Navier-Stokes equations with several independent pressure laws and explicit predictor-corrector schemes

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Abstract

This work is concerned with the numerical capture of stiff viscous shock solutions of Navier-Stokes equations for complex compressible materials, in the regime of large Reynolds numbers. After [2] and [6], a relevant numerical capture is known to require the satisfaction of an extended set of non classical Rankine-Hugoniot conditions due to the non conservation form of the governing PDE model. Here, we show how to enforce their validity at the discrete level without the need for solving local non linear algebraic problems. Non linearities are bypassed when introducing new averaging techniques which are proved to satisfy all the desirable stability properties when invoking suitable approximate Riemann solutions. A relaxation procedure is proposed to that purpose with the benefit of a fairly simple overall numerical method.

1 Introduction

In this paper, we are interested in the numerical approximation of the solutions of the following system in non conservation form :

$$\begin{cases} \partial_t \rho^\epsilon + \partial_x (\rho u)^\epsilon = 0, \\ \partial_t (\rho u)^\epsilon + \partial_x (\rho u^2 + \sum_{i=1}^N p_i)^\epsilon = \epsilon \partial_x (\sum_{i=1}^N \mu_i \partial_x u^\epsilon), \\ \partial_t p_i^\epsilon + \partial_x (p_i u)^\epsilon + (\gamma_i - 1) p_i^\epsilon \partial_x u^\epsilon = \epsilon \mu_i (\partial_x u^\epsilon)^2, \quad i = 1, ..., N, \end{cases} \tag{1}$$

for a given fixed but small $\epsilon > 0$ under the assumption of a large Reynolds number. This PDE system readily stands as a natural extension of the usual Navier-Stokes equations when a single pressure is involved in the momentum equation. Here, N independent pressure laws occur and are governed by their own PDE, in complete symmetry with the equation governing a single pressure. Then the sum of all these pressures, $p := \sum_{i=1}^N p_i$, enters the momentum equation. Several PDE models from the physics can be identified under this form. Berthon [2] was the first to recognize that the so-called two transport models (namely the $k - \epsilon$ or related models) from the compressible turbulence setting must be actually understood within the present framework for extended Navier-Stokes equations with $N = 2$. Chalons [5] has then shown that more sophisticated turbulent models (the so-called muti-scale models) also naturally fall within the present frame with $N \geq 2$ arbitrary pressures. Let us mention that other models from the physics also enter the proposed setting :

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namely the multi-fluid models introduced by Lagoutière [18] but also other complex gas mixtures analyzed by Truesdell [27].

Here we assume that each of the independent pressures p_i follows a polytropic law with its own adiabatic exponent $\gamma_i > 1,\ i = 1, ..., N$. Next, the viscosity coefficients μ_i entering the system (1) denote non negative real numbers but with the requirement of a positive sum, namely $\mu := \sum_{i=1}^{N} \mu_i > 0$.

As a consequence of the thermodynamic closure equations, it can be easily checked (for the details, see [2] with $N = 2$ and [5] with $N > 2$) that the first order underlying system in (1) is non linear hyperbolic over a natural phase space. More precisely, three distinct eigenvalues are in order, namely $u - c < u < u + c$ where the sound speed follows from $c^2 := \sum_{i=1}^{N} \gamma_i p_i/\rho > 0$. The two extreme fields are seen to be genuinely non linear from the assumption $\gamma_i > 1,\ i = 1, ..., N$. By this property, the solutions of the hyperbolic system with viscous perturbations (1) are known to generally develop stiff zones of transition often referred in the literature as to viscous shock profiles (see Gilbarg [12] for instance). The numerical approximation of these solutions is of primary importance in the present work. The assumption of a large Reynolds number makes this issue to be particularly challenging since the PDE model (1) naturally writes in non conservation form. Berthon [2] and Chalons [5] have proved that (1) cannot recast equivalently for smooth solutions in full conservation form, unless restrictive modelling assumptions are specifically addressed. Actually, the only non trivial additional conservation is associated with the total energy ρE and reads

$$\partial_t(\rho E)^\epsilon + \partial_x(\rho H u)^\epsilon = \epsilon \partial_x(\mu u^\epsilon \partial_x u^\epsilon), \tag{2}$$

with

$$\rho E = \frac{(\rho u)^2}{2\rho} + \sum_{i=1}^{N} \frac{p_i}{\gamma_i - 1}, \quad \rho H = \rho E + p.$$

This lack of a fully conservative reformulation of (1) has been shown in [2] when $N = 2$ and then in [5] for arbitrary $N \geq 2$ to be responsible for the negative issue that classical splitting methods for (1) produce approximate solutions which grossly disagree with the expected exact ones. More precisely, the origin of such a failure has been understood when first observing that smooth solutions of (1) obey the following N entropy balance equations :

$$\partial_t s_i^\epsilon + u^\epsilon \partial_x s_i{}^\epsilon = \epsilon \mu_i \frac{\gamma_i - 1}{\rho^{\epsilon \gamma_i}} (\partial_x u^\epsilon)^2 \geq 0, \quad s_i = p_i/\rho^{\gamma_i}, \quad i = 1, ..., N, \tag{3}$$

and then noticing that the corresponding entropy production rates must be kept in balance according to

$$\begin{gathered} \mu_N \tfrac{\rho^{\epsilon\gamma_i}}{\gamma_i - 1} \{\partial_t s_i^\epsilon + u^\epsilon \partial_x s_i^\epsilon\} = \mu_i \tfrac{\rho^{\epsilon\gamma_N}}{\gamma_N - 1} \{\partial_t s_N + u^\epsilon \partial_x s_N\}, \\ 1 \leq i \leq N - 1. \end{gathered} \tag{4}$$

A failure in satisfying (4) has been proved to result in large errors between approximate and exact solutions while the authors in [2], [5] and [6] have established that solving (1) when enforcing for validity both the total energy equation (2) and the $(N - 1)$ relations (4) yields approximate solutions in fairly good agreement with exact ones. Roughly speaking, the numerical method proposed in these works can be understood as a predictor-corrector strategy which consists first in approximating the solutions of (1) (actually some equivalent form of (1)) and then, in a second step, in correcting the updated unknown so as to satisfy the following equivalent form of (1) :

$$\left\{\begin{array}{l}\partial_t \rho^\epsilon + \partial_x (\rho u)^\epsilon = 0,\\ \partial_t (\rho u)^\epsilon + \partial_x (\rho u^2 + p)^\epsilon = \epsilon \partial_x (\mu \partial_x u^\epsilon),\\ \mu_N \frac{\rho^{\epsilon \gamma_i}}{\gamma_i - 1}\{\partial_t s_i^\epsilon + u^\epsilon \partial_x s_i^\epsilon\} = \mu_i \frac{\rho^{\epsilon \gamma_N}}{\gamma_N - 1}\{\partial_t s_N + u^\epsilon \partial_x s_N\},\ 1 \leq i \leq N-1,\\ \partial_t (\rho E) + \partial_x (\rho H u) = \epsilon \partial_x (\mu u^\epsilon \partial_x u^\epsilon).\end{array}\right. \tag{5}$$

Let us underline that the above description of the predictor-corrector strategy is rough. Indeed and to prove the relevance of this procedure, namely the positivity of all the independent pressures together with entropy inequalities, it was necessary to deal with non linear transforms in the s_i, namely $S_i = f_i(s_i)$, so as to obey some convexity properties. For such an aim, let us point out that the natural choice $f_i = Id$ for any given $i \in \{1, ..., N\}$ is precluded. Non linear transforms in the s_i have been thus considered in [2] and [6] for the sake of stability properties but at the expense of the need for solving a non linear algebraic problem at the correction step.

In the present work, we prove that the non linear algebraic problem can be bypassed when choosing $f_i = Id$ for all $i = 1, ..., N$ while still achieving the required stability properties. To circumvent the lack of convexity, we shall introduce suitable averaging procedures which no longer rely on the classical Lebesgue measure involved in the usual Godunov approach. Such averaging procedures will be easily seen to receive fairly simple evaluations when considering classical approximate Riemann solvers for the hyperbolic underlying system in (1). We shall in particular derive such a suitable Riemann solver when approximating the solutions of the first order underlying system in (1) by those of a relaxation system.

2 Averaging procedures and stability estimates

The present section aims at describing a new predictor-corrector procedure for approximating the solutions of (1). By comparison with the method introduced in [6], the procedure we propose shares the same stability estimates but with the benefit of being far less expensive since the non linear algebraic problems to be solved in the correction step are now replaced by explicit formulae.
We propose to approximate the smooth solutions of (1) when addressing the equivalent formulation :

$$\left\{\begin{array}{l}\partial_t \rho + \partial_x \rho u = 0,\\ \partial_t \rho u + \partial_x (\rho u^2 + p) = \epsilon \partial_x (\mu \partial_x u),\\ \partial_t s_i + u \partial_x s_i = \epsilon \mu_i \frac{\gamma_i - 1}{\rho^{\gamma_i}} (\partial_x u)^2, \quad i = 1, ..., N.\end{array}\right. \tag{6}$$

Let us define $\mathbf{u} := (\rho, \rho u, \{s_i\}_{i=1,...,N})$ and its associated phase space

$$\Omega = \{\mathbf{u} \in \mathbb{R}^{N+2}, \rho > 0, u \in \mathbb{R}, s_i > 0, i = 1, ..., N\}.$$

In (6), entropy equations take the place of pressure equations and the total energy is understood as a non linear function of the unknown $\mathbf{u}$, so that smooth solutions of (6) obey the following non trivial additional equation :

$$\partial_t \{\rho E\}(\mathbf{u}) + \partial_x \{\rho H u\}(\mathbf{u}) = \epsilon \partial_x (\mu u \partial_x u).$$

The reason for considering the total energy as a non linear function of the unknown will arise naturally in the next of the section. Note that for simplicity in the notations, the superscript ϵ has been (and will be hereafter) omitted.

Let be given $\mathbf{u}_h(x, t^n)$ some approximate solution of (6) at time t^n. This approximate solution will be classically assumed to be piecewise constant with :

$$\mathbf{u}_h(x, t^n) = \mathbf{u}_j^n, \quad \text{for all} \quad x \in (x_{j-1/2}, x_{j+1/2}), \quad j \in \mathbb{Z},$$

where for simplicity $(x_{j+1/2} - x_{j-1/2}) = \Delta x > 0$. In order to advance it to the next time level t^{n+1}, we first propose two steps based on a splitting strategy between the convective and the diffusive part. Despite this procedure sounds quite natural, we are proving that a correction step is necessarily needed in order to properly capture shock solutions of (1). The relevance of the correction step will only follow from energy estimates.

Step 1 ($t^n \to t^{n+1=}$)
Noticing that the smooth solutions of (6) obey :

$$\partial_t \rho s_i + \partial_x \rho s_i u = \epsilon \mu_i \frac{\gamma_i - 1}{\rho^{\gamma_i - 1}} (\partial_x u)^2, \quad i = 1, ..., N, \tag{7}$$

then solving the underlying first order system in (6) clearly amounts to consider weak solutions of the following non linear hyperbolic system in conservation form :

$$\begin{cases} \partial_t \rho + \partial_x \rho u = 0, \\ \partial_t \rho u + \partial_x (\rho u^2 + p) = 0, \\ \partial_t \rho s_i + \partial_x \rho s_i u = 0, \quad i = 1, ..., N. \end{cases} \tag{8}$$

Usefull properties of the discontinuous solutions of (8) are summarized in the following statement :

Lemma 1 *Let be given two states $\mathbf{u}_-, \mathbf{u}_+$ in the phase space Ω and $\sigma \in I\!R$ solving the Rankine-Hugoniot conditions in (8). Then, the discontinuous solution $\mathbf{u}(x,t) = \mathbf{u}_- + (\mathbf{u}_+ - \mathbf{u}_-)\mathbf{H}(x - \sigma t)$ of (8) obeys in the usual sense of the distributions the following transport equations :*

$$\partial_t s_i + u \partial_x s_i = 0, \quad \mathcal{D}', \quad i = 1, ..., N. \tag{9}$$

Let us select the discontinuous solutions of (8) associated with the first (respectively the last) GNL field, according to the Lax entropy conditions :

$$(u - c)(\mathbf{u}_+) < \sigma < (u - c)(\mathbf{u}_-), \quad (resp. \quad (u + c)(\mathbf{u}_+) < \sigma < (u + c)(\mathbf{u}_-)), \tag{10}$$

while necessarily $u(\mathbf{u}_-) = \sigma = u(\mathbf{u}_+)$ concerning the remaining LD fields. Then the entropy discontinuous solutions of (8)–(10) obey equivalently :

$$\partial_t \{\rho E\}(\mathbf{u}) + \partial_x \{\rho H u\}(\mathbf{u}) \leq 0, \quad \mathcal{D}', \tag{11}$$

with strict inequalities concerning the shocks of the two extreme GNL fields.

From (11), the pair $(\rho E, \rho H u)$ plays the role of an entropy pair for selecting shock solutions, despite the lack of convexity in the mapping $\mathbf{u} \to \{\rho E\}(\mathbf{u})$. However, such a lack of convexity will rise difficulty at the discrete level since a control over the total energy is actually needed to enforce for validity central positivity properties in the numerical procedure. To achieve such a control, well-suited approximations of the transport equations (9) will be introduced hereafter.

Proof First, classical arguments (see [14] for instance) ensure from the Rankine-Hugoniot conditions associated with (8) that each of the s_i stays constant accross the shock waves of (8) while u

is constant at contact discontinuities. The transport equations (9) thus follow without ambiguity. Next, the continuity property $s_i^- = s_i^+ := s_i^0$ valid along the shock-curve associated with the first (respectively the last) GNL field makes such a curve to coincide with the first (resp. second) shock curve of a 2×2 isentropic Euler model which pressure law reads $\mathcal{P}(\rho) = \sum_{1 \leq i \leq N} s_i^0 \rho_i^\gamma$. Thus, well-known considerations (see again [14]) immediately apply to prove (11) from (10) and conversely. The validity of (11) for contact discontinuities easily follows from the continuity of the velocity u and the total pressure $\sum_{1 \leq i \leq N} p_i$. This concludes the proof.

In order to solve the Cauchy problem (8)-(11) for small times, we make use of the celebrated Godunov method so as to consider a sequence of non interacting exact Riemann solutions under the CFL restriction

$$\frac{\Delta t}{\Delta x} \max_{\mathbf{u}}(|u-c|(\mathbf{u}), |u+c|(\mathbf{u})) \leq \frac{1}{2}, \tag{12}$$

for all the $\mathbf{u}$ under consideration. For the times under consideration, the solution is then obtained as the superposition of these Riemann solutions. With this solution, we suggest to invoke suitable averagings of $s_i(\mathbf{u}(x, t^{n+1=}))$ in order to produce a piecewise constant update $\mathbf{u}_j^{n+1=}$:

Definition 1 *Under the CFL condition (12), the solution* $\mathbf{u}$ *of the Cauchy problem (8)-(11) is given at time* $t^{n+1=}$ *the following averaging formulae over each cell* $(x_{j-1/2}, x_{j+1/2})$, $j \in \mathbb{Z}$ *:*

$$\begin{aligned} \rho_j^{n+1=} &= \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} \{\rho\}(\mathbf{u}(x, t^{n+1=}))dx, \\ (\rho u)_j^{n+1=} &= \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} \{\rho u\}(\mathbf{u}(x, t^{n+1=}))dx, \end{aligned} \tag{13}$$

while the specific entropies are projected onto the constants according to :

$$(s_i)_j^{n+1=} = \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} \{s_i\}(\mathbf{u}(x, t^{n+1=})) \times \frac{\rho^{\gamma_i}}{\overline{\rho^{\gamma_i}}_j^{n+1=}}(\mathbf{u}(x, t^{n+1=}))dx, \tag{14}$$

$j \in \mathbb{Z}$, *where we have set :*

$$\overline{\rho^{\gamma_i}}_j^{n+1=} = \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} \{\rho^{\gamma_i}\}(\mathbf{u}(x, t^{n+1=}))dx.$$

In other words, both the density and the momentum receive classical averaging formulae based on the Lebesgue measure, while by contrast the specific entropies are averaged when invoking probability-like measures : *i.e.* positive Borel measures with unit total mass. Consequently, the positivity of each of the specific entropies is preserved by construction. A well-known consequence of the classical averagings (13) is that both density and momentum can be given an equivalent updating formula in full conservation form :

$$\begin{aligned} \rho_j^{n+1=} &= \rho_j^n - \tfrac{\Delta t}{\Delta x} \Delta\{\rho u\}_{j+1/2}^n, \\ (\rho u)_j^{n+1=} &= (\rho u)_j^n - \tfrac{\Delta t}{\Delta x} \Delta\{\rho u^2 + p\}_{j+1/2}^n, \end{aligned} \tag{15}$$

where classically $\{\rho u\}_{j+1/2}^n = \{\rho u\}(\mathbf{u}(0^+, \mathbf{u}_j^n, \mathbf{u}_{j+1}^n))$ with a similar definition for $\{\rho u^2 + p\}_{j+1/2}^n$. By contrast, the original projection formulae (14) cannot give rise to a discrete conservation form for governing the specific entropies since these variables already obey at the PDE level the transport equations (9). We shall prove later on that the averaging formulae (14) actually yield a discrete version of (9) under the form :

$$(s_i)_j^{n+1=} = (s_i)_j^n - \frac{\Delta t}{\Delta x} v_i^{j-1/2,+}\{(s_i)_j^n - (s_i)_{j-1}^n\} - \frac{\Delta t}{\Delta x} v_i^{j+1/2,-}\{(s_i)_{j+1}^n - (s_i)_j^n\}, \tag{16}$$

where the discrete velocity $v_i^{j+1/2*}$ will be given a consistent definition.
The unusual projection formulae (14) will be given further insight hereafter but they are primarily dictated by the following result :

Lemma 2 *Let the approximate solution be advanced to the next time level $t^{n+1=}$ according to (13)-(14). Then, under the CFL condition (12), the following in-cell energy inequalities are satisfied for all $j \in \mathbb{Z}$:*

$$\{\rho E\}(\mathbf{u}_j^{n+1=}) - (\rho E)_j^n + \lambda\Delta\{\rho\mathcal{H}u\}(\mathbf{u}_j^n, \mathbf{u}_{j+1}^n) \leq 0, \tag{17}$$

where classically :

$$\{\rho\mathcal{H}u\}(\mathbf{u}_j^n, \mathbf{u}_{j+1}^n) = \{(\rho E + p)u\}(\mathbf{u}(0^+, \mathbf{u}_j^n, \mathbf{u}_{j+1}^n)). \tag{18}$$

Proof First, classical considerations (see [14]) imply from the inequality (11) the following in-cell inequality :

$$\frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} \{\rho E\}(x, t^{n+1=})dx \leq (\rho E)_j^n - \lambda\Delta\{\rho\mathcal{H}u\}(\mathbf{u}_j^n, \mathbf{u}_{j+1}^n), \tag{19}$$

where the numerical energy flux is given by (18). Let us now notice that the definitions of $(s_i)_j^{n+1=}$ and $\overline{\rho^{\gamma_i}}_j^{n+1=}$ for all $i = 1, ..., N$ lead to

$$\begin{aligned}
&\int_{x_{j-1/2}}^{x_{j+1/2}} \{\rho E\}(x, t^{n+1=})dx = \\
&\int_{x_{j-1/2}}^{x_{j+1/2}} \{\frac{(\rho u)^2}{2\rho}\}(\mathbf{u}(x, t^{n+1=}))dx + \sum_{i=1}^{N}\int_{x_{j-1/2}}^{x_{j+1/2}} \frac{\{p_i\}}{\gamma_i - 1}(\mathbf{u}(x, t^{n+1=}))dx = \\
&\int_{x_{j-1/2}}^{x_{j+1/2}} \{\frac{(\rho u)^2}{2\rho}\}(\mathbf{u}(x, t^{n+1=}))dx + \sum_{i=1}^{N}\frac{(s_i)_j^{n+1=}}{\gamma_i - 1}\int_{x_{j-1/2}}^{x_{j+1/2}} \{\rho^{\gamma_i}\}(\mathbf{u}(x, t^{n+1=}))dx.
\end{aligned} \tag{20}$$

In the one hand, the convexity of the mapping $(\rho, \rho u) \to (\rho u)^2/2\rho$ ensures from the well-known Jensen's inequality :

$$\frac{((\rho u)_j^{n+1=})^2}{2\rho_j^{n+1=}} \leq \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} \{\frac{(\rho u)^2}{2\rho}\}(\mathbf{u}(x, t^{n+1=}))dx. \tag{21}$$

In the second hand, the convexity of the mapping $\rho \to \rho^{\gamma_i}$ ($\gamma_i > 1$) gives again because of the Jensen's inequality :

$$(\rho_j^{n+1=})^{\gamma_i} \leq \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} \{\rho^{\gamma_i}\}(\mathbf{u}(x, t^{n+1=}))dx, \quad i = 1, ..., N.$$

Since the averaging formulae (14) preserve the positivity of the specific entropies, we deduce the required estimate :

$$\begin{aligned}
\{\rho E\}(\mathbf{u}_j^{n+1=}) \;&:=\; \frac{((\rho u)_j^{n+1=})^2}{2\rho_j^{n+1=}} + \sum_{i=1}^{N}\frac{(\rho_j^{n+1=})^{\gamma_i}}{\gamma_i - 1}(s_i)_j^{n+1=} \\
&\leq\; \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} \{\rho E\}(x, t^{n+1=})dx.
\end{aligned} \tag{22}$$

This completes the proof.

The energy inequality (17) thus makes relevant the unusual averagings (13)-(14) to define a discrete method for approximating the weak solutions of the non linear hyperbolic system (8) in the first step of our splitting strategy. By contrast, note that the classical projections onto the constants :

$$(s_i)_j^{n+1=} = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} \{s_i\}(\mathbf{u}(x,t^{n+1=}))dx, \quad i=1,...,N,$$

or possibly

$$(\rho s_i)_j^{n+1=} = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} \{\rho s_i\}(\mathbf{u}(x,t^{n+1=}))dx, \quad i=1,...,N,$$

cannot provide us with a relevant definition to make the total energy to decrease simply because of the lack of convexity of the mapping $\mathbf{u} \to \{\rho E\}(\mathbf{u})$.
We are now in a position to address the second step devoted to the viscous perturbations.

Step 2 ($t^{n+1=} \to t^{n+1-}$)
Let us consider $\mathbf{u}_h(x,t^{n+1=})$ the solution of (8) at time Δt. This one now naturally serves as initial data for solving the viscous perturbations for times $t \in [0,\Delta t]$:

$$\begin{cases} \partial_t \rho = 0, \\ \partial_t \rho u = \epsilon \partial_x(\mu \partial_x u), \\ \partial_t s_i = \epsilon \mu_i \frac{\gamma_i - 1}{\rho^{\gamma_i}} (\partial_x u)^2, \quad i=1,...,N. \end{cases} \tag{23}$$

The numerical approximation of the solution of (23) has received a particular attention in [6]. Extending the work by Berthon in [2], these authors have proved that convenient finite differences formulae can be proposed so that the total energy, understood as a function of the updated unknown, evolves in time consistently with :

$$\partial_t \rho E = \epsilon \partial_x(\mu u \partial_x u), \tag{24}$$

while the N specific entropies obey by construction a discrete version of :

$$\mu_N \frac{\rho^{\gamma_i}}{\gamma_i - 1}\partial_t s_i = \mu_i \frac{\rho^{\gamma_N}}{\gamma_N - 1}\partial_t s_N, \quad i=1,...,N-1. \tag{25}$$

More precisely, the solution of (23) at time Δt is approximated by :

$$\begin{cases} \rho_j^{n+1-} = \rho_j^{n+1=}, \\ (\rho u)_j^{n+1-} = (\rho u)_j^{n+1=} + \Delta t \overline{\partial_x(\mu \partial_x u)}_j^{n+1-}, \\ (s_i)_j^{n+1-} = (s_i)_j^{n+1=} + \Delta t \mu_i \frac{\gamma_i - 1}{(\rho_j^{n+1-})^{\gamma_i}} \overline{(\partial_x u)^2}_j^{n+1-}, \quad i=1,...,N, \end{cases} \tag{26}$$

where the precise definition of the discrete operator $\overline{\partial_x(\mu\partial_x u)}_j^{n+1-}$, given in [6] (see also [2]), results from a conservative implicit in time method for solving u. Once this implicit scheme is solved, the discrete operator $\overline{(\partial_x u)^2}_j^{n+1-}$ can receive an explicit definition so that in one hand $\overline{(\partial_x u)^2}_j^{n+1-} \geq 0$, while on the second hand :

$$\{\rho E\}(\rho_j^{n+1-}, (\rho u)_j^{n+1-}, \{(s_i)_j^{n+1-}\}_{i=1,...,N}) = (\rho E)_j^{n+1=} + \Delta t \overline{\partial_x(\mu u \partial_x u)}_j^{n+1-}, \tag{27}$$

where the resulting discrete quantity $\overline{\partial_x(\mu u \partial_x u)}_j^{n+1-}$ is a consistent and conservative discretization of the expected exact operator (see [6] for the details).
To conclude this brief presentation, let us emphasize that the N last updating formulae in (26) easily imply for all $i = 1, ..., N-1$:

$$\mu_N \frac{(\rho_j^{n+1-})^{\gamma_i}}{\gamma_i - 1}\{(s_i)_j^{n+1-} - (s_i)_j^{n+1=}\} = \mu_i \frac{(\rho_j^{n+1-})^{\gamma_N}}{\gamma_N - 1}\{(s_N)_j^{n+1-} - (s_N)_j^{n+1=}\}, \tag{28}$$

which is nothing but a consistent discrete form of (25).

Step 3 ($t^{n+1-} \to t^{n+1}$) : Correction procedure
As they stand, the first two steps we have just described provide us with a formally consistent approximation of the solutions of (1). But these two steps cannot result in a relevant numerical method since by construction, the L^1 norm of the total energy is (strictly) decreasing in time by virtue of the next inequality :

$$\{\rho E\}(\mathbf{u}_j^{n+1-}) \leq (\rho E)_j^n - \frac{\Delta t}{\Delta x}\Delta\{(\rho E + p)u\}_{j+1/2}^n + \Delta t \overline{\partial_x(\mu u \partial_x u)}_j^{n+1-}, \tag{29}$$

as a direct consequence of the inequality (17) and the equality (27). Numerical evidences of this failure are proposed in section 4. A correction procedure for restoring the conservation of the total energy at the discret level is therefore needed. Despite that inequality (29) is responsible for this need, the same inequality (29) actually authorizes the correction step to take place, that is to say when preserving the positivity of each of the entropies. Our main motivation here, after [2] and [6], is to propose a correction step so that the final updated unknown satisfies the $(N-1)$ identities (28) at the discrete level while preserving the conservation of the total energy. In that aim, we are led to keep unchanged the definitions of the density and the momentum :

$$\rho_j^{n+1} = \rho_j^{n+1-}, \quad (\rho u)_j^{n+1} = (\rho u)_j^{n+1-}, \quad \text{for all } j \in \mathbb{Z}, \tag{30}$$

while defining the N specific entropies $(s_i)_j^{n+1}$ as the solutions of the following $(N-1)$ equations :

$$\mu_N \frac{(\rho_j^{n+1})^{\gamma_i}}{\gamma_i - 1}\{(s_i)_j^{n+1} - (s_i)_j^{n+1=}\} = \mu_i \frac{(\rho_j^{n+1})^{\gamma_N}}{\gamma_N - 1}\{(s_N)_j^{n+1} - (s_N)_j^{n+1=}\}, \tag{31}$$

supplemented with

$$\sum_{i=1}^{N} \frac{(\rho_j^{n+1})^{\gamma_i}}{\gamma_i - 1}(s_i)_j^{n+1} = (\rho E)_j^{n+1} - \frac{((\rho u)_j^{n+1})^2}{2\rho_j^{n+1}}. \tag{32}$$

Here the total energy $(\rho E)_j^{n+1}$ is defined by

$$(\rho E)_j^{n+1} := (\rho E)_j^n - \lambda \Delta\{\rho \mathcal{H} u\}(\mathbf{u}_j^n, \mathbf{u}_{j+1}^n) + \Delta t \overline{\partial_x(\mu u \partial_x u)}_j^{n+1-}, \tag{33}$$

so as to restore its conservation property at the discrete level. We first underline that the algebraic problem (31)-(32) is linear so that the next easy result, which proof is left to the reader, holds :

Proposition 1 *The linear system (31) in the unknown $\{(s_i)_j^{n+1}\}_{1 \leq i \leq N}$ admits an unique solution explicitly given by :*

$$(s_i)_j^{n+1} = (s_i)_j^{n+1=} + \frac{\mu_i(\gamma_i - 1)}{(\rho_j^{n+1})^{\gamma_i}\sum_{l=1}^N \mu_l}\left((\rho E)_j^{n+1} - \frac{((\rho u)_j^{n+1})^2}{2\rho_j^{n+1}} - \sum_{l=1}^{N} \frac{(\rho_j^{n+1})^{\gamma_l}}{\gamma_l - 1}(s_l)_j^{n+1=}\right), \tag{34}$$

for all $i = 1, \ldots, N$.

The correction procedure is thus fully explicit by contrast to the technique developed in [2] and [6]. In particular, no iterative procedure is required to solve (31). To prove the reliability of the present correction step, we now state that the same stability properties are met thanks to the discrete energy inequality (29).

Theorem 1 *Under the CFL condition (12), the solution (34) satisfies the following maximum principles :*

$$(s_i)_j^{n+1} \geq (s_i)_j^{n+1=} \geq \min((s_i)_{j-1}^n, (s_i)_j^n, (s_i)_{j+1}^n), \quad j \in \mathbb{Z}, \quad i = 1, ..., N. \tag{35}$$

As a consequence, the numerical method (13)-(14)-(26)-(30)-(33)-(34) preserves the positivity of the specific entropies and hence the positivity of the partial pressures :

$$(p_i)_j^{n+1} := (\rho_j^{n+1})^{\gamma_i} (s_i)_j^{n+1} \geq 0, \quad i = 1, ..., N.$$

Proof Note first that

$$\frac{((\rho u)_j^{n+1})^2}{2\rho_j^{n+1}} + \sum_{l=1}^{N} \frac{(\rho_j^{n+1})^{\gamma_l}}{\gamma_l - 1} (s_l)_j^{n+1=} \leq \{\rho E\}(\mathbf{u}_j^{n+1-}),$$

since we have $(s_i)_j^{n+1=} \leq (s_i)_j^{n+1-}$ by the positivity property of $\overline{(\partial_x u)^2}_j^{n+1-}$ reported in the description of *Step 2*. As a consequence, the energy inequality (29) valid under the CFL condition (12) just recasts as :

$$\frac{((\rho u)_j^{n+1})^2}{2\rho_j^{n+1}} + \sum_{l=1}^{N} \frac{(\rho_j^{n+1})^{\gamma_l}}{\gamma_l - 1} (s_l)_j^{n+1=} \leq \{\rho E\}(\mathbf{u}_j^{n+1-}) \leq (\rho E)_j^{n+1},$$

by definition (33) of $(\rho E)_j^{n+1}$. Invoking the explicit formula (34) for defining each of the specific entropies $(s_i)_j^{n+1}$, the positivity of the viscosity coefficients easily ensures that

$$(s_i)_j^{n+1} \geq (s_i)_j^{n+1=}.$$

The proof will be completed under some suitable CFL condition when proving for validity the convex decomposition (16). This is exactly the matter of the rest of the present section (the final result is given in Proposition 3).

Let us now turn defining the discrete velocities entering the identities (16). In that aim, it is convenient to adopt an Harten, Lax, Van Leer type of analysis (see [15]) when introducing the following half averagings at a given interface $x_{j-1/2}$ under the CFL (12) :

$$(s_i)_{j+1/2,L}^{n+1=} = \frac{2}{\Delta x} \int_{x_j}^{x_{j+1/2}} s_i(\mathbf{u}(x, t^{n+1=})) \times \frac{\rho^{\gamma_i}}{\overline{\rho^{\gamma_i}}_{j+1/2,L}^{n+1=}} (\mathbf{u}(x, t^{n+1=})) dx, \tag{36}$$

$j \in \mathbb{Z}$, where we have set :

$$\overline{\rho^{\gamma_i}}_{j+1/2,L}^{n+1=} = \frac{2}{\Delta x} \int_{x_j}^{x_{j+1/2}} \{\rho^{\gamma_i}\}(\mathbf{u}(x, t^{n+1=})) dx.$$

Symmetrically we define :

$$(s_i)_{j+1/2,R}^{n+1=} = \frac{2}{\Delta x} \int_{x_{j+1/2}}^{x_{j+1}} s_i(\mathbf{u}(x, t^{n+1=})) \times \frac{\rho^{\gamma_i}}{\overline{\rho^{\gamma_i}}_{j+1/2,R}^{n+1=}} (\mathbf{u}(x, t^{n+1=})) dx, \tag{37}$$

$j \in \mathbb{Z}$, where :

$$\overline{\rho^{\gamma_i}}^{n+1=}_{j+1/2,R} = \frac{2}{\Delta x}\int_{x_{j+1/2}}^{x_{j+1}} \{\rho^{\gamma_i}\}(\mathbf{u}(x, t^{n+1=}))dx.$$

Notice that the averaging procedures (14) for defining each of the $(s_i)_j^{n+1=}$ equivalently rewrite in terms of the above half averages as follows :

$$(s_i)_j^{n+1=} = \frac{\overline{\rho^{\gamma_i}}^{n+1=}_{j-1/2,R} \times (s_i)^{n+1=}_{j-1/2,R} + \overline{\rho^{\gamma_i}}^{n+1=}_{j+1/2,L} \times (s_i)^{n+1=}_{j+1/2,L}}{\overline{\rho^{\gamma_i}}^{n+1=}_{j-1/2,R} + \overline{\rho^{\gamma_i}}^{n+1=}_{j+1/2,L}}, \tag{38}$$

since by construction

$$\overline{\rho^{\gamma_i}}^{n+1=}_j = \frac{1}{2}(\overline{\rho^{\gamma_i}}^{n+1=}_{j-1/2,R} + \overline{\rho^{\gamma_i}}^{n+1=}_{j+1/2,L}). \tag{39}$$

Equipped with these notations, let us state :

Proposition 2 *Under the CFL condition (12), the half averages (36) and (37) respectively read :*

$$\begin{aligned}
\overline{\rho^{\gamma_i}}^{n+1=}_{j+1/2,L}(s_i)^{n+1=}_{j+1/2,L} &= \overline{\rho^{\gamma_i}}^{n+1=}_{j+1/2,L}(s_i)^n_j - 2\frac{\Delta t}{\Delta x} w_i^{j+1/2,-}((s_i)^n_{j+1} - (s_i)^n_j),\\
\overline{\rho^{\gamma_i}}^{n+1=}_{j+1/2,R}(s_i)^{n+1=}_{j+1/2,R} &= \overline{\rho^{\gamma_i}}^{n+1=}_{j+1/2,R}(s_i)^n_{j+1} - 2\frac{\Delta t}{\Delta x} w_i^{j+1/2,+}((s_i)^n_{j+1} - (s_i)^n_j),
\end{aligned} \tag{40}$$

where the discrete quantities $w_i^{j+1/2}$ are given by

$$w_i^{j+1/2,\pm} = w_i^{\pm}(\mathbf{u}_j^n, \mathbf{u}_{j+1}^n) = \int_0^{(u^\star_{j+1/2})^{\pm}} \rho^{\gamma_i}(\xi)d\xi, \tag{41}$$

with

$$(u^\star_{j+1/2})^+ = \max(u^\star_{j+1/2}, 0), \quad (u^\star_{j+1/2})^- = \min(u^\star_{j+1/2}, 0).$$

Here $\xi \to \rho(\xi)$ denotes the density function in the self-similar exact Riemann solution at the interface $x_{j+1/2}$ while $u^\star_{j+1/2}$ is the speed of propagation of the contact discontinuity. By contruction, the mappings $(\mathbf{u}_L, \mathbf{u}_R) \in \Omega^2 \to w_i^{\pm}(\mathbf{u}_L, \mathbf{u}_R) \in \mathbb{R}$ are locally Lipschitz-continuous with the property that

$$w_i^{\pm}(\mathbf{u}, \mathbf{u}) = \rho^{\gamma_i} u^{\pm} \quad \textit{for all} \quad \mathbf{u} \in \Omega.$$

In the definition (41), it is tacitly assumed that the Riemann solution under consideration is made of three distinct waves so that the velocity $u^\star$ of the intermediate contact wave is clearly well-defined. In order to extend its definition to the case of a Riemann solution made of a single wave (either a 1-wave or a 3-wave with classical definitions), we propose the following natural convention. When $\mathbf{u}_j$ and $\mathbf{u}_{j+1}$ are connected by a 1-wave, we set $u^\star_{j+1/2} = u^n_{j+1}$ while when connected by a 3-wave, we choose $u^\star_{j+1/2} = u^n_j$. This convention ensures the reported Lipschitz-continuity.

Proof Under the restriction CFL (12), the solution $\mathbf{u}(x,t)$ coincides for all $(x,t) \in (x_j, x_{j+1}) \times (t^n, t^{n+1=})$ with the exact Riemann solution $\mathbf{u}((x - x_{j+1/2})/(t - t^n), \mathbf{u}_j^n, \mathbf{u}_{j+1}^n)$. Up to some shift in space and time, each of the specific entropies is easily seen to be given as the following step function propagating with speed $u^\star = u^\star_{j+1/2}$, namely the speed of the contact discontinuity :

$$s_i(x,t) = (s_i)^n_j + ((s_i)^n_{j+1} - (s_i)^n_j)\mathbf{H}(x - u^\star t), \tag{42}$$

where $\mathbf{H}$ denotes the usual Heavyside function. Let us restrict ourselves to a positive value $u^\star$. The case of a non positive value will follow from straightforward adaptations. Under this assumption, we clearly get when evaluating the formula definition (36) :

$$(s_i)^{n+1=}_{j+1/2,L} = (s_i)^n_j$$

while

$$\overline{\rho^{\gamma_i}}^{n+1=}_{j+1/2,R} \times (s_i)^{n+1=}_{j+1/2,R} =$$
$$\frac{2}{\Delta x}\int_0^{\Delta x/2} \rho^{\gamma_i}(\frac{x}{\Delta t}) \times \{(s_i)^n_j + ((s_i)^n_{j+1} - (s_i)^n_j)\mathbf{H}(x - u^\star \Delta t)\}dx =$$
$$\frac{2\Delta t}{\Delta x}\{(s_i)^n_j \int_0^{u^\star} \rho^{\gamma_i}(\xi)d\xi + (s_i)^n_{j+1} \int_{u^\star}^{\frac{\Delta x}{2\Delta t}} \rho^{\gamma_i}(\xi)d\xi\}.$$

We thus easily deduce that

$$\overline{\rho^{\gamma_i}}^{n+1=}_{j+1/2,R} \times (s_i)^{n+1=}_{j+1/2,R} =$$
$$(s_i)^n_{j+1} \times \frac{2\Delta t}{\Delta x}\int_0^{\frac{\Delta x}{2\Delta t}} \rho^{\gamma_i}(\xi)d\xi - \frac{2\Delta t}{\Delta x}\int_0^{u^\star} \rho^{\gamma_i}(\xi)d\xi \times ((s_i)^n_{j+1} - (s_i)^n_j),$$

where

$$\overline{\rho^{\gamma_i}}^{n+1=}_{j+1/2,R} = \frac{2\Delta t}{\Delta x}\int_0^{\frac{\Delta x}{2\Delta t}} \rho^{\gamma_i}(\xi)d\xi.$$

The conclusion easily follows.

We are now in position to establish the next heavily awaited result :

Proposition 3 *Under the CFL condition (12), the updating formulae (14) read :*

$$(s_i)^{n+1=}_j = (s_i)^n_j - \frac{\Delta t}{\Delta x} v_i^{j-1/2,+}((s_i)^n_j - (s_i)^n_{j-1}) - \frac{\Delta t}{\Delta x} v_i^{j+1/2,-}((s_i)^n_{j+1} - (s_i)^n_j), \tag{43}$$

where we have set :

$$v_i^{j+1/2,\pm} = v_i^\pm(\mathbf{u}^n_j, \mathbf{u}^n_{j+1}) = \int_0^{(u^\star_{j+1/2})^\pm} \rho^{\gamma_i}(\xi)d\xi / \overline{\rho^{\gamma_i}}^{n+1=}_j. \tag{44}$$

By contruction, the mappings $(\mathbf{u}_L, \mathbf{u}_R) \in \Omega^2 \to v_i^\pm(\mathbf{u}_L, \mathbf{u}_R) \in \mathbb{R}$ are locally Lipschitz-continuous with the property that

$$v_i^\pm(\mathbf{u}, \mathbf{u}) = u^\pm \quad \textit{for all} \quad \mathbf{u} \in \Omega.$$

Proof Invoking (38) and (39), equalities (40) readily imply

$$2\overline{\rho^{\gamma_i}}^{n+1=}_j \times (s_i)^{n+1=}_j = \overline{\rho^{\gamma_i}}^{n+1=}_{j-1/2,R} \times (s_i)^n_j - 2\frac{\Delta t}{\Delta x} w_i^{j-1/2,+}((s_i)^n_j - (s_i)^n_{j-1})$$
$$+\overline{\rho^{\gamma_i}}^{n+1=}_{j+1/2,L} \times (s_i)^n_j - 2\frac{\Delta t}{\Delta x} w_i^{j+1/2,-}((s_i)^n_{j+1} - (s_i)^n_j),$$

that is to say, again thanks to (39)

$$(s_i)^{n+1=}_j = (s_i)^n_j - \frac{\Delta t}{\Delta x}\frac{w_i^{j-1/2,+}}{\overline{\rho^{\gamma_i}}^{n+1=}_j}((s_i)^n_j - (s_i)^n_{j-1}) - \frac{\Delta t}{\Delta x}\frac{w_i^{j+1/2,-}}{\overline{\rho^{\gamma_i}}^{n+1=}_j}((s_i)^n_{j+1} - (s_i)^n_j).$$

This concludes the proof.

The following statement now summarizes the main properties of the new prediction-correction method we have proposed.

Corollary 1 *The method (13)-(14)-(26)-(30)-(33)-(34) provides an approximate discrete solution which satisfies the following identities :*

$$\begin{cases} \rho_j^{n+1} = \rho_j^n - \frac{\Delta t}{\Delta x}\Delta\{\rho u\}_{j+1/2}^n, \\ (\rho u)_j^{n+1} = (\rho u)_j^n - \frac{\Delta t}{\Delta x}\Delta\{\rho u^2 + p\}_{j+1/2}^n + \Delta t\overline{\partial_x(\mu\partial_x u)}_j^{n+1-}, \\ (\rho E)_j^{n+1} = (\rho E)_j^n - \frac{\Delta t}{\Delta x}\Delta\{(\rho E + p)u\}_{j+1/2}^n + \Delta t\overline{\partial_x(\mu u\partial_x u)}_j^{n+1-}, \\ \mu_N \dfrac{(\rho_j^{n+1})^{\gamma_i}}{\gamma_i - 1}\{(s_i)_j^{n+1} - (s_i)_j^{n+1=}\} = \mu_i \dfrac{(\rho_j^{n+1})^{\gamma_N}}{\gamma_N - 1}\{(s_N)_j^{n+1} - (s_N)_j^{n+1=}\}, \end{cases} \tag{45}$$

where with $i = 1, ..., N$:

$$(s_i)_j^{n+1=} := (s_i)_j^n - \frac{\Delta t}{\Delta x} v_i^{j-1/2,+}\{(s_i)_j^n - (s_i)_{j-1}^n\} - \frac{\Delta t}{\Delta x} v_i^{j+1/2,-}\{(s_i)_{j+1}^n - (s_i)_j^n\}, \tag{46}$$

for some consistent averaged values $v_i^{j+1/2}$ of the velocity u defined in (44) above. In addition, the following N entropy-like inequalities are valid under the CFL condition (12) :

$$(s_i)_j^{n+1} - (s_i)_j^n + \frac{\Delta t}{\Delta x}\Big\{v_i^{j-1/2,+}\{(s_i)_j^n - (s_i)_{j-1}^n\} + v_i^{j+1/2,-}\{(s_i)_{j+1}^n - (s_i)_j^n\}\Big\} \geq 0. \tag{47}$$

In other words, the proposed method yields a consistent discretization of the equivalent system (5) based on the $(N-1)$ generalized jump conditions (4). We emphasize that inequalities (47) are just a discrete version of the following non standard entropy inequalities :

$$\partial_t s_i + u\partial_x s_i = \mu_i \frac{\gamma_i - 1}{\rho^{\gamma_i}}(\partial_x u)^2 \geq 0, \quad i = 1, ..., N.$$

Let us however stress that the satisfaction of the maximum principles (35) on each of the specific entropies s_i are known after Tadmor [25] (see also Khobalatte and Perthame [17]) to hold as a consequence of an infinite number of classical entropy inequalities.
The purpose of the following section is to describe a relevant approximate Riemann solver which will yield a fairly simple evaluation of the discrete velocities defined in (44). As far as the exact Riemann solver is concerned, we may end up with somehow cumbersome formulae essentially because of the possible rarefaction waves. Our main objective is thus to suitably get rid of such waves when introducing approximate Riemann solutions systematically made of discontinuities separating constant states.

3 A Relaxation model

In order to meet the above requirement, we suggest to approximate the solutions of the system (8)-(11) by those of a suitable extended first order system with singular perturbation. Motivated by the work of Liu [21] and Chen, Levermore and Liu [7], such a system aims at restoring not only the original PDE model but also its entropy inequality in the regime of an infinite relaxation parameter. Here the extended first order system will have the property of being non linear hyperbolic with only linearly degenerate fields. Linear degeneracy is responsible for Riemann solutions uniquely made of discontinuities. Here and by contrast to the approach proposed by Jin and Xin [16], most of the non linearities in the original PDE model are kept for the sake of a better accuracy. Indeed, the relaxation system admits as natural solutions the contact discontinuities of the original PDE. Motivated by the work of Suliciu [23] (see also [8]), we suggest to suitably modify the total pressure law entering the original model since this law turns to concentrate all the genuine non linearities. The proposed approach has deep relationships with relaxation techniques already introduced in the literature, but is actually different in its design principle. Let us mention the work [8] and also the book by Bouchut [3] in the 3×3 Euler setting. We also refer the reader to [4] for a general approach devoted to flux vector splitting methods. Here we no longer consider the pressure as an independent variable equipped with its own PDE, but we keep this pressure as a non linear function of the unknown when suitably shadowing the exact role played by the specific volume τ. The strategy we propose actually considerably simplify the analysis of the relaxation model. The associated finite volumes method will actually turn to be algebraically equivalent to the ones described in [8], [4]. In the Lagrangian setting, the corresponding finite volumes method also coincides with the scheme proposed by Desprès [10] via another formalism (see also Gallice [11]). All these contributions can be understood as a fruitful reinterpretation of a scheme proposed by Toro [26] but which can be actually traced back to Viviand and Veuillot [28] in the mid seventies.

3.1 The model and its associated relaxation energy

We propose to approximate the entropy weak solutions of (8)-(11) by those of the following non linear first order system with singular perturbation :

$$\begin{cases} \partial_t \rho_\lambda + \partial_x (\rho u)_\lambda = 0, \\ \partial_t (\rho u)_\lambda + \partial_x (\rho u^2 + \Pi)_\lambda = 0, \\ \partial_t (s_i)_\lambda + u_\lambda \partial_x (s_i)_\lambda = 0, \quad i = 1, ..., N, \\ \partial_t (\rho \mathcal{T})_\lambda + \partial_x (\rho \mathcal{T} u)_\lambda = \lambda \rho_\lambda (\tau_\lambda - \mathcal{T}_\lambda), \end{cases} \tag{48}$$

with the following closure equation :

$$\begin{gathered} \Pi = \Pi(\mathcal{T}, \{s_i\}_{1 \leq i \leq N}) = p(\mathcal{T}, \{s_i\}_{1 \leq i \leq N}) + a^2 (\mathcal{T} - \tau), \quad \tau = \tfrac{1}{\rho}, \\ p(\mathcal{T}, \{s_i\}_{1 \leq i \leq N}) := \sum_{i=1}^{N} p_i(\mathcal{T}, s_i). \end{gathered} \tag{49}$$

¿From this definition, the equilibrium system (8) is clearly formally recovered in the regime of an infinite relaxation parameter $\lambda > 0$ since we get $\mathcal{T} = \tau$. In this loose sense, solutions of the relaxation system (48) may serve to approximate the solutions of the equilibrium system (8). But to prevent the proposed relaxation procedure from instabilities in the asymptotic regime $\lambda = +\infty$, it is well-known after the work by Liu [21] and Chen, Levermore and Liu [7], that the equilibrium system and the relaxation system must meet some compatibility conditions : eigenvalues of both systems must be properly interlaced. These so-called Whitham conditions will be seen hereafter to be satisfied for properly prescribed values of the positive real number a entering the definition of the relaxation pressure law (49).
Setting $\mathbf{U} =^T (\rho, \rho u, \{s_i\}_{i=1,...,N}, \rho \mathcal{T})$, the relaxation system (48)–(49) is associated with the following natural phase space :

$$\Omega_{\mathbf{U}} = \{\mathbf{U} \in \mathbb{R}^{N+3}, \rho > 0,\ s_i > 0,\ i = 1, ..., N,\ \rho \mathcal{T} > 0\}.$$

For simplicity in the notations, (48) will be given the next condensed form when $\lambda = 0$:

$$\partial_t \mathbf{U} + \mathcal{A}(\mathbf{U}) \partial_x \mathbf{U} = 0, \tag{50}$$

where $\mathcal{A}$ finds an immediate definition. The first statement motivates the interest of the proposed relaxation procedure :

Proposition 4 *Let be given $a > 0$. Then the first order system (50) is hyperbolic for all $\mathbf{U}$ in $\Omega_{\mathbf{U}}$ with the following distinct eigenvalues :*

$$\lambda_1(\mathbf{U}) = u - a\tau = u - \frac{a}{\rho}, \quad \lambda_2(\mathbf{U}) = u, \quad \lambda_3(\mathbf{U}) = u + a\tau = u + \frac{a}{\rho}.$$

The eigenvalue $\lambda_2(\mathbf{U})$ has $(N+1)$ order of multiplicity while $\lambda_1(\mathbf{U})$ and $\lambda_3(\mathbf{U})$ are simple. All the fields under consideration are linearly degenerate.

Easy calculations left to the reader give the required results. As a consequence of the linear degeneracy of all the fields, the Riemann problem associated with (50) will be seen below to be trivially solvable. In addition, contact discontinuity solutions of the equilibrium system will be seen to be exactly preserved by the proposed relaxation procedure. These two highly desirable properties have actually dictated the derivation model (48)-(49).
The next statement proves that the relaxation system (48)–(49) enters the general framework proposed by Chen, Levermore and Liu [7] :

Theorem 2
Let us introduce the relaxation energy defined for all $\mathbf{U}$ in $\Omega_{\mathbf{U}}$ by :

$$\{\rho\Sigma\}(\mathbf{U}) = \frac{1}{2}\frac{(\rho u)^2}{\rho} + \rho\phi(\mathcal{T}, \{s_i\}_{1\le i\le N}), \tag{51}$$

where

$$\phi(\mathcal{T}, \{s_i\}_{1\le i\le N}) = \sum_{i=1}^{N} \varepsilon_i(\mathcal{T}, s_i) + \frac{1}{2a^2}\{\Pi^2(\mathcal{T}, \{s_i\}_{1\le i\le N}) - (\sum_{i=1}^{N} p_i(\mathcal{T}, s_i))^2\}. \tag{52}$$

This relaxation energy is consistent with the equilibrium energy in the sense that :

$$\rho\Sigma(\rho, \rho u, \{s_i\}_{1\le i\le N}, \rho\tau) = \rho E,$$

and

$$\{(\rho\Sigma + \Pi)u\}(\rho, \rho u, \{s_i\}_{1\le i\le N}, \rho\tau) = (\rho E + p)u.$$

In addition, smooth solutions of (48)–(49) satisfy the following energy equation for all $\lambda \ge 0$:

$$\partial_t\{\rho\Sigma\}(\mathbf{U}_\lambda) + \partial_x\{(\rho\Sigma + \Pi)u\}(\mathbf{U}_\lambda) = -\lambda\rho_\lambda(a^2 - \{\rho c\}^2(\mathbf{U}_\lambda))(\tau_\lambda - \mathcal{T}_\lambda)^2, \tag{53}$$

which yields an energy inequality under the following Whitham condition :

$$a^2 > \{\rho c\}^2(\mathcal{T}, \{s_i\}_{1\le i\le N}) = -\sum_{i=1}^{N}\{\partial_\tau p_i\}(\mathcal{T}, s_i), \tag{54}$$

for all the $\mathcal{T}$ and $\{s_i\}_{1\le i\le N}$ under consideration.
Moreover, $(\rho,\ \rho u,\ \{s_i\}_{1\le i\le N})$ being fixed, for all T-compact $\mathcal{K}$ including $\mathcal{T} = \tau$ so as to consider a finite value $a^2(\mathcal{K})$ in the Whitham condition (54), then the following minimization principle holds :

$$\rho E = \frac{1}{2}\frac{(\rho u)^2}{\rho} + \sum_{i=1}^{N} \rho\varepsilon_i(\tau, s_i) = \min_{\mathcal{T}\in\mathcal{K}}\{\{\rho\Sigma\}(\mathbf{U})\}. \tag{55}$$

In other words and under the Whitham condition (54), the relaxation energy spontaneously decreases in L^1 norm in the relaxation procedure. Moreover, its minimum coincides with the equilibrium energy ρE. With this respect, the relaxation energy plays the role of an entropy compatible with the relaxation procedure in the sense of Chen, Levermore and Liu [7]. According to the general framework proposed by these authors, one should have required the relaxation energy to be in addition strictly convex in $\mathbf{U}$. But here, such a convexity property is not useful since we do not need for a Lax entropy selection principle for the weak solutions of the PDE system (48)–(49). Indeed, all its fields are linearly degenerate and such a strong property is known (see [14], [22]) to imply that the equation (53), valid for smooth solutions, is still valid for weak solutions with equality in the sense of the distributions. The validity of (53) in $\mathcal{D}'$ together with the reported minimization principle will suffice for our forthcoming purposes.

Lemma 3 *Smooth solutions of (48)-(49) obey the following pressure-like equation for all $\lambda \geq 0$:*

$$\partial_t \Pi(\mathbf{U}_\lambda) + u_\lambda \partial_x \Pi(\mathbf{U}_\lambda) + a^2 \tau_\lambda \partial_x u_\lambda = \lambda(a^2 - \{\rho c\}^2(\mathbf{U}_\lambda)) \times (\tau_\lambda - \mathcal{T}_\lambda), \tag{56}$$

where $\{\rho c\}^2(\mathbf{U})$ is defined with little abuse in the notation in (54).

Proof The validity of (56) stems from the property that smooth solutions of (48) satisfy the following transport equations :

$$\begin{cases} \partial_t s_i + u\partial_x s_i = 0, \quad i = 1, ..., N, \\ \partial_t \mathcal{T} + u\partial_x \mathcal{T} = \lambda(\tau - \mathcal{T}), \end{cases} \tag{57}$$

where the subscript λ has been (and will be from now on) omitted for simplicity in the notations. We thus easily infer the following identity :

$$\partial_t p(\mathcal{T}, \{s_i\}_{1\leq i\leq N}) + u\partial_t p(\mathcal{T}, \{s_i\}_{1\leq i\leq N}) = \lambda(\{\sum_{i=1}^{N} \partial_\tau p_i\}(\mathcal{T}, s_i))(\tau - \mathcal{T}). \tag{58}$$

Next and by the density equation, the specific volume τ is known to be solution of :

$$\partial_t \tau + u\partial_x \tau - \tau \partial_x u = 0,$$

so that we deduce :

$$\partial_t a^2(\mathcal{T} - \tau) + u\partial_x a^2(\mathcal{T} - \tau) + a^2 \tau \partial_x u = \lambda a^2 (\tau - \mathcal{T}). \tag{59}$$

The required pressure like equation (56) just follows from the definition (49) of $\Pi(\mathcal{T}, \{s_i\}_{1\leq i\leq N})$ when summing (58) and (59).

With this result, let us prove Theorem 2.

Proof Classical considerations imply from the density and momentum equations that smooth solutions obey the following equation for governing the kinetic energy :

$$\partial_t \rho \frac{u^2}{2} + \partial_x \rho \frac{u^3}{2} + u\partial_x \Pi = 0. \tag{60}$$

In order to derive an equation in conservation form, let us notice that the additional pressure-like equation (56) yields after easy manipulations :

$$\partial_t \rho \frac{\Pi^2}{2a^2} + \partial_x \rho u \frac{\Pi^2}{2a^2} + \Pi \partial_x u = \lambda \rho \Pi (1 - \frac{\{\rho c\}^2}{a^2})(\tau - \mathcal{T}), \tag{61}$$

so as to arrive at :

$$\partial_t \rho\{\frac{u^2}{2} + \frac{\Pi^2}{2a^2}\} + \partial_x(\rho u\{\frac{u^2}{2} + \frac{\Pi^2}{2a^2}\} + \Pi u) = \lambda\rho\Pi(1 - \frac{\{\rho c\}^2}{a^2})(\tau - \mathcal{T}). \tag{62}$$

The energy involved in (62) is obviously not consistent with the equilibrium total energy so that (62) must receive extra contributions as follows. Invoking again the transport equations (57), any given smooth nonlinear combination of $\mathcal{T}$ and the $\{s_i\}_{i=1,\ldots,N}$, say

$$\theta(\mathcal{T}, \{s_i\}_{i=1,\ldots,N}) = \sum_{i=1}^{N} \varepsilon_i(\mathcal{T}, s_i) - \frac{(\sum_{i=1}^{N} p_i(\mathcal{T}, s_i))^2}{2a^2},$$

is easily seen to satisfy :

$$\begin{array}{c} \partial_t \rho\theta(\mathcal{T}, \{s_i\}_{i=1,\ldots,N}) + \partial_x \rho\theta(\mathcal{T}, \{s_i\}_{i=1,\ldots,N})u \\ = \\ -\lambda\rho\sum_{i=1}^{N} p_i(\mathcal{T}, s_i)(1 - \frac{\{\rho c\}^2}{a^2})(\tau - \mathcal{T}). \end{array} \tag{63}$$

Therefore, summing (62) and (63) gives in view of the definition of the relaxation energy :

$$\partial_t \rho\Sigma + \partial_x(\rho\Sigma u + \Pi u) = \lambda\rho(1 - \frac{\{\rho c\}^2}{a^2}) \times (\Pi - \sum_{i=1}^{N} p_i(\mathcal{T}, s_i))(\tau - \mathcal{T}),$$

which is nothing but the required inequality (53) when invoking the definition (49) of the relaxation pressure law.
Let us now turn establishing the minimization principle (55). Let us fix $(\rho, \rho u, \{s_i\}_{1\leq i\leq N})$ and let us consider a fixed compact $\mathcal{K} = [\mathcal{T}_{min}, \mathcal{T}_{max}]$ with the property that $\mathcal{T} = \tau \in \mathcal{K}$. Then, easy calculations yield for all $\mathcal{T} \in \mathcal{K}$:

$$\{\partial_{\mathcal{T}} \rho\Sigma\}(\mathbf{U}) = \rho\partial_{\mathcal{T}}\phi(\mathcal{T}, \{s_i\}_{1\leq i\leq N}) = (a^2(\mathcal{K}) - \{\rho c\}^2(\mathcal{T}, \{s_i\}_{1\leq i\leq N})) \times (\mathcal{T} - \tau), \tag{64}$$

where $a^2(\mathcal{K})$ denotes some finite values satisfying the Whitham condition on $\mathcal{K}$. As a consequence, this first order derivative exactly vanishes once on $\mathcal{K}$: namely for $\mathcal{T} = \tau$. But the second $\mathcal{T}$-derivative evaluated at this solution gives :

$$\{\partial^2_{\mathcal{T}\mathcal{T}} \rho\Sigma\}(\rho, \rho u, \{s_i\}_{1\leq i\leq N}, \rho\tau) = (a^2(\mathcal{K}) - \{\rho c\}^2(\tau, \{s_i\}_{1\leq i\leq N})) > 0, \tag{65}$$

again because of the Whitham condition. This suffices to prove that $\mathcal{T} = \tau$ is a local minimum on $\mathcal{K}$. Since the first derivative vanishes only once and this with a positive second derivative, the endpoints $\mathcal{T}_{min}$ and $\mathcal{T}_{max}$ of $\mathcal{K}$ cannot yield lower values of $\rho\Sigma$. Therefore the local minimum $\mathcal{T} = \tau$ is necessarily global. This completes the proof since by construction one has :

$$\{\rho\Sigma\}(\rho, \rho u, \{s_i\}_{1\leq i\leq N}, \rho\tau) = \rho E.$$

Equipped with these stability results valid under the Whitham condition (54), we now turn describing a relevant finite volumes method for approximating the entropy weak solutions of the hyperbolic system (8)-(11).

3.2 The numerical method

The numerical procedure we propose naturally falls within the classical framework for relaxation methods, which we briefly recall for the sake of completeness. After Jin and Xin [16], a given piecewise constant approximate solution $\mathbf{u}_h$ at time t^n for (8)-(11) is evolved to the next time level $t^n + \Delta t$ when adopting the following two steps.

1. *Evolution in time*
   For small times $t \in [0, \Delta t)$, we solve the Cauchy problem for the relaxation system (48) when setting λ to zero and prescribing the initial data $\mathbf{U}_h(x, t^n)$ at equilibrium from $\mathbf{u}_h(x, t^n)$:

$$\mathbf{U}_h(x, t^n) =^T (\rho_h, (\rho u)_h, \{(s_i)_h\}_{i=1,...N}, (\rho \mathcal{T})_h)(x, t^n), \tag{66}$$

   where by definition :

$$(\rho \mathcal{T})_h(x, t^n) = 1, \quad \text{for all} \quad x \in \mathbb{R}. \tag{67}$$

   When Δt is small enough, e.g. under the CFL condition :

$$\frac{\Delta t}{\Delta x} \max_{\mathbf{U}}(|u - a\tau|(\mathbf{U}), |u + a\tau|(\mathbf{U})) \leq \frac{1}{2}, \tag{68}$$

   the solution $\mathbf{U}_h(x, t^n + \Delta t^-)$ is obtained when solving a sequence of local non interacting Riemann problems.

2. *Projection*
   Since $\lambda = 0$, the above solution generally no longer belongs to the equilibrium manifold, and this motivates the following pointwise in x projection step at time $t^n + \Delta t$:

$$\mathcal{T}_h(x, t^n + \Delta t) = \tau_h(x, t^n + \Delta t^-), \quad \text{for all} \quad x \in \mathbb{R} \tag{69}$$

   when keeping unchanged the other variables. This second step just amounts to solve the following ODE system :

$$\begin{cases} \partial_t \rho_\lambda = 0, \\ \partial_t (\rho u)_\lambda = 0, \\ \partial_t (s_i)_\lambda = 0, \quad i = 1, ..., N, \\ \partial_t (\rho \mathcal{T})_\lambda = \lambda \rho_\lambda (\tau_\lambda - \mathcal{T}_\lambda), \end{cases} \tag{70}$$

   with initial data $\mathbf{U}_h(x, t^n + \Delta t^-)$ while sending λ to infinity.

This two steps procedure will be seen to yield a entropy consistent finite volumes method for approximating the weak solutions of (8)-(11) under a relevant choice of the parameter a.

3.3 The total energy stability estimate

After [9] and [1], we briefly motivate an adaptative in space choice of a. To that purpose, let us observe that the relaxation energy inequality (53) strongly suggests to choose the smallest value of a that satisfies the Whitham condition (54), in order to lower the energy rate of dissipation in the relaxation procedure. It is actually possible to optimize locally in space the choice of a under the CFL restriction (68) while achieving a consistent finite volume method. Indeed, let us define $\mathbf{U}_h(x, t)$ with $t \in (t^n, t^n + \Delta t)$ locally in space when setting for $x \in (x_j, x_{j+1})$:

$$\mathbf{U}_h(x, t) = \mathbf{U}_{a_{j+1/2}}(\frac{x - x_{j+1/2}}{t - t^n}; \mathbf{U}_j^n, \mathbf{U}_{j+1}^n), \tag{71}$$

where the self-similar function $\mathbf{U}_{a_{j+1/2}}$ denotes the exact Riemann solution of the relaxation system at interface $x_{j+1/2}$. Here the relaxation pressure law is locally defined by

$$\Pi(\mathcal{T}, \{s_i\}_{1\le i\le N}) = \sum_{1\le i\le N} p(\mathcal{T}, \{s_i\}_{1\le i\le N}) + a_{j+1/2}^2(\mathcal{T} - \tau),$$

for some relevant definition of $a_{j+1/2}$ specified later on, but solely depending on the two states $\mathbf{U}_j^n$ and $\mathbf{U}_{j+1}^n$. Then, it is crucial to observe that under the CFL condition (68), the following continuity property holds for all $j \in \mathbb{Z}$:

$$\mathbf{U}_j^n = \mathbf{U}_{a_{j-1/2}}(\frac{\Delta x}{2\Delta t}; \mathbf{U}_{j-1}^n, \mathbf{U}_j^n) = \mathbf{U}_{a_{j+1/2}}(-\frac{\Delta x}{2\Delta t}; \mathbf{U}_j^n, \mathbf{U}_{j+1}^n), \tag{72}$$

where $\Pi(\mathbf{U}_j^n) = \sum_{1\le i\le N} p(\tau, \{s_i\}_{1\le i\le N})$ in view of the equilibrium property (67). In orther words, $\mathbf{U}_h(x, t^n + \Delta t^-)$ built from (71) still serves to define a consistent approximate Riemann solution for the equilibrium system (8)-(11) (see [5] for the details).
The simplicity in the evaluation of $\mathbf{U}_h(x, t^n + \Delta t^-)$ comes from the property that the Riemann problem for the relaxation system is trivially solved according to :

Lemma 4 *Let be given $\mathbf{U}_L$ and $\mathbf{U}_R$ two states in $\Omega_{\mathbf{U}}$ at equilibrium : i.e with $\mathcal{T}_L = \tau_L$ and $\mathcal{T}_R = \tau_R$. Let us then define $a := a(\mathbf{U}_L, \mathbf{U}_R) > 0$ in (49) large enough such that the following natural ordering holds :*

$$\lambda_1(\mathbf{U}_L) = u_L - a\tau_L < u^\star < \lambda_3(\mathbf{U}_R) = u_R + a\tau_R, \tag{73}$$

where $u^\star$ stands for some intermediate velocity given by :

$$u^\star = \frac{1}{2}(u_L + u_R) - \frac{1}{2a}(p_R - p_L).$$

Then, the Riemann solution $\mathbf{U}_a(x/t; \mathbf{U}_L, \mathbf{U}_R)$ for the relaxation system is made of at most four constant states, $\mathbf{U}_L, \mathbf{U}_1, \mathbf{U}_2, \mathbf{U}_R$ systematically separated by three discontinuities, respectively propagating with speed $\lambda_1(\mathbf{U}_L)$, $\lambda_2(\mathbf{U}_1) = \lambda_2(\mathbf{U}_2) = u^\star$ and $\lambda_3(\mathbf{U}_R)$. The intermediate states $\mathbf{U}_1$ and $\mathbf{U}_2$ are defined from :

$$\begin{array}{l} u_1 = u_2 = u^\star, \\ (s_i)_1 = (s_i)_L, \quad (s_i)_2 = (s_i)_R, \quad 1 \le i \le N, \\ \mathcal{T}_1 = \tau_L, \quad \mathcal{T}_2 = \tau_R, \end{array}$$

together with

$$\rho_1 = a/(u^\star - \lambda_1(\mathbf{U}_L)), \quad \rho_2 = a/(\lambda_3(\mathbf{U}_R) - u^\star),$$

so that under the ordering condition (73), both ρ_1 and ρ_2 are positive. Finally, $\mathbf{U}_1$ and $\mathbf{U}_2$ share the same total pressure, given by :

$$\Pi(\mathbf{U}_1) = p_L + a^2(\tau_L - \tau_1) = p_R + a^2(\tau_R - \tau_2) = \Pi(\mathbf{U}_2).$$

Proof Since all the fields are linearly degenerate, the Riemann solution we seek for is uniquely made of discontinuities, the i^{th} one propagating at the characteristic speed λ_i of the field under consideration. These discontinuities separate at most four constant states $\mathbf{U}_L$, $\mathbf{U}_1$, $\mathbf{U}_2$ and $\mathbf{U}_R$, two neighbouring states being solutions of the Rankine-Hugoniot conditions associated with the relaxation system. Once the parameter a obeys the natural ordering condition (73) in the three wave velocities, these jump conditions are easily seen to yield the conclusion.

Let us finally address the proper definition of the parameter $a_{j+1/2}$ entering the Riemann solution $\mathbf{U}_{a_{j+1/2}}(.; \mathbf{U}_j^n, \mathbf{U}_{j+1}^n)$ at each interface $x_{j+1/2}$. To this purpose, let us define from this solution the following self-similar function, setting for $(x,t) \in (x_j, x_{j+1}) \times (t^n, t^n + \Delta t)$:

$$\mathbf{u}_{a_{j+1/2}}(x,t) := {}^T(\rho, \rho u, \{s_i\}_{i=1,\dots,N})_{a_{j+1/2}}(\frac{x - x_{j+1/2}}{t - t^n}).$$

Then under the CFL restriction (68), $\mathbf{u}_{a_{j+1/2}}(x, t^n + \Delta t)$ is nothing but the pointwise projection (69) of $\mathbf{U}_{a_{j+1/2}}$ on the equilibrium manifold. Equipped with this suitable function, we now state the main result of this section :

Theorem 3 *Let us assume that $a_{j+1/2} := a_{j+1/2}(\mathbf{U}_j^n, \mathbf{U}_{j+1}^n)$ is chosen large enough so as to satisfy the ordering condition (73) together with the following Whitham conditions :*

$$\begin{aligned} &\sum_{i=1}^N \frac{\partial p_i}{\partial \tau}(\tau_j^n, (s_i)_j^n) + a_{j+1/2}^2 > 0, \ \sum_{i=1}^N \frac{\partial p_i}{\partial \tau}((\tau_1)_{j+1/2}^n, (s_i)_j^n) + a_{j+1/2}^2 > 0, \\ &\sum_{i=1}^N \frac{\partial p_i}{\partial \tau}(\tau_{j+1}^n, (s_i)_{j+1}^n) + a_{j+1/2}^2 > 0, \ \sum_{i=1}^N \frac{\partial p_i}{\partial \tau}((\tau_2)_{j+1/2}^n, (s_i)_{j+1}^n) + a_{j+1/2}^2 > 0, \end{aligned} \tag{74}$$

where $(\tau_1)_{j+1/2}^n$ and $(\tau_2)_{j+1/2}^n$ denote the specific volumes in the intermediate states of the Riemann solution $\mathbf{U}_{a_{j+1/2}}(., \mathbf{U}_j^n, \mathbf{U}_{j+1}^n)$. Then, under the CFL restriction (68), the following energy-like inequalities hold :

$$\begin{gathered} \frac{2}{\Delta x} \int_{x_j}^{x_{j+1/2}} \{\rho E\}(\mathbf{u}_{a_{j+1/2}}(x, t^n + \Delta t))dx \\ \leq \\ (\rho E)_j^n - 2\frac{\Delta t}{\Delta x}(\{(\rho\Sigma + \Pi)u\}_{j+1/2}^n - (\rho H u)_j^n), \end{gathered} \tag{75}$$

$$\begin{gathered} \frac{2}{\Delta x} \int_{x_{j+1/2}}^{x_{j+1}} \{\rho E\}(\mathbf{u}_{a_{j+1/2}}(x, t^n + \Delta t))dx \\ \leq \\ (\rho E)_{j+1}^n - 2\frac{\Delta t}{\Delta x}((\rho H u)_{j+1}^n - \{(\rho\Sigma + \Pi)u\}_{j+1/2}^n), \end{gathered} \tag{76}$$

where by definition,

$$\{(\rho\Sigma + \Pi)u\}_{j+1/2}^n = \{(\rho\Sigma + \Pi)u\}(\mathbf{U}_{a_{j+1/2}}(0^+; \mathbf{U}_j^n, \mathbf{U}_{j+1}^n)).$$

Observe that for sufficiently large values of a, $(\tau_1)_{j+1/2}^n$ (respectively $(\tau_2)_{j+1/2}^n$) becomes arbitrarily close to τ_j^n (respectively τ_{j+1}^n). As an immediate consequence, it is easy to check that if a is chosen large enough, the Whitham conditions (74) (as well as the ordering property (73)) necessarily hold. We refer the reader to Bouchut [4] for simplified Whitham conditions which are more convenient to deal with from a numerical point of view.

Proof We prove below the inequality (75). The companion one will follow from simple adaptations. Let us first recall that the Riemann solution $\mathbf{U}_{a_{j+1/2}}(.; \mathbf{U}_j^n, \mathbf{U}_{j+1}^n)$ is obtained when choosing $\lambda = 0$ in the relaxation procedure. So that it obeys in the sense of the distributions the following total energy additional conservation law in view of Theorem 2 :

$$\partial_t\{\rho\Sigma\}(\mathbf{U}_{a_{j+1/2}}) + \partial_x\{(\rho\Sigma + \Pi)u\}(\mathbf{U}_{a_{j+1/2}}) = 0.$$

Let us indeed recall that the property of linear degeneracy of all the fields implies the validity of the above conservation law not only for smooth but also weak solutions of the relaxation model. Now classical arguments give :

$$\begin{aligned}&\frac{2}{\Delta x}\int_{x_j}^{x_{j+1/2}}\{\rho\Sigma\}(\mathbf{U}_{a_{j+1/2}}(x,t^n+\Delta t))dx = \\ &(\rho E)_j^n-\frac{2}{\Delta x}(\Delta t\{(\rho\Sigma+\Pi)u\}_{j+1/2}^n-\int_{t^n}^{t^n+\Delta t}\{(\rho\Sigma+\Pi)u\}(\mathbf{U}_{a_{j+1/2}}(x_j,t))dt).\end{aligned} \tag{77}$$

But the continuity property (72) is easily seen to yield :

$$\int_{t^n}^{t^n+\Delta t}\{(\rho\Sigma+\Pi)u\}(\mathbf{U}_{a_{j+1/2}}(x_j,t))dt=\Delta t((\rho\Sigma+\Pi)u)_j^n=\Delta t(\rho Hu)_j^n,$$

since and again $\mathbf{U}_j^n$ is by construction at equilibrium.
Let us next prove the following inequality for all $x\in(x_j,x_{j+1})$:

$$\{\rho E\}(\mathbf{u}_{a_{j+1/2}}(x,t^n+\Delta t))\leq\rho\Sigma(\mathbf{U}_{a_{j+1/2}}(x,t^n+\Delta t)). \tag{78}$$

This inequality will then easily imply the required entropy-like inequality from (77). To prove (78), we first observe that for all $x\in(x_j,x_{j+1/2}+\lambda_1(\mathbf{U}_j^n)\Delta t)$, one trivially has :

$$\{\rho E\}(\mathbf{u}_{a_{j+1/2}}(x,t^n+\Delta t))=\{\rho\Sigma\}(\mathbf{U}_{a_{j+1/2}}(x,t^n+\Delta t))=(\rho E)_j^n.$$

Let us now prove that for all x in $\mathcal{I}_1=(x_{j+1/2}+\lambda_1(\mathbf{U}_j^n)\Delta t,x_{j+1/2}+u^\star\Delta t)$, namely when $\mathbf{U}_{a_{j+1/2}}(x,t^n+\Delta t)=(\mathbf{U}_1)_{j+1/2}^n$, that

$$\{\rho E\}(\mathbf{u}_{a_{j+1/2}}(x,t^n+\Delta t))\leq\{\rho\Sigma\}(\mathbf{U}_{a_{j+1/2}}(x,t^n+\Delta t)). \tag{79}$$

To that purpose, let us define the following compact $\mathcal{K}=[\mathcal{T}_{min},\mathcal{T}_{max}]$ where $\mathcal{T}_{min}=\min(\tau_j^n,(\tau_1)_{j+1/2}^n)$ and $\mathcal{T}_{max}=\max(\tau_j^n,(\tau_1)_{j+1/2}^n)$, so that both $\mathcal{T}(x,t^n+\Delta t)=\tau_j^n$ and $\tau(x,t^n+\Delta t)=(\tau_1)_{j+1/2}^n$ in the Riemann solution belongs to this compact for all $x\in\mathcal{I}_1$. But, the first two discrete Whitham conditions just express that $a_{j+1/2}^2$ satisfies the requirement for applying the minimization principle stated in Theorem 2 over the compact $\mathcal{K}$ for any given fixed $x\in\mathcal{I}_1$. This indeed follows from the convexity of each of the partial pressure laws in τ, which ensures the monotonicity of $\tau\to\partial_\tau\sum_{i=1}^N p_i$. The validity of (79) thus follows. Exactly in the same way, one can prove that :

$$\{\rho E\}(\mathbf{u}_{a_{j+1/2}}(x,t^n+\Delta t))\leq\{\rho\Sigma\}(\mathbf{U}_{a_{j+1/2}}(x,t^n+\Delta t)),$$

for all $x\in(x_{j+1/2}+u^\star\Delta t,x_{j+1/2}+\lambda_3(\mathbf{U}_{j+1}^n)\Delta t)$ while :

$$\{\rho E\}(\mathbf{u}_{a_{j+1/2}}(x,t^n+\Delta t))=\{\rho\Sigma\}(\mathbf{U}_{a_{j+1/2}}(x,t^n+\Delta t))=(\rho E)_{j+1}^n,$$

when $x\in(x_{j+1/2}+\lambda_3(\mathbf{U}_{j+1}^n)\Delta t,x_{j+1})$. This completes the proof.

We conclude when applying the above relaxation procedure within the frame of the new averaging technics introduced in Definition 1. We first notice the representation formula (42) applies in the present relaxation setting. Consequently, exactly the same definitions (44) of the discrete velocities $v_i^{j+1/2,\pm}$ follow to get updating formulae for the $\{s_i\}_{1\leq i\leq N}$ under the form (43). Here the very benefit comes from the very simple structure of each of the Riemann solutions which makes fairly straightforward the evaluation of the required velocities $v_i^{j+1/2,\pm}$. The relevance of the Relaxation approach with respect to this new averaging technique is assessed by this last statement :

Proposition 5 *Under the CFL restriction (68), the following energy-like inequalities hold :*

$$\{\rho E\}(\mathbf{u}_j^{n+1=}) - (\rho E)_j^n + \lambda\Delta\{\rho\mathcal{H}u\}_{j+1/2}^n \leq 0, \quad j \in \mathbb{Z}, \tag{80}$$

where with the notations of the previous section, the numerical energy flux reads :

$$\{\rho\mathcal{H}u\}_{j+1/2}^n = \{(\rho\Sigma + \Pi)u\}(\mathbf{U}_{a_{j+1/2}}(0^+; \mathbf{U}_j^n, \mathbf{U}_{j+1}^n)).$$

Proof As an immediate consequence of the inequalities (75)-(76), we get under the CFL restriction (68) :

$$\frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} \{\rho E\}(\mathbf{u}_h(x, t^n + \Delta t))dx \leq (\rho E)_j^n - \frac{\Delta t}{\Delta x}\Delta\{\rho\mathcal{H}u\}_{j+1/2}^n,$$

where we have set :

$$\mathbf{u}_h(x, t^n + \Delta t) = \begin{cases} \mathbf{u}_{a_{j-1/2}}(x, t^n + \Delta t), & x \in (x_{j-1/2}, x_j), \\ \mathbf{u}_{a_{j+1/2}}(x, t^n + \Delta t), & x \in (x_j, x_{j+1/2}). \end{cases}$$

We have thus arrived at exactly the same form of inequality (19) which is the starting point of the proof of the Lemma 2. The conclusion follows when applying exactly the same steps.

4 Numerical experiments

This section aims at illustrating that the new averaging procedures we have described succeed in capturing numerical solutions of (1) in fairly good agreement with exact ones. We consider constant viscosity laws $\{\mu_i\}_{i=1,\dots,N}$ with a Reynolds number equal to 10^5 which is typical in aerodynamic. The numerical experiments we propose consist in approximating Riemann solutions using an uniform mesh made of 300 points.The corresponding numerical solutions are compared with the exact ones. To assess the importance of the correction step, we also display the approximate solutions obtained by a classical splitting approach (without correction procedure).

Experiment 1

We set $N = 3$ and choose

γ_1	γ_2	γ_3	μ_2/μ_1	μ_3/μ_1
1.4	1.6	1.4	1.	1.

while the left and right states entering the initial data are expressed in terms of their density ρ, velocity u and partial pressures $\{p_i\}_{1\leq i\leq 3}$:

$$\begin{array}{llllll} (\rho, u, \{p_i\}_{1\leq i\leq 3})_L = (3.0, & 2.0, & 1.5, & 1.0, & 1.0), \\ (\rho, u, \{p_i\}_{1\leq i\leq 3})_R = (2.6848, & -2.1586, & 1.1930, & 0.7086, & 0.7649). \end{array} \tag{81}$$

We observe on figures 1, 2 and 3 that despite the exact solution exhibits large pressure jumps, the numerical solution produced by our scheme fully agrees with the exact one, contrary to the approximate solution given by the classical splitting approach.

Experiment 2

We now deal with ratios of the viscosity laws $\{\mu_i/\mu_1\}_{i=2,3}$ large with respect to 1. More precisely, we consider :

γ_1	γ_2	γ_3	μ_2/μ_1	μ_3/μ_1
1.4	1.6	1.6	100.	100.

The left and right states in the initial data read as follows :

$$\begin{aligned}(\rho, u, \{p_i\}_{1\le i\le 3})_L &= (1.0, \quad 1.0, \quad 1.0, \quad 1.0, \quad 0.6),\\ (\rho, u, \{p_i\}_{1\le i\le 3})_R &= (1.4291, \quad -0.5477, \quad 0.2277, \quad 0.4355, \quad 0.0929).\end{aligned} \tag{82}$$

Solutions are plotted on figures 4, 5 and 6. Once more, let us notice the excellent agreement between the solutions given by our procedure and the exact ones.

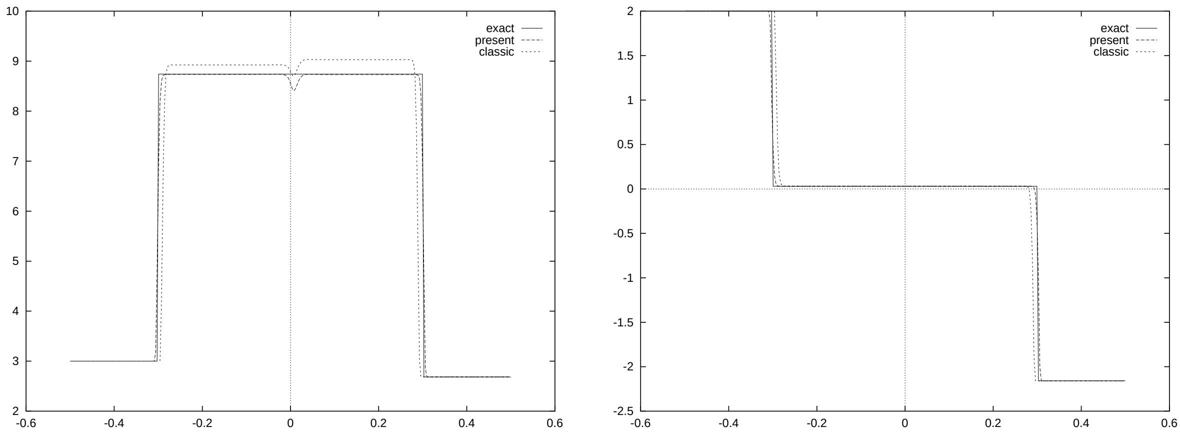

Figure 1: Experiment 1 : density, velocity

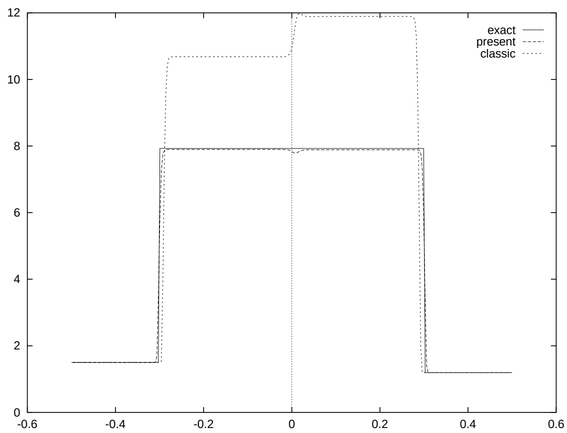

Figure 2: Experiment 1 : pressure 1

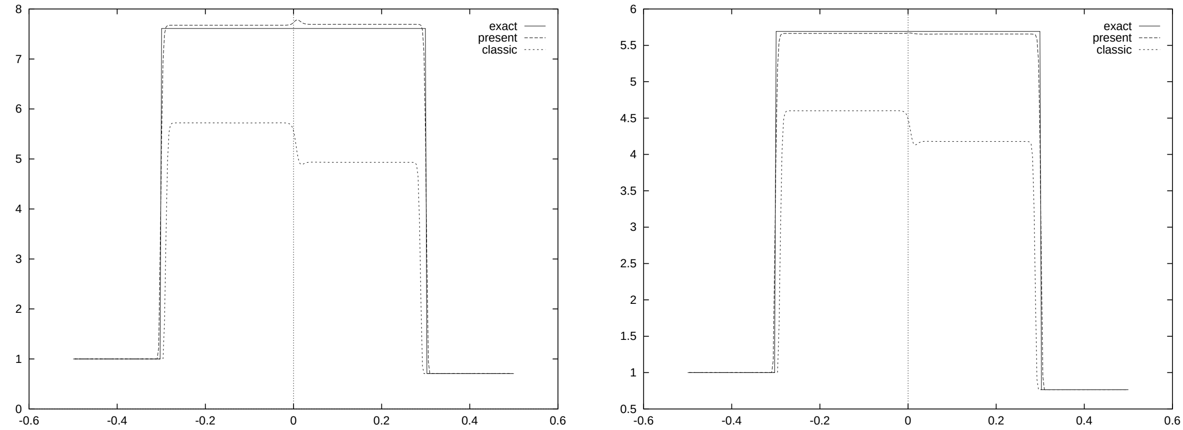

Figure 3: Experiment 1 : pressure 2, pressure 3

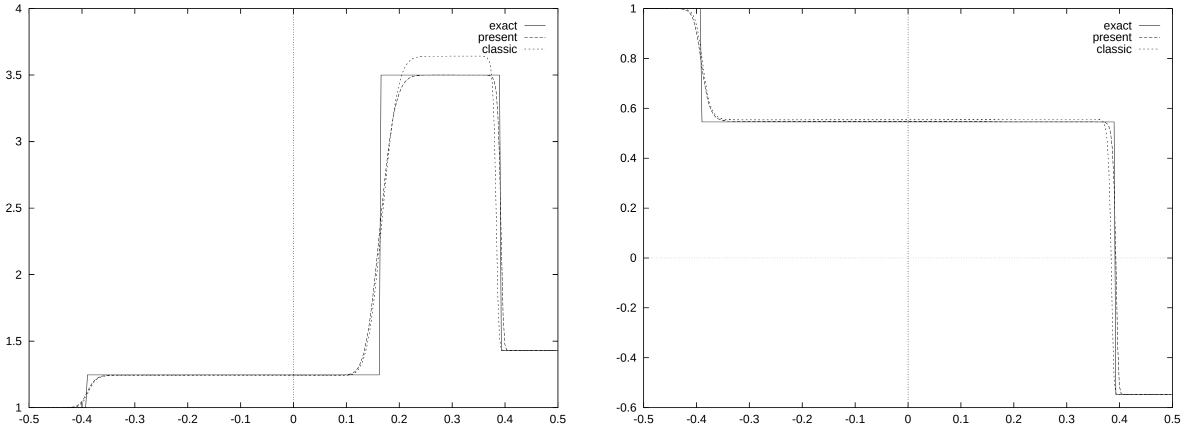

Figure 4: Experiment 2 : density, velocity

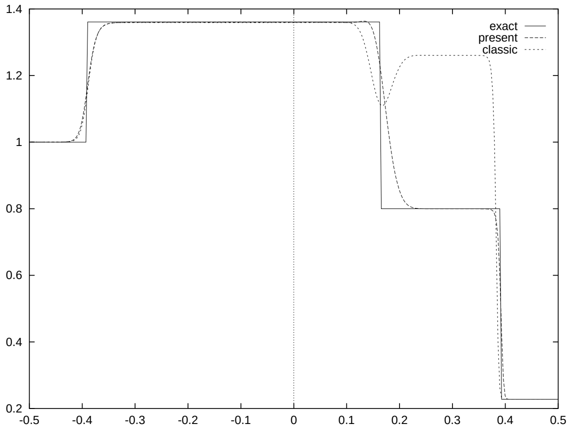

Figure 5: Experiment 2 : pressure 1

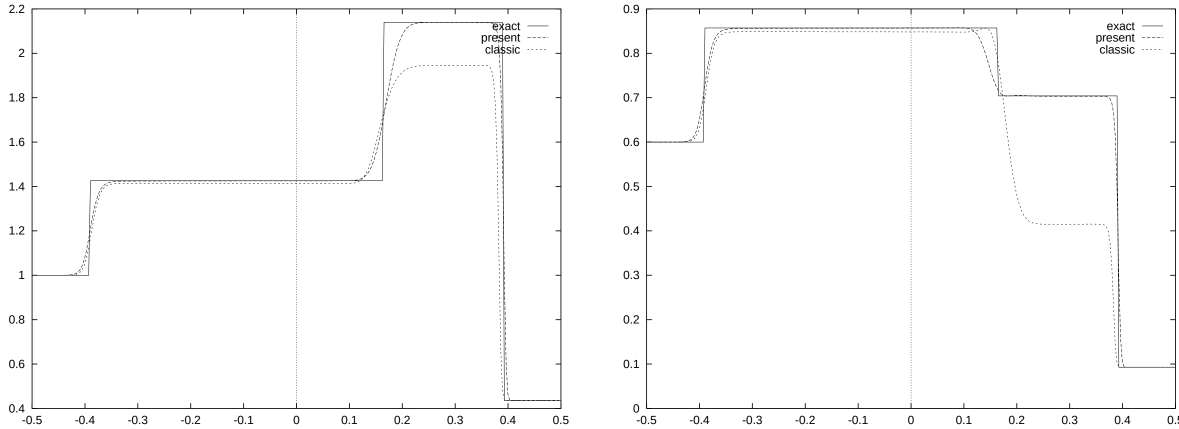

Figure 6: Experiment 2 : pressure 2, pressure 3

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