Spectral discretization of a Nagdhi shell model

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Abstract: We consider the Nagdhi equations which model a thin three-dimensional shell. We propose a spectral discretization of this problem in the case where the midsurface of the shell is weakly regular. We perform the numerical analysis of the discrete problem and prove optimal error estimates.

Résumé: Nous considérons les équations de Nagdhi qui modélisent une coque tridimensionnelle de faible épaisseur. Nous proposons une discrétisation spectrale de ce problème dans le cas où la surface moyenne de la coque a une régularité faible. Nous effectuons l'analyse numérique du problème discret et prouvons des majorations optimales de l'erreur.

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1. Introduction.

We consider a formulation of Nagdhi's shell model in Cartesian coordinates which is appropriate for linearly elastic shells that present curvature discontinuities. The aim of this paper is to propose a spectral discretization of the mixed formulation of this problem and to perform its numerical analysis.

The formulation of Naghdi's model which is used here was introduced by Blouza [7] and Blouza and Le Dret [11]. This formulation relies on the idea of using a local basis-free formulation in which the unknowns are described in Cartesian coordinates instead of covariant or contravariant components as is usually done in shell theory, see for example [2]. Such a formulation is able to handle shells with a $W^{2,\infty}$-midsurface, thus allowing for curvature discontinuities, as opposed to $\mathscr{C}^3$ in the classical formalism (see [14, Chap. 7] and the references therein), and leads to much simpler expressions. Even though it is proved in [11] to be well-posed and to be the natural limit of the classical formulation when a sequence of regular midsurfaces converges to a $W^{2,\infty}$-midsurface, the new formulation has not been used in a numerical spectral setting to the best of our knowledge. For simplicity, we only consider the case of a shell with a $W^{2,\infty}$-midsurface.

The literature on finite element approximation of two-dimensional shell models is large. Let us mention a few approaches. Concerning conforming methods, the Ganev and Argyris triangles provide interpolation by polynomials of degree 4 and 5, with high order convergence in $c\,h^4$ when the solution is smooth enough. These elements are used for example to study the linear Koiter model for $\mathscr{C}^3$-shells in the classical covariant formulation, see [1, Part II, Chap. 1]. This method is applied to approximate geometrically exact shell models in [12]. The Argyris elements are also used in [17] for numerical analysis of Koiter's model with little regularity in the Cartesian formulation proposed in [10]. We also mention the 3-dimensional shell element approach, see [13]. Still in the context of shells with little regularity, i.e. when the midsurface is of $W^{2,\infty}$-regularity, a non conforming DKT (discrete Kirchhoff triangle) element is used in [20] to approximate a Koiter model similar to one introduced in [10]. Other works [18][19] concern the finite element discretization of shell problems with domain decomposition. The main difficulty of all these discretizations is that, in most situations, a locking phenomenon appears when the choice of the discretization parameter is not compatible with the thickness of the shell.

In this paper, we propose a spectral discretization of Naghdi's model. It relies on a mixed variational formulation of the corresponding equation proposed by Blouza, Hecht and Le Dret [9]: A Lagrange multiplier is introduced to enforce the tangency requirement on one of the unknowns. A further penalization term is also added in order to stabilize the system. We first describe the discrete problem which is constructed from the variational formulation of the model by the Galerkin method with numerical integration (see [4, §15] or [6, Chap. V] for a detailed presentation of this procedure). Under some further but likely regularity assumptions on the midsurface of the shell, we prove that it is well-posed. Finally, relying on standard polynomial approximation and interpolation results, we prove error estimates which are fully optimal.

The extension of this study to the case of a piecewise regular shell discretized by the spectral element method is under consideration. Numerical experiments should confirm the interest of this discretization.

An outline of the paper is as follows.

- In Section 2, we recall the geometry of the midsurface and Nagdhi's shell formulation. We introduce a mixed version of Nagdhi's model intended to approximate the above mentioned tangency. We prove the existence and uniqueness of the solution of the mixed model and establish its convergence to the solution of the original Nagdhi problem when the penalization parameter tends to 0.
- Section 3 is devoted to the description of the spectral discrete problem. We also prove its well-posedness.
- Error estimates are derived in Section 4.

2. Presentation of the model.

Greek indices and exponents take their values in the set $\{1,2\}$ and Latin indices and exponents take their values in the set $\{1,2,3\}$. Unless otherwise specified, the summation convention for repeated indices and exponents according to this set of values is assumed. Let $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ be the canonical orthonormal basis of the Euclidean space $\mathbb{R}^3$. We denote by $\boldsymbol{u} \cdot \boldsymbol{v}$ the inner product of $\mathbb{R}^3$, $|\boldsymbol{u}| = \sqrt{\boldsymbol{u} \cdot \boldsymbol{u}}$ the associated Euclidean norm and $\boldsymbol{u} \wedge \boldsymbol{v}$ the vector product of $\boldsymbol{u}$ and $\boldsymbol{v}$.

Let ω be a bounded connected domain of $\mathbb{R}^2$ with a Lipschitz–continuous boundary $\partial\omega$. We consider a shell whose midsurface is given by $S = \boldsymbol{\varphi}(\bar{\omega})$ where $\boldsymbol{\varphi}$ is a one-to-one mapping in $W^{2,\infty}(\omega)^3$ such that the two vectors

$$\boldsymbol{a}_\alpha(\boldsymbol{x}) = (\partial_\alpha \boldsymbol{\varphi})(\boldsymbol{x})$$

are linearly independent at each point $\boldsymbol{x}$ of $\bar{\omega}$. Thus,

$$\boldsymbol{a}_3(\boldsymbol{x}) = \frac{\boldsymbol{a}_1(\boldsymbol{x}) \wedge \boldsymbol{a}_2(\boldsymbol{x})}{|\boldsymbol{a}_1(\boldsymbol{x}) \wedge \boldsymbol{a}_2(\boldsymbol{x})|}$$

is the unit normal vector on the midsurface at point $\boldsymbol{\varphi}(\boldsymbol{x})$. The vectors $\boldsymbol{a}_i(\boldsymbol{x})$ define the local covariant basis at point $\boldsymbol{\varphi}(\boldsymbol{x})$. The contravariant basis $\boldsymbol{a}^i(\boldsymbol{x})$ is defined by the relations $\boldsymbol{a}_i \cdot \boldsymbol{a}^j = \delta_i^j$ where δ_i^j is the Kronecker symbol. In particular $\boldsymbol{a}_3(\boldsymbol{x})$ coincides with $\boldsymbol{a}^3(\boldsymbol{x})$. Note that all these vectors belong to $W^{1,\infty}(\omega)^3$. We set $a(\boldsymbol{x}) = |\boldsymbol{a}_1(\boldsymbol{x}) \wedge \boldsymbol{a}_2(\boldsymbol{x})|^2$ so that $\sqrt{a(\boldsymbol{x})}$ is the area element of the midsurface in the chart $\boldsymbol{\varphi}$.

The first and second fundamental forms of the surface are given in covariant components by

$$a_{\alpha\beta} = \boldsymbol{a}_\alpha \cdot \boldsymbol{a}_\beta \qquad \text{and} \qquad b_{\alpha\beta} = \boldsymbol{a}_3 \cdot \partial_\beta \boldsymbol{a}_\alpha.$$

Let $\boldsymbol{u}$ be a midsurface displacement in $H^1(\omega)^3$ and $\boldsymbol{r}$ be a rotation vector in $H^1(\omega)^3$ such that $\boldsymbol{r}$ is tangential to the midsurface. These functions are given in covariant and Cartesian components by

$$\boldsymbol{u}(\boldsymbol{x}) = u_i(\boldsymbol{x})\boldsymbol{a}^i(\boldsymbol{x}) = u_i^c(\boldsymbol{x})\boldsymbol{e}_i, \qquad \text{with} \quad u_i = \boldsymbol{u} \cdot \boldsymbol{a}_i \quad \text{and} \quad u_i^c = \boldsymbol{u} \cdot \boldsymbol{e}_i,$$

and

$$\boldsymbol{r}(\boldsymbol{x}) = r_\alpha(\boldsymbol{x})\boldsymbol{a}^\alpha(\boldsymbol{x}) = r_i^c(\boldsymbol{x})\boldsymbol{e}_i \qquad \text{with} \quad r_\alpha = \boldsymbol{r} \cdot \boldsymbol{a}_\alpha \quad \text{and} \quad r_i^c = \boldsymbol{r} \cdot \boldsymbol{e}_i,$$

Note that the tangency requirement is easily expressed in covariant coordinates, as it simply reads $r_3 = 0$, whereas it becomes

$$r_i^c(\boldsymbol{x}) a_{3,i}^c(\boldsymbol{x}) = 0 \quad \text{in } \omega, \tag{2.1}$$

in Cartesian coordinates.

Let $a^{\alpha\beta\rho\sigma}$ denote the coefficients of the elasticity tensor. In the case of homogeneous, isotropic material with Young modulus $E > 0$ and Poisson coefficient ν, $0 \leq \nu < \frac{1}{2}$, these coefficients are given by

$$a^{\alpha\beta\rho\sigma} = \frac{E}{2(1+\nu)}(a^{\alpha\rho}a^{\beta\sigma} + a^{\alpha\sigma}a^{\beta\rho}) + \frac{E\nu}{1-\nu^2}a^{\alpha\beta}a^{\rho\sigma}, \tag{2.2}$$

where $a^{\alpha\beta} = \boldsymbol{a}^\alpha \cdot \boldsymbol{a}^\beta$ are the contravariant components of the first fundamental form. We note that each coefficient of this tensor belongs to $L^\infty(\omega)$. Moreover, it satisfies the usual symmetry properties

$$a^{\alpha\beta\rho\sigma}(\boldsymbol{x}) = a^{\rho\sigma\alpha\beta}(\boldsymbol{x}) = a^{\beta\alpha\rho\sigma}(\boldsymbol{x}), \qquad \text{for a.e. } \boldsymbol{x} \in \omega, \tag{2.3}$$

and is uniformly strictly positive: There exists a positive constant c_0 such that, for all symmetric tensors $\boldsymbol{\tau} = (\tau_{\alpha\beta})$ in $\mathbb{R}^{2\times 2}$,

$$a^{\alpha\beta\rho\sigma}(\boldsymbol{x})\tau_{\alpha\beta}\tau_{\rho\sigma} \geq c_0\, |\boldsymbol{\tau}|^2 \qquad \text{for a.e. } \boldsymbol{x} \in \omega. \tag{2.4}$$

In this context, the covariant components of the change of metric tensor read

$$\gamma_{\alpha\beta}(\boldsymbol{u}) = \frac{1}{2}(\partial_\alpha \boldsymbol{u} \cdot \boldsymbol{a}_\beta + \partial_\beta \boldsymbol{u} \cdot \boldsymbol{a}_\alpha), \tag{2.5}$$

the covariant components of the change of transverse shear tensor read

$$\delta_{\alpha 3}(\boldsymbol{u}, \boldsymbol{r}) = \frac{1}{2}(\partial_\alpha \boldsymbol{u} \cdot \boldsymbol{a}_3 + \boldsymbol{r} \cdot \boldsymbol{a}_\alpha), \tag{2.6}$$

and the covariant components of the change of curvature tensor read

$$\chi_{\alpha\beta}(\boldsymbol{u}, \boldsymbol{r}) = \frac{1}{2}(\partial_\alpha \boldsymbol{u} \cdot \partial_\beta \boldsymbol{a}_3 + \partial_\beta \boldsymbol{u} \cdot \partial_\alpha \boldsymbol{a}_3 + \partial_\alpha \boldsymbol{r} \cdot \boldsymbol{a}_\beta + \partial_\beta \boldsymbol{r} \cdot \boldsymbol{a}_\alpha), \tag{2.7}$$

see [7] and [11]. Note that all these quantities make sense for shells with little regularity, and are easily expressed with the Cartesian coordinates of the unknowns and geometrical data. For instance, we have

$$\partial_\alpha \boldsymbol{u} \cdot \boldsymbol{a}_\beta = (\partial_\alpha u_i^c)\, a_{\beta,i}^c,$$

and so on.

We assume that the boundary $\partial\omega$ of the chart domain is divided into two parts: γ_0 which has a strictly positive 1-dimensional measure and on which the shell is clamped and the complementary part $\gamma_1 = \partial\omega \setminus \gamma_0$ on which the shell is subjected to applied tractions and moments. We also assume that $\partial\gamma_0 = \partial\gamma_1$ is a Lipschitz–continuous submanifold of $\partial\omega$.

To take into account the boundary conditions, we define the space

$$H^1_{\gamma_0}(\omega) = \big\{\mu \in H^1(\omega);\ \mu = 0 \text{ on } \gamma_0\big\}. \tag{2.8}$$

We also denote by $H^{\frac{1}{2}}_{00}(\gamma_1)$ the space of functions in $H^{\frac{1}{2}}(\gamma_1)$ such that their extension by zero to $\partial\omega$ belongs to $H^{\frac{1}{2}}(\partial\omega)$, see [21, Chap. 1, §11]. Let us now consider the function space, introduced in [7] and [11], which is appropriate in the context of shells with little regularity

$$\mathbb{V}(\omega) = \big\{(\boldsymbol{v}, \boldsymbol{s}) \in H^1_{\gamma_0}(\omega)^3 \times H^1_{\gamma_0}(\omega)^3;\ \boldsymbol{s} \cdot \boldsymbol{a}_3 = 0 \text{ in } \omega\big\}. \tag{2.9}$$

This space is endowed with the natural Hilbert norm

$$\|(\boldsymbol{v},\boldsymbol{s})\|_{\mathbb{V}(\omega)} = \big(\|\boldsymbol{v}\|^2_{H^1(\omega)^3} + \|\boldsymbol{s}\|^2_{H^1(\omega)^3}\big)^{1/2}. \tag{2.10}$$

We now recall the variational formulation of the problem corresponding to the linear Nagdhi model for shells with little regularity. It reads

Find $(\boldsymbol{u},\boldsymbol{r})$ *in* $\mathbb{V}(\omega)$ *such that*

$$\forall (\boldsymbol{v},\boldsymbol{s}) \in \mathbb{V}(\omega), \quad a\big((\boldsymbol{u},\boldsymbol{r});(\boldsymbol{v},\boldsymbol{s})\big) = \mathcal{L}((\boldsymbol{v},\boldsymbol{s})), \tag{2.11}$$

where the bilinear form $a(\cdot;\cdot)$ is defined by

$$\begin{aligned} a\big((\boldsymbol{u},\boldsymbol{r});(\boldsymbol{v},\boldsymbol{s})\big) = \int_\omega \Big\{ e a^{\alpha\beta\rho\sigma} \Big[\gamma_{\alpha\beta}(\boldsymbol{u})\gamma_{\rho\sigma}(\boldsymbol{v}) + \frac{e^2}{12}\chi_{\alpha\beta}(\boldsymbol{u},\boldsymbol{r})\chi_{\rho\sigma}(\boldsymbol{v},\boldsymbol{s}) \Big] \\ + 2e\frac{E}{1+\nu} a^{\alpha\beta}\delta_{\alpha 3}(\boldsymbol{u},\boldsymbol{r})\delta_{\beta 3}(\boldsymbol{v},\boldsymbol{s}) \Big\} \sqrt{a}\, d\boldsymbol{x}, \end{aligned} \tag{2.12}$$

and the linear form $\mathcal{L}(\cdot)$ is given by

$$\mathcal{L}((\boldsymbol{v},\boldsymbol{s})) = \int_\omega \boldsymbol{f}\cdot\boldsymbol{v}\,\sqrt{a}\,d\boldsymbol{x} + \int_{\gamma_1} (\boldsymbol{N}\cdot\boldsymbol{v} + \boldsymbol{M}\cdot\boldsymbol{s})\,d\tau. \tag{2.13}$$

The three terms in $a(\cdot,\cdot)$ represent the membrane, bending and shear deformations, respectively. The data $\boldsymbol{f}$, $\boldsymbol{M}$ and $\boldsymbol{N}$ represent a given resultant force density, an applied traction density and an applied moment density, respectively. Finally the thickness of the shell which is assumed to be constant is denoted by e, $0 < e < 1$.

We refer to [7] and [11] for the proof of the well-posedness of this problem which is stated in the next theorem. Since $\boldsymbol{a}_3$ belongs to $W^{1,\infty}(\omega)^3$, the form $a(\cdot;\cdot)$ is obviously continuous on $\mathbb{V}(\omega)\times\mathbb{V}(\omega)$, with norm smaller than $c\,e$. Similarly, the form $\mathcal{L}$ is continuous on $\mathbb{V}(\omega)$ and its norm satisfies, with obvious notation,

$$\|\mathcal{L}\| \le c\,\big(\|\boldsymbol{f}\|_{H^1_{\gamma_0}(\omega)^{3\prime}} + \|\boldsymbol{N}\|_{H^{\frac12}_{00}(\gamma_1)^{3\prime}} + \|\boldsymbol{M}\|_{H^{\frac12}_{00}(\gamma_1)^{3\prime}}\big). \tag{2.14}$$

So the well-posedness mainly relies on the following ellipticity property. In view of their discrete analogues, we recall from [11] the main arguments for its proof:

- It follows from the properties of the coefficients $a^{\alpha\beta\rho\sigma}$ and $a^{\alpha\beta}$ that

$$\forall (\boldsymbol{v},\boldsymbol{s}) \in \mathbb{X}(\omega), \quad a\big((\boldsymbol{v},\boldsymbol{s});(\boldsymbol{v},\boldsymbol{s})\big) \ge c\,e^3\,[(\boldsymbol{v},\boldsymbol{s})]^2, \tag{2.15}$$

where the quantity $[\cdot]$ is defined by

$$\begin{aligned} [(\boldsymbol{v},\boldsymbol{s})] = \Big(\sum_{\alpha=1}^2\sum_{\beta=1}^2 \|\gamma_{\alpha\beta}(\boldsymbol{v})\|^2_{L^2(\omega)^{2\times 2}} + \sum_{\alpha=1}^2\sum_{\beta=1}^2 \|\chi_{\alpha\beta}(\boldsymbol{v},\boldsymbol{s})\|^2_{L^2(\omega)^{2\times 2}} \\ + \sum_{\alpha=1}^2 \|\delta_{\alpha 3}(\boldsymbol{v},\boldsymbol{s})\|^2_{L^2(\omega)^2}\Big)^{\frac12}. \end{aligned} \tag{2.16}$$

• If a pair $(\boldsymbol{v}, \boldsymbol{s})$ belongs to $H^1(\omega)^3 \times H^1(\omega)^3$ and satisfies $\boldsymbol{s} \cdot \boldsymbol{a}_3 = 0$, the condition $[(\boldsymbol{v}, \boldsymbol{s})] = 0$ implies that there exist two constants $\boldsymbol{\psi}$ and $\boldsymbol{\mu}$ in $\mathbb{R}^3$ such that, for a.e. $\boldsymbol{x}$ in ω,

$$\boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{\psi} \wedge \boldsymbol{\varphi}(\boldsymbol{x}) + \boldsymbol{\mu}, \qquad \boldsymbol{s}(\boldsymbol{x}) = \frac{1}{\sqrt{a^*(\boldsymbol{x})}} \big((\boldsymbol{\psi} \cdot \boldsymbol{a}^2)(\boldsymbol{x}) \boldsymbol{a}^1(\boldsymbol{x}) - (\boldsymbol{\psi} \cdot \boldsymbol{a}^1)(\boldsymbol{x}) \boldsymbol{a}^2(\boldsymbol{x}) \big), \tag{2.17}$$

where $a^*(\boldsymbol{x})$ stands for the quantity $|\boldsymbol{a}^1(\boldsymbol{x}) \wedge \boldsymbol{a}^2(\boldsymbol{x})|^2$. Then, it follows from the nullity conditions on γ_0 that any pair $(\boldsymbol{v}, \boldsymbol{s})$ in $\mathbb{V}(\omega)$ such that $[(\boldsymbol{v}, \boldsymbol{s})] = 0$ is zero.
• Thanks to the regularity assumptions on the $\boldsymbol{a}_k$, the norm $[\cdot]$ on $\mathbb{V}(\omega)$ is clearly smaller than $c \| \cdot \|_{\mathbb{V}(\omega)}$. The equivalence between these two norms follows by contradiction, see [11, Lemma 3.6]: There exists a constant $c_1 > 0$ such that

$$\forall (\boldsymbol{v}, \boldsymbol{s}) \in \mathbb{V}(\omega), \quad [(\boldsymbol{v}, \boldsymbol{s})] \geq c_1 \, \|(\boldsymbol{v}, \boldsymbol{s})\|_{\mathbb{V}(\omega)}. \tag{2.18}$$

Proposition 2.1. *There exists a constant $c_* > 0$ such that*

$$\forall (\boldsymbol{v}, \boldsymbol{s}) \in \mathbb{V}(\omega), \quad a\big((\boldsymbol{v}, \boldsymbol{s}); (\boldsymbol{v}, \boldsymbol{s})\big) \geq c_* \, e^3 \, \|(\boldsymbol{v}, \boldsymbol{s})\|^2_{\mathbb{V}(\omega)}. \tag{2.19}$$

Theorem 2.2. *For any data $(\boldsymbol{f}, \boldsymbol{N}, \boldsymbol{M})$ in $H^1_{\gamma_0}(\omega)^{3\prime} \times H^{\frac{1}{2}}_{00}(\gamma_1)^{3\prime} \times H^{\frac{1}{2}}_{00}(\gamma_1)^{3\prime}$, problem* (2.11) *admits a unique solution $(\boldsymbol{u}, \boldsymbol{r})$ in $\mathbb{V}(\omega)$. Moreover this solution satisfies*

$$\|(\boldsymbol{u}, \boldsymbol{r})\|_{\mathbb{V}(\omega)} \leq c \, e^{-3} \, \|\mathcal{L}\|. \tag{2.20}$$

Remark. It follows from the asymptotic analysis performed in [23] (see also [8]) that the norm $\|\mathcal{L}\|$, more precisely the right-hand side of (2.14), behaves like $c\,e$ in the case of membrane dominated shells and like $c\,e^3$ in the case of bending dominated shells. So the right-hand side of estimate (2.20) does not necessarily tend to $+\infty$ when e tends to zero.

However, since the purpose of the present work is to approximate the solution of problem (2.11) with a spectral method and to proceed in the simplest possible way, we immediately encounter a problem: The tangency constraint $\boldsymbol{s} \cdot \boldsymbol{a}_3 = 0$ which appears in the definition of $\mathbb{V}(\omega)$ clearly cannot be implemented in a standard way for a general shell. So the idea, already proposed in [9], consists in handling this constraint via the introduction of a Lagrange multiplier. We thus introduce a mixed Naghdi problem in which the unknowns are the displacement $\boldsymbol{u}$ and the rotation $\boldsymbol{r}$ which belong to $H^1_{\gamma_0}(\omega)^3$ without any orthogonality constraint on $\boldsymbol{r}$, and the Lagrange multiplier λ which belongs to the space $H^1_{\gamma_0}(\omega)$ and is aimed to enforce the tangency constraint $\boldsymbol{r} \cdot \boldsymbol{a}_3 = 0$. In view of the discretization, we also possibly add a stabilizing term. Its usefulness appears in the next section.

Let us introduce the relaxed function space

$$\mathbb{X}(\omega) = H^1_{\gamma_0}(\omega)^3 \times H^1_{\gamma_0}(\omega)^3, \tag{2.21}$$

still equipped with the norm defined in (2.10) which is now denoted by $\|\cdot\|_{\mathbb{X}(\omega)}$. We also set $\mathbb{M}(\omega) = H^1_{\gamma_0}(\omega)$. For simplicity, we use an extension of the forms $a(\cdot;\cdot)$ and $\mathcal{L}(\cdot)$ defined in (2.12) and (2.13), respectively, to $\mathbb{X}(\omega)\times\mathbb{X}(\omega)$ and $\mathbb{X}(\omega)$, with the notation

$$\forall U = (\boldsymbol{u},\boldsymbol{r}) \in \mathbb{X}(\omega), \forall V = (\boldsymbol{v},\boldsymbol{s}) \in \mathbb{X}(\omega),\qquad a(U;V) = a\big((\boldsymbol{u},\boldsymbol{r});(\boldsymbol{v},\boldsymbol{s})\big) \quad\text{and}\quad \mathcal{L}(V) = \mathcal{L}((\boldsymbol{v},\boldsymbol{s})).$$

For a nonnegative parameter η, we consider the variational problem

Find (U,ψ) in $\mathbb{X}(\omega)\times\mathbb{M}(\omega)$ such that

$$\begin{aligned} &\forall V \in \mathbb{X}(\omega), \quad a(U;V) + \eta\,\tilde{a}(U;V) + b(V;\psi) = \mathcal{L}(V),\\ &\forall \chi \in \mathbb{M}(\omega), \quad b(U;\chi) = 0, \end{aligned} \tag{2.22}$$

where the bilinear forms $\tilde{a}(\cdot;\cdot)$ and $b(\cdot;\cdot)$ are defined by

$$\tilde{a}(U;V) = \int_\omega \partial_\alpha(\boldsymbol{r}\cdot\boldsymbol{a}_3)\partial_\alpha(\boldsymbol{s}\cdot\boldsymbol{a}_3)\,d\boldsymbol{x}, \tag{2.23}$$

and

$$b(V;\chi) = \int_\omega \partial_\alpha(\boldsymbol{s}\cdot\boldsymbol{a}_3)\partial_\alpha\chi\,d\boldsymbol{x}. \tag{2.24}$$

It must be observed that, since $\boldsymbol{a}_3$ belongs to $W^{1,\infty}(\omega)^3$, the forms $\tilde{a}(\cdot;\cdot)$ and $b(\cdot;\cdot)$ are continuous on $\mathbb{X}(\omega)\times\mathbb{X}(\omega)$ and $\mathbb{X}(\omega)\times\mathbb{M}(\omega)$, respectively. Moreover, the following identity is readily checked

$$\mathbb{V}(\omega) = \big\{V = (\boldsymbol{v},\boldsymbol{s}) \in \mathbb{X}(\omega);\ \forall\chi\in\mathbb{M}(\omega),\ b(V;\chi) = 0\big\}. \tag{2.25}$$

The next ellipticity property

$$\forall V \in \mathbb{V}(\omega), \quad a(V;V) + \eta\,\tilde{a}(V;V) \geq c_*\,e^3\,\|V\|^2_{\mathbb{X}(\omega)}, \tag{2.26}$$

is an obvious consequence of (2.19), and it can be checked thanks to exactly the same argument as in [9] that an analogous property still holds for all V in $\mathbb{X}(\omega)$ whenever η is positive. We now investigate the inf-sup condition on the form $b(\cdot;\cdot)$.

Proposition 2.3. *There exists a positive constant $c_\sharp$ such that the following inf-sup condition holds*

$$\forall\chi\in\mathbb{M}(\omega), \quad \sup_{V\in\mathbb{X}(\omega)} \frac{b(V;\chi)}{\|V\|_{\mathbb{X}(\omega)}} \geq c_\sharp\,\|\chi\|_{H^1(\omega)}. \tag{2.27}$$

Proof: Let χ be an arbitrary element of $\mathbb{M}(\omega)$. Since χ vanishes on γ_0 and $\boldsymbol{a}_3$ belongs to $W^{1,\infty}(\omega)^3$, it is readily checked that $V = (\boldsymbol{0},\chi\boldsymbol{a}_3)$ belongs to $\mathbb{X}(\omega)$. Using the fact that $\chi\,\boldsymbol{a}_3\cdot\boldsymbol{a}_3$ is equal to χ, we have with this choice of V

$$b(V;\chi) \geq |\chi|^2_{H^1(\omega)},$$

so that, thanks to a generalized Poincaré–Friedrichs inequality,

$$b(V;\chi) \geq c\,\|\chi\|^2_{H^1(\omega)}.$$

On the other hand, we observe that, owing to the regularity of $\boldsymbol{a}_3$,

$$\|V\|_{\mathbb{X}(\omega)} \leq \|\chi\boldsymbol{a}_3\|_{H^1(\omega)^3} \leq c\,\|\chi\|_{H^1(\omega)}.$$

Combining the last two inequalities gives the desired inf-sup condition.

The next corollary is a direct consequence of Proposition 2.3.

Corollary 2.4. *When η is equal to zero, problems* (2.11) *and* (2.22) *are fully equivalent, in the sense that*
(i) for any solution (U,ψ) of problem (2.22), *$U = (\boldsymbol{u},\boldsymbol{r})$ is a solution of problem* (2.11);
(ii) for any solution $U = (\boldsymbol{u},\boldsymbol{r})$ of problem (2.11), *there exists a unique ψ in $\mathbb{M}(\omega)$ such that (U,ψ) is a solution of problem* (2.22).

The next theorem is an immediate consequence of properties (2.26) and (2.27), see [15, Chap. I, Cor. 4.1] for instance.

Theorem 2.5. *For any data $(\boldsymbol{f},\boldsymbol{N},\boldsymbol{M})$ in $H^1_{\gamma_0}(\omega)^{3\prime} \times H^{\frac{1}{2}}_{00}(\gamma_1)^{3\prime} \times H^{\frac{1}{2}}_{00}(\gamma_1)^{3\prime}$, problem* (2.22) *admits a unique solution (U,ψ) in $\mathbb{X}(\omega)\times\mathbb{M}(\omega)$. Moreover this solution satisfies*

$$\|U\|_{\mathbb{X}(\omega)} + \|\psi\|_{H^1(\omega)} \leq c\,e^{-3}\,\|\mathcal{L}\|. \tag{2.28}$$

Remark. Since we are aiming for simplicity of implementation, we have made no attempt to make the duality term intrinsic. In fact, it does depend on the chart, whereas the other terms do not. This could arguably be considered to be a poor choice, especially if a chart was used that gave much more weight to one part of the shell compared to the rest. An intrinsic choice that obviously works is

$$\tilde{b}(V;\chi) = \int_\omega a^{\alpha\beta}\partial_\alpha(\boldsymbol{s}\cdot\boldsymbol{a}_3)\partial_\beta\chi\,\sqrt{a}\,d\boldsymbol{x}.$$

To conclude, we check the consistency of the stabilizing term. For a while, we denote by (U^η,ψ^η) the solution of problem (2.22) for a fixed value of η and by (U,ψ) the solution of this same problem for $\eta = 0$.

Proposition 2.6. *For any data $(\boldsymbol{f},\boldsymbol{N},\boldsymbol{M})$ in $\in H^1_{\gamma_0}(\omega)^{3\prime} \times H^{\frac{1}{2}}_{00}(\gamma_1)^{3\prime} \times H^{\frac{1}{2}}_{00}(\gamma_1)^{3\prime}$, the following convergence property holds*

$$\lim_{\eta\to 0}\Big(\|U^\eta - U\|_{\mathbb{X}(\omega)} + \|\psi^\eta - \psi\|_{\mathbb{M}(\omega)}\Big) = 0. \tag{2.29}$$

Proof: Since both constants c_* in (2.26) and $c_\sharp$ in (2.27) are independent of η, standard arguments yield the following bound for the solution (U^η, ψ^η)

$$\|U^\eta\|_{\mathbb{X}(\omega)} + \|\psi^\eta\|_{\mathbb{M}(\omega)} \le c\, e^{-3}\, \|\mathcal{L}\|.$$

On the other hand the pair $(U^\eta - U, \psi^\eta - \psi)$ belongs to $\mathbb{X}(\omega) \times \mathbb{M}(\omega)$ and satisfies

$$\begin{aligned} &\forall V \in \mathbb{X}(\omega), \quad a(U^\eta - U; V) + b(V; \psi^\eta - \psi) = -\eta\, \tilde{a}(U^\eta; V), \\ &\forall \chi \in \mathbb{M}(\omega), \quad b(U^\eta - U; \chi) = 0, \end{aligned}$$

Using again (2.26) and (2.27) combined with the previous estimate leads to

$$\|U^\eta - U\|_{\mathbb{X}(\omega)} + \|\psi^\eta - \psi\|_{\mathbb{M}(\omega)} \le c\,\eta\, e^{-3}\, \|\mathcal{L}\|, \tag{2.30}$$

whence the convergence property.

Note that estimate (2.30) provides an explicit estimate of the error issued from the addition of a stabilizing term.

3. The spectral discrete problem and its well-posedness.

To describe the discrete problem, we now assume that ω is the square $]-1,1[^2$ (this can induce a further diffeomorphism, however for simplicity we keep the notation $\boldsymbol{\varphi}$ for the chart). We assume that γ_0 is the union of one, two, three or four whole edges of ω.

For each nonnegative integer n, we denote by $\mathbb{P}_n(\omega)$ the space of restrictions to ω of polynomials with two variables and degree $\leq n$ with respect to each variable. In order to take into account the boundary conditions of the problem, we introduce the space $\mathbb{P}_n^{\gamma_0}(\omega) = \mathbb{P}_n(\omega) \cap H^1_{\gamma_0}(\omega)$. Next, for a fixed integer $N \geq 2$ and another integer L, $2 \leq L \leq N$, we define the discrete spaces

$$\mathbb{X}_N = \mathbb{P}_N^{\gamma_0}(\omega)^3 \times \mathbb{P}_N^{\gamma_0}(\omega)^3, \qquad \mathbb{M}_N = \mathbb{P}_L^{\gamma_0}(\omega). \tag{3.1}$$

The reason for using two different parameters L and N is explained later on.

We also make use of the Gauss-Lobatto formula on the interval $]-1,1[$. Let $\mathbb{P}_n(-1,1)$ denote the space of restrictions to $]-1,1[$ of polynomials with degree $\leq n$. For a third integer $M \geq N$, we set: $\xi_0 = -1$ and $\xi_M = 1$. We recall that there exists $M-1$ nodes ξ_j, $1 \leq j \leq M-1$, in $]-1,1[$, with $\xi_0 < \xi_1 < \ldots < \xi_M$, and $M+1$ positive weights ρ_j, $0 \leq j \leq M$, such that

$$\forall \Phi \in \mathbb{P}_{2M-1}(-1,1), \quad \int_{-1}^{1} \Phi(\zeta)\, d\zeta = \sum_{j=0}^{M} \Phi(\xi_j)\, \rho_j. \tag{3.2}$$

Moreover the following property holds [4, form. (13.20)]

$$\forall \varphi \in \mathbb{P}_M(-1,1), \quad \|\varphi\|^2_{L^2(-1,1)} \leq \sum_{j=0}^{M} \varphi^2(\xi_j)\, \rho_j \leq 3\, \|\varphi\|^2_{L^2(-1,1)}. \tag{3.3}$$

The interest of using "over-integration", i.e., taking $M > N$, in the case of non constant coefficients has been fully brought to light in [22].

This leads to define a discrete product on ω: For any continuous functions u and v on $\bar{\omega}$,

$$(u,v)_M = \sum_{i=0}^{M} \sum_{j=0}^{M} u(\xi_i,\xi_j) v(\xi_i,\xi_j)\, \rho_i \rho_j. \tag{3.4}$$

It follows from (3.3) that this product is a scalar product on $\mathbb{P}_M(\omega)$. We also introduce the Lagrange interpolation operator $\mathcal{I}_M$ at the nodes (ξ_i,ξ_j), $0 \leq i,j \leq M$, with values in $\mathbb{P}_M(\omega)$. Finally, the discrete product $(\cdot,\cdot)_M^{\gamma_1}$ is defined according to the geometry of γ_1: For any continuous functions u and v on $\bar{\gamma}_1$, if γ_1 is the edge $\{-1\}\times]-1,1[$,

$$(u,v)_M^{\gamma_1} = \sum_{j=0}^{M} u(-1,\xi_j) v(-1,\xi_j)\, \rho_j, \tag{3.5}$$

while if γ_1 is the union of the two edges $\{-1\}\times]-1,1]$ and $[-1,1[\times\{1\}$,

$$(u,v)_M^{\gamma_1} = \sum_{j=0}^{M} u(-1,\xi_j)v(-1,\xi_j)\,\rho_j + \sum_{j=0}^{M} u(\xi_j,1)v(\xi_j,1)\,\rho_j, \tag{3.6}$$

and so on. The Lagrange interpolation operator $i_M^{\gamma_1}$ is simply defined as the trace of $\mathcal{I}_M$ on γ_1.

From now on, we make the further non restrictive assumption that the $\boldsymbol{a}_\alpha$ belong to $H^{s_0}(\omega)^3$ and that $\boldsymbol{a}_3$ belongs to $H^{s_0+1}(\omega)^3$ for a real number $s_0 > 1$. Moreover, in order to take into account this rather weak regularity, we introduce the $H^1(\omega)$-projection $\boldsymbol{a}_{kN}$ of each $\boldsymbol{a}_k$ onto $\mathbb{P}_N(\omega)^3$, which satisfies

$$\begin{aligned} &\forall \boldsymbol{v}_N \in \mathbb{P}_N(\omega)^3, \qquad (\partial_\alpha \boldsymbol{a}_{kN}, \partial_\alpha \boldsymbol{v}_N)_M = \int_\omega (\partial_\alpha \boldsymbol{a}_k)(\partial_\alpha \boldsymbol{v}_N)\,d\boldsymbol{x} \\ &(\boldsymbol{a}_{kN}, 1)_M = \int_\omega \boldsymbol{a}_k(\boldsymbol{x})\,d\boldsymbol{x}. \end{aligned} \tag{3.7}$$

Note that, in the previous problem, $(\cdot,\cdot)_M$ can be replaced by $(\cdot,\cdot)_N$ if preferred without any modification on the properties of $\boldsymbol{a}_{kN}$. In a similar way, we define the $\boldsymbol{c}_{\alpha N}$ as the solution of the same problem with $\boldsymbol{a}_k$ replaced by $\partial_\alpha \boldsymbol{a}_3$. It follows from (3.3) that the $\boldsymbol{a}_{kN}$ and $\boldsymbol{c}_{\alpha N}$ are uniquely defined from these equations. This leads to the following discrete forms of the tensors

$$\gamma_{\alpha\beta}^N(\boldsymbol{u}) = \frac{1}{2}(\partial_\alpha \boldsymbol{u}\cdot \boldsymbol{a}_{\beta N} + \partial_\beta \boldsymbol{u}\cdot \boldsymbol{a}_{\alpha N}), \tag{3.8}$$

$$\delta_{\alpha 3}^N(U) = \frac{1}{2}(\partial_\alpha \boldsymbol{u}\cdot \boldsymbol{a}_{3N} + \boldsymbol{r}\cdot \boldsymbol{a}_{\alpha N}), \tag{3.9}$$

$$\chi_{\alpha\beta}^N(U) = \frac{1}{2}(\partial_\alpha \boldsymbol{u}\cdot \boldsymbol{c}_{\beta N} + \partial_\beta \boldsymbol{u}\cdot \boldsymbol{c}_{\alpha N} + \partial_\alpha \boldsymbol{r}\cdot \boldsymbol{a}_{\beta N} + \partial_\beta \boldsymbol{r}\cdot \boldsymbol{a}_{\alpha N}). \tag{3.10}$$

Up to the replacement of the $\boldsymbol{a}_k$ by $\boldsymbol{a}_{kN}$ and also of the $\partial_\alpha \boldsymbol{a}_3$ by $\boldsymbol{c}_{\alpha N}$, the discrete problem is now constructed from (2.22) by the Galerkin method with numerical integration. It reads

Find (U_N, ψ_N) *in* $\mathbb{X}_N \times \mathbb{M}_N$ *such that*

$$\begin{aligned} &\forall V_N \in \mathbb{X}_N, \quad a_M(U_N;V_N) + \eta\,\tilde{a}_M(U_N;V_N) + b_M(V_N;\psi_N) = \mathcal{L}_M(V_N), \\ &\forall \chi_N \in \mathbb{M}_N, \quad b_M(U_N;\chi_N) = 0, \end{aligned} \tag{3.11}$$

where the bilinear forms $a_M(\cdot;\cdot)$, $\tilde{a}_M(\cdot;\cdot)$ and $b_M(\cdot;\cdot)$ are defined, with the notation $U_N = (\boldsymbol{u}_N, \boldsymbol{r}_N)$ and $V_N = (\boldsymbol{v}_N, \boldsymbol{s}_N)$ and obvious extension of the discrete products to vector-valued functions, by

$$\begin{aligned} a_M(U_N;V_N) =\,& e\,\big(a^{\alpha\beta\rho\sigma}\,\gamma_{\alpha\beta}^N(\boldsymbol{u}_N), \gamma_{\rho\sigma}^N(\boldsymbol{v}_N)\sqrt{a}\big)_M \\ &+ \frac{e^3}{12}\,\big(a^{\alpha\beta\rho\sigma}\,\chi_{\alpha\beta}^N(U_N), \chi_{\rho\sigma}^N(V_N)\sqrt{a}\big)_M \\ &+ 2e\frac{E}{1+\nu}\,\big(a^{\alpha\beta}\delta_{\alpha 3}^N(U_N), \delta_{\beta 3}^N(V_N)\sqrt{a}\big)_M, \\ \tilde{a}_M(U_N;V_N) =\,& \big(\partial_\alpha \mathcal{I}_M(\boldsymbol{r}_N\cdot \boldsymbol{a}_{3N}), \partial_\alpha \mathcal{I}_M(\boldsymbol{s}_N\cdot \boldsymbol{a}_{3N})\big)_M, \\ b_M(V_N;\chi_N) =\,& \big(\partial_\alpha \mathcal{I}_M(\boldsymbol{s}_N\cdot \boldsymbol{a}_{3N}), \partial_\alpha \chi_N\big)_M. \end{aligned} \tag{3.12}$$

The linear form $\mathcal{L}_M(\cdot)$ is given by

$$\mathcal{L}_M(V_N) = \big(\boldsymbol{f}, \boldsymbol{v}_N \sqrt{a}\big)_M + (\boldsymbol{N}, \boldsymbol{v}_N)_M^{\gamma_1} + (\boldsymbol{M}, \boldsymbol{s}_N)_M^{\gamma_1}. \tag{3.13}$$

Remark. The discrete problem (3.11) differs from the standard spectral discretization of elliptic problems in two ways.

• The idea for the replacement of the $\boldsymbol{a}_\alpha$ by $\boldsymbol{a}_{\alpha N}$ and of the $\partial_\alpha \boldsymbol{a}_3$ by $\boldsymbol{c}_{\alpha N}$ in the definition of the forms $a_M(\cdot,\cdot)$ and $b_M(\cdot,\cdot)$ comes from the lack of regularity of the $\boldsymbol{a}_k$. Indeed, if one of the $\boldsymbol{a}_\alpha$ is not replaced by $\boldsymbol{a}_{\alpha N}$, the continuity of these forms would require the boundedness of $\mathcal{I}_M \boldsymbol{a}_\alpha$ at least in $H^1(\omega)^3$, which would require that $\boldsymbol{a}_\alpha$ belongs to $H^{s_0}(\omega)^3$ for $s_0 > \frac{3}{2}$. We prefer to avoid this assumption.

• It is usual in spectral methods to take M equal to N, in order that the mass matrix is diagonal. The choice of an M possibly larger than N here is due to the fact that the coefficients involved in the previous forms depend on the space variable and are not very smooth (see [22] for more details). However, if ξ_j^* denote the nodes ξ_j for M equal to N and φ_j^* are the associated Lagrange polynomials, the unknown U_N admits the expansion

$$U_N(x,y) = \sum_{i=0}^{N} \sum_{j=0}^{N} U_N(\xi_i^*, \xi_j^*)\, \varphi_i^*(x) \varphi_j^*(y). \tag{3.14}$$

So computing the values of U_N and $\partial_\alpha U_N$ at the nodes (ξ_j, ξ_j) only requires the knowledge of the two matrices made of the $\varphi_i^*(\xi_k)$ and of the $\varphi_i^{*\prime}(\xi_k)$, respectively.

Remark. Note that only the values of the coefficients at the nodes (ξ_i, ξ_j) are involved in the definition of the discrete forms. For instance, if φ_j denote the Lagrange polynomials associated with the nodes ξ_j, we have

$$\mathcal{I}_M(\boldsymbol{s}_N \cdot \boldsymbol{a}_{3N})(x,y) = \sum_{i=0}^{M} \sum_{j=0}^{M} \boldsymbol{s}_N(\xi_i, \xi_j) \cdot \boldsymbol{a}_{3N}(\xi_i, \xi_j)\, \varphi_i(x) \varphi_j(y). \tag{3.15}$$

This makes the implementation of the previous problem rather easy (the values of the $\varphi_i'(\xi_j)$ are given in [6, §V.4] for instance).

The analysis of problem (3.11) relies on a number of properties of the previous forms. We begin with their continuity. In a preliminary step, we establish some results concerning the new coefficients $\boldsymbol{a}_{kN}$ and $\boldsymbol{c}_{\alpha N}$.

Lemma 3.1. *There exists a constant c independent of N such that*

$$\|\boldsymbol{a}_\alpha - \boldsymbol{a}_{\alpha N}\|_{L^\infty(\omega)^3} \le c\, N^{1-s_0} (\log N)^{\frac{1}{2}} \|\boldsymbol{a}_\alpha\|_{H^{s_0}(\omega)^3}. \tag{3.16}$$

There exists a constant c independent of N such that

$$\|\boldsymbol{a}_3 - \boldsymbol{a}_{3N}\|_{L^\infty(\omega)^3} \le c\, N^{-s_0} (\log N)^{\frac{1}{2}} \|\boldsymbol{a}_3\|_{H^{s_0+1}(\omega)^3}, \tag{3.17}$$

and that, for any real number p, $2 \le p \le \frac{2}{2-s_0}$,

$$\|\boldsymbol{a}_3 - \boldsymbol{a}_{3N}\|_{W^{1,p}(\omega)^3} \le c\, N^{4(\frac{1}{2}-\frac{1}{p})-s_0} \|\boldsymbol{a}_3\|_{H^{s_0+1}(\omega)^3}. \tag{3.18}$$

There exists a constant c independent of N such that

$$\|\partial_\alpha \boldsymbol{a}_3 - \boldsymbol{c}_{\alpha N}\|_{L^\infty(\omega)^3} \le c\, N^{1-s_0}\, (\log\, N)^{\frac{1}{2}}\, \|\boldsymbol{a}_3\|_{H^{s_0+1}(\omega)^3}. \tag{3.19}$$

Proof: There exists [4, Thm 7.4] a polynomial $\tilde{\boldsymbol{a}}_{kN}$ in $\mathbb{P}_N(\omega)^3$ such that, for all real numbers s, $0 \le s \le s_0$,

$$\|\boldsymbol{a}_\alpha - \tilde{\boldsymbol{a}}_{\alpha N}\|_{H^s(\omega)^3} \le c\, N^{s-s_0}\, \|\boldsymbol{a}_\alpha\|_{H^{s_0}(\omega)^3}. \tag{3.20}$$

To prove the first estimate, we use the triangle inequality

$$\|\boldsymbol{a}_\alpha - \boldsymbol{a}_{\alpha N}\|_{L^\infty(\omega)^3} \le \|\boldsymbol{a}_\alpha - \tilde{\boldsymbol{a}}_{\alpha N}\|_{L^\infty(\omega)^3} + \|\boldsymbol{a}_{\alpha N} - \tilde{\boldsymbol{a}}_{\alpha N}\|_{L^\infty(\omega)^3}.$$

To bound the first quantity, we recall from [16] that, for all ε, $0 < \varepsilon < s_0 - 1$, the norm of the Sobolev imbedding of $H^{1+\varepsilon}(\omega)$ into $L^\infty(\omega)$ is bounded by $c\,\varepsilon^{-\frac{1}{2}}$ for a constant c independent of ε. Combining this result with (3.20) with $s = 1+\varepsilon$ gives

$$\|\boldsymbol{a}_\alpha - \tilde{\boldsymbol{a}}_{\alpha N}\|_{L^\infty(\omega)^3} \le c\,\varepsilon^{-\frac{1}{2}}\, \|\boldsymbol{a}_\alpha - \tilde{\boldsymbol{a}}_{\alpha N}\|_{H^{1+\varepsilon}(\omega)^3} \le c'\,\varepsilon^{-\frac{1}{2}}\, N^{\varepsilon+1-s_0}\, \|\boldsymbol{a}_\alpha\|_{H^{s_0}(\omega)^3}.$$

Evaluating the second one relies on the inverse inequality (see [3, Chap. III, Prop. 3.1]), valid for any $p < +\infty$,

$$\|\boldsymbol{a}_{\alpha N} - \tilde{\boldsymbol{a}}_{\alpha N}\|_{L^\infty(\omega)^3} \le c\, N^{\frac{4}{p}}\, \|\boldsymbol{a}_{\alpha N} - \tilde{\boldsymbol{a}}_{\alpha N}\|_{L^p(\omega)^3}.$$

We recall from [24] that the norm of the imbedding of $H^1(\omega)$ into $L^p(\omega)$ behaves like $c\,p^{\frac{1}{2}}$ for a constant c independent of p. This yields

$$\begin{aligned}\|\boldsymbol{a}_{\alpha N} - \tilde{\boldsymbol{a}}_{\alpha N}\|_{L^\infty(\omega)^3} &\le c\,p^{\frac{1}{2}}\, N^{\frac{4}{p}}\, \|\boldsymbol{a}_{\alpha N} - \tilde{\boldsymbol{a}}_{\alpha N}\|_{H^1(\omega)^3} \\ &\le c\,p^{\frac{1}{2}}\, N^{\frac{4}{p}}\, \big(\|\boldsymbol{a}_\alpha - \boldsymbol{a}_{\alpha N}\|_{H^1(\omega)^3} + \|\boldsymbol{a}_\alpha - \tilde{\boldsymbol{a}}_{\alpha N}\|_{H^1(\omega)^3}\big).\end{aligned}$$

It follows from the definition of $\boldsymbol{a}_{\alpha N}$ that

$$\|\boldsymbol{a}_\alpha - \boldsymbol{a}_{\alpha N}\|_{H^1(\omega)^3} \le c\, N^{1-s_0}\, \|\boldsymbol{a}_\alpha\|_{H^{s_0}(\omega)^3}. \tag{3.21}$$

Combining this with (3.20) for $s = 1$ thus leads to

$$\|\boldsymbol{a}_{\alpha N} - \tilde{\boldsymbol{a}}_{\alpha N}\|_{L^\infty(\omega)^3} \le c\,p^{\frac{1}{2}}\, N^{\frac{4}{p}+1-s_0}\, \|\boldsymbol{a}_\alpha\|_{H^{s_0}(\omega)^3}.$$

Thus estimate (3.16) follows from the previous lines by taking $\varepsilon = \frac{4}{p} = \frac{1}{\log\, N}$. Estimate (3.17) relies on exactly the same arguments with s_0 replaced by $s_0 + 1$. To prove (3.18), we first take s such that $\frac{1}{p} = \frac{2-s}{2}$, so that $H^s(\omega)$ is imbedded in $W^{1,p}(\omega)$, and use an analogous inverse inequality as previously (see again [3, Chap. III, Prop. 3.1]), which gives

$$\|\boldsymbol{a}_3 - \boldsymbol{a}_{3N}\|_{W^{1,p}(\omega)^3} \le c\, \|\boldsymbol{a}_3 - \tilde{\boldsymbol{a}}_{3N}\|_{H^s(\omega)^3} + c'\, N^{4(\frac{1}{2}-\frac{1}{p})}\, \|\boldsymbol{a}_{3N} - \tilde{\boldsymbol{a}}_{3N}\|_{H^1(\omega)^3}.$$

Thus, we deduce from the analogues of (3.20) and (3.21), with s_0 replaced by s_0+1, that

$$\|\boldsymbol{a}_3 - \boldsymbol{a}_{3N}\|_{W^{1,p}(\omega)^3} \le c\,(N^{s-s_0-1} + N^{4(\frac{1}{2}-\frac{1}{p})-s_0})\|\boldsymbol{a}_3\|_{H^{s_0+1}(\omega)^3}.$$

Thus, noting that $s-s_0-1$ is equal to $2(\frac{1}{2}-\frac{1}{p})-s_0$, hence is smaller than $4(\frac{1}{2}-\frac{1}{p})-s_0$, gives the desired result. The proof of (3.19) relies on exactly the same arguments as for (3.16).

As a consequence of the previous lemma, the norms of the coefficients $\boldsymbol{a}_{kN}$ and $\boldsymbol{c}_{\alpha N}$. in $L^\infty(\omega)^3$ and also of $\boldsymbol{a}_{3N}$ in $W^{1,p}(\omega)^3$ are bounded independently of N. This leads to the following continuity results.

Lemma 3.2. *There exists a constant c independent of N and $M \ge N$ such that the following continuity property holds*

$$\forall U_N \in \mathbb{X}_N, \forall V_N \in \mathbb{X}_N, \quad |a_M(U_N;V_N)| \le c\,e\,\|U_N\|_{\mathbb{X}(\omega)}\|V_N\|_{\mathbb{X}(\omega)}. \tag{3.22}$$

Proof: Since the coefficients $a^{\alpha\beta\rho\sigma}$ and also $\sqrt{a}$ are bounded, we derive by a Cauchy–Schwarz inequality

$$e\,|\big(a^{\alpha\beta\rho\sigma}\,\gamma^N_{\alpha\beta}(\boldsymbol{u}_N),\gamma^N_{\rho\sigma}(\boldsymbol{v}_N)\,\sqrt{a}\big)_M| \le c\,e\,(\gamma^N_{\alpha\beta}(\boldsymbol{u}_N),\gamma^N_{\alpha\beta}(\boldsymbol{u}_N))^{\frac{1}{2}}_M(\gamma^N_{\rho\sigma}(\boldsymbol{v}_N),\gamma^N_{\rho\sigma}(\boldsymbol{v}_N))^{\frac{1}{2}}_M.$$

Thanks to Lemma 3.1, we observe that, at each node (ξ_i,ξ_j), $0 \le i,j \le M$,

$$\gamma^N_{\alpha\beta}(\boldsymbol{u}_N)(\xi_i,\xi_j) \le c\,\big(|(\partial_\alpha u_N)(\xi_i,\xi_j)| + |(\partial_\beta u_N)(\xi_i,\xi_j)|\big),$$

so that

$$(\gamma^N_{\alpha\beta}(\boldsymbol{u}_N),\gamma^N_{\alpha\beta}(\boldsymbol{u}_N))_M \le c'\,\big((\partial_\alpha\boldsymbol{u}_N,\partial_\alpha\boldsymbol{u}_N)_M + (\partial_\beta\boldsymbol{u}_N,\partial_\beta\boldsymbol{u}_N)_M\big).$$

Using (3.3) and a similar estimate for $(\gamma^N_{\rho\sigma}(\boldsymbol{v}_N),\gamma^N_{\rho\sigma}(\boldsymbol{v}_N))_M$ leads to

$$e\,|\big(a^{\alpha\beta\rho\sigma}\,\gamma^N_{\alpha\beta}(\boldsymbol{u}_N),\gamma^N_{\rho\sigma}(\boldsymbol{v}_N)\,\sqrt{a}\big)_M| \le c\,e\,\|\boldsymbol{u}_N\|_{H^1(\omega)^3}\|\boldsymbol{v}_N\|_{H^1(\omega)^3}.$$

Similar arguments also yield

$$\frac{e^3}{12}\,|\big(a^{\alpha\beta\rho\sigma}\,\chi^N_{\alpha\beta}(U_N),\chi^N_{\rho\sigma}(V_N)\,\sqrt{a}\big)_M| \le c\,e^3\,\|U_N\|_{\mathbb{X}(\omega)}\|V_N\|_{\mathbb{X}(\omega)},$$

and also

$$2e\frac{E}{1+\nu}\,|\big(a^{\alpha\beta}\delta^N_{\alpha 3}(U_N)\delta^N_{\rho 3}(V_N)\,\sqrt{a}\big)_M| \le c\,e\,\big(\|\boldsymbol{u}_N\|_{H^1(\omega)^3} + \|\boldsymbol{r}_N\|_{L^2(\omega)^3}\big)\big(\|\boldsymbol{v}_N\|_{H^1(\omega)^3} + \|\boldsymbol{s}_N\|_{L^2(\omega)^3}\big).$$

Property (3.22) is then derived from the last three bounds.

Lemma 3.3. *There exists a constant c independent of N and $M \ge N$ such that the following continuity property holds*

$$\forall U_N \in \mathbb{X}_N, \forall V_N \in \mathbb{X}_N, \quad |\tilde{a}_M(U_N;V_N)| \le c\,\|U_N\|_{\mathbb{X}(\omega)}\|V_N\|_{\mathbb{X}(\omega)}. \tag{3.23}$$

Proof: We derive from (3.3) that

$$|\tilde{a}_M(U_N;V_N)| \le 3\,|\mathcal{I}_M(\boldsymbol{r}_N\cdot\boldsymbol{a}_{3N})|_{H^1(\omega)}|\mathcal{I}_M(\boldsymbol{s}_N\cdot\boldsymbol{a}_{3N})|_{H^1(\omega)}.$$

We recall from [4, form. (13.27) & (13.28)] the following property, valid for all integers K,

$$\forall w_K\in\mathbb{P}_K(\omega),\quad |\mathcal{I}_M w_K|_{H^1(\omega)} \le c\,(1+\frac{K}{M})\,|w_K|_{H^1(\omega)}. \tag{3.24}$$

Since both $\boldsymbol{r}_N\cdot\boldsymbol{a}_{3N}$ and $\boldsymbol{s}_N\cdot\boldsymbol{a}_{3N}$ are polynomials with degree smaller than $2N$, this implies

$$|\tilde{a}_M(U_N;V_N)| \le c\,|\boldsymbol{r}_N\cdot\boldsymbol{a}_{3N}|_{H^1(\omega)}|\boldsymbol{s}_N\cdot\boldsymbol{a}_{3N}|_{H^1(\omega)}.$$

Then, we observe that

$$\partial_\alpha(\boldsymbol{r}_N\cdot\boldsymbol{a}_{3N}) = \partial_\alpha\boldsymbol{r}_N\cdot\boldsymbol{a}_{3N}+\boldsymbol{r}_N\cdot\partial_\alpha\boldsymbol{a}_{3N}.$$

This yields, for the p such that $4(\frac{1}{2}-\frac{1}{p})<s_0$ and q given by $\frac{1}{p}+\frac{1}{q}=\frac{1}{2}$,

$$|\boldsymbol{r}_N\cdot\boldsymbol{a}_{3N}|_{H^1(\omega)} \le |\boldsymbol{r}_N|_{H^1(\omega)^3}\|\boldsymbol{a}_{3N}\|_{L^\infty(\omega)^3}+\|\boldsymbol{r}_N\|_{L^q(\omega)^3}\|\boldsymbol{a}_{3N}\|_{W^{1,p}(\omega)^3}.$$

By combining Lemma 3.1 with the embedding of $H^1(\omega)$ into $L^q(\omega)$, we obtain

$$|\boldsymbol{r}_N\cdot\boldsymbol{a}_{3N}|_{H^1(\omega)} \le c\,\|\boldsymbol{r}_N\|_{H^1(\omega)^3}\|\boldsymbol{a}_3\|_{H^{s_0+1}(\omega)^3}.$$

A similar estimate holds for $|\boldsymbol{s}_N\cdot\boldsymbol{a}_{3N}|_{H^1(\omega)}$, which yields the desired continuity property.

The continuity of $b_M(\cdot;\cdot)$ relies on the same arguments, so that we omit the proof of the next lemma.

Lemma 3.4. *There exists a constant c independent of N and $M\ge N$ such that the following continuity property holds*

$$\forall V_N\in\mathbb{X}_N,\forall\chi_N\in\mathbb{M}_N,\quad |b_M(V_N;\chi_N)|\le c\,\|V_N\|_{\mathbb{X}(\omega)}\|\chi_N\|_{H^1(\omega)}. \tag{3.25}$$

Finally we derive from (3.3) that, if $\|\mathcal{L}_M\|_N$ denotes the norm of the form $\mathcal{L}_M$ in the space of linear forms on $\mathbb{X}_N$,

$$\|\mathcal{L}_M\|_N \le c\,\big(\|\mathcal{I}_M\boldsymbol{f}\|_{L^2(\omega)^3}+\|i_M^{\gamma_1}\boldsymbol{N}\|_{L^2(\gamma_1)^3}+\|i_M^{\gamma_1}\boldsymbol{M}\|_{L^2(\gamma_1)^3}\big). \tag{3.26}$$

To go further, we introduce the kernel

$$\mathbb{V}_N=\big\{V_N=(\boldsymbol{v}_N,\boldsymbol{s}_N)\in\mathbb{X}_N;\ \forall\chi_N\in\mathbb{M}_N,\ b_M(V_N;\chi_N)=0\big\}. \tag{3.27}$$

It is readily checked that $\mathbb{V}_N$ is not contained in $\mathbb{V}(\omega)$ in the general case. So the proof of the next ellipticity property relies on the following result, due to [9, Lemma 3.3] and already hinted in Section 2: For a constant $c_\flat > 0$,

$$\forall V \in \mathbb{X}(\omega), \quad [V]^2 + \tilde{a}(V;V) \geq c_\flat \, \|V\|^2_{\mathbb{X}(\omega)}, \tag{3.28}$$

where the seminorm $[\cdot]$ is defined in (2.16). Let us also consider its discrete analogue on $\mathbb{X}_N$:

$$[V_N]_M = \Big(\big(\gamma^N_{\alpha\beta}(\boldsymbol{v}_N), \gamma^N_{\alpha\beta}(\boldsymbol{v}_N)\big)_M + \big(\chi^N_{\alpha\beta}(\boldsymbol{v}_N, \boldsymbol{s}_N), \chi^N_{\alpha\beta}(\boldsymbol{v}_N, \boldsymbol{s}_N)\big)_M \\ + \big(\delta^N_{\alpha 3}(\boldsymbol{v}_N, \boldsymbol{s}_N), \delta^N_{\alpha 3}(\boldsymbol{v}_N, \boldsymbol{s}_N)\big)_M \Big)^{\frac{1}{2}}. \tag{3.29}$$

From now on, we choose L and M such that, for fixed real numbers λ and μ, $0 < \lambda < 1$ and $0 < \mu \leq 1$,

$$L = \lfloor (1-\lambda)\, N \rfloor \qquad \text{and} \qquad M = \lfloor (1+\mu)\, N \rfloor, \tag{3.30}$$

where $\lfloor \cdot \rfloor$ denotes the integer part.

Proposition 3.5. *There exists a positive integer N_* and a positive constant $\tilde{c}_*$ such that, for all $N \geq N_*$, the following ellipticity property holds*

$$\forall V_N \in \mathbb{X}_N, \quad a_M(V_N;V_N) + \eta\, \tilde{a}_M(V_N;V_N) \geq \tilde{c}_* \, \min\{e^3, \eta\} \, \|V_N\|^2_{\mathbb{X}(\omega)}. \tag{3.31}$$

Proof: It is readily checked from (2.4) that, for all V_N in $\mathbb{X}_N$,

$$a_M(V_N;V_N) + \eta\, \tilde{a}_M(V_N;V_N) \geq \tilde{c}_* \, \min\{e^3, \eta\} \, \Big([V_N]^2_M + \tilde{a}_M(V_N;V_N) \Big). \tag{3.32}$$

On the other hand, since $\mathbb{X}_N$ is contained in $\mathbb{X}(\omega)$, it follows from (3.28) that

$$[V_N]^2 + \tilde{a}(V_N;V_N) \geq c_\flat \, \|V_N\|^2_{\mathbb{X}(\omega)}. \tag{3.33}$$

So it remains to compare $[V_N]$ and $[V_N]_M$ and also $\tilde{a}(V_N, V_N)$ and $\tilde{a}_M(V_N;V_N)$. Let K denote the integer part of $\mu\, N - 1$ (we assume N large enough for K to be positive).
1) Let $\boldsymbol{a}_{\alpha K}$ denote an approximation of $\boldsymbol{a}_\alpha$ in $\mathbb{P}_K(\omega)^3$ which still satisfies (3.16), and $\gamma^K_{\alpha\beta}(\boldsymbol{v}_N)$ be defined as in (3.8) with $\boldsymbol{a}_{\alpha N}$ replaced by $\boldsymbol{a}_{\alpha K}$. It follows from the exactness property (3.2) that

$$\big(\gamma^K_{\alpha\beta}(\boldsymbol{v}_N), \gamma^K_{\alpha\beta}(\boldsymbol{v}_N)\big)_M = \|\gamma^K_{\alpha\beta}(\boldsymbol{v}_N)\|^2_{L^2(\omega)^{2\times 2}}.$$

Then, we use the inequalities

$$\big(\gamma^N_{\alpha\beta}(\boldsymbol{v}_N), \gamma^N_{\alpha\beta}(\boldsymbol{v}_N)\big)_M \geq \big(\gamma^K_{\alpha\beta}(\boldsymbol{v}_N), \gamma^K_{\alpha\beta}(\boldsymbol{v}_N)\big)_M + 2\big((\gamma^N_{\alpha\beta} - \gamma^K_{\alpha\beta})(\boldsymbol{v}_N), \gamma^K_{\alpha\beta}(\boldsymbol{v}_N)\big)_M,$$

and

$$\|\gamma^K_{\alpha\beta}(\boldsymbol{v}_N)\|^2_{L^2(\omega)^{2\times 2}} \geq \|\gamma_{\alpha\beta}(\boldsymbol{v}_N)\|^2_{L^2(\omega)^{2\times 2}} - 2\int_\Omega \big(\gamma_{\alpha\beta}(\boldsymbol{v}_N) - \gamma^K_{\alpha\beta}(\boldsymbol{v}_N)\big)\gamma_{\alpha\beta}(\boldsymbol{v}_N)\, d\boldsymbol{x}.$$

On the other hand, the same arguments as for Lemma 3.2 yield that

$$|\big((\gamma^N_{\alpha\beta}-\gamma^K_{\alpha\beta})(\boldsymbol{v}_N),\gamma^K_{\alpha\beta}(\boldsymbol{v}_N)\big)_M| \le c\sum_{\alpha=1}^{2}\|\boldsymbol{a}_{\alpha N}-\boldsymbol{a}_{\alpha K}\|_{L^\infty(\omega)^3}\,\|\boldsymbol{v}_N\|^2_{H^1(\omega)^3},$$

while it is readily checked that

$$|\int_\Omega \big(\gamma_{\alpha\beta}(\boldsymbol{v}_N)-\gamma^K_{\alpha\beta}(\boldsymbol{v}_N)\big)\gamma_{\alpha\beta}(\boldsymbol{v}_N)\,d\boldsymbol{x}| \le c\sum_{\alpha=1}^{2}\|\boldsymbol{a}_{\alpha}-\boldsymbol{a}_{\alpha K}\|_{L^\infty(\omega)^3}\,\|\boldsymbol{v}_N\|^2_{H^1(\omega)^3}.$$

All this yields

$$\begin{aligned}\big(\gamma^N_{\alpha\beta}(\boldsymbol{v}_N),\gamma^N_{\alpha\beta}(\boldsymbol{v}_N)\big)_M &\ge \|\gamma_{\alpha\beta}(\boldsymbol{v}_N)\|^2_{L^2(\omega)^{2\times 2}}\\ &\quad - c\sum_{\alpha=1}^{2}\big(\|\boldsymbol{a}_\alpha-\boldsymbol{a}_{\alpha N}\|_{L^\infty(\omega)^3}+\|\boldsymbol{a}_\alpha-\boldsymbol{a}_{\alpha K}\|_{L^\infty(\omega)^3}\big)\,\|\boldsymbol{v}_N\|^2_{H^1(\omega)^3}.\end{aligned}$$

Using the same arguments for estimating the two other terms together with Lemma 3.1 leads to

$$[V_N]^2_M \ge [V_N]^2 - c\,N^{1-s_0}\,(\log N)^{\frac12}\,\|V_N\|^2_{\mathbb{X}(\omega)}. \tag{3.34}$$

2) Similarly, let $\boldsymbol{a}_{3K}$ denote an approximation of $\boldsymbol{a}_3$ in $\mathbb{P}_K(\omega)^3$ which still satisfies (3.17). Since $\boldsymbol{s}_N\cdot\boldsymbol{a}_{3K}$ now belongs to $\mathbb{P}_M(\omega)$, it is equal to $\mathcal{I}_M(\boldsymbol{s}_N\cdot\boldsymbol{a}_{3K})$ and, moreover,

$$\big(\partial_\alpha\mathcal{I}_M(\boldsymbol{s}_N\cdot\boldsymbol{a}_{3K}),\partial_\alpha\mathcal{I}_M(\boldsymbol{s}_N\cdot\boldsymbol{a}_{3K})\big)_M = \int_\omega \partial_\alpha(\boldsymbol{s}_N\cdot\boldsymbol{a}_{3K})\partial_\alpha(\boldsymbol{s}_N\cdot\boldsymbol{a}_{3K})\,d\boldsymbol{x}.$$

Thus, the same arguments as for Lemma 3.3 (see (3.24)) yield that

$$\begin{aligned}\tilde a_M(V_N;V_N) &\ge \tilde a(V_N,V_N)\\ &\quad - c\sum_{\alpha=1}^{2}\big(\|\partial_\alpha(\boldsymbol{s}_N\cdot(\boldsymbol{a}_3-\boldsymbol{a}_{3N}))\|_{L^2(\omega)^3}+\|\partial_\alpha(\boldsymbol{s}_N\cdot(\boldsymbol{a}_3-\boldsymbol{a}_{3K}))\|_{L^2(\omega)^3}\big)\,\|\boldsymbol{s}_N\|_{H^1(\omega)^3}.\end{aligned}$$

Using once more Lemma 3.1 (with p such that $4(\frac12-\frac1p)\le 1$) leads to

$$\tilde a_M(V_N;V_N) \ge \tilde a(V_N,V_N) - c\,N^{1-s_0}\,(\log N)^{\frac12}\,\|\boldsymbol{s}_N\|^2_{H^1(\omega)^3}. \tag{3.35}$$

Combining (3.33) to (3.35) gives

$$[V_N]^2_M + \tilde a_M(V_N;V_N) \ge \big(c_\flat - c\,N^{1-s_0}\,(\log N)^{\frac12}\big)\,\|V_N\|^2_{\mathbb{X}(\omega)}.$$

We choose N_* such that $c\,N_*^{1-s_0}\,(\log N_*)^{\frac12}\le\frac{c_\flat}{2}$. Thus, the desired property follows from (3.32).

Proving the inf-sup condition on $b_M(\cdot,\cdot)$ is performed in two steps. Note that it requires the choice of L made in (3.30) (we refer to [5, §3] for proving an inf-sup condition with such a choice in a completely different context).

Lemma 3.6. *There exist a positive integer $N_\sharp$ and a positive constant $\tilde{c}_\sharp$ such that, for all $N \geq N_\sharp$, the following inf-sup condition holds*

$$\forall \chi_N \in \mathbb{M}_N, \quad \sup_{V \in \mathbb{X}_N} \frac{b(V_N; \chi_N)}{\|V_N\|_{\mathbb{X}(\omega)}} \geq \tilde{c}_\sharp \, \|\chi_N\|_{H^1(\omega)}. \tag{3.36}$$

Proof: Let χ_N be any element of $\mathbb{M}_N$. Let now K' stand for the integer part of $\lambda N - 1$. We introduce an approximation $\boldsymbol{a}_{3K'}$ of $\boldsymbol{a}_3$ in $\mathbb{P}_{K'}(\omega)^3$ which satisfies (3.17) and (3.18). It follows from the definition (3.1) of $\mathbb{M}_N$ and the choice (3.30) of L that, for any χ_N in $\mathbb{M}_N$, the function $\boldsymbol{s}_N = \chi_N \, \boldsymbol{a}_{3K'}$ belongs to $\mathbb{P}_N(\omega)^3$. Since χ_N vanishes on γ_0, the function $V_N = (\boldsymbol{0}, \boldsymbol{s}_N)$ belongs to $\mathbb{X}_N$, and satisfies

$$b(V_N; \chi_N) = \int_\omega \partial_\alpha(\chi_N \, \boldsymbol{a}_{3K'} \cdot \boldsymbol{a}_3) \partial_\alpha \chi_N \, d\boldsymbol{x}.$$

Since $\boldsymbol{a}_3 \cdot \boldsymbol{a}_3$ is equal to 1, this gives

$$b(V_N; \chi_N) = |\chi_N|^2_{H^1(\omega)} - \int_\omega \partial_\alpha(\chi_N \, (\boldsymbol{a}_3 - \boldsymbol{a}_{3K'}) \cdot \boldsymbol{a}_3) \partial_\alpha \chi_N \, d\boldsymbol{x}.$$

Combining the Poincaré-Friedrichs inequality with the continuity of $b(\cdot;\cdot)$ yields

$$b(V_N; \chi_N) \geq c \, \|\chi_N\|^2_{H^1(\omega)} - |\chi_N \, (\boldsymbol{a}_3 - \boldsymbol{a}_{3K'}) \cdot \boldsymbol{a}_3)|_{H^1(\omega)} |\chi_N|_{H^1(\omega)}.$$

We have, for an appropriate value of p and with $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$,

$$\begin{aligned} \|\partial_\alpha(\chi_N \, (\boldsymbol{a}_3 - \boldsymbol{a}_{3K'}) \cdot \boldsymbol{a}_3)\|_{L^2(\omega)} \leq{}& |\chi_N|_{H^1(\omega)} \|\boldsymbol{a}_3\|_{L^\infty(\omega)^3} \|\boldsymbol{a}_3 - \boldsymbol{a}_{3K'}\|_{L^\infty(\omega)^3} \\ &+ \|\chi_N\|_{L^2(\omega)} \|\boldsymbol{a}_3\|_{W^{1,\infty}(\omega)^3} \|\boldsymbol{a}_3 - \boldsymbol{a}_{3K'}\|_{L^\infty(\omega)^3} \\ &+ \|\chi_N\|_{L^q(\Omega)} \|\boldsymbol{a}_3\|_{L^\infty(\omega)^3} \|\boldsymbol{a}_3 - \boldsymbol{a}_{3K'}\|_{W^{1,p}(\omega)^3}. \end{aligned}$$

Using the fact [24] that the norm of the imbedding of $H^1(\omega)$ into $L^q(\omega)$ behaves like $c \, q^{\frac{1}{2}}$, combined with (3.17) and (3.18), and taking q equal to $\log N$ lead to

$$b(V_N; \chi_N) \geq \big(c - c' \, N^{-s_0} \, (\log \, N)^{\frac{1}{2}}\big) \, \|\chi_N\|^2_{H^1(\omega)},$$

whence, for N large enough,

$$b(V_N; \chi_N) \geq \frac{c}{2} \, \|\chi_N\|^2_{H^1(\omega)}.$$

On the other hand, usual arguments combined with (3.17) and (3.18) give

$$\|\boldsymbol{s}_N\|_{H^1(\omega)^3} \leq c \, \|\chi_N\|_{H^1(\omega)},$$

which yields the desired inf-sup condition.

Proposition 3.7. *There exist a positive integer* $N_{\sharp\sharp}$ *and a positive constant* $\tilde{c}_{\sharp\sharp}$ *such that, for all* $N \geq N_{\sharp\sharp}$*, the following inf-sup condition holds*

$$\forall \chi_N \in \mathbb{M}_N, \quad \sup_{V \in \mathbb{X}_N} \frac{b_M(V_N; \chi_N)}{\|V_N\|_{\mathbb{X}(\omega)}} \geq \tilde{c}_{\sharp\sharp} \, \|\chi_N\|_{H^1(\omega)}. \tag{3.37}$$

Proof: Thanks to Lemma 3.6, there exists a V_N in $\mathbb{X}_N$ such that

$$b(V_N; \chi_N) \geq c \, \|\chi_N\|^2_{H^1(\omega)} \quad \text{and} \quad \|V_N\|_{\mathbb{X}(\omega)} \leq \tilde{c}_\sharp^{-1} \, \|\chi_N\|_{H^1(\omega)}. \tag{3.38}$$

On the other hand, we have

$$b_M(V_N; \chi_N) = b(V_N; \chi_N) - (b - b_M)(V_N; \chi_N),$$

so that it remains to estimate $(b - b_M)(V_N; \chi_N)$. If K denotes the integer part of $\mu\, N - 1$ and $\boldsymbol{a}_{3K}$ an approximation of $\boldsymbol{a}_3$ in $\mathbb{P}_K(\omega)^3$ which satisfies (3.17) and (3.18), $\mathcal{I}_M(\boldsymbol{s}_N \cdot \alpha_{3K})$ is equal to $\boldsymbol{s}_N \cdot \alpha_{3K}$, whence

$$(\partial_\alpha \mathcal{I}_M(\boldsymbol{s}_N \cdot \boldsymbol{a}_{3K}), \partial_\alpha \chi_N)_M = \int_\omega \partial_\alpha(\boldsymbol{s}_N \cdot \boldsymbol{a}_{3K}) \partial_\alpha \chi_N \, d\boldsymbol{x}. \tag{3.39}$$

Adding and subtracting this quantity and using (3.3) and (3.24) lead to , with $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$,

$$\begin{aligned} |(b - b_M)(V_N; \chi_N)| \leq c \, \|\boldsymbol{s}_N\|_{H^1(\omega)^3} \|\chi_N\|_{H^1(\omega)} & \\ \Big(\|\boldsymbol{a}_3 - \boldsymbol{a}_{3K}\|_{L^\infty(\omega)^3} + q^{\frac{1}{2}} \, \|\boldsymbol{a}_3 - \boldsymbol{a}_{3K}\|_{W^{1,p}(\omega)^3} & \\ + \|\boldsymbol{a}_3 - \boldsymbol{a}_{3N}\|_{L^\infty(\omega)^3} + q^{\frac{1}{2}} \, \|\boldsymbol{a}_3 - \boldsymbol{a}_{3N}\|_{W^{1,p}(\omega)^3} \Big). & \end{aligned}$$

By using (3.17) and (3.18) and taking $q = \log N$, this gives

$$|(b - b_M)(V_N; \chi_N)| \leq c \, N^{-s_0} \, (\log N)^{\frac{1}{2}} \, \|V_N\|_{\mathbb{X}(\omega)} \|\chi_N\|_{H^1(\omega)}. \tag{3.40}$$

Combining this last inequality with (3.38) yields the desired condition for N large enough.

The well-posedness result is now a direct consequence of Propositions 3.5 and 3.7. The stability estimate also requires (3.26).

Theorem 3.8. *There exists a positive integer* N_0 *such that, for any data* $(\boldsymbol{f}, \boldsymbol{N}, \boldsymbol{M})$ *in* $\mathscr{C}^0(\overline{\omega})^3 \times \mathscr{C}^0(\overline{\gamma}_1)^3 \times \mathscr{C}^0(\overline{\gamma}_1)^3$ *and for* $N \geq N_0$*, problem* (3.11) *admits a unique solution* (U_N, ψ_N) *in* $\mathbb{X}_N \times \mathbb{M}_N$*. Moreover this solution satisfies*

$$\|U_N\|_{\mathbb{X}(\omega)} + \|\psi_N\|_{H^1(\omega)} \leq c \, \max\{e^{-3}, \eta^{-1}\} \, \|\mathcal{L}_M\|_N. \tag{3.41}$$

It can be noted that the condition $N \geq N_0$ is not necessary for the existence of a solution to problem (3.11). However it is not restrictive since we intend to work with large values of N.

4. Error estimates.

The error estimate that we now prove is derived from Proposition 3.5 and requires the integer N_* introduced there. Indeed we are not interested in the evaluation of the error concerning the Lagrange multiplier ψ. We first prove the following version of the second Strang's lemma.

Proposition 4.1. *For any integer $N \geq N_*$, the following error estimate holds between the solution (U, ψ) of problem (2.22) and the solution (U_N, ψ_N) of problem (3.11)*

$$\begin{aligned} \|U - U_N\|_{\mathbb{X}(\omega)} \leq c \max\{e^{-3}, \eta^{-1}\} \\ \Bigg(\inf_{W_N \in \mathbb{V}_N} \Big(\max\{e, \eta\} \, \|U - W_N\|_{\mathbb{X}(\omega)} + \sup_{V_N \in \mathbb{X}_N} \frac{E^a_M(W_N; V_N) + \eta \, \tilde{E}^a_M(W_N; V_N)}{\|V_N\|_{\mathbb{X}(\omega)}} \Big) \\ + \inf_{\chi_N \in \mathbb{M}_N} \Big(\|\psi - \chi_N\|_{H^1(\omega)} + \sup_{V_N \in \mathbb{X}_N} \frac{E^b_M(V_N; \chi_N)}{\|V_N\|_{\mathbb{X}(\omega)}} \Big) \\ + \sup_{V_N \in \mathbb{X}_N} \frac{E^{\mathcal{L}}_M(V_N)}{\|V_N\|_{\mathbb{X}(\omega)}} \Bigg), \end{aligned} \tag{4.1}$$

where the four quantities E^a_M, $\tilde{E}^a_M$, E^b_M and $E^{\mathcal{L}}_M$ are defined by

$$\begin{aligned} &E^a_M(W_N; V_N) = (a - a_M)(W_N; V_N), \qquad \tilde{E}^a_M(W_N; V_N) = (\tilde{a} - \tilde{a}_M)(W_N; V_N), \\ &E^b_M(V_N; \chi_N) = (b - b_M)(V_N; \chi_N), \qquad E^{\mathcal{L}}_M(V_N) = (\mathcal{L} - \mathcal{L}_M)(V_N). \end{aligned} \tag{4.2}$$

Proof: Let V_N and W_N be any functions in $\mathbb{V}_N$. We derive from problem (3.11) that

$$a_M(U_N - W_N; V_N) + \eta \, \tilde{a}_M(U_N - W_N; V_N) = \mathcal{L}_M(V_N) - a_M(W_N; V_N) - \eta \, \tilde{a}_M(W_N; V_N).$$

Then, using problem (2.22) yields for any χ_N in $\mathbb{M}_N$ (note that $b_M(V_N; \chi_N)$ is equal to zero)

$$\begin{aligned} a_M(U_N - W_N; V_N) &+ \eta \, \tilde{a}_M(U_N - W_N; V_N) \\ &= -E^{\mathcal{L}}_M(V_N) + a(U - W_N; V_N) + E^a_M(W_N; V_N) \\ &\quad + \eta \, \tilde{a}(U - W_N; V_N) + \eta \, \tilde{E}^a_M(W_N; V_N) + b(V_N; \psi - \chi_N) + E^b_M(V_N; \chi_N). \end{aligned}$$

Next, we take V_N equal to $U_N - W_N$ (which belongs to $\mathbb{V}_N$) and use the ellipticity property (3.31). Since the norm of $a(\cdot; \cdot)$ is smaller than $c\, e$, this gives the desired estimate for the term $\|U_N - W_N\|_{\mathbb{X}(\omega)}$. We conclude thanks to a triangle inequality by noting that $e \max\{e^{-3}, \eta^{-1}\}$ is larger than 1.

The two terms

$$\inf_{W_N \in \mathbb{V}_N} \|U - W_N\|_{\mathbb{X}(\omega)} \qquad \text{and} \qquad \inf_{\chi_N \in \mathbb{M}_N} \|\psi - \chi_N\|_{H^1(\omega)},$$

represent the approximation errors while the four other terms are issued from numerical integration and the replacement of the coefficients of the initial problem by discrete ones. We begin with the first approximation error.

Lemma 4.2. *For any integer $N \geq N_{\sharp\sharp}$, there exists a constant c independent of N such that, for all U in $\mathbb{V}(\omega)$,*

$$\inf_{W_N\in\mathbb{V}_N} \|U-W_N\|_{\mathbb{X}(\omega)} \leq c \inf_{Z_N\in\mathbb{X}_N} \Big(\|U-Z_N\|_{\mathbb{X}(\omega)} + \sup_{\omega_N\in\mathbb{M}_N} \frac{E_M^b(Z_N;\omega_N)}{\|\omega_N\|_{H^1(\omega)}}\Big). \tag{4.3}$$

Proof: Let Z_N be any element of $\mathbb{X}_N$. It follows from the inf-sup condition (3.37), see [15, Chap. I, Lemma 4.1] that there exists a $\tilde{Z}_N$ in $\mathbb{X}_N$ such that

$$\forall \omega_N \in \mathbb{M}_N, \quad b_M(\tilde{Z}_N;\omega_N) = b_M(Z_N;\omega_N),$$

and

$$\|\tilde{Z}_N\|_{\mathbb{X}(\omega)} \leq \tilde{c}_{\sharp\sharp}^{-1} \sup_{\omega_N\in\mathbb{M}_N} \frac{b_M(Z_N;\omega_N)}{\|\omega_N\|_{H^1(\omega)}}. \tag{4.4}$$

Then, the function $W_N = Z_N - \tilde{Z}_N$ belongs to $\mathbb{V}_N$. Moreover, we obtain the desired estimate by using the triangle inequality

$$\|U-W_N\|_{\mathbb{X}(\omega)} \leq \|U-Z_N\|_{\mathbb{X}(\omega)} + \|\tilde{Z}_N\|_{\mathbb{X}(\omega)},$$

combined with (4.4), the identity

$$b_M(Z_N;\omega_N) = -b(U-Z_N;\omega_N) - E_M^b(Z_N;\omega_N),$$

and the continuity of $b(\cdot;\cdot)$.

Since γ_0 is the union of whole edges of ω, the following estimtates are standard, see [4, §7] for instance: If the solution (U,ψ) belongs to $H^S(\omega)^{3\times 3} \times H^S(\omega)$ for a real number $S \geq 1$,

$$\begin{aligned} &\inf_{Z_N\in\mathbb{X}_N} \|U-Z_N\|_{\mathbb{X}(\omega)} \leq c\, N^{1-S}\, \|U\|_{H^S(\omega)^{3\times 3}}, \\ &\qquad\qquad \inf_{\chi_N\in\mathbb{M}_N} \|\psi-\chi_N\|_{H^1(\omega)} \leq c\, N^{1-S}\, \|\psi\|_{H^S(\omega)}. \end{aligned} \tag{4.5}$$

So it remains to investigate the four terms defined in (4.2).

Lemma 4.3. *There exists a constant c only depending on the norms of the $\boldsymbol{a}_\alpha$ in $H^{s_0}(\omega)^3$ and of $\boldsymbol{a}_3$ in $H^{s_0+1}(\omega)^3$ such that*

$$\forall W_N \in \mathbb{X}_N, \quad \sup_{V_N\in\mathbb{X}_N} \frac{E_M^a(W_N;V_N)}{\|V_N\|_{\mathbb{X}(\omega)}} \leq c\, e\, N^{1-s_0}\, (\log N)^{\frac{1}{2}}\, \|W_N\|_{\mathbb{X}(\omega)}. \tag{4.6}$$

Proof: Let K'' now denote the integral part of $\frac{\mu N-1}{2}$. It follows from the definition of the $a^{\alpha\beta\rho\sigma}$ and also of a that all these coefficients and also $\sqrt{a}$ belong to $H^{s_0}(\omega)$. We denote by $\boldsymbol{a}_{\alpha K''}$, $a_{K''}^{\alpha\beta\rho\sigma}$ and $(\sqrt{a})_{K''}$ approximations of the $\boldsymbol{a}_\alpha$, $a^{\alpha\beta\rho\sigma}$ and $\sqrt{a}$ in $\mathbb{P}_{K''}(\omega)^3$

or in $\mathbb{P}_{K''}(\omega)$ which still satisfies (3.16). We then derive form the exactness property (3.2) and with obvious notation

$$\big(a_{K''}^{\alpha\beta\rho\sigma}\,\gamma_{\alpha\beta}^{K''}(\boldsymbol{w}_N),\gamma_{\rho\sigma}^{K''}(\boldsymbol{v}_N)\,(\sqrt{a})_{K''}\big)_M = \int_\omega a_{K''}^{\alpha\beta\rho\sigma}\gamma_{\alpha\beta}^{K''}(\boldsymbol{w}_N)\gamma_{\rho\sigma}^{K''}(\boldsymbol{v}_N)\,(\sqrt{a})_{K''}\,d\boldsymbol{x}.$$

Inserting this equality in the definition of $a^M(\cdot;\cdot)$ and similar ones for the other terms of $a(\cdot;\cdot)$ and $a_M(\cdot;\cdot)$ leads to the following bound

$$E_M^a(W_N;V_N) \le c\,\kappa_N\,\|W_N\|_{\mathbb{X}(\omega)}\|V_N\|_{\mathbb{X}(\omega)},$$

where the quantity κ_N is equal to

$$\begin{aligned}\kappa_N = \max\Big\{&\|\boldsymbol{a}_k - \boldsymbol{a}_{kK''}\|_{L^\infty(\omega)^3}, \|\boldsymbol{a}_k - \boldsymbol{a}_{kN}\|_{L^\infty(\omega)^3},\\ &\|\partial_\alpha\boldsymbol{a}_3 - \boldsymbol{c}_{\alpha K''}\|_{L^\infty(\omega)^3}, \|\partial_\alpha\boldsymbol{a}_3 - \boldsymbol{c}_{\alpha N}\|_{L^\infty(\omega)^3},\\ &\|a^{\alpha\beta\rho\sigma} - a_{K''}^{\alpha\beta\rho\sigma}\|_{L^\infty(\omega)}, \|\sqrt{a} - (\sqrt{a})_{K''}\|_{L^\infty(\omega)}\Big\}.\end{aligned}$$

So the desired estimate is obviously derived from (3.16), (3.17) and (3.19).

Lemma 4.4. *There exists a constant c only depending on the norms of $\boldsymbol{a}_3$ in $H^{s_0+1}(\omega)^3$ such that*

$$\forall W_N \in \mathbb{X}_N,\quad \sup_{V_N\in\mathbb{X}_N}\frac{\tilde{E}_M^a(W_N;V_N)}{\|V_N\|_{\mathbb{X}(\omega)}} \le c\,N^{-s_0}\,(\log\,N)^{\frac{1}{2}}\,\|W_N\|_{\mathbb{X}(\omega)}. \tag{4.7}$$

Proof: We set: $W_N = (\boldsymbol{w}_N, \boldsymbol{t}_N)$ and $V_N = (\boldsymbol{v}_N, \boldsymbol{s}_N)$. As in the proof of Proposition 3.5, we take K equal to the integer part of $\mu\,N-1$ and consider an approximation $\boldsymbol{a}_{3K}$ of $\boldsymbol{a}_3$ in $\mathbb{P}_K(\omega)^3$ which still satisfies (3.17) and (3.18). Thus, $\mathcal{I}_M(\boldsymbol{t}_N\cdot\boldsymbol{a}_{3K})$ and $\mathcal{I}_M(\boldsymbol{s}_N\cdot\boldsymbol{a}_{3K})$ are equal to $\boldsymbol{t}_N\cdot\boldsymbol{a}_{3K}$ and $\boldsymbol{s}_N\cdot\boldsymbol{a}_{3K}$, respectively, whence

$$\big(\partial_\alpha\mathcal{I}_M(\boldsymbol{t}_N\cdot\boldsymbol{a}_{3K}),\partial_\alpha\mathcal{I}_M(\boldsymbol{s}_N\cdot\boldsymbol{a}_{3K})\big)_M = \int_\omega \partial_\alpha(\boldsymbol{t}_N\cdot\boldsymbol{a}_{3K})\partial_\alpha(\boldsymbol{s}_N\cdot\boldsymbol{a}_{3K})\,d\boldsymbol{x}.$$

Adding and subtracting this equality and using the continuity of $\tilde{a}_M(\cdot;\cdot)$ proved in Lemma 3.3, we derive

$$\begin{aligned}\frac{\tilde{E}_M^a(W_N;V_N)}{\|V_N\|_{\mathbb{X}(\omega)}} \le c\,\Big(&\|W_N\|_{\mathbb{X}(\omega)}\big(\|\boldsymbol{s}_N\cdot(\boldsymbol{a}_{3N}-\boldsymbol{a}_{3K})\|_{H^1(\omega)} + \|\boldsymbol{s}_N\cdot(\boldsymbol{a}_3-\boldsymbol{a}_{3K})\|_{H^1(\omega)}\big)\\ &+ \|V_N\|_{\mathbb{X}(\omega)}\big(\|\boldsymbol{t}_N\cdot(\boldsymbol{a}_{3N}-\boldsymbol{a}_{3K})\|_{H^1(\omega)} + \|\boldsymbol{t}_N\cdot(\boldsymbol{a}_3-\boldsymbol{a}_{3K})\|_{H^1(\omega)}\big)\Big).\end{aligned}$$

Then, we use the inequality, with $\frac{1}{p}+\frac{1}{q}=\frac{1}{2}$,

$$|\boldsymbol{s}_N\cdot(\boldsymbol{a}_3-\boldsymbol{a}_{3K})\|_{H^1(\omega)} \le \|\boldsymbol{s}_N\|_{H^1(\omega)^3}\|\boldsymbol{a}_3-\boldsymbol{a}_{3K}\|_{L^\infty(\omega)^3} + \|\boldsymbol{s}_N\|_{L^q(\omega)^3}\|\boldsymbol{a}_3-\boldsymbol{a}_{3K}\|_{W^{1,p}(\omega)^3},$$

and similar ones for the other terms. Combining this with (3.17) and (3.18), using once more the fact that the norm of the imbedding of $H^1(\omega)$ into $L^q(\omega)$ is smaller that $c\,q^{\frac{1}{2}}$ and taking q equal to $\log\,N$ give the desired result.

The proof of the next lemma relies on exactly the same arguments as for Proposition 3.7, see (3.40). So we omit it.

Lemma 4.5. *There exists a constant c only depending on the norms of $\boldsymbol{a}_3$ in $H^{s_0+1}(\omega)^3$ such that*

$$\forall \chi_N \in \mathbb{M}_N, \qquad \sup_{V_N \in \mathbb{X}_N} \frac{E_M^b(V_N;\chi_N)}{\|V_N\|_{\mathbb{X}(\omega)}} \leq c\, N^{-s_0}\,(\log N)^{\frac{1}{2}}\, \|\chi_N\|_{H^1(\omega)}, \tag{4.8}$$

and

$$\forall Z_N \in \mathbb{X}_N, \qquad \sup_{\omega_N \in \mathbb{M}_N} \frac{E_M^b(Z_N;\omega_N)}{\|\omega_N\|_{H^1(\omega)}} \leq c\, N^{-s_0}\,(\log N)^{\frac{1}{2}}\, \|Z_N\|_{\mathbb{X}(\omega)}. \tag{4.9}$$

Lemma 4.6. *Assume that the data $(\boldsymbol{f}, \boldsymbol{N}, \boldsymbol{M})$ belongs to $H^{s_1}(\omega)^3 \times H^{s_1}(\gamma_1)^3 \times H^{s_1}(\gamma_1)^3$ for a real number $s_1 > 1$. There exists a constant $\tilde{c}$ only depending on the norms of the $\boldsymbol{a}_\alpha$ in $H^{s_0}(\omega)^3$ such that*

$$\sup_{V_N \in \mathbb{X}_N} \frac{E_M^L(V_N)}{\|V_N\|_{\mathbb{X}(\omega)}} \leq c\,\Big(\tilde{c}\, N^{1-s_0}\,(\log N)^{\frac{1}{2}} \,+ c(\boldsymbol{f}, \boldsymbol{N}, \boldsymbol{M})\, N^{-s_1}\Big), \tag{4.10}$$

where the quantity $c(\boldsymbol{f}, \boldsymbol{N}, \boldsymbol{M})$ is defined by

$$c(\boldsymbol{f}, \boldsymbol{N}, \boldsymbol{M}) = \|\boldsymbol{f}\|_{H^{s_1}(\omega)^3} + \|\boldsymbol{N}\|_{H^{s_1}(\gamma_1)^3} + \|\boldsymbol{M}\|_{H^{s_1}(\gamma_1)^3}. \tag{4.11}$$

Proof: If K denotes the integer part of $\mu N - 1$ and $(\sqrt{a})_K$ an approximation of $\sqrt{a}$ in $\mathbb{P}_K(\omega)$ which satisfies (3.16) (we recall that $\sqrt{a}$ belongs to $H^{s_0}(\omega)$), we derive from (3.2) the identity, for all $V_N = (\boldsymbol{v}_N, \boldsymbol{s}_N)$ in $\mathbb{X}_N$

$$\big(\boldsymbol{f}, \boldsymbol{v}_N\,(\sqrt{a})_K\big)_M = \int_\omega \mathcal{I}_M \boldsymbol{f} \cdot \boldsymbol{v}_N (\sqrt{a})_K \,d\boldsymbol{x},$$

whence

$$\begin{aligned} &\Big|\int_\omega \boldsymbol{f} \cdot \boldsymbol{v}_N \sqrt{a}\,d\boldsymbol{x} - \big(\boldsymbol{f}, \boldsymbol{v}_N\,\sqrt{a}\big)_M\Big| \\ &\qquad \leq \big(\|\sqrt{a} - (\sqrt{a})_K\|_{L^\infty(\omega)} \|\boldsymbol{f}\|_{L^2(\omega)^3} + \|(\sqrt{a})_K\|_{L^\infty(\omega)} \|\boldsymbol{f} - \mathcal{I}_M \boldsymbol{f}\|_{L^2(\omega)^3}\big) \|\boldsymbol{v}_N\|_{L^2(\omega)^3}. \end{aligned}$$

Similar but simpler arguments also yield

$$\Big|\int_{\gamma_1} \boldsymbol{N} \cdot \boldsymbol{v}_N \,d\tau - (\boldsymbol{N}, \boldsymbol{v}_N)_M^{\gamma_1}\Big| \leq c\, \|\boldsymbol{N} - i_M^{\gamma_1} \boldsymbol{N}\|_{L^2(\gamma_1)^3} \|\boldsymbol{v}_N\|_{H^1(\omega)^3},$$

and

$$\Big|\int_{\gamma_1} \boldsymbol{M} \cdot \boldsymbol{v}_N \,d\tau - (\boldsymbol{M}, \boldsymbol{v}_N)_M^{\gamma_1}\Big| \leq c\, \|\boldsymbol{M} - i_M^{\gamma_1} \boldsymbol{M}\|_{L^2(\gamma_1)^3} \|\boldsymbol{v}_N\|_{H^1(\omega)^3}.$$

So estimate (4.10) is a direct consequence of (3.16) and the approximation properties of the operators $\mathcal{I}_M$ and $i_M^{\gamma_1}$, see [4, Thms 13.4 & 14.2].

To conclude, we insert (4.3) into (4.1). Next, we use (4.5) and Lemmas 4.3 to 4.6 to bound all the terms in the right-hand side.

Theorem 4.7. *Assume that:*
(i) the solution (U,ψ) *of problem* (2.22) *belongs to* $H^S(\omega)^{3\times 3}\times H^S(\omega)$ *for a real number* $S\geq 1$,
(ii) the data $(\boldsymbol{f},\boldsymbol{N},\boldsymbol{M})$ *belongs to* $H^{s_1}(\omega)^3\times H^{s_1}(\gamma_1)^3\times H^{s_1}(\gamma_1)^3$ *for a real number* $s_1>1$.
Then, for any integer $N\geq N_0$, *the following error estimate holds between this solution* (U,ψ) *and the solution* (U_N,ψ_N) *of problem* (3.11)

$$\|U-U_N\|_{\mathbb{X}(\omega)}\leq c\,\max\{e^{-3},\eta^{-1}\}\\ \Big(c(U,\psi)\,\max\{e,\eta\}\,N^{1-S}+\tilde{c}\,N^{1-s_0}\,(\log\,N)^{\frac{1}{2}}+c(\boldsymbol{f},\boldsymbol{N},\boldsymbol{M})\,N^{-s_1}\Big),\tag{4.12}$$

where the quantity $c(U,\psi)$ *is defined by*

$$c(U,\psi)=\|U\|_{H^S(\omega)^{3\times 3}}+\|\psi\|_{H^S(\omega)},\tag{4.13}$$

the constant $\tilde{c}$ *only depends on the coefficients involved in problem* (2.11) *and the quantity* $c(\boldsymbol{f},\boldsymbol{N},\boldsymbol{M})$ *is defined in* (4.11).

Estimate (4.12) is optimal and proves the convergence of the method without any restriction. Moreover, when combined with (2.30), it provides an estimate of the error between the solution of the initial problem (2.11) and the part U_N of the solution (U_N,ψ_N) of problem (3.11), of order

$$\eta\boldsymbol{e}^{-3}+\max\{e^{-3},\eta^{-1}\}\,\max\{e\,N^{1-S},\eta\,N^{1-S},N^{1-s_0}\,(\log\,N)^{\frac{1}{2}},N^{-s_1}\}.\tag{4.14}$$

In order to optimize the choice of η, it seems reasonable to assume that s_0 is smaller than S and that s_1-1, so that taking η such that

$$\eta=c\,e^{\frac{3}{2}}\,N^{\frac{1-s_0}{2}},$$

which leads to an error of order $e^{-\frac{3}{2}}\,N^{\frac{1-s_0}{2}}\,(\log\,N)^{\frac{1}{2}}$. So taking N large enough gives an error smaller than any fixed tolerance. A consequence is that the locking phenomenon is not directly linked to the discretization (and could not appear when the ratio N^{1-s_0}/e^3 tends to zero). In contrast, it can be thought that the condition number of the linear system resulting from problem (3.11) highly increases with N, so that the this phenomenon could be due to the use of bad preconditioners in the iterative solution of the system. Numerical experiments should confirm this idea.

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