy has been derived by constructing a test function with an appropriate location of its zeroes. The lower bound is still open. Let us point out that here, we use a scaling which makes the presentation closer to the standard semiclassical asymptotics, which amounts to rescaling distances by $\sqrt{\Omega_h h}$ with respect to [AfBl].

Further properties of the minimizer are the topic of this paper. Some results have been announced in a note intended for physicists [ABN]. The structure of the Bargmann space (1.2) was introduced by physicists in the N-body problem (see [GiJa] for instance) but it has apparently never been used to derive qualitative properties of minimizers of an energy such as E^h_{LLL}. Here, we would like to understand some properties of the zeroes and recover the slow varying profile suggested by the experiments and numerical computations. As a first understanding, the two scales can be decoupled. Indeed, let us consider the minimization of (1.1) under $\int |u|^2 = 1$, without the holomorphic constraint on f. The Euler-Lagrange equation is

$$|z|^2\bar{u} + Na\Omega_h^2 \left|\bar{u}\right|^2 \bar{u} - \lambda\bar{u} = 0,$$

where λ is the Lagrange multiplier due to the L^2 constraint. Any solution has an inverted parabola shape, that is, satisfies

$$\left|u_{min}\right|^2(z) = \frac{2}{\pi R_h^2}\left(1 - \frac{|z|^2}{R_h^2}\right)\mathbf{1}_{\{|z|\leq R_h\}}, \quad R_h = \sqrt{\lambda} = \left(\frac{2Na\Omega_h^2}{\pi}\right)^{1/4}. \tag{1.6}$$

When $h > 0$ is fixed, this compactly supported function u_{min} cannot be in any weak sense a limit of $P_n(z)e^{-|z|^2/2h}$, where $(P_n)_{n\in\mathbb{N}}$ is a sequence of polynomials, since a limit of holomorphic functions in the distributional sense is holomorphic. Nevertheless, we expect the minimizers f^h of (1.4) to have a specific location of zeroes, such that its slow varying envelope approaches, in a weak sense, an inverted parabola, when h tends to 0. Its radius is not given by (1.6), but is modified to take into account the energy contribution of the zeroes. Understanding this issue is one of the aims of this paper. The minimization of the full problem (1.4) displays properties related to these two simplified problems in an approximate sense that we will try to understand.

Instead of dropping the holomorphy constraint in (1.4), one may think of minimizing (1.1) without the first term. This must be done with a slightly different constraint, as we will see in the sequel. The problem is then reduced to the one studied by Abrikosov [Abr, KRA] for a type II superconductor and the minimizer is expected to be a wave function whose modulus is periodic over a hexagonal lattice of size $\sqrt{h}$ and vanishes exactly once in each cell.

Let us present our main results. Firstly, we prove the existence of minimizers of (1.4) and derive an Euler-Lagrange equation which allows us to get some properties of the zeroes of the minimizers. This uses the explicit expression of the orthogonal projection from $L^2(\mathbb{C}, e^{-|z|^2/h}L(dz))$ onto $\mathcal{F}_h$:

$$[\Pi_h f](z) = \frac{1}{\pi h}\int_{\mathbb{C}} e^{\frac{z\overline{z'}}{h}} e^{-\frac{|z'|^2}{h}} f(z')\ L(dz')$$

for all $f \in L^2(\mathbb{C}, e^{-|z|^2/h}L(dz))$. We will also need for any s the spaces

$$\mathcal{F}_h^s = \left\{ f \text{ entire, s.t. } \int_{\mathbb{C}} \langle z\rangle^{2s} |f(z)|^2 e^{-\frac{|z|^2}{h}} L(dz) < \infty \right\}, \tag{1.7}$$

where $\langle z\rangle = \sqrt{1+|z|^2}$.

Theorem 1.1. *For fixed $h>0$, the minimization problem (1.4) admits a solution in $\mathcal{F}_h^1$. Any minimizer is a solution to the Euler-Lagrange equation*

$$\Pi_h\left[\left(|z|^2 + Na\Omega_h^2 e^{-\frac{|z|^2}{h}}|f|^2 - \lambda\right) f\right] = 0, \tag{1.8}$$

where $\lambda \in \mathbb{R}$ is the Lagrange multiplier satisfying the uniform estimates $e_{LLL}^h \le \lambda \le 2e_{LLL}^h$ with

$$e_{LLL}^h = \inf\left\{ E_{LLL}^h\left(e^{-\frac{|z|^2}{2h}} f\right), \quad f\in\mathcal{F}_h, \quad \|f\|_{\mathcal{F}_h} = 1 \right\}.$$

The Euler-Lagrange equation can also be written as

$$zh\partial_z f + \frac{Na\Omega_h^2}{2}\bar{f}(h\partial_z)[f^2(2^{-1}.)] - (\lambda - h) f = 0, \ \text{in } \mathcal{F}_h^{-1}, \tag{1.9}$$

the operator $\bar{f}(h\partial_z)$ being defined as the limit $\lim_{K\to\infty}\sum_{k=0}^K \overline{a_k}(h\partial_z)^k$ if $f(z) = \sum_{k=0}^\infty a_k z^k$.

This Theorem will be proved in Section 3. The existence of a solution to the minimization problem, as well as the explicit expression of the Euler Lagrange equations, are obtained using the properties of the space $\mathcal{F}_h$ and the projector Π_h. Moreover the minimum e_{LLL}^h satisfies

$$\frac{2\Omega_h}{3}\sqrt{\frac{2Na}{\pi}} < e_{LLL}^h \le \frac{2\Omega_h}{3}\sqrt{\frac{2Nab}{\pi}} + o_{Na}(h^0), \tag{1.10}$$

where the parameter b describes the contribution of the vortex lattice and is related to a minimization problem which is described in Theorem 1.4. The lower bound in (1.10) comes from the energy of the inverted parabola (1.6). The upper bound is a result of [AfBl]. We are going to provide a new proof of this upper bound in Theorem 1.5, as well as a more precise estimate on the remainder term . We will relate the coefficient $b \sim 1.1596$ to the Abrikosov problem. Let us point out the gap between the lower bound and the upper bound due to the coefficient b: this coefficient takes into account the contribution of the vortex lattice and does not appear in the lower bound, given the way we have computed it. Theorems 1.4 and 1.5 will provide more details on b.

The Euler-Lagrange equation allows us to derive that a minimizer has an infinite number of zeroes and in particular cannot be achieved by a function f which is a polynomial:

Theorem 1.2. *Assume that f is a solution to (1.9) with the condition*

$$\lambda^3 h < 27 \left(\frac{N a \Omega_h^2 e^{-\frac{1}{12}}}{4\pi} \right)^2 .$$

Then f has an infinite number of zeroes.

The proof of Theorem 1.2 consists in using equation (1.9) and checking that if the number of zeroes is finite, the highest degree term in the equation provides a contradiction. Since (1.10) and the remark following (1.8) imply that the Lagrange multiplier associated with the minimizer of (1.4) is bounded, we deduce:

Corollary 1.3. *Let f be a minimizer of (1.4). If h is sufficiently small, then f has an infinite number of zeroes.*

This question arouse in the physics community since neither experimental nor numerical evidence could predict the existence of zeroes in the region where the modulus of the wave function is very small. Our result implies that vortices do not lie in a bounded region but extend to infinity.

The next step, which is the core of Section 4, is to derive more information on the distribution of zeroes. We are first going to restrict the minimization to the second term of (1.1) only, under the condition that $f(z) = u(z) e^{|z|^2/2h}$ is holomorphic. In this reduced problem, there is no confining potential which provides compactness and thus no decay of the function u. But we are going to consider the problem for functions whose modulus is periodic on a lattice $\mathfrak{L}$. Then f is no longer in $\mathcal{F}_h$ but is still in $\mathcal{F}_h^s$ for $s < -1$. We replace integrals by averages and intend to minimize $⨍ |u|^4$ under $⨍ |u|^2 = 1$ where $⨍ |u|^n$ denotes the average of the periodic function $|u|^n$ (here, Q is any cell of the lattice):

$$⨍ |u|^n = \frac{\int_Q |u|^n(z)\ L(dz)}{\int_Q L(dz)} = \lim_{R \to \infty} \frac{\int_{|z| \le R} |u|^n\ L(dz)}{\int_{|z| \le R} L(dz)} .$$

Let us call $1/\nu$ the smallest period of the lattice $\mathfrak{L}$ and choose this direction to define the real axis. In fact, geometrical properties of lattices and the invariance of the problem under isometries allow to reduce further the parameters defining $\mathfrak{L}$: one can restrict to lattices defined by

$$\mathfrak{L} = \frac{1}{\nu} (\mathbb{Z} \oplus \tau \mathbb{Z}), \quad \nu \in \mathbb{R}_+^* \tag{1.11}$$
$$\tau = \tau_R + i\tau_I, \quad \tau_I > 0, \quad |\tau| \ge 1, \quad -\frac{1}{2} \le \tau_R < \frac{1}{2} \quad (\tau_R \le 0 \text{ if } |\tau| = 1)$$

and this provides a description for all lattices with smallest period $1/\nu$, up to direct isometries.
The functions u such that $f(z) = u(z)e^{|z|^2/2h}$ is holomorphic, $|u|$ is periodic over the lattice $\mathfrak{L}$ and u vanishes exactly on $\mathfrak{L}$ with simple zeroes, form a one dimensional space spanned by a function called u_τ. This function is completely determined by the complex number τ of the lattice: indeed, the periodicity of $|u|$ imposes the value of ν in terms of h and τ_I. The function u_τ can be expressed in terms of the Theta function according to:

$$u_\tau(z) = e^{-\frac{|z|^2}{2h}} f_\tau(z) \ , \qquad f_\tau(z) = e^{\frac{z^2}{2h}} \Theta\left(\sqrt{\frac{\tau_I}{\pi h}} z, \tau\right)$$

$$\text{with} \qquad \Theta(v,\tau) = \frac{1}{i} \sum_{n=-\infty}^{+\infty} (-1)^n e^{i\pi\tau(n+1/2)^2} e^{(2n+1)\pi i v}, \quad v \in \mathbb{C}.$$

Such an ansatz was introduced by Abrikosov in [Abr] to minimize the quantity

$$\gamma(\tau) = \frac{\fint |u_\tau|^4}{\left(\fint |u_\tau|^2\right)^2} \tag{1.12}$$

which also arises in the study of superconductors. It turns out that the square ($\tau = i$) and hexagonal lattices ($\tau = e^{2i\pi/3}$) are critical points of the function $\tau \to \gamma(\tau)$. The fact that the hexagonal lattice is a minimizer was numerically checked by Kleiner, Roth and Autler in [KRA]. An explicit computation of the quantity $\gamma(\tau)$, with the help of a result by Nonnenmacher and Voros [NoVo] about quantum chaos, provides a complete proof that $\tau = e^{2i\pi/3}$ is the global minimizer. In addition, we shall see that for any τ the function f_τ solves an equation close to our Euler-Lagrange equation (1.8). This can be summarized in the following theorem:

Theorem 1.4. *Let $\mathfrak{L}$ be a lattice given by its parameters ν and τ through (1.11). If the function f is entire, satisfies $f^{-1}(\{0\}) = \mathfrak{L}$ with simple zeroes, and $\left|e^{-\frac{|z|^2}{2h}} f(z)\right|$ is $\mathfrak{L}$-periodic, then the parameter ν and the function f are determined by τ through:*

$$\nu = \sqrt{\frac{\tau_I}{\pi h}} \quad \textit{and} \quad f(z) = c f_\tau(z), \quad c \in \mathbb{C}^*$$

$$\textit{with} \qquad f_\tau(z) = e^{\frac{z^2}{2h}} \Theta\left(\frac{\sqrt{\tau_I}}{\sqrt{\pi h}} z, \tau\right) . \tag{1.13}$$

The function $f_\tau(z)$ solves the equation

$$\Pi_h\left(e^{-\frac{|z|^2}{h}} |f_\tau|^2 f_\tau\right) = \lambda_\tau f_\tau, \quad \textit{in } \mathcal{F}_h^s, \ s < -1\,, \tag{1.14}$$

$$\textit{with} \qquad \lambda_\tau = \frac{\fint |u_\tau|^4}{\fint |u_\tau|^2} = \frac{1}{\sqrt{2\tau_I}} \sum_{k,\ell\in\mathbb{Z}} e^{-\frac{\pi}{\tau_I}|k\tau-\ell|^2} \tag{1.15}$$

where $u_\tau(z) = e^{-\frac{|z|^2}{2h}} f_\tau(z)$. *Moreover, the quantity* $\gamma(\tau)$ *defined in (1.12) satisfies*

$$\gamma(\tau) = \sum_{k,\ell\in\mathbb{Z}} e^{-\frac{\pi}{\tau_I}|k\tau-\ell|^2} . \tag{1.16}$$

The complex number $\tau = j = e^{\frac{2i\pi}{3}}$, *corresponding to the hexagonal lattice, is the unique minimizer of* $\gamma(\tau)$ *in the fundamental domain*

$$\left\{\tau = \tau_R + i\tau_I \in \mathbb{C},\ \tau_I > 0,\ |\tau| \geq 1,\ -\frac{1}{2} \leq \tau_R < \frac{1}{2}, (\tau_R \leq 0 \text{ if } |\tau| = 1)\right\}$$

and $b = \gamma(j) \sim 1.1596$.

Let us point out that (1.14) is similar to (1.8) without the confining term. We expect that, when h is sufficiently small, any minimizer of (1.4) is close in some sense to $f_\tau(z)\alpha(z)$, where f_τ is the function described in Theorem 1.4 which varies on a characteristic size $\sqrt{h}$, and α is a slow varying profile which optimizes the energy and takes into account the confining potential. We are not able to prove such a result but the converse: $f_\tau(z)\alpha(z)$ can be approximated, as h tends to 0, by the function $\Pi_h(\alpha f_\tau)$ of $\mathcal{F}_h$, which is almost a solution to (1.8):

Theorem 1.5. *Let* $\tau \in \mathbb{C}\setminus\mathbb{R}$, *let* $\tilde\alpha \in C^{0,\frac{1}{2}}(\mathbb{C};\mathbb{C})$ *be such that* $\operatorname{supp}(\tilde\alpha) \subset K$ *for some compact set* K *and* $\int |\tilde\alpha|^2 = 1$. *For* f_τ *defined by (1.13), we set*

$$g^h_{\tilde\alpha,\tau} = \|\Pi_h(\tilde\alpha f_\tau)\|^{-1}_{\mathcal{F}_h} \Pi_h(\tilde\alpha f_\tau), \quad \text{and} \quad v^h_{\tilde\alpha,\tau}(z) = g^h_{\tilde\alpha,\tau}(z) e^{-\frac{|z|^2}{2h}} . \tag{1.17}$$

Then we have

$$E^h_{\mathrm{LLL}}\left(v^h_{\tilde\alpha,\tau}\right) = \int_{\mathbb{C}} \left(|z|^2|\tilde\alpha(z)|^2 + \frac{Na\gamma(\tau)}{2}|\tilde\alpha(z)|^4\right) L(dz) + O(h^{1/4}) \tag{1.18}$$

where $\gamma(\tau)$ *is given by (1.12) or (1.16) and* $O(h^{1/4})$ *depends only on* $\|\tilde\alpha\|_{C^{0,1/2}}$, τ *and* K. *Moreover, for any* $\lambda \in \mathbb{C}$,

$$\Pi_h\left(\left(|z|^2 - \lambda + Na\Omega_h^2\left|v^h_{\tilde\alpha,\tau}\right|^2\right) g^h_{\tilde\alpha,\tau}\right) = \Pi_h\left(\left(|z|^2 - \lambda + Na\gamma(\tau)|\tilde\alpha|^2\right) g^h_{\tilde\alpha,\tau}\right) + R_h\,, \tag{1.19}$$

where $\|R_h\|_{\mathcal{F}_h} \leq C(\tilde\alpha,\tau,\lambda,K)h^{1/4}$, *and* $C(\tilde\alpha,\tau,\lambda,K)$ *depends only on* $\|\tilde\alpha\|_{C^{0,1/2}}$, τ, λ *and* K.

In order to approximate a minimizer of (1.4), we need to pick the optimal function $\tilde\alpha$. Minimizing the right-hand side of (1.18) with respect to τ and $\tilde\alpha$ under the constraint $\int |\tilde\alpha|^2 = 1$ yields

$$\tau = j \text{ and } |\tilde\alpha(z)|^2 = \frac{1}{Na\gamma(\tau)}\left(\sqrt{\frac{2Na\gamma(\tau)}{\pi}} - |z|^2\right)_+ , \tag{1.20}$$

where the first equality is a consequence of Theorem 1.4. This provides in particular a test function for the upper bound of the energy, and (1.18) makes precise the remainder estimate in the upper bound of (1.10), which is an improvement of the results of [AfBl].

With this choice of $\tilde{\alpha}$ and τ, and if in addition λ in (1.19) is such that $\lambda = \sqrt{2Na\gamma(\tau)/\pi}$, (1.19) implies that

$$\Pi_h\left(\left(|z|^2 - \lambda + Na\Omega_h^2 \left|v_{\tilde{\alpha},\tau}^h\right|^2\right) g_{\tilde{\alpha},\tau}^h\right) = O\left(h^{1/4}\right) \quad \text{in} \quad \mathcal{F}_h.$$

In other words, $g_{\tilde{\alpha},\tau}^h$ is a solution to (1.8) up to an error term of order $h^{1/4}$. We will prove that, as h tends to 0, $g_{\tilde{\alpha},\tau}^h$ is very close to $f_\tau(z)\tilde{\alpha}(z)$. This implies that, inside the support of $\tilde{\alpha}$, the zeroes of $g_{\tilde{\alpha},\tau}^h$ are located on an almost regular triangular lattice. We do not have much information though, on the zeroes located outside the support of $\tilde{\alpha}$, the "invisible vortices". An open question is to derive that there is a solution to (1.8) close to $g_{\tilde{\alpha},\tau}^h$, for the specific choice of τ and $\tilde{\alpha}$ given by (1.20). One may hope to prove such a result by an analogue of a Newton method.

Finally, the relevance of numerical approximations is considered in Section 6: the numerical simulations in [ABD] consist in minimizing the energy G^h on a space of polynomials with bounded degree, instead of the space $\mathcal{F}_h$. A natural question, for which we provide a partial answer, is to check that when the degree of the polynomials is taken sufficiently large, the solution to the finite dimensional problem provides a suitable approximation of the minimizer in $\mathcal{F}_h$.

The appendix gathers standard tools about Bargmann transforms, pseudo-differential calculus, and fixes notations and normalizations.

We end this introduction with a remark about the confining potential. In most current experiments, the potential trapping the atoms is harmonic, which gives rise to the term $|z|^2|u|^2$ in the energy. Nevertheless, in some recent experiments [SBCD], the potential contains a quartic contribution. The results of Theorem 1.1 and 1.5 can be extended when the quadratic potential term $|z|^2|u|^2$ in the energy is replaced by $V(|z|^2)|u|^2$, if $\sqrt{(c - V(|z|^2))_+}$ is $C^{0,1/2}$ for any real number c. This includes in particular the case of the combined harmonic and quartic trapping potential of the experiments described in [SBCD]. The particularity of the potential of [SBCD] is that the support of the slow varying profile is no longer a disc (as in our case of the inverted parabola) but an annulus. Interesting questions arise concerning the number and distribution of "invisible" vortices in the inner disc.

2 Preliminaries: Bargmann transform and hypercontractivity

In this section, we give some preliminary results which are useful to define properly the Euler-Lagrange equation of (1.4). We first recall the definition of the Bargmann transform and some of its properties, and then a hypercontractivity property for spaces of entire functions. Some details are added in Appendix A.

2.1 Bargmann transform and Fock-Bargmann space

We will use the semiclassical Bargmann transform with the following normalization

$$[B_h\varphi](z) = \frac{1}{(\pi h)^{3/4}} e^{\frac{z^2}{2h}} \int_{\mathbb{R}} e^{-\frac{(\sqrt{2}z-y)^2}{2h}} \varphi(y)\ dy\,,$$

with $z = \frac{x-i\xi}{\sqrt{2}} \in \mathbb{C}$ and $\varphi \in \mathcal{S}'(\mathbb{R})$. The operator B_h enjoys the following properties (details may be found in Appendix A or can be adapted from [Bar, Fol, Mar]):

a) Isometry property: for any $h > 0$, B_h defines a unitary transform from $L^2(\mathbb{R}, dy)$ onto $\mathcal{F}_h$ (note that our normalization gives $L(dz) = \frac{dxd\xi}{2}$).

b) Reproducing Kernel: The product $B_h^* B_h$ is the identity on $L^2(\mathbb{R}, dy)$ while $B_h B_h^* = \Pi_h$ is the orthogonal projection from $L^2(\mathbb{C}, e^{-|z|^2/h} L(dz))$ onto $\mathcal{F}_h$. The adjoint of B_h is given by

$$[B_h^* f](y) = \frac{1}{(\pi h)^{3/4}} \int_{\mathbb{C}} e^{\frac{\overline{z'}^2}{2h}} e^{\frac{-\left(y-\sqrt{2}\overline{z'}\right)^2}{2h}} e^{-\frac{|z'|^2}{h}} f(z')\ L(dz')\ .$$

A simple Gaussian integration w.r.t. $y \in \mathbb{R}$ yields

$$[\Pi_h f](z) = [B_h B_h^* f](z) = \frac{1}{\pi h} \int_{\mathbb{C}} e^{\frac{z\overline{z'}}{h}} e^{-\frac{|z'|^2}{h}} f(z')\ L(dz')$$

for all $f \in L^2(\mathbb{C}, e^{-|z|^2/h} L(dz))$.

c) Harmonic oscillator: The harmonic oscillator (or number operator in the Fock representation) is the self adjoint operator on $L^2(\mathbb{R}, dy)$ given by:

$$\tilde{N}_h = \frac{1}{2}(-h^2\partial_y^2 + y^2 - h)$$
$$D(\tilde{N}_h) = \left\{u \in L^2(\mathbb{R}, dy),\ y^\alpha D_y^\beta u \in L^2(\mathbb{R}, dy),\ \alpha + \beta \leq 2\right\},$$

where $D_y = -i\partial_y$. We then define N_h by

$$N_h = B_h \tilde{N}_h B_h^* = z(h\partial_z)\,.$$

An element $f = B_h\varphi$ of $\mathcal{F}_h$ considered as an element of $L^2(\mathbb{C}, e^{-\frac{|z|^2}{h}} L(dz))$, satisfies

$$h\partial_z f = h\partial_z(\Pi_h f) = \Pi_h(\overline{z} f)\ .$$

We also note

$$z(h\partial_z)f = h\partial_z(zf) - hf = h\partial_z\Pi_h(zf) - hf = \Pi_h(|z|^2 - h)\Pi_h f.$$

Since $B_h = \Pi_h B_h$, this provides another useful writing of the operator $\tilde{N}_h$:

$$\tilde{N}_h = B_h^*[|z|^2 - h]B_h\ , \quad N_h = \Pi_h(|z|^2 - h)\Pi_h\ .$$

2.2 Hypercontractivity

We will use the notation A_h^p, $0 < p < \infty$ introduced by Carlen [Car] for the spaces of entire functions f such that

$$\int_{\mathbb{C}} |f(z)|^p \, e^{-|z|^2/h} \, L(dz) < \infty \,.$$

The corresponding quantity

$$\|f\|_{A_h^p} = \left(\int_{\mathbb{C}} |f(z)|^p \, e^{-|z|^2/h} \, \frac{L(dz)}{\pi h} \right)^{1/p}$$

is a norm when $p \geq 1$.

Note that our normalization corresponds to the one of Carlen [Car] by taking ($h = h_{Carlen}/2\pi$) with the relation $\|f\|_{\mathcal{F}_h} = \sqrt{\pi h} \, \|f\|_{A_h^2}$. The number operator $N_1 = z\partial_z = h^{-1}N_h$ defines a semigroup on any A_h^p given by

$$[P_t f](z) = [e^{-tN_1} f](z) = f(e^{-t}z) \,.$$

With these notations, according to [Car, Theorem 4] (see also [Nel]), the hypercontractivity property of P_t is

$$\|P_t f\|_{A_h^q} \leq \|f\|_{A_h^p} \,, \quad 0 < p < q, \quad e^{-t} \leq (p/q)^{1/2} \,. \tag{2.1}$$

Thus, we can deduce the next result.

Lemma 2.1. *The quantity*

$$\int_{\mathbb{C}} \overline{f_1(z) f_2(z)} f_3(z) f_4(z) e^{\frac{-2|z|^2}{h}} \, L(dz)$$

defines a continuous $(2,2)$*-linear functional* [1] *on* $\mathcal{F}_h$ *with norm smaller than* $\frac{1}{2\pi h}$. *Hence for any* $\alpha, \beta \in \{0,1,2\}$, *the* $\partial_{\bar{f}}^{\alpha} \partial_f^{\beta}$ *derivative of the functional*

$$f \to \int_{\mathbb{C}} |f(z)|^4 \, e^{\frac{-2|z|^2}{h}} \, L(dz) = \frac{\pi h}{2} \, \|P_{t_0} f\|_{A_h^4}^4 \,, \quad t_0 = \frac{\ln 2}{2}$$

defines a continuous $(2-\alpha, 2-\beta)$*-linear mapping from* $\mathcal{F}_h$ *into* $\overline{\mathcal{F}_h}^{\hat{\otimes}\alpha} \hat{\otimes} \mathcal{F}_h^{\hat{\otimes}\beta}$ *with norm* $\frac{4}{2\pi h (2-\alpha)!(2-\beta)!}$.

Proof: Because of the Hölder inequality, it is enough to consider the case $f_1 = f_2 = f_3 = f_4 = f$. We apply the hypercontractivity inequality with $t_0 = \frac{\ln 2}{2}$, $p = 2$ and $q = 4$:

$$\int_{\mathbb{C}} |f(z)|^4 \, e^{-2|z|^2/h} \, L(dz) =$$

$$\frac{\pi h}{2} \int_{\mathbb{C}} \left| f(2^{-1/2} z) \right|^4 e^{-|z|^2/h} \, \frac{L(dz)}{\pi h} = \frac{\pi h}{2} \, \|P_{t_0} f\|_{A_h^4}^4 \leq \frac{\pi h}{2} \, \|f\|_{A_h^2}^4 = \frac{1}{2\pi h} \, \|f\|_{\mathcal{F}_h}^4 \,.$$

The rest refers to standard notions for continuous multilinear functionals. □

[1] A $(2,2)$-linear functional is an $\mathbb{R}$-quadrilinear functional which is $\overline{\mathbb{C}}$-linear with respect to the two first arguments and $\mathbb{C}$-linear with respect to the two last arguments.

3 Existence of a minimum for $h > 0$

This section is devoted to the proof of Theorems 1.1 and 1.2. We first give an elementary proof of the existence of a minimizer (Subsection 3.1), then study the corresponding Euler-Lagrange equation (Subsection 3.2). In Subsection 3.3, we prove that any minimizer of (1.4) necessarily has an infinite number of zeroes.

3.1 Existence theorem

In this paragraph, h is a fixed positive number.

Proof of Theorem 1.1:

Let $(f_n)_{n\in\mathbb{N}}$ be a minimizing sequence for (1.4) and $u_n = f_n e^{-\frac{|z|^2}{2h}}$. In general, (1.1) is not defined for all f in $\mathcal{F}_h$, but only for $f \in \mathcal{F}_h^1$. We assume that the energy is infinite if $f \notin \mathcal{F}_h^1$. The sequence $(|z|u_n)_{n\in\mathbb{N}}$ is bounded in $L^2(\mathbb{C}, L(dz))$ and the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in $L^4(\mathbb{C}, L(dz)) \cap L^2(\mathbb{C}, L(dz))$. Let u_∞ be its weak limit (possibly after extracting a subsequence), the convexity of the functional E^h_{LLL} implies

$$E^h_{LLL}(u_\infty) \leq \inf_n E^h_{LLL}(u_n) = e^h_{LLL}\ .$$

Furthermore, f_n converges to $f_\infty = e^{\frac{|z|^2}{h}} u_\infty$ in $\mathcal{D}'(\mathbb{C})$ and the holomorphy of f_n, $\partial_{\bar{z}} f_n = 0$, implies the holomorphy of f_∞.

The estimates

$$\int_{\mathbb{C}} |z|^2 |f_n(z)|^2 e^{-\frac{|z|^2}{h}} L(dz) \leq E^h_{LLL}(u_n)$$

$$\text{and} \qquad \|f_n\|_{\mathcal{F}_h} = 1$$

imply that the sequence is bounded in the form domain $Q(N_h) = D(N_h^{1/2})$ of the number operator N_h. After a conjugation with the Bargmann transform it is the form domain of the harmonic oscillator

$$B_h^* Q(N_h) B_h = \left\{ u \in L^2(\mathbb{R}, dy),\ \ yu, h\partial_y u \in L^2(\mathbb{R}, dy) \right\}\ .$$

Hence the form domain $Q(N_h)$ is compactly embedded in $\mathcal{F}_h$ and we get

$$1 = \lim_{n\to\infty} \|f_n\|_{\mathcal{F}_h} = \|f_\infty\|_{\mathcal{F}_h}\ .$$

We have proved that $E^h_{LLL}(e^{-\frac{|z|^2}{2h}} f_\infty) = e^h_{LLL}$. The rest of the proof is contained in Subsection 3.2 below (Propositions 3.1 and 3.2). $\square$

3.2 The Euler-Lagrange equation

The functional to be minimized can be rewritten as

$$E_{LLL}^h(e^{-\frac{|z|^2}{h}}f) = \langle f \mid (N_h+h)f\rangle_{\mathcal{F}_h} + \frac{Na\Omega_h^2\pi h}{4}\left\|P_{t_0}f\right\|_{A_h^4}^4\,, \quad t_0 = \frac{\ln 2}{2}\,. \tag{3.1}$$

A natural space for the analysis of the variational problem is

$$\left\{f\in Q(N_h), \quad P_{t_0}f\in A_h^4\right\}$$

and we recall that it is (compactly) embedded in $\mathcal{F}_h$. The hypercontractivity property also implies that this space equals $Q(N_h)=\mathcal{F}_h^1$.

Proposition 3.1. *The minimization problem* (1.4) *is equivalent to*

$$\min_{f\in\mathcal{F}_h^1, \|f\|_{\mathcal{F}_h}=1} \langle f \mid (N_h+h)f\rangle_{\mathcal{F}_h} + \frac{Na\Omega_h^2\pi h}{4}\left\|P_{t_0}f\right\|_{A_h^4}^4\,.$$

A minimum $f\in\mathcal{F}_h^1$ must satisfy the Euler-Lagrange equation

$$zh\partial_z f + Na\Omega_h^2\Pi_h\left(e^{-\frac{|z|^2}{h}}\left|f\right|^2 f\right) - (\lambda-h)f = 0 \quad \text{in } \mathcal{F}_h^{-1} \tag{3.2}$$

with the Lagrange multiplier $\lambda\in\mathbb{R}$. Moreover the Lagrange multiplier associated with a minimum satisfies the uniform estimates

$$\frac{2\Omega_h}{3}\sqrt{\frac{2Na}{\pi}} < e_{LLL}^h \le \lambda \le 2e_{LLL}^h \le 2\frac{2\Omega_h}{3}\sqrt{\frac{2bNa}{\pi}} + o_{Na}(h^0)\,. \tag{3.3}$$

Proof: According to Lemma 2.1 all the quantities are weakly differentiable on $\mathcal{F}_h^1 = Q(N_h)$. The computation of the complex derivative at a point f gives

$$\begin{aligned}\partial_{\overline{f}}E_{LLL}^h(f).\delta f &= \langle f \mid (N_h+h)\delta f\rangle_{\mathcal{F}_h} + Na\Omega_h^2\int_{\mathbb{C}}\left|f(z)\right|^2\overline{f(z)}\delta f(z)e^{-\frac{2|z|^2}{h}}\,L(dz)\\ &= \left\langle zh\partial_z f + Na\Omega_h^2\Pi_h\left(e^{-\frac{|z|^2}{h}}\left|f\right|^2 f\right)\,\middle|\,\delta f\right\rangle_{\mathcal{F}_h}\,,\end{aligned}$$

where the introduction of the orthogonal projector Π_h is made possible by $\Pi_h\delta f=\delta f$ for $\delta f\in\mathcal{F}_h$. Since the gradient of the constraint is simply f, we find (3.2).

Taking the $\mathcal{F}_h$-scalar product of each side of the Euler-Lagrange equation with f gives

$$\lambda = \langle f, (N_h+h)f\rangle_{\mathcal{F}_h} + \frac{Na\Omega_h^2\pi h}{2}\left\|P_{t_0}f\right\|_{A_h^4}^4\,.$$

This implies that $e_{LLL}^h\le\lambda\le 2e_{LLL}^h$. The upper bound of e_{LLL}^h, uniform with respect to h, comes from the estimates of [AfBl]. The strict lower bound comes from the fact that a holomorphic function cannot have a compact support. □

We recall that polynomials are dense in any A_p^h (see [Car, Gro]). In fact, we will simply use the density of $\mathbb{C}[z]$ in any $\mathcal{F}_h^s$, $s \in \mathbb{R}$, which is a consequence of the spectral theorem for the number operator N_h since the range of the spectral projector $1_{[0,M]}(N_h)$ is the set of polynomials of degree bounded by Mh^{-1}.
We thus deduce the

Proposition 3.2. *The Euler-Lagrange equation (3.2) satisfied by a function f minimizing (1.4) in $\mathcal{F}_h^1$ can be rewritten as*

$$zh\partial_z f + Na\Omega_h^2 \Pi_h \left(e^{-\frac{|z|^2}{h}} |f|^2 \right) \Pi_h f - (\lambda - h) f = 0, \tag{3.4}$$

$$\Pi_h \left[|z|^2 + Na\Omega_h^2 e^{-\frac{|z|^2}{h}} |f|^2 - \lambda \right] \Pi_h f = 0, \tag{3.5}$$

or $$zh\partial_z f + \frac{Na\Omega_h^2}{2} \bar{f}(h\partial_z)[f^2(2^{-1}.)] - (\lambda - h) f = 0, \tag{3.6}$$

where the operator $\bar{f}(h\partial_z)$ is defined as the limit $\lim_{K\to\infty} \sum_{k=0}^K \overline{a_k}(h\partial_z)^k$ if $f(z) = \sum_{k=0}^\infty a_k z^k$, and where all the quantities make sense in $\mathcal{F}_h^{-1}$.

Proof: Equation (3.5) is a consequence of

$$N_h = zh\partial_z = \Pi_h(|z|^2 - h)\Pi_h.$$

For (3.6), we actually want other expressions of the middle term

$$\Pi_h \left(e^{-\frac{|z|^2}{h}} |f|^2 f \right).$$

We recall that according to Lemma 2.1 the mapping

$$(f_1, f_2, f_3) \to \Pi_h \left(e^{-\frac{|z|^2}{h}} \overline{f_1(z)} f_2(z) f_3(z) \right)$$

is a $(1,2)$-linear continuous mapping from $\mathcal{F}_h$ into $\mathcal{F}_h$. By the density of $\mathbb{C}[z]$ in $\mathcal{F}_h$, we only need to check the new expressions on polynomials f. If $f \in \mathbb{C}[z]$, we have

$$\begin{aligned} \Pi_h \left(e^{-\frac{|z|^2}{h}} |f|^2 f \right) &= \Pi_h \left(e^{-\frac{|z|^2}{h}} |f|^2 \right) \Pi_h f \\ &= \Pi_h(\overline{f(z)}) \Pi_h \Pi_h (e^{-\frac{|z|^2}{h}} f^2) \\ &= \overline{f}(h\partial_z) \Pi_h (e^{-\frac{|z|^2}{h}} f^2). \end{aligned}$$

A simple change of variable gives

$$\begin{aligned} \Pi_h \left(e^{-\frac{|z|^2}{h}} f^2 \right)(z) &= (\pi h)^{-1} \int_{\mathbb{C}} e^{\frac{z\overline{z'} - 2|z'|^2}{h}} f^2(z')\ L(dz') \\ &= 2^{-1} \Pi_h \left(f^2(2^{-1/2}.) \right) (2^{-1/2} z) = \frac{1}{2} f^2(z/2). \end{aligned}$$

□

Remark 3.3. *Equation (3.6) translates easily in the orthonormal basis of monomials* $\frac{1}{\sqrt{\pi h^{n+1} n!}} z^n = B_h H_n^h$ *(see Appendix A): if*

$$f(z) = \sum_{n \geq 0} a_n z^n,$$

then (3.6) is equivalent to

$$\forall n \geq 0, \quad [(n+1)h - \lambda] a_n + \frac{Na\Omega_h^2}{2^n} \sum_{k \geq 0} \frac{\overline{a_k} h^k}{2^k} \sum_{p=0}^{n+k} a_p a_{n+k-p} = 0,$$

with the normalization condition $\sum_{n \geq 0} \pi h^{n+1} n! |a_n|^2 = 1.$

3.3 Properties of the minimizer

We shall prove Theorem 1.2. We start with excluding the possibility that the minimizer is a polynomial.

Lemma 3.4. *Let f be a solution to (1.9), with the condition*

$$\lambda^3 h < 27 \left(\frac{Na\Omega_h^2 e^{-\frac{1}{12}}}{4\pi} \right)^2, \tag{3.7}$$

then f cannot be a polynomial.

Proof: We argue by contradiction, and assume that f is a polynomial of degree n. Since f satisfies (1.9), it follows that $P(z) = \overline{f}(h\partial_z) \left[f\left(\frac{z}{2}\right)^2 \right]$ is a polynomial of degree n. On the other hand, $(h\partial_z)^k [f\left(\frac{z}{2}\right)^2]$ is a polynomial of degree $2n-k$, so that f must be equal to αz^n for some $\alpha \in \mathbb{C}$. Plugging this equality in (1.9), we find that $nh + Na\Omega_h^2 |\alpha|^2 \frac{h^n}{2^{2n+1}} \frac{(2n)!}{n!} - \lambda + h = 0$. Since f is of norm 1, we necessarily have $|\alpha|^2 = \frac{1}{\pi h^{n+1} n!}$, so

$$(n+1)h + Na\Omega_h^2 \frac{(2n)!}{h\pi 2^{2n+1} (n!)^2} - \lambda = 0.$$

Besides, the improved Stirling formula [Rob] implies that

$$\frac{e^{-\frac{1}{12}}}{\sqrt{n}} \leq \frac{(2n)!}{2^{2n} (n!)^2} \leq \frac{e^{\frac{1}{12}}}{\sqrt{n}}.$$

Thus,

$$\lambda - h \geq nh + Na\Omega_h^2 \frac{e^{-\frac{1}{12}}}{2h\pi\sqrt{n}} \geq \frac{c_1}{h^{\frac{1}{3}}},$$

where $c_1 > 0$ is given by

$$c_1 = 3\left(\frac{Na\Omega_h^2 e^{-\frac{1}{12}}}{4\pi}\right)^{\frac{2}{3}}.$$

If $h^{1/3} < c_1/\lambda$, this is a contradiction. $\square$

The proof of Theorem 1.2 relies on a similar but more involved argument.

Proof of Theorem 1.2: We assume that f has a finite number of zeroes. One can thus find an entire function ϕ and a polynomial P such that

$$f(z) = P(z)e^{\phi(z)}.$$

We now apply [Car, Theorem 3], with $p = 2$,

$$|f(z)| \leq e^{\frac{|z|^2}{2h}}\|f\|_{A_h^2} = \frac{e^{\frac{|z|^2}{2h}}}{\sqrt{2\pi h}}\|f\|_{\mathcal{F}_h}.$$

We thus necessarily have $\mathbb{R}\mathrm{e}(\phi(z)) \leq \frac{|z|^2}{2h} + C$, for some constant C (possibly depending on $h > 0$). It is a classical result, used in Hadamard's factorization theorem, that this implies that ϕ is a polynomial of degree less than 2 (see [Boa]-page 3 for example). We provide the short proof for the sake of completeness: it is a consequence of the following estimate

$$M(r) \leq \frac{2r}{R-r}A(R) + \frac{R+r}{R-r}|\phi(0)| \text{ for } r < R$$

where $A(r) = \sup_{\theta\in(0,2\pi)} Re\left(\phi(re^{i\theta})\right)$, and $M(r) = \sup_{\theta\in(0,2\pi)} |\phi(re^{i\theta})|$. This inequality can be proved first when $\phi(0) = 0$ applying the Schwarz Lemma to $\phi(z)/(2A(R) - \phi(z))$ and then extended to any $\phi(0)$. Thus $\phi(z)$ is an analytic function which has a modulus estimated by $1 + |z|^2$. It is a polynomial of degree ≤ 2. This yields

$$f(z) = P(z)e^{\beta z + \gamma z^2},$$

where $f \in \mathcal{F}_h$ implies $|\gamma| < \frac{1}{2h}$. Note also that the rotational invariance of (3.2) allows to assume

$$\gamma \in \mathbb{R}, \quad |\gamma| < \frac{1}{2h},$$

after possibly changing β and P. We first compute the linear terms of (3.2):

$$\begin{aligned} zh\partial_z f(z) - (\lambda - h)f(z) &= \left(zhP'(z) + (\beta z + 2\gamma z^2)P(z) - (\lambda - h)P(z)\right)e^{\beta z + \gamma z^2} \\ &= Q(z)e^{\beta z + \gamma z^2}, \quad (3.8) \end{aligned}$$

where $Q \in \mathbb{C}[z]$. We next compute the nonlinear term:

$$\Pi_h\left(e^{-\frac{|z|^2}{h}}|f(z)|^2 f(z)\right) = \frac{1}{\pi h}\int_{\mathbb{C}} e^{\frac{z\overline{z'}}{h} - 2\frac{|z'|^2}{h} + \gamma\overline{z'}^2 + 2\gamma z'^2} e^{\overline{\beta z'} + 2\beta z'}\overline{P(z')}P(z')^2 L(dz'). \quad (3.9)$$

By setting $z'=x'+iy'$, the Gauss decomposition of the argument of the first exponential gives

$$\frac{z\overline{z'}}{h}-2\frac{|z'|^2}{h}+\gamma\overline{z'}^2+2\gamma z'^2=$$
$$\frac{1}{h}\left[(3h\gamma-2)\left(x'+\frac{z+2ih\gamma y'}{2(3h\gamma-2)}\right)^2+\frac{4-8h^2\gamma^2}{(3h\gamma-2)}\left(y'+iz\frac{1-2h\gamma}{4-8h^2\gamma^2}\right)^2\right.$$
$$\left.+\frac{h\gamma}{2(1-2h^2\gamma^2)}z^2\right]. \quad (3.10)$$

Hence a Gaussian integration, (3.2) and (3.8) lead to

$$-Q(z)e^{\beta z+\gamma z^2}=\Pi_h\left(e^{-\frac{|z|^2}{h}}|f(z)|^2f(z)\right)=Q_1(z)e^{\frac{h\gamma}{2h(1-2h^2\gamma^2)}z^2}e^{\beta' z}$$

where $Q_1(z)$ is a polynomial and $\beta'\in\mathbb{C}$. This is possible only when

$$\gamma=\frac{\gamma}{2(1-2h^2\gamma^2)},$$

which, with the condition $|\gamma|<\frac{1}{2h}$, implies $\gamma=0$.
The total argument of the exponentials in (3.9), now equals

$$\frac{z\overline{z'}}{h}-2\frac{|z'|^2}{h}+\overline{\beta z'}+2\beta z'=$$
$$\frac{1}{h}\left[2\left(x'-z-\frac{h\beta}{2}-\frac{h\overline{\beta}}{4}\right)^2+2\left(y'+iz-i\frac{h\beta}{2}+i\frac{h\overline{\beta}}{4}\right)^2+4h\beta z+C_{\beta,h}\right],$$

with $C_{\beta,h}\in\mathbb{C}$ is independent of z. The Gaussian integration now gives

$$-Q(z)e^{\beta z}=Q_1(z)e^{4\beta z}$$

for some polynomial $Q_1(z)$. This is possible only when $\beta=0$.
Hence if the solution f of (3.2) has a finite number of zeroes, it has to be a polynomial. Lemma 3.4 implies that it is not possible if (3.7) is satisfied. □

Proof of Corollary 1.3: The upper bound of the Lagrange multiplier (3.3) which provides

$$\lambda\leq 2e_{LLL}^h\leq\frac{4\Omega_h}{3}\sqrt{2Nab}\pi+o_{Na}(h^0)$$

implies that the minimizer cannot have a finite number of zeroes when h is small, as it is stated in Corollary 1.3.
In fact, Theorem 1.2 and the above control on the Lagrange multiplier also imply that the minimizer has an infinite number of vortices as soon as

$$8E_0^3h<27\left(\frac{Na\Omega_h^2e^{-\frac{1}{12}}}{4\pi}\right)^2,$$

where E_0 is any number larger than e_{LLL}^h. Such an upper bound of e_{LLL}^h can be computed numerically. Our general statement is not as precise due to the implicit error term $o_{Na}(h^0)$.

4 The Theta function and Abrikosov lattices

This section is devoted to the proof of Theorem 1.4 as well as some refinements. After recalling some classical properties of the Theta function Θ (subsection 4.1), we check that the Abrikosov ansatz (1.13), which involves the Theta function, is the only possible one when assuming that the wave function u has an $\mathfrak{L}$-periodic modulus and admits exactly one zero per cell (subsection 4.2). Then, we check that the Euler Lagrange equation (1.14) holds and find the expression (1.16) for $\gamma(\tau)$ (subsection 4.3). The optimality of the hexagonal lattice is finally deduced from a previous result about this quantity in [NoVo] obtained in the framework of quantum chaos (subsection 4.4).

4.1 The Theta function

We give a quick summary of some mathematical properties of the Theta function defined by

$$\Theta(v,\tau)=\frac{1}{i}\sum_{n=-\infty}^{+\infty}(-1)^n e^{i\pi\tau(n+1/2)^2}e^{(2n+1)\pi iv},\quad v\in\mathbb{C} \tag{4.1}$$

which will be used here and refer the reader to [Cha] for details and further information. The Theta function satisfies

$$\Theta(v,\tau)=-\Theta(-v,\tau) \tag{4.2}$$
$$\Theta(v+1,\tau)=-\Theta(v,\tau) \tag{4.3}$$
$$\Theta(v+\tau,\tau)=-q^{-1}e^{-2\pi iv}\Theta(v,\tau)\ , \tag{4.4}$$

where $q=e^{i\pi\tau}$. It is naturally associated with the lattice $\mathbb{Z}\oplus\tau\mathbb{Z}$. In fact, in order to describe all lattices, one can restrict the value of τ: there is a one to one mapping (see [Cha]) between lattices in $\mathbb{C}$ such that 1 is the smallest period and the Poincaré half-space quotiented by the action of the modular group:

$$\tau_I=\mathcal{I}\mathrm{m}\,\tau>0,\quad |\tau|\geq 1,\quad -1/2\leq\tau_R=\mathbb{R}\mathrm{e}\,\tau<1/2 \tag{4.5}$$

with

$$\mathbb{R}\mathrm{e}\,\tau\leq 0,\quad \text{if}\quad |\tau|=1\ . \tag{4.6}$$

For some integral quantities associated with the Theta function, the transformation formula

$$\sqrt{\frac{\tau}{i}}\Theta(v,\tau)=ie^{-\frac{\pi iv^2}{\tau}}\Theta(\frac{v}{\tau},-\frac{1}{\tau}) \tag{4.7}$$

combined with the invariance with respect to $\tau\to\tau+1$ provides a way to check their modular invariance.

4.2 The Abrikosov Ansatz

In this section, we prove the first part of Theorem 1.4.

Proposition 4.1. *Let $\mathfrak{L} = (1/\nu)\left(\mathbb{Z} \oplus \tau\mathbb{Z}\right)$ be a lattice with $\nu \in \mathbb{R}^{+*}$ and let f be an entire function such that $f^{-1}(\{0\}) = \mathfrak{L}$ with simple zeroes. If the modulus of $u(z) = e^{-\frac{|z|^2}{2h}} f(z)$ is $\mathfrak{L}$-periodic according to*

$$\left| u(z + \frac{1}{\nu}) \right| = |u(z)| \tag{4.8}$$

$$\text{and} \qquad \left| u(z + \frac{\tau}{\nu}) \right| = |u(z)| \ , \tag{4.9}$$

then $\nu = \sqrt{\frac{\tau_I}{\pi h}}$ and f (resp. u) is proportional to

$$f_\tau(z) = e^{\frac{z^2}{2h}} \Theta\left(\frac{\sqrt{\tau_I}}{\sqrt{\pi h}} z, \tau \right) \quad (\textit{resp. } u_\tau(z) = e^{-\frac{|z|^2}{2h}} f_\tau(z)) \, . \tag{4.10}$$

Proof: This is a specific version of Hadamard's factorization theorem ([Boa]). Since f and the function $\Theta(\nu v, \tau)$ have the same zeroes, their quotient is an analytic function which does not vanish. Hence one can find an analytic function ϕ such that

$$f(z) = e^{\phi(z)} \Theta(\nu z, \tau) \, .$$

The $\mathfrak{L}$-periodicity of $|u| = e^{-\frac{|z|^2}{2h}} |f(z)|$ implies the upper bounds

$$\forall z \in \mathbb{C}, \ \ e^{\mathbb{R}\mathrm{e}(\phi(z)) - \frac{|z|^2}{h}} \left| \Theta(\nu z, \tau) \right| \leq C_1 \, .$$

Therefore when the periodicity cell Q is chosen such that $\mathfrak{L} \cap \partial Q = \emptyset$, there exists a constant $C > 0$ such that

$$\forall z \in \partial Q + \mathfrak{L}, \quad \mathbb{R}\mathrm{e}\left(\phi(z)\right) \leq \left(C + (2h)^{-1} \right) |z|^2 + \ln(CC_1) \, .$$

Since $\mathbb{R}\mathrm{e}\left(\phi(z)\right)$ is a harmonic function on any cell $\frac{m}{\nu} + \frac{n\tau}{\nu} + Q$, $(m,n) \in \mathbb{Z}^2$, the maximum principle implies

$$\forall z \in \mathbb{C}, \quad \mathbb{R}\mathrm{e}\left(\phi(z)\right) \leq C'(|z|^2 + 1)$$

for some constant $C' > 0$. As in the proof of Theorem 1.2, we find that there exists $(\delta, \eta, \beta) \in \mathbb{C}^3$ such that

$$f(z) = e^{\delta + \eta z + \beta z^2} \Theta(\nu z, \tau) \, , \quad u(z) = e^{-\frac{|z|^2}{h}} f(z)$$

where the constant δ can be set to 0. We get

$$\begin{aligned} u\left(z + \frac{1}{\nu}\right) &= e^{-\frac{|z+1/\nu|^2}{2h}} e^{\beta(z+1/\nu)^2 + \eta(z+1/\nu)} \Theta(\nu z + 1, \tau) \\ &= -e^{-\frac{|z+1/\nu|^2}{2h}} e^{\beta(z+1/\nu)^2 + \eta(z+1/\nu)} \Theta(\nu z, \tau) \\ &= -e^{-\frac{|z+1/\nu|^2 - |z|^2}{2h} + \beta[(z+1/\nu)^2 - z^2] + \eta/\nu} u(z). \end{aligned}$$

The first periodicity constraint (4.8) implies

$$\mathbb{R}\mathrm{e}\left[-\frac{1}{2h\nu}-\frac{\mathbb{R}\mathrm{e}(\overline{z})}{h}+\beta\left(\frac{1}{\nu}+2z\right)+\eta\right]=0.$$

Let $\beta=\beta_R+i\beta_I$, $z=z_R+iz_I$, and $\eta=\eta_R+i\eta_I$, we have

$$\left(2\beta_R-\frac{1}{h}\right)z_R-2\beta_I z_I-\frac{1}{2h\nu}+\frac{\beta_R}{\nu}+\eta_R=0,\quad \forall z_R,z_I\in\mathbb{R}.$$

This implies $\beta=\frac{1}{2h}$ and $\eta_R=0$. We also have

$$\begin{aligned} f(z+\tau/\nu) &= e^{\frac{(z+\tau/\nu)^2}{2h}+i\eta_I(z+\tau/\nu)}\Theta(\nu z+\tau,\tau)\\ &= -e^{\frac{2\nu^{-1}z\tau+\nu^{-2}\tau^2}{2h}+i\eta_I\tau/\nu-i\pi(2\nu z+\tau)}f(z)\,,\\ \text{and}\quad u(z+\tau/\nu) &= -e^{-\left[\frac{2\,\mathbb{R}\mathrm{e}(\overline{\tau}z)}{2h\nu}+\frac{|\tau|^2}{2h\nu^2}\right]}e^{\frac{2z\tau}{2h\nu}+\frac{\tau^2}{2h\nu^2}+i\eta_I\frac{\tau}{\nu}-i\pi(2\nu z+\tau)}u(z)\,. \end{aligned}$$

Let $\tau=\tau_R+i\tau_I$. The second periodicity constraint (4.9) leads to

$$\begin{aligned} -\frac{1}{h\nu}(\tau_R z_R+\tau_I z_I)-\frac{\tau_R^2+\tau_I^2}{2h\nu^2}+\frac{2}{2h\nu}(\tau_R z_R-\tau_I z_I)&\\ +\frac{\tau_R^2-\tau_I^2}{2h\nu^2}+\pi(2\nu z_I+\tau_I)-\frac{\eta_I\tau_I}{\nu}&=0\,, \end{aligned}$$

and hence to

$$2z_I\left(-\frac{\tau_I}{h\nu}+\pi\nu\right)-\frac{\tau_I^2}{h\nu^2}+\pi\tau_I-\frac{\eta_I\tau_I}{\nu}=0\,.$$

The only possibility is

$$\begin{aligned} &\pi\nu^2 h=\tau_I,\quad \text{i.e}\quad \nu=\sqrt{\frac{\tau_I}{\pi h}}\\ \text{and}\quad &\eta_I=0\,. \end{aligned}$$

Conversely, the previous calculations also show that

$$u(z)=e^{-\frac{|z|^2}{2h}}e^{\frac{z^2}{2h}}\Theta(\sqrt{\frac{\tau_I}{\pi h}}z,\tau)$$

has an $\mathfrak{L}$-periodic modulus. □

The next statement ensures that specifying the position of the vortices on $\mathfrak{L}$ has no effect on the quantity $\fint|u|^4/(\fint|u|^2)^2$.

Proposition 4.2. *Let $\mathfrak{L}=(1/\nu)\,(\mathbb{Z}\oplus\tau\mathbb{Z})$ with $\tau=\tau_R+i\tau_I$ and $\nu>0$. Let f be an entire function such that $u(z)=e^{-\frac{|z|^2}{2h}}f(z)$ has an $\mathfrak{L}$-periodic modulus according to (4.8)-(4.9). If $f^{-1}(\{0\})=z_0+\mathfrak{L}$ with simple zeroes, then $\nu=\sqrt{\tau_I/(\pi h)}$ and the function $|u(.+z_0)|$ is proportional to $|u_\tau|$.*

Proof: It is enough to notice that

$$\left|\frac{u(z+z_0)}{u_\tau(z)}\right| = e^{-\frac{2\,\mathbb{R}\mathrm{e}(\overline{z_0}z)+|z_0|^2}{2h}}\left|\frac{f(z+z_0)}{f_\tau(z)}\right| = \left|\frac{e^{-\frac{2\overline{z_0}z+|z_0|^2}{2h}}f(z)}{f_\tau(z)}\right|$$

is a continuous periodic function. Hence $\frac{e^{-\frac{2\overline{z_0}z+|z_0|^2}{2h}}f(z)}{f_\tau(z)}$ is a bounded holomorphic function. Thus it is constant and so is its modulus. □

4.3 The Abrikosov function as a solution to an Euler-Lagrange equation

Although the variational problem (1.4) loses compactness when the confining potential is removed, the Euler-Lagrange equation (3.2) makes sense without the linear term. In this section, we check that the function f_τ defined in (4.10) provides a solution and we find an explicit expression for the mean energy $\gamma(\tau)$ introduced in (1.12).

Proposition 4.3. *For any $\tau \in \mathbb{C}$ such that $\tau_I > 0$, the function f defined by*

$$f_\tau(z) = e^{\frac{z^2}{2h}}\Theta\left(\frac{\sqrt{\tau_I}}{\sqrt{\pi h}}z, \tau\right)$$

satisfies

$$\Pi_h\left(|u_\tau|^2 f_\tau\right) = \lambda_\tau f_\tau, \quad \textit{in } \mathcal{F}_h^s,\ s<-1$$

with u_τ defined by

$$u_\tau(z) = f_\tau(z)e^{-\frac{|z|^2}{2h}},$$

and

$$\lambda_\tau = \frac{⨍|u_\tau|^4}{⨍|u_\tau|^2} = \frac{1}{\sqrt{2\tau_I}}\sum_{k,\ell\in\mathbb{Z}} e^{-\frac{\pi}{\tau_I}|k\tau-\ell|^2}. \tag{4.11}$$

Thus, the mean energy $\gamma(\tau) = \frac{⨍|u_\tau|^4}{(⨍|u_\tau|^2)^2}$ introduced in (1.12) has an explicit expression:

$$\gamma(\tau) = \frac{\lambda_\tau}{⨍|u_\tau|^2} = \sum_{k,\ell\in\mathbb{Z}} e^{-\frac{\pi}{\tau_I}|k\tau-\ell|^2}. \tag{4.12}$$

Proof: Let $Z = \sqrt{\frac{\tau_I}{\pi h}}z$, we have

$$f_\tau(z) = e^{\frac{\pi Z^2}{2\tau_I}}\Theta(Z,\tau), \quad u_\tau(z) = e^{\frac{\pi}{2\tau_I}(Z^2-|Z|^2)}\Theta(Z,\tau).$$

Moreover, using the definition of Θ, we have:

$$|\Theta(Z,\tau)|^2 = \sum_{n,n'\in\mathbb{Z}}(-1)^{n+n'}e^{i\pi\left(\tau(n+\frac{1}{2})^2-\overline{\tau}(n'+\frac{1}{2})^2\right)}e^{i\pi\left((2n+1)Z-(2n'+1)\overline{Z}\right)}.$$

We next set $Z = x + y\tau$, where $x, y \in \mathbb{R}$, and compute:

$$|\Theta(Z,\tau)|^2 = \sum_{n,n'\in\mathbb{Z}} (-1)^{n+n'} e^{i\pi\left(\tau(n+\frac{1}{2}+y)^2 - \overline{\tau}(n'+\frac{1}{2}+y)^2\right)} e^{2i\pi x(n-n')} e^{-i\pi(\tau-\overline{\tau})y^2} .$$

We also have $|u_\tau(z)|^2 = \left| e^{\frac{\pi}{2\tau_I}\left(Z^2 - |Z|^2\right)} \right|^2 |\Theta(Z,\tau)|^2 = e^{-2\pi\tau_I y^2} \left|\Theta(Z,\tau)\right|^2$. Hence,

$$|u_\tau(z)|^2 = \sum_{n,n'\in\mathbb{Z}} (-1)^{n+n'} e^{i\pi\left(\tau(n+\frac{1}{2}+y)^2 - \overline{\tau}(n'+\frac{1}{2}+y)^2\right)} e^{2i\pi x(n-n')} .$$

We know that $|u_\tau|^2$ satisfies (4.8) and (4.9), so that the above expression is periodic of period one in both x and y. Following [Tor], we compute the Fourier coefficients of this periodic function, by computing the Fourier transform

$$\int_{\mathbb{C}} e^{-iy.\eta} e^{i\pi\left[\tau(n+1/2+y)^2 - \overline{\tau}(n'+1/2+y)^2\right]}\, dy = \frac{1}{\sqrt{2\tau_I}} e^{-\frac{\pi}{2\tau_I}\left|(n-n')\tau - \frac{\eta}{2\pi}\right|^2 + i\frac{\eta}{2}(n-n')} e^{i\eta n'}$$

The change of indices $k = n - n'$ and the Poisson formula

$$\sum_{n'\in\mathbb{Z}} e^{i\eta n'} = 2\pi \sum_{\ell\in\mathbb{Z}} \delta(\eta - 2\pi\ell)$$

lead to

$$|u_\tau(z)|^2 = \frac{1}{\sqrt{2\tau_I}} \sum_{k,\ell\in\mathbb{Z}} (-1)^{k\ell+k+\ell} e^{-\frac{\pi}{2\tau_I}|k\tau-\ell|^2} e^{2i\pi(kx+\ell y)}, \tag{4.13}$$

Thus, using that $kx + \ell y = kZ + (\ell - k\tau)\frac{Z-\overline{Z}}{2i\tau_I}$ and $Z = \sqrt{\frac{\tau_I}{\pi h}}z$, (4.13) yields

$$\begin{aligned} |u_\tau(z)|^2 f(z) &= \frac{1}{\sqrt{2\tau_I}} \sum_{k,\ell\in\mathbb{Z}} (-1)^{k\ell+k+\ell} e^{-\frac{\pi}{2\tau_I}|k\tau-\ell|^2} e^{2i\pi(kx+\ell y) + \frac{z^2}{2h}} \Theta\left(\sqrt{\frac{\tau_I}{\pi h}}z, \tau\right) \\ &= \frac{1}{\sqrt{2\tau_I}} \sum_{k,\ell\in\mathbb{Z}} (-1)^{k\ell+k+\ell} e^{-\frac{\pi}{2\tau_I}|k\tau-\ell|^2} e^{\frac{z^2}{2h}} e^{2i\pi\sqrt{\frac{\tau_I}{\pi h}}\left(kz + (\ell-k\tau)\frac{z-\overline{z}}{2i\tau_I}\right)} \Theta\left(\sqrt{\frac{\tau_I}{\pi h}}z, \tau\right). \end{aligned}$$

Up to now, all the series were considered as (rapidly) pointwise convergent series. We shall use the next lemma, the proof of which is postponed.

Lemma 4.4. *Let*

$$f_\tau(z) = e^{\frac{z^2}{2h}} \Theta\left(\frac{\sqrt{\tau_I}}{\sqrt{\pi h}}z, \tau\right) ,$$

and

$$u_\tau(z) = f_\tau(z) e^{-\frac{|z|^2}{2h}},$$

then

$$\Pi_h\left(|u_\tau(z)|^2 f_\tau(z)\right) = \frac{1}{\sqrt{2\tau_I}} \sum_{k,\ell\in\mathbb{Z}} e^{-\frac{\pi}{\tau_I}\left((k\tau_R-\ell)^2 + k^2\tau_I^2\right)} e^{\frac{z^2}{2h}} \Theta\left(\sqrt{\frac{\tau_I}{\pi h}}z, \tau\right).$$

holds in $\mathcal{F}_h^s$ *for any* $s < -1$.

We thus find (1.14), with $\lambda_\tau = \frac{1}{\sqrt{2\tau_I}} \sum_{k,\ell\in\mathbb{Z}} e^{-\frac{\pi}{\tau_I}|k\tau-\ell|^2}$. The Fourier series (4.13) has a cell integral equal to

$$\int_Q |u_\tau(z)|^2 \; L(dz) = \pi h \int_{[0,1]^2} \frac{1}{\sqrt{2\tau_I}} \, dxdy = \frac{\pi h}{\sqrt{2\tau_I}}.$$

Thus,

$$\fint |u_\tau|^2 = \frac{\pi h}{\pi h\sqrt{2\tau_I}} = \frac{1}{\sqrt{2\tau_I}},$$

and the L^4 norm of u is deduced from (4.13) by the Parseval identity,

$$\fint |u_\tau|^4 = \frac{1}{\pi h}\frac{\pi h}{2\tau_I} \sum_{k,\ell\in\mathbb{Z}} e^{-\frac{\pi}{\tau_I}|k\tau-\ell|^2} = \frac{1}{2\tau_I} \sum_{k,\ell\in\mathbb{Z}} e^{-\frac{\pi}{\tau_I}|k\tau-\ell|^2},$$

which yields the result. □

Proof of Lemma 4.4: First, the periodicity of $|u_\tau(z)|$ implies

$$\forall\delta>0, \quad \int_{\mathbb{C}} \langle z\rangle^{-2(1+\delta)} \, |u_\tau(z)|^2 \; L(dz) < \infty\,,$$

which means $f_\tau \in \mathcal{F}_h^s$ for any $s<-1$. The series (4.13) is a rapidly convergent one in the symbol class $S(1, dx^2+d\xi^2)$ ($\left|\partial_x^\alpha\partial_\xi^\beta\sigma(x,\xi)\right| \le C_{\alpha,\beta}$), by taking $z = \frac{x-i\xi}{\sqrt{2}}$. Hence the same convergence holds after applying the convolution with $\frac{1}{\pi h}e^{-\frac{|z|^2}{h}}$ (recall $L(dz) = \frac{1}{2}dxd\xi$). The conjugation with the Bargmann transform gives according to Appendix A

$$B_h^*\Pi_h \, |u_\tau|^2 \, \Pi_h B_h = \left(|u_\tau|^2\right)^{\text{A-Wick}}(y, hD_y) = \left(\frac{1}{\pi h}e^{-\frac{|z|^2}{h}} * |u_\tau|^2\right)^W (y, hD_y)\,.$$

The global pseudo-differential calculus (see [Hor2, BoLe, BoCh] or [Hel, HeNi] for related specific applications) implies that the convergence

$$\left(\frac{1}{\pi h}e^{-\frac{|z|^2}{h}} * |u_\tau|^2\right)^W (y, hD_y) =$$
$$\frac{1}{\sqrt{2\tau_I}} \lim_{N\to\infty} \sum_{|k|+|\ell|\le N} (-1)^{k\ell+k+\ell} e^{-\frac{\pi}{2\tau_I}|k\tau-\ell|^2} \left(\frac{1}{\pi h}e^{-\frac{|z|^2}{h}} * e^{2i\pi\sqrt{\frac{\tau_I}{\pi h}}\left(kz+(\ell-k\tau)\frac{z-\bar z}{2i\tau_I}\right)}\right)^W (y, hD_y)$$

holds in $\mathcal{L}(D(\tilde N_h^s))$ with $D(\tilde N_h^s) = B_h^*\mathcal{F}_h^s$, for any $s\in\mathbb{R}$. By conjugating back with the Bargmann transform B_h and by applying the result with $s<-1$ this leads to

$$\Pi_h\left(|u_\tau|^2 f\right) = \Pi_h\left(|u_\tau|^2\right)\Pi_h f_\tau$$
$$= \frac{1}{\sqrt{2\tau_I}} \lim_{N\to\infty} \sum_{|k|+|\ell|\le N} (-1)^{k\ell+k+\ell} e^{-\frac{\pi}{2\tau_I}|k\tau-\ell|^2} \Pi_h\left[e^{\frac{z^2}{2h}} e^{2i\pi\sqrt{\frac{\tau_I}{\pi h}}\left(kz+(\ell-k\tau)\frac{z-\bar z}{2i\tau_I}\right)} \Theta\left(\sqrt{\frac{\tau_I}{\pi h}}z, \tau\right)\right]$$

We now use the equality $\Pi_h\left(e^{\beta\overline{z}}p(z)\right) = p(z+h\beta)$, for any $p \in \mathcal{F}_h^s$, $s \in \mathbb{R}$, and compute

$$\begin{aligned}
&\frac{1}{\sqrt{2\tau_I}} \sum_{|k|+|\ell|\leq N} (-1)^{k\ell+k+\ell} e^{-\frac{\pi}{2\tau_I}|k\tau-\ell|^2} e^{\frac{1}{2h}\left(z+\sqrt{\frac{\pi h}{\tau_I}}(k\tau-\ell)\right)^2} \\
&\qquad\qquad \times e^{2i\pi\sqrt{\frac{\tau_I}{\pi h}}\left(k+\frac{\ell-k\tau}{2i\tau_I}\right)\left(z+\sqrt{\frac{\pi h}{\tau_I}}(k\tau-\ell)\right)} \Theta\left(\sqrt{\frac{\tau_I}{\pi h}}z + k\tau - \ell, \tau\right) \\
&= \frac{1}{\sqrt{2\tau_I}} \sum_{|k|+|\ell|\leq N} (-1)^{k\ell+k+\ell} e^{-\frac{\pi}{2\tau_I}\left(|k\tau-\ell|^2+(k\tau-\ell)^2\right)} e^{\frac{z^2}{2h}} \\
&\qquad\qquad \times e^{2i\pi\sqrt{\frac{\tau_I}{\pi h}}kz+2i\pi k(k\tau-\ell)} \Theta\left(\sqrt{\frac{\tau_I}{\pi h}}z + k\tau - \ell, \tau\right).
\end{aligned}$$

Using the equalities

$$\begin{aligned}
& e^{2i\pi k(k\tau-\ell)} = e^{-2\pi k^2\tau_I} e^{2i\pi k^2\tau_R}, \\
& |k\tau-\ell|^2 + (k\tau-\ell)^2 = 2\left((k\tau_R-\ell)^2 + i\left(k^2\tau_R\tau_I - k\ell\tau_I\right)\right) \\
\text{and}\quad & \Theta\left(Z + k\tau - \ell\right) = (-1)^{k+\ell} e^{-2i\pi kZ} e^{-i\pi k^2\tau} \Theta(Z,\tau),
\end{aligned}$$

we infer

$$\begin{aligned}
\Pi_h\left(|u_\tau(z)|^2 f_\tau(z)\right) &= \lim_{N\to\infty} \frac{1}{\sqrt{2\tau_I}} \sum_{|k|+|\ell|\leq N} (-1)^{k\ell+k+\ell} e^{-\frac{\pi}{\tau_I}\left((k\tau_R-\ell)^2+i(k^2\tau_R\tau_I-k\ell\tau_I)\right)} e^{\frac{z^2}{2h}} \\
&\qquad\qquad \times e^{i\pi k(k\tau-2\ell)} (-1)^{k+\ell} \Theta\left(\sqrt{\frac{\tau_I}{\pi h}}z, \tau\right). \\
&= \frac{1}{\sqrt{2\tau_I}} \left[\lim_{N\to\infty} \sum_{|k|+|\ell|\leq N} e^{-\frac{\pi}{\tau_I}\left((k\tau_R-\ell)^2+k^2\tau_I^2\right)}\right] e^{\frac{z^2}{2h}} \Theta\left(\sqrt{\frac{\tau_I}{\pi h}}z, \tau\right).
\end{aligned}$$

□

4.4 Optimal lattice

The optimization of the mean energy $\gamma(\tau)$ with respect to the lattice parameter τ asserting that the minimum is reached for the hexagonal lattice, was first verified numerically [KRA] after a one-dimensional reduction: they state that the set of admissible τ can be restricted to a line ($\tau_R = -1/2$, $\tau_I \in (0,1)$) and then provide some numerical evidence that for this restricted minimization, the hexagonal lattice is a local minimum and the square lattice a local maximum.

Our explicit expression (4.12) yields that $\gamma(\tau)$ is the square of the L^2-norm of a Hussimi function on the torus studied in [NoVo]. A more complete explanation of the relationship between the two problems is given at the end of this subsection. We now provide a brief account of the proof by Nonnenmacher and Voros in [NoVo] that $\tau = e^{2i\pi/3}$ is the global minimizer of $\gamma(\tau)$. This relies on a similar one dimensional reduction as [KRA], but contains a rigorous proof of the local extrema on the arc circle $|\tau| = 1$.

Proposition 4.5 (Optimal lattice, [NoVo]). *The function* $\gamma(\tau) = \sum_{k,\ell\in\mathbb{Z}} e^{-\frac{\pi}{\tau_I}|k\tau-\ell|^2}$ *has the modular invariance* $\gamma(\tau+1) = \gamma(\tau)$, $\gamma(\frac{-1}{\tau}) = \gamma(\tau)$, *and the symmetry* $\gamma(-\overline{\tau}) = \gamma(\tau)$. *It has exactly two critical points in the intersection of the fundamental domain (4.5)-(4.6) with* $\{\tau_I \leq 1.65\}$. *The critical point at* $\tau = i$ *(square lattice) is a saddle point with the critical value* $\gamma(i) \sim 1.1803$ *and the critical point at* $\tau = j = e^{2i\pi/3}$ *is the unique global minimizer (up to modular symmetry) with value* $b = \gamma(e^{2i\pi/3}) \sim 1.1596$.

Remark 4.6. *The values* $\gamma(i)$ *and* $\gamma(e^{2i\pi/3})$ *can be expressed in terms of the Gamma function. The values given in [NoVo] are* $\gamma(i) = \frac{\Gamma(1/4)^2}{2\pi^{3/2}}$ *and* $b = \gamma(e^{2i\pi/3}) = \frac{3\Gamma(1/3)^3}{2^{7/3}\pi^2}$.

Sketch of the proof: We give here a brief account on the proof of [NoVo]-Appendix A and refer the reader to this reference for details. The modular invariance of $\gamma(\tau)$ is a consequence of its definition (1.12) and the properties of Theta functions, or can be checked directly on the series expression. This allows to restrict the analysis to the fundamental domain (4.5)-(4.6). The Poisson formula and a change of variables yield

$$\gamma(\tau) = \frac{1}{\tau_I} \sum_{k',\ell'\in\mathbb{Z}} \int_{\mathbb{R}^2} e^{-\frac{\pi}{\tau_I}(y^2+x^2)} e^{2i\pi k'\frac{x}{\tau_I}} e^{2i\pi\ell'(x\tau_R-y)}\, dxdy\,.$$

Since the function γ is real valued, the real part of $\partial_\tau\gamma(\tau)$ equals

$$\partial_{\tau_R}\gamma(\tau) = \frac{1}{\tau_I} \sum_{k',\ell'\in\mathbb{Z}} \int_{\mathbb{R}^2} e^{-\frac{\pi}{\tau_I}(y^2+x^2)} (2i\pi\ell' x) e^{2i\pi\ell'(x\tau_R/\tau_I-y)} e^{2i\pi k'\frac{x}{\tau_I}}\, dydx\,.$$

By using the Poisson formula for the sum on k', and a Gaussian integration, we find that $\partial_{\tau_R}\gamma(\tau) = 0$ is equivalent to

$$\sum_{k,\ell'>0} e^{-\pi\tau_I(k^2+\ell'^2)} k\ell' \sin(2\pi k\ell'\tau_R) = 0\,. \tag{4.14}$$

Estimating the decay of the exponential term provides that critical points in $\{\mathcal{I}\mathrm{m}\,\tau \geq 0.31\}$, (which contains the fundamental domain (4.5)-(4.6)), belong to $\{\tau_R \in \{-1/2, 0\}\}$. Next, we find the formula

$$\frac{d}{dt}\log(\gamma(it))\Big|_{t=\tau_I} = \frac{1}{2\tau_I\Theta_3(0,i\tau_I)}(1 + 2\sum_{k\geq 1} e^{-\pi\tau_I k^2}(1-4\pi\tau_I k^2)) \tag{4.15}$$

which is a consequence of $\gamma(it) = \Theta_3(0,it)\Theta_3(0,\frac{i}{t})$ and $\Theta_3(0,\frac{i}{t}) = \sqrt{t}\Theta_3(0,it)$ for $\Theta_3(0,it) = \sum_{n\in\mathbb{Z}} e^{-\pi t n^2}$. The monotonicity of $e^{-x}(1-4x)$ on $[5/4,+\infty)$ implies that the derivative (4.15) increases for $\tau_I \geq 1$. On the arc circle $\mathcal{AC}_0 = \left\{|\tau| = 1,\ -\frac{1}{2} \leq \tau_R \leq \frac{1}{2}\right\}$, the symmetries of $\gamma(\tau)$ imply that the radial derivative is 0. Hence the derivative (4.15) is positive on the half-line $\{\mathbb{R}\mathrm{e}\,\tau = 0\} \cap \{\tau_I > 1\}$. Along the arc circle $\mathcal{AC}_0$, the function γ has to be strictly monotone between $\tau = j$ and $\tau = i$. The point $\tau = i$ is thus a saddle point.

By applying the transformation $z \to \frac{-1}{z+1}$, the half line $\{\tau_R = -1/2, \tau_I \geq 1/2\}$ is sent to the arc circle $\mathcal{AC}_1$ passing through the points $-1+i$, $j = \frac{1}{2}(-1+\sqrt{3}i)$ and 0 centered at

-1 with radius 1. On $\{\tau \in \mathcal{AC}_1, \tau_I \geq 0.31\}$ the fact that $\mathbb{R}\mathrm{e}\, \partial_\tau \gamma(\tau)$ only vanishes on $\tau_R = -1/2$ implies that there is no other critical point than j in $\{\tau_R = -1/2, 1/2 \leq \tau_I \leq 1.65\}$. The lower bound $\gamma(\tau) \geq 1 + 2e^{-\frac{\pi}{\tau_I}}$ then implies that j is the unique global minimum in $\{-1/2 \leq \tau_R < 1/2, |\tau| \geq 1\}$. $\square$

The relationship between the minimization of $\gamma(\tau)$ with respect to τ and the quantities studied by Nonnenmacher and Voros in the framework of quantum chaos has some reasons. The Theta function which appears naturally in the Abrikosov Ansatz, is also the key ingredient of the explicit geometric quantization of the torus considered as a compact Kaehler space. This provides a basic example for the analysis of quantum chaos and we refer the reader for example to [NoVo, ShZe, BoGu]. Let us specify this relationship. One can start with the general variational problem,

$$\inf_{\substack{e^{\frac{|z|^2}{2h}} u(z) \text{ entire} \\ |u(z)| \ \mathfrak{L}\text{-periodic}}} \frac{\fint |u|^4}{\left(\fint |u|^2\right)^2}, \tag{4.16}$$

without assuming that u admits exactly one zero per fundamental cell. One is lead to consider entire functions $f(z)$ such that $u(z) = e^{-\frac{|z|^2}{2h}} f(z)$ has an $\mathfrak{L}$-periodic modulus according to (4.8)-(4.9), with N_0 zeroes per fundamental cell, $N_0 \in \mathbb{N}^*$. The same arguments as in Proposition 4.1 provide the form of f (and u):

- The parameters of the lattice ν and τ_I are related with N_0 and h according to

$$\nu = \sqrt{\frac{\tau_I}{\pi N_0 h}}. \tag{4.17}$$

- If $z_1, \dots z_{N_0}$ denote the zeroes of f (repeated with multiplicity) in a fundamental cell, then f is proportional to

$$e^{\frac{(z-z_0)^2}{2h}} \prod_{k=1}^{N_0} \Theta\left(\sqrt{\frac{\tau_I}{\pi N_0 h}}(z - z_k), \tau\right), \tag{4.18}$$

  with $z_0 = 2i\,\mathcal{I}\mathrm{m}\left(\frac{\sum_{k=1}^{N_0} z_k}{N_0}\right)$.

Up to some constraint about $\frac{\sum_{k=1}^{N_0} z_k}{N_0}$, the expression (4.18) is exactly the one of [NoVo] for the general elements of the Bargmann space constructed on the torus as an N_0-dimensional Hilbert space of holomorphic sections of some line bundle. Then $|u(z)|^2$ appears as the Hussimi function associated with f and is a well defined function on the torus $\mathbb{C}/\mathfrak{L}$. The L^2-norm of this Hussimi function studied in [NoVo] is proportional to $\fint |u|^4$.

The constraint $\frac{\sum_{k=1}^{N_0} z_k}{N_0} = c_1 + ic_2 \mod (\mathfrak{L})$ in the geometric quantization of the torus comes from some quasiperiodic conditions involved in the construction of the line bundle. In the minimization of $\fint |u|^4 / \left(\fint |u|^2\right)^2$ such a condition has no effect owing to the same translational invariance argument as in Proposition 4.2. In comparison with the framework of geometric quantization or quantum chaos, this additional flexibility could be useful for further developments in the analysis of modulated Abrikosov lattices which occurs in Bose-Einstein condensates.

Finally, an interesting question which does not seem completely solved is about the infimum of (4.16). This could be attacked by considering the limit as the number of zeroes per fundamental cell N_0 goes to infinity, with h and τ fixed, in the spirit of [NoVo, ShZe].

5 Almost critical points

In this section, we prove Theorem 1.5.

Proposition 5.1. *We assume that the complex number τ belongs to the fundamental domain (4.5)-(4.6). Let $\alpha \in C^{0,\frac{1}{2}}(\mathbb{C};\mathbb{C})$ be such that* $\operatorname{supp}\alpha \subset K$*, where K is a fixed compact subset of $\mathbb{C}$. Consider*

$$u_\tau(z) = f_\tau(z)e^{-\frac{|z|^2}{2h}}, \quad \textit{with} \quad f_\tau(z) = e^{\frac{z^2}{2h}}\Theta\left(\frac{\sqrt{\tau_I}}{\sqrt{\pi h}}z, \tau\right),$$

and set

$$f_{\alpha,\tau} = \Pi_h(\alpha f_\tau), \quad \textit{and} \quad u_{\alpha,\tau}(z) = f_{\alpha,\tau}(z)e^{-\frac{|z|^2}{2h}}. \tag{5.1}$$

Then, for any $\lambda \in \mathbb{R}$,

$$\begin{aligned}(N_h + h - \lambda)f_{\alpha,\tau} + Na\Omega_h^2\Pi_h\left(|u_{\alpha,\tau}|^2 f_{\alpha,\tau}\right)&\\ = \Pi_h\left[\left(|z|^2 - \lambda + \gamma(\tau)Na\Omega_h^2|\alpha|^2\right)\alpha\right]\Pi_h f_\tau + R(\alpha,\tau,h)\,,&\end{aligned} \tag{5.2}$$

$$\textit{with} \qquad \|R(\alpha,\tau,h)\|_{\mathcal{F}_h} \leq C(\|\alpha\|_{C^{0,1/2}}, \tau, \lambda, K)h^{1/4}\,, \tag{5.3}$$

where $\gamma(\tau)$ is defined by (4.12). The constant $C(\|\alpha\|_{C^{0,1/2}}, \tau, \lambda, K)$ can be uniformly bounded when K is fixed and $\|\alpha\|_{C^{0,1/2}}$, τ and λ depend on h but stay in some fixed compact set.

Remark 5.2. *Note that $f_{1,\tau} = f_\tau$. More generally, the function $f_{\alpha,\tau}$ defined in Proposition 5.1 belongs to $\cup_{s\in\mathbb{R}}\mathcal{F}_h^s$ when α is polynomially bounded.*

Before giving the proof of Proposition 5.1, we need a few technical lemmas:

Lemma 5.3. *Assume either that g belongs to $\mathcal{F}_h^s$ for some s or $v = e^{-\frac{|z|^2}{2h}}g$ is in $L^p(\mathbb{C}, L(dz))$ with $1 \leq p \leq \infty$. Let $\alpha \in L^\infty(\mathbb{C})$, be such that* $\operatorname{supp}(\alpha) \subset K$*, where K is compact. The function $v_\alpha = e^{-\frac{|z|^2}{2h}}\Pi_h(\alpha g)$ satisfies*

$$|v_\alpha(z)| \leq \left(\int_K \langle z\rangle^{-2s}L(dz)\right)^{\frac{1}{2}}\|\alpha\|_{L^\infty}\|g\|_{\mathcal{F}_h^s}\frac{e^{-\frac{d(z,K)^2}{2h}}}{\pi h}, \quad \textit{if } g \in \mathcal{F}_h^s. \tag{5.4}$$

$$\textit{(resp.)} \qquad |v_\alpha(z)| \leq \left(\int_K 1\ L(dz)\right)^{1-\frac{1}{p}}\|\alpha\|_{L^\infty}\|v\|_{L^p}\frac{e^{-\frac{d(z,K)^2}{2h}}}{\pi h} \quad \textit{if } v \in L^p(\mathbb{C}, L(dz)) \tag{5.5}$$

Proof: The definition of Π_h implies

$$v_\alpha(z) = \int_{\mathbb{C}} \frac{e^{-\frac{|z-z'|^2}{2h}+i\frac{\mathcal{I}\mathrm{m}(z\overline{z'})}{h}}}{\pi h}\alpha(z')v(z')L(dz').$$

The Cauchy-Schwarz or Hölder inequality yields

$$|v_\alpha(z)| \le \|\alpha\|_{L^\infty}\frac{1}{\pi h}\int_K e^{-\frac{|z-z'|^2}{2h}}|v(z')|L(dz')$$

which provides the result. $\square$

Lemma 5.4. *Assume either that g belongs to $\mathcal{F}_h^s$ for some s or $v = e^{-\frac{|z|^2}{2h}}g$ is in $L^p(B_{R_0}, L(dz))$ with $1 \le p \le \infty$. Let $\alpha \in C^{0,\beta}(\mathbb{C};\mathbb{C})$ be such that* $\mathrm{supp}(\alpha) \subset K \subset B_{R_0}$, *where $R_0 > 0$ and $\beta \in (0,1)$. We take $C^0(\mathbb{C};\mathbb{C})$ for $\beta = 0$ and $C^1(\mathbb{C};\mathbb{C})$ for $\beta = 1$. Let $g_\alpha = \Pi_h(\alpha g)$ and $v_\alpha(z) = e^{-\frac{|z|^2}{h}}g_\alpha$. The function g_α belongs to $\mathcal{F}_h^{s'}$ (resp. v_α belongs to $L^p(B_{R_0}, L(dz))$) for all $s' \in \mathbb{R}$ with*

$$\left\|\langle z\rangle^{s'}(v_\alpha - \alpha v)\right\|_{L^2} \le C_{R_0,s,s'}\|\alpha\|_{C^{0,\beta}}\|g\|_{\mathcal{F}_h^s}h^{\frac{\beta}{2}} \tag{5.6}$$

$$\text{(resp.)}\qquad \|v_\alpha - \alpha v\|_{L^p} \le C_{R_0,p}\|\alpha\|_{C^{0,\beta}}\|v\|_{L^p(B_{R_0})}h^{\frac{\beta}{2}}.$$

Proof: We use here again the definition of Π_h, and write

$$v_\alpha(z) = \int_{\mathbb{C}} \frac{e^{-\frac{|z-z'|^2}{2h}+i\frac{\mathcal{I}\mathrm{m}(z\overline{z'})}{h}}}{\pi h}\alpha(z')v(z')L(dz').$$

It is sufficient to consider the case $s' \ge 0$. Using Young's inequality, we have, for any $R > R_0$,

$$\begin{aligned}\|\langle z\rangle^{s'}v_\alpha\|_{L^2(B_R)} &\le (1+R)^{2s'}\left[\int_{\mathbb{C}}\frac{1}{\pi h}e^{-\frac{|z|^2}{2h}}L(dz)\right]\|\alpha v\|_{L^2}\\ &\le (1+R)^{2s'}\max\left\{1,(1+R)^{-2s}\right\}\|\alpha\|_{L^\infty}\|g\|_{\mathcal{F}_h^s},\end{aligned}$$

and using Lemma 5.3 with $K = B_{R_0}$, we also have

$$\|\langle z\rangle^{s'}v_\alpha\|^2_{L^2(B_R^c)} \le \left(\int_{B_{R_0}}\langle z\rangle^{-2s}\,L(dz)\right)\|\alpha\|^2_{L^\infty}\|g\|^2_{\mathcal{F}_h^s}\left(\int_{B_R^c}\langle z\rangle^{2s'}\frac{e^{-\frac{(|z|-R_0)^2}{2h}}}{\pi^2h^2}\,L(dz)\right).$$

The last integral is bounded independently of h, for instance if $R = R_0 + 1$. We thus have, for some constant C depending only on s, s', R_0,

$$\|\langle z\rangle^{s'}(v_\alpha - \alpha v)\|_{L^2} \le C\|\alpha\|_{L^\infty}\|\langle z\rangle^s v\|_{L^2} = C\|\alpha\|_{L^\infty}\|g\|_{\mathcal{F}_h^s}. \tag{5.7}$$

We next assume that $\alpha \in C^1(\mathbb{C};\mathbb{C})$. We then take $R \geq R_0+1$, and let χ_R be a smooth function such that $\chi_R = 1$ in B_R and $\chi_R = 0$ in B_R^c. Then,

$$v_\alpha - \alpha v = \chi_R(v_\alpha - \alpha v) + (1-\chi_R)v_\alpha \,. \tag{5.8}$$

The second term is treated exactly as above, and is thus bounded as follows:

$$\begin{aligned}\|\langle z\rangle^s v_\alpha\|^2_{L^2(B_R^c)} &\leq \left(\int_{B_{R_0}} \langle z\rangle^{2s}\ L(dz)\right)\|\alpha\|^2_{L^\infty}\|g\|^2_{\mathcal{F}_h^s}\left(\int_{B_R^c}\langle z\rangle^{2s'}\frac{e^{-\frac{(|z|-R_0)^2}{2h}}}{\pi^2h^2}\ L(dz)\right)\\ &\leq C_{s',R,R_0}\|\alpha\|^2_{C^1}\|g\|^2_{\mathcal{F}_h^s}e^{-\frac{1}{4h}}.\end{aligned} \tag{5.9}$$

Turning to the first term of (5.8), we point out that $\frac{h}{z-z'}\partial_{\overline{z'}}e^{\frac{z-z'}{h}\overline{z'}} = e^{\frac{z-z'}{h}\overline{z'}}$. Therefore, an integration by parts gives (here ε is any positive number):

$$\begin{aligned}g_\alpha(z) &= \frac{1}{\pi h}\int_{B_\varepsilon(z)^c}\frac{h}{z-z'}\partial_{\overline{z'}}e^{\frac{z-z'}{h}\overline{z'}}\alpha(z')g(z')L(dz') + \frac{1}{\pi h}\int_{B_\varepsilon(z)} e^{\frac{z-z'}{h}\overline{z'}}\alpha(z')g(z')L(dz')\\ &= \frac{1}{\pi h}\int_{\partial B_\varepsilon(z)}\frac{h}{z-z'}e^{\frac{z-z'}{h}\overline{z'}}\alpha(z')g(z') - \frac{1}{\pi h}\int_{B_\varepsilon(z)^c}\frac{h}{z-z'}e^{\frac{z-z'}{h}\overline{z'}}\partial_{\overline{z'}}\alpha(z')g(z')L(dz')\\ &\qquad + \frac{1}{\pi h}\int_{B_\varepsilon(z)} e^{\frac{z-z'}{h}\overline{z'}}\alpha(z')g(z')L(dz').\end{aligned}$$

Therefore, letting ε go to zero, we find

$$g_\alpha(z) = \alpha(z)g(z) - \frac{1}{\pi h}\int_{\mathbb{C}}\frac{h}{z-z'}e^{\frac{z-z'}{h}\overline{z'}}\partial_{\overline{z'}}\alpha(z')g(z')L(dz'),$$

hence

$$|v_\alpha(z) - \alpha(z)v(z)| \leq \frac{1}{\pi}\frac{e^{-\frac{|z|^2}{2h}}}{|z|} * \left(|(\partial_{\overline{z}}\alpha)v|\right).$$

Using Young's inequality, one thus gets

$$\|v_\alpha - \alpha v\|_{L^2(B_R)} \leq \frac{\|\alpha\|_{C^1}}{\pi}\left\|\frac{e^{-\frac{|z|^2}{2h}}}{|z|}\right\|_{L^1}\|v\|_{L^2(B_{R_0})}.$$

This clearly implies

$$\left\|\langle z\rangle^{s'}(v_\alpha - \alpha v)\right\|_{L^2(B_R)} \leq C_{R_0,s,s'}\|g\|_{\mathcal{F}_h^s}\|\alpha\|_{C^1}\sqrt{h}\,. \tag{5.10}$$

Collecting (5.9) and (5.10), we thus have

$$\left\|\langle z\rangle^{s'}(v_\alpha - \alpha v)\right\|_{L^2} \leq C_{R_0,s,s'}\|g\|_{\mathcal{F}_h^s}\|\alpha\|_{C^1}\sqrt{h}\,.$$

This inequality and (5.7) imply (5.6) by real interpolation argument[2]. The L^p-estimates follow by the same line. $\square$

Proof of Proposition 5.1: Let

$$A=(N_h+h)f_{\alpha,\tau}-\Pi_h\left(|z|^2\alpha\right)\Pi_h f_\tau\,,$$

and associate with any polynomial p the function $v=e^{-\frac{|z|^2}{2h}}p$. The scalar product $\langle p\mid A\rangle_{\mathcal{F}_h}$ is estimated by

$$\begin{aligned}|\langle p|A\rangle_{\mathcal{F}_h}| &= \left|\int_{\mathbb{C}}|z|^2\overline{v(z)}u_{\alpha,\tau}(z)L(dz)-\int_{\mathbb{C}}\overline{v(z)}|z|^2\alpha(z)u_\tau(z)L(dz)\right|\\ &\le \|v\|_{L^2}\|\langle z\rangle^2(u_{\alpha,\tau}-\alpha u_\tau)\|_{L^2}\\ &\le C\|p\|_{\mathcal{F}_h}\|f_\tau\|_{\mathcal{F}_h}h^{\frac14}.\end{aligned}$$

The last inequality comes from Lemma 5.4 applied to α and $g=f_\tau$. The density of polynomials in $\mathcal{F}_h$ then allows to conclude that this inequality is valid for any $p\in\mathcal{F}_h$. We next define $B=\Pi_h\left(|u_{\alpha,\tau}|^2f_{\alpha,\tau}\right)-\gamma(\tau)\Pi_h\left(|\alpha|^2\alpha\right)\Pi_hf_\tau$, and compute, using (1.14):

$$\begin{aligned}\langle p|B\rangle_{\mathcal{F}_h} &= \int_{\mathbb{C}}\overline{v(z)}|u_{\alpha,\tau}(z)|^2u_{\alpha,\tau}(z)L(dz)-\left\langle\Pi_h\left(|\alpha|^2\overline{\alpha}p\right)\mid\Pi_h\left(|u_\tau|^2f_\tau\right)\right\rangle_{\mathcal{F}_h}\\ &= \int_{\mathbb{C}}\overline{v(z)}|u_{\alpha,\tau}(z)|^2u_{\alpha,\tau}(z)L(d)z\\ &\qquad -\int_{\mathbb{C}}\langle z\rangle^2e^{-\frac{|z|^2}{2h}}\overline{\Pi_h\left(|\alpha|^2\overline{\alpha}p\right)}\langle z\rangle^{-2}|u_\tau|^2u_\tau\;L(dz)\,.\end{aligned}\tag{5.11}$$

We apply Lemma 5.4 to $|\alpha|^2\alpha$ and p to get

$$\left\|\langle z\rangle^2\left(\Pi_h(|\alpha|^2\alpha p)-|\alpha|^2\alpha p\right)e^{-\frac{|z|^2}{2h}}\right\|_{L^2}\le C\|p\|_{\mathcal{F}_h}h^{\frac14}\,,$$

while the periodicity of the function $|u_\tau|$ implies

$$\left\|\langle z\rangle^{-2}|u_\tau|^2u_\tau\right\|_{L^2}\le C.$$

Hence the second term of (5.11) differs from $\int|\alpha|^2\alpha|u_\tau|^2u_\tau\overline{v}$, by an error term bounded by $C\|p\|_{\mathcal{F}_h}h^{\frac14}$. For the first term we write

$$\begin{aligned}\left|\int_{\mathbb{C}}\overline{v}|u_{\alpha,\tau}|^2u_{\alpha,\tau}\;L(dz)-\int_{\mathbb{C}}\overline{v}|\alpha|^2|u|^2\alpha u_\tau\;L(dz)\right| &\le \|p\|_{\mathcal{F}_h}\left\||u_{\alpha,\tau}|^2-|\alpha|^2|u_\tau|^2\right\|_{L^\infty}\|u_{\alpha,\tau}\|_{L^2}\\ &\quad+\|p\|_{\mathcal{F}_h}\|\alpha\|_{L^\infty}^2\|u_\tau\|_{L^\infty}^2\|u_{\alpha,\tau}-\alpha u_\tau\|_{L^2}\;.\end{aligned}$$

[2] We recall that the real interpolation works for non integer Hölder spaces. See [Che, Hor1] for details on the Hölder-Zygmund classes.

Here again, Lemma 5.4 applied to α and f_τ implies

$$\left|\int_{\mathbb{C}} \overline{v}|u_{\alpha,\tau}|^2 u_{\alpha,\tau}\ L(dz) - \int_{\mathbb{C}} \overline{v}|\alpha|^2\alpha|u_\tau|^2 u_\tau\ L(dz)\right| \leq C\,\|p\|_{\mathcal{F}_h}\, h^{1/4}\,,$$

and finally $|\langle p|B\rangle_{\mathcal{F}_h}| \leq C\|p\|_{\mathcal{F}_h} h^{\frac{1}{4}}$. These estimates hold for any $p \in \mathcal{F}_h$. The proof of $\|A+B\|_{\mathcal{F}_h} \leq Ch^{1/4}$ is complete. □

Before giving the proof of Theorem 1.5, let us point out that the definition $f_{\alpha,\tau}$ in the previous lemmas and propositions differs from $g^h_{\tilde{\alpha},\tau} = \|\Pi_h(\tilde{\alpha} f_\tau)\|^{-1}_{\mathcal{F}_h}\,\Pi_h(\tilde{\alpha} f_\tau)$ in Theorem 1.5. Both functions, $f_{\alpha,\tau}$ and $g^h_{\tilde{\alpha},\tau}$, actually depend on $h > 0$ through Π_h, even when α, $\tilde{\alpha}$ and τ are fixed. The notation $g^h_{\tilde{\alpha},\tau}$ insists on the additional dependence due to the normalization factor. We will see that, though $\tilde{\alpha}$ and τ are prescribed, the previous results will be used with an h-dependent α but with norm estimates uniform with respect to h.

Proof of Theorem 1.5: First note that if $\tilde{\alpha} \in C^{0,1/2}(\mathbb{C};\mathbb{C})$, Lemma 5.4 yields

$$\left|\|\Pi_h(\tilde{\alpha} f_\tau)\|_{\mathcal{F}_h} - \|\tilde{\alpha} u_\tau\|_{L^2}\right| \leq Ch^{1/4},$$

while real interpolation between C^0 and C^1 provides

$$\|\tilde{\alpha} u_\tau\|^2_{L^2} = \int_{\mathbb{C}} |\tilde{\alpha}|^2 |u_\tau|^2\ L(dz) = \fint |u_\tau|^2 \int_{\mathbb{C}} |\tilde{\alpha}|^2\ L(dz) + O(h^{1/4})$$

since $|u_\tau|^2$ is periodic with a period bounded by $O(h^{1/2})$. For $\tilde{\alpha} \in C^{0,1/2}(\mathbb{C};\mathbb{C})$ satisfying $\|\tilde{\alpha}\|_{L^2} = 1$, this gives

$$\|\Pi_h(\tilde{\alpha} f_\tau)\|^2_{\mathcal{F}_h} = \fint |u_\tau|^2\, \|\tilde{\alpha}\|^2_{L^2}\,(1 + O(h^{1/4})) = \frac{\lambda_\tau}{\gamma(\tau)}(1 + O(h^{1/4}))\,. \tag{5.12}$$

The last equality comes from (4.12). We define $g^h_{\tilde{\alpha},\tau} = \|\Pi_h(\tilde{\alpha} f_\tau)\|^{-1}_{\mathcal{F}_h}\,\Pi_h(\tilde{\alpha} f_\tau)$ and $v^h_{\tilde{\alpha},\tau}(z) = g^h_{\tilde{\alpha},\tau} e^{-|z|^2/2h}$. Hence by applying Proposition 5.1 with $\alpha = \frac{\tilde{\alpha}}{\|\Pi_h(\tilde{\alpha} f_\tau)\|_{\mathcal{F}_h}}$, we get

$$\begin{aligned}(N_h + h - \lambda) g^h_{\tilde{\alpha},\tau} + Na\Omega_h^2 \Pi_h\left(|v^h_{\tilde{\alpha},\tau}|^2 g^h_{\tilde{\alpha},\tau}\right) &= \\ \Pi_h\left[\left(|z|^2 - \lambda + \lambda_\tau Na\Omega_h^2 \frac{|\tilde{\alpha}|^2}{\|\Pi_h(\tilde{\alpha} f_\tau)\|^2_{\mathcal{F}_h}}\right)\alpha f_\tau\right] &+ O_{\mathcal{F}_h}(h^{1/4}),\end{aligned} \tag{5.13}$$

which together with (5.12) implies

$$\begin{aligned}(N_h + h - \lambda) g^h_{\tilde{\alpha},\tau} + Na\Omega_h^2 \Pi_h\left(|v^h_{\tilde{\alpha},\tau}|^2 g^h_{\tilde{\alpha},\tau}\right) &= \\ \Pi_h\left[\left(|z|^2 - \lambda + \gamma(\tau) Na\Omega_h^2 |\tilde{\alpha}|^2\right)\alpha f_\tau\right] &+ O_{\mathcal{F}_h}(h^{1/4})\,.\end{aligned} \tag{5.14}$$

We then apply Lemma 5.4 and find (1.19). Next, computing the energy of $v^h_{\tilde{\alpha},\tau}$, we have

$$E^h_{\mathrm{LLL}}\left(v^h_{\tilde{\alpha},\tau}\right) = \int_{\mathbb{C}} \left(|z|^2 |v^h_{\tilde{\alpha},\tau}|^2 + \frac{Na\Omega_h^2}{2} |v^h_{\tilde{\alpha},\tau}|^4\right) L(dz)\,.$$

Hence, using (5.12) and applying Lemma 5.4 with $\alpha = \frac{\tilde{\alpha}}{\|\Pi_h(\tilde{\alpha} f_\tau)\|_{\mathcal{F}_h}}$, $\beta = 1/2$, $s' = 2$, $s < -1$ and $p = 4$, we find

$$E^h_{\mathrm{LLL}}\left(v^h_{\tilde{\alpha},\tau}\right) = \int_{\mathbb{C}} \left(|z|^2 |\tilde{\alpha}(z)|^2 + \frac{Na\gamma(\tau)}{2} |\tilde{\alpha}(z)|^4 \right) L(dz) + O(h^{1/4}),$$

which is (1.18). □

Although we are not yet able to check that such an ansatz is really close to a true critical point, Theorem 1.5 can be stated to provide necessary conditions to approximate a minimum: the lattice has to be hexagonal and the envelope has to be an inverted parabola.

Theorem 5.5. *Let the complex number τ belong to the fundamental domain (4.5)-(4.6), and let $\tilde{\alpha} \in C^{0,\frac{1}{2}}(\mathbb{C};\mathbb{C})$, $\tilde{\alpha} \neq 0$, have a compact support (τ and $\tilde{\alpha}$ do not depend on $h > 0$). Let f_τ be defined according to (1.13) and*

$$g^h_{\tilde{\alpha},\tau} = \|\Pi_h(\tilde{\alpha} f_\tau)\|^{-1}_{\mathcal{F}_h} \Pi_h(\tilde{\alpha} f_\tau), \quad \text{and} \quad v^h_{\tilde{\alpha},\tau}(z) = g^h_{\tilde{\alpha},\tau}(z) e^{-\frac{|z|^2}{2h}}. \tag{5.15}$$

a) *If there exists a family $(f^h, \lambda^h)_{h>0}$ of solutions of the Euler-Lagrange equation (3.2) such that λ^h is uniformly bounded and*

$$\lim_{h\to 0} \left(\left\|g^h_{\tilde{\alpha},\tau} - f^h\right\|_{\mathcal{F}^2_h} + \left\|\Pi_h \left|\left|v^h_{\alpha,\tau}\right|^2 - \left|u^h\right|^2\right| \Pi_h\right\|_{\mathcal{L}(\mathcal{F}_h)} \right) = 0, \quad (u^h(z) = f(z) e^{-\frac{|z|^2}{2h}}), \tag{5.16}$$

then

$$\lim_{h\to 0} \lambda^h = \lim_{h\to 0} \sqrt{\frac{2Na\Omega_h^2 \gamma(\tau)}{\pi}} = \sqrt{\frac{2Na\gamma(\tau)}{\pi}}$$

$$\text{and} \quad |\tilde{\alpha}|^2 = C \left(\sqrt{\frac{2Na\gamma(\tau)}{\pi}} - |z|^2 \right)_+, \quad C > 0.$$

With such a choice of $\tilde{\alpha}$, the LLL-energy of $g^h_{\tilde{\alpha},\tau}$ is

$$G^h(g^h_{\tilde{\alpha},\tau}) = E^h_{LLL}(e^{-\frac{|z|^2}{2h}} g^h_{\tilde{\alpha},\tau}) = \frac{2}{3}\sqrt{\frac{2Na\gamma(\tau)}{\pi}} + O(h^{1/4}).$$

b) *More specifically when $(f^h)_{h>0}$ is a family of solutions of the minimization problem (1.4) such that (5.16) holds, then*

$$\tau = j = e^{\frac{2i\pi}{3}}, \quad \gamma(\tau) = \gamma(j) = b \sim 1.1596,$$

$$\lim_{h\to 0} \lambda^h = \lim_{h\to 0} \sqrt{\frac{2Na\Omega_h^2 b}{\pi}} = \sqrt{\frac{2Nab}{\pi}}$$

$$\text{and} \quad |\tilde{\alpha}|^2 = C \left(\sqrt{\frac{2Nab}{\pi}} - |z|^2 \right)_+, \quad C > 0.$$

With such a choice of $(\tilde{\alpha},\tau)$, the energy of $g^h_{\tilde{\alpha},\tau}$ is

$$G^h(g^h_{\tilde{\alpha},\tau}) = E^h_{LLL}(e^{-\frac{|z|^2}{2h}} g^h_{\tilde{\alpha},\tau}) = \frac{2}{3}\sqrt{\frac{2Nab}{\pi}} + O(h^{1/4})\,.$$

Proof of Theorem 5.5: The same argument as in the proof of Theorem 1.5 leads to (5.13), with $\lambda = \lambda^h$, as soon as λ^h remains in a bounded set:

$$(N_h + h - \lambda^h) g^h_{\tilde{\alpha},\tau} + Na\Omega_h^2 \Pi_h \left(|v^h_{\tilde{\alpha},\tau}|^2 g^h_{\tilde{\alpha},\tau}\right) = \\ \Pi_h \left[\left(|z|^2 - \lambda^h + \lambda_\tau Na\Omega_h^2 \frac{|\tilde{\alpha}|^2}{\|\Pi_h(\tilde{\alpha} f_\tau)\|^2_{\mathcal{F}_h}}\right) \frac{\tilde{\alpha}}{\|\Pi_h(\tilde{\alpha} f_\tau)\|^2_{\mathcal{F}_h}}\right] \Pi_h f_\tau + O_{\mathcal{F}_h}(h^{1/4}).$$

a) With the uniform bound $\lambda^h \leq \Lambda$, one can extract a subsequence λ^{h_n} such that $\lim_{n\to\infty} \lambda^{h_n} = \lambda^0$. The assumption (5.16) implies that $g^{h_n}_{\alpha,\tau}$ almost solves the Euler-Lagrange equation (3.2) with $\lambda = \lambda^{h_n}$. This leads to

$$\lim_{n\to\infty} \left\| \Pi_{h_n} \left[\left(|z|^2 - \lambda^{h_n} + \lambda_\tau Na\Omega_{h_n}^2 \frac{|\tilde{\alpha}|^2}{\|\Pi_{h_n}(\tilde{\alpha} f_\tau)\|^2_{\mathcal{F}_{h_n}}}\right) \frac{\tilde{\alpha}}{\|\Pi_{h_n}(\tilde{\alpha} f_\tau)\|^2_{\mathcal{F}_{h_n}}}\right] f_\tau \right\|^2_{\mathcal{F}_{h_n}} = 0$$

Owing to Proposition 5.1 applied to

$$\alpha_n = \left[\left(|z|^2 - \lambda^{h_n} + \lambda_\tau Na\Omega_{h_n}^2 \frac{|\tilde{\alpha}|^2}{\|\Pi_{h_n}(\tilde{\alpha} f_\tau)\|^2_{\mathcal{F}_{h_n}}}\right) \frac{\tilde{\alpha}}{\|\Pi_{h_n}(\tilde{\alpha} f_\tau)\|^2_{\mathcal{F}_{h_n}}}\right],$$

we obtain

$$\lim_{n\to\infty} \int_{\mathbb{C}} |\alpha_n|^2 |u_\tau(z)|^2 ~ L(dz) = 0$$

Since $|\alpha_n|^2$ inherits a uniform Hölder continuity from the one of $\tilde{\alpha}$, we have

$$\lim_{n\to\infty} \int_{\mathbb{C}} |\alpha_n|^2 |u_\tau|^2 ~ L(dz) - \int_{\mathbb{C}} |\alpha_n|^2 ~ L(dz) \fint |u_\tau|^2 = 0\,.$$

while

$$\lim_{n\to\infty} \|\Pi_{h_n} \tilde{\alpha} f_\tau\|^2 = \int_{\mathbb{C}} |\tilde{\alpha}|^2 ~ L(dz) \fint |u_\tau|^2\,.$$

This leads to

$$\int_{\mathbb{C}} \left| \left(|z|^2 - \lambda^0 + \frac{\lambda_\tau}{\fint |u_\tau|^2} Na \frac{|\tilde{\alpha}|^2}{\|\tilde{\alpha}\|^2_{L^2}}\right) \frac{\tilde{\alpha}}{\|\tilde{\alpha}\|^2_{L^2}} \right|^2 ~ L(dz) = 0\,.$$

The quotient $\frac{\lambda_\tau}{\fint |u_\tau|^2}$ equals $\gamma(\tau)$, so that the function $\tilde{\alpha} \in C^{0,1/2}(\mathbb{C};\mathbb{C})$, $\tilde{\alpha} \neq 0$ has to satisfy

$$\frac{|\tilde{\alpha}|^2}{\|\tilde{\alpha}\|^2_{L^2}} = \frac{(\lambda^0 - |z|^2)_+}{Na\gamma(\tau)}\,. \tag{5.17}$$

Computing the integral of the right-hand side leads to

$$1 = \frac{\pi(\lambda^0)^2}{2Na\gamma(\tau)}\,,$$

which yields $\lim_{h\to 0}\lambda^h = \sqrt{2Na\gamma(\tau)/\pi}\,.$
Once $\tilde{\alpha}$ is chosen according to (5.17), the computation of the energy $G^h(g^h_{\tilde{\alpha},\tau})$ again makes use of Proposition 5.1:

$$\begin{aligned} G^h(g^h_{\tilde{\alpha},\tau}) + O(h^{1/4}) &= \int_{\mathbb{C}} \frac{|z|^2|\tilde{\alpha}|^2}{\|\tilde{\alpha}\|^2_{L^2}} + \frac{Na\gamma(\tau)|\tilde{\alpha}|^2}{2\,\|\tilde{\alpha}\|^4_{L^2}}\; L(dz) \\ &= \frac{1}{\pi(\lambda^0)^2/2}\int_{\mathbb{C}} |z|^2\left(\lambda^0 - |z|^2\right)_+ + \frac{1}{2}\left(\lambda_0 - |z|^2\right)^2_+\; L(dz) \\ &= \frac{(\lambda^0)^3}{\pi(\lambda^0)^2/2}\int_0^1 t(1-t) + \frac{(1-t)^2}{2}\;dt = \frac{2}{3}\sqrt{\frac{2Na\gamma(\tau)}{\pi}} \end{aligned}$$

b) It is a direct consequence of a) by noticing that the assumption that $(f^h)_{h>0}$ is a family of minimizers, imply that the Lagrange multipliers λ^h are uniformly bounded with respect to h (see Proposition 3.1). The limit of the LLL-energy being proportional to $\sqrt{\gamma(\tau)}$, Proposition 4.5 implies $\tau = j = e^{\frac{2i\pi}{3}}$. □

Remark 5.6. *The condition (5.16) could be replaced by* $\left\|f^h - g^h_{\tilde{\alpha},\tau}\right\|_{\mathcal{F}_h} = o(h^{3/4})$ *in addition to* $\lim_{h\to 0}\left\|f^h - g^h_{\tilde{\alpha},\tau}\right\|_{\mathcal{F}^2_h} = 0$. *The condition* $\lim_{h\to 0}\left\|f^h - g^h_{\tilde{\alpha},\tau}\right\|_{\mathcal{F}^2_h} = 0$ *does not seem to be sufficient because the continuity of the nonlinear term is h-dependent according to Lemma 2.1.*

6 Approximation by the minimization over polynomials for fixed h

In this section, we consider the approximation of the optimization problem (1.4) by finite dimensional ones, that is, $\mathcal{F}_h$ is replaced by a set of polynomials with bounded degree.

For $K \in \mathbb{N}$, $\mathbb{C}_K[z]$ denotes the set of polynomials with degree smaller than or equal to K. Since $(c_{n,h}z^n)_{n\in\mathbb{N}}$, $c_{n,h} = \frac{1}{(\pi h)^{1/2}h^{n/2}\sqrt{n!}}$, is an orthonormal spectral basis for N_h with $N_h z^n = hnz^n$, the orthogonal projection $\Pi_{h,K}$ onto $\mathbb{C}_K[z]$ coincides with the orthogonal spectral projection:

$$\Pi_{h,K} = 1_{[0,hK]}(N_h)\,.$$

We shall use the notation $\Pi'_{h,K}$ for the imbedding from $\mathbb{C}_K[z]$ into $\mathcal{F}_h$:

$$\Pi_{h,K}\circ\Pi'_{h,K} = \mathrm{Id}_{\mathbb{C}_K[z]} \qquad \Pi'_{h,K}\circ\Pi_{h,K} = \Pi_{h,K}\,. \tag{6.1}$$

We introduce the reduced minimum of the finite dimensional optimization problem:

$$e^h_{LLL,K} = \min_{P\in\mathbb{C}_K[z],\|P\|_{\mathcal{F}_h}=1} E^h_{LLL}(e^{-|z|^2/2h}P)\,. \tag{6.2}$$

Theorem 6.1. *1) The minima e^h_{LLL} and $e^h_{LLL,K}$ satisfy*

$$\forall K \in \mathbb{N} \cap (h^{-1}C_2(h), +\infty), \quad 0 < e^h_{LLL,K} - e^h_{LLL} \leq \frac{C_2(h)^2 + C_2(h)^3}{(1 - C_2(h)(hK)^{-1})^4}(hK)^{-1}$$

where $C_2(h) = \frac{8\Omega_h}{3\pi}\sqrt{\frac{2bNa}{h}} + o_{Na}(h^{-1/2})$ does not depend on K.
If f solves the minimization problem (1.4) then the sequence $(f_K)_{K\in\mathbb{N}}$ defined by $f_K = \|\Pi_{h,K}f\|^{-1}\,\Pi_{h,K}f$, which satisfies $f_K \in \mathbb{C}_K[z]$ is a minimizing sequence for (1.4).
***2)** If for any $K \in \mathbb{N}$, $P_K \in \mathbb{C}_K[z]$ denotes any solution to (6.2) then the sequence $(P_K)_{K\in\mathbb{N}}$ is a minimizing sequence for (1.4). Its accumulation points for the $\|\cdot\|_{\mathcal{F}_h}$ topology are solutions of (1.4). Moreover if a subsequence $(P_{K_n})_{n\in\mathbb{N}}$ converges to f in $\mathcal{F}_h$ then the convergence also holds in $\mathcal{F}^2_h$ according to:*

$$\lim_{n\to\infty} \|f - P_K\|_{\mathcal{F}_h} + \|N_h(f - P_{K_n})\|_{\mathcal{F}_h} = 0\,.$$

Proof: **1)** Let $f \in \mathcal{F}_h$ be a solution to (1.4). Equation (3.4) implies

$$N_h f = -Na\Omega_h^2\Pi_h\left(e^{-\frac{|z|^2}{h}}\,|f|^2\right)\Pi_h f + (\lambda - h)f, \tag{6.3}$$

while the solution f and the Lagrange multiplier λ fulfill the uniform bound

$$\frac{Na\Omega_h^2}{2}\left\|e^{-|z|^2/h}\,|f|^2\right\|_{L^2(\mathbb{C},L(dz))} \leq C_1(h) \quad \text{and} \quad \lambda \leq 2C_1(h)\,,$$

with $C_1(h) = \frac{2\Omega_h}{3}\sqrt{\frac{2bNa}{\pi}} + O_{Na}(h^{1/4})$. After conjugating with the Bargmann transform, the Toeplitz operator $\Pi_h\left(e^{-\frac{|z|^2}{h}}\,|f|^2\right)\Pi_h$ is the Anti-Wick quantization of a symbol bounded in $L^2(\mathbb{C}, L(dz))$ (see Appendix A). We deduce

$$\frac{Na\Omega_h^2}{2}\left\|\Pi_h\left(e^{-\frac{|z|^2}{h}}\,|f|^2\right)\Pi_h\right\|_{\mathcal{L}(\mathcal{F}_h)} \leq \frac{Na\Omega_h^2}{2\sqrt{2\pi h}}\left\|e^{-\frac{|z|^2}{h}}\,|f|^2\right\|_{L^2} \leq \frac{C_1(h)}{\sqrt{2\pi h}}$$

and

$$Na\Omega_h^2\left\|\Pi_h\left(e^{-\frac{|z|^2}{h}}\,|f|^2\right)\Pi_h f\right\|_{\mathcal{F}_h} \leq \frac{2C_1(h)}{\sqrt{2\pi h}}\,.$$

We deduce the estimates $\|N_h f\|_{\mathcal{F}_h} \leq \frac{4C_1(h)}{\sqrt{2\pi h}}$. The spectral theorem now provides the estimates

$$\|f - g_K\|_{\mathcal{F}_h} \leq \frac{4C_1(h)}{\sqrt{2\pi h}(hK)} \quad \text{with} \quad g_K = \Pi_{h,K}f \in \mathbb{C}_K[z]\,.$$

We obtain

$$\begin{aligned} 0 \leq \langle f \mid N_h f\rangle_{\mathcal{F}_h} - \langle g_K \mid N_h g_K\rangle_{\mathcal{F}_h} &= \langle f - g_K \mid N_h(f - g_K)\rangle_{\mathcal{F}_h} \\ &\leq \|f - g_K\|_{\mathcal{F}_h}\,\|N_h f\|_{\mathcal{F}_h} \leq \frac{16C_1(h)^2}{2\pi h(hK)}\,. \end{aligned}$$

For the second term of the energy, the Minkowski inequality and the hypercontractivity property (2.1) give

$$\left| \|P_{t_0} f\|_{A^4_h} - \|P_{t_0} g_K\|_{A^4_h} \right| \le \|P_{t_0}(f-g_K)\|_{A^4_h} \le \frac{1}{(\pi h)^{1/2}} \|f-g_K\|_{\mathcal{F}_h} \le \frac{4C_1(h)}{(\pi h)^{1/2}\sqrt{2\pi h}(hK)} \,.$$

The inequality

$$\|P_{t_0} g_K\|_{A_4^h} \le \frac{\|g_K\|_{\mathcal{F}_h}}{(\pi h)^{1/2}} \le \frac{\|f\|_{\mathcal{F}_h}}{(\pi h)^{1/2}}$$

finally leads to

$$\left| \left\| P_{t_0} f \right\|^4_{A^4_h} - \left\| P_{t_0} g_K \right\|^4_{A^4_h} \right| \le \frac{16 C_1(h)}{(\pi h)^2 \sqrt{2\pi h}(hK)}$$

According to (3.1), the addition of the two terms gives

$$\begin{aligned} E^h_{LLL}(e^{-|z|^2/2h} g_K) - E^h_{LLL}(e^{-|z|^2/2h} f) &\le \frac{16 C_1(h)^2}{2\pi h (hK)} + \frac{4Na\Omega_h^2 C_1(h)}{\sqrt{2\pi h}(\pi h)(hK)} \\ &\le \left[\frac{8C_1(h)^2}{\pi h} + 9\pi \frac{C_1(h)^3}{(\pi h)^{3/2}}\right] \frac{1}{hK} \end{aligned}$$

The normalization $f_K = \|g_K\|^{-1}_{\mathcal{F}_h} g_K$ and $|\, \|g_K\|_{\mathcal{F}_h} - 1| \le \frac{4C_1(h)}{\sqrt{2\pi h}(hK)}$ yields

$$E^h_{LLL}(e^{-|z|^2/2h} f_K) - E^h_{LLL}(e^{-|z|^2/2h} f) \le \frac{C_2(h)^2 + C_2(h)^3}{(1 - C_2(h)(hK)^{-1})^4}(hK)^{-1}$$

with $C_2(h) = 4C_1(h)/\sqrt{\pi h}$. The strict lower bound is a consequence of Lemma 3.4.

2) According to Part 1), the sequence $(P_K)_{K\in\mathbb{N}}$ is a minimizing sequence for (1.4). The compactness of the imbedding $\mathcal{F}^1_h \mapsto \mathcal{F}_h$ already used in the proof of Theorem 1.1 implies the first result about accumulation points in $\mathcal{F}_h$.
Assume now that $\lim_{n\to\infty} P_{K_n} = f$ is a solution to (1.4). According to Proposition 3.1, there exists a Lagrange multiplier $\lambda > 0$ such that

$$(N_h + h) f = \lambda f - Na\Omega_h^2 \Pi_h \left(e^{-\frac{|z|^2}{h}} |f|^2 f \right) \,. \tag{6.4}$$

Similarly the Euler-Lagrange equation for P_K, a solution to (6.2) writes

$$(N_h + h) P_K = \lambda_K P_K - Na\Omega_h^2 \Pi_{h,K} \left(e^{-\frac{|z|^2}{h}} |P_K|^2 P_K \right) \,. \tag{6.5}$$

Owing to Lemma 2.1, the mapping $\varphi \in \mathcal{F}_h \to \Pi_h \left(e^{-\frac{|z|^2}{h}} |\varphi|^2 \varphi \right) \in \mathcal{F}_h$ is continuous so that:

$$\lim_{n\to\infty} \Pi_h \left(e^{-\frac{|z|^2}{h}} |P_{K_n}|^2 P_{K_n} \right) = \Pi_h \left(e^{-\frac{|z|^2}{h}} |f|^2 f \right) \quad \text{in } \mathcal{F}_h \,.$$

For $Q_{K_0} \in \mathbb{C}_{K_0}[z]$ with $K_0 \in \mathbb{N}$ fixed, we take the scalar product with (6.5) with $K \geq K_0$:

$$\langle (N_h + h) Q_{K_0} \,|\, P_K \rangle_{\mathcal{F}_h} + N a \Omega_h^2 \left\langle Q_{K_0} \,|\, \Pi_h \left(e^{-\frac{|z|^2}{h}} |P_K|^2 P_K \right) \right\rangle_{\mathcal{F}_h} = \lambda_K \langle Q_{K_0} \,|\, P_K \rangle_{\mathcal{F}_h} .$$

Since $f \neq 0$, one can find a polynomial Q_{K_0} such that $\langle Q_{K_0} \,|\, f \rangle_{\mathcal{F}_h} \neq 0$. The convergence $\| f - P_{K_n} \|_{\mathcal{F}_h} \to 0$ then implies

$$\lim_{n\to\infty} \lambda_{K_n} = \lim_{n\to\infty} \frac{\langle (N_h + h) Q_{K_0} \,|\, P_{K_n} \rangle_{\mathcal{F}_h} + N a \Omega_h^2 \left\langle Q_{K_0} \,|\, \Pi_h \left(e^{-\frac{|z|^2}{h}} |P_{K_n}|^2 P_{K_n} \right) \right\rangle_{\mathcal{F}_h}}{\langle Q_{K_0} \,|\, P_{K_n} \rangle_{\mathcal{F}_h}} = \lambda$$

We next write the difference (6.5)-(6.4) in the form

$$\begin{aligned}(N_h + h)(P_{K_n} - f) &= [\lambda_{K_n} P_{K_n} - \lambda f] \\ &+ \Pi_{h,K_n} \Pi_h \left(e^{-\frac{|z|^2}{h}} \left(|P_{K_n}|^2 P_{K_n} - |f|^2 f \right) \right) + (1 - \Pi_{h,K_n}) \Pi_h \left(e^{-\frac{|z|^2}{h}} |f|^2 f \right) .\end{aligned}$$

The convergence to 0 of each term of the right-hand side yields

$$\lim_{n\to\infty} \| N_h (P_{K_n} - f) \|_{\mathcal{F}_h} = 0 .$$

□

A Bargmann transform and Fock-Bargmann space

Along this article, we use the semiclassical Bargmann transform with the following normalization

$$[B_h \varphi](z) = \frac{1}{(\pi h)^{3/4}} e^{\frac{z^2}{2h}} \int_{\mathbb{R}} e^{-\frac{(\sqrt{2} z - y)^2}{2h}} \varphi(y) \; dy \,,$$

with $z = \frac{x - i\xi}{\sqrt{2}} \in \mathbb{C}$ and $\varphi \in \mathcal{S}'(\mathbb{R})$. Other normalizations are possible:

- In [Mar], the standard semiclassical FBI transform is defined as:

$$[T_h \varphi](x, \xi) = \frac{1}{2^{1/2} (\pi h)^{3/4}} e^{-\xi^2/2h} \int_{\mathbb{R}} e^{-\frac{(x - i\xi - y)^2}{2h}} \varphi(y) \; dy.$$

  Other normalizations or extensions can be found in [Del, Sjo].

- In [Fol], the Bargmann transform is defined as

$$[B\varphi](z) = 2^{1/4} \int_{\mathbb{R}} e^{2\pi z y - \pi y^2 - (\pi/2) z^2} \varphi(y) \; dy.$$

Elementary calculations lead to:

$$[B_h\varphi](z) = 2^{1/2}e^{\frac{x^2+\xi^2}{4h}}e^{-i\frac{x\xi}{2h}}[T_h\varphi](z) = 2^{1/2}e^{|z|^2/2h}e^{-i\frac{x\xi}{2h}}[T_h\varphi](z)$$

and $$[B_h\varphi](z) = \frac{1}{(\pi h)^{1/4}}[B(2\pi h)^{1/4}\varphi((2\pi h)^{1/2}.)]\left(\frac{z}{(\pi h)^{1/2}}\right) .$$

We mainly refer to the presentation of Martinez which already contains the small parameter $h > 0$, but the reader can make the relationship with other results by applying the previous change of variables. We simply list the classical properties of the Bargmann transform and the Fock-Bargmann space and refer to these references for proofs.

a) Isometry property: For any $h > 0$, the transform T_h defines an isometry between $L^2(\mathbb{R}, dy)$ into $L^2(\mathbb{C}, dxd\xi)$ and onto the space $L^2(\mathbb{C}, dxd\xi) \cap e^{-\xi^2/2h}\mathcal{H}(\mathbb{C})$ where $\mathcal{H}(\mathbb{C})$ denotes the space of entire functions. Here the holomorphy of B_h directly comes from its definition and B_h defines a unitary transform from $L^2(\mathbb{R}, dy)$ onto $\mathcal{F}_h$ (note that our normalization gives $L(dz) = \frac{dxd\xi}{2}$).

b) Reproducing Kernel: The product $B_h^*B_h$ is nothing but the identity on $L^2(\mathbb{R}, dy)$ while $B_hB_h^* = \Pi_h$ is the orthogonal projection from $L^2(\mathbb{C}, e^{-|z|^2/h}L(dz))$ onto $\mathcal{F}_h$. The adjoint of B_h is given by

$$[B_h^*f](y) = \frac{1}{(\pi h)^{3/4}}\int_{\mathbb{C}} e^{\frac{\overline{z'}^2}{2h}}e^{\frac{-(y-2^{1/2}\overline{z'})^2}{2h}}e^{-\frac{|z'|^2}{h}}f(z')\ L(dz') .$$

A simple Gaussian integration w.r.t. $y \in \mathbb{R}$ yields

$$[\Pi_hf](z) = [B_hB_h^*f](z) = \frac{1}{\pi h}\int_{\mathbb{C}} e^{\frac{z\overline{z'}}{h}}e^{-\frac{|z'|^2}{h}}f(z')\ L(dz')$$

for all $f \in L^2(\mathbb{C}, e^{-|z|^2/h}L(dz))$.

c) Coherent states: The phase space $\mathbb{R}^2_{x,\xi}$ is endowed with the symplectic form

$$\sigma((x_1,\xi_1),(x_2,\xi_2)) = \xi_1x_2 - x_1\xi_2,$$

The associated unitary phase translations on $L^2(\mathbb{R}, dy)$ are given by

$$[\tau^h_{(x,\xi)}u](y) = e^{i\frac{\xi(2y-x)}{2h}}u(y-x), \quad \tau^h_{(x,\xi)} = e^{i\frac{\xi y - x(hD_y)}{h}},$$

where $D_y = -i\partial_y$. The unitary phase translations satisfy the Weyl relation

$$\tau^h_{X_1}\circ\tau^h_{X_2} = e^{i\frac{\sigma(X_1,X_2)}{h}}\tau^h_{X_1+X_2}, \quad X_k = (x_k, \xi_k) .$$

The coherent states are the normalized $L^2(\mathbb{R}, dy)$ functions given by:

$$\Phi^h_0(y) = \frac{1}{(\pi h)^{1/4}}e^{-\frac{y^2}{2h}}$$

and $$\Phi^h_{(x,\xi)}(y) = [\tau^h_{(x,\xi)}\Phi_0](y) = \frac{1}{(\pi h)^{1/4}}e^{i\frac{\xi(2y-x)}{2h}}e^{-\frac{(y-x)^2}{2h}} .$$

By recalling $z = \frac{x - i\xi}{\sqrt{2}}$ we get

$$[B_h\varphi](z) = \frac{1}{(\pi h)^{3/4}} e^{\frac{(x-i\xi)^2}{4h}} \int_{\mathbb{R}} e^{-\frac{(y-x+i\xi)^2}{2h}} \varphi(y)\ dy$$

$$= \frac{1}{(\pi h)^{1/2}} e^{\frac{|z|^2}{2h}} \int_{\mathbb{R}} \overline{\Phi^h_{x,\xi}(y)} \varphi(y)\ dy = \frac{1}{(\pi h)^{1/2}} e^{\frac{|z|^2}{2h}} \langle \Phi^h_{(x,\xi)} \mid \varphi \rangle_{\mathcal{F}_h}\ .$$

The identity $B_h^* B_h = \mathrm{Id}$ becomes the standard identity resolution on $L^2(\mathbb{R}, dy)$

$$\int_{\mathbb{R}^2} |\Phi^h_{x,\xi}\rangle \langle \Phi^h_{x,\xi}|\ \frac{dx d\xi}{(2\pi h)}\ .$$

From the previous relation, we conjugate the action of $\tau^h_{(x_0,\xi_0)}$ via B_h:

$$\begin{aligned}
[B_h \tau^h_{(x_0,\xi_0)} \varphi](z) &= \frac{e^{\frac{|z|^2}{2h}}}{(\pi h)^{1/2}} \langle \Phi^h_{(x,\xi)} \mid \tau^h_{(x_0,\xi_0)} \varphi \rangle_{\mathcal{F}_h} \\
&= \frac{e^{\frac{|z|^2}{2h}}}{(\pi h)^{1/2}} \langle \tau^h_{-(x_0,\xi_0)} \tau^h_{(x,\xi)} \Phi^h_0 \mid \varphi \rangle_{\mathcal{F}_h} \\
&= \frac{e^{\frac{|z|^2}{2h}}}{(\pi h)^{1/2}} e^{-i\frac{\sigma((-x_0,-\xi_0),(x,\xi))}{2h}} \langle \Phi^h_{(x-x_0,\xi-\xi_0)} \mid \varphi \rangle_{\mathcal{F}_h} \\
&= e^{\frac{|z|^2}{2h}} e^{\frac{2i(\xi_0 x - x_0 \xi)}{4h}} e^{-\frac{|z-z_0|^2}{2h}} [B_h\varphi](z - z_0) = e^{\frac{\overline{z_0}(2z - z_0)}{2h}} [B_h \varphi](z - z_0)\ ,
\end{aligned}$$

with $z = \frac{x - i\xi}{\sqrt{2}}$ and $z_0 = \frac{x_0 - i\xi_0}{\sqrt{2}}$. The Bargmann transform of the function Φ^h_0 equals with our normalization the constant function $(\pi h)^{-1/2}$ and we get more generally

$$[B_h \Phi^h_{(x_0,\xi_0)}](z) = (\pi h)^{-1/2} e^{\frac{\overline{z_0}(2z - z_0)}{2h}}\ .$$

Hence the relation

$$h\partial_z [B_h \Phi^h_{(x_0,\xi_0)}] = \overline{z_0} [B_h \Phi^h_{(x_0,\xi_0)}]\ ,$$

holds for all $z_0 = \frac{x_0 - i\xi_0}{\sqrt{2}} \in \mathbb{C}$.

d) Harmonic oscillator: The harmonic oscillator (or number operator in the Fock representation) is the self adjoint operator on $L^2(\mathbb{R}, dy)$ given by:

$$\tilde{N}_h = \frac{1}{2}(-h^2 \partial_y^2 + y^2 - h) = a_h^* a_h$$

$$D(\tilde{N}_h) = \left\{ u \in L^2(\mathbb{R}, dy),\ y^\alpha D_y^\beta u \in L^2(\mathbb{R}, dy),\ \alpha + \beta \leq 2 \right\},$$

where $D_y = -i\partial_y$ and the annihilation and creation operators, $a_h = \frac{1}{\sqrt{2}}(h\partial_y + y)$ and $a_h^* = \frac{1}{\sqrt{2}}(-h\partial_y + y)$, satisfy the CCR $[a_h, a_h^*] = h$. The normalized Hermite functions are then given by

$$H_0^h(y) = \frac{1}{(\pi h)^{1/4}} e^{-\frac{y^2}{h}} \qquad H_n^h = \frac{1}{h^{n/2}\sqrt{n!}} (a_h^*)^n H_0 \quad \text{for } n \in \mathbb{N},$$

and form an orthonormal basis of eigenfunctions with

$$\tilde{N}_h H_n^h = nhH_n^h\,.$$

An integration by part shows

$$[B_h h\partial_y\varphi](z) = [B_h y\varphi](z) - \sqrt{2}z[B_h\varphi](z)$$

which yields

$$zB_h = B_h \circ (\frac{-h\partial_y + y}{\sqrt{2}}) = B_h \circ a_h^*\ .$$

By differentiating $B_h\varphi$ with respect to z, we obtain

$$h\partial_z[B_h\varphi](z) = -z[B_h\varphi](z) + \sqrt{2}[B_h y\varphi](z)$$

which leads to

$$(h\partial_z)\circ B_h = B_h \circ (\frac{h\partial_y + y}{\sqrt{2}}) = B_h \circ a_h\ .$$

From this we recover

$$a_h\Phi^h_{(x_0,\xi_0)} = \overline{z_0}\Phi^h_{(x_0,\xi_0)} \qquad \text{with } z_0 = \frac{x_0 - i\xi_0}{\sqrt{2}}\ ,$$

$$B_h[H_n^h] = \frac{1}{(\pi h)^{1/2}h^{n/2}\sqrt{n!}}z^n\ ,$$

$$\tilde{N}_h = B_h^*[z(h\partial_z)]B_h\ ,$$

and we set

$$N_h = B_h\tilde{N}_h B_h^* = z(h\partial_z)\,.$$

An element $f = B_h\varphi$ of $\mathcal{F}_h$ considered as an element of $L^2(\mathbb{C}, e^{-\frac{|z|^2}{h}}L(dz))$, satisfies

$$h\partial_z f = h\partial_z(\Pi_h f) = \Pi_h(\overline{z}f)\ .$$

We also note

$$z(h\partial_z)f = h\partial_z(zf) - hf = h\partial_z\Pi_h(zf) - hf = \Pi_h(|z|^2 - h)\Pi_h f.$$

Since $B_h = \Pi_h B_h$, this provides another useful writing of the operator $\tilde{N}_h$:

$$\tilde{N}_h = B_h^*[|z|^2 - h]B_h\ , \quad N_h = \Pi_h(|z|^2 - h)\Pi_h\ .$$

e) h-Pseudo-differential operators: We simply recall the link with the Anti-Wick quantization[3]. The Anti-Wick quantization of a symbol $b(x,\xi)$ can be defined as

$$b^{A-Wick}(y, hD_y) = \int_{\mathbb{R}^2} b(x,\xi)|\Phi^h_{(x,\xi)}\rangle\langle\Phi^h_{(x,\xi)}|\ \frac{dxd\xi}{2\pi h}\ .$$

[3]The “Anti-Wick” name corresponds to the fact that the quantized symbol $|z|^2 = z\overline{z} = \overline{z}z$ equals $a_h a_h^*$.

It is a positive quantization in the sense that

$$(b \geq 0) \Rightarrow (b^{A-Wick}(y, hD_y) \geq 0)$$

and this implies

$$\left\| b^{A-Wick}(y, hD_y) \right\|_{\mathcal{L}(L^2)} \leq \|b\|_{L^\infty} .$$

Another simple consequence of its definition

$$\left\| b^{A-Wick}(y, hD_y) \right\|_{\mathcal{L}^1(L^2)} \leq \frac{1}{2\pi h} \|b\|_{L^1}$$

where $\mathcal{L}^1(L^2(\mathbb{R}, dy))$ denotes the space of trace-class operators.
The Anti-Wick quantization is close to the Weyl quantization due to the relation

$$b^{A-Wick}(y, hD_y) = (\frac{e^{-|z|^2/h}}{\pi h} * b)^W(y, hD_y)$$

For symbols in $S(1, dx^2 + d\xi^2)$ this leads to

$$\left\| b^{A-Wick}(y, hD_y) - b^W(y, hD_y) \right\|_{\mathcal{L}(L^2)} = O(h)$$

which allows to write in this class of symbols

$$\begin{aligned} & b_1^{A-Wick}(y, hD_y) \circ b_2^{A-Wick}(y, hD_y) = (b_1 b_2)(y, hD_y) + O_{\mathcal{L}(L^2)}(h) \\ & \frac{i}{h}[b_1^{A-Wick}(y, hD_y), b_2(y, hD_y)] = \{b_1, b_2\}^{A-Wick}(y, hD_y) + O_{\mathcal{L}(L^2)}(h) \ . \end{aligned}$$

Such results can be extended to some Hörmander classes (see [Mar, Hor2][Chap XVIII]) or even to symbols with low regularity (see [Ler]). We note also the estimates

$$\left\| b^{A-Wick}(y, hD_y) \right\|_{\mathcal{L}(L^2)} \leq \|b(y, hD_y)\|_{\mathcal{L}^2(L^2)} \leq (2\pi h)^{-1/2} \|b\|_{L^2}$$

deduced from the relation with the Weyl quantization.
Finally we translate the action of $b^{A-Wick}(y, hD_y)$ on the Fock space $\mathcal{F}_h$. From the relationship between the Bargmann transform and the coherent states, we get the relations

$$\begin{aligned} & b^{A-Wick}(y, hD_y) = B_h^* \circ b(x, \xi) \circ B_h \\ \text{and} \qquad & B_h b^{A-Wick}(y, hD_y) B_h^* = \Pi_h \circ b(x, \xi) \circ \Pi_h \end{aligned}$$

where $b(x, \xi)$ simply denotes the multiplication by the function $b(x, \xi)$ in the space $L^2(\mathbb{C}, e^{-|z|^2/h} \ L(dz))$. Hence $b^{A-Wick}(y, hD_y)$ acts on $\mathcal{F}_h$ as a Toeplitz operator.

It is also possible to introduce an analytic pseudodifferential calculus which is not required here.

We refer for this aspect the reader to [Mar, Sjo, Del].

The spaces $\mathcal{F}_h^s$ can also be defined using the self-adjoint number operator N_h by setting for real s, $\mathcal{F}_h^s = D(N_h^{s/2})$. Note that $Q(N_h) = \mathcal{F}_h^1$, $D(N_h) = \mathcal{F}_h^2$, while $\mathcal{F}_h^{-1}$ and $\mathcal{F}_h^{-2}$ are their dual with respect the scalar product on $\mathcal{F}_h$. All those spaces $\mathcal{F}_h^s$ are spaces of weighted holomorphic functions (see [Mar] or use the spectral theorem with the help of the orthonormal basis $(c_{n,h}z^n)_{n\in\mathbb{N}}$).

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