Asymptotic behavior of Structures made of Plates

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Abstract. The aim of this work is to study the asymptotic behavior of a structure made of plates of thickness 2δ when $\delta\to0$. This study is carried on within the frame of linear elasticity by using the unfolding method. It is based on several decompositions of displacements of the structure and on the passing to the limit in fixed domains.

We begin with studying the displacements of a plate. We show that any displacement is the sum of an elementary displacement concerning the normal lines on the middle surface of the plate and a warping. An elementary displacement is linear with respect to the variable x_3. It is written $\mathcal{U}(\hat{x})+\mathcal{R}(\hat{x})\wedge x_3\mathbf{e}_3$ where $\mathcal{U}$ is a displacement of the mid-surface of the plate. We show a priori estimates and convergence results when $\delta\to0$. We characterize the limits of the unfolded displacements of a plate as well as the limits of the unfolded strained tensor.

Then we extend these results to structures made of plates. We show that any displacement of a structure is the sum of an elementary displacement of each plate and of a residual displacement. The elementary displacements of the structure (e.p.s.d.) coincide with elementary rods displacements in the junctions. Any e.p.s.d. is given by two functions belonging to $H^1(\mathbf{S};\mathbb{R}^3)$ where $\mathbf{S}$ is the skeleton of the structure (the set formed by the mid-surfaces of the plates constituting the surface). One of these functions, $\mathcal{U}$, is the skeleton displacement. We show that $\mathcal{U}$ is the sum of an extensional displacement and of an inextensional one. The first one characterizes the membrane displacements and the second one is a rigid displacement in the direction of the plates and it characterizes the flexion of the plates.

Eventually we pass to the limit as $\delta\to0$ in the linearized elasticity system. On the one hand we obtain a variational problem that is satisfied by the limit extensional displacement, and on the other hand, a variational problem satisfied by the limit of inextensional displacements.

Résumé. L'objectif de ce travail est l'étude du comportement asymptotique d'une structure formée de plaques d'épaisseur 2δ lorsque $\delta\to0$. Cette étude se place dans le cadre de l'élasticité linéaire et utilise la méthode de l'éclatement. Cette méthode est basée sur plusieurs décompositions des déplacements de la structure et sur le passage à la limite dans des domaines fixes.

On commence par étudier les déplacements d'une plaque. On montre que tout déplacement est la somme d'un déplacement élémentaire concernant les fibres transverses de la plaque et d'un gauchissement. Un déplacement élémentaire est linéaire par rapport à la variable x_3. Il s'écrit $\mathcal{U}(\hat{x})+\mathcal{R}(\hat{x})\wedge x_3\mathbf{e}_3$ où $\mathcal{U}$ est le déplacement de la surface moyenne de la plaque. On donne des estimations a priori et des résultats de convergence quand $\delta\to0$. On caractérise les limites des éclatés des déplacements ainsi que les limites des éclatés des composantes du tenseur des déformations.

Nous étendons ensuite ces résultats aux structures formées de plaques. On montre que tout déplacement de la structure est la somme d'un déplacement élémentaire de chaque plaque de la structure et d'un déplacement résiduel. Le déplacement élémentaire de la structure (d.e.s.) coïncide avec un déplacement élémentaire d'une structure-poutres dans les jonctions. Tout d.e.s. est donné par deux fonctions appartenant à $H^1(\mathbf{S};\mathbb{R}^3)$ où $\mathbf{S}$ est le squelette de la structure (l'ensemble formé par les surfaces moyennes des plaques). L'une de ces fonctions $\mathcal{U}$, est le déplacement du squelette. On montre que $\mathcal{U}$ est la somme d'un déplacement inextensionnel et d'un déplacement extensionnel. Le premier caractérise les déplacements membranaires des plaques et les déplacements des arêtes. Le second déplacement est rigide dans la direction des plaques; il caractérise la flexion des plaques.

Finalement on passe à la limite $\delta\to0$ dans les équations linéarisées de l'élasticité. On obtient d'une part un problème variationnel vérifié par le déplacement extensionnel limite et d'autre part un problème vérifié par le déplacement inextensionnel limite.

KEY WORDS: linear elasticity, plates, junctions, unfolding method.

1. Introduction

Many articles and books have been dedicated to the mathematical justification of plates models (see for example [1,4,5,8]). A first study concerning the asymptotic behavior of a structure made of two thin plates of thickness ε, is due to Le Dret [15]. The obtained asymptotic model derives from the three-dimensional system of elasticity thanks to a thin domain standard technique (the plates are transformed into a fixed domain). At the limit, Le Dret obtains a two-dimensional system coupling the flexion displacements of the two mid-surfaces of the plates.

Our study is a sequel to [9] and [10]. We extend the decompositions of a displacement using elementary, extensional and inextensional displacements to the plates and to the structures made of plates displacements. This is a new method to study the linearized elasticity problems in thin domains. We call it the unfolding method. It has been first introduced in periodic homogenization [7]. It is characterized by decompositions, estimates, transformations of the thin domains in fixed domains and some simplifications in the studied operators. This method can be used to solve problems posed in any domains with a thin dimension.

Our paper is organised into three parts. In the first one we study the displacements of a plate, the second one is devoted to the displacements of a structure made of plates from which we deduce the behavior of the strain tensor components in each reference plates. In the third part we give the asymptotic behavior of a structure made of thin plates.

In Section 2 we consider a plate of thickness 2δ. We first introduce the elementary displacements of a plate (Definition 2.1). These are the displacements of the normal lines of the mid-surface of the plate. An elementary displacement is linear with respect to the variable x_3. It is written $\mathcal{U}(\hat{x}) + \mathcal{R}(\hat{x}) \wedge x_3\mathbf{e}_3$ where $\mathcal{U}$ is a displacement of the mid-surface. By such a displacement the normal line is transformed into a line which is generally no longer perpendicular to the mid-surface. With each displacement u of the plate we associate an elementary displacement U_e and a warping $\overline{u}$ (Definition 2.2). Theorem 2.3 gives estimates of appropriate norms of U_e and of the displacement $\overline{u}$ in terms of δ. We are now equipped to obtain the asymptotic behavior of a displacements sequence $\big(u_\delta\big)_{\delta>0}$ with strain energy of order δ. This is the main result of this section and it is given in Theorem 2.6. The previous decomposition allows us to give a simple interpretation (see Theorem 2.6) of the limits of the unfolded $\mathcal{T}_\delta\big(\gamma_{ij}(u_\delta)\big)$ of the strain tensor $\gamma_{ij}(u_\delta)$ (where the unfolding operator $\mathcal{T}_\delta$ is given in Definition 2.5) in terms of the derivatives of a formal displacement of the reference plate. There is not a unique associated elementary displacement that satisfies estimates (2.2). In Definition 2.2 we give the simplest one. But the one we give in Definition 2.8 is more suitable for the study of a structure made of plates.

The structure $\mathcal{S}_\delta$ made of plates of thickness 2δ is introduced in Section 3. Our hypotheses about the skeleton of the structure $\mathbf{S}$ (i.e. the plates mid-surfaces set) allow us to consider a wide range of structures. We extend to them the notions and decompositions of Section 2. Definition 3.1 gives us the elementary plates structure displacements (e.p.s.d.). These displacements coincide with elementary plate displacements in each plate and there are elementary rod displacements in the junctions (see [9,10,11]). Any e.p.s.d. is known by two functions belonging to $H^1(\mathbf{S};\mathrm{I\!R}^3)$. The first one $\mathcal{U}$ is the skeleton displacement, the second one gives the rotations of the normal lines of the mid-surfaces. We show that $\mathcal{U}$ is the sum of an extensional displacement and of an inextensional one (Definitions 3.6 and 3.5). The first one characterizes the membrane displacements and the second one is a rigid displacement in the direction of the plates and it characterizes the plates flexion. Corollary of Lemma 3.7 gives estimates for them with an appropriate norm. In subsection 3.4 we consider an e.p.s.d. sequence $\big(u_\delta\big)_{\delta>0}$ with strain energy of order δ. Thanks to all these decompositions we give the limits of the unfolded $\mathcal{T}_\delta\big(\gamma_{ij}(u_\delta)\big)$ of the strain tensor as in the case of a plate. We also characterize the space of the inextensional limits displacements.

In the last section, we give the limit for $\delta \to 0$ of the linearized elasticity system (4.1), written in $\mathcal{S}_\delta$, where the applied forces F_δ satisfy assumptions (4.3). The main results are Theorem 4.1 and Theorem 4.3. In the first one we show that the extensional displacement limit is the solution of a second-order system, and in the second one we show that the limit of the inextensional displacement is the solution of a fourth-order system.

In this work we use the Einstein convention of summation over repeated indices. As a rule, the Greek indices α and β take values in $\{1,2\}$ and the Latin indices i, $i^{'}$, j and $j^{'}$ take values in $\{1,2,3\}$.

2. The plate displacements

2.1 The elementary plate displacements

The Euclidian space $\mathbb{R}^3$ is related to the frame $(O;\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3)$. Let ω be a bounded domain in $\mathbb{R}^2$ with a lipschitzian boundary. The plate $\Omega_\delta = \omega\times]-\delta,\delta[$, $\delta > 0$, is the open set having as middle surface ω and as thickness 2δ. The direction of the normal lines of ω is given by $\mathbf{e}_3$.The reference plate is the open set $\Omega = \omega\times]-1,1[$.

The running point of Ω_δ (respectively Ω) is denoted $x = (x_1,x_2,x_3) = (\widehat{x},x_3)$, (resp. $(\widehat{x},X_3)$) where $\widehat{x}\in\omega$ and $X_3 \in]-1,1[$.

For any open set $\omega^{'}$ of $\mathbb{R}^n$, $n\in\{2,3\}$, and any displacement u belonging to $H^1(\omega^{'};\mathbb{R}^n)$, we put

$$\mathcal{E}(u,\omega^{'}) = \int_{\omega^{'}} \gamma_{ij}(u)\gamma_{ij}(u), \qquad \gamma_{ij}(u) = \frac{1}{2}\Big\{\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big\}, \qquad \mathcal{D}(u,\omega^{'}) = \int_{\omega^{'}} \frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}.$$

Definition 2.1 : An **elementary plate displacement** (e.p.d.) is an element Φ belonging to $H^1(\Omega_\delta;\mathbb{R}^3)$, such that

$$\Phi(x) = \mathcal{A}(\widehat{x}) + \mathcal{B}(\widehat{x})\wedge x_3\mathbf{e}_3, \qquad \text{a.e. } x = (\widehat{x},x_3)\in\omega\times]-\delta,\delta[=\Omega_\delta,$$

where $\mathcal{A}$ and $\mathcal{B}$ belong to $H^1(\omega;\mathbb{R}^3)$; $\mathcal{A}$ is the mid-surface displacement and $\mathcal{B}$ is the rotation angles of the normal lines of the mid-surface.

Elementary plate displacement associated with a displacement of $H^1(\Omega_\delta;\mathbb{R}^3)$.

Definition 2.2 : With any displacement $u\in H^1(\Omega_\delta;\mathbb{R}^3)$, we associate the elementary plate displacement U_e and the warping $\overline{u}$ defined as

$$(2.1) \qquad \begin{cases} u(x) = U_e(x) + \overline{u}(x) & U_e(x) = \mathcal{U}(\widehat{x}) + \mathcal{R}(\widehat{x})\wedge x_3\mathbf{e}_3, \qquad x=(\widehat{x},x_3)\in\Omega_\delta,\\ \mathcal{U}(\widehat{x}) = \dfrac{1}{2\delta}\displaystyle\int_{-\delta}^{\delta} u(\widehat{x},x_3)dx_3, & \mathcal{R}(\widehat{x}) = \dfrac{3}{2\delta^3}\displaystyle\int_{-\delta}^{\delta} x_3\mathbf{e}_3\wedge u(\widehat{x},x_3)dx_3.\end{cases}$$

We get

$$\begin{cases} u_1(x) = \mathcal{U}_1(\widehat{x}) + \mathcal{R}_2(\widehat{x})x_3 + \overline{u}_1(x)\\ u_2(x) = \mathcal{U}_2(\widehat{x}) - \mathcal{R}_1(\widehat{x})x_3 + \overline{u}_2(x), \qquad x=(\widehat{x},x_3)\in\Omega_\delta,\\ u_3(x) = \mathcal{U}_3(\widehat{x}) + \overline{u}_3(x)\end{cases}$$

The displacement $\overline{u}$ gives us information on the deformation of the normal lines of the mid-surface.

Theorem 2.3 : *The elementary plate displacement U_e and the warping $\overline{u}$ verify*

$$(2.2) \qquad \mathcal{E}(U_e,\Omega_\delta) + \mathcal{D}(\overline{u},\Omega_\delta) + \frac{1}{\delta^2}||\overline{u}||^2_{L^2(\Omega_\delta;\mathbb{R}^3)} \le C\mathcal{E}(u,\Omega_\delta).$$

The constant depends only on ω.

The proof of Theorem 2.3 is based on Lemma 2.3 in [9] and on Lemma 2.4.

We denote

$$\omega_\eta = \big\{\widehat{x} \in \mathbb{R}^2 \mid dist(\widehat{x}, \omega) < \eta\big\}, \qquad \eta > 0.$$

Lemma 2.4 : *There exist $R \geq 1$ and $\delta_0 > 0$, depending only on ω, such that for any $\delta \in]0, \delta_0]$, $\omega_{2\delta}$ is covered by a family of open sets, of diameter less than $R\delta$, star-shaped with respect to a disc of radius $\delta/2$ and such that any point of $\omega_{2\delta}$ belongs to a finite number (independent of δ) of open sets of that family.*

Proof : The open set $\mathcal{A}_{pq} =](p-1/2)\delta, (p+3/2)\delta[\times](q-1/2)\delta, (q+3/2)\delta[$, $(p,q) \in \mathbb{Z}^2$, has a diameter of $2\sqrt{2}\delta$ and is star-shaped with respect to the disc of center $\big((p+1/2)\delta, (q+1/2)\delta)\big)$ and of radius $\delta/2$. Let $\mathcal{I}_\delta$ be the set of the pairs (p,q) of $\mathbb{Z}^2$ such that $\mathcal{A}_{pq} \subset \omega$. The distance between the boundary of ω and $\bigcup_{(p,q)\in\mathcal{I}_\delta} \mathcal{A}_{pq}$ is less than 3δ.

Let us proceed now to the covering of the neighborhood of the boundary of ω.
The boundary of ω is lipschitzian. Hence there exist constants $C < B < A$ and M strictly positive, a finite number N of local coordinate systems (x_{1r}, x_{2r}) in $(O_r; \mathbf{e}_{1r}, \mathbf{e}_{2r})$ and mappings $f_r : [-A, A] \longrightarrow \mathbb{R}$, Lipschitz continuous with ratio M, $1 \leq r \leq N$, such that

$$\begin{cases} \partial\omega = \displaystyle\bigcup_{r=1}^{N} \Big\{(x_{1r}, x_{2r}) \mid x_{2r} = f_r(x_{1r}), \quad |x_{1r}| < A - B\Big\}, \\ \big\{x \in \omega \,|\, dist(x, \partial\omega) < C\big\} \subset \displaystyle\bigcup_{r=1}^{N} \Big\{(x_{1r}, x_{2r}) \mid f_r(x_{1r}) < x_{2r} < f_r(x_{1r}) + B,\ |x_{1r}| \leq A\Big\} \subset \omega, \\ \omega_C \setminus \omega \subset \displaystyle\bigcup_{r=1}^{N} \Big\{(x_{1r}, x_{2r}) \mid f_r(x_{1r}) - B < x_{2r} < f_r(x_{1r}),\ |x_{1r}| \leq A\Big\} \subset \mathbb{R}^2 \setminus \omega. \end{cases}$$

Through the use of easy geometrical arguments we show that if $\delta \leq \inf\{C/3, B/4(1+M)\}$, we have

$$\omega_{2\delta} \setminus \omega \subset \bigcup_{r=1}^{N} \Big\{(x_{1r}, x_{2r}) \mid f_r(x_{1r}) - 2\delta\sqrt{1+M^2} < x_{2r} < f_r(x_{1r}), \quad |x_{1r}| \leq A\Big\},$$

$$\big\{x \in \omega \,|\, dist(x, \partial\omega) < 3\delta\big\} \subset \bigcup_{r=1}^{N} \Big\{(x_{1r}, x_{2r}) \mid f_r(x_{1r}) < x_{2r} < f_r(x_{1r}) + 3\delta\sqrt{1+M^2},\ |x_{1r}| \leq A\Big\}.$$

For any $a \in]-A, A-2\delta[$, the domains

$$\mathcal{B}_{\delta,a,r} = \big\{(x_{1r}, x_{2r}) \mid f_r(x_{1r}) - \delta(6M+2) < x_{2r} < f_r(x_{1r}) + \delta(6M+2),\ \ x_{1r} \in]a, a+2\delta[\big\}$$

and $\mathcal{B}_{\delta,a,r} \cap \omega$ are star-shaped with respect to the disc of center $(a+\delta, f_r(a) + (3M+1)\delta)$ and of radius $\delta/2$. These open sets have a diameter less than $(14M+6)\delta = R\delta$.
Eventually if $0 < \delta \leq \delta_0 = \inf\{C/3, B/(6M+4)\}$ the open sets $\mathcal{A}_{pq}$ $\big((p,q) \in \mathcal{I}_\delta\big)$, $\mathcal{B}_{\delta,a_p,r}$, where $a_p = p\delta \in [-A, A[$ $(p \in \mathbb{Z})$, $\mathcal{B}_{\delta,-A,r}$ and $\mathcal{B}_{\delta,A-2\delta,r}$ $(r \in \{1, \ldots, N\})$ cover $\omega_{2\delta}$; their diameter are less than $R\delta$ and they are star-shaped with respect to a disc of radius $\delta/2$. Any point of $\omega_{2\delta}$ belongs to a finite number (depending only on ω) of open sets of that family. □

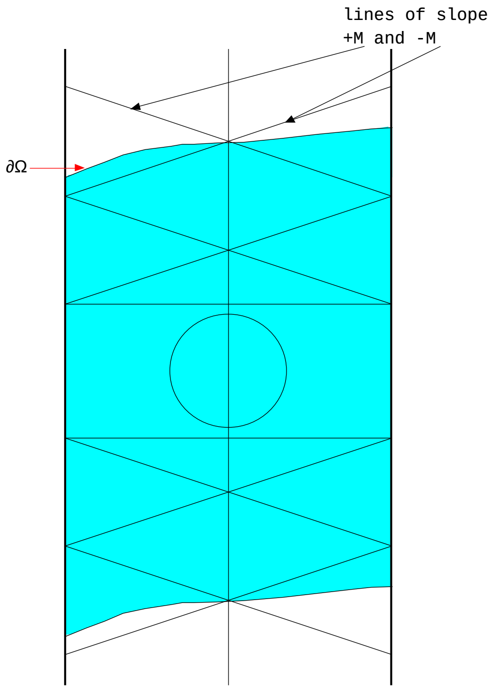

Figure 1. The domain $\mathcal{B}_{\delta,a,r}\cap\omega$

We denote $\{\omega^{'}_{\delta,n}\}_{n\in N_\delta}$ the covering of $\omega_{2\delta}$ obtained in Lemma 2.4 and $\{\omega_{\delta,n}\}_{n\in N_\delta}$ the covering of ω defined by $\omega_{\delta,n}=\omega^{'}_{\delta,n}\cap\omega$, $n\in N_\delta$.

Proof of Theorem 2.3 : The open set $\omega_{\delta,n}$ is star-shaped with respect to a disc of center A_n and of radius $\delta/2$. We put $\mathcal{O}_{\delta,n}=\omega_{\delta,n}\times]-\delta,\delta[\subset\Omega_\delta$, $n\in N_\delta$. The domain $\mathcal{O}_{\delta,n}$ has a diameter less than $(R+2)\delta$, and is star-shaped with respect to a ball of center A_n and of radius $\delta/2$. From Lemma 2.3 of [9], there exists a rigid displacement r_n such that

$$(2.3)\quad \mathcal{D}(u-r_n,\mathcal{O}_{\delta,n})+\frac{1}{\delta^2}||u-r_n||^2_{L^2(\mathcal{O}_{\delta,n};\mathbb{R}^3)}\le C\mathcal{E}(u,\mathcal{O}_{\delta,n}),\quad r_n(x)=a_n+b_n\wedge(x-A_n),\quad (a_n,b_n)\in\mathbb{R}^3.$$

The constant depends only on R.

We calculate the mean of $(u-r_n)(x)$ and of $x_3\mathbf{e}_3\wedge(u-r_n)(x)$ on the intervals $\{\widehat{x}\}\times]-\delta,\delta[$, $\widehat{x}\in\omega_{\delta,n}$, then we integrate on $\omega_{n,\delta}$ the inequalities we have obtained. Thanks to (2.3), we have

$$(2.4)\qquad \int_{\omega_{\delta,n}}|\mathcal{U}(\widehat{x})-a_n-b_n\wedge\overrightarrow{A_n\widehat{x}}|^2d\widehat{x}\le C\delta\mathcal{E}(u,\mathcal{O}_{\delta,n}),\qquad \int_{\omega_{\delta,n}}|\mathcal{R}_\alpha(\widehat{x})-b_{\alpha,n}|^2d\widehat{x}\le\frac{C}{\delta}\mathcal{E}(u,\mathcal{O}_{\delta,n}).$$

In (2.3) we eliminate the displacement r_n thanks to the estimations (2.4). Hence we have $||u-U_e||^2_{L^2(\mathcal{O}_{\delta,n};\mathbb{R}^3)}\le C\delta^2\mathcal{E}(u,\mathcal{O}_{\delta,n})$, then we add these inequalities and we obtain

$$||u-U_e||^2_{L^2(\Omega_\delta;\mathbb{R}^3)}\le C\delta^2\mathcal{E}(u,\Omega_\delta).$$

Both components of e.p.d. U_e belong to $H^1(\omega;\mathbb{R}^3)$. We calculate the mean of the gradient of $(u-r_n)(x)$, then the mean of $x_3\mathbf{e}_3\wedge\dfrac{\partial(u-r_n)}{\partial x_\alpha}$ on the intervals $\{\widehat{x}\}\times]-\delta,\delta[$, $\widehat{x}\in\omega_{\delta,n}$. Using (2.3) we obtain

$$(2.5)\qquad \Big\|\frac{\partial\mathcal{U}}{\partial x_\alpha}-b_n\wedge\mathbf{e}_\alpha\Big\|^2_{L^2(\omega_{\delta,n};\mathbb{R}^3)}+\delta^2\Big\|\frac{\partial\mathcal{R}}{\partial x_\alpha}\Big\|^2_{L^2(\omega_{\delta,n};\mathbb{R}^2)}\le\frac{C}{\delta}\mathcal{E}(u,\mathcal{O}_{\delta,n}),$$

hence, after elimination of b_n in the first inequality,

$$(2.6)\qquad \begin{cases} \Big\|\frac{\partial \mathcal{U}_1}{\partial x_1}\Big\|^2_{L^2(\omega_{\delta,n})}+\Big\|\frac{\partial \mathcal{U}_2}{\partial x_2}\Big\|^2_{L^2(\omega_{\delta,n})}+\Big\|\frac{\partial \mathcal{U}_1}{\partial x_2}+\frac{\partial \mathcal{U}_2}{\partial x_1}\Big\|^2_{L^2(\omega_{\delta,n})}\\ +\Big\|\frac{\partial \mathcal{U}_3}{\partial x_1}+\mathcal{R}_2\Big\|^2_{L^2(\omega_{\delta,n})}+\Big\|\frac{\partial \mathcal{U}_3}{\partial x_2}-\mathcal{R}_1\Big\|^2_{L^2(\omega_{\delta,n})}\le \frac{C}{\delta}\mathcal{E}(u,\mathcal{O}_{\delta,n})\end{cases}$$

From (2.5) and (2.6) we deduce the estimate of $\mathcal{E}(U_e,\mathcal{O}_{\delta,n})$

$$\mathcal{E}(U_e,\mathcal{O}_{\delta,n})\le C\mathcal{E}(u,\mathcal{O}_{\delta,n})\qquad \text{hence}\qquad \mathcal{E}(U_e,\Omega_\delta)\le C\mathcal{E}(u,\Omega_\delta).$$

From (2.3), (2.5), (2.6), and after elimination of the gradient of r_n we also deduce

$$\mathcal{D}(u-U_e,\mathcal{O}_{\delta,n})\le C\mathcal{E}(u,\mathcal{O}_{\delta,n}),\quad \text{hence}\quad \mathcal{D}(u-U_e,\Omega_\delta)\le C\mathcal{E}(u,\Omega_\delta).$$

Theorem 2.3 is proved. □

2.2 Limit of a plate displacements sequence.

Definition 2.5 : The unfolding operator $\mathcal{T}_\delta$ from $L^2(\Omega_\delta)$ into $L^2(\Omega)$ is defined by

$$\mathcal{T}_\delta(\phi)(\widehat{x},X_3)=\phi(\widehat{x},\delta X_3),\qquad \text{a.e. in }\ \Omega$$

For any element $\phi\in H^1(\Omega_\delta)$, we have $\mathcal{T}_\delta(\phi)\in H^1(\Omega)$ and

$$\frac{\partial \mathcal{T}_\delta(\phi)}{\partial x_\alpha}=\mathcal{T}_\delta\Big(\frac{\partial\phi}{\partial x_\alpha}\Big),\qquad \frac{\partial \mathcal{T}_\delta(\phi)}{\partial X_3}=\delta\mathcal{T}_\delta\Big(\frac{\partial\phi}{\partial x_3}\Big).$$

□

Let u be a displacement in $H^1(\Omega_\delta;\mathbb{R}^3)$. We decompose u into a sum of an elementary displacement and a warping, $u=U_e+\overline{u}$. We put

$$\overline{U}_1=\frac{1}{\delta}\mathcal{T}_\delta(\overline{u}_1)+X_3\Big(\frac{\partial \mathcal{U}_3}{\partial x_1}+\mathcal{R}_2\Big)\qquad \overline{U}_2=\frac{1}{\delta}\mathcal{T}_\delta(\overline{u}_2)+X_3\Big(\frac{\partial \mathcal{U}_3}{\partial x_2}-\mathcal{R}_1\Big)\qquad \overline{U}_3=\frac{1}{\delta}\mathcal{T}_\delta(\overline{u}_3)$$

From theorem (2.3) we have

$$\|\overline{U}\|_{L^2(\omega;H^1(-1,1;\mathbb{R}^3))}\le C\frac{|u|_{\mathcal{E}}}{\sqrt{\delta}}$$

Theorem 2.6 : *Let $\big(u_\delta\big)_{\delta>0}$ be a sequence of displacements of $H^1(\Omega_\delta;\mathbb{R}^3)$ verifying*

$$\mathcal{E}(u_\delta,\Omega_\delta)\le C\delta.$$

There exist $(a_\delta,b_\delta)\in\mathbb{R}^3\times\mathbb{R}^3$ and extracted sequences (still denoted in the same way), such that

$$(2.7)\qquad \begin{cases}\mathcal{U}_{1,\delta}-a_{1,\delta}+x_2b_{3,\delta}\rightharpoonup U_1,\qquad \mathcal{U}_{2,\delta}-a_{2,\delta}-x_1b_{3,\delta}\rightharpoonup U_2\qquad \textit{weakly in}\quad H^1(\omega)\\ \delta\big\{\mathcal{U}_{3,\delta}-a_{3,\delta}+x_1b_{2,\delta}-x_2b_{1,\delta}\big\}\rightharpoonup U_3\quad \textit{weakly in}\quad H^1(\omega).\end{cases}$$

Moreover U_3 belongs to $H^2(\omega)$. We have the following weak convergences of the unfolded of u_δ, $\overline{U}_\delta$ and of the components of the strain tensor :

$$
(2.8)\qquad \begin{cases}
\mathcal{T}_\delta\big(u_{1,\delta}-a_{1,\delta}+x_2b_{3,\delta}\big)\rightharpoonup U_1-X_3\dfrac{\partial U_3}{\partial x_1},\qquad \mathcal{T}_\delta\big(u_{2,\delta}-a_{2,\delta}-x_1b_{3,\delta}\big)\rightharpoonup U_2-X_3\dfrac{\partial U_3}{\partial x_2},\\
\delta\mathcal{T}_\delta\big(u_{3,\delta}-a_{3,\delta}+x_1b_{2,\delta}-x_2b_{1,\delta}\big)\rightharpoonup U_3,\quad \textit{weakly in}\quad H^1(\Omega),\\
\overline{U}_\delta\rightharpoonup\overline{U}\qquad \textit{weakly in}\quad L^2(\omega;H^1(-1,1;\mathbb{R}^3)),\\
\mathcal{T}_\delta\big(\gamma_{\alpha\beta}(u_\delta)\big)\rightharpoonup\dfrac{1}{2}\Big\{\dfrac{\partial U_\alpha}{\partial x_\beta}+\dfrac{\partial U_\beta}{\partial x_\alpha}\Big\}-X_3\dfrac{\partial^2U_3}{\partial x_\alpha\partial x_\beta},\qquad \mathcal{T}_\delta\big(\gamma_{\alpha3}(u_\delta)\big)\rightharpoonup\dfrac{1}{2}\dfrac{\partial\overline{U}_\alpha}{\partial X_3},\\
\mathcal{T}_\delta\big(\gamma_{33}(u_\delta)\big)\rightharpoonup\dfrac{\partial\overline{U}_3}{\partial X_3}\qquad \textit{weakly in}\quad L^2(\Omega).
\end{cases}
$$

Proof : With each u_δ we associate the e.p.d. $U_{e,\delta}$ with components $\mathcal{U}_\delta$ and $\mathcal{R}_\delta$. From (2.4) the displacement $\mathcal{U}_{M,\delta}=\mathcal{U}_{1,\delta}\mathbf{e}_1+\mathcal{U}_{2,\delta}\mathbf{e}_2$ has a strain energy $\mathcal{E}(\mathcal{U}_{M,\delta},\omega)\le\dfrac{C}{\delta}\mathcal{E}(U_{e,\delta},\Omega_\delta)\le\dfrac{C}{\delta}\mathcal{E}(u_\delta,\Omega_\delta)\le C$. The Korn inequality applied to $\mathcal{U}_{M,\delta}$ affirms the existence of a rigid displacement $r_{M,\delta}(\widehat{x})=\begin{pmatrix}a_{1,\delta}-b_{3,\delta}x_2\\ a_{2,\delta}+b_{3,\delta}x_1\end{pmatrix}$, such that

$$
||\mathcal{U}_{M,\delta}-r_{M,\delta}||^2_{L^2(\omega;\mathbb{R}^2)}+\mathcal{D}(\mathcal{U}_{M,\delta}-r_{M,\delta},\omega)\le C\mathcal{E}(\mathcal{U}_{M,\delta},\omega)\le\frac{C}{\delta}\mathcal{E}(u_\delta,\Omega_\delta)\le C.
$$

If $b_{\alpha,\delta}$ is the mean of $\mathcal{R}_{\alpha,\delta}$ on ω, we obtain from the Poincaré-Wirtinger inequality

$$
||\mathcal{R}_{\alpha,\delta}-b_{\alpha,\delta}||^2_{L^2(\omega)}\le C||\nabla\mathcal{R}_{\alpha,\delta}||^2_{[L^2(\omega)]^2}\le\frac{C}{\delta^3}\mathcal{E}(u_\delta,\Omega_\delta)\le\frac{C}{\delta^2}
$$

The estimate of $\mathcal{E}(U_{e,\delta},\Omega_\delta)$ obtained in Theorem 2.3, gives

$$
(2.9)\qquad \Big\|\frac{\partial\mathcal{U}_{3,\delta}}{\partial x_1}+\mathcal{R}_{2,\delta}\Big\|^2_{L^2(\omega)}+\Big\|\frac{\partial\mathcal{U}_{3,\delta}}{\partial x_2}-\mathcal{R}_{1,\delta}\Big\|^2_{L^2(\omega)}\le\frac{C}{\delta}\mathcal{E}(u_\delta,\Omega_\delta)\le C,
$$

hence

$$
\Big\|\frac{\partial\mathcal{U}_{3,\delta}}{\partial x_1}+b_{2,\delta}\Big\|^2_{L^2(\omega)}+\Big\|\frac{\partial\mathcal{U}_{3,\delta}}{\partial x_2}-b_{1,\delta}\Big\|^2_{L^2(\omega)}\le\frac{C}{\delta^3}\mathcal{E}(u_\delta,\Omega_\delta)\le\frac{C}{\delta^2}.
$$

We apply the Poincaré-Wirtinger inequality to the function $\mathcal{U}_{3,\delta}+b_{2,\delta}x_1-b_{1,\delta}x_2$. There exists $a_{3,\delta}$ such that

$$
||\mathcal{U}_{3,\delta}-a_{3,\delta}+b_{2,\delta}x_1-b_{1,\delta}x_2||^2_{L^2(\omega)}\le\frac{C}{\delta^3}\mathcal{E}(u_\delta,\Omega_\delta)\le\frac{C}{\delta^2}.
$$

The sequences $\mathcal{U}_{M,\delta}-r_{M,\delta}$, $\delta\big\{\mathcal{U}_{3,\delta}-a_{3,\delta}+b_{2,\delta}x_1-b_{1,\delta}x_2\big\}$, $\delta(\mathcal{R}_{\alpha,\delta}-b_{\alpha,\delta})$ and $\overline{U}_\delta$ are bounded in $H^1(\omega;\mathbb{R}^2)$ (respectively $H^1(\omega)$ and $L^2(\omega;H^1(-1,1;\mathbb{R}^3))$. We extract from these sequences some subsequences, still denoted in the same way, such that

$$
(2.10)\qquad \begin{cases}
\mathcal{U}_{1,\delta}-a_{1,\delta}+x_2b_{3,\delta}\rightharpoonup U_1,\quad \mathcal{U}_{2,\delta}-a_{2,\delta}-x_1b_{3,\delta}\rightharpoonup U_2\quad \text{weakly in}\quad H^1(\omega),\\
\delta\big\{\mathcal{U}_{3,\delta}-a_{3,\delta}+x_1b_{2,\delta}-x_2b_{1,\delta}\big\}\rightharpoonup U_3\quad \text{weakly in}\quad H^1(\omega),\\
\delta(\mathcal{R}_{\alpha,\delta}-b_{\alpha,\delta})\rightharpoonup\mathcal{R}_\alpha\quad \text{weakly in}\quad H^1(\omega),\\
\overline{U}_\delta\rightharpoonup\overline{U}\qquad \text{weakly in}\quad L^2(\omega;H^1(-1,1;\mathbb{R}^3)).
\end{cases}
$$

The limits of the sequences $\delta\Big\{\dfrac{\partial\mathcal{U}_{3,\delta}}{\partial x_1}+\mathcal{R}_{2,\delta}\Big\}$ and $\delta\Big\{\dfrac{\partial\mathcal{U}_{3,\delta}}{\partial x_2}-\mathcal{R}_{1,\delta}\Big\}$ are equal to zero by (2.9), hence the equalities

$$
(2.11)\qquad \frac{\partial U_3}{\partial x_1}=-\mathcal{R}_2,\qquad \frac{\partial U_3}{\partial x_2}=\mathcal{R}_1,
$$

and the belonging of U_3 to $H^2(\omega)$. From the limits (2.10) and from the equalities (2.11) we immediately deduce the limits of the unfolded $\mathcal{T}_\delta\big(u_{1,\delta}-a_{1,\delta}+x_2b_{3,\delta}\big)$, $\mathcal{T}_\delta\big(u_{2,\delta}-a_{2,\delta}-x_1b_{3,\delta}\big)$ and $\delta\mathcal{T}_\delta\big(u_{3,\delta}-a_{3,\delta}+x_1b_{2,\delta}-x_2b_{1,\delta}\big)$ in $H^1(\Omega)$.
Now we calculate the components of the strain tensor and we transform them by unfolding

$$\begin{aligned}
&\mathcal{T}_\delta\big(\gamma_{11}(u_\delta)\big)=\frac{\partial\mathcal{U}_{1,\delta}}{\partial x_1}+\frac{X_3}{\delta}\frac{\partial\mathcal{R}_{2,\delta}}{\partial x_1}+\mathcal{T}_\delta\Big(\frac{\partial\overline{u}_{1,\delta}}{\partial x_1}\Big),\qquad \mathcal{T}_\delta\big(\gamma_{22}(u_\delta)\big)=\frac{\partial\mathcal{U}_{2,\delta}}{\partial x_2}-\frac{X_3}{\delta}\frac{\partial\mathcal{R}_{1,\delta}}{\partial x_2}+\mathcal{T}_\delta\Big(\frac{\partial\overline{u}_{2,\delta}}{\partial x_2}\Big),\\
&\mathcal{T}_\delta\big(\gamma_{12}(u_\delta)\big)=\frac{1}{2}\Big\{\frac{\partial\mathcal{U}_{1,\delta}}{\partial x_2}+\frac{\partial\mathcal{U}_{2,\delta}}{\partial x_1}\Big\}-\frac{X_3}{2\delta}\Big\{\frac{\partial\mathcal{R}_{1,\delta}}{\partial x_1}-\frac{\partial\mathcal{R}_{2,\delta}}{\partial x_2}\Big\}+\frac{1}{2}\mathcal{T}_\delta\Big(\frac{\partial\overline{u}_{1,\delta}}{\partial x_2}+\frac{\partial\overline{u}_{2,\delta}}{\partial x_1}\Big),\\
&\mathcal{T}_\delta\big(\gamma_{\alpha3}(u_\delta)\big)=\frac{1}{2}\frac{\partial\overline{U}_{\alpha,\delta}}{\partial X_3}+\frac{1}{2}\mathcal{T}_\delta\Big(\frac{\partial\overline{u}_{3,\delta}}{\partial x_\alpha}\Big),\qquad\qquad \mathcal{T}_\delta\big(\gamma_{33}(u_\delta)\big)=\frac{\partial\overline{U}_{3,\delta}}{\partial X_3}.
\end{aligned}$$

Thanks to the previous limits and estimates of $\overline{u}_\delta$ we obtain the last limits of (2.8). □

Corollary : For the warping $\overline{u}_\delta$ we have the following convergences

$$\begin{aligned}
&\frac{1}{\delta}\mathcal{T}_\delta(\overline{u}_{\alpha,\delta})\rightharpoonup\overline{U}_\alpha-\frac{3X_3}{2}\int_{-1}^{1}\overline{U}_\alpha(.,s)s\,ds\qquad\text{weakly in}\quad L^2(\Omega)\\
&\frac{1}{\delta}\mathcal{T}_\delta(\overline{u}_{3,\delta})\rightharpoonup\overline{U}_3\qquad\text{weakly in}\quad L^2(\Omega)
\end{aligned}$$

2.3 A second decomposition of a plate displacement

The decomposition of a displacement given in Definition 2.2 is the simplest and the most natural. It is perfectly adequate to study a plate fixed on a part of its lateral boundary. In the case of a structure made of several plates we must take into account the rotation of the plates around an axis perpendicular to the mid-surface. These rotations will be obtained by introducing a new decomposition of a displacement into the sum of an elementary displacement and a residual one. In this decomposition the third component of the rotation vector will not be null any longer.

In this subsection we modify the plates geometry by making their rims round. This modification allows us to anticipate the plates shape of the structures we study in the following section.

We consider now a round-rimmed plate $\Omega^{'}_\delta$ with a middle surface ω_δ. We denote

$$\begin{aligned}
&\Omega^{'}_\delta=\big\{x\in\mathbb{R}^3\mid dist(x,\omega)<\delta\ \big\},\qquad \Gamma_\delta=\big\{x\in\mathbb{R}^3\mid dist(x,\partial\omega)<\delta)\big\}\\
&\Omega^{''}_\delta=\big\{x\in\mathbb{R}^2\mid dist(x,\omega)<2\delta\ \big\}\times]-\delta,\delta[=\omega_{2\delta}\times]-\delta,\delta[.
\end{aligned}$$

where ω is a <u>*polygonal*</u> bounded domain in $\mathbb{R}^2$.

The boundary of ω is made of a finite number of segments. Let $\mathcal{C}$ be a connected component of $\partial\omega$. There exists $\delta_0>0$ such that for any $\delta\in]0,\delta_0]$ the domains

$$\mathcal{C}_\delta=\Big\{x\in\mathbb{R}^3\mid dist(x,\mathcal{C})<\delta\Big\},\quad\text{and}\quad \mathcal{C}_{3\delta}=\Big\{x\in\mathbb{R}^3\mid dist(x,\mathcal{C})<3\delta\Big\}$$

are rods structures. Then there exists $\eta_1>0$ such that all the balls centered in a vertex of $\mathcal{C}$, and of radius $3\eta_1\delta$ contain the junctions of the rods belonging to $\mathcal{C}_{3\delta}$.
We recall that for any $\delta\in]0,\delta_0]$, there exists an extension operator, linear and continuous, P_δ from $H^1(\mathcal{C}_\delta)$ into $H^1(\mathcal{C}_{3\delta})$ such that for any $\phi\in H^1(\mathcal{C}_\delta)$,

$$P_\delta(\phi)_{|\mathcal{C}_\delta}=\phi,\qquad ||P_\delta(\phi)||_{L^2(\mathcal{C}_{3\delta})}+\delta||\nabla P_\delta(\phi)||_{[L^2(\mathcal{C}_{3\delta})]^3}\le C\Big\{||\phi||_{L^2(\mathcal{C}_\delta)}+\delta||\nabla\phi||_{[L^2(\mathcal{C}_\delta)]^3}\Big\}.$$

The constant does not depend on δ.

Lemma 2.7 : *For any $\delta\in]0,\delta_0]$, there exists an extension operator $\mathcal{P}_\delta$, linear and continuous from $H^1(\Omega^{'}_\delta;\mathbb{R}^3)$ into $H^1(\Omega^{''}_\delta;\mathbb{R}^3)$ such that*

$$\mathcal{P}_\delta(u)_{|\Omega^{'}_\delta}=u,\qquad \begin{cases}\mathcal{E}(\mathcal{P}_\delta(u),\Omega^{''}_\delta)\le C\mathcal{E}(u,\Omega^{'}_\delta)\\ \mathcal{E}(\mathcal{P}_\delta(u),\Omega^{''}_\delta\setminus\Omega_\delta)\le C\mathcal{E}(u,\Gamma_\delta)\end{cases}$$

The constants do not depend on δ.

Proof : We begin with extending u in the neighborhood of a connected component of $\partial\omega$.
Let $\mathcal{C}$ be a connected component of $\partial\omega$. The restriction of u to $\mathcal{C}_\delta$ is a displacement belonging to $H^1(\mathcal{C}_\delta;\mathbb{R}^3)$. Hence there exists an elementary rods structure displacement $U_{e,R}$ (see [9]) which coincides with a rigid displacement in each set $B(A,3\eta_1\delta)\cap\mathcal{C}_\delta$ where A is a vertex of $\mathcal{C}$ and which verifies

$$\mathcal{E}(U_{e,R},\mathcal{C}_\delta)+\mathcal{D}(u-U_{e,R},\mathcal{C}_\delta)+\frac{1}{\delta^2}||u-U_{e,R}||^2_{L^2(\mathcal{C}_\delta,\mathbb{R}^3)}\le C\mathcal{E}(u,\mathcal{C}_\delta).\tag{2.12}$$

The displacement $U_{e,R}$ is also an elementary rod structure displacement of $\mathcal{C}_{3\delta}$ and $\mathcal{E}(U_{e,R},\mathcal{C}_{3\delta})\le C\mathcal{E}(u,\mathcal{C}_\delta)$.
The displacement

$$u^{'}_{\mathcal{C}_{3\delta}}=U_{e,R}+P_\delta(u-U_{e,R})$$

is an extension of u to the set $\mathcal{C}_{3\delta}$. From (2.12) we have the following inequalities:

$$\mathcal{E}(u^{'}_{\mathcal{C}_{3\delta}},\mathcal{C}_{3\delta})\le 2\big\{\mathcal{E}(U_{e,R},\mathcal{C}_{3\delta})+\mathcal{E}(P_\delta(u-U_{e,R}),\mathcal{C}_{3\delta})\big\}\le C\mathcal{E}(u,\mathcal{C}_\delta)\tag{2.13}$$

In the same way we build an extension of u in the neighborhood of the other connected components of $\partial\omega$. The extension $\mathcal{P}_\delta(u)$ is then the displacement which coincides with u in $\Omega^{'}_\delta$ and which is equal to one of the previous extensions in $\Omega^{''}_\delta\setminus\Omega^{'}_\delta$. The estimates of the lemma are the immediate consequences of the inequalities (2.13) obtained in the neighborhood of each connected components of ω. □

Remarks : If one of the edges of the boundary of ω is fixed we can take an elementary rods structure displacement with its two components equal to zero on this edge without modifying the estimates (2.12) and then extend u beyond this edge by 0.
We also can construct an extension operator P_δ when ω is of lipschitzian boundary with the help of a few changes. □

The extension of u to $\Omega^{''}_\delta$ is still denoted u.

A second elementary plate displacement associated with a displacement of $H^1(\Omega^{'}_\delta;\mathbb{R}^3)$.

Definition 2.8 : With any $u\in H^1(\Omega^{'}_\delta;\mathbb{R}^3)$ we associate the elementary plate displacement $U^{'}_e$ and the residual displacement $\overline{u}^{'}$ defined by

$$\begin{cases}u(x)=U^{'}_e(\widehat{x})+\overline{u}^{'}(x) & U^{'}_e(x)=\mathcal{U}^{'}(\widehat{x})+\mathcal{R}^{'}(\widehat{x})\wedge x_3\mathbf{e}_3, \qquad x\in\Omega^{'}_\delta,\\ \mathcal{U}^{'}(\widehat{x})=\dfrac{6}{\pi\delta^3}\displaystyle\int_{B(O;\delta/2)}u(\widehat{x}+\mathbf{z})d\mathbf{z}, & \mathcal{R}^{'}(\widehat{x})=\dfrac{60}{\pi\delta^5}\displaystyle\int_{B(O;\delta/2)}\mathbf{z}\wedge u(\widehat{x}+\mathbf{z})d\mathbf{z}, \qquad \widehat{x}\in\omega_\delta.\end{cases}\tag{2.14}$$

Now, similarly to the elementary rod displacements (see [9,10,11]), the third component of the rotation angles $\mathcal{R}^{'}$ is not null.

Theorem 2.9 : *We have the following inequalities :*

$$\mathcal{E}(U^{'}_e,\Omega^{'}_\delta)+\mathcal{D}(\overline{u}^{'},\Omega^{'}_\delta)+\frac{1}{\delta^2}||\overline{u}^{'}||^2_{L^2(\Omega^{'}_\delta;\mathbb{R}^3)}\le C\mathcal{E}(u,\Omega^{'}_\delta).\tag{2.15}$$

The constant depends only on ω.

Proof : We now consider the covering $\{\omega^{'}_{\delta,n}\}_{n\in N_\delta}$ of $\omega_{2\delta}$ obtained in Lemma 2.4.
We put $\omega^{''}_{\delta,n}=\{\widehat{x}\in\omega^{'}_{\delta,n}\mid dist(\widehat{x},\partial\omega^{'}_{\delta,n})>\delta/2\}$ and $\mathcal{O}^{'}_{\delta,n}=\omega^{'}_{\delta,n}\times]-\delta,\delta[$, $\mathcal{O}^{''}_{\delta,n}=\omega^{''}_{\delta,n}\times]-\delta,\delta[$, $n\in N_\delta$. The family $\{\omega^{''}_{\delta,n}\}_{1\le n\le N_\delta}$ verifies

$$measure\Big(\bigcup_{n\in N_\delta}\omega^{''}_{\delta,n}\setminus\omega_\delta\Big)=0$$

From Lemma 2.3 in [9], for any open set $\mathcal{O}^{'}_{\delta,n}$ there exists a rigid displacement r_n such that

$$\mathcal{D}(u-r_n,\mathcal{O}^{'}_{\delta,n})+\frac{1}{\delta^2}||u-r_n||^2_{L^2(\mathcal{O}^{'}_{\delta,n};\mathbb{R}^3)}\le C\mathcal{E}(u,\mathcal{O}^{'}_{\delta,n}),\quad r_n(x)=a_n+b_n\wedge(x-A_n),\quad (a_n,b_n)\in\mathbb{R}^3.$$

The constants do not depend on n nor δ. Now we go on as in Theorem 2.3 (see also Theorem 4.3 in [9]) to obtain the estimates of Theorem 2.9. □

Corollary : From the expression of $U^{'}_e$ and thanks to the estimates of Theorem 2.9 we have

$$||\nabla\mathcal{R}^{'}||^2_{L^2(\omega_\delta;\mathbb{R}^6)}\le C\frac{\mathcal{E}(u,\Omega^{'}_\delta)}{\delta^3}\qquad\Big\|\frac{\partial\mathcal{U}^{'}}{\partial x_\alpha}-\mathcal{R}^{'}\wedge\mathbf{e}_\alpha\Big\|^2_{L^2(\omega_\delta;\mathbb{R}^3)}\le C\frac{\mathcal{E}(u,\Omega^{'}_\delta)}{\delta}$$

□

Remark : We consider again the sequence of displacements $(u_\delta)_{\delta>0}$ of Theorem 2.6. Now we take the second decomposition of the displacements : $u_\delta=U^{'}_{e,\delta}+\overline{u}^{'}_\delta$ given by (2.14). We get

$$\mathcal{E}(U^{'}_{e,\delta},\Omega^{'}_\delta)+\mathcal{D}(\overline{u}^{'}_\delta,\Omega^{'}_\delta)+\frac{1}{\delta^2}||\overline{u}^{'}_\delta||^2_{L^2(\Omega^{'}_\delta;\mathbb{R}^3)}\le C\mathcal{E}(u_\delta,\Omega^{'}_\delta)\le C\delta.$$

Then we obtain

$$||U_{e,\delta}-U^{'}_{e,\delta}||^2_{L^2(\Omega^{'}_\delta;\mathbb{R}^3)}\le C\delta^3\quad\Longrightarrow\quad||\mathcal{U}_\delta-\mathcal{U}^{'}_\delta||_{L^2(\omega;\mathbb{R}^3)}\le C\delta\qquad||\mathcal{R}_{\alpha,\delta}-\mathcal{R}^{'}_{\alpha,\delta}||_{L^2(\omega)}\le C.$$

After extraction of subsequences expressed by the same notation, we obtain the convergences

$$\begin{cases}\mathcal{U}_\delta-\mathcal{U}^{'}_\delta\longrightarrow 0\qquad\delta(\mathcal{R}_\delta-\mathcal{R}^{'}_\delta)\longrightarrow 0\qquad\text{strongly in}\quad L^2(\omega;\mathbb{R}^3)\\ \dfrac{1}{\delta}\mathcal{T}_\delta(\overline{u}^{'}_\delta)\rightharpoonup\overline{U}^{'}\qquad\text{weakly in}\quad L^2(\omega;H^1(-1,1;\mathbb{R}^3)).\end{cases}$$

The limits of the unfolded $\mathcal{T}_\delta(\gamma_{i3}(u_\delta))$ give $\dfrac{\partial\overline{U}^{'}}{\partial X_3}=\dfrac{\partial\overline{U}}{\partial X_3}$. Except for the limit of the sequence of the unfolded $\dfrac{1}{\delta}\mathcal{T}_\delta(\overline{u}^{'}_\delta)$, the limits (2.7) and (2.8) do not depend on the decomposition of the displacement u_δ into the sum of an elementary plate displacement and a residual one. What matters is to be able to approximate u_δ with the help of an elementary plate displacement that verifies the estimates (2.15).

3. The displacements of a structure made of plates

3.1 The structure made of plates

We work on a set of N plane bounded domains with polygonal boundary, included in $\mathbb{R}^3$, $(\omega_l)_{1\le l\le N}$. The skeleton $\mathbf{S}$ is the union of $(\overline{\omega}_l)_{1\le l\le N}$. A *face* of $\mathbf{S}$ is a closed set $\overline{\omega}_l$. An *edge* of $\mathbf{S}$ is a maximal segment shared by a set of faces or a maximal segment belonging to the boundary of a face. A *vertex* of $\mathbf{S}$ is an extremity of an edge.

Hypotheses : We suppose

H1 • for any pair of faces $(\overline{\omega}_l, \overline{\omega}_p)$, there exists a sequence of faces $\overline{\omega}_l = \overline{\omega}_{l_0},\ \overline{\omega}_{l_1},\ \dots\ \overline{\omega}_{l_k} = \overline{\omega}_p$ such that $\overline{\omega}_{l_r}$ and $\overline{\omega}_{l_{r+1}}$ have an edge in common, $0 \le r \le k-1$,

H2 • for any vertex A and any pair of faces $(\overline{\omega}_l, \overline{\omega}_p)$ containing A, there exists a sequence of faces $\overline{\omega}_l = \overline{\omega}_{l_0},\ \overline{\omega}_{l_1},\ \dots\ \overline{\omega}_{l_k} = \overline{\omega}_p$ such that $\overline{\omega}_{l_r}$ and $\overline{\omega}_{l_{r+1}}$ have an edge in common containing A, $0 \le r \le k-1$,

H3 • the skeleton $\mathbf{S}$ is fixed all along some edges.

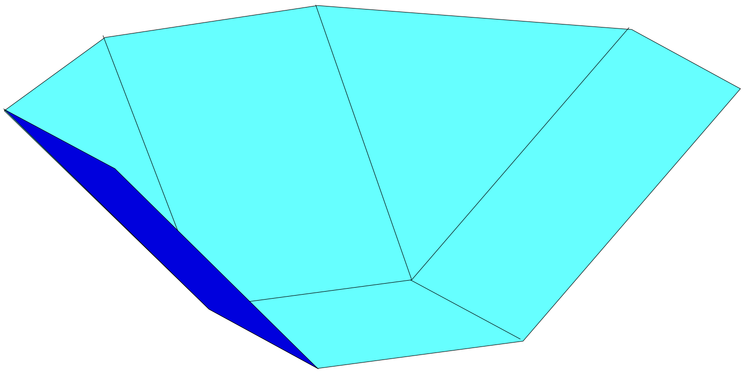

Figure 2. The skeleton $\mathbf{S}$.

We denote

- Γ_0 the fixed part of the skeleton,
- $\mathcal{J}$ the set of edges common to several faces,
- $\mathcal{N}$ the set of vertexes common to several faces.

The structure made of plates is the domain $\mathcal{S}_\delta = \big\{x \in \mathbb{R}^3 \mid dist(x, \mathbf{S}) < \delta\big\}$. This structure is made of the gathering of the plates $\Omega^{'}_{l,\delta}$ with thickness 2δ, with middle surface $\omega_{l,\delta}$ and with rounded rim. Each domain $\Omega^{'}_{l,\delta}$ is equipped with a local frame $(O^{(l)}; \mathbf{e}_1^{(l)}, \mathbf{e}_2^{(l)}, \mathbf{e}_3^{(l)})$, $O^{(l)} \in \omega_l$, $\mathbf{e}_3^{(l)}$ is the normal direction to the face $\overline{\omega}_l$. The plate $\Omega^{'}_{l,\delta}$ contains the plate $\Omega_{l,\delta} = \omega_l \times]-\delta, \delta[$ ($\mathbb{R}^3$ being equipped with the above local frame). The reference plate is the open set $\Omega_l = \omega_l \times]-1, 1[$ obtained through the transformation of $\Omega_{l,\delta}$ by the orthogonal affiniy of ratio $1/\delta$.

The structure $\mathcal{S}_\delta$ is fixed to part $\Gamma_{0,\delta} = \big\{x \in \partial\mathcal{S}_\delta \mid dist(x, \Gamma_0) = \delta\big\}$ of its boundary. For each edge $J \in \mathcal{J}$ we choose a unit vector $\mathbf{e}_J$ in the direction of the edge.

We consider $\mathbf{S}_\delta = \bigcup\limits_{l=1}^{N} \overline{\omega}_{l,\delta}$, the set of the middle surfaces of the plates.

There exist two constants $\delta_0 > 0$ and $\eta_0 \ge 1$ depending only on the skeleton $\mathbf{S}$ such that for any $\delta \in]0, \delta_0]$ the parts common to several plates are in the union of the junctions

$$\bigcup_{J \in \mathcal{J}} \Big\{x \in \mathbb{R}^3 \mid dist(x, J) < \eta_0 \delta\Big\}.$$

The real η_0 depends on the smallest angle between two faces having in common a same edge.

There exists also $\eta_1 \ge \eta_0$ such that for any vertex $A \in \mathcal{N}$ the ball $B(A; \eta_1\delta)$ contains the common part of the plates having the point A in common.

We choose the real δ_0 such that the balls centered in the extremities of the edges and of radius $(2\eta_1 + 1)\delta_0$ are disjoint.

The restriction of a function ϕ, defined on $\mathbf{S}$, (resp. $\mathbf{S}_\delta$, $\mathcal{S}_\delta$), to ω_l, (resp. $\omega_{l,\delta}$, $\Omega^{'}_{l,\delta}$) is denoted $\phi^{(l)}$. In the same way, we denote $x_\alpha^{(l)} = x^{(l)} \cdot \mathbf{e}_\alpha^{(l)}$ and $x_3^{(l)} = x^{(l)} \cdot \mathbf{e}_3^{(l)}$ the local variables $\big(x^{(l)} = (\widehat{x}^{(l)}, x_3^{(l)}) = (x_1^{(l)}, x_2^{(l)}, x_3^{(l)})\big)$.

The space $H^1(\mathbf{S};\mathbb{R}^n)$ (resp. $H^1(\mathbf{S}_\delta;\mathbb{R}^n)$) is the set of the functions ϕ defined a.e. in $\mathbf{S}$ (resp. $\mathbf{S}_\delta$), with values in $\mathbb{R}^n$, such that the restriction $\phi^{(l)}$ belongs to $H^1(\omega_l;\mathbb{R}^n)$ (resp. $H^1(\omega_{l,\delta};\mathbb{R}^n)$) and such that for any edge $J\in\mathcal{J}$ and any pair of faces $(\overline{\omega}_l,\overline{\omega}_k)$ containing J, we have the equality of the restrictions to J, $(\phi^{(l)})_{|J}=(\phi^{(k)})_{|J}$ in $H^{1/2}(J,\mathbb{R}^n)$.

The space $H^1_{\Gamma_0}(\mathbf{S};\mathbb{R}^n)$ is the subspace of $H^1(\mathbf{S};\mathbb{R}^n)$ the elements of which are a.e. equal to zero on Γ_0. We equip $H^1(\mathbf{S}_\delta;\mathbb{R}^3)$ with the inner product

$$[\mathcal{U},\mathcal{V}]=\sum_{l=1}^N\int_{\omega_{l,\delta}}\Big\{\mathcal{U}^{(l)}\cdot\mathcal{V}^{(l)}+\frac{\partial\mathcal{U}^{(l)}}{\partial x^{(l)}_\alpha}\cdot\frac{\partial\mathcal{V}^{(l)}}{\partial x^{(l)}_\alpha}\Big\}.$$

The associated norm is denoteted $||\cdot||$.

3.2 The elementary displacements of plates structure

Definition 3.1 : An **elementary plates structure displacement** (e.p.s.d.) is a displacement Φ belonging to $H^1(\mathcal{S}_\delta;\mathbb{R}^3)$, such that there exist two elements $\mathcal{A}$ and $\mathcal{B}$ in $H^1(\mathbf{S}_\delta;\mathbb{R}^3)$, such that for any $l\in\{1,\dots,N\}$,

$$\Phi^{(l)}(x)=\mathcal{A}^{(l)}(\widehat{x}^{(l)})+\mathcal{B}^{l)}(\widehat{x}^{(l)})\wedge x_3^{(l)}\mathbf{e}_3^{(l)}$$

is an elementary plate displacement of the plate $\Omega^{'}_{l,\delta}$.

The functions $\mathcal{A}$ and $\mathcal{B}$ are respectively the first component and the second component of the e.p.s.d. Φ. The function $\mathcal{A}$ accounts for the displacement of the skeleton faces while $\mathcal{B}$ accounts for the rotation of the normal directions to the plates and of the rotation of the faces around the edges.

Theorem 3.2 : *For any displacement $u\in H^1_{\Gamma_0}(\mathcal{S}_\delta;\mathbb{R}^3)$ there exists an elementary plates structure displacement U_e of components $(\mathcal{U},\mathcal{R})\in H^1_{\Gamma_0}(\mathbf{S}_\delta;\mathbb{R}^3)\times H^1_{\Gamma_0}(\mathbf{S}_\delta;\mathbb{R}^3)$ such that*

$$\left\{\begin{aligned}&\sum_{l=1}^N\Big\{\delta^3||\nabla\mathcal{R}^{(l)}||^2_{L^2(\omega_{l,\delta};\mathbb{R}^6)}+\delta\Big\|\frac{\partial\mathcal{U}^{(l)}}{\partial x^{(l)}_\alpha}-\mathcal{R}^{(l)}\wedge\mathbf{e}^{(l)}_\alpha\Big\|^2_{L^2(\omega_{l,\delta};\mathbb{R}^3)}\Big\}\le C\mathcal{E}(u,\mathcal{S}_\delta),\\&\mathcal{E}(U_e,\mathcal{S}_\delta)+\mathcal{D}(u-U_e,\mathcal{S}_\delta)\le C\mathcal{E}(u,\mathcal{S}_\delta),\quad ||u-U_e||^2_{L^2(\mathcal{S}_\delta;\mathbb{R}^3)}\le C\delta^2\mathcal{E}(u,\mathcal{S}_\delta).\end{aligned}\right.\tag{3.1}$$

Proof : Let u be in $H^1_{\Gamma_0}(\mathcal{S}_\delta;\mathbb{R}^3)$. For any l belonging to $\{1,\dots,N\}$, we extend the restriction $u^{(l)}$ to the plate $\Omega^{'}_{l,\delta}$, into a displacement of $\Omega^{''}_{l,\delta}$ (the plate of thickness 2δ and of middle surface $\omega_{l,2\delta}$). Therefore, using the formulas (2.14), we can define an elementary plate displacement $U_e^{'(l)}$ of the plate $\Omega^{''}_{l,\delta}$ verifying (2.15). Both components $\mathcal{U}^{'(l)}$ and $\mathcal{R}^{'(l)}$ of $U_e^{'(l)}$ are the restrictions of an element belonging to $H^1(\mathbf{S}_{2\delta};\mathbb{R}^3)$.

Step 1 We consider a plate $\Omega^{'}_\delta$ of mid-surface ω_δ. Let u be a displacement of this plate, thanks to Lemma 2.7 we extend u to a displacement of the plate $\Omega^{''}_\delta$. We decompose u into the sum of an elementary plate displacement $U^{'}_{e,P}$, given by (2.14), and of a residual displacement $\overline{u}^{'}_P$,

$$u(x)=U^{'}_{e,P}(x)+\overline{u}^{'}_P(x)\qquad\qquad U^{'}_{e,P}(x)=\mathcal{U}^{'}_P(\widehat{x})+\mathcal{R}^{'}_P(\widehat{x})\wedge x_3\mathbf{e}_3,\qquad x\in\Omega^{''}_\delta.$$

Now we modify the elementary plate displacement $U^{'}_{e,P}$ in the neighborhood of an edge J contained in $\overline{\omega}$. Without being detrimental to the general case we can suppose that the edge's direction is $\mathbf{e}_1$ and that one of these extremities is the chosen origin on the face, so that J is identified with the segment $[0,L]\times\{0\}$ where L is the edge's length. Let J_δ and $J^{'}_\delta$ be the rods

$$\begin{aligned}J_\delta&=]-2\eta_1\delta,L+2\eta_1\delta[\times]-2\eta_1\delta,2\eta_1\delta[\times]-\delta,\delta[\cap\Omega^{''}_\delta\\J^{'}_\delta&=]-(2\eta_1+1)\delta,L+(2\eta_1+1)\delta[\times]-(2\eta_1+1)\delta,(2\eta_1+1)\delta[\times]-\delta,\delta[\cap\Omega^{''}_\delta\end{aligned}$$

The restriction of the displacement u to the rod J_δ can be decomposed into the sum of an elementary rod displacement $U_{e,R}$ and of a residual displacement $\overline{u}_R$ (see [9,10,11])

$$u(x) = U_{e,R}(x) + \overline{u}_R(x) \qquad\qquad U_{e,R}(x) = \mathcal{U}_R(x_1) + \mathcal{R}_R(x_1) \wedge \big(x_2\mathbf{e}_2 + x_3\mathbf{e}_3\big) \qquad x \in J_\delta.$$

The two components $\mathcal{U}_R$ and $\mathcal{R}_R$ of $U_{e,R}$ belong to $H^1(-2\eta_1\delta, L + 2\eta_1\delta; \mathbb{R}^3)$ and we have (see [9,10,11])

$$\mathcal{E}(U_{e,R}, J_\delta) + \mathcal{D}(\overline{u}_R, J_\delta) + \frac{1}{\delta^2}||\overline{u}_R||^2_{L^2(J_\delta;\mathbb{R}^3)} \le C\mathcal{E}(u, J_\delta). \tag{3.2}$$

We can choose the displacement $U_{e,R}$ such that $\mathcal{U}_R$ (respectively $\mathcal{R}_R$) depends only on the restriction of $\mathcal{U}^{'}_P$ (respectively $\mathcal{R}^{'}_P$) to the line $]-2\eta_1\delta, L + 2\eta_1\delta[\times\{0\}$ and moreover such that

$$\begin{aligned}
&\mathcal{U}_R(x_1) = \mathcal{U}^{'}_P(x_1, 0) \qquad \mathcal{R}_R(x_1) = \mathcal{R}^{'}_P(x_1, 0) \qquad x_1 \in]2\eta_1\delta, L - 2\eta_1\delta[\\
&\mathcal{U}_R(0) = \mathcal{U}^{'}_P(0, 0) \qquad \mathcal{R}_R(0) = \mathcal{R}^{'}_P(0, 0) \qquad \mathcal{U}_R(L) = \mathcal{U}^{'}_P(L, 0) \qquad \mathcal{R}_R(L) = \mathcal{R}^{'}_P(L, 0).
\end{aligned}$$

The displacement $U_{e,R}$ coincides with a rigid displacement in the balls centered in the extremities of J and of radius $\eta_1\delta$ (see [9]). The functions $\mathcal{U}_R$ and $\mathcal{R}_R$ are extended into functions belonging to $H^1_{loc}(\mathbb{R}; \mathbb{R}^3)$. These extensions are then identified with elements belonging to $H^1_{loc}(\mathbb{R}^2; \mathbb{R})$ depending only on the variable x_1. From (2.15), we have

$$\mathcal{E}(U_{e,P}, J_\delta) + \mathcal{D}(\overline{u}^{'}_P, J_\delta) + \frac{1}{\delta^2}||\overline{u}^{'}_P||^2_{L^2(J_\delta;\mathbb{R}^3)} \le C\mathcal{E}(u, J^{'}_\delta). \tag{3.3}$$

Hence from (3.2) and (3.3) we obtain

$$\mathcal{D}(U^{'}_{e,P} - U_{e,R}, J_\delta) + \frac{1}{\delta^2}||U^{'}_{e,P} - U_{e,R}||^2_{L^2(J_\delta;\mathbb{R}^3)} \le C\mathcal{E}(u, J^{'}_\delta). \tag{3.4}$$

The constant depends only on η_1.
We are now going to modify the elementary plate displacement $U_{e,P}$ in the neighborhood of J.

We consider a function m belonging to $\mathcal{C}^\infty(\mathbb{R}^+; \mathbb{R})$ such that

$$m(t) = 1 \quad \forall t \ge 2, \qquad m(t) = 0 \quad \forall t \le 1, \qquad |m^{'}(t)| \le 2 \quad \forall t \in \mathbb{R}. \tag{3.5}$$

We define the components, $\mathcal{U}$ and $\mathcal{R}$, of a new elementary plate displacement U_e by

$$\begin{cases}
\mathcal{U}(\widehat{x}) = U_{e,R}(\widehat{x})\Big[1 - m\Big(\dfrac{dist(\widehat{x}, J)}{\eta_1\delta}\Big)\Big] + \mathcal{U}^{'}_P(\widehat{x})m\Big(\dfrac{dist(\widehat{x}, J)}{\eta_1\delta}\Big),\\
\mathcal{R}(\widehat{x}) = \mathcal{R}_R(\widehat{x})\Big[1 - m\Big(\dfrac{dist(\widehat{x}, J)}{\eta_1\delta}\Big)\Big] + \mathcal{R}^{'}_P(\widehat{x})m\Big(\dfrac{dist(\widehat{x}, J)}{\eta_1\delta}\Big), \qquad \widehat{x} \in \omega_\delta,\\
U_e(x) = \mathcal{U}(\widehat{x}) + \mathcal{R}(\widehat{x}) \wedge x_3\mathbf{e}_3, \qquad x \in \Omega^{''}_\delta.
\end{cases}$$

Hence by definition of U_e we have

$$\begin{aligned}
&\text{if } x \in \Omega^{''}_\delta \text{ and } dist(\widehat{x}, J) < \eta_1\delta \quad \text{then} \quad U_e(x) = U_{e,R}(x)\\
&\text{if } x \in \Omega^{''}_\delta \text{ and } dist(\widehat{x}, J) > 2\eta_1\delta \quad \text{then} \quad U_e(x) = U^{'}_{e,P}(x).
\end{aligned}$$

Thanks to (3.2), (3.3) and (3.4) the elementary plate displacement U_e verifies

$$\mathcal{E}(U_e,\Omega^{''}_\delta)\le C\mathcal{E}(u,\Omega^{'}_\delta),\qquad \mathcal{D}(U_e-U^{'}_{e,P},\Omega^{''}_\delta)\le C\mathcal{E}(u,J^{'}_\delta),\qquad ||U_e-U^{'}_{e,P}||^2_{L^2(\Omega^{''}_\delta;\mathbb{R}^3)}\le C\delta^2\mathcal{E}(u,J^{'}_\delta).$$

The constants depend only on ω, J and η_1.

Step 2 Now we modify $U_e^{'(l)}$ in the neighborhood of each junction in order to obtain an elementary plates structure displacement belonging to $H^1_{\Gamma_0}(\mathcal{S}_\delta;\mathbb{R}^3)$.
Let J be an edge belonging to the faces $\overline{\omega}_{l_1},\ \dots,\overline{\omega}_{l_p}$. We consider the elementary plate displacement $U_e^{(l_k)}$, $k\in\{1,\dots,p\}$, given in Step 1. In the neighborhood of J we have $U_e^{(l_1)}=\dots=U_e^{(l_p)}$. Now we go on with all the edges belonging to $\mathcal{J}$. We obtain an elementary plates structure displacement U_e which is an elementary rods structure displacement in the neighborhood of the edges and which is equal to $U_e^{'(l)}$ in $\Omega^{'}_{l,\delta}$ and outside the junctions. We then deduce (3.1). □

Proposition 3.3 (Korn inequality) : *For any displacement* $u\in H^1_{\Gamma_0}(\mathcal{S}_\delta;\mathbb{R}^3)$, *we have*

$$\delta||\mathcal{R}||^2+\delta||\mathcal{U}||^2+\mathcal{D}(u,\mathcal{S}_\delta)+||u||^2_{L^2(\mathcal{S}_\delta;\mathbb{R}^3)}\le\frac{C}{\delta^2}\mathcal{E}(u,\mathcal{S}_\delta).$$

The constant does not depend on δ.
Proof : The estimates (3.1) of the gradients of the $\mathcal{R}^{(l)}$ functions, the nullity of $\mathcal{R}$ on Γ_0 and the hypothesis **H1** allow us to obtain step by step $||\mathcal{R}||^2_{L^2(\mathbf{S}_\delta;\mathbb{R}^3)}\le C/\delta^3\mathcal{E}(u,\mathcal{S}_\delta)$. This inequality and (3.1) give then an upperbound of the L^2 norms of the functions gradients $\mathcal{U}^{(l)}$. The nullity of $\mathcal{U}$ on Γ_0 and the hypothesis **H1** imply then that $||\mathcal{U}||^2_{L^2(\mathbf{S}_\delta;\mathbb{R}^3)}\le C/\delta^3\mathcal{E}(u,\mathcal{S}_\delta)$. From these estimates of $\mathcal{U}$ and $\mathcal{R}$ follow $\mathcal{D}(U_e,\mathcal{S}_\delta)\le C/\delta^2\mathcal{E}(u,\mathcal{S}_\delta)$ and $||U_e||^2_{L^2(\mathcal{S}_\delta;\mathbb{R}^3)}\le C/\delta^2\mathcal{E}(u,\mathcal{S}_\delta)$. Then again, thanks to (3.1), we obtain the estimates of the L^2 norm of u and of its gradient. □

3.3 Inextensional displacements, extensional displacements

We first recall some classical results on a Sobolev space with weight. Let ω be a bounded domain in $\mathbb{R}^2$ with lipschitzian boundary and I, J two closed lines included in $\overline{\omega}$ with a common point O. We denote

$$H^1_\rho(\omega)=\Big\{\phi\in L^2(\omega)\mid\sqrt{\rho}\nabla\phi\in[L^2(\omega)]^2\Big\}$$

where $\rho(x)$ is the distance between x and O. We have

- $H^1_\rho(\omega)$ is continuously imbedded in $W^{1,p}(\omega)$ for all $p\in[1,4/3[$ and is compactly imbedded in $L^2(\omega)$,
- $H^1(\omega)$ is dense in $H^1_\rho(\omega)$,
- for any $u\in H^{1/2}(I)$ and any $v\in H^{1/2}(J)$, there exists a lifting $w\in H^1_\rho(\omega)$ of u and v such that

$$w_{|I}=u,\qquad w_{|J}=v,\qquad\text{and}\qquad ||w||^2_{L^2(\omega)}+\int_\omega\rho|\nabla w|^2\le C\big\{||u||^2_{H^{1/2}(I)}+||v||^2_{H^{1/2}(J)}\big\}.$$

The constant depends only on I and J.

The space $H^1_{\rho,\Gamma_0}(\mathbf{S};\mathbb{R}^3)$ is the set of the functions ϕ defined a.e. in $\mathbf{S}$, with values in $\mathbb{R}^3$, such that

- for any $l\in\{1,\dots,N\}$, the restrictions $\phi_1^{(l)}$ and $\phi_2^{(l)}$ belong to $H^1(\omega_l)$ and $\phi_3^{(l)}$ belongs to

$$H^1_\rho(\omega_l)=\Big\{\psi\in L^2(\omega_l)\mid\sqrt{\rho}\,\nabla\psi\in[L^2(\omega_l)]^2\ \Big\}$$

where $\rho(\widehat{x})=dist(x,\mathcal{N})$ (distance from the point $x\in\mathbb{R}^3$ to the vertexes belonging to several faces),

- for any edge $J \in \mathcal{J}$ and any pair of faces $(\overline{\omega}_l, \overline{\omega}_k)$ containing J, we have $(\phi^{(l)})_{|J} = (\phi^{(k)})_{|J}$ in $H^{1/2}(J, \mathrm{I\!R}^3)$,
- the function ϕ is equal to zero on Γ_0.

We equip $H^1_{\rho,\Gamma_0}(\mathbf{S}; \mathrm{I\!R}^3)$ with the inner product

$$< \mathcal{U}, \mathcal{V} >_\rho = \sum_{l=1}^{N} \int_{\omega_l} \Big\{ \gamma_{\alpha\beta}(\mathcal{U}^{(l)}) \gamma_{\alpha\beta}(\mathcal{V}^{(l)}) + \rho \nabla \mathcal{U}^{(l)}_3 \cdot \nabla \mathcal{V}^{(l)}_3 \Big\},$$

and with the norm $|\mathcal{U}|_\rho = \sqrt{< \mathcal{U}, \mathcal{U} >_\rho}$. The usual norm on $H^1_{\rho,\Gamma_0}(\mathbf{S}; \mathrm{I\!R}^3)$ is

$$||\mathcal{U}||_\rho = \sqrt{\sum_{l=1}^{N} \int_{\omega_l} \Big\{ |\nabla \mathcal{U}^{(l)}_1|^2 + |\nabla \mathcal{U}^{(l)}_2|^2 + \rho |\nabla \mathcal{U}^{(l)}_3|^2 \Big\}}$$

Lemma 3.4 : *The norms $|| \cdot ||_\rho$ and $| \cdot |_\rho$ are equivalent in the space $H^1_{\rho,\Gamma_0}(\mathbf{S}; \mathrm{I\!R}^3)$. Moreover $H^1_{\Gamma_0}(\mathbf{S}; \mathrm{I\!R}^3)$ is dense in $H^1_{\rho,\Gamma_0}(\mathbf{S}; \mathrm{I\!R}^3)$.*

Proof : Let be V in $H^1_{\rho,\Gamma_0}(\mathbf{S}; \mathrm{I\!R}^3)$. We apply the bidimentional Korn inequality to the membrane displacements $V^{(l)}_M = V^{(l)}_1 \mathbf{e}^{(l)}_1 + V^{(l)}_2 \mathbf{e}^{(l)}_2$ and then we add all the inequalities to obtain

$$\sum_{l=1}^{N} \big\{ ||\nabla V^{(l)}_1||^2_{[L^2(\omega_l)]^2} + ||\nabla V^{(l)}_2||^2_{[L^2(\omega_l)]^2} \big\} \le C \big(|V|^2_\rho + ||V||^2_{L^2(\mathbf{S}; \mathrm{I\!R}^3)} \big)$$

hence

$$||V||_\rho \le C \big\{ |V|_\rho + ||V||_{L^2(\mathbf{S}; \mathrm{I\!R}^3)} \big\}.$$

The space $H^1_{\rho,\Gamma_0}(\mathbf{S}; \mathrm{I\!R}^3)$ is embedded in $L^2(\mathbf{S}; \mathrm{I\!R}^3)$. Then we prove by contradiction that there exists a constant C_0 such that $||V||_\rho \le C_0 |V|_\rho$. Moreover we can immediately see that there exists C_1 such that $|V|_\rho \le C_1 ||V||_\rho$. The norms $| \cdot |_\rho$ and $|| \cdot ||_\rho$ are therefore equivalent. □

Definition 3.5 : An **inextensional displacement of the skeleton** is an element U belonging to $H^1_{\rho,\Gamma_0}(\mathbf{S}; \mathrm{I\!R}^3)$ such that

$$\forall l \in \{1, \dots, N\}, \qquad \gamma_{\alpha\beta}(U^{(l)}) = 0, \qquad \text{in} \qquad \omega_l.$$

The membrane component $U^{(l)}_M = U^{(l)}_1 \mathbf{e}^{(l)}_1 + U^{(l)}_2 \mathbf{e}^{(l)}_2$ of an inextensional displacement is a rigid displacement of the face $\overline{\omega}_l$. The inextensional displacements space of the skeleton is denoted $D_I(\mathbf{S})$.

Definition 3.6 : An **extensional displacement of the skeleton** is an element of the orthogonal $D_E(\mathbf{S})$ of $D_I(\mathbf{S})$ in $H^1_{\rho,\Gamma_0}(\mathbf{S}; \mathrm{I\!R}^3)$.

The set of extensional displacements is equipped with the semi-norm

$$||U||_E = \sqrt{\sum_{l=1}^{N} \int_{\omega_l} \gamma_{\alpha\beta}(U^{(l)}) \gamma_{\alpha\beta}(U^{(l)})}, \qquad U \in D_E(\mathbf{S}).$$

The semi-norm $|| \cdot ||_E$ is a norm, because if $U \in D_E(\mathbf{S})$ is such that $||U||_E = 0$ then, $\gamma_{\alpha\beta}(U^{(l)}) = 0$ for any l. The displacement U is then of inextensional type and is equal to zero.

Lemma 3.7 : *The norms $|| \cdot ||_E$ and $| \cdot |_\rho$ are equivalent in $D_E(\mathbf{S})$.*

Proof : Let V be in $D_E(\mathbf{S})$. As in the proof of Lemma 3.4 we get

$$\sum_{l=1}^{N}\big\{||\nabla V_1^{(l)}||^2_{[L^2(\omega_l)]^2}+||\nabla V_2^{(l)}||^2_{[L^2(\omega_l)]^2}\big\}\le C\big(||V||^2_E+||V||^2_{L^2(\mathbf{S};\mathbb{R}^3)}\big)$$

We put $J\in\mathcal{J}$ a common edge to the faces $\overline{\omega}_l$ and $\overline{\omega}_k$. The restrictions to J of the membrane displacements $V_M^{(l)}$ and $V_M^{(k)}$ completely define the restriction $V_{|J}$. Hence we get

$$\sum_{J\in\mathcal{J}}||V_{|J}||^2_{H^{1/2}(J,\mathbb{R}^3)}\le C\big(||V||^2_E+||V||^2_{L^2(\mathbf{S};\mathbb{R}^3)}\big)$$

There exists a displacement $W\in H^1_{\rho,\Gamma_0}(\mathbf{S};\mathbb{R}^3)$ such that

$$W_\alpha^{(l)}=V_\alpha^{(l)},\ \ \forall l\in\{1,\dots,N\},\qquad W_{|J}=V_{|J},\ \ \forall J\in\mathcal{J},$$

and verifying

$$|W|^2_\rho\le C\sum_{l=1}^{N}\big\{||\nabla V_1^{(l)}||^2_{[L^2(\omega_l)]^2}+||\nabla V_2^{(l)}||^2_{[L^2(\omega_l)]^2}\big\}+C\sum_{J\in\mathcal{J}}||V_{|J}||^2_{H^{1/2}(J;\mathbb{R}^3)}\le C\big(||V||^2_E+||V||^2_{L^2(\mathbf{S};\mathbb{R}^3)}\big).$$

The displacement $V-W$ is of inextensional type and hence orthogonal to V, hence

$$|V|_\rho\le|W|_\rho\le C\big(||V||_E+||V||_{L^2(\mathbf{S};\mathbb{R}^3)}\big).\tag{3.6}$$

Now we show that the norm $||\cdot||_E$ is equivalent to the norm $|\cdot|_\rho$ in $D_E(\mathbf{S})$. We already have $||V||_E\le|V|_\rho$ for any $V\in D_E(\mathbf{S})$. We suppose that the norms are not equivalent. For any $n\in\mathbb{N}^*$, we can find $V_n\in D_E(\mathbf{S})$ such that $||V_n||_E\le 1/n$ and $|V_n|_\rho=1$. The sequence $\big(V_n\big)_{n\in\mathbb{N}^*}$ being bounded in $H^1_\rho(\mathbf{S};\mathbb{R}^3)$, we can then extract a sub-sequence, still denoted in the same way, such that

$$V_n\rightharpoonup V\quad\text{weakly in}\quad H^1_{\rho,\Gamma_0}(\mathbf{S};\mathbb{R}^3).$$

The limit V belongs also to $D_E(\mathbf{S})$. Let us make n tend to infinity in the inequality $||V_n||_E\le 1/n$, we obtain $\gamma_{\alpha\beta}(V^{(l)})=0$. The displacement V is of inextensional type, and hence is equal to zero.
The space $H^1_\rho(\mathbf{S};\mathbb{R}^3)$ is compactly imbedded in $L^2(\mathbf{S};\mathbb{R}^3)$. Hence the sequence $\big(V_n\big)_{n\in\mathbb{N}^*}$ converges strongly to 0 in $L^2(\mathbf{S};\mathbb{R}^3)$, hence $||V_n||_{L^2(\mathbf{S};\mathbb{R}^3)}\longrightarrow 0$. From (3.6) follows that the sequence $\big(V_n\big)_{n\in\mathbb{N}^*}$ converges strongly to 0 in $H^1_{\rho,\Gamma_0}(\mathbf{S};\mathbb{R}^3)$. This stands in contradiction with $|V_n|_\rho=1$. □

Corollary : Let u be a displacement belonging to $H^1_{\Gamma_0}(\mathcal{S}_\delta;\mathbb{R}^3)$ and U_e the e.p.s.d. given by Theorem 3.2. The restriction to $\mathbf{S}$ of the first component $\mathcal{U}$ of U_e can be written as the sum of an extensional displacement and an inextensional displacement,

$$\mathcal{U}=U_E+U_I,\qquad U_E\in D_E(\mathbf{S}),\quad U_I\in D_I(\mathbf{S}).$$

According to the inequalities (3.1) and Lemma 3.7 we have

$$|U_E|^2_\rho\le C||U_E||^2_E\le\frac{C}{\delta}\mathcal{E}(u,\mathcal{S}_\delta),\qquad |U_I|^2_\rho\le\frac{C}{\delta^3}\mathcal{E}(u,\mathcal{S}_\delta).\tag{3.7}$$

The constants are independent of δ.

3.4 The limit displacements

Let $\big(u_\delta\big)_{\delta>0}$ be a sequence of displacements belonging to $H^1_{\Gamma_0}(\mathcal{S}_\delta;\mathbb{R}^3)$ and verifying

$$\mathcal{E}(u_\delta,\mathcal{S}_\delta)\le C\delta,$$

where the constant is independent of δ. Thanks to the estimates of Proposition 3.3, (3.7) and (2.4), from the sequences $\delta\mathcal{U}_\delta$, $\delta\mathcal{R}_\delta$, $\delta U_{I,\delta}$ and $U_{E,\delta}$ we extract some sub-sequences, still denoted in the same way and which weakly converge,

$$\begin{cases}\delta\mathcal{U}_\delta\rightharpoonup U_I, \qquad \delta\mathcal{R}_\delta\rightharpoonup\mathcal{R}\qquad\text{weakly in}\quad H^1_{\Gamma_0}(\mathbf{S};\mathbb{R}^3),\\ \delta U_{I,\delta}\rightharpoonup U_I\qquad\text{weakly in}\quad D_I(\mathbf{S}),\\ U_{E,\delta}\rightharpoonup U_E\qquad\text{weakly in}\quad D_E(\mathbf{S}).\end{cases}$$

The sequences $\delta\mathcal{U}_\delta$ and $\delta U_{I,\delta}$ have the same limit in $H^1_{\rho,\Gamma_0}(\mathbf{S};\mathbb{R}^3)$ because the sequence $\delta U_{E,\delta}$ converges to 0 in $H^1_\rho(\mathbf{S};\mathbb{R}^3)$. After passing to the limit and from (3.1) comes

$$\forall l\in\{1,\dots,N\},\quad\forall\alpha\in\{1,2\},\qquad\frac{\partial U_I^{(l)}}{\partial x_\alpha^{(l)}}=\mathcal{R}^{(l)}\wedge\mathbf{e}_\alpha^{(l)}.$$

The membrane displacement $U_{I,1}^{(l)}\mathbf{e}_1^{(l)}+U_{I,2}^{(l)}\mathbf{e}_2^{(l)}$ of the face $\omega^{(l)}$ is rigid. We have

$$\mathcal{R}_3^{(l)}=\frac{\partial U_{I,2}^{(l)}}{\partial x_1^{(l)}}=-\frac{\partial U_{I,1}^{(l)}}{\partial x_2^{(l)}}$$

The function $\mathcal{R}^{(l)}\cdot\mathbf{e}_3^{(l)}$ is constant on the face $\omega^{(l)}$. This component of $\mathcal{R}^{(l)}$ gives the rotation angle of the face $\omega^{(l)}$ around a normal line to the face.

Now we define the space of the inextensional displacements limits. We put

$$\mathcal{D}_I(\mathbf{S})=\Big\{\mathcal{A}\in D_I(\mathbf{S})\cap H^1_{\Gamma_0}(\mathbf{S};\mathbb{R}^3)\mid\exists\mathcal{B}\in H^1_{\Gamma_0}(\mathbf{S};\mathbb{R}^3),\ \frac{\partial\mathcal{A}^{(l)}}{\partial x_\alpha^{(l)}}=\mathcal{B}^{(l)}\wedge\mathbf{e}_\alpha^{(l)},\quad\forall l\in\{1,\dots,N\}\Big\}$$

For any $\mathcal{A}\in\mathcal{D}_I(\mathbf{S})$, there is only one $\mathcal{B}$ which we denote $\widehat{\nabla}\mathcal{A}$. Then we have

$$\forall l\in\{1,\dots,N\},\qquad\forall\alpha\in\{1,2\},\qquad\frac{\partial\mathcal{A}^{(l)}}{\partial x_\alpha^{(l)}}=\widehat{\nabla}\mathcal{A}^{(l)}\wedge\mathbf{e}_\alpha^{(l)}.$$

We equip $\mathcal{D}_I(\mathbf{S})$ with the norm $||\mathcal{A}||_I=||\widehat{\nabla}\mathcal{A}||_{H^1(\mathbf{S};\mathbb{R}^3)}$. The inextensional limit displacement U_I belongs to $\mathcal{D}_I(\mathbf{S})$.

3.5 Limit of the unfolded displacements and limit of the unfolded strain tensor components

Using (2.7) and after transformation by unfolding we have the following limits in the refering plates :

$$
(3.8)\qquad \begin{cases} \delta\mathcal{T}_\delta(u_\delta^{(l)}) \rightharpoonup U_I^{(l)} \qquad \text{weakly in} \quad H^1(\Omega_l;\mathbb{R}^3) \\ \mathcal{T}_\delta(u_{\alpha,\delta}^{(l)}-U_{I,\alpha,\delta}^{(l)}) \rightharpoonup U_{E,\alpha}^{(l)} - X_3^{(l)}\dfrac{\partial U_{I,3}^{(l)}}{\partial x_\alpha^{(l)}} \qquad \text{weakly in} \quad H^1(\Omega_l), \\ \mathcal{T}_\delta\big(\gamma_{\alpha\beta}(u_\delta^{(l)})\big) \rightharpoonup \dfrac{1}{2}\Big\{\dfrac{\partial U_{E,\alpha}^{(l)}}{\partial x_\beta^{(l)}} + \dfrac{\partial U_{E,\beta}^{(l)}}{\partial x_\alpha^{(l)}}\Big\} - X_3^{(l)}\dfrac{\partial^2 U_{I,3}^{(l)}}{\partial x_\alpha^{(l)}\partial x_\beta^{(l)}} \quad \text{weakly in} \quad L^2(\Omega_l), \\ \mathcal{T}_\delta\big(\gamma_{\alpha 3}(u_\delta^{(l)})\big) \rightharpoonup \dfrac{1}{2}\dfrac{\partial \overline{U}_\alpha^{(l)}}{\partial X_3^{(l)}}, \qquad \mathcal{T}_\delta\big(\gamma_{33}(u_\delta^{(l)})\big) \rightharpoonup \dfrac{\partial \overline{U}_3^{(l)}}{\partial X_3^{(l)}} \qquad \text{weakly in} \quad L^2(\Omega_l). \end{cases}
$$

where $\overline{U}^{(l)}$ belongs to $L^2(\omega^{(l)};H^1(-1,1;\mathbb{R}^3))$. We choose $\overline{U}^{(l)}$ such that $\displaystyle\int_{-1}^1 \overline{U}^{(l)}(\cdot,s)ds = 0$.

3.6 The inextensional displacements of $D_I(\mathbf{S})$ and of $\mathcal{D}_I(\mathbf{S})$.

The inextensional displacements of $D_I(\mathbf{S})$

Let V be an inextensional displacement. From the definition of the inextensional displacements we have $\gamma_{\alpha\beta}(V^{(l)}) = 0$ in $\overline{\omega}_l$. Hence, in each face the membrane displacement $V_M^{(l)} = V_1^{(l)}\mathbf{e}_1^{(l)} + V_2^{(l)}\mathbf{e}_2^{(l)}$ is a rigid displacement. The restriction of V to an edge $J \in \mathcal{J}$ is then

$$
(3.9)\qquad V_{|J}(M) = \overrightarrow{A}_J + \overrightarrow{B}_J \wedge \overrightarrow{A_J M} \qquad \forall M \in J
$$

where A_J is an vertex of the edge. The vectors $\overrightarrow{A}_J$ and $\overrightarrow{B}_J$ depend only on the edge. We choose $\overrightarrow{B}_J$ orthogonal to $\mathbf{e}_J$ (an unit vector in the direction of the edge) to have the unicity of this vector.

The inextensional displacements of $\mathcal{D}_I(\mathbf{S})$

A displacement $V \in \mathcal{D}_I(\mathbf{S})$ verifies $\dfrac{\partial V^{(l)}}{\partial x_\alpha^{(l)}} = \widehat{\nabla} V^{(l)} \wedge \mathbf{e}_\alpha^{(l)}$. This displacement belongs also to $D_I(\mathbf{S})$, hence $\widehat{\nabla} V_3^{(l)}$ is constant in each face. Let A be a vertex belonging to $\mathcal{N}$ and let J and L be two edges sharing the vertex A. The functions V and $\widehat{\nabla} V$ belong to $H^1(\mathbf{S};\mathbb{R}^3)$. The hypothesis **H2** implies that

$$
(3.10)\qquad ||V_{|J\cup L}||_{H^{1/2}(J\cup L,\mathbb{R}^3)} \le C||V||_I, \qquad ||\widehat{\nabla} V_{|J\cup L}||_{H^{1/2}(J\cup L,\mathbb{R}^3)} \le C||V||_I.
$$

From (3.9) we have $V_{|J}(M) = \overrightarrow{A}_J + \overrightarrow{B}_J \wedge \overrightarrow{AM}$ and $V_{|L}(M) = \overrightarrow{A}_L + \overrightarrow{B}_L \wedge \overrightarrow{AM}$, hence $\overrightarrow{A}_J = \overrightarrow{A}_L$. We denote $V(A)$ this value which is common to all the edges containing the vertex A.
We also have $\overrightarrow{B}_J \wedge \mathbf{e}_J = \widehat{\nabla} V_{|J} \wedge \mathbf{e}_J$, $\geq_J$Q is an unit vector in the direction of the edge. The vector $\widehat{\nabla} V_{|J} \wedge \mathbf{e}_J$ is constant along the edge J, hence

$$
\overrightarrow{B}_J \cdot (\mathbf{e}_J \wedge \mathbf{e}_L) = \overrightarrow{B}_L \cdot (\mathbf{e}_J \wedge \mathbf{e}_L).
$$

There exists a vector $\overrightarrow{B}_{JL} \in \mathbb{R}^3$ (depending on $\overrightarrow{B}_J$ and $\overrightarrow{B}_L$) such that

$$
\overrightarrow{B}_J \wedge \mathbf{e}_J = \overrightarrow{B}_{JL} \wedge \mathbf{e}_J, \qquad \overrightarrow{B}_L \wedge \mathbf{e}_L = \overrightarrow{B}_{JL} \wedge \mathbf{e}_L.
$$

Since $\widehat{\nabla} V_{|L}(M) \wedge \mathbf{e}_L = \overrightarrow{B}_L \wedge \mathbf{e}_L = \overrightarrow{B}_{JL} \wedge \mathbf{e}_L$ for any $M \in L$. From (3.10) we deduce that

$$
\int_0^{L_J} \frac{|\widehat{\nabla} V_{|J}(A+t\mathbf{e}_J) \wedge \mathbf{e}_L - \overrightarrow{B}_{JL} \wedge \mathbf{e}_L|^2}{t} dt \le C||V||_I,
$$

hence

$$\int_0^{L_J} \frac{|\widehat{\nabla} V_{|J}(A+t\mathbf{e}_J) - \overrightarrow{B}_{JL})|^2}{t} dt \le C||V||_I,$$

because we have $\widehat{\nabla} V_{|J}(M) \wedge \mathbf{e}_J = \overrightarrow{B}_{JL} \wedge \mathbf{e}_J$ for any $M \in J$. The vector $\overrightarrow{B}_{JL}$ does not depend on the edge L. Hence this vector is independant from the edges that go via A, and is denoted $\widehat{\nabla} V(A)$.
The restriction of the displacement V to any edge that goes via A is the restriction to that edge of a rigid displacement depending only on the vertex A,

$$\forall J \in \mathcal{J}, \qquad \forall M \in J, \qquad V_{|J}(M) = V(A) + \widehat{\nabla} V(A) \wedge \overrightarrow{AM}.$$

4. The asymptotic behavior of structures made by plates

4.1 Elasticity problem

The plates are made of an homogeneous and isotropic material. Our equations are given within the framework of linearised elasticity. In $\mathcal{S}_\delta$ let the elasticity system be

$$(4.1) \qquad \begin{cases} -\dfrac{\partial}{\partial x_j}\{ \, a_{iji'j'} \dfrac{\partial u_{i',\delta}}{\partial x_j'} \} = F_{i,\delta} \quad \text{in} \quad \mathcal{S}_\delta, \\ u_\delta = 0 \qquad \text{on} \quad \Gamma_{0,\delta}, \\ a_{iji'j'} \dfrac{\partial u_{i',\delta}}{\partial x_j'} n_j = 0 \quad \text{in} \quad \Gamma_\delta, \qquad \Gamma_\delta = \partial \mathcal{S}_\delta \setminus \Gamma_{0,\delta}. \end{cases}$$

The variational formulation of the problem (4.1) is

$$(4.2) \qquad \begin{cases} u_\delta \in H^1_{\Gamma_0}(\mathcal{S}_\delta; \mathbb{R}^3) \\ \displaystyle\int_{\mathcal{S}_\delta} a_{ij'j'} \gamma_{ij}(u_\delta)\gamma_{i'j'}(v) = \int_{\mathcal{S}_\delta} F_\delta \cdot v \qquad \forall v \in H^1_{\Gamma_0}(\mathcal{S}_\delta; \mathbb{R}^3) \end{cases}$$

where $a_{iji'j'} = \lambda \delta_{ij}\delta_{i'j'} + \mu(\delta_{ii'}\delta_{jj'} + \delta_{ij'}\delta_{ji'})$. The constants λ and μ are the Lamé constants of the material. The plates $\Omega'_{l,\delta}$ are submitted to volume applied forces . Among these forces we make a distinction between those concerning the extensional displacements and those concerning the inextensional displacements.

$$(4.3) \qquad F_\delta(x) = \sum_{l=1}^{N} \{ \delta f_I(\widehat{x}^{(l)}) + f_E(\widehat{x}^{(l)}) \} \mathbf{1}_{\Omega_{l,\delta}}(x), \qquad f_I, \ f_E \in L^2(\mathbf{S}; \mathbb{R}^3),$$

where $\Omega_{l,\delta} = \omega_l \times]-\delta, \delta[$ (in the local frame) and where $\mathbf{1}_{\Omega_{l,\delta}}$ is the characteristic function of the open set $\Omega_{l,\delta}$. Hence several volume forces are stacked up in the junctions.
The function f_E verifies the condition of orthogonality

$$(4.4) \qquad \forall V \in D_I(\mathbf{S}), \qquad \int_{\mathbf{S}} f_E \cdot V = 0.$$

Let $(\mathcal{U}_\delta, \mathcal{R}_\delta)$ be the two components of the e.p.s.d. associated to the solution u_δ of the problem (4.2). In the plate $\Omega_{l,\delta}$, the displacement $u_\delta^{(l)}$ is the sum of the e.p.d. $\mathcal{U}_\delta^{(l)}(\widehat{x}^{(l)}) + \mathcal{R}^{(l)}(\widehat{x}^{(l)}) \wedge x_3^{(l)} \mathbf{e}_3^{(l)}$ and a residual displacement. The displacement $\mathcal{U}_\delta$ is the sum of an extensional displacement $U_{E,\delta}$ and of an inextensional displacement $U_{I,\delta}$. Then, thanks to (3.1) and (3.7), we have

$$(4.5) \qquad \Big| \frac{1}{2\delta} \int_{\mathcal{S}_\delta} F_\delta \cdot u_\delta - \int_{\mathbf{S}} f_E \cdot U_{E,\delta} - \delta \int_{\mathbf{S}} f_I \cdot U_{I,\delta} \Big| \le C \big\{ ||f_E||_{L^2(\mathbf{S};\mathbb{R}^3)} + ||f_I||_{L^2(\mathbf{S};\mathbb{R}^3)} \big\} \sqrt{\mathcal{E}(u_\delta, \mathcal{S}_\delta)}$$

hence

$$\Big|\int_{\mathcal{S}_\delta} F_\delta\cdot u_\delta\Big| \le C\big\{||f_E||_{L^2(\mathbf{S};\mathrm{I\!R}^3)}+||f_I||_{L^2(\mathbf{S};\mathrm{I\!R}^3)}\big\}\sqrt{\delta}\sqrt{\mathcal{E}(u_\delta,\mathcal{S}_\delta)}.$$

We deduce that the solution of the variational problem (4.2) verifies the estimation

$$\mathcal{E}(u_\delta,\mathcal{S}_\delta)\le C\delta\big\{||f_E||^2_{L^2(\mathbf{S};\mathrm{I\!R}^3)}+||f_I||^2_{L^2(\mathbf{S};\mathrm{I\!R}^3)}\big\}.$$

4.2 Asymptotic behavior of the stress tensor

We begin with determining the displacements $\overline{U}^{(l)}$.

Let ϕ be a displacement in $H^1(\Omega_l;\mathrm{I\!R}^3)$, equal to zero in the neighborhood of all the sets $J\times]-1,1[$ where J is an edge included in the face $\overline{\omega}_l$. For δ sufficiently small, the displacement $\phi_\delta(x)=\delta\phi\big(\widehat{x}^{(l)},\dfrac{x_3^{(l)}}{\delta}\big)$ is an acceptable displacement of the full structure $\mathcal{S}_\delta$. We have the following strong convergences of the unfolded strain tensor components of ϕ_δ :

$$\begin{cases}\mathcal{T}_\delta\big(\gamma_{\alpha\beta}(\phi_\delta)\big)\longrightarrow 0 & \text{strongly in}\quad L^2(\Omega_l),\\ \mathcal{T}_\delta\big(\gamma_{\alpha3}(\phi_\delta)\big)\longrightarrow \dfrac{1}{2}\dfrac{\partial\phi_\alpha}{\partial X_3^{(l)}} & \text{strongly in}\quad L^2(\Omega_l),\\ \mathcal{T}_\delta\big(\gamma_{33}(\phi_\delta)\big)\longrightarrow \dfrac{\partial\phi_3}{\partial X_3^{(l)}} & \text{srtongly in}\quad L^2(\Omega_l).\end{cases}$$

We now take ϕ_δ as a test-displacement in (4.2), we transform by unfolding the integral on $\Omega_{l,\delta}$ into an integral on Ω_l and after dividing by the thickness of the plate we pass to the limit. We obtain

$$\int_{\Omega_l}\Big[\lambda\Big\{\frac{\partial U^{(l)}_{E,1}}{\partial x_1^{(l)}}-X_3^{(l)}\frac{\partial^2 U^{(l)}_{I,3}}{\partial x_1^{(l),2}}+\frac{\partial U^{(l)}_{E,2}}{\partial x_2^{(l)}}-X_3^{(l)}\frac{\partial^2 U^{(l)}_{I,3}}{\partial x_2^{(l),2}}\Big\}+(\lambda+2\mu)\frac{\partial\overline{U}^{(l)}_3}{\partial X_3^{(l)}}\Big]\frac{\partial\phi_3}{\partial X_3^{(l)}}+\mu\Big[\frac{\partial\overline{U}^{(l)}_1}{\partial X_3^{(l)}}\frac{\partial\phi_1}{\partial X_3^{(l)}}+\frac{\partial\overline{U}^{(l)}_2}{\partial X_3^{(l)}}\frac{\partial\phi_2}{\partial X_3^{(l)}}\Big]=0$$

because the right member of (4.2) tends to 0 $\big(\dfrac{1}{2\delta}\Big|\displaystyle\int_{\Omega_{l,\delta}}F_\delta\cdot\phi_\delta\Big|\le C\delta||\phi||_{L^2(\Omega_l;\mathrm{I\!R}^3)}\big)$.

The set of these test-displacements is a dense subset in $L^2(\omega_l;H^1(-1,1;\mathrm{I\!R}^3))$. Hence the above equality is verified for any element in $L^2(\omega_l;H^1(-1,1;\mathrm{I\!R}^3))$. We deduce the partial derivatives $\dfrac{\partial\overline{U}^{(l)}}{\partial X_3^{(l)}}$ in terms of the first partial derivatives of $U_E^{(l)}$ and of the second partial derivatives of $U_I^{(l)}$,

$$\frac{\partial\overline{U}^{(l)}_1}{\partial X_3^{(l)}}=\frac{\partial\overline{U}^{(l)}_2}{\partial X_3^{(l)}}=0,\qquad \frac{\partial\overline{U}^{(l)}_3}{\partial X_3^{(l)}}=\frac{\lambda}{\lambda+2\mu}\Big\{-\frac{\partial U^{(l)}_{E,1}}{\partial x_1^{(l)}}-\frac{\partial U^{(l)}_{E,2}}{\partial x_2^{(l)}}+X_3^{(l)}\Delta U^{(l)}_{I,3}\Big\}$$

$$\overline{U}^{(l)}_1=\overline{U}^{(l)}_1=0,\qquad \overline{U}^{(l)}_3=-\frac{\lambda X_3^{(l)}}{\lambda+2\mu}\Big\{\frac{\partial U^{(l)}_{E,1}}{\partial x_1^{(l)}}+\frac{\partial U^{(l)}_{E,2}}{\partial x_2^{(l)}}\Big\}+\frac{\lambda}{\lambda+2\mu}\Big(\frac{(X_3^{(l)})^2}{2}-\frac{1}{6}\Big)\Delta U^{(l)}_{I,3}$$

We give now the weak limits in $L^2(\Omega_l)$ of the unfolded stress tensor components

$$\begin{cases}\mathcal{T}_\delta\big(\sigma_{11}(u_\delta^{(l)})\big)\rightharpoonup\dfrac{E}{1-\nu^2}\Big[\dfrac{\partial U^{(l)}_{E,1}}{\partial x_1^{(l)}}-X_3^{(l)}\dfrac{\partial^2U^{(l)}_{I,3}}{\partial x_1^{(l),2}}+\nu\Big\{\dfrac{\partial U^{(l)}_{E,2}}{\partial x_2^{(l)}}-X_3^{(l)}\dfrac{\partial^2U^{(l)}_{I,3}}{\partial x_2^{(l),2}}\Big\}\Big],\\ \mathcal{T}_\delta\big(\sigma_{12}(u_\delta^{(l)})\big)\rightharpoonup\mu\Big\{\dfrac{\partial U^{(l)}_{E,1}}{\partial x_2^{(l)}}+\dfrac{\partial U^{(l)}_{E,2}}{\partial x_2^{(l)}}-2X_3^{(l)}\dfrac{\partial^2U^{(l)}_{I,3}}{\partial x_1^{(l)}\partial x_2^{(l)}}\Big\},\\ \mathcal{T}_\delta\big(\sigma_{22}(u_\delta^{(l)})\big)\rightharpoonup\dfrac{E}{1-\nu^2}\Big[\dfrac{\partial U^{(l)}_{E,2}}{\partial x_2^{(l)}}-X_3^{(l)}\dfrac{\partial^2U^{(l)}_{I,3}}{\partial x_2^{(l),2}}+\nu\Big\{\dfrac{\partial U^{(l)}_{E,1}}{\partial x_1^{(l)}}-X_3^{(l)}\dfrac{\partial^2U^{(l)}_{I,3}}{\partial x_1^{(l),2}}\Big\}\Big],\\ \mathcal{T}_\delta\big(\sigma_{i3}(u_\delta^{(l)})\big)\rightharpoonup 0.\end{cases}$$

4.3 The extensional displacement U_E or the problem of coupled membrane plates

Theorem 4.1 : *The extensional displacement U_E is the solution of the variational problem*

$$
(4.6)\qquad \frac{E}{1-\nu^2}\sum_{l=1}^{N}\int_{\omega_l}\big[(1-\nu)\gamma_{\alpha\beta}(U_E^{(l)})\gamma_{\alpha\beta}(V^{(l)})+\nu\gamma_{\alpha\alpha}(U_E^{(l)})\gamma_{\beta\beta}(V^{(l)})\big]=\int_{\mathbf{S}}f_E\cdot V,\qquad \forall V\in D_E(\mathbf{S}).
$$

where E is the Young modulus and ν the Poisson constant. □

The proof of Theorem 4.1 requires the next lemma.

Lemma 4.2 : *For any element $V\in H^1_{\Gamma_0}(\mathbf{S};\mathbb{R}^3)$, there exists a sequence of displacements $\big(V_\delta\big)_{0<\delta\le\delta_0}$ belonging to $H^1_{\Gamma_0}(\mathbf{S}_\delta;\mathbb{R}^3)\cap H^1_{\Gamma_0}(\mathcal{S}_\delta;\mathbb{R}^3)$ such that*

$$
V_\delta\longrightarrow V\qquad \textit{strongly in}\quad H^1_{\Gamma_0}(\mathbf{S};\mathbb{R}^3).
$$

Proof : Let V be an element belonging to $H^1_{\Gamma_0}(\mathbf{S};\mathbb{R}^3)$. We extend V into an element still denoted V belonging to $H^1_{\Gamma_0}(\mathbf{S}_\delta;\mathbb{R}^3)$.

Step 1 For any δ in the interval $]0,\delta_0]$, we build a displacement $V_\delta^{'}$ constant in the neighborhood of each vertex belonging to $\mathcal{N}$ and approaching V.

We begin with modifying V in the neighborhood of a vertex. Let A be a vertex common to the faces $\overline{\omega}_{l_1},\dots,\overline{\omega}_{l_p}$; $\overrightarrow{A}^{(l_i)}$ the mean value of $V^{(l_i)}$ in the disc $B(A;\delta)\cap\omega_{l_i,\delta_0}$ and $\overrightarrow{A}$ the mean value of the vectors $\overrightarrow{A}^{(l_i)}$. If the vertex A belongs to Γ_0 we replace $\overrightarrow{A}$ by $\vec{0}$.

We define the displacement $V_{\delta,A}$ in $\overline{\omega}_{l,\delta}$ by

$$
V^{(l)}_{\delta,A}(\widehat{x}^{(l)})=V(\widehat{x}^{(l)})m\Big(\frac{dist(\widehat{x}^{(l)},A)}{(\eta_1+1)\delta}\Big)+\Big\{1-m\Big(\frac{dist(\widehat{x}^{(l)},A)}{(\eta_1+1)\delta}\Big)\Big\}\overrightarrow{A},\qquad l\in\{l_1,\dots,l_p\}.
$$

where m has been introduced by (3.5). In the disc $B(A;(\eta_1+1)\delta)\cap\overline{\omega}_{l,\delta}$ the displacement $V^{(l)}_{\delta,A}$ is by construction constant and equal to $\overrightarrow{A}$. Let us estimate the L^2 norm of $V^{(l)}_{\delta,A}-V^{(l)}$, $l\in\{l_1,\dots,l_p\}$. The Poincaré-Wirtinger inequality allows us to estimate the L^2 norm of $V^{(l)}-\overrightarrow{A}^{(l)}$ in the disc $B(A;\delta)\cap\omega_{l,\delta}$,

$$
||V^{(l)}-\overrightarrow{A}^{(l)}||^2_{L^2(B(A;\delta)\cap\overline{\omega}_{l,\delta};\mathbb{R}^3)}\le C\delta^2||\nabla V^{(l)}||^2_{[L^2(B(A;\delta)\cap\overline{\omega}_{l,\delta};\mathbb{R}^3)]^2}.
$$

Hence, if J is an edge of vertex A contained in $\overline{\omega}_l$ we have

$$
||V_{|J}-\overrightarrow{A}^{(l)}||^2_{L^2(B(A;\delta)\cap J;\mathbb{R}^3)}\le C\delta||\nabla V^{(l)}||^2_{[L^2(B(A;\delta)\cap\overline{\omega}_{l,\delta};\mathbb{R}^3)]^2}
$$

For any other face $\overline{\omega}_{k,\delta}$ containing $J\cap B(A;\delta)$, we also have the above estimate, hence

$$
||\overrightarrow{A}^{(k)}-\overrightarrow{A}^{(l)}||^2_2\le C\big\{||\nabla V^{(l)}||^2_{[L^2(B(A;\delta)\cap\overline{\omega}_{l,\delta};\mathbb{R}^3)]^2}+||\nabla V^{(k)}||^2_{[L^2(B(A;\delta)\cap\overline{\omega}_{k,\delta};\mathbb{R}^3)]^2}\big\}.
$$

We deduce that

$$
\begin{aligned}
&||V^{(l)}-\overrightarrow{A}||^2_{L^2(B(A;\delta)\cap\omega_{l,\delta};\mathbb{R}^3)}\le C\delta^2\sum_{i=1}^{p}||\nabla V^{(l_i)}||^2_{[L^2(B(A;\delta)\cap\omega_{l_i,\delta};\mathbb{R}^3)]^2}\\
\Longrightarrow\quad &||V^{(l)}-\overrightarrow{A}||^2_{L^2(B(A;(2\eta_1+1)\delta)\cap\omega_{l,\delta};\mathbb{R}^3)}\le C\delta^2\sum_{k=1}^{p}||\nabla V^{(l_k)}||^2_{[L^2(B(A;(2\eta_1+1)\delta)\cap\omega_{l_k,\delta};\mathbb{R}^3)]^2}
\end{aligned}
$$

And eventually

$$||V_{\delta,A}-V||_{H^1(\mathbf{S}_\delta;\mathbb{R}^3)}\le C\sum_{k=1}^{p}||\nabla V^{(l_k)}||_{[L^2(B(A;(2\eta_1+1)\delta)\cap\omega_{l_k,\delta};\mathbb{R}^3)]^2}.$$

We can do the same with all the structure vertexes. We denote $V_\delta^{'}$ the displacement obtained after having modified V in a neighborhood of each vertex. Hence we have

$$||V_\delta^{'}-V||_{H^1(\mathbf{S}_\delta;\mathbb{R}^3)}\le C\sum_{A\in\mathcal{N}}||V||_{H^1(B(A;(2\eta_1+1)\delta)\cap\mathbf{S}_\delta;\mathbb{R}^3)}.$$

Step 2 Let J be an edge belonging to the faces $\overline{\omega}_{l_1},\ \dots,\overline{\omega}_{l_p}$. We suppose that the direction of J is $\mathbf{e}_1=\mathbf{e}_1^{(l)}$, $l\in\{l_1,\dots,l_p\}$. We denote $W_{\delta,J}^{(l_k)}$ the element of $H^1(J;\mathbb{R}^3)$ defined by

$$W_{\delta,J}^{(l_k)}(x_1)=\frac{1}{2\delta}\int_{-\delta}^{\delta}V_\delta^{'(l_k)}(x_1\mathbf{e}_1+s\mathbf{e}_2^{(l_k)})ds$$

and we denote $W_{\delta,J}$ the mean values of $W_{\delta,J}^{(l_k)}$. We have $W_{\delta,J}^{(l_k)}\in H^1(J;\mathbb{R}^3)$ and

$$\left\{\begin{aligned}&||V-W_{\delta,J}||_{L^2(\widehat{J}_\delta;\mathbb{R}^3)}\le C\delta||V_\delta^{'}||_{H^1(\widehat{J}_\delta;\mathbb{R}^3)}\le C\delta||V||_{H^1(\widehat{J}_\delta;\mathbb{R}^3)}\\&||V-W_{\delta,J}||_{H^1(\widehat{J}_\delta;\mathbb{R}^3)}\le C||V_\delta^{'}||_{H^1(\widehat{J}_\delta;\mathbb{R}^3)}\le C||V||_{H^1(\widehat{J}_\delta;\mathbb{R}^3)}\end{aligned}\right.\tag{4.7}$$

where $\widehat{J}_\delta=\big\{\widehat{x}\in\mathbf{S}_\delta\mid dist(\widehat{x},J)<\eta_0\delta\big\}$, $\widehat{J}_\delta$ is the union of two-dimensional sets of breadth $2\eta_0\delta$ and of length $L_J+2\eta_0\delta$. If the edge J is contained in Γ_0, we take $W_{\delta,J}=0$, in this case we have the estimate (4.7) again. The displacement

$$V_\delta=\sum_{J\in\mathcal{J}}V_\delta^{'}\Big\{1-m\Big(\frac{dist(\cdot,J)}{\eta_0\delta}\Big)\Big\}+\sum_{J\in\mathcal{J}}W_{\delta,J}m\Big(\frac{dist(\cdot,J)}{\eta_0\delta}\Big)$$

belongs to $H^1_{\Gamma_0}(\mathcal{S}_\delta;\mathbb{R}^3)$ and verifies

$$||V_\delta-V||_{H^1(\mathbf{S}_{\delta_0};\mathbb{R}^3)}\le C\sum_{A\in\mathcal{N}}||V||_{H^1(B(A;(2\eta_1+1)\delta)\cap\mathbf{S}_{\delta_0};\mathbb{R}^3)}+C\sum_{J\in\mathcal{J}}||V||_{H^1(\widehat{J}_\delta;\mathbb{R}^3)}$$

The constant is independant of δ. We can do the same with all edges. We denote V_δ the displacement obtained after having modified V in a neighborhood of each edge. □

Proof of Theorem 4.1 : Let V be an element of $H^1_{\Gamma_0}(\mathbf{S};\mathbb{R}^3)$. We take the displacement V_δ given by Lemma 4.2 as test-displacement in (4.2). Then we transform by unfolding the integrals on the plates into integrals on the reference plates and we divide by 2δ. Thus we are led to take into account again the neighborhoods of the edges belonging to $\mathcal{J}$.

Let J be an edge common to several faces. For any face $\overline{\omega}_l$ containing J, we have

$$\mathbf{1}_{\big\{\hat{x}^{(l)}\in\omega_{l,\delta}\ \mid\ dist(\hat{x}^{(l)},J)<\eta_1\delta\big\}}\mathcal{T}_\delta\big(\gamma_{ij}(V_\delta^{(l)})\big)\longrightarrow 0\qquad\text{strongly in}\quad L^2(\Omega_l).\tag{4.8}$$

The part of $\mathcal{S}_\delta$ which is neighbor of the edge J and common to several plates is contained into the cylinder $\big\{x\in\mathbb{R}^3\mid dist(x,J)<\eta_1\delta\big\}$. Thanks to the convergences (4.8) its contribution in the limit problem is equal to zero. Then we can make δ tend to 0 in order to obtain (4.6) with the displacement V. The limit of the right handside term of (4.2) is given thanks to (4.5).

The space $H^1_{\Gamma_0}(\mathbf{S};\mathbb{R}^3)$ is dense in $H^1_{\rho,\Gamma_0}(\mathbf{S};\mathbb{R}^3)$ which gives (4.6) with any displacement of $D_E(\mathbf{S})$. □

4.3 The inextensional displacement U_I or the problem of coupled bending plates

Theorem 4.3 : *The inextensional displacement U_I is the solution of the variational problem*

$$\frac{E}{3(1-\nu^2)}\sum_{l=1}^{N}\int_{\omega_l}\Big[(1-\nu)\frac{\partial^2 U^{(l)}_{I,3}}{\partial x^{(l)}_\alpha\partial x^{(l)}_\beta}\frac{\partial^2 V^{(l)}_3}{\partial x^{(l)}_\alpha\partial x^{(l)}_\beta}+\nu\Delta U^{(l)}_{I,3}\Delta V^{(l)}_3\Big]=\int_{\mathbf{S}} f_I\cdot V,\qquad \forall V\in\mathcal{D}_I(\mathbf{S}). \tag{4.9}$$

The proof of Theorem 4.3 requires the next lemma.

Lemma 4.4 : *For any element $V\in\mathcal{D}_I(\mathbf{S})$, there exists a sequence of displacements $\big(W_\delta\big)_{0<\delta\le\delta_0}$ such that*

$$W_\delta\in H^1_{\Gamma_0}(\mathcal{S}_\delta;\mathbb{R}^3),\quad \textit{and}\quad \begin{cases}\delta\mathcal{T}_\delta\big(W^{(l)}_\delta\big)\longrightarrow V^{(l)}\quad \textit{strongly in}\quad L^2(\Omega_l;\mathbb{R}^3),\\ \mathcal{T}_\delta\big(\gamma_{\alpha\beta}(W^{(l)}_\delta)\big)\longrightarrow -X^{(l)}_3\dfrac{\partial^2 V^{(l)}_3}{\partial x^{(l)}_\alpha\partial x^{(l)}_\beta}\quad \textit{strongly in}\quad L^2(\Omega_l),\\ \mathcal{T}_\delta\big(\gamma_{k3}(W^{(l)}_\delta)\big)\longrightarrow 0\quad \textit{strongly in}\quad L^2(\Omega_l).\end{cases}$$

Proof : Let be $V\in\mathcal{D}_I(\mathbf{S})$. We recall that there exists only one function in $H^1_{\Gamma_0}(\mathbf{S};\mathbb{R}^3)$ denoted $\widehat{\nabla}V$ such that $\dfrac{\partial V^{(l)}}{\partial x^{(l)}_\alpha}=\widehat{\nabla}V^{(l)}\wedge\mathbf{e}^{(l)}_\alpha$, $l\in\{1,\dots,N\}$, $\alpha\in\{1,2\}$.

Step 1 Extension of $V^{(l)}$ and of $\widehat{\nabla}V^{(l)}$ to $\omega_{l,\delta}$.

As in Lemma 2.7 we extend the displacement

$$v^{(l)}(x)=\frac{1}{\delta}\Big\{V^{(l)}(\widehat{x}^{(l)})+\widehat{\nabla}V^{(l)}(\widehat{x}^{(l)})\wedge x^{(l)}_3\mathbf{e}^{(l)}_3\Big\},\qquad x\in\Omega_{l,\delta}=\omega_l\times]-\delta,\delta[,$$

of the plate $\Omega_{l,\delta}$ into a displacement still denoted $v^{(l)}$ of the plate $\Omega^{''}_{l,\delta}=\omega_{l,2\delta}\times]-\delta,\delta[$. The extension $v^{(l)}$ is by construction equal to zero on $\Gamma_{0,\delta}\cap\Omega^{''}_{l,\delta}$. We have

$$\mathcal{E}(v^{(l)},\Omega^{''}_{l,\delta})\le C\mathcal{E}(v^{(l)},\Omega_{l,\delta})\le C\delta||\widehat{\nabla}V^{(l)}||^2_{H^1(\omega_l;\mathbb{R}^3)}\qquad \mathcal{E}(v^{(l)},\Omega^{''}_{l,\delta}\setminus\Omega_{l,\delta})\le C\delta||\widehat{\nabla}V^{(l)}||^2_{H^1(\mathbf{B}_{l,\delta};\mathbb{R}^3)}$$

where $\mathbf{B}_{l,\delta}=\big\{\widehat{x}^{(l)}\in\omega_l\mid dist(\widehat{x}^{(l)},\partial\omega_l)<\delta)\big\}$. Let

$$V^{(l)}_\delta(x)=\frac{1}{\delta}\Big\{\mathcal{V}^{(l)}(\widehat{x}^{(l)})+\mathcal{A}^{(l)}(\widehat{x}^{(l)})\wedge x^{(l)}_3\mathbf{e}^{(l)}_3\Big\},\qquad x\in\Omega''_{l,\delta}$$

be the elementary plate displacement associated to $v^{(l)}$ by the formulas (2.1). Its components $\mathcal{V}^{(l)}$ and $\mathcal{A}^{(l)}$ are the restrictions to $\omega_{l,\delta}$ of elements $\big($denoted $\mathcal{V}$, $\mathcal{A}\big)$ belonging to $H^1(\mathbf{S}_\delta;\mathbb{R}^3)$. They verify $\mathcal{V}^{(l)}_{|\omega_l}=V^{(l)}$, $\mathcal{A}^{(l)}_{|\omega_l}=\widehat{\nabla}V^{(l)}$ and we have

$$\mathcal{E}(v^{(l)}-V^{(l)}_\delta,\Omega^{''}_{l,\delta})+\mathcal{D}(v^{(l)}-V^{(l)}_\delta,\Omega^{''}_{l,\delta})+\frac{1}{\delta^2}||v^{(l)}-V^{(l)}_\delta||^2_{L^2(\Omega^{''}_{l,\delta};\mathbb{R}^3)}\le C\delta||\widehat{\nabla}V^{(l)}||_{H^1(\mathbf{B}_{l,\delta};\mathbb{R}^3)}. \tag{4.10}$$

Step 2 We modify $V^{(l)}_\delta$ in the neighborhood of the vertexes belonging to $\mathcal{N}$.

Let A be a vertex belonging to $\mathcal{N}$, for any edge J containing A, we have (see subsection 3.6)

$$\widehat{\nabla}V(x)\wedge\mathbf{e}_J=\widehat{\nabla}V(A)\wedge\mathbf{e}_J\qquad\qquad V_{|J}(x)=V(A)+\widehat{\nabla}V(A)\wedge(x-A)=r_A(x),\qquad x\in J$$

where $\mathbf{e}_J$ is a unit vector in the direction of the edge. Then we have

$$(4.11)\qquad \begin{cases} ||v^{(l)} - \frac{1}{\delta} r_A||^2_{L^2(B(A;2\eta_1\delta);\mathbb{R}^3)} \leq C\delta^2 \mathcal{E}(v^{(l)}, B(A;2\eta_1\delta)) \leq C\delta^3 ||\widehat{\nabla} V^{(l)}||_{H^1(B(A;2\eta_1\delta)\cap \mathbf{S}_\delta;\mathbb{R}^3)} \\ \mathcal{D}(v^{(l)} - \frac{1}{\delta} r_A, B(A;2\eta_1\delta)) \leq C\mathcal{E}(v^{(l)}, B(A;2\eta_1\delta)) \leq C\delta ||\widehat{\nabla} V^{(l)}||_{H^1(B(A;2\eta_1\delta)\cap \mathbf{S}_\delta;\mathbb{R}^3)} \end{cases}$$

For any face $\overline{\omega}_l$ containing the vertex A, we define the displacement $V^{(l)}_{\delta,A}$ by

$$V^{(l)}_{\delta,A}(x) = V^{(l)}_\delta(x) m\Big(\frac{dist(x,A)}{\eta_1\delta}\Big) + \frac{1}{\delta}\Big\{1 - m\Big(\frac{dist(x,A)}{\eta_1\delta}\Big)\Big\} r_A(x), \qquad x \in \Omega'_{l,\delta},$$

where m has been introduced by (3.5). Thanks to (4.10) and (4.11), we obtain the following inequality :

$$\begin{aligned} ||V^{(l)}_{\delta,A} - V^{(l)}_\delta||^2_{L^2(B(A;2\eta_1\delta)\cap\Omega'_{l,\delta};\mathbb{R}^3)} &\leq C\delta^3\big\{||\widehat{\nabla} V||_{H^1(B(A;2\eta_1\delta)\cap\mathbf{S}_\delta;\mathbb{R}^3)} + ||\widehat{\nabla} V^{(l)}||_{H^1(\mathbf{B}_{l,\delta};\mathbb{R}^3)}\big\}, \\ \text{and} \qquad \mathcal{E}(V^{(l)}_{\delta,A} - V^{(l)}_\delta, \Omega'_{l,\delta}) &\leq C\delta\big\{||\widehat{\nabla} V||_{H^1(B(A;2\eta_1\delta)\cap\mathbf{S}_\delta;\mathbb{R}^3)} + ||\widehat{\nabla} V^{(l)}||_{H^1(\mathbf{B}_{l,\delta};\mathbb{R}^3)}\big\}. \end{aligned}$$

The displacement $V^{'(l)}_\delta = \sum\limits_{A\in\overline{\omega}_l\cap\mathcal{N}} V^{(l)}_{\delta,A}$ coincides with a rigid displacement independent of l in the neighborhood of each vertex contained in the face $\overline{\omega}_l$. It verifies

$$\mathcal{E}(V^{'(l)}_\delta - V^{(l)}_\delta, \Omega'_{l,\delta}) \leq C\delta \sum_{A\in\overline{\omega}_l\cap\mathcal{N}} ||\widehat{\nabla} V||_{H^1(B(A;2\eta_1\delta)\cap\mathbf{S}_\delta;\mathbb{R}^3)} + C\delta||\widehat{\nabla} V^{(l)}||_{H^1(\mathbf{B}_{l,\delta};\mathbb{R}^3)}.$$

Step 3 We modify $V^{'(l)}_\delta$ in the neighborhood of each edge belonging to $\mathcal{J}$.
Let J be an edge belonging to the faces $\overline{\omega}_{l_1},\ \ldots, \overline{\omega}_{l_p}$. We suppose that the direction of J is $\mathbf{e}_1 = \mathbf{e}^{(l)}_1$, $l \in \{l_1, \ldots, l_p\}$. One of the extremities of J is chosen as origin on each face, so that J is identified with the segment $[0, L_J] \times \{0\}$ where L_J is the edge's length. In each plate $\Omega^{'}_{l,\delta}$ we consider the neighborhood of J

$$J^{(l)}_{\eta_0\delta} =]0, L_J[\times] - 2\eta_0\delta, 2\eta_0\delta[\times] - \delta, \delta[, \qquad \eta \geq 1.$$

The restriction of $V^{'(l)}_\delta$ to $J^{(l)}_{\eta_0\delta}$ is decomposed into the sum of an elementary rod displacement $V^{(l)}_{e,J}$ and of a residual displacement,

$$V^{(l)}_{e,J}(x) = \mathcal{V}^{(l)}_J(x_1) + \mathcal{A}^{(l)}_J(x_1) \wedge \big(x^{(l)}_2\mathbf{e}^{(l)}_2 + x^{(l)}_3\mathbf{e}^{(l)}_3\big) \qquad x \in \Omega^{'}_{l,\delta}.$$

Let us remind that (see [9]) the components $\mathcal{V}^{(l)}_J$ and $\mathcal{A}^{(l)}_J$ of $V^{(l)}_{e,J}$ belong to $H^1(J;\mathbb{R}^3)$, and verify

$$\mathcal{E}(V^{(l)}_{e,J}, J^{(l)}_{\eta_0\delta}) + \mathcal{D}(V^{'(l)}_\delta - V^{(l)}_{e,J}, J^{(l)}_{\eta_0\delta}) + \frac{1}{\delta^2}||V^{'(l)}_\delta - V^{(l)}_{e,J}||^2_{L^2(J^{(l)}_{\eta_0\delta};\mathbb{R}^3)} \leq C\mathcal{E}(V^{'(l)}_\delta, J^{(l)}_{\eta_0\delta}) \leq C\mathcal{E}(v^{(l)}, J^{(l)}_{\eta_0\delta}).$$

By construction, the displacement $V^{(l)}_{e,J}$ coincides with a rigid displacement in the neighborhood of the edge extremities. We deduce that

$$||V_{|J} - \mathcal{V}^{(l)}_J||^2_{L^2(J;\mathbb{R}^3)} \leq C\mathcal{E}(v^{(l)}, J^{(l)}_{\eta_0\delta}), \qquad ||\widehat{\nabla} V_{|J} - \mathcal{A}^{(l)}_J||^2_{L^2(J;\mathbb{R}^3)} \leq \frac{C}{\delta^2}\mathcal{E}(v^{(l)}, J^{(l)}_{\eta_0\delta})$$

Let $V_{e,J}$ be the elementary rod displacement equal to the mean value of the displacements $V^{(l_1)}_{e,J}, \ldots, V^{(l_p)}_{e,J}$. We have

$$\sum_{i=1}^p\Big\{\mathcal{E}(V_{e,J}, J^{(l_i)}_{2\eta_0\delta}) + \mathcal{E}(V^{(l_i)}_{\delta,1} - V_{e,J}, J^{(l_i)}_{2\eta_0\delta}) + \frac{1}{\delta^2}||V^{(l_i)}_{\delta,1} - V_{e,J}||^2_{L^2(J^{(l_i)}_{2\eta_0\delta};\mathbb{R}^3)}\Big\} \leq C\sum_{i=1}^p \mathcal{E}(v^{(l_i)}, J^{(l_i)}_{2\eta_0\delta}).$$

We now modify the displacement $V_\delta^{'(l)}$ in the neighborhood $J^{(l)}_{2\eta_0\delta}$ of the edge J,

$$V^{(l)}_{\delta,J}(x) = V_\delta^{'(l)}(x)\Big\{1 - m\Big(\frac{dist(\widehat{x}^{(l)}, J)}{\eta_0\delta}\Big)\Big\} + V_{e,J}(x)m\Big(\frac{dist(\widehat{x}^{(l)}, J)}{\eta_0\delta}\Big)\Big\}, \qquad x \in \Omega'_{l,\delta}.$$

Then we have

$$\sum_{i=1}^{p}\mathcal{E}(V^{(l_i)}_{\delta,J} - V_\delta^{'(l_i)}, \Omega'_{l_i,\delta}) \le C\sum_{i=1}^{p}\mathcal{E}(u^{(l_i)}, J^{(l_i)}_{2\eta_0\delta})$$

Eventually the displacement W_δ obtained by modifying $V_\delta^{'(l)}$ in the neighborhood of each edge of $\mathcal{J}$ belongs to $H^1_{\Gamma_0}(\mathcal{S}_\delta;\mathrm{I\!R}^3)$ and for any $l \in \{1,\dots,N\}$ verifies the properties of Lemma 4.4. □

Proof of Theorem 4.3 : Let V be an element of $\mathcal{D}_I(\mathbf{S})$. We take the displacement W_δ given by Lemma 4.4 as a test-displacement in (4.2). For any edge J and any face $\overline{\omega}_l$ containing J we have

$$\mathbf{1}_{\big\{\hat{x}^{(l)}\in\omega_{l,\delta}\ \mid\ dist(\hat{x}^{(l)},J)<\eta_0\delta\big\}}\mathcal{T}_\delta\big(\gamma_{ij}(W^{(l)}_\delta)\big) \longrightarrow 0 \qquad \text{strongly in} \quad L^2(\Omega_l). \tag{4.12}$$

We transform by unfolding the integrals on the plates into integrals on the reference plates, then we divide by 2δ. We pass to the limit. Thanks to (4.12) the contribution of the immediate junction neighborhoods tends to 0. We obtain (4.9) with the test-displacement V. □

Remark : The operators of problems (4.6) and (4.9) are coercive. It results that the whole encountered sequences converge to their limit. We are going to show now that these convergences are strong. We consider the formal displacement U of the structure $\mathcal{S}_\delta$ defined in each plate by

$$U^{(l)}(x) = U^{(l)}_E(\widehat{x}^{(l)}) + \frac{1}{\delta}U^{(l)}_I(\widehat{x}^{(l)}) + \widehat{\nabla}U^{(l)}_I(\widehat{x}^{(l)}) \wedge x^{(l)}_3\mathbf{e}^{(l)}_3, \qquad x \in \Omega_{l,\delta}.$$

Let $\mathbf{1}_{\mathcal{J}^c_\delta}$ be the characteristic function of the complement in $\mathcal{S}_\delta$ of the union of the edges neighborhoods $\bigcup_{J\in\mathcal{J}}\big\{x \in \mathcal{S}_\delta \mid dist(x,J) < \eta_0\delta\big\}$. In the reference plate Ω_l, we have the convergences

$$\mathcal{T}_\delta(\gamma_{ij}(u^{(l)}_\delta))\mathcal{T}_\delta(\mathbf{1}_{\mathcal{J}^c_\delta}) \rightharpoonup 0 \quad \text{weakly in} \quad L^2(\Omega_l).$$

Hence

$$\left\{\begin{aligned}
&\sum_{l=1}^{N}\int_{\Omega_l}\sigma_{ij}(U^{(l)})\gamma_{ij}(U^{(l)}) \le \liminf_{\delta\to0}\sum_{l=1}^{N}\int_{\Omega_l}\mathcal{T}_\delta(\sigma_{ij}(u^{(l)}_\delta))\mathcal{T}_\delta(\gamma_{ij}(u^{(l)}_\delta))\mathcal{T}_\delta(\mathbf{1}_{\mathcal{J}^c_\delta})\\
&= \liminf_{\delta\to0}\frac{1}{2\delta}\int_{\mathcal{S}_\delta}\sigma_{ij}(u_\delta)\gamma_{ij}(u_\delta)\mathbf{1}_{\mathcal{J}^c_\delta} \le \limsup_{\delta\to0}\frac{1}{2\delta}\int_{\mathcal{S}_\delta}\sigma_{ij}(u_\delta)\gamma_{ij}(u_\delta)\\
&= \limsup_{\delta\to0}\frac{1}{2\delta}\int_{\mathcal{S}_\delta}F_\delta\cdot u_\delta = \lim_{\delta\to0}\frac{1}{2\delta}\int_{\mathcal{S}_\delta}F_\delta\cdot u_\delta = \int_{\mathbf{S}}f_E\cdot U_E + \int_{\mathbf{S}}f_I\cdot U_I
\end{aligned}\right. \tag{4.13}$$

The first term of (4.13) is the sum of the left handside members of (4.6) and (4.9). Hence the above inequalities are equalities. Besides

$$\liminf_{\delta\to0}\frac{1}{2\delta}\int_{\mathcal{S}_\delta}\sigma_{ij}(u_\delta)\gamma_{ij}(u_\delta)1_{\mathcal{J}^c_\delta} \le \liminf_{\delta\to0}\frac{1}{2\delta}\int_{\mathcal{S}_\delta}\sigma_{ij}(u_\delta)\gamma_{ij}(u_\delta) \le \limsup_{\delta\to0}\frac{1}{2\delta}\int_{\mathcal{S}_\delta}\sigma_{ij}(u_\delta)\gamma_{ij}(u_\delta)$$

$$\text{and} \quad \liminf_{\delta\to0}\frac{1}{2\delta}\int_{\mathcal{S}_\delta}\sigma_{ij}(u_\delta)\gamma_{ij}(u_\delta)1_{\mathcal{J}^c_\delta} \le \limsup_{\delta\to0}\frac{1}{2\delta}\int_{\mathcal{S}_\delta}\sigma_{ij}(u_\delta)\gamma_{ij}(u_\delta)1_{\mathcal{J}^c_\delta} \le \limsup_{\delta\to0}\frac{1}{2\delta}\int_{\mathcal{S}_\delta}\sigma_{ij}(u_\delta)\gamma_{ij}(u_\delta)$$

We deduce that $\lim_{\delta\to 0}\frac{1}{2\delta}\int_{\mathcal{S}_\delta}\sigma_{ij}(u_\delta)\gamma_{ij}(u_\delta)\big(1-\mathbf{1}_{\mathcal{J}^c_\delta}\big)=0$. All the sequences of the unfolded of the strained tensor components strongly converge in $L^2(\Omega_l)$. We have also the strong convergences

$$\begin{cases} \delta\mathcal{U}_\delta \longrightarrow U_I, \quad \delta\mathcal{R}_\delta \longrightarrow \mathcal{R} \quad \text{strongly in} \quad H^1(\mathbf{S};\mathrm{I\!R}^3),\\ \delta U_{I,\delta} \longrightarrow U_I \qquad \text{strongly in} \quad D_I(\mathbf{S}),\\ U_{E,\delta} \longrightarrow U_E \qquad \text{strongly in} \quad D_E(\mathbf{S}). \end{cases}$$

4.4 Complements

The orthogonal condition (4.4) requires an explanation . First, for any function $\phi\in H^1(\omega_l)$ equal to zero on the edges, the displacement Φ defined by

$$\Phi^{(l)}=\phi\mathbf{e}_3^{(l)}, \quad \text{in } \omega_l, \text{ and by 0 in the other faces,}$$

belongs to $D_I(\mathbf{S})$. We deduce that the function $f_{E,3}^{(l)}$ is orthogonal to ϕ and then, by density of these test-functions in $L^2(\omega_l)$, we get

$$\forall l\in\{1,\dots,N\}, \qquad f_{E,3}^{(l)}=0.$$

Let $D_{I,0}(\mathbf{S})$ be the space of the inextensional displacements equal to zero on the edges belonging to $\mathcal{J}$ and let $\big(D_{I,0}(\mathbf{S})\big)^\perp$ be its orthogonal in $D_I(\mathbf{S})$ for the inner product $<\cdot,\cdot>_\rho$. The subset $\big(D_{I,0}(\mathbf{S})\big)^\perp$ is of finite dimension. The condition (4.4) is then equivalent to

$$\forall V\in\big(D_{I,0}(\mathbf{S})\big)^\perp, \qquad \int_{\mathbf{S}} f_E\cdot V=0.$$

This last condition results in a finite number of equalities related to the means in the faces $\overline{\omega}_l$ of the components $f_{E,\alpha}^{(l)}$ of f_E.

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