Periodic reiterated homogenization for elliptic functions

Nicolas Meunier*, Jean Van Schaftingen†

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Abstract

In this paper, we study reiterated homogenization for equations of the form $-\text{div}\ (a_\epsilon(x, Du_\epsilon)) = f$. We assume that a_ϵ is a Carathéodory function and satisfies some monotonicity and growth conditions and its reiterated unfolding converges almost everywhere to a Carathéodory type function. Under these assumptions, we show that the sequence of solutions converges to the solution of a limit variational problem. In particular this contains the case $a_\epsilon(x, \xi) = (x, \frac{x}{\epsilon}, \frac{x}{\epsilon\delta(\epsilon)}, \xi)$, where a is periodic in the second and third arguments, and continuous in each argument.

1 Introduction

This article is devoted to reiterated homogenization for nonlinear partial differential equations with oscillating coefficients and multiscales. This type of equation models various physical problems arising in media with holes, heterogeneous materials with several length-scales.

Consider partial differential equations of the form:

$$\begin{cases} -\text{div}\ (a_\epsilon(x, \nabla u_\epsilon)) = f & \text{in } \Omega, \\ u_\epsilon \in W_0^{1,p}(\Omega), \end{cases} \tag{1.1}$$

where Ω is a Lipschitz open bounded subset of $\mathbf{R}^N$, $1 < p < \infty$, $p^{-1} + q^{-1} = 1$ and $f \in W^{-1,q}(\Omega)$. The interesting case is when the function a_ϵ is increasingly oscillating as ϵ goes to zero. The homogenization study of equation (1.1) consists in examining the

*Laboratoire Jacques-Louis Lions, Université Pierre et Marie Curie, 175 rue du Chevaleret, Paris 75013, France, meunier@ann.jussieu.fr,

†Département de Mathématique, Université catholique de Louvain, 2 chemin du Cyclotron, Louvain-la-Neuve, 1348, Belgique, vanschaftingen@math.ucl.ac.be

behavior of the sequence of the solution $(u_\epsilon)_\epsilon$, as ϵ tends to zero. In Bensoussan, Lions and Papanicolaou [2] and Sanchez-Palencia [14], the terminology of reiterated homogenization was introduced when a_ϵ is of the following form:

$$a_\epsilon(x,\xi) = a(x, x/\epsilon, x/\epsilon^2, \xi).$$

In [2], [14], the function a_ϵ is linear ($p = 2$) and it is proved that under suitable assumptions, the sequence $(u_\epsilon)_\epsilon$ converges weakly in $H_0^1(\Omega)$ to the solution u_0 of the reiterated homogenization problem

$$\begin{cases} -\text{div } (a_{\text{hom}}(x, \nabla u_0)) = f & \text{in } \Omega, \\ u_0 \in W_0^{1,p}(\Omega). \end{cases} \tag{1.2}$$

More recently, Lions, Lukkassen and Persson in [9] considered non-linear periodic monotone a_ϵ's where $a_\epsilon(x,\xi) = a(x, \frac{x}{\epsilon}, \frac{x}{\epsilon^2}, \xi)$, with $\epsilon = \frac{1}{h}$ and $h \in \mathbb{N}^*$, such that a is periodic in the second and third variables and measurable for all $\xi \in \mathbf{R}^N$. Moreover it was assumed that a satisfies a continuity condition of the form $|a(x,y,z,\xi) - a(x,y_1,z,\xi)|^p \leq \omega(|y_1 - y_2|)(1 + |\xi|)^p$, and the following conditions in ξ:

- there exists $\beta \geq p$ and $c > 0$ such that for all $x \in \Omega$, $\xi_1, \xi_2 \in \mathbf{R}^N$,

$$\frac{|\xi_1 - \xi_2|^\beta}{(1 + |\xi_1| + |\xi_2|)^{\beta - p}} \leq c(a_\epsilon(x,\xi_1) - a_\epsilon(x,\xi_2)|\xi_1 - \xi_2), \tag{1.3}$$

- there exists $C > 0$ such that for all $x \in \Omega$ et $\xi \in \mathbf{R}^N$,

$$|a_\epsilon(x,\xi)| \leq C(1 + |\xi|^{p-1}), \tag{1.4}$$

- and

$$a(x,y,z,0) = 0. \tag{1.5}$$

The homogenization result is obtained using a method of energy and multiscale convergence (cf Allaire and Briane [1]).

In this paper, we extend the previous results to the case where a_ϵ is of the form: $a_\epsilon(x,\xi) = a(x, x/\epsilon, x/(\epsilon\delta(\epsilon)), \xi)$, where $\delta(\epsilon)$ tends to zero along with ϵ, a satisfies similar condition (1.3), with $\beta \geq \max(2,p)$, (1.4) and (1.5). We only assume that a is of Carathéodory type. We obtain this result using an extension of the periodic unfolding method introduced by Cioranescu, Damlamian and Griso [4], suitable for reiterated problems (see [13] and [15] for homogenization in the non periodic case). Moreover, in order to apply the method for a_ϵ as above we define an unfolding operator with a microscopic correction.

The plan of the paper is the following. First, we recall the standard results of the unfolding method and also prove some refinements needed for our case in section two. In the third section, we define a periodic unfolding operator with a microscopic correction. In section four, we introduce the reiterated unfolding operator. Then, in sections five and six, we give the homogenization results in the case of the simple scale and in the reiterated case. Part of the results of this article were announced in [11].

2 The periodic unfolding operator

The periodic unfolding operator was introduced by Cioranescu, Damlamian and Griso [4]. We recall the definitions and properties that we will need to define a reiterated unfolding operator with microscopic correction in order to treat the reiterated homogenization problem. The proofs can be found in Cioranescu, Damlamian and Griso [4] for the $L^p(\Omega)$ case and were exposed to the authors for the $L^p(\mathbf{R}^N)$ case, [5]. We state them and also include some variants which will be useful for the reiterated case.

2.1 Definition of the periodic unfolding operator and L^p properties

In $\mathbf{R}^n$, let Y be a reference cell (ex. $]0,1[^n$, or more generally a set having the paving property with respect to a basis $(b_1, ..., b_n)$ defining the periods). For $y \in \mathbf{R}^n$, we denote $[y]_Y$ the unique integer combination $\sum_{j=1}^n k_j b_j$ of the periods such that $y - [y]_Y$ belongs to Y and we set

$$\{y\}_Y = y - [y]_Y \ \in Y.$$

Definition 2.1. Let $\epsilon > 0$, Y be a reference cell and $u \,:\, \mathbf{R}^N \to S$, where S is a set, the unfolding operator $\mathcal{T}_\epsilon^Y$ is defined as follows

$$\mathcal{T}_\epsilon^Y(u) \,:\, \mathbf{R}^N \times \mathbf{R}^N \to S$$
$$(x,y) \mapsto \mathcal{T}_\epsilon^Y(u)(x,y) = u(\epsilon\big[\frac{x}{\epsilon}\big]_Y + \epsilon y).$$

One readily sees that

$$\forall x \in \mathbf{R}^N,\ \mathcal{T}_\epsilon^Y(u)(x, \{x/\epsilon\}) = u(x).$$

Moreover, $\mathcal{T}_\epsilon^Y(u)$ is invariant under the action of $\mathbf{Z}^N$:

$$\mathcal{T}_\epsilon^Y(u)(x + \epsilon\xi, y - \xi) = \mathcal{T}_\epsilon^Y(u)(x,y)$$

If $u : \mathbf{R}^N \to S$ and $f : S \to S'$, then

$$\mathcal{T}_\epsilon^Y(f \circ u) = f \circ \mathcal{T}_\epsilon^Y(u).$$

In particular if $u \,:\, \mathbf{R}^N \to S$ and $v \,:\, \mathbf{R}^N \to T$, the preceding property applied to the projections $P : (u,v) \mapsto u$ and $Q : (u,v) \mapsto v$ yields

$$\mathcal{T}_\epsilon^Y((u,v)) = (\mathcal{T}_\epsilon^Y(u), \mathcal{T}_\epsilon^Y(v)).$$

Therefore, if $F : S \times T \to R$,

$$\mathcal{T}_\epsilon^Y(F(u,v)) = F(\mathcal{T}_\epsilon^Y(u), \mathcal{T}_\epsilon^Y(v)) \tag{2.1}$$

Useful particular cases are when $S = \mathbf{R}$, $T = \mathbf{R}$ and $F : (s,t) \to st$ and where $S = \mathbf{R}^N$, $T = \mathbf{R}^N$ and F is the dot product.

Remark 1. The previous statements allow to define the unfolded of an operator $\mathcal{T}_\epsilon^Y(a)$.

Proposition 2.2. *For every* $u \in L^1(\mathbf{R}^N)$,

$$\int_{\mathbf{R}^N} u(x)\,\mathrm{d}x = \frac{1}{|Y|}\int_{\mathbf{R}^N \times Y} \mathcal{T}_\epsilon^Y(u)(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

In particular, if $1 \le p < +\infty$ *and* $u \in L^p(\mathbf{R}^N)$, *then* $\mathcal{T}_\epsilon^Y(u) \in L^p(\mathbf{R}^N \times Y)$, *and*

$$\|\mathcal{T}_\epsilon^Y(u)\|_{L^p(\mathbf{R}^N \times Y)} = |Y|^{1/p}\|u\|_{L^p(\mathbf{R}^N)}.$$

If χ_A denotes the characteristic function of a measurable set A, the combination of Proposition 2.2 together with (2.1) yields

Proposition 2.3. *Let* $A \subset \mathbf{R}^N$ *be measurable. if* $u \in L^1(A)$, *then* $\mathcal{T}_\epsilon^Y(\chi_A)\mathcal{T}_\epsilon^Y(u)$ *is well-defined on* $\mathbf{R}^N \times \mathbf{R}^N$, $\mathcal{T}_\epsilon^Y(\chi_A)\mathcal{T}_\epsilon^Y(u) \in L^1(\mathbf{R}^N \times Y)$, *and*

$$\int_A u(x)\,\mathrm{d}x = \frac{1}{|Y|}\int_{\mathbf{R}^N \times Y} \mathcal{T}_\epsilon^Y(\chi_A)\mathcal{T}_\epsilon^Y(u)\,\mathrm{d}x\,\mathrm{d}y.$$

In particular, if $1 \le p < +\infty$ *and* $u \in L^p(A)$, *then* $\mathcal{T}_\epsilon^Y(\chi_A)\mathcal{T}_\epsilon^Y(u)$ *is well-defined on* $\mathbf{R}^N \times \mathbf{R}^N$, $\mathcal{T}_\epsilon^Y(\chi_A)\mathcal{T}_\epsilon^Y(u) \in L^p(\mathbf{R}^N \times Y)$ *and*

$$\|\mathcal{T}_\epsilon^Y(\chi_A)\mathcal{T}_\epsilon^Y(u)\|_{L^p(\mathbf{R}^N \times Y)} = |Y|^{1/p}\|u\|_{L^p(A)}.$$

Remark 2. Proposition 2.3 shows that the natural domain of the unfolding $\mathcal{T}_\epsilon^Y(u)$ of a function u defined on A is the unfolding of the set A. As we shall see later,

$$\mathcal{T}_\epsilon^Y(\chi_A) \to \chi_{A \times Y}$$

in $L^1_{\mathrm{loc}}(\mathbf{R}^N \times Y)$ if $|\partial A| = 0$. The convergence is in $L^1(\mathbf{R}^N \times Y)$ if A has a finite measure. If Ω is a bounded open set with a Lipschitz boundary, then there is a constant C such that when ϵ is sufficiently small,

$$\|\mathcal{T}_\epsilon^Y(\chi_\Omega) - \chi_{\Omega \times Y}\|_{L^1(\mathbf{R}^N \times Y)} \le C\epsilon.$$

2.2 Unfolding locally summable functions

Since the unfolding operator has a local action, it is natural to examine its effect on locally summable functions.

Lemma 2.4. *For every bounded open set $\Omega \subset \mathbf{R}^N \times \mathbf{R}^N$, and every $\epsilon_0 > 0$, there is $C \geq 1$ and a bounded open set $\Omega' \subset \mathbf{R}^N$, such that for every $u \in L^p_{\rm loc}(\mathbf{R}^N)$, $1 \leq p < +\infty$, for every $\epsilon < \epsilon_0$,*

$$\|\mathcal{T}_\epsilon^Y(u)\|_{L^p(\Omega)} \leq C|Y|^{1/p}\|u\|_{L^p(\Omega')}. \tag{2.2}$$

Proof. The case $p > 1$ follows from the case $p = 1$ applied to the function $|u|^p$. We can thus consider that $p = 1$.

Assume first that $\Omega = \Omega_1 \times (\zeta + Y)$. Define

$$\Omega' = \{x \in \mathbf{R}^N \text{ s.t. } d(\Omega_1, x) < 2\epsilon_0(\text{ diam } (Y) + |\zeta|)\}. \tag{2.3}$$

Note that if $x + \epsilon\zeta \in \Omega_1$ and $y \in Y$, then $|x - \epsilon[x/\epsilon]_Y - \epsilon y| = |\epsilon\{x/\epsilon\}_Y - \epsilon y| < \epsilon$, and therefore $\epsilon[x/\epsilon]_Y + \epsilon y \in \Omega$. This means that $\mathcal{T}_\epsilon^Y(\chi\Omega') \geq \chi_{(\Omega_1 - \epsilon\zeta)\times Y}$. Therefore, by the group invariance of the unfolding and by Proposition 2.3

$$\begin{aligned}\int_{\Omega_1\times(Y+\zeta)} |\mathcal{T}_\epsilon^Y(u)|\,\mathrm{d}x\,\mathrm{d}y &\leq \int_{(\Omega_1-\epsilon\zeta)\times Y} |\mathcal{T}_\epsilon^Y(u)|\,\mathrm{d}x\,\mathrm{d}y \\ &\leq \int_{\mathbf{R}^N\times Y} \mathcal{T}_\epsilon^Y(\chi_\Omega)|\mathcal{T}_\epsilon^Y(u)|\,\mathrm{d}x\,\mathrm{d}y = \int_\Omega |u|\,\mathrm{d}x.\end{aligned}$$

In general consider $\Omega = \Omega_1 \times \Omega_2$. since Ω_2 is bounded, there is a finite collection $(\xi_i)_{i=1}^m$ in $\mathbf{Z}^N$ such that

$$\Omega_2 \subset \bigcup_{i=1}^m (Y + \xi_i)$$

The Proposition is applicable to each of the sets $\xi_i + Y$ with $C = 1$ and yields a set Ω^i. Let $\Omega' = \cup_i \Omega^i$, $C = m$. for every $\epsilon < \epsilon_0$, one has finally

$$\begin{aligned}\int_{\Omega_1\times\Omega_2} |\mathcal{T}_\epsilon^Y(u)|\,\mathrm{d}\,\mathrm{d}y &\leq \sum_{i=1}^m \int_{\Omega_1\times(\xi_i+Y)} |\mathcal{T}_\epsilon^Y(u)|\,\mathrm{d}x\,\mathrm{d}y \\ &\leq |Y| \sum_{i=1}^m \int_{\Omega^i} |u|\,\mathrm{d}x \\ &\leq C|Y| \int_{\Omega'} |u|\,\mathrm{d}x.\end{aligned}$$

□

Proposition 2.5. *For every $1 \leq p < \infty$, $\mathcal{T}_\epsilon^Y$ is a linear and continuous operator from $L^p_{\rm loc}(\mathbf{R}^N)$ to $L^p_{\rm loc}(\mathbf{R}^N \times \mathbf{R}^N)$.*

We turn now on to the L^p_{loc} convergence properties for $1 \leq p < +\infty$.

Theorem 2.6. *Let $(u_\epsilon)_\epsilon, u$ in $L^p_{\text{loc}}(\mathbf{R}^N)$, $1 \leq p < +\infty$. If $u_\epsilon \to u$ strongly in $L^p_{\text{loc}}(\mathbf{R}^N)$ then*

$$\mathcal{T}^Y_\epsilon(u_\epsilon) \to u \otimes 1 \qquad \textit{strongly in } L^p_{\text{loc}}(\mathbf{R}^N \times \mathbf{R}^N) \textit{ as } \epsilon \to 0.$$

Global convergences follow easily

Theorem 2.7. *Let $A \subset \mathbf{R}^N$ be measurable, $\Omega \subset \mathbf{R}^N$ be a lipschitz open bounded subset, $(u_\epsilon)_\epsilon, u$ in $L^p(\Omega)$, $1 \leq p < +\infty$. If $u_\epsilon \to u$ strongly in $L^p(\Omega)$, then*

$$\mathcal{T}^Y_\epsilon(\chi_\Omega)\mathcal{T}^Y_\epsilon(u_\epsilon) \to (\chi_\Omega u) \otimes 1 \textit{ strongly in } L^p(\mathbf{R}^N \times Y) \textit{ as } \epsilon \to 0.$$

In the following result, we prove that the limit (if it exists) of an unfolding sequence is periodic.

Lemma 2.8. *Let $u_\epsilon \in L^1_{\text{loc}}(\mathbf{R}^N)$ ($L^p_{\text{loc}}(\mathbf{R}^N)$) and $\hat{u} \in L^1_{\text{loc}}(\mathbf{R}^N \times \mathbf{R}^N)$ ($L^p_{\text{loc}}(\mathbf{R}^N \times \mathbf{R}^N)$). If*

$$\mathcal{T}^Y_\epsilon(u_\epsilon) \rightharpoonup \hat{u} \ *\textit{–weakly in } \mathcal{M}(\mathbf{R}^N \times \mathbf{R}^N),\ (\textit{ weakly in } L^p_{\text{loc}}(\mathbf{R}^N)).$$

where $\mathcal{M}(\mathbf{R}^N \times \mathbf{R}^N)$ denotes the Radon measure space, then $\hat{u}$ is Y-periodic.

Proof. Let $\varphi \in C(\mathbf{R}^N \times \mathbf{R}^N)$ be compactly supported and let $\xi \in \mathbf{Z}^N$. By the $*$–weak convergence If (u_ϵ), and the uniform continuity of φ,

$$\begin{aligned}
&\int_{\mathbf{R}^N \times \mathbf{R}^N} \varphi(x,y)\big(\hat{u}(x,y+\xi) - \hat{u}(x,y)\big)\,\mathrm{d}x\,\mathrm{d}y \\
&\qquad \lim_{\epsilon\to 0}\int_{\mathbf{R}^N \times \mathbf{R}^N} \varphi(x,y)\big(\mathcal{T}^Y_\epsilon(u_\epsilon)(x,y+\xi) - \mathcal{T}^Y_\epsilon(u_\epsilon)(x,y)\big)\,\mathrm{d}x\,\mathrm{d}y \\
&\qquad \lim_{\epsilon\to 0}\int_{\mathbf{R}^N \times \mathbf{R}^N} \varphi(x,y)\big(\mathcal{T}^Y_\epsilon(u_\epsilon)(x+\epsilon\xi,y) - \mathcal{T}^Y_\epsilon(u_\epsilon)(x,y)\big)\,\mathrm{d}x\,\mathrm{d}y \\
&\qquad\qquad = \lim_{\epsilon\to 0}\int_{\mathbf{R}^N \times \mathbf{R}^N} \big(\varphi(x-\epsilon\xi,y) - \varphi(x,y)\big)\mathcal{T}^Y_\epsilon(u_\epsilon)(x,y)\,\mathrm{d}x\,\mathrm{d}y = 0.
\end{aligned}$$

Therefore, $\hat{u}(x,y+\xi) = \hat{u}(x,y)$ for every $\xi \in \mathbf{Z}^N$. □

Remark 3. In particular if

$$\mathcal{T}^Y_\epsilon(u_\epsilon) \to \hat{u} \text{ a.e.},$$

then $\hat{u}$ is Y-periodic. Indeed, in this case we can assume that $|u_\epsilon| < 1$ (otherwise we consider the sequence $\operatorname{arctg}(u_\epsilon)$) and $u_\epsilon \rightharpoonup \hat{u}$ $*$–weakly in $\mathcal{M}(\mathbf{R}^N \times \mathbf{R}^N)$, therefore we can apply the previous Lemma 2.8.

2.3 Unfolding operator and gradients

In this section we study the properties of the unfolding operator applied on the gradient of some functions.

If $u \in W^{1,p}_{\text{loc}}$ then by Proposition 2.5, $\mathcal{T}^Y_\epsilon(u) \in L^p_{\text{loc}}(\mathbf{R}^N \times \mathbf{R}^N)$ and $\mathcal{T}^Y_\epsilon(\nabla u) \in L^p_{\text{loc}}(\mathbf{R}^N \times \mathbf{R}^N)$. Moreover, for every test function $\varphi \in \mathcal{D}(\mathbf{R}^N \times \mathbf{R}^N)$

$$\int_{\mathbf{R}^N\times\mathbf{R}^N} \nabla_y\varphi\ \mathcal{T}^Y_\epsilon(u)\ \mathrm{d}x\ \mathrm{d}y = \int_{\mathbf{R}^N\times\mathbf{R}^N} \nabla_y\varphi(x,y)\ (\epsilon[x/\epsilon]_Y + \epsilon y)\ \mathrm{d}x\ \mathrm{d}y$$
$$= -\int_{\mathbf{R}^N\times\mathbf{R}^N} \varphi(x,y)\ \epsilon\nabla(\epsilon[x/\epsilon]_Y + \epsilon y)\ \mathrm{d}x\ \mathrm{d}y = -\int_{\mathbf{R}^N\times\mathbf{R}^N} \varphi\ \epsilon\mathcal{T}(\nabla u)\ \mathrm{d}\ \mathrm{d}y,$$

Therefore $\mathcal{T}^Y_\epsilon(u)$ is weakly differentiable with respect to y, and

$$\epsilon\mathcal{T}^Y_\epsilon(\nabla u) = \nabla_y(\mathcal{T}^Y_\epsilon(u)). \tag{2.4}$$

The following proposition is an important tool for the sequel.

Proposition 2.9. *Let $(u_\epsilon)_\epsilon$ be a sequence of $W^{1,p}_{\text{loc}}(\mathbf{R}^N)$ and let $\hat{u} \in^p_{\text{loc}} (\mathbf{R}^N;\mathbf{R}^N)$. If $(u_\epsilon)_\epsilon$ is bounded in $L^p_{\text{loc}}(\mathbf{R}^N)$, $(\epsilon\nabla u_\epsilon)_\epsilon$ is bounded in $L^p_{\text{loc}}(\mathbf{R}^N)$ and*

$$\mathcal{T}^Y_\epsilon(u_\epsilon) \rightharpoonup \hat{u} \text{ weakly in } L^p_{\text{loc}}(\mathbf{R}^N \times \mathbf{R}^N) \text{ as } \epsilon \to 0,$$

then

$$\epsilon\mathcal{T}^Y_\epsilon(\nabla u_\epsilon) \rightharpoonup \nabla_y\hat{u} \text{ weakly in } L^p_{\text{loc}}(\mathbf{R}^N \times \mathbf{R}^N) \text{ as } \epsilon \to 0.$$

Moreover $\hat{u}$ is Y-periodic in y.

For every $u \in L^1_{\text{loc}}(\mathbf{R}^n)$, the functions $\mathcal{Q}^Y_\epsilon(u)$ and $\mathcal{R}^Y_\epsilon(u)$ are defined as follows:

$$u = \mathcal{Q}^Y_\epsilon(u) + \mathcal{R}^Y_\epsilon(u), \tag{2.5}$$

There $\mathcal{Q}^Y_\epsilon(w)$ is a Q^1 function such that

$$\mathcal{Q}^Y_\epsilon(u)(\epsilon\xi_k) = \frac{1}{|Y|}\int_Y u(\epsilon\xi_k + y)\ \mathrm{d}y. \tag{2.6}$$

The following properties of $\mathcal{Q}^Y_\epsilon(w)$ and $\mathcal{R}^Y_\epsilon(w)$ follow from the Poincaré-Wirtinger inequality together with a scaling of order ϵ and a change of coordinate, see [7, 4]:

Proposition 2.10. *Let $\epsilon_0 > 0$, $u \in W^{1,p}_{\text{loc}}(\mathbf{R}^N)$, $\Omega \subset \mathbf{R}^N$ be bounded and open. There is a bounded open set $\Omega' \subset \Omega$ such that for every $\epsilon < \epsilon_0$,*

$$\|\mathcal{Q}^Y_\epsilon(u)\|_{W^{1,p}(\Omega)} \leq C\|u\|_{W^{1,p}(\Omega')}, \tag{2.7a}$$
$$\|\mathcal{R}^Y_\epsilon(u)\|_{L^p(\Omega)} \leq C\epsilon\|\nabla u\|_{L^p(\Omega')}, \tag{2.7b}$$
$$\|\mathcal{R}^Y_\epsilon(u)\|_{L^p(\Omega)} \leq C\|u\|_{W^{1,p}(\Omega')}, \tag{2.7c}$$
$$\|\mathcal{R}^Y_\epsilon(u)\|_{W^{1,p}(\Omega)} \leq C\|u\|_{W^{1,p}(\Omega')}. \tag{2.7d}$$

2.4 Averaging operator

Definition 2.11. Let $\epsilon > 0$, Y be a reference cell and u in $L^1_{loc}(\mathbf{R}^N \times \mathbf{R}^N)$, the averaging operator $\mathcal{U}^Y_\epsilon$ is defined as follows

$$\mathcal{U}^Y_\epsilon(u)(x) = \frac{1}{|Y|}\int_Y u(\epsilon\big[\frac{x}{\epsilon}\big]_Y + \epsilon z, \epsilon\big\{\frac{x}{\epsilon}\big\}_Y)\ \mathrm{d}y.$$

The following proposition easily follows from a change of coordinate together with Fubini Theorem.

Proposition 2.12. *For every $u \in L^1_{loc}(\mathbf{R}^N)$, one has*

$$\mathcal{U}^Y_\epsilon\Big(\mathcal{T}^Y_\epsilon(u)\Big)(x) = u(x),$$

and for every $v \in L^1_{loc}(\mathbf{R}^N \times \mathbf{R}^N)$,

$$\mathcal{T}^Y_\epsilon\Big(\mathcal{U}^Y_\epsilon(v)\Big)(x,y) = \frac{1}{|Y|}\int_Y v(\epsilon\big[\frac{x}{\epsilon}\big]_Y + \epsilon z)\ \mathrm{d}z.$$

Remark 4. Since we consider $L^p_{loc}(\mathbf{R}^N \times \mathbf{R}^N)$ functions, we can not define the averaging operator as the adjoint of the unfolding operator. This is the case in $L^p(\mathbf{R}^N \times \mathbf{R}^N)$.

Rememberring Proposition 2.2 and Proposition 2.12, we deduce that:

Proposition 2.13. *for every $u \in L^1(\mathbf{R}^N \times Y)$,*

$$\frac{1}{|Y|}\int_{\mathbf{R}^N\times Y} u(x,y)\ \mathrm{d}x\ \mathrm{d}y = \int_{\mathbf{R}^N} \mathcal{U}^Y_\epsilon(u)(x)\ \mathrm{d}x.$$

In particular, if $1 \le p < +\infty$ and $u \in L^p(\mathbf{R}^N \times Y)$, then $\mathcal{U}^Y_\epsilon(u) \in L^p(\mathbf{R}^N)$, and

$$\|\mathcal{U}^Y_\epsilon(u)\|_{L^p(\mathbf{R}^N)} = |Y|^{-1/p}\|u\|_{L^p(\mathbf{R}^N\times Y)}.$$

Similarly to Lemma 2.4, we prove

Lemma 2.14. *for every bounded open set $\Omega \subset \mathbf{R}^N$, and every $\epsilon_0 > 0$, there is $C \ge 1$ and a bounded open set $\Omega \times Y \subset \Omega' \subset \mathbf{R}^N \times \mathbf{R}^N$, such that for every $u \in L^p_{\mathrm{loc}}(\mathbf{R}^N \times \mathbf{R}^N)$, $1 \le p < +\infty$, for every $\epsilon < \epsilon_0$,*

$$\|\mathcal{U}^Y_\epsilon(u)\|_{L^p(\Omega)} \le C|Y|^{-1/p}\|u\|_{L^p(\Omega')}. \tag{2.8}$$

Therefore,

Proposition 2.15. *For every $1 \le p < \infty$, $\mathcal{U}^Y_\epsilon$ is a linear and continuous operator from $L^p_{\mathrm{loc}}(\mathbf{R}^N \times \mathbf{R}^N)$ to $L^p_{\mathrm{loc}}(\mathbf{R}^N)$.*

Proposition 2.16. *If* $1 \leq p < \infty$ *and* $w \in L^p_{\mathrm{loc}}(\mathbf{R}^N \times \mathbf{R}^N)$ *then*

$$\mathcal{T}_\epsilon^Y\left(\mathcal{U}_\epsilon^Y(w)\right) \to w$$

strongly in $L^p_{\mathrm{loc}}(\mathbf{R}^N \times \mathbf{R}^N)$ *as* $\epsilon \to 0$.

Proof. By Proposition 2.13, this is clear for uniformly continuous functions. Proposition 2.15 allows to conclude with a density argument. □

We turn now on to the L^p locally convergence equivalence properties for $1 \leq p < +\infty$. The following result is in [4].

Theorem 2.17. *Let* $(u_\epsilon)_\epsilon$ *in* $L^p_{\mathrm{loc}}(\mathbf{R}^N)$ *and* $\hat{u} \in L^p_{\mathrm{loc}}(\mathbf{R}^N \times \mathbf{R}^N)$, $1 \leq p < +\infty$. *The following strong equivalences are equivalent:*

$$\begin{aligned} \text{i) } \mathcal{T}_\epsilon^Y(u_\epsilon) &\to \hat{u} \text{ strongly in } L^p_{\mathrm{loc}}(\mathbf{R}^N \times \mathbf{R}^N) \text{ as } \epsilon \to 0, \\ \text{ii) } u_\epsilon - \mathcal{U}_\epsilon^Y(\hat{u}) &\to 0 \text{ strongly in } L^p_{\mathrm{loc}}(\mathbf{R}^N) \text{ as } \epsilon \to 0. \end{aligned}$$

3 Unfolding operator with microscopic translations

In order to obtain a powerful reiterated homogeneization technique, we should understand small perturbation of the unfolding obtained by microscopic translation.

Let $\gamma : \mathbf{R}^N \to \mathbf{R}^N$. The translation operator τ_γ associated to γ, is defined for $w : \mathbf{R}^N \times \mathbf{R}^N \to S$ by

$$\tau_\gamma(w)(x, y) = w(x, y - \gamma(x)).$$

The next Propositions follow from measure theory.

Proposition 3.1. *If* γ *and* w *are both Borel measurable, then* $\tau_\gamma(w)$ *is Borel measurable.*

If $w \in L^1_{\mathrm{loc}}(\mathbf{R}^N)$ *and* $\varphi \in L^\infty_c(\mathbf{R}^N)$, *then*

$$\int_{\mathbf{R}^N \times \mathbf{R}^N} \varphi \tau_\gamma(w) \,\mathrm{d}x \,\mathrm{d}y = \int_{\mathbf{R}^N \times \mathbf{R}^N} \tau_{-\gamma}(\varphi) w \,\mathrm{d}x \,\mathrm{d}y$$

If $w \in L^1_{\mathrm{loc}}(\mathbf{R}^N)$ *and* $\nabla_y w \in L^1_{\mathrm{loc}}(\mathbf{R}^N)$, *then* $\nabla_y \tau_\gamma w = \tau_\gamma \nabla_y w \in L^1_{\mathrm{loc}}(\mathbf{R}^N \times \mathbf{R}^N)$.

Proposition 3.2. *For every open bounded set* $\Omega_2 \subset \mathbf{R}^N$ *and* $C \geq 0$, *there is a bounded open set* $\Omega_2' \subset \mathbf{R}^N$ *such that for every bounded open set* $\Omega_1 \subset \mathbf{R}^N$, *every Borel measurable function* $\gamma : \mathbf{R}^N \to \mathbf{R}^N$ *and every* $w \in L^p_{\mathrm{loc}}(\mathbf{R}^N \times \mathbf{R}^N)$, *if* $|\gamma| \leq C$ *on* Ω_1 *then*

$$\|\tau_\gamma w\|_{L^p(\Omega_1 \times \Omega_2)} \leq \|w\|_{L^p(\Omega_1 \times \Omega_2')}.$$

Now we can consider sequences

Proposition 3.3. *Let $w, w_\epsilon \in L^p_{\text{loc}}(\mathbf{R}^N \times \mathbf{R}^N)$, $1 \leq p < +\infty$. Let $\gamma, \gamma_\epsilon : \mathbf{R}^N \to \mathbf{R}^N$ be Borel measurable such that the sequence (γ_ϵ) is bounded in $L^\infty_{loc}(\mathbf{R}^N)$ and $\gamma_\epsilon \to \gamma$ almost everywhere. If $w_\epsilon \to w$ strongly in $L^p_{\text{loc}}(\mathbf{R}^N \times \mathbf{R}^N)$, then $\tau_{\gamma_\epsilon} w_\epsilon \to \tau_\gamma w$ strongly in $L^p_{\text{loc}}(\mathbf{R}^N \times \mathbf{R}^N)$. If $w_\epsilon \rightharpoonup w$ weakly in $L^p_{\text{loc}}(\mathbf{R}^N \times \mathbf{R}^N)$, then $\tau_{\gamma_\epsilon} w_\epsilon \rightharpoonup \tau_\gamma w$ weakly in $L^p_{\text{loc}}(\mathbf{R}^N \times \mathbf{R}^N)$.*

Proof. Using Proposition 3.2, we deduce that for all ϵ,

$$\|\tau_{\gamma_\epsilon} w_\epsilon\|_{L^p(\Omega_1 \times \Omega_2)} \leq \|w_\epsilon\|_{L^p(\Omega_1 \times \Omega_2')},$$

hence this gives the strong convergence.

Let φ be in $\mathcal{D}(\Omega_1 \times \Omega_2)$. We have

$$\int_{\mathbf{R}^N} \tau_{\gamma_\epsilon}(w_\epsilon)\varphi \, \mathrm{d}x = \int_{\mathbf{R}^N} w_\epsilon \tau_{-\gamma_\epsilon}(\varphi) \, \mathrm{d}x.$$

Consequently, using that $\tau_{-\gamma_\epsilon}\varphi \to \tau_{-\gamma}\varphi$ strongly in $L^q(\Omega_1 \times \Omega_2)$, we obtain that

$$\int_{\mathbf{R}^N} \tau_{\gamma_\epsilon}(w_\epsilon)\varphi \, \mathrm{d}x \to \int_{\mathbf{R}^N} w\tau_{-\gamma}(\varphi) \, \mathrm{d}x = \int_{\mathbf{R}^N} \tau_\gamma(w)\varphi \, \mathrm{d}x.$$

□

Now let us see how the microscopic translation interacts with the unfolding. Let

$$\mathcal{T}^Y_{\epsilon,\gamma_\epsilon}(u)(x,y) = \tau_{\gamma_\epsilon}\mathcal{T}^Y_\epsilon(u)(x,y) = u\big(\epsilon[x/\epsilon]_Y + \epsilon(y - \gamma_\epsilon(x))\big).$$

Proposition 3.4 (Relationships with unfolding). *If $u \in L^1(\mathbf{R}^N)$, then*

$$\int_{\mathbf{R}^N} u \, \mathrm{d}x = \frac{1}{|Y|}\int_{\mathbf{R}^N \times \mathbf{R}^N} \tau_{\gamma_\epsilon}(\chi_{\mathbf{R}^N \times Y})\mathcal{T}^Y_{\epsilon,\gamma_\epsilon}(u) \, \mathrm{d}x \, \mathrm{d}y.$$

Suppose the sequence (γ_ϵ) is bounded in $L^\infty_{loc}(\mathbf{R}^N)$ and $\gamma_\epsilon \to 0$ almost everywhere. If $u_\epsilon \to u$ strongly in $L^p_{\text{loc}}(\mathbf{R}^N)$, then

$$\mathcal{T}^Y_{\epsilon,\gamma_\epsilon}(u_\epsilon) \to u \otimes 1$$

strongly in $L^p_{\text{loc}}(\mathbf{R}^N)$ as $\epsilon \to 0$.

Proof. The integration formula follows from Proposition 3.1 and Proposition 3.2. The convergence result comes from Proposition 3.3, Theorem 2.6 together with Proposition 3.2. □

Remark 5. Proposition 3.2 and Proposition 3.4 implie that we have the equivalent of Lemma 2.4 and Proposition 2.5 for $\mathcal{T}^Y_{\epsilon,\gamma_\epsilon}$.

If $f \in L^1_{\text{loc}}(\Omega_1 \times \Omega_2)$ and $g \in \mathcal{D}(\Omega_1)$, let

$$f \llcorner g = \int_{\mathbf{R}^N} f(x,y)g(x)\,\mathrm{d}x. \tag{3.1}$$

The following result is the main result of this section. In the case where there is no microscopic translation, this is a result of [4]. In order to have a reiterated operator we add the strong convergence (3.4).

Theorem 3.5. *Suppose the sequence (γ_ϵ) is bounded in $L^\infty_{loc}(\mathbf{R}^N)$ and $\gamma_\epsilon \to 0$ almost everywhere. Let $(u_\epsilon)_\epsilon$ be a sequence of $W^{1,p}_{\text{loc}}(\mathbf{R}^N)$. If $u_\epsilon \rightharpoonup u$ weakly in $W^{1,p}_{\text{loc}}(\mathbf{R}^N)$, then,*

$$\mathcal{Q}^Y_\epsilon(u_\epsilon) \rightharpoonup u \text{ weakly in } W^{1,p}_{\text{loc}}(\mathbf{R}^N), \tag{3.2}$$

$$\mathcal{T}^Y_{\epsilon,\gamma_\epsilon}(\nabla \mathcal{Q}^Y_\epsilon(u_\epsilon)) \rightharpoonup \nabla u \otimes 1 \text{ weakly in } L^p_{\text{loc}}(\mathbf{R}^N \times \mathbf{R}^N), \tag{3.3}$$

More precisely, for every $\varphi \in \mathcal{D}(\mathbf{R}^N)$

$$\mathcal{T}^Y_{\epsilon,\gamma_\epsilon}(\nabla \mathcal{Q}^Y_\epsilon(u_\epsilon)) \llcorner \varphi \to \int_{\mathbf{R}^N} \varphi \nabla u \,\mathrm{d}x \text{ strongly in } L^p_{\text{loc}}(\mathbf{R}^N), \tag{3.4}$$

Moreover there exists $\hat{u}$ in $L^p_{\text{loc}}(\mathbf{R}^N \times \mathbf{R}^N)$ such that for a subsequence (not relabeled), the following convergences hold:

$$\frac{1}{\epsilon}\mathcal{T}^Y_{\epsilon,\gamma_\epsilon}(\mathcal{R}^Y_\epsilon(u_\epsilon)) \rightharpoonup \hat{u} \text{ weakly in } L^p_{\text{loc}}(\mathbf{R}^N \times \mathbf{R}^N), \tag{3.5}$$

$$\mathcal{T}^Y_{\epsilon,\gamma_\epsilon}(\nabla \mathcal{R}^Y_\epsilon(u_\epsilon)) \rightharpoonup \nabla_y \hat{u} \text{ weakly in } L^p_{\text{loc}}(\mathbf{R}^N \times \mathbf{R}^N), \tag{3.6}$$

$$\mathcal{T}^Y_{\epsilon,\gamma_\epsilon}(\nabla u_\epsilon) \rightharpoonup \nabla u \otimes 1 + \nabla_y \hat{u} \text{ weakly in } L^p_{\text{loc}}(\mathbf{R}^N \times \mathbf{R}^N), \tag{3.7}$$

as ϵ goes to zero. Additionally, $\hat{u}$ is a Y-periodic function in y.

Proof. By (2.7b), we have

$$\mathcal{Q}^Y_\epsilon(u_\epsilon) \to u \text{ strongly in } L^p_{\text{loc}}(\mathbf{R}^N). \tag{3.8}$$

Therefore, since $\mathcal{Q}^Y_\epsilon(u_\epsilon)$ is bounded in $W^{1,p}_{\text{loc}}(\mathbf{R}^N)$,

$$\mathcal{Q}^Y_\epsilon(u_\epsilon) \rightharpoonup u \text{ weakly in } W^{1,p}_{\text{loc}}(\mathbf{R}^N).$$

With Proposition 2.5 and Proposition 3.3, the convergence (3.8) also yields

$$\mathcal{T}^Y_{\epsilon,\gamma_\epsilon}(\mathcal{Q}^Y_\epsilon(u_\epsilon)) \to u \otimes 1 \text{ strongly in } L^p_{\text{loc}}(\mathbf{R}^N \times \mathbf{R}^N).$$

Let us now establish the second convergence (3.3). For $\varphi \in \mathcal{D}(\mathbf{R}^N)$, define the auxiliary sequence (w_ϵ) in $L^p_{\text{loc}}(\mathbf{R}^N)$ by

$$w_\epsilon(y) = \mathcal{T}^Y_{\epsilon,\gamma_\epsilon}(\nabla \mathcal{Q}^Y_\epsilon(u_\epsilon)) \llcorner \varphi.$$

The elements of the sequence (w_ϵ) are Q^1 affine functions. The space of Q^1 affine functions is locally finite dimensional, so that any bounded set is strongly relatively compact for the $L^p_{\text{loc}}(\mathbf{R}^N)$ topology. By (2.7a) and Proposition 2.5, the sequence (w_ϵ) is bounded and thus relatively compact in $L^p_{\text{loc}}(\mathbf{R}^N)$. Let w be an accumulation point of the sequence (w_ϵ). The function w is Q^1 affine. Moreover recalling Lemma 2.8. We see that w is periodic. Since w is Q^1 affine and periodic, w must be constant. Using the convergence (3.2), one has

$$\begin{aligned} w(x) = \frac{1}{|Y|}\int_Y w\,\mathrm{d}y &= \lim_{\epsilon\to 0}\frac{1}{|Y|}\int_Y w_\epsilon\,\mathrm{d}y \\ &= \lim_{\epsilon\to 0}\int_{\mathbf{R}^N\times Y}\varphi(x)\mathcal{T}^Y_{\epsilon,\gamma_\epsilon}(\nabla\mathcal{Q}^Y_\epsilon(u_\epsilon))\,\mathrm{d}x\,\mathrm{d}y \\ &= \lim_{\epsilon\to 0}\int_{\mathbf{R}^N\times Y}\mathcal{T}^Y_{\epsilon,\gamma_\epsilon}(\varphi)\mathcal{T}^Y_{\epsilon,\gamma_\epsilon}(\nabla\mathcal{Q}^Y_\epsilon(u_\epsilon))\,\mathrm{d}x\,\mathrm{d}y \\ &= \lim_{\epsilon\to 0}\int_{\mathbf{R}^N}\varphi\nabla\mathcal{Q}^Y_\epsilon(u_\epsilon) = \int_{\mathbf{R}^N}\varphi\nabla u\,\mathrm{d}x. \end{aligned}$$

This shows that w_ϵ has a unique accumulation point, therefore, as $\epsilon\to 0$,

$$w_\epsilon \to \int_{\mathbf{R}^N}\varphi\nabla u\,\mathrm{d}x.$$

This establishes (3.4). Since $\mathcal{T}^Y_{\epsilon,\gamma_\epsilon}(\nabla\mathcal{Q}^Y_\epsilon(u_\epsilon))$ is bounded in $L^p_{\text{loc}}(\mathbf{R}^N\times\mathbf{R}^N)$ this gives the convergence (3.3).

The convergences (3.5) and (3.6) come as follows. By (2.7d), the sequence $(\nabla R^Y_\epsilon(u_\epsilon))_\epsilon$ is bounded in $L^p_{\text{loc}}(\mathbf{R}^N)$ and by (2.7b) the sequence $(\epsilon^{-1}R^Y_\epsilon(u_\epsilon))_\epsilon$ is bounded in $L^p_{\text{loc}}(\mathbf{R}^N)$. Therefore, Proposition 2.9 is applicable. This gives (3.6) and the Y-periodicity of $\hat{u}$.

The last convergence (3.7) comes from the decomposition

$$\mathcal{T}^Y_{\epsilon,\gamma_\epsilon}(\nabla u_\epsilon) = \mathcal{T}^Y_{\epsilon,\gamma_\epsilon}(\nabla Q^Y_\epsilon(u_\epsilon)) + \mathcal{T}^Y_{\epsilon,\gamma_\epsilon}(\nabla R^Y_\epsilon(u_\epsilon)). \qquad \square$$

4 Composition of unfolding operators

In this section we compose unfolding operators with the following convention:

Any unfolding operator acts on the two last variables of a function.

For simplicity, we will state the result for a composition of two unfolding operators, but the composition of any number of unfolding operators would be similar.

Let Y and Z be two reference cells (sets having the paving property with respect to basis, defining the periods, $(b_1,...,b_N)$ and $(c_1,...,c_N)$, respectively). We define the reiterated unfolding operator as follows:

For $\epsilon > 0$ and $\delta(\epsilon) > 0$, with $\lim_{\epsilon\to 0}\delta(\epsilon) = 0$, Y and Z two reference cells and $u : \mathbf{R}^N \to X$, where X is a set, the composition of the unfolding operators associated to Y and Z with τ_{γ_ϵ} in between, where γ_ϵ is a bounded sequence of $L^\infty_{\text{loc}}(\mathbf{R}^N)$, gives

$$\mathcal{T}^Z_{\delta(\epsilon)}(\tau_{\gamma_\epsilon}(\mathcal{T}^Y_\epsilon(u))) = \mathcal{T}^Z_{\delta(\epsilon)}(\mathcal{T}^Y_{\epsilon,\gamma_\epsilon}(u))$$

$$\begin{aligned} \mathcal{T}^Z_{\delta(\epsilon)}(\mathcal{T}^Y_{\epsilon,\gamma_\epsilon}(u)) \, : \mathbf{R}^N \times \mathbf{R}^N \times \mathbf{R}^N &\to X \\ (x,y,z) &\mapsto u\big(\epsilon\big[\tfrac{x}{\epsilon}\big]_Y + \epsilon\delta(\epsilon)\big[\tfrac{y}{\delta(\epsilon)}\big]_Z + \epsilon\delta(\epsilon)z - \epsilon\gamma_\epsilon(x)\big). \end{aligned}$$

We immediately see that for all $x \in \Omega$, we have

$$\mathcal{T}^Z_{\delta(\epsilon)}(\mathcal{T}^Y_{\epsilon,\gamma_\epsilon}(u))(x, \{\frac{x}{\epsilon}\}_Y, \frac{\{\frac{x}{\epsilon}\}_Y}{\delta(\epsilon)}\}_Z + \frac{\gamma_\epsilon(x)}{\delta(\epsilon)}) = u(x).$$

We also have the integration formula

$$\int_{\mathbf{R}^N} u \,\mathrm{d}x = \frac{1}{|Y||Z|}\int_{\mathbf{R}^N\times\mathbf{R}^N\times Z} \mathcal{T}^Z_{\delta(\epsilon)}\big(\tau_{\gamma_\epsilon}(\chi_{\mathbf{R}^N\times Y})\big)\mathcal{T}^Z_{\delta(\epsilon)}\big(\mathcal{T}^Y_{\epsilon,\gamma_\epsilon}(u)\big)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

Remark 6. Proposition 3.2 and Proposition 3.4 implie that we have the equivalent of Lemma 2.4 and Proposition 2.5 for $\mathcal{T}^Z_{\delta(\epsilon)}\big(\mathcal{T}^Y_{\epsilon,\gamma_\epsilon}(u)\big)$.

The following result is the main result of this section.

Theorem 4.1. *Let $(u_\epsilon)_\epsilon$ be a sequence of $W^{1,p}_{\text{loc}}(\mathbf{R}^N)$ such that, $u_\epsilon \rightharpoonup u$ weakly in $W^{1,p}_{\text{loc}}(\mathbf{R}^N)$. Suppose the sequence (γ_ϵ) is bounded in $L^\infty_{loc}(\mathbf{R}^N)$ and $\gamma_\epsilon \to 0$ almost everywhere. Then,*

$$\mathcal{T}^Z_{\delta(\epsilon)}(\mathcal{T}^Y_{\epsilon,\gamma_\epsilon}(\nabla\mathcal{Q}^Y_\epsilon(u_\epsilon))) \to \nabla u \otimes 1 \otimes 1. \tag{4.1a}$$

strongly in $L^p_{\text{loc}}(\mathbf{R}^N\times\mathbf{R}^N\times\mathbf{R}^N)$. Moreover, there exists a subsequence (not relabeled), $\hat{u}$ in $L^p_{\text{loc}}(\mathbf{R}^N\times\mathbf{R}^N)$ and $\tilde{u}$ in $L^p_{\text{loc}}(\mathbf{R}^N\times\mathbf{R}^N\times\mathbf{R}^N)$ such that the following convergences hold:

$$\tfrac{1}{\delta(\epsilon)}\mathcal{T}^Z_{\delta(\epsilon)}(\tfrac{1}{\epsilon}\mathcal{T}^Y_{\epsilon,\gamma_\epsilon}\mathcal{R}^Y_\epsilon(u_\epsilon)) \rightharpoonup \tilde{u}, \tag{4.1b}$$

$$\mathcal{T}^Z_{\delta(\epsilon)}(\mathcal{T}^Y_{\epsilon,\gamma_\epsilon}(\nabla u_\epsilon) \rightharpoonup \nabla u \otimes 1 \otimes 1 + \nabla_y\hat{u}\otimes 1 + \nabla_z\tilde{u}, \tag{4.1c}$$

weakly in $L^p_{\text{loc}}(\mathbf{R}^N\times\mathbf{R}^N\times\mathbf{R}^N)$ as ϵ goes to zero. Additionally, $\hat{u}$ is Y-periodic and $\tilde{u}$ is Z-periodic.

Proof. Let $\varphi \in \mathcal{D}(\mathbf{R}^N)$. By Theorem 3.5, we know that

$$\mathcal{T}^Y_{\epsilon,\gamma_\epsilon}(\nabla Q^Y_\epsilon(u_\epsilon)) \llcorner \varphi \to \nabla u \otimes 1 \llcorner \varphi \qquad \text{strongly in } L^p_{\text{loc}}(\mathbf{R}^N),\ \text{as } \epsilon \to 0.$$

Therefore, by Theorem 2.6,

$$\mathcal{T}^Z_{\delta(\epsilon)}(\mathcal{T}^Y_{\epsilon,\gamma_\epsilon}(\nabla \mathcal{Q}^Y_\epsilon(u_\epsilon)) \llcorner \varphi) \to (\nabla u \otimes 1 \llcorner \varphi) \otimes 1 \quad \text{strongly in } L^p_{\rm loc}(\mathbf{R}^N), \text{ as } \epsilon \to 0.$$

By Fubini's Theorem,

$$\mathcal{T}^Z_{\delta(\epsilon)}(\mathcal{T}^Y_{\epsilon,\gamma_\epsilon}(\nabla \mathcal{Q}^Y_\epsilon(u_\epsilon))) \llcorner \varphi \to \nabla u \otimes 1 \otimes 1) \llcorner \varphi \quad \text{strongly in } L^p_{\rm loc}(\mathbf{R}^N \times \mathbf{R}^N), \text{ as } \epsilon \to 0.$$

Since by Fubini's Theorem, Proposition 2.5, Proposition 2.10 together with Proposition 3.2, the sequence $(\mathcal{T}^Z_{\delta(\epsilon)}(\mathcal{T}^Y_{\epsilon,\gamma_\epsilon}(\nabla Q^Y_\epsilon(u_\epsilon))))_\epsilon$ is bounded in $L^p_{\rm loc}(\mathbf{R}^N \times \mathbf{R}^N \times \mathbf{R}^N)$, we obtain (4.1a).

Next apply Theorem 3.5 to the sequence $(u_\epsilon)_\epsilon$ in order to obtain $\hat{u}$ and the corresponding converging subsequence (u_ϵ). Then applying Fubini's Theorem, and twice Proposition 2.5 and Proposition 2.10, (2.7d) and Proposition 3.2, we obtain that the sequence $\frac{1}{\delta(\epsilon)}\mathcal{T}^Z_{\delta(\epsilon)}(\mathcal{R}^Z_{\delta(\epsilon)}(\frac{1}{\epsilon}\mathcal{T}^Y_{\epsilon,\gamma_\epsilon}\mathcal{R}^Y_\epsilon(u_\epsilon))))_\epsilon$ is bounded in $L^p_{\rm loc}(\mathbf{R}^N \times \mathbf{R}^N \times \mathbf{R}^N)$. Therefore, there exists $\tilde{u} \in L^p_{\rm loc}(\mathbf{R}^N \times \mathbf{R}^N \times \mathbf{R}^N)$ and a subsequence of our subsequence such that

$$\frac{1}{\delta(\epsilon)}\mathcal{T}^Z_{\delta(\epsilon)}(\mathcal{R}^Z_{\delta(\epsilon)}(\tfrac{1}{\epsilon}\mathcal{T}^Y_{\epsilon,\gamma_\epsilon}\mathcal{R}^Y_\epsilon(u_\epsilon))) \rightharpoonup \tilde{u}, \tag{4.2}$$

weakly in $L^p_{\rm loc}(\mathbf{R}^N \times \mathbf{R}^N \times \mathbf{R}^N)$ as ϵ goes to zero. This is (4.1b).

Finally, let $\varphi \in \mathcal{D}(\mathbf{R}^N)$ be a test function.
Consider the sequence

$$v_\epsilon = \tfrac{1}{\epsilon}\mathcal{T}^Y_{\epsilon,\gamma_\epsilon}\mathcal{R}^Y_\epsilon(u_\epsilon) \llcorner \varphi.$$

By Theorem 3.5,

$$v_\epsilon \rightharpoonup v = \hat{u} \llcorner \varphi \text{ weakly in } W^{1,p}_{\rm loc}(\mathbf{R}^N). \tag{4.3}$$

Consequently, we can apply again Theorem 3.5 to the sequence $(v_\epsilon)_\epsilon$ for the unfolding operator $\mathcal{T}^Z_{\delta(\epsilon)}$, this gives the existence of $\hat{v}$ and the following weak convergence for a subsequence

$$\frac{1}{\delta(\epsilon)}\mathcal{T}^Z_{\delta(\epsilon)}(\mathcal{R}^Z_{\delta(\epsilon)}(v_\epsilon)) \rightharpoonup \hat{v} \text{ weakly in } L^p_{\rm loc}(\mathbf{R}^N \times \mathbf{R}^N). \tag{4.4}$$

Recalling next (4.2), we deduce that

$$\hat{v} = \tilde{u} \llcorner \varphi. \tag{4.5}$$

Additionally, the convergence (3.7) in Theorem 3.5 gives that

$$\mathcal{T}^Z_{\delta(\epsilon)}(\nabla_y v_\epsilon) \rightharpoonup \nabla_y v \otimes 1 + \nabla_z \hat{v} \text{ weakly in } L^p_{\rm loc}(\mathbf{R}^N \times \mathbf{R}^N; \mathbf{R}^N), \tag{4.6}$$

hence using (4.5), the fact that $\nabla_y(\hat{u} \llcorner \varphi) \otimes 1 = \nabla_y v \otimes 1$ together with the definition of v_ϵ, we obtain

$$\frac{1}{\delta(\epsilon)}\nabla_z \mathcal{T}^Z_{\delta(\epsilon)}(\tfrac{1}{\epsilon}\mathcal{T}^Y_{\epsilon,\gamma_\epsilon}\mathcal{R}^Y_\epsilon(u_\epsilon)) \llcorner \varphi = \mathcal{T}^Z_{\delta(\epsilon)}(\nabla_y \tfrac{1}{\epsilon}\mathcal{T}^Y_{\epsilon,\gamma_\epsilon}\mathcal{R}^Y_\epsilon(u_\epsilon)) \llcorner \varphi \rightharpoonup \nabla_y \hat{u} \otimes 1 \llcorner \varphi \nabla_z \tilde{v} \llcorner \varphi, \tag{4.7}$$

weakly in $L^p_{\mathrm{loc}}(\mathbf{R}^N \times \mathbf{R}^N; \mathbf{R}^N)$. Since the sequence $\mathcal{T}^Z_{\delta(\epsilon)}(\frac{1}{\epsilon}\mathcal{T}^Y_{\epsilon,\gamma_\epsilon}\mathcal{R}^Y_\epsilon(u_\epsilon))_\epsilon$ is bounded in $L^p_{\mathrm{loc}}(\mathbf{R}^N \times \mathbf{R}^N \times \mathbf{R}^N)$, we have

$$\frac{1}{\delta(\epsilon)}\nabla_z \mathcal{T}^Z_{\delta(\epsilon)}(\frac{1}{\epsilon}\mathcal{T}^Y_{\epsilon,\gamma_\epsilon}(\mathcal{R}^Y_\epsilon(u_\epsilon))) = \mathcal{T}^Z_{\delta(\epsilon)}(\nabla_y \frac{1}{\epsilon}\mathcal{T}^Y_{\epsilon,\gamma_\epsilon}(\mathcal{R}^Y_\epsilon(u_\epsilon))) \rightharpoonup \nabla_y \hat{u} \otimes 1 + \nabla_z \tilde{v}$$

weakly in $L^p_{\mathrm{loc}}(\mathbf{R}^N \times \mathbf{R}^N \times \mathbf{R}^N)$. The convergence (4.1c) follows. □

We can compose the averaging operators with a translation in between: for u in $L^1_{\mathrm{loc}}(\mathbf{R}^N \times \mathbf{R}^N \times \mathbf{R}^N)$, we can define

$$\mathcal{U}^Y_\epsilon\Big(\tau_{-\gamma_\epsilon}\mathcal{U}^Z_{\delta(\epsilon)}(u)\Big),$$

with the convention that the averaging operator acts on the two last variables. Then we have for $1 \le p < +\infty$,

Theorem 4.2. *Let $(u_\epsilon)_\epsilon$ be in $L^p_{\mathrm{loc}}(\mathbf{R}^N)$ and $\hat{u} \in L^p_{\mathrm{loc}}(\mathbf{R}^N \times \mathbf{R}^N)$, $1 \le p < +\infty$. Suppose the sequence (γ_ϵ) is bounded in $L^\infty_{loc}(\mathbf{R}^N)$ and $\gamma_\epsilon \to 0$ almost everywhere. The following strong equivalences are equivalent:*

i) $\mathcal{T}^Z_{\delta(\epsilon)}\Big(\mathcal{T}^Y_{\epsilon,\gamma_\epsilon}(u_\epsilon)\Big) \to \hat{u}$ *strongly in $L^p_{\mathrm{loc}}(\mathbf{R}^N \times \mathbf{R}^N \times \mathbf{R}^N)$ as $\epsilon \to 0$,*

ii) $u_\epsilon - \mathcal{U}^Y_\epsilon\Big(\tau_{-\gamma_\epsilon}\mathcal{U}^Z_{\delta(\epsilon)}(\hat{u})\Big) \to 0$ *strongly in $L^p_{\mathrm{loc}}(\mathbf{R}^N)$ as $\epsilon \to 0$.*

Proof. We apply Theorem 2.17 with $\mathcal{U}^Z_{\delta(\epsilon)}$ and we obtain the equivalence between

$$\mathcal{T}^Z_{\delta(\epsilon)}\Big(\mathcal{T}^Y_{\epsilon,\gamma_\epsilon}(u_\epsilon)\Big) \to \hat{u} \quad \text{strongly in } L^p_{\mathrm{loc}}(\mathbf{R}^N \times \mathbf{R}^N \times \mathbf{R}^N) \text{ as } \epsilon \to 0, \text{ and}$$
$$\mathcal{T}^Y_{\epsilon,\gamma_\epsilon}(u_\epsilon) - \mathcal{U}^Z_{\delta(\epsilon)}(\hat{u}) \to 0 \quad \text{strongly in } L^p_{\mathrm{loc}}(\mathbf{R}^N \times \mathbf{R}^N) \text{ as } \epsilon \to 0.$$

Then we obtain the equivalence with

$$\mathcal{T}^Y_\epsilon(u_\epsilon) - \tau_{-\gamma_\epsilon}\mathcal{U}^Z_{\delta(\epsilon)}(\hat{u}) \to 0 \text{ strongly in } L^p_{\mathrm{loc}}(\mathbf{R}^N \times \mathbf{R}^N) \text{ as } \epsilon \to 0.,$$

and applying again Theorem 2.17 with $\mathcal{U}^Y_\epsilon$, this gives the equivalence with

$$u_\epsilon - \mathcal{U}^Y_\epsilon\Big(\tau_{-\gamma_\epsilon}\mathcal{U}^Z_{\delta(\epsilon)}(\hat{u})\Big) \to 0 \text{ strongly in } L^p_{\mathrm{loc}}(\mathbf{R}^N) \text{ as } \epsilon \to 0.$$

□

5 Homogenization results

In this section, we give an homogenization result (cf [13], [15] for details about homogenization techniques) in the case of one scale, applying the unfolding method together with the approach of [9]. For the linear case this is the approach of [4].

Theorem 5.1. *Let* $1 < p < \infty$, $p^{-1} + q^{-1} = 1$, Ω *be a bounded open set,* $a_\epsilon : \Omega \times \mathbf{R}^N \to \mathbf{R}^N$, *with* $a_\epsilon(., \xi)$ *mesurable for all* $\xi \in \mathbf{R}^N$ *and* $a_\epsilon(x, .)$ *continuous for almost all* $x \in \Omega$, *be such that*

- *there exists* $\beta \geq \max(2, p)$ *and* $c > 0$ *such that for all* $x \in \Omega$, $\xi_1, \xi_2 \in \mathbf{R}^N$,

$$\frac{|\xi_1 - \xi_2|^\beta}{(1 + |\xi_1| + |\xi_2|)^{\beta - p}} \leq c(a_\epsilon(x, \xi_1) - a_\epsilon(x, \xi_2)|\xi_1 - \xi_2), \tag{5.1}$$

- *there exists* $C > 0$ *such that for all* $x \in \Omega$ *et* $\xi \in \mathbf{R}^N$,

$$a_\epsilon(x, \xi)| \leq C(1 + |\xi|^{p-1}), \tag{5.2}$$

- *and*

$$a_\epsilon(x, 0) = 0 \text{ for all } x \in \Omega. \tag{5.3}$$

Furthermore, we assume that there exists a cell Y *such that for almost every* $(x, y) \in \mathbf{R}^N \times \mathbf{R}^N$,

$$\mathcal{T}_\epsilon^Y(\chi_\Omega a_\epsilon)(x, y, \xi) \to \chi_{\Omega \times \mathbf{R}^N} a_0(x, y, \xi), \text{ as } \epsilon \to 0, \tag{5.4}$$

where $a_0(x, y, \xi)$ *is of Carathéodory type.*

Let $f_\epsilon \in W^{-1,q}(\Omega)$ *be such that* $f_\epsilon \to f$ *strongly in* $W^{-1,q}(\Omega)$.

If $u_\epsilon \in W_0^{1,p}(\Omega)$ *is the unique solution of the problem*

$$\begin{cases} \displaystyle\int_\Omega (a_\epsilon(x, \nabla u_\epsilon)|\nabla \varphi) \, \mathrm{d}x = \int_\Omega f_\epsilon \varphi \, \mathrm{d}x, \\ \forall \varphi \in W_0^{1,p}(\Omega), \end{cases} \tag{5.5}$$

Then

$$u_\epsilon \rightharpoonup u_0 \text{ weakly in } W_0^{1,p}(\Omega),$$

where u_0 *is the first term of the unique solution* $(u_0, \hat{u})$ *of the following variational problem:*

$$\begin{cases} u_0 \in W_0^{1,p}(\Omega), \ \hat{u} \in L^p(\Omega; W_{per}^{1,p}(Y)), \text{ with } \displaystyle\int_Y \hat{u}(x, y) \, \mathrm{d}y = 0, \\ \forall \Psi \in W_0^{1,p}(\Omega), \forall \Phi \in L^p(\Omega; W_{per}^{1,p}(Y)), \\ \displaystyle\frac{1}{|Y|} \int_{\Omega \times Y} (a_0(x, y, \nabla_x u_0 + \nabla_y \hat{u})|\nabla_x \Psi(x) + \nabla_y \Phi(x, y)) \, \mathrm{d}x \, \mathrm{d}y = \int_\Omega f \Psi \, \mathrm{d}x. \end{cases} \tag{5.6}$$

Moreover, the following strong convergences hold

$$\mathcal{T}_\epsilon^Y(\chi_\Omega)\mathcal{T}_\epsilon^Y(\nabla u_\epsilon) \to \chi_{\Omega\times\mathbf{R}^N}(\nabla_x u_0 + \nabla_y \hat{u}) \qquad \text{in } L^p(\mathbf{R}^N \times Y), \text{ as } \epsilon \to 0.$$

$$\nabla u_\epsilon - \nabla_x u_0 - \mathcal{U}_\epsilon^Y(\chi_{\Omega\times\mathbf{R}^N}\nabla_y \hat{u}) \to 0 \qquad \text{in } L^p(\Omega), \text{ as } \epsilon \to 0.$$

Remark 7. Theorem 5.1 implies for example the homogenization result obtained by Wall [16], and Lukkasen and Wall [10].

The previous result easily extends to the case of equations with different boundary conditions for which a variational formulation holds. For example, for the Neumann problem, we state the following:

Theorem 5.2. *Under the same assumptions on a_ϵ, a_0 and f_ϵ, if moreover, Ω has a Lipschitz boundary and*

$$\int_\Omega f_\epsilon \,\mathrm{d}x = 0,$$

if $u_\epsilon \in W^{1,p}(\Omega)$ denotes the unique solution of

$$\begin{cases} \displaystyle\int_\Omega u_\epsilon \,\mathrm{d}x = 0, \\ \displaystyle\int_\Omega (a_\epsilon(x, \nabla u_\epsilon) | \nabla\varphi) \,\mathrm{d}x = \int_\Omega f_\epsilon \varphi \,\mathrm{d}x, \\ \forall \varphi \in W^{1,p}(\Omega), \end{cases} \tag{5.7}$$

Then we have,

$$u_\epsilon \rightharpoonup u_0 \text{ weakly in } W^{1,p}(\Omega),$$

where u_0 is the first term of the unique solution $(u_0, \hat{u})$ of the following variational problem:

$$\begin{cases} u_0 \in W^{1,p}(\Omega),\ \hat{u} \in L^p(\Omega; W^{1,p}_{per}(Y)),\ \text{with} \displaystyle\int_Y \hat{u}(x,y)\,\mathrm{d}y = 0,\ \text{and} \int_\Omega u_0 \,\mathrm{d}x = 0, \\ \forall \Psi \in W_0^{1,p}(\Omega), \forall \Phi \in L^p(\Omega; W^{1,p}_{per}(Y)), \\ \displaystyle\frac{1}{|Y|}\int_{\Omega\times Y} (a_0(x, y, \nabla_x u_0 + \nabla_y \hat{u}) | \nabla_x \Psi(x) + \nabla_y \Phi(x,y)) \,\mathrm{d}x\,\mathrm{d}y = \int_\Omega f \Psi \,\mathrm{d}x. \end{cases} \tag{5.8}$$

Moreover, the following strong convergences hold

$$\mathcal{T}_\epsilon^Y(\chi_\Omega)\mathcal{T}_\epsilon^Y(\nabla u_\epsilon) \to \chi_{\Omega\times\mathbf{R}^N}(\nabla_x u_0 + \nabla_y \hat{u}) \qquad \text{in } L^p(\mathbf{R}^N \times Y), \text{ as } \epsilon \to 0.$$

$$\nabla u_\epsilon - \nabla_x u_0 - \mathcal{U}_\epsilon^Y(\chi_{\Omega\times\mathbf{R}^N}\nabla_y \hat{u}) \to 0 \qquad \text{in } L^p(\Omega), \text{ as } \epsilon \to 0.$$

where u_0 has been extended to $\mathbf{R}^N$ by a continuous extension operator.

Proof of Theorem 5.1. We know from [8], that there exists a unique solution of problem (5.5) for each ϵ. Let u_ϵ be the extension of the solution to problem (5.5).

First we establish the weak convergence of the unfolded sequences. It is classical, see [9], to deduce that the sequence $(u_\epsilon)_\epsilon$ is bounded in $W^{1,p}(\mathbf{R}^N)$. To see this, let $a, b, \eta > 0$, Young's inequality gives

$$ab \leq \frac{p}{\beta}\eta^{-\frac{\beta}{p}}a^{\frac{\beta}{p}} + \frac{\beta - p}{\beta}\eta^{\frac{\beta}{\beta-p}}a^{\frac{\beta}{\beta-p}},$$

hence using (5.3) and (5.1), we deduce that

$$\begin{aligned} |\xi|^p &= (1+|\xi|)^{p(p-\beta)}|\xi|^p(1+|\xi|)^{p(\beta-p)} \\ &\leq \frac{p}{\beta}\eta^{\frac{-\beta}{p}}(1+|\xi|)^{p-\beta}|\xi|^\beta + \frac{\beta-p}{\beta}\eta^{\frac{\beta}{\beta-p}}(1+|\xi|)^p \\ &\leq \frac{p}{\beta}\eta^{\frac{-\beta}{p}}(a_\epsilon(x,\xi)|\xi) + 2^{p-1}\frac{\beta-p}{\beta}\eta^{\frac{\beta}{\beta-p}}(1+|\xi|^p). \end{aligned}$$

Next, by choosing η sufficiently small, we deduce that there exits $C > 0$ such that for all $x \in \Omega$ and $\xi \in \mathbf{R}^N$, the following inequality holds

$$|\xi|^p \leq C(1 + (a_\epsilon(x,\xi)|\xi)). \tag{5.9}$$

The sequence $(u_\epsilon)_\epsilon$ is bounded in $W^{1,p}(\Omega)$. Indeed, choose u_ϵ in (5.5), it follows from (5.9) that

$$\begin{aligned} \int_\Omega |\nabla u_\epsilon|^p \,\mathrm{d}x &\leq C\int_\Omega (1 + (a_\epsilon(x,\nabla u_\epsilon)|\nabla u_\epsilon))\,\mathrm{d}x \\ &\leq C(|\Omega| + \|f_\epsilon\|_{W^{-1,q}(\Omega)}\|\nabla u_\epsilon\|_{W^{1,p}(\Omega)}) \\ &\leq C(1 + \|\nabla u_\epsilon\|_{W^{1,p}(\Omega)}). \end{aligned} \tag{5.10}$$

If $\|\nabla u_\epsilon\|_{W^{1,p}(\Omega)} \leq 1$, then it is clear that $(u_\epsilon)_\epsilon$ is bounded in $W^{1,p}(\Omega)$. Next, if $\|\nabla u_\epsilon\|_{W^{1,p}(\Omega)} \leq 1$, using that the two norms $\|.\|_{W^{1,p}(\Omega)}$ and $\|\nabla .\|_{L^p(\Omega)}$ are equivalent on $W_0^{1,p}(\Omega)$, together with (5.10), we deduce that

$$\|u_\epsilon\|_{W^{1,p}(\Omega)} \leq C.$$

Up to a subsequence, $u_\epsilon \rightharpoonup u_0$ weakly in $W^{1,p}(\Omega)$. By Theorem 3.5, there is $\hat{u} \in L^p(\Omega; W^{1,p}_{\text{per}}(Y))$ such that

$$\begin{cases} \mathcal{T}_\epsilon^Y(u_\epsilon) \rightharpoonup u_0 & \text{weakly in } L^p(\mathbf{R}^N; W^{1,p}(Y)), \\ \mathcal{T}_\epsilon^Y(\nabla u_\epsilon) \rightharpoonup \nabla_x u_0 + \nabla_y \hat{u} & \text{weakly in } L^p(\mathbf{R}^N \times Y; \mathbf{R}^N). \end{cases}$$

Similarly, the sequence $\mathcal{T}_\epsilon^Y(\chi_\Omega)(\mathcal{T}_\epsilon^Y(a_\epsilon)(x,y,\mathcal{T}_\epsilon^Y(\nabla u_\epsilon)))$ is bounded in $L^q(\mathbf{R}^N \times Y)$. Up to a subsequence, there is $\eta \in L^q(\mathbf{R}^N \times Y)$ such that

$$\mathcal{T}_\epsilon^Y(\chi_\Omega)\mathcal{T}_\epsilon^Y(a_\epsilon)(x,y,\mathcal{T}_\epsilon^Y(\nabla u_\epsilon)) \rightharpoonup \eta \quad \text{weakly in } L^q(\mathbf{R}^N \times Y).$$

We are now ready to obtain a first homogenized equation. For the test function $v_\epsilon \in W_0^{1,p}(\Omega)$, that we extend to $\mathbf{R}^N$, we have by Proposition 2.3

$$\frac{1}{|Y|}\int_{\mathbf{R}^N\times Y} \mathcal{T}_\epsilon^Y(\chi_\Omega)(\mathcal{T}_\epsilon^Y(a_\epsilon)(x,y,\mathcal{T}_\epsilon^Y(\nabla u_\epsilon))|\mathcal{T}_\epsilon^Y(\nabla u_\epsilon))\,\mathrm{d}x\,\mathrm{d}y = \int_\Omega f_\epsilon v_\epsilon\,\mathrm{d}x.$$

If $v_\epsilon = \Psi \in \mathcal{D}(\Omega)$, and if $\epsilon \to 0$, since by Theorem 2.6, $\mathcal{T}_\epsilon^Y(\nabla\Psi) \to \nabla\Psi \otimes 1$ strongly in $L^p(\mathbf{R}^N \times Y)$, we obtain

$$\frac{1}{|Y|}\int_{\Omega\times Y} (\eta(x,y)|\nabla\Psi(x))\,\mathrm{d}x\,\mathrm{d}y = \int_\Omega f\Psi\,\mathrm{d}x.$$

For $\varphi \in \mathcal{D}(\Omega)$ and $\psi \in W^{1,p}_{\mathrm{per}}(Y)$, consider $v_\epsilon(x) = \epsilon\varphi(x)\psi(x/\epsilon)$. The sequence v_ϵ converges weakly to 0 in $W^{1,p}(\Omega)$ while the unfolded sequence $\mathcal{T}_\epsilon^Y(\nabla v_\epsilon)$ converges strongly to $\varphi(x)\nabla\psi(y)$ in $L^p(\Omega; W^{1,p}_{\mathrm{per}}(Y))$, hence

$$\frac{1}{|Y|}\int_{\Omega\times Y} (\eta(x,y)|\varphi(x)\nabla\psi(y))\,\mathrm{d}x\,\mathrm{d}y = 0.$$

From the previous statement and the density of test functions v_ϵ in $L^p(\Omega; W^{1,p}_{\mathrm{per}}(Y))$, we deduce that for all $\Phi \in L^p(\Omega; W^{1,p}_{\mathrm{per}}(Y))$,

$$\frac{1}{|Y|}\int_{\Omega\times Y} (\eta(x,y)|\nabla_y\Phi(x,y))\,\mathrm{d}x\,\mathrm{d}y = 0.$$

We are now going to prove that the sequence $\mathcal{T}_\epsilon^Y(\nabla u_\epsilon)$ converges in fact strongly. First, we see that

$$\begin{aligned}
&\int_{\mathbf{R}^N\times Y} \mathcal{T}_\epsilon^Y(\chi_\Omega)|\nabla_x u_0 + \nabla_y\hat{u} - \mathcal{T}_\epsilon^Y(\nabla u_\epsilon)|^p\,\mathrm{d}x\,\mathrm{d}y\\
&\quad\le \left(\int_{\mathbf{R}^N\times Y} \mathcal{T}_\epsilon^Y(\chi_\Omega)\big(1 + |\nabla_x u_0 + \nabla_y\hat{u}| + |\mathcal{T}_\epsilon^Y(\nabla u_\epsilon)|\big)^p\,\mathrm{d}x\,\mathrm{d}y\right)^{1-p/\beta}\\
&\qquad\qquad \left(\int_{\mathbf{R}^N\times Y} \frac{\mathcal{T}_\epsilon^Y(\chi_\Omega)|\nabla_x u_0 + \nabla_y\hat{u} - \mathcal{T}_\epsilon^Y(\nabla u_\epsilon)|^\beta}{(1 + |\nabla_x u_0 + \nabla_y\hat{u}| + |\mathcal{T}_\epsilon^Y(\nabla u_\epsilon)|)^{\beta-p}}\,\mathrm{d}x\,\mathrm{d}y\right)^{p/\beta}.
\end{aligned}$$

Since $\mathcal{T}_\epsilon^Y(\chi_\Omega)\mathcal{T}_\epsilon^Y(\nabla u_\epsilon)$ converges weakly in $L^p(\mathbf{R}^N \times Y; \mathbf{R}^N)$, thanks to the fact that $(\mathcal{T}_\epsilon^Y(\chi_\Omega))_\epsilon$ is bounded in $L^\infty(\mathbf{R}^N \times \mathbf{R}^N)$ and converges almost everywhere, the first factor in the right-hand side is bounded as $\epsilon \to 0$ since the unfolded operator $\mathcal{T}_\epsilon^Y(a_\epsilon)$ enjoys the same monotonicity property (5.1) as a_ϵ, we have

$$\begin{aligned}
&\int_{\mathbf{R}^N\times Y} \frac{\mathcal{T}_\epsilon^Y(\chi_\Omega)|\nabla_x u_0 + \nabla_y\hat{u} - \mathcal{T}_\epsilon^Y(\nabla u_\epsilon)|^\beta}{(1 + |\nabla_x u_0 + \nabla_y\hat{u}| + |\mathcal{T}_\epsilon^Y(\nabla u_\epsilon)|)^{\beta-p}}\,\mathrm{d}x\,\mathrm{d}y\\
&\quad\le c\int_{\mathbf{R}^N\times Y} \mathcal{T}_\epsilon^Y(\chi_\Omega)\big(\mathcal{T}_\epsilon^Y(a_\epsilon)(x,y,\nabla_x u_0 + \nabla_y\hat{u}) - \mathcal{T}_\epsilon^Y(a_\epsilon)(x,y,\mathcal{T}_\epsilon^Y(\nabla u_\epsilon))\big|\\
&\qquad\qquad\qquad\qquad \nabla_x u_0 + \nabla_y\hat{u} - \mathcal{T}_\epsilon^Y(\nabla u_\epsilon)\big)\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

We need to prove that the right-hand side goes to 0 as $\epsilon \to 0$. First, by Proposition 2.3,

$$\frac{1}{|Y|}\int_{\mathbf{R}^N\times Y} \mathcal{T}_\epsilon^Y(\chi_\Omega)(\mathcal{T}_\epsilon^Y(a_\epsilon)(x,y,\mathcal{T}_\epsilon^Y(\nabla u_\epsilon))|\mathcal{T}_\epsilon^Y(\nabla u_\epsilon))\,\mathrm{d}x\,\mathrm{d}y \\ = \int_\Omega f_\epsilon u_\epsilon\,\mathrm{d}x \to \int_\Omega f u_0\,\mathrm{d}x.$$

while the weak convergence of $\mathcal{T}_\epsilon^Y(\chi_\Omega)\mathcal{T}_\epsilon^Y(a_\epsilon)(x,y,\mathcal{T}_\epsilon^Y(\nabla u_\epsilon)))$ to η previously established yields

$$\frac{1}{|Y|}\int_{\mathbf{R}^N\times Y} \mathcal{T}_\epsilon^Y(\chi_\Omega)(\mathcal{T}_\epsilon^Y(a_\epsilon)(x,y,\mathcal{T}_\epsilon^Y(\nabla u_\epsilon))|\nabla_x u_0+\nabla_y\hat{u})\,\mathrm{d}x\,\mathrm{d}y \\ \to \frac{1}{|Y|}\int_{\Omega\times Y}(\eta(x,y)|\nabla_x u_0\nabla_y\hat{u})\,\mathrm{d}x\,\mathrm{d}y = \int_\Omega f u_0\,\mathrm{d}x.$$

Now, it remains to prove that

$$\int_{\mathbf{R}^N\times Y} \mathcal{T}_\epsilon^Y(\chi_\Omega)(\mathcal{T}_\epsilon^Y(a_\epsilon)(x,y,\nabla_x u_0+\nabla_y\hat{u})|\nabla_x u_0+\nabla_y\hat{u}-\mathcal{T}_\epsilon^Y(\nabla u_\epsilon))\,\mathrm{d}x\,\mathrm{d}y \to 0.$$

Since $\nabla_x u_0+\nabla_y\hat{u}-\mathcal{T}_\epsilon^Y(\nabla u_\epsilon)\rightharpoonup 0$ weakly in $L^p(\mathbf{R}^N\times Y;\mathbf{R}^N)$, the strong convergence of

$$\mathcal{T}_\epsilon^Y(a_\epsilon)(x,y,\nabla_x u_0+\nabla_y\hat{u})$$

in $L^q(\mathbf{R}^N\times Y)$ would be sufficient. This follows from Lebesgue dominated convergence Theorem. One has

$$\chi_{\Omega\times Y}a_{\text{hom}}(x,y,\nabla_x u_0+\nabla_y\hat{u})-\mathcal{T}_\epsilon^Y(\chi_\Omega)\mathcal{T}_\epsilon^Y(a_\epsilon)(x,y,\nabla_x u_0+\nabla_y\hat{u})^q \to 0,$$

for almost every $(x,y)\in\mathbf{R}^N\times Y$, while

$$|\chi_{\Omega\times Y}a_{\text{hom}}(x,y,\nabla_x u_0+\nabla_y\hat{u})-\mathcal{T}_\epsilon^Y(\chi_\Omega)\mathcal{T}_\epsilon^Y(a_\epsilon)(x,y,\nabla_x u_0+\nabla_y\hat{u})|^q \\ \le C(1+|\nabla_x u_0+\nabla_y\hat{u}|^{p-1})^q.$$

Hence we have just proved that $\mathcal{T}_\epsilon^Y(\chi_\Omega)\mathcal{T}_\epsilon^Y(\nabla u_\epsilon)\to\chi_{\Omega\times Y}(\nabla_x u_0+\nabla_y\hat{u})$ strongly in $L^p(\mathbf{R}^N\times Y)$ as $\epsilon\to 0$.

Finally we prove that $\eta(x,y)=a_{\text{hom}}(x,y,\nabla_x u_0+\nabla_y\hat{u})$, this will follow from the strong convergence

$$\mathcal{T}_\epsilon^Y(\chi_\Omega)\mathcal{T}_\epsilon^Y(a_\epsilon)(x,y,\mathcal{T}_\epsilon^Y(\nabla u_\epsilon))\to\chi_{\Omega\times Y}a_{\text{hom}}(x,y,\nabla_x u_0+\nabla_y\hat{u}) \tag{5.11}$$

in $L^q(\mathbf{R}^N\times\mathbf{R}^N)$. Up to a subsequence, $\mathcal{T}_\epsilon^Y(\chi_\Omega)\mathcal{T}_\epsilon^Y(\nabla u_\epsilon)\to\chi_{\Omega\times Y}(\nabla_x u_0+\nabla_y\hat{u})$ almost everywhere and there is $g\in L^p(\mathbf{R}^N\times\mathbf{R}^N)$ such that

$$\mathcal{T}_\epsilon^Y(\chi_\Omega)\mathcal{T}_\epsilon^Y(\nabla u_\epsilon)\le g.$$

By Lemma 5.3, (5.11) occurs for almost every $x, y \in \mathbf{R}^N \times \mathbf{R}^N$. Moreover,

$$\mathcal{T}_\epsilon^Y(\chi_\Omega)\mathcal{T}_\epsilon^Y(a_\epsilon)(x, y, \mathcal{T}_\epsilon^Y(\nabla u_\epsilon)) \leq C(1 + |g|^{p-1}).$$

The strong convergence (5.11) in $L^q(\mathbf{R}^N \times \mathbf{R}^N)$ follows from Lebesgue's monotone convergence Theorem.

It remains to prove the corrector result. Recalling Theorem 2.12, and the strong convergence $\mathcal{T}_\epsilon^Y(\chi_\Omega)\mathcal{T}_\epsilon^Y(\nabla u_\epsilon) \to \chi_{\Omega\times Y}(\nabla_x u_0 + \nabla_y \hat{u})$ in $L^p(\mathbf{R}^N \times Y)$ as $\epsilon \to 0$, we deduce that

$$\chi_\Omega \nabla u_\epsilon - \mathcal{U}_\epsilon^Y(\chi_{\Omega\times\mathbf{R}^N}(\nabla u_0 \otimes 1 + \nabla_y \hat{u})) \to 0,$$

strongly in $L^p(\mathbf{R}^N)$ as ϵ goes to zero. Since,

$$\begin{aligned} & \|\mathcal{U}_\epsilon^Y(\chi_{\Omega\times\mathbf{R}^N}\nabla u_0 \otimes 1) - \chi_\Omega \nabla u_0\|_{L^p(\mathbf{R}^N)} \\ = \; & \|\mathcal{U}_\epsilon^Y(\chi_{\Omega\times\mathbf{R}^N}\nabla u_0 \otimes 1) - \mathcal{U}_\epsilon^Y\Big(\mathcal{T}_\epsilon^Y(\chi_\Omega \nabla u_0)\Big)\|_{L^p(\mathbf{R}^N)} \\ = \; & \frac{1}{|Y|^{1/p}}\|\chi_{\Omega\times\mathbf{R}^N}\nabla u_0 \otimes 1 - \mathcal{T}_\epsilon^Y(\chi_\Omega \nabla u_0)\|_{L^p(\mathbf{R}^N\times Y)}, \end{aligned}$$

therefore, recalling Theorem 2.7, we deduce that the previous expression tends to zero as ϵ goes to zero. □

Lemma 5.3. *Let $G \subset \mathbf{R}^N$ be open, $a_n : G \to \mathbf{R}^N$ and $a : G \to \mathbf{R}^N$. Suppose for every $\xi_1, \xi_2 \in G$,*

$$(a_n(\xi_1) - a_n(\xi_2)|\xi_1 - \xi_2) \geq 0,$$

and a is continuous. If for every $\xi \in G$, $a_n(\xi) \to a(\xi)$ as $n \to \infty$, then for every $\xi \in G$, there is a neighboorhood $V \ni \xi$ and n_0 such that for every $n \geq n_0$,

$$|a_n(\xi) - a(\xi)| \leq \epsilon.$$

Proof. Let $\zeta \in G$ and $\epsilon > 0$. Since a is continuous, there is $\delta > 0$ such that for every $\xi_1, \xi_2 \in B(\zeta, \delta)$,

$$|a(\xi_1) - a(\xi_2)| < \epsilon.$$

Let $(\zeta_0, \cdots, \zeta_N)$ be the vertices of a regular simplex S centered around ζ and contained in $B(\zeta, \delta)$. Since a_n converges pointwise, there is n_0 such that if $n \geq 0$, then for $0 \leq k \leq N$,

$$|a_n(\zeta_k) - a(\zeta_k)| \leq \epsilon.$$

Let V be the open simplex generated by $((\zeta+\zeta_k)/2)_{k=0}^N$. For every $\xi \in V$ and $1 \leq k \leq N$, by the monotonicity assumption on a

$$(a_n(\xi) - a(\xi)|\zeta_k - \xi) \leq (a_n(\zeta_k) - a(\zeta_k)|\zeta_k - \xi) + (a(\zeta_k) - a(\zeta)|\zeta_k - \xi) \leq 2\delta\epsilon.$$

By convexity, for every $\theta \in S$,

$$(a_n(\xi) - a(\xi)|\zeta_k - \theta) \le 2\delta\epsilon.$$

There is $c_N > 0$ which depends only on N such that for $\xi \in S$, $B(\xi, c_N\delta) \subset V$. Hence, for every $\theta \in B(0,1)$,

$$(a_n(\xi) - a(\xi)|\theta) = \frac{1}{c_N\delta}(a_n(\xi) - a(\xi)|(\zeta + c_N\delta\theta) - \zeta) \le 2\epsilon/c_N,$$

therefore $|a_n(\xi) - a(\xi)| \le 2\epsilon/c_N$. □

Remark 8. It was noticed by Damlamian that the continuity of a is not necessary since we can use maximal monotonicity arguments and this is the object of article [12] in which we deal with periodic homogenization for maximal monotone graph which are not necessarly univalued .

Remark 9. An important example is the case where

$$a_\epsilon(x,\xi) = a(x, x/\epsilon, \xi),$$

where a is Y-periodic with respect to its second variable and is continuous with respect to its first variable. In this case for almost every $(x,y,z) \in \mathbf{R}^N \times \mathbf{R}^N$, we easily see that

$$\mathcal{T}_\epsilon^Y(\chi_\Omega a_\epsilon)(x,y,\xi) \to \chi_{\Omega\times\mathbf{R}^N} a(x,y,\xi),$$

as ϵ goes to zero.

6 Reiterated homogenization

Now we show how to use the reiterated unfolding operator with some microscopic translation and apply it to the case of several small scales. For simplicity, we treat the case of two small scales.

Theorem 6.1. *Let $1 < p < \infty$, $p^{-1} + q^{-1} = 1$, Ω be a bounded open set. Let $a_\epsilon : \Omega \times \mathbf{R}^N \to \mathbf{R}^N$, with $a_\epsilon(.,\xi)$ mesurable for all $\xi \in \mathbf{R}^N$ and $a_\epsilon(x,.)$ continuous for almost all $x \in \Omega$, be such that*

- *there exists $\beta \ge \max(2,p)$ and $c > 0$ such that for all $x \in \Omega$, $\xi_1, \xi_2 \in \mathbf{R}^N$,*

$$\frac{|\xi_1 - \xi_2|^\beta}{(1 + |\xi_1| + |\xi_2|)^{\beta-p}} \le c(a_\epsilon(x,\xi_1) - a_\epsilon(x,\xi_2)|\xi_1 - \xi_2), \tag{6.1}$$

- *there exists $C > 0$ such that for all $x \in \Omega$ and $\xi \in \mathbf{R}^N$,*

$$|a_\epsilon(x,\xi)| \le C(1 + |\xi|^{p-1}), \tag{6.2}$$

- *and*

$$a_\epsilon(x,0) = 0 \text{ for all } x \in \Omega. \tag{6.3}$$

Let us assume that there exist two cells Y and Z such that for almost every $(x,y,z) \in \mathbf{R}^N \times \mathbf{R}^N \times \mathbf{R}^N$,

$$\mathcal{T}^Z_{\delta(\epsilon)}(\mathcal{T}^Y_{\epsilon,\gamma_\epsilon}(\chi_\Omega a_\epsilon))(x,y,z,\xi) \to \chi_{\Omega\times\mathbf{R}^N\times\mathbf{R}^N} a_0(x,y,z,\xi), \tag{6.4}$$

as ϵ goes to zero, with $\lim_{\epsilon\to 0}\delta(\epsilon) = 0$ and $a_0(x,y,z,\xi)$ is Carathéodory. Moreover, we assume that the sequence γ_ϵ is as follows

$$\gamma_\epsilon(x) = \delta(\epsilon)\{[x/\epsilon]_Y/\delta(\epsilon)\}_Z.$$

Let $f_\epsilon \in W^{-1,q}(\Omega)$ be such that $f_\epsilon \to f$ strongly in $W^{-1,q}(\Omega)$.
If $u_\epsilon \in W^{1,p}_0(\Omega)$ is the unique solution of the problem

$$\begin{cases} \displaystyle\int_\Omega (a_\epsilon(x,\nabla u_\epsilon)|\nabla\varphi)\,\mathrm{d}x = \int_\Omega f_\epsilon\varphi\,\mathrm{d}x, \\ \forall\varphi \in W^{1,p}_0(\Omega), \end{cases} \tag{6.5}$$

then,

$$u_\epsilon \rightharpoonup u_0 \text{ weakly in } W^{1,p}_0(\Omega), \text{ as } \epsilon \to 0.$$

Moreover, u_0 is the first term of the unique solution $(u_0,\hat{u},\tilde{u})$ of the variational problem

$$\begin{cases} u_0 \in W^{1,p}_0(\Omega),\ \hat{u} \in L^p(\Omega; W^{1,p}_{per}(Y)),\ \text{with } \displaystyle\int_Y \hat{u}(x,y)\,\mathrm{d}y = 0, \\ \text{and } \tilde{u} \in L^p(\Omega\times Y; W^{1,p}_{per}(Z)),\ \text{with } \displaystyle\int_Z \tilde{u}(x,y,z)\,\mathrm{d}z = 0, \\ \forall\Psi \in W^{1,p}_0(\Omega), \forall\Phi \in L^p(\Omega; W^{1,p}_{\mathrm{per}}(Y)), \forall\Theta \in L^p(\Omega\times Y; W^{1,p}_{\mathrm{per}}(Z)) \\ \displaystyle\frac{1}{|Y||Z|}\int_{\Omega\times Y\times Z}(b(x,y,z,\nabla_x u_0 + \nabla_y\hat{u} + \nabla_z\tilde{u})| \\ \qquad \nabla_x\Psi(x) + \nabla_y\Phi(x,y) + \nabla_z\Theta(x,y,z))\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \displaystyle\int_\Omega f\Psi\,\mathrm{d}x. \end{cases} \tag{6.6}$$

Furthermore, the following strong convergence holds, when ϵ goes to zero,

$$\mathcal{T}^Z_{\delta(\epsilon)}(\mathcal{T}^Y_{\epsilon,\gamma_\epsilon}(\nabla u_\epsilon)) \to \nabla_x u_0 + \nabla_y\hat{u} + \nabla_z\tilde{u} \text{ in } L^p(\Omega\times Y\times Z; \mathbf{R}^N).$$

Proof. The proof is done exacly as in Theorem 5.1 except for the choice of the test function in the first step. If $v_\epsilon = \Psi \in \mathcal{D}(\Omega)$, and if $\epsilon \to 0$, we obtain

$$\frac{1}{|Y||Z|}\int_{\Omega\times Y\times Z}(\eta(x,y,z)|\nabla\Psi(x))\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_\Omega f\Psi\,\mathrm{d}x.$$

For $\varphi \in \mathcal{D}(\Omega)$ and $\psi \in W^{1,p}_{\rm per}(Y)$, consider $v_\epsilon(x) = \epsilon\varphi(x)\psi(x/\epsilon)$. The sequence v_ϵ converges weakly to 0 in $W^{1,p}(\Omega)$ while the unfolded sequence $\mathcal{T}^Z_{\delta(\epsilon)}\Big(\mathcal{T}^Y_{\epsilon,\gamma_\epsilon}(\nabla v_\epsilon)\Big)$ converges strongly to $\varphi(x)\nabla\psi(y)$ in $L^p(\Omega; W^{1,p}_{\rm per}(Y))$, hence

$$\frac{1}{|Y||Z|}\int_{\Omega\times Y\times Z}(\eta(x,y,z)|\varphi(x)\nabla\psi(y))\,{\rm d}x\,{\rm d}y\,{\rm d}z = 0.$$

Next, if $\varphi \in \mathcal{D}(\Omega)$, $\psi \in \mathcal{D}(Y)$ and $\theta \in W^{1,p}_{\rm per}(Z)$, consider

$$w_\epsilon(x) = \epsilon\delta(\epsilon)\varphi(x)\psi(x/\epsilon)\theta(\frac{x+\epsilon\gamma_\epsilon - \epsilon\delta(\epsilon)\{[x/\epsilon]_Y/\delta(\epsilon)\}_Z}{\epsilon\delta(\epsilon)}).$$

We immediately see that

$$w_\epsilon(x) = \epsilon\delta(\epsilon)\varphi(x)\psi(x/\epsilon)\theta(\frac{x}{\epsilon\delta(\epsilon)}).$$

The sequence w_ϵ converges weakly to 0 in $W^{1,p}(\Omega)$ while the unfolded sequence $\mathcal{T}^Z_{\delta(\epsilon)}\Big(\mathcal{T}^Y_{\epsilon,\gamma_\epsilon}(\nabla w_\epsilon)\Big)$ converges strongly to $\varphi(x)\psi(y)\nabla\theta(z)$ in $L^p(\Omega; W^{1,p}_{\rm per}(Y))$, hence

$$\frac{1}{|Y||Z|}\int_{\Omega\times Y\times Z}(\eta(x,y,z)|\varphi(x)\nabla\psi(y))\,{\rm d}x\,{\rm d}y\,{\rm d}z = 0.$$

From the previous statement and the density of test functions v_ϵ in $L^p(\Omega; W^{1,p}_{\rm per}(Y))$ and w_ϵ in $L^p(\Omega\times Y; W^{1,p}_{\rm per}(Z))$ we deduce that for all $\Phi \in L^p(\Omega; W^{1,p}_{\rm per}(Y))$ and $\Theta \in L^p(\Omega\times Y; W^{1,p}_{\rm per}(Z))$

$$\frac{1}{|Y||Z|}\int_{\Omega\times Y\times Z}(\eta(x,y,z)|\nabla_y\Phi(x,y))\,{\rm d}x\,{\rm d}y\,{\rm d}z = 0,$$

and

$$\frac{1}{|Y||Z|}\int_{\Omega\times Y\times Z}(\eta(x,y,z)|\nabla_z\Theta(x,y,z))\,{\rm d}x\,{\rm d}y\,{\rm d}z = 0,$$

□

Remark 10. An important example is the case where

$$a_\epsilon(x,\xi) = a(x, x/\epsilon, x/(\epsilon\delta(\epsilon)), \xi),$$

where a is Y-periodic with respect to its second variable and Z-periodic with respect to its third variable, and is continuous with respect to its first two variables. We recall that

$$\gamma_\epsilon(x) = \delta(\epsilon)\{[x/\epsilon]_Y/\delta(\epsilon)\}_Z.$$

In this case, for almost every $(x,y,z)\in\mathbf{R}^N$, we easily see that

$$\mathcal{T}^Z_{\delta(\epsilon)}(\mathcal{T}^Y_{\epsilon,\gamma_\epsilon}(\chi_\Omega a_\epsilon))(x,y,z,\xi) \to \chi_{\Omega\times\mathbf{R}^N\times\mathbf{R}^N}a(x,y,z,\xi),$$

This generalizes the results of [9] in which it was necessary to assume a stronger condition on $a(x,.,z,\xi)$.

Furthermore, from Theorem 4.2,we deduce the following result for the correctors:

Theorem 6.2. *We have the following strong convergences in $L^p(\Omega)$:*

$$\nabla_x u_\epsilon - \nabla_x u_0 \otimes 1 \otimes 1 - \mathcal{U}_\epsilon^Y\Big(\tau_{-\gamma_\epsilon}\mathcal{U}^Z_{\delta(\epsilon)}(\chi_{\Omega\times Y\times\mathbf{R}^N}\nabla_y\hat{u}\otimes 1)\Big)$$
$$- \quad \mathcal{U}_\epsilon^Y\Big(\tau_{-\gamma_\epsilon}\mathcal{U}^Z_{\delta(\epsilon)}(\chi_{\Omega\times Y\times\mathbf{R}^N}\nabla_z\tilde{u})\Big) \to 0.$$

Remark 11. As a particular case, Theorem 6.1 applies to the following situation: $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$, with Ω_1, Ω_2 disjoint lipschitzian open sets and $a_\epsilon(x,\xi)$ is such that $a_\epsilon(x,\xi) = a^1(x,\xi)$ if $x \in \Omega_1$ and

$$a_\epsilon(x,\xi) = a^2(x, \frac{x}{\epsilon}, \frac{x}{\epsilon\delta(\epsilon)}, \xi)$$

if $x \in \Omega_2$, where a^1 and a^2 are continuous with respect to every argument and satisfy (5.1), (5.2). This is more general than what was treated in [9].

Remark 12. As in the linear case, see [4, 6], Theorem 6.1 can be generalized to perforated domains.

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