Improved ad hoc Interface Conditions for Schwarz Solution Procedure tuned to Highly Heterogeneous Media

Yvon Maday* Frédéric Magoulès†

27 April 2005

*Laboratoire Jacques-Louis Lions, Université Pierre et Marie Curie, BP 187, 75252 Paris Cedex 05, France, maday@ann.jussieu.fr, Division of Applied Mathematics, Brown University, 182 George Street, Providence RI 02912, USA

†Institut Elie Cartan de Nancy, Université Henri Poincaré, BP 239, 54506 Vandoeuvre-les-Nancy Cedex, France, frederic.magoules@iecn.u-nancy.fr

Abstract

In this paper, an improved method to derive efficient interface conditions for the Schwarz solution procedure tuned to highly heterogeneous media is presented. This method, based on two parameters optimization, involves new interface conditions specially designed to keep the heterogeneity between the subdomains on the interface. The mathematical analysis of these interface conditions is first presented. Then the asymptotic analysis upon the mesh size parameter together with the heterogeneity ratio is detailed. These interface conditions lead to better asymptotic convergence rate than other available interface conditions. Numerical experiments illustrate the dependence of the proposed method upon several parameters, and confirm the robustness and efficiency of the Schwarz algorithm when equipped with such interface conditions.

Keywords

Schwarz algorithm, optimized Schwarz method, interface condition, transmission condition, heterogeneous media

Résumé

Dans cet article, une méthode pour construire des conditions d'interface adéquates pour l'algorithme de Schwarz dans le cas de matériaux très hétérogènes est présentée. Cette méthode, basée sur l'optimisation de deux paramètres, utilise de nouvelles conditions d'interface, spécialement conçues pour prendre en compte l'hétérogénéité des matériaux au niveau de l'interface entre les sous-domaines. Une analyse mathématique de ces conditions est d'abord effectuée. Une analyse asymptotique en fonction de la finesse du maillage et du rapport d'hétérogénéité est ensuite réalisée. Ces conditions d'interface conduisent à un taux de convergence asymptotique bien meilleur à celui obtenue par d'autres conditions d'interface existant dans la littérature. Des expériences numériques illustrent la dépendance de la méthode proposée en fonction de nombreux paramètres, et confirment l'efficacité et la robustesse de cette méthode équipée avec de telles conditions d'interface.

Mots clefs

algorithm de Schwarz, méthode de Schwarz optimisée, condition d'interface, condition de transmission, matériaux hétérogènes

1 Introduction

The interface conditions in the Schwarz algorithm without overlap [1, 2, 3] have a strong influence on the convergence of the algorithm. Optimal interface conditions which lead to the best possible convergence of the Schwarz algorithm can be derived at the continuous level [4] or at the discrete level [5]. These optimal interface conditions however are non-local in nature and several works continue to be investigated in order to use them efficiently in numerical simulations (see [6] for recent contributions). Other approaches consist in approximating these optimal interface conditions with local operators. These local operators depending on few parameters are optimized in order to improve the performances of the Schwarz algorithm. When dealing with heterogeneous media, the heterogeneity of the media between the subdomains requires a special treatment as shown in recent works [7, 8, 9, 10] in order to get convergence results similar to the previous works performed on other equations [11, 12, 13, 14, 15, 16, 17].

In this paper, a technique to design efficient interface conditions for highly heterogeneous media is presented. This technique is based on two parameters optimization, and leads to an asymptotic convergence rate equal to $\zeta_1 - \zeta_2\sqrt{h} + O(h)$, where $0 < \zeta_1 \leq 1$, $0 < \zeta_2$, and where h denotes the mesh size. For small values of h, and for strong heterogeneity between the subdomains, the coefficient ζ_1 ($\zeta_1 \ll 1$) tends to the heterogeneity ratio. The improvement on the convergence of the algorithm is better, with an order of magnitude, than the best results obtained with the interface conditions based on one parameter which lead to an asymptotic convergence rate equal to $1 - \zeta\sqrt{h} + O(h)$, where $\zeta \in \mathbb{R}$ as first analyzed in [10].

The structure of this paper is the following. Section 2 presents a brief review of the Schwarz algorithm considered in this paper. Section 3 presents the mathematical analysis of a technique to design efficient approximations of the optimal interface conditions. This technique is based on the solution of an optimization problem involving two independent parameters. Section 4 presents the asymptotic analysis of this method together with the mesh size and the heterogeneity ratio. Section 5 shows some numerical experiments and illustrates the convergence of the Schwarz algorithm equipped with these original interface conditions. Even though the optimization is performed on a model problem in the whole plane, the derived optimized interface conditions can be used for general partitions involving cross-points. Finally, in Section 6 the conclusion of this paper is presented.

2 Review of the Schwarz method

The following equation in an heterogenous media is considered

$$(-\nabla \cdot (\mu \nabla))\, u(x,y) = f(x,y), \quad x,y \in \Omega$$

in the domain $\Omega = \mathbb{R}^2$ with the condition at infinity,

$$\lim_{r\to\infty} u = 0,$$

where $r = \sqrt{x^2+y^2}$ and $\mu \in \mathbb{R}^+$. The domain Ω is decomposed into two non-overlapping subdomains $\Omega^{(1)} = (-\infty, 0] \times \mathbb{R}$ and $\Omega^{(2)} = [0,\infty) \times \mathbb{R}$. For simplicity the coefficient μ is assumed to be constant per subdomain. The Schwarz algorithm with 'absorbing' interface condition reads

$$\begin{cases} \left(-\nabla \cdot (\mu^{(1)}\nabla)\right) u^{(1)}_{2n+2}(x,y) = f(x,y), \quad x,y \in \Omega^{(1)} & (2.1) \\ \left(\mu^{(1)}\partial_x + \mathcal{A}^{(1)}\right) u^{(1)}_{2n+2}(0,y) = \left(\mu^{(2)}\partial_x + \mathcal{A}^{(1)}\right) u^{(2)}_{2n+1}(0,y) & (2.2) \end{cases}$$

$$\begin{cases} \left(-\nabla \cdot (\mu^{(2)}\nabla)\right) u^{(2)}_{2n+1}(x,y) = f(x,y), \quad x,y \in \Omega^{(2)} & (2.3) \\ \left(\mu^{(2)}\partial_x + \mathcal{A}^{(2)}\right) u^{(2)}_{2n+1}(0,y) = \left(\mu^{(1)}\partial_x + \mathcal{A}^{(2)}\right) u^{(1)}_{2n}(0,y) & (2.4) \end{cases}$$

where n represents the iteration parameter. The operators $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$ are to be determined for the best performance of the algorithm. They are obtained in order to minimize the convergence rate of the algorithm. To analyze the convergence, it suffices to consider by linearity the case of $f(x,y) = 0$ and to analyze the convergence to zero. For this purpose a Fourier transform is applied to the systems of equations (2.1)-(2.4) in the y direction. Solving the associated system of ordinary differential equations, taking into account the condition at infinity, and using the interface conditions leads to the expression of the solution in a subdomain at the iteration $n+1$ upon the solution in this subdomain at the iteration $n-1$. This relation involves a quantity called convergence rate $\kappa(k)$ defined by

$$\kappa(k) = \left| \frac{\mu^{(2)}\Lambda^{(2)}(k) - \Theta^{(1)}(k)}{\mu^{(1)}\Lambda^{(1)}(k) + \Theta^{(1)}(k)} \; \frac{\mu^{(1)}\Lambda^{(1)}(k) + \Theta^{(2)}(k)}{\mu^{(2)}\Lambda^{(2)}(k) - \Theta^{(2)}(k)} \right| \tag{2.5}$$

where k represents the Fourier variable, $\Theta^{(s)}$ denotes the Fourier transform of $\mathcal{A}^{(s)}$ (assumed to be diagonal), and $\Lambda^{(s)} = |k|$, for $s = 1, 2$. With the following selection $\Theta^{(1)}(k) = \mu^{(2)}\Lambda^{(2)}(k)$, and $\Theta^{(2)}(k) = -\mu^{(1)}\Lambda^{(1)}(k)$ the convergence rate (2.5) vanishes and the Schwarz algorithm converges in two steps independently of the initial estimate. This optimal selection of the operators $\Theta^{(1)}$ and $\Theta^{(2)}$ in the Fourier space, leads to the convergence of the algorithm in two iterations. This choice corresponds to $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$ in the physical space, equal to Steklov-Poincaré operators. Such operators are non-local by definition and several research, see e.g. [11, 12, 13, 14, 15, 16, 17], have been carried out these last ten years in order to build local approximation of these optimal operators.

3 Improved interface conditions

The non-local optimal operators, involving the symbol $|k|$ in the Fourier space, are here approximated by some constants which represent differential operators in the physical space and are thus local. Different techniques of approximation have been analyzed in the recent years for other equations. The first results presented in references [18, 19] for the Helmholtz equation, use a simple zeroth order Taylor expansion of the Steklov-Poincaré operator. In reference [11] for the Maxwell equation, in reference [12] for the convection diffusion equations, in reference [13] for the Euler equations, and in references [16, 17, 20] for the Helmholtz equation, the approximations involve polynomial expressions of degree two upon the variable k in the Fourier space, which represent tangential differential operators in the physical space. For heterogeneous media, in [10] the case $\Theta^{(1)} = -\Theta^{(2)} = \alpha$ has been considered, and in [8, 10] the case $\Theta^{(1)} = \mu^{(2)}\alpha$ and $\Theta^{(2)} = -\mu^{(1)}\alpha$ has been analyzed. In this paper the approximation considered is of the form

$$\Theta^{(1)} = \mu^{(2)}\alpha, \quad \Theta^{(2)} = -\mu^{(1)}\beta \tag{3.1}$$

where α and β are two constants. These constants are selected in such a way that they minimize the convergence rate of the Schwarz algorithm i.e., that the optimized parameters $\alpha^\dagger$ and $\beta^\dagger$ are solution of the min-max problem

$$\min_{\alpha\in\mathbb{R},\beta\in\mathbb{R}} \max_{k\in(k_{\min},k_{\max})} \kappa(\alpha,\beta,k) \tag{3.2}$$

where $k_{\min}$ and $k_{\max}$ denotes the smallest and highest frequencies considered that we are interested in and/or that the discretization allows us to capture. In order to find these optimized parameters, several Lemmas are now introduced.

Lemma 3.1 *Under the assumptions*

$$0 < k_{\min} < \beta < \alpha < k_{\max}, \tag{3.3}$$

and if α and β are supposed to be known, the function $\kappa(\alpha, \beta, k)$ as a function of $k \in (k_{\min}, k_{\max})$, reaches its minimum at the frequency $k^* = \beta$ and $k^* = \alpha$.

Proof 3.1 *Substituting $\Theta^{(1)}$ and $\Theta^{(2)}$ given by (3.1) in the expression of the convergence rate (2.5) leads to:*

$$\kappa(\alpha, \beta, k) = \frac{\mu^{(1)}\mu^{(2)}(-|k| + \alpha)(-|k| + \beta)}{(\mu^{(1)}|k| + \mu^{(2)}\alpha)(\mu^{(2)}|k| + \mu^{(1)}\beta)}.$$

It is clear that $\kappa(\alpha, \beta, k)$ vanishes for the frequency $k^* = \beta$ and $k^* = \alpha$. Because the function $\kappa(\alpha, \beta, k)$ has positive values, the frequency k^* is obviously the frequency where the function reaches its minimum.

Lemma 3.2 *Under the assumptions*

$$0 < \mu^{(1)} < \mu^{(2)}, \quad \text{and} \quad 0 < k_{\min} < \beta < \alpha < k_{\max}, \tag{3.4}$$

and if α and β are supposed to be known, the function $\kappa(\alpha, \beta, k)$ defined for $k \in (k_{\min}, k_{\max})$ reaches its maximum either at the frequency $k^* = k_{\min}$, $k^* = k_0 = \sqrt{\alpha\beta}$, or $k^* = k_{\max}$.

Proof 3.2 *The derivative of the function $\kappa(\alpha, \beta, k)$ upon the variable k can be computed as:*

$$\kappa_k(\alpha, \beta, k) = -\epsilon\ \frac{\mu^{(1)}\mu^{(2)}(\mu^{(1)} + \mu^{(2)})(\mu^{(1)}\beta + \mu^{(2)}\alpha)(\alpha\beta - |k|^2)}{(\mu^{(1)}|k| + \mu^{(2)}\alpha)^2(\mu^{(2)}|k| + \mu^{(1)}\beta)^2}$$

with

$$\epsilon = sign\left(\frac{\mu^{(1)}\mu^{(2)}(-|k| + \alpha)(-|k| + \beta)k}{(\mu^{(1)}|k| + \mu^{(2)}\alpha)(\mu^{(2)}|k| + \mu^{(1)}\beta)}\right)$$

where $sign(.)$ represents the sign of the quantity surrounded by the brackets. Under the assumptions given by the Lemma, the sign of the function $\kappa_k(\alpha, \beta, k)$ can be analyzed:

$$\begin{cases} \kappa_k(\alpha, \beta, k) < 0, \ \text{for } k \in (k_{\min}, \beta) & (3.5) \\ \kappa_k(\alpha, \beta, k) > 0, \ \text{for } k \in (\beta, \sqrt{\alpha\beta}) & (3.6) \\ \kappa_k(\alpha, \beta, k) = 0, \ \text{for } k = \sqrt{\alpha\beta} & (3.7) \\ \kappa_k(\alpha, \beta, k) < 0, \ \text{for } k \in (\sqrt{\alpha\beta}, \alpha) & (3.8) \\ \kappa_k(\alpha, \beta, k) > 0, \ \text{for } k \in (\alpha, k_{\max}). & (3.9) \end{cases}$$

From the previous analysis, and because $\kappa(\alpha,\beta,k)$ is a continuous function, it is clear, that the maximum is obtained at the frequency $k^* = k_{\min}$, $k^* = k_0 = \sqrt{\alpha\beta}$, or $k^* = k_{\max}$.

How to choose the optimized parameters α^* and β^* is given by the following analysis under the hypotheses

$$0 < \mu^{(1)} < \mu^{(2)}, \quad \text{and} \quad 0 < k_{\min} < \beta < \alpha < k_{\max}.$$

As shown in the previous Lemma, if α and β are supposed to be known, the maximum of the convergence rate is obtained at the frequency $k^* = k_{\min}$, $k^* = k_0 = \sqrt{\alpha\beta}$, or $k^* = k_{\max}$. This means that the maximum corresponds to one of the following (positive) value

$$\begin{cases} \kappa(\alpha,\beta,k_{\min}) = \dfrac{\mu^{(1)}\mu^{(2)}(k_{\min}-\alpha)(k_{\min}-\beta)}{(\mu^{(1)}k_{\min}+\mu^{(2)}\alpha)(\mu^{(2)}k_{\min}+\mu^{(1)}\beta)} & (3.10) \\ \kappa(\alpha,\beta,k_0) = \dfrac{\left|(-\mu^{(2)}\sqrt{\alpha\beta}+\mu^{(2)}\alpha)(\mu^{(1)}\sqrt{\alpha\beta}-\mu^{(1)}\beta)\right|}{(\mu^{(1)}\sqrt{\alpha\beta}+\mu^{(2)}\alpha)(\mu^{(2)}\sqrt{\alpha\beta}+\mu^{(1)}\beta)} & (3.11) \\ \kappa(\alpha,\beta,k_{\max}) = \dfrac{\mu^{(1)}\mu^{(2)}(-k_{\max}+\alpha)(-k_{\max}+\beta)}{(\mu^{(1)}k_{\max}+\mu^{(2)}\alpha)(\mu^{(2)}k_{\max}+\mu^{(1)}\beta)}. & (3.12) \end{cases}$$

Let

$$\aleph(\alpha,\beta) = \max_{k\in\{k_{\min},k_0,k_{\max}\}} \kappa(\alpha,\beta,k),$$

we denote by $\alpha^\dagger$ and $\beta^\dagger$ the optimized parameters obtained from the solution of the minimization problem

$$\aleph(\alpha^\dagger,\beta^\dagger) = \min_{\alpha\in\mathbb{R},\beta\in\mathbb{R}} \aleph(\alpha,\beta).$$

Let also α^* and β^* denote the parameters obtained from the solution of the equation

$$\kappa(\alpha^*,\beta^*,k_{\min}) = \kappa(\alpha^*,\beta^*,k_0) = \kappa(\alpha^*,\beta^*,k_{\max}) \tag{3.13}$$

it is clear that

$$\aleph(\alpha^\dagger,\beta^\dagger) \leq \aleph(\alpha^*,\beta^*).$$

In the following, the expression of the parameters α^* and β^* is sought. Solving the equation $\kappa(\alpha^*,\beta^*,k_{\min}) = \kappa(\alpha^*,\beta^*,k_{\max})$, leads to a first relation between α^* and β^*

$$\beta^* = \frac{k_{\min}k_{\max}}{\alpha^*} \tag{3.14}$$

Substituting β^* in (3.10) and in (3.11) gives

$$\begin{cases} \kappa(\alpha^*,\beta^*,k_{\min}) = \dfrac{-\mu^{(1)}\mu^{(2)}(k_{\min}-\alpha^*)(k_{\max}-\alpha^*)}{(\mu^{(1)}k_{\min}+\mu^{(2)}\alpha^*)(\mu^{(1)}k_{\max}+\mu^{(2)}\alpha^*)} & (3.15) \\ \kappa(\alpha^*,\beta^*,k_0) = \dfrac{\mu^{(1)}\mu^{(2)}\left|(\sqrt{k_{\min}k_{\max}}-\alpha^*)(\alpha^*\sqrt{k_{\min}k_{\max}}-k_{\min}k_{\max})\right|}{(\mu^{(1)}\sqrt{k_{\min}k_{\max}}+\mu^{(2)}\alpha^*)(\mu^{(2)}\alpha^*\sqrt{k_{\min}k_{\max}}+\mu^{(1)}k_{\min}k_{\max})} & (3.16) \end{cases}$$

Solving the equation $\kappa(\alpha^*,\beta^*,k_{\min}) = \kappa(\alpha^*,\beta^*,k_0)$, with the previous expressions, leads to a fourth order equation

$$A_1\ X^4 + B_1\ X^3 + C_1\ X^2 + D_1\ X + E_1 = 0 \tag{3.17}$$

where the real coefficients A_1, B_1, C_1, D_1, and E_1 are given by

$$\begin{aligned}
A_1 &= 2{\mu^{(2)}}^2 k_{\max}^{\frac{1}{2}} k_{\min}^{\frac{1}{2}} \\
B_1 &= \mu^{(2)}\mu^{(1)}\left(2k_{\min}k_{\max} + k_{\min}^{\frac{3}{2}}k_{\max}^{\frac{1}{2}} + k_{\min}^{\frac{1}{2}}k_{\max}^{\frac{3}{2}}\right) \\
&\quad + {\mu^{(2)}}^2\left(-k_{\max}^{\frac{3}{2}}k_{\min}^{\frac{1}{2}} - 2k_{\min}k_{\max} - k_{\min}^{\frac{3}{2}}k_{\max}^{\frac{1}{2}}\right) \\
C_1 &= 2k_{\min}^{\frac{3}{2}}k_{\max}^{\frac{3}{2}}{\mu^{(2)}}^2 + 2{\mu^{(1)}}^2 k_{\min}^{\frac{3}{2}}k_{\max}^{\frac{3}{2}} \\
&\quad + \mu^{(2)}\mu^{(1)}\left(-4k_{\min}^2k_{\max} - 4k_{\max}^2k_{\min}\right) \\
D_1 &= {\mu^{(1)}}^2\left(-2k_{\min}^2k_{\max}^2 - k_{\max}^{\frac{5}{2}}k_{\min}^{\frac{3}{2}} - k_{\min}^{\frac{5}{2}}k_{\max}^{\frac{3}{2}}\right) \\
&\quad + \mu^{(1)}\mu^{(2)}\left(+k_{\min}^{\frac{5}{2}}k_{\max}^{\frac{3}{2}} + 2k_{\min}^2k_{\max}^2 + k_{\min}^{\frac{3}{2}}k_{\max}^{\frac{5}{2}}\right) \\
E_1 &= 2k_{\min}^{\frac{5}{2}}k_{\max}^{\frac{5}{2}}{\mu^{(1)}}^2.
\end{aligned}$$

The solution of this equation gives the possible values of α^*.

4 Asymptotic analysis

In the classical Schwarz method with overlap of order h, which is often all that one can afford in real applications, the convergence rate is equal to $1-\zeta\ h + O(h^2)$, where $\zeta \in \mathbb{R}$. In our previous work [10], we have analyzed a variant of the Schwarz algorithm without overlap and with particular interface conditions based on one parameter optimization. These interface conditions were specially design to keep the heterogeneity of the media between the subdomains on the interface. We have shown that the asymptotic convergence rate of the algorithm was equal to $1-\zeta\ \sqrt{h}+O(h)$, where $\zeta \in \mathbb{R}$. Here, we show that with the proposed interface conditions based on two parameters optimization, the asymptotic convergence rate of the algorithm is equal to $\zeta_1-\zeta_2\ \sqrt{h}+O(h)$, where $\zeta_1, \zeta_2 \in \mathbb{R}^2$. For small values of the mesh size h and for high heterogeneous media, the coefficient ζ_1 ($\zeta_1 \ll 1$) tends to the value of the heterogeneity ratio. The following Theorem and its corollary makes precise the previous statement and gives the asymptotic convergence rate in the mesh parameter h of the discretized optimized Schwarz method.

Theorem 4.1 *Under the assumptions*

$$0<\mu^{(1)}<\mu^{(2)}, \qquad \textit{and} \qquad 0<k_{\min}<k_{\max}, \tag{4.1}$$

the asymptotic convergence rate of the non-overlapping Schwarz algorithm (2.1)-(2.4) with optimized interface conditions (3.1) obtained from (3.14) and (3.17) discretized with mesh parameter h is satisfied

$$\kappa \leq \zeta_1-\zeta_2\ \sqrt{h}+O(h)$$

where the constants ζ_1 and ζ_2 are given by

$$\begin{aligned}
\zeta_1 &= \frac{\mu^{(1)}k_{\min}-\frac{2\mu^{(1)}\mu^{(2)}k_{\min}}{\mu^{(2)}-\mu^{(1)}}}{-\mu^{(2)}k_{\min}-\frac{2\mu^{(1)}\mu^{(2)}k_{\min}}{\mu^{(2)}-\mu^{(1)}}}\\
\zeta_2 &= -\left(\frac{\frac{4\mu^{(1)}\mu^{(2)}k_{\min}^{\frac{3}{2}}}{(-\mu^{(1)}+\mu^{(2)})\sqrt{\pi}}-\frac{4{\mu^{(1)}}^2k_{\min}^{\frac{3}{2}}}{(-\mu^{(1)}+\mu^{(2)})\sqrt{\pi}}}{-\mu^{(2)}k_{\min}-\frac{2\mu^{(1)}\mu^{(2)}k_{\min}}{\mu^{(2)}-\mu^{(1)}}}\right).
\end{aligned}$$

Corollary 4.1 *In case $\mu^{(1)} \ll \mu^{(2)}$, the asymptotic convergence rate (in the limit $h \ll \frac{\mu^{(1)}}{\mu^{(2)}} \to 0$) is equal to*

$$\kappa = \frac{\mu^{(1)}}{\mu^{(2)}}-\left(\frac{4\sqrt{k_{\min}}}{\sqrt{\pi}}\right)\frac{\mu^{(1)}}{\mu^{(2)}}\ \sqrt{h}+O(h)+O\left(\frac{{\mu^{(1)}}^2}{{\mu^{(2)}}^2}\right).$$

To prove Theorem 4.1 several Lemmas are now introduced.

Lemma 4.1 *Under the assumptions of the Theorem 4.1 and assuming that the roots of the equation (3.17) are real, two of these roots are positive and two of them are negative.*

Proof 4.1 *In the minimization problem, the highest frequencies are bounded by* $k_{\max}$*, and choosing a numerical grid with grid spacing* h *such as* $k_{\max} = \frac{\pi}{h}$*, allows to substitute* $k_{\max}$ *in the coefficients of the equation (3.17) and leads to*

$$A_{1,h}\ X^4 + B_{1,h}\ X^3 + C_{1,h}\ X^2 + D_{1,h}\ X + E_{1,h} = 0 \tag{4.2}$$

where

$$\begin{aligned}
A_{1,h} &= \frac{2{\mu^{(2)}}^2\sqrt{k_{\min}}\sqrt{\pi}}{\sqrt{h}} \\
B_{1,h} &= \frac{\sqrt{k_{\min}}\pi^{\frac{3}{2}}\mu^{(2)}\mu^{(1)} - \pi^{\frac{3}{2}}{\mu^{(2)}}^2\sqrt{k_{\min}}}{h^{\frac{3}{2}}} \\
&+ \frac{-2k_{\min}\pi{\mu^{(2)}}^2 + 2\mu^{(2)}\mu^{(1)}k_{\min}\pi}{h} \\
&+ \frac{-k_{\min}^{\frac{3}{2}}{\mu^{(2)}}^2\sqrt{\pi} + k_{\min}^{\frac{3}{2}}\sqrt{\pi}\mu^{(1)}\mu^{(2)}}{\sqrt{h}} \\
C_{1,h} &= \frac{-4\pi^2\mu^{(2)}\mu^{(1)}k_{\min}}{h^2} \\
&+ \frac{2k_{\min}^{\frac{3}{2}}\pi^{\frac{3}{2}}{\mu^{(2)}}^2 + 2{\mu^{(1)}}^2k_{\min}^{\frac{3}{2}}\pi^{\frac{3}{2}}}{h^{\frac{3}{2}}} \\
&+ \frac{-4k_{\min}^2\mu^{(2)}\mu^{(1)}\pi}{h} \\
D_{1,h} &= \frac{k_{\min}^{\frac{3}{2}}\pi^{\frac{5}{2}}\mu^{(2)}\mu^{(1)} - \pi^{\frac{5}{2}}{\mu^{(1)}}^2k_{\min}^{\frac{3}{2}}}{h^{\frac{5}{2}}} \\
&+ \frac{2k_{\min}^2\pi^2\mu^{(2)}\mu^{(1)} - 2k_{\min}^2\pi^2{\mu^{(1)}}^2}{h^2} \\
&+ \frac{-k_{\min}^{\frac{5}{2}}{\mu^{(1)}}^2\pi^{\frac{3}{2}} + k_{\min}^{\frac{5}{2}}\pi^{\frac{3}{2}}\mu^{(1)}\mu^{(2)}}{h^{\frac{3}{2}}} \\
E_{1,h} &= \frac{2k_{\min}^{\frac{5}{2}}\pi^{\frac{5}{2}}{\mu^{(1)}}^2}{h^{\frac{5}{2}}}.
\end{aligned}$$

Let us denote by r_1*,* r_2*,* r_3*, and* r_4 *the roots ranked in increasing order of equation (4.2), or similarly the root of the normalized equation:*

$$X^4 + B_{2,h}\ X^3 + C_{2,h}\ X^2 + D_{2,h}\ X + E_{2,h} = 0 \tag{4.3}$$

where

$$B_{2,h} = \frac{B_{1,h}}{A_{1,h}}, \quad C_{2,h} = \frac{C_{1,h}}{A_{1,h}}, \quad D_{2,h} = \frac{D_{1,h}}{A_{1,h}}, \quad E_{2,h} = \frac{E_{1,h}}{A_{1,h}}.$$

Under the assumptions of Theorem 4.1, and for small values of h, it is clear that

$$B_{2,h} < 0, \quad C_{2,h} < 0, \quad D_{2,h} > 0, \quad E_{2,h} > 0.$$

From the relation $B_{2,h} = -\sum_{i=1}^{4} r_i < 0$, it follows that $r_4 > 0$. From relation $C_{2,h} = \sum_{i,j=1,i\neq j}^{4} r_i r_j < 0$, it follows that $r_1 < 0$. From the previous results and from relation $E_{2,h} = \prod_{i=1}^{4} r_i > 0$, it follows that the number of negative roots is even. In conclusion, the following relation occurs

$$r_1 < r_2 < 0 < r_3 < r_4. \tag{4.4}$$

It can be shown that the roots r_3, and r_4 satisfy the property

$$\aleph(r_4, \frac{k_{\min}k_{\max}}{r_4}) < \aleph(r_3, \frac{k_{\min}k_{\max}}{r_3}).$$

Figure 1 and Figure 2 illustrate the roots r_3 and r_4, obtained as the intersection points between the function $\kappa(\alpha, \frac{k_{\min}k_{\max}}{\alpha}, k_0)$, and the function $\kappa(\alpha, \frac{k_{\min}k_{\max}}{\alpha}, k_{\min}) = \kappa(\alpha, \frac{k_{\min}k_{\max}}{\alpha}, k_{\max})$ as functions of α.

Lemma 4.2 *Under the assumptions of the Theorem 4.1 and assuming that the roots are real, the roots behave asymptotically as $r_1 = -O(\frac{1}{\sqrt{h}})$, $r_2 = -O(1)$, $r_3 = O(\frac{1}{\sqrt{h}})$, $r_4 = O(\frac{1}{h})$.*

Proof 4.2 *Considering the asymptotic behavior of the coefficients when h tends to zero, it is clear that $B_{2,h} = -\sum_{i=1}^{4} r_i = -O(\frac{1}{h})$, and from equation (4.4) it follows $r_4 \geq O(\frac{1}{h})$. The coefficient $C_{2,h} = \sum_{i,j=1,i\neq j}^{4} r_i r_j = O(\frac{1}{h^{\frac{3}{2}}})$, implies that $|r_1| \geq O(\frac{1}{\sqrt{h}})$. The coefficient $D_{h,2} = \sum_{i,j,k=1,i\neq j\neq k}^{4} r_i r_j r_k = O(\frac{1}{h^2})$, implies that $|r_3| \geq O(\frac{1}{\sqrt{h}})$. Finally, the coefficient $E_{h,2} = \prod_{i=1}^{4} r_i = O(\frac{1}{h^2})$ implies that $|r_2| \leq O(1)$. Furthermore, it can be shown that $r_2 \approx \frac{-D_{2,h}}{E_{2,h}}$, hence equality holds in all previous lower bounds.*

The complete proof of the Theorem is now presented.

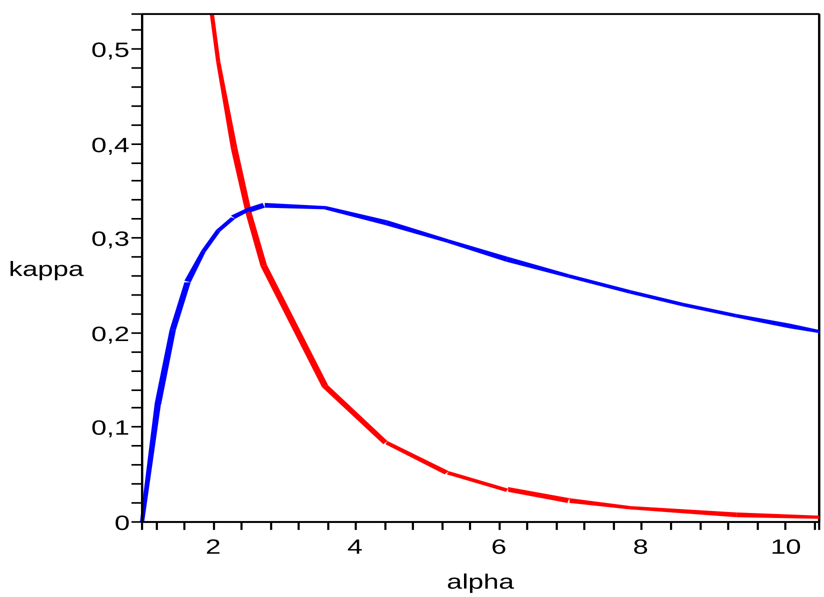

Figure 1: Maximum of the convergence rate upon the parameter α to be optimized when $\beta = \frac{k_{\min} k_{\max}}{\alpha}$. The intersection between the curve corresponds to the root r_3.

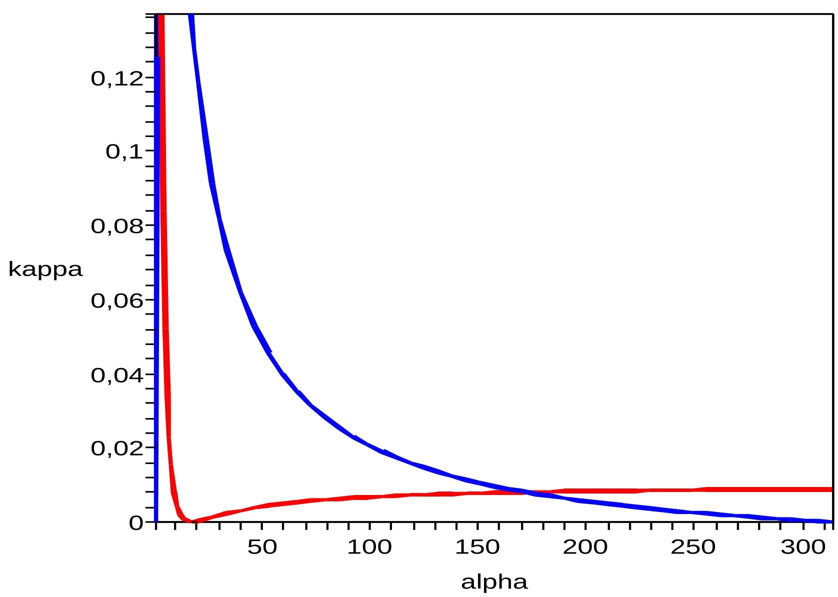

Figure 2: Maximum of the convergence rate upon the parameter α to be optimized when $\beta = \frac{k_{\min} k_{\max}}{\alpha}$. The intersection between the curve corresponds to the root r_4.

Proof 4.3 (of Theorem 4.1) *Between the possible solution of the equation of degree four, we decide to choose* $\alpha^* = r_4$*. This choice will be considered in the numerical experiments. In order to get the asymptotic behavior of the convergence rate, the analytic expression of* α^* *solution of the equation of degree four is quite complex and we prefer to approximate it. Because* $r_4 \gg \{|r_1|, |r_2|, |r_3|\}$ *and* $\frac{-B_1}{A_1} = r_1 + r_2 + r_3 + r_4 \approx r_4$*, we choose as the optimized parameters* $\alpha^* = \frac{-B_1}{A_1}$ *and* $\beta^* = \frac{k_{\min} k_{\max}}{\alpha^*}$*. The expansion in* h *of the optimized convergence rate* $\aleph(\alpha^*, \beta^*)$ *for* $k_{\max} = \frac{\pi}{h}$*, leads to the stated result of the Theorem 4.1.*

Remark 4.1 *When the heterogeneity between the two subdomains increases, the constant* ζ_1 *tends to* $\frac{\mu^{(1)}}{\mu^{(2)}}$*. The associated asymptotic convergence rate is thus very small compared to the asymptotic convergence rate of other approaches available in the literature.*

Remark 4.2 *Even though, if* h *is small, the constant* α^* *can be quite large, the scaling that we propose in (3.1) is the good one as a function of* $\mu^{(1)}$ *and* $\mu^{(2)}$*.*

5 Numerical experiments

5.1 Numerical value of the optimized parameters

In all the numerical experiments, it is interesting to notice that $\alpha^\dagger = \alpha^*$ and $\beta^\dagger = \beta^*$. In order to obtain the best value of the convergence rate, even for non asymptotic values of h, the exact value of the optimized parameter α^* solution of the polynomial equation of degree four (3.17), is evaluated. For this purpose, equation (3.17) is written in the normalized form

$$X^4 + a\ X^3 + b\ X^2 + c\ X + d = 0$$

where a, b, c, d are non zero coefficients. Setting $X = Z - \frac{a}{4}$ reduces the equation to

$$Z^4 + p\ Z^2 + q\ Z + r = 0$$

where

$$p = b - \frac{3\ a^2}{8}, \quad q = c - \frac{a\ b}{2} + \frac{a^3}{8}, \quad r = d - \frac{a\ c}{4} + \frac{b\ a^2}{16} - \frac{3\ a^4}{256}.$$

Two cases should be considered (i) $q = 0$ or (ii) $q \neq 0$.

(i) $q = 0$. The equation reduces to $Z^4 + p\ Z^2 + r = 0$. Setting $Y = Z^2$ transforms the equation to $Y^2 + p\ Y + r = 0$. The solutions are

$$Y_1 = \frac{-p}{2} + \sqrt{\frac{p^2}{4} - r}, \quad \text{and} \quad Y_2 = \frac{-p}{2} - \sqrt{\frac{p^2}{4} - r}.$$

This leads to the values of Z

$$Z_1 = \sqrt{Y_1}, \quad Z_2 = -\sqrt{Y_1}, \quad Z_3 = \sqrt{Y_2}, \quad Z_4 = -\sqrt{Y_2}.$$

(ii) $q \neq 0$. The equation reduces to $Z^4 + p\ Z^2 + q\ Z + r = 0$. Setting $2P - Q^2 = p$, $-2QR = q$ and $P^2 - R^2 = r$, transforms the equation to $(Z^2 + P)^2 - (QZ + R)^2 = 0$. If the values (P_0, Q_0, R_0) can be determined, solving the equation

$$(Z^2 + P_0) + (Q_0\ Z + R_0) = 0 \quad \text{or} \quad (Z^2 + P_0) - (Q_0\ Z + R_0) = 0$$

is equivalent to solve the reduced equation. The solution Z can thus be obtained. To find P, Q and R, the following system of equations must be solved

$$2P - Q^2 = p, \quad -2QR = q, \quad P^2 - R^2 = r. \tag{5.1}$$

This system is equivalent to

$$Q^2 = \frac{q^2}{4P^2 - r}, \quad R^2 = P^2 - r, \quad QR = -\frac{q}{2}.$$

This is equivalent to solve upon the variable P the following equation

$$p^3 - (\frac{p}{2})\ P^2 - r\ P + p\ \frac{r}{2} - \frac{1}{8}\ q^2 = 0.$$

The solution P_0 can thus be obtained. From equations (5.1) the variables (Q_0, R_0) can be obtained and so far the variable Z.

Figure 3 illustrates the maximum of the convergence rate upon the parameters α and β to be optimized, for a mesh size $h = 10^{-2}$ and a density $\mu^{(1)} = 1$ and $\mu^{(2)} = 10^2$. Figure 4 represents a zoom for the small values of β. Figure 5 shows the optimized convergence rate obtained with the optimized parameters $\alpha^\dagger = \alpha^* = 172.3580$ and $\beta^\dagger = \beta^* = 1.8887$. As already mentioned, and despite the existence of a Theorem, the optimized parameters $\alpha^\dagger$ and $\beta^\dagger$ solution of the minimization problem are here numerically equal to α^* and β^*.

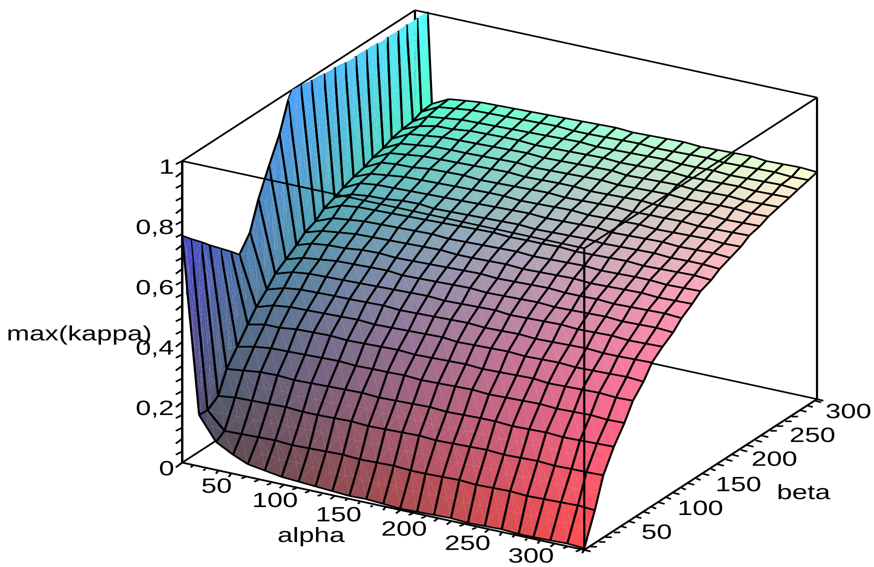

Figure 3: Maximum of the convergence rate upon the parameters α and β to be optimized.

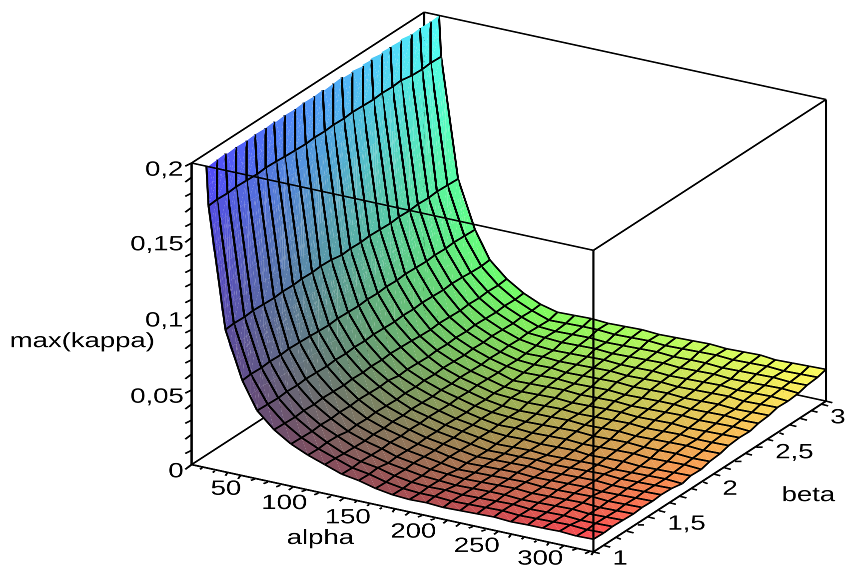

Figure 4: Zoom (for the small values of β) of the maximum of the convergence rate upon the parameters α and β to be optimized.

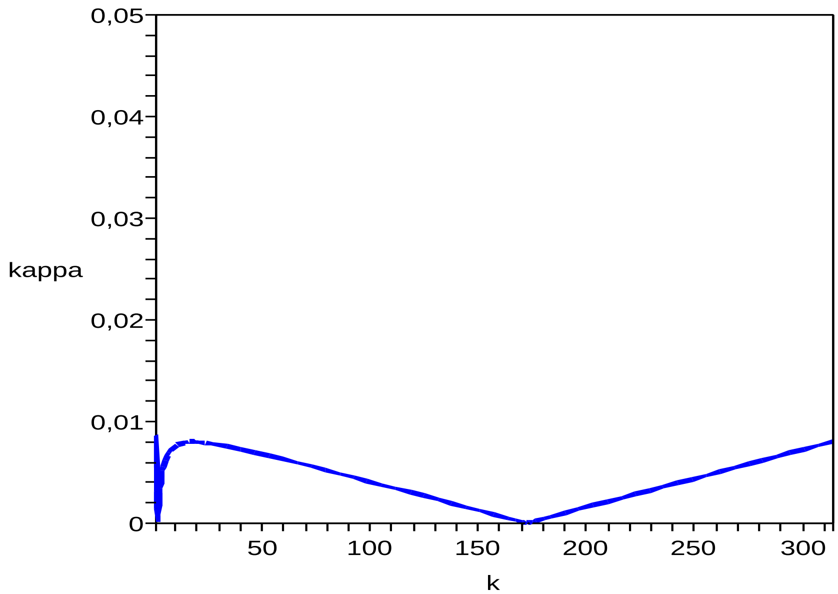

Figure 5: Optimized convergence rate upon the frequency.

5.2 Convergence analysis

A two dimensional shaped square domain Ω is considered. The boundary conditions obtained from the trace of the function $u_0(x,y) = 16((2x-1)^2 - (2y-1)^2)$ are of Dirichlet type on the left, and of Neumann type on the right, on the top and on the bottom. An regular mesh is used for the discretization, and the unit square is decomposed into two subdomains of equal size, as shown Figure 6. Two analysis are now presented.

The first analysis considers the parameters $\mu^{(1)} = 1$ and $\mu^{(2)} = 10^2$. Table 1 shows the number of iterations upon the mesh parameter h for optimized interface conditions involving respectively one (OO1p)[1], and one and half (OO1.5p)[2] optimized parameters, as presented in [10], and the two (OO2p)[3] optimized parameters proposed in this paper. As expected, the iterative Schwarz algorithm requires more iterations when equipped with the OO1p interface condition than with the OO1.5p, or the OO2p. The fantastic improvement of the

[1] OO1p: $\Theta^{(1)} = \alpha$ and $\Theta^{(2)} = -\alpha$

[2] OO1.5p: $\Theta^{(1)} = \mu^{(2)}\alpha$ and $\Theta^{(2)} = -\mu^{(1)}\alpha$

[3] OO2p: $\Theta^{(1)} = \mu^{(2)}\alpha$ and $\Theta^{(2)} = -\mu^{(1)}\beta$

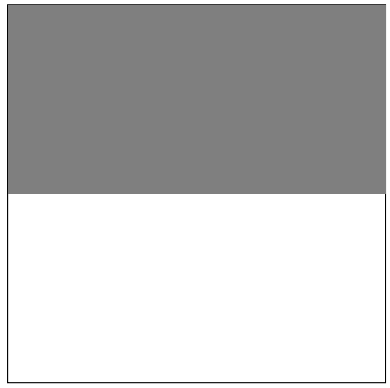

Figure 6: Mesh partition into two subdomains.

Mesh size h	Schwarz method with Optimized 1p	Schwarz method with Optimized 1.5p	Schwarz method with Optimized 2p
1/50	45	10	6
1/100	48	12	6
1/200	46	13	6

Table 1: Number of iterations for different interface conditions and different mesh parameter. ($N_s = 2$, $\mu^{(1)} = 1$, $\mu^{(2)} = 10^2$).

OO2p interface condition is mainly due to the fact that two parameters are used for the optimization, and that the property of the heterogeneity between the subdomains on the interface appears in such interface conditions.

The second analysis considers different parameters $\mu^{(1)}$ and $\mu^{(2)}$. The Schwarz algorithm equipped with OO1p interface conditions leads to more iterations when the heterogeneity between the subdomains increases. Opposite when equipped with the OO1.5p and the OO2p interface conditions, the algorithm requires less and less iterations. In other words, the strongest is the heterogeneity, the fewer iterations are required to solve the problem.

The robustness of the Schwarz algorithm equipped with the proposed OO2p interface conditions is now analyzed when more than two subdomains are involved in the mesh partition, as shown Figure 7. It is important to notice the existence of several cross-points in the mesh partition. The OO1.5p and the OO2p interface conditions reduces the number of iteration by a factor four in comparison to the OO1p interface conditions, see Table 3 when different ratio

Heterogeneity ratio $\mu^{(2)}/\mu^{(1)}$	Schwarz method with Optimized 1p	Schwarz method with Optimized 1.5p	Schwarz method with Optimized 2p
10	41	28	12
10^2	48	12	6
10^3	65	6	4
10^4	66	4	3

Table 2: Number of iterations for different interface conditions and different ratio of heterogeneity between the media. ($N_s = 2$, $h = 1/100$).

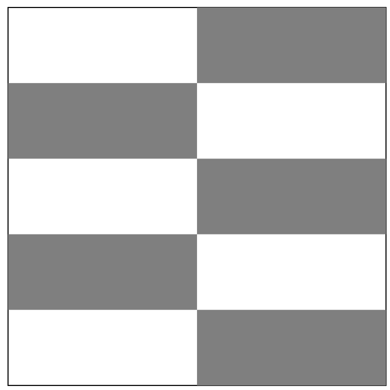

Figure 7: Mesh partition into two subdomains.

of heterogeneity are considered.

It is important to notice that one iteration of the Schwarz algorithm requires exactly the same number operations and the same CPU time for all the interface conditions considered in this paper i.e. OO1p, OO1.5p or OO2p, as already explained in [10].

6 Conclusions

In this paper, an original method to derive efficient interface conditions for the Schwarz solution procedure tuned to highly heterogeneous media is presented. These interface conditions are based on two parameters optimization and keep the property of the heterogeneity of the media on the interface. The optimized parameters are theoretically obtained from the solution of a minimization problem. In this paper, we propose to obtained the optimized parameters from the

Heterogeneity ratio $\mu^{(2)}/\mu^{(1)}$	Schwarz method with Optimized 1p	Schwarz method with Optimized 1.5p	Schwarz method with Optimized 2p
10^2	73	21	15
10^3	111	12	11
10^4	118	9	7

Table 3: Number of iterations for different interface conditions and different ratio of heterogeneity between the media. ($N_s = 10$, $h = 1/100$).

solution of an equation of degree four. An asymptotic approximation of these parameters when the mesh size tends to zero allows us to provide an upper bound for the asymptotic convergence rate of the Schwarz algorithm equipped with these interface conditions. This asymptotic convergence rate is equal to $\zeta_1 - \zeta_2\sqrt{h} + O(h)$, with $0 < \zeta_1 \ll 1$, and $0 < \zeta_2$. This result is better than the asymptotic convergence rate equal to $1 - \zeta\sqrt{h} + O(h)$ obtained in previous work when the optimization procedure was based on one single parameter. Numerical experiments illustrate the efficiency and robustness of the Scwharz algorithm equipped with such interface conditions.

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